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Bounds for eigenvalues of the Dirichlet problem for the logarithmic Laplacian¹

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Abstract

We provide bounds for the sequence of eigenvalues $\{\lambda_i(\Omega)\}_i$ of the Dirichlet problem

$$L_\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where L_Δ is the logarithmic Laplacian operator with Fourier transform symbol $2 \ln |\zeta|$. The logarithmic Laplacian operator is not positively defined if the volume of the domain is large enough. In this article, we obtain the upper and lower bounds for the sum of the first k eigenvalues by extending the Li-Yau method and Kröger's method respectively. Moreover, we show the limit of the quotient of the sum of the first k eigenvalues by $k \ln k$, is independent of the volume of the domain. Finally, we discuss the lower and upper bounds of the k -th principle eigenvalue, the asymptotic behavior of the limit of eigenvalues.

Keywords: Dirichlet eigenvalues; Logarithmic Laplacian.

MSC2010: 35P15; 35R09.

1 Introduction and main results

The logarithmic Laplacian L_Δ in \mathbb{R}^N ($N \geq 1$) is defined by

$$L_\Delta u(x) = c_N \int_{\mathbb{R}^N} \frac{u(x)1_{B_1(x)}(y) - u(y)}{|x - y|^N} dy + \rho_N u(x), \quad (1.1)$$

where

$$c_N := \pi^{-N/2} \Gamma(N/2) = \frac{2}{\omega_{N-1}}, \quad \rho_N := 2 \ln 2 + \psi\left(\frac{N}{2}\right) - \gamma, \quad (1.2)$$

$\omega_{N-1} := H^{N-1}(S^{N-1}) = \int_{S^{N-1}} dS$, $\gamma = -\Gamma'(1)$ is the Euler Mascheroni constant and $\psi = \frac{\Gamma'}{\Gamma}$ is the Digamma function associated to the Gamma function Γ . The natural domain of definition of L_Δ is the set of uniformly Dini continuous functions u in \mathbb{R}^N such that

$$\int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^N} dx < \infty.$$

The aim of this article is to provide estimates of the eigenvalues of the operator L_Δ in a bounded domain $\Omega \subset \mathbb{R}^N$, which are the real numbers λ such that there exists a solution to the Dirichlet problem

$$\begin{cases} L_\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.3)$$

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In recent years, there has been a renewed and increasing interest in the study of boundary value problems involving linear and nonlinear integro-differential operators. This growing interest is justified both by seminal advances in the understanding of nonlocal phenomena from a PDE or a probabilistic point of view, see e.g. [2–5, 12, 13, 16, 27, 30, 31] and the references therein, and by important applications. Among nonlocal differential order operators, the simplest and most studied examples, are the fractional powers of the Laplacian which exhibit many phenomenological properties. Recall that for $s \in (0, 1)$ the fractional Laplacian of a function $u \in C_c^\infty(\mathbb{R}^N)$ is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) := \int_{\mathbb{R}^N} e^{ix \cdot \xi} ((-\Delta)^s u)(x) dx = |\xi|^{2s} \widehat{u}(\xi) \quad \text{for all } \xi \in \mathbb{R}^N,$$

where and in the sequel both \mathcal{F} and $\widehat{\cdot}$ denote the Fourier transform. Equivalently, $(-\Delta)^s$ can be written as a singular integral operator under the following form

$$(-\Delta)^s u(x) = c_{N,s} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad (1.4)$$

where $c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$ and Γ is the Gamma function, see e.g. [31].

The fractional Laplacian has the following limiting properties when s approaches the values 0 and 1:

$$\lim_{s \rightarrow 1^-} (-\Delta)^s u(x) = -\Delta u(x) \quad \text{and} \quad \lim_{s \rightarrow 0^+} (-\Delta)^s u(x) = u(x) \quad \text{for } u \in C_c^2(\mathbb{R}^N),$$

see e.g. [12]. It is proved in [7] a remarkable expansion at $s = 0$ valid for $u \in C_c^2(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$,

$$(-\Delta)^s u(x) = u(x) + s L_\Delta u(x) + o(s) \quad \text{as } s \rightarrow 0^+$$

where, formally, the operator

$$L_\Delta := \left. \frac{d}{ds} \right|_{s=0} (-\Delta)^s \quad (1.5)$$

is given as a *logarithmic Laplacian*; indeed,

- (i) for $1 < p \leq \infty$, we have $L_\Delta u \in L^p(\mathbb{R}^N)$ and $\frac{(-\Delta)^s u - u}{s} \rightarrow L_\Delta u$ in $L^p(\mathbb{R}^N)$ as $s \rightarrow 0^+$;
- (ii) $\mathcal{F}(L_\Delta u)(\xi) = 2 \ln |\xi| \widehat{u}(\xi)$ for a.e. $\xi \in \mathbb{R}^N$.

Note that the problems with integral-differential operators given by kernels with a singularity of order $-N$ have received growing interest recently, as they give rise to interesting limiting regularity properties and Harnack inequalities without scaling invariance, see e.g. [20]. Another important domain of study consists in solving the Dirichlet problem with zero exterior value [7]. We refer to [15, 19] for more topics related to the logarithmic Laplacian and also [14, 18] for general nonlocal operator and related embedding results. Let $\mathbb{H}(\Omega)$ denote the space of all measurable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ with $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and

$$\iint_{\substack{x, y \in \mathbb{R}^N \\ |x - y| \leq 1}} \frac{(u(x) - u(y))^2}{|x - y|^N} dx dy < +\infty.$$

As we shall see it, $\mathbb{H}(\Omega)$ is a Hilbert space under the inner product

$$\mathcal{E}(u, w) = \frac{c_N}{2} \iint_{\substack{x, y \in \mathbb{R}^N \\ |x - y| \leq 1}} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^N} dx dy$$

and with the associated norm $\|u\|_{\mathbb{H}(\Omega)} = \sqrt{\mathcal{E}(u, u)}$, where c_N is given in (1.2). By [11, Theorem 2.1], the embedding $\mathbb{H}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Throughout this article we identify $L^2(\Omega)$ with

the space of functions in $L^2(\mathbb{R}^N)$ which vanish a.e. in $\mathbb{R}^N \setminus \Omega$. The quadratic form associated with L_Δ is well-defined on $\mathbb{H}(\Omega)$ by

$$\mathcal{E}_L : \mathbb{H}(\Omega) \times \mathbb{H}(\Omega) \rightarrow \mathbb{R}, \quad \mathcal{E}_L(u, w) = \mathcal{E}(u, w) - c_N \iint_{\substack{x, y \in \mathbb{R}^N \\ |x-y| \geq 1}} \frac{u(x)w(y)}{|x-y|^N} dx dy + \rho_N \int_{\mathbb{R}^N} uw dx,$$

where ρ_N is defined in (1.2). A function $u \in \mathbb{H}(\Omega)$ is an eigenfunction of (1.3) corresponding to the eigenvalue λ if

$$\mathcal{E}_L(u, \phi) = \lambda \int_{\Omega} u \phi dx \quad \text{for all } \phi \in \mathbb{H}(\Omega).$$

Proposition 1.1. [7, Theorem 1.4] *Let Ω be a bounded domain in \mathbb{R}^N . Then problem (1.3) admits a sequence of eigenvalues*

$$\lambda_1(\Omega) < \lambda_2(\Omega) \leq \dots \leq \lambda_i(\Omega) \leq \lambda_{i+1}(\Omega) \leq \dots$$

with corresponding eigenfunctions ϕ_i , $i \in \mathbb{N}$ such that the following holds:

(a) $\lambda_i(\Omega) = \min\{\mathcal{E}_L(u, u) : u \in \mathbb{H}_i(\Omega) : \|u\|_{L^2(\Omega)} = 1\}$, where

$$\mathbb{H}_1(\Omega) := \mathbb{H}(\Omega) \quad \text{and} \quad \mathbb{H}_i(\Omega) := \{u \in \mathbb{H}(\Omega) : \int_{\Omega} u \phi_j dx = 0 \text{ for } j = 1, \dots, i-1\} \quad \text{for } i > 1;$$

(b) $\{\phi_i : i \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\Omega)$;

(c) ϕ_1 is positive in Ω . Moreover, $\lambda_1(\Omega)$ is simple, i.e., if $u \in \mathbb{H}(\Omega)$ satisfies (1.3) in weak sense with $\lambda = \lambda_1(\Omega)$, then $u = t\phi_1$ for some $t \in \mathbb{R}$;

(d) $\lim_{i \rightarrow \infty} \lambda_i(\Omega) = +\infty$.

Due to lack of homogeneity of the logarithmic Laplacian there is no scaling property. As a consequence homothety of the domain is inoperative for studying the variations of principle eigenvalue as it is the case for the Laplacian or the fractional Laplacian. Secondly, the logarithmic Laplacian operator is no longer positively defined if $|\Omega|$ is too large: it is proved in [7] that the positivity of the principle eigenvalue is equivalent to the comparison principle, which does not hold in balls with large radius. These properties of the logarithmic Laplacian operator make more complicated the study of the asymptotics of the Dirichlet eigenvalues as we will see below, but also make more difficult the obtention of bounds for eigenvalues.

It is well-known that the Hilbert-Pólya conjecture is to associate the zero of the Riemann Zeta function with the eigenvalue of a Hermitian operator. This quest initiated the mathematical interest for estimating the sum of Dirichlet eigenvalues of the Laplacian while in physics the question is related to count the number of bound states of a one body Schrödinger operator and to estimate their asymptotic distribution. In 1912, Weyl in [33] shows that the k -th eigenvalue $\mu_k(\Omega)$ of Dirichlet problem with the Laplacian operator

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

has the asymptotic behavior $\mu_k(\Omega) \sim C_N(k|\Omega|)^{\frac{2}{N}}$ as $k \rightarrow +\infty$, where $C_N = (2\pi)^2|B_1|^{-\frac{2}{N}}$. Later on, Pólya [28] (in 1960) proved that

$$\mu_k(\Omega) \geq C \left(\frac{k}{|\Omega|} \right)^{\frac{2}{N}} \quad (1.7)$$

holds for $C = C_N$ and any "plane-covering domain" D in \mathbb{R}^2 , (his proof also works in dimension $N \geq 3$) and he also conjectured that (1.7) holds with $C = C_N$ for any bounded domain in \mathbb{R}^N .

Rozenbljum [29] and independently Lieb [25] proved (1.7) with a positive constant C for general bounded domain. Li-Yau [24] improved the value of the constant C obtaining $C = \frac{N}{N+2}C_N$, and with that constant (1.7) is also called Berezin-Li-Yau inequality because this constant is achieved with the help of Legendre transform as in the Berezin's earlier paper [1]. The Berezin-Li-Yau inequality then is generalized in [?, 9–11, 21, 25], for degenerate elliptic operators in [8, 17, 34] for the fractional Laplacian $(-\Delta)^s$ defined in (1.4) and the inequality reads

$$\mu_{s,k}(\Omega) \geq \frac{N}{N+2s} C_N \left(\frac{k}{|\Omega|} \right)^{\frac{2s}{N}}. \quad (1.8)$$

For eigenvalues of the Dirichlet problem (1.3), Laptev and Weth obtain in [23, Corollary 6.2] the following sharp estimate:

$$\lim_{\lambda \rightarrow +\infty} e^{-\frac{\lambda}{2}N} \mathcal{N}(\lambda) = \frac{|\Omega| \omega_{N-1}}{N(2\pi)^N},$$

where \mathcal{N} is the counting function of eigenvalues, i.e.

$$\mathcal{N}(t) = \sum_{j \in \mathbb{N}} \text{sgn}_+(t - \lambda_j(\Omega)) = \sum_{j \in \mathbb{N}} (t - \lambda_j(\Omega))_+^0.$$

Here $\text{sgn}_+(r) = 1$ if $r > 0$, $\text{sgn}_+(r) = 0$ if $r \leq 0$ and $r_{\pm} = (|r| \pm r)/2$ denote the positive and negative part of $x \in \mathbb{R}$. This estimate could give the Weyl's formula

$$\lim_{k \rightarrow +\infty} \frac{\lambda_k(\Omega)}{\ln k} = \frac{2}{N}$$

by some fundamental calculation. Furthermore lower bounds for the first eigenvalue are considered there by a particular scaling property.

Our purpose in this article is to provide the lower and upper bounds for the sum of the first k eigenvalues by developing the Berezin-Li-Yau methods and Kröger's result for the Laplacian.

The fundamental point for the Berezin-Li-Yau method for the lower bound is based on the homogeneity of $|\xi|^2$, the expression of the Fourier symbol of Laplacian ($\mathcal{F}(-\Delta)$). So this method could be extended for Dirichlet eigenvalues involving the fractional Laplacian. Thanks to the inhomogeneity of the expression of the Fourier symbol of L_{Δ} , the Berezin-Li-Yau method can not be applied to our problem (1.3) directly. To this end, we provide some appropriate estimates for the solutions of equations:

$$r \ln r = c \quad \text{and} \quad \frac{r}{\ln r - \ln \ln r} = t.$$

The estimates that we obtain provide a uniform lower bound of the sum of the first k -eigenvalues, independently of k , an estimate which has a particular interest when these eigenvalues are negative. More precisely, we prove the following inequalities:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\{\lambda_i(\Omega)\}_{i \in \mathbb{N}}$ be the sequence of eigenvalues of problem (1.3) obtained in Proposition 1.1 and define*

$$d_N = \frac{2\omega_{N-1}}{N^2(2\pi)^N}. \quad (1.9)$$

Then there holds

(i) for any $k \in \mathbb{N}^$,*

$$\sum_{i=1}^k \lambda_i(\Omega) \geq -d_N |\Omega|;$$

(ii) if $k > \frac{eNd_N}{2} |\Omega|$,

$$\sum_{i=1}^k \lambda_i(\Omega) > 0;$$

(iii) if $k \geq \frac{e^{e+1}Nd_N}{2}|\Omega|$,

$$\sum_{i=1}^k \lambda_i(\Omega) \geq \frac{2k}{N} \left(\ln k + \ln \left(\frac{2}{eNd_N|\Omega|} \right) - \ln \ln \left(\frac{2k}{eNd_N|\Omega|} \right) \right). \quad (1.10)$$

Our second interest is to give an upper bound for the sum of the first k eigenvalues. Motivated by Kröger's result for the Laplacian [21], we shall construct an upper bound by computing the related Rayleigh quotient via a particular complex valued function. Together with the lower bound (1.10), we can derive the limit of the sum of the first k eigenvalues as $k \rightarrow +\infty$. The results state as following.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\{\lambda_i(\Omega)\}_{i \in \mathbb{N}}$ be the sequence of eigenvalues of problem (1.3). Then for $k > \frac{eNd_N}{2}|\Omega|$,*

$$\sum_{i=1}^k \lambda_i(\Omega) \leq \frac{2k}{N} \left(\ln(k+1) + \ln \left(\frac{p_N}{|\Omega|} \right) + \frac{\omega_{N-1}}{\sqrt{|\Omega|}} \ln \ln \left(\frac{p_N(k+1)}{|\Omega|} \right) \right), \quad (1.11)$$

and

$$\lim_{k \rightarrow +\infty} (k \ln k)^{-1} \sum_{i=1}^k \lambda_i(\Omega) = \frac{2}{N}, \quad (1.12)$$

where $p_N = \frac{2(2\pi)^N N}{\omega_{N-1}}$.

Note that the assumption that $k > \frac{eNd_N}{2}|\Omega|$ is required to make sure that $\lambda_{k_0} > 0$, here k_0 is the smallest positive integer $k_0 \geq \frac{eNd_N}{2}|\Omega|$.

From the bounds of sum of eigenvalues from our main results, we can provide an analogous of the Wely's formula for the logarithmic Laplacian.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\{\lambda_i(\Omega)\}_{i \in \mathbb{N}}$ be the sequence of eigenvalues of problem (1.3). Then,*

$$(i) \quad \lim_{k \rightarrow +\infty} \frac{\lambda_k(\Omega)}{\ln k} = \frac{2}{N}, \quad (1.13)$$

and

$$(ii) \quad \frac{2}{N} \left(\ln k + \ln \frac{2}{eNd_N|\Omega|} \right) \leq \lambda_k(\Omega) \leq \frac{2}{N} \ln k + c_0 (\ln \ln(k+e))^2 + \frac{2}{N} \ln \frac{|B_1|}{|\Omega|}, \quad (1.14)$$

where $c_0 > 0$ is independent of k and Ω .

Remark 1.5. (a) Notice that the two limits of $\frac{\lambda_k(\Omega)}{\ln k}$ and of $(k \ln k)^{-1} \sum_{i=1}^k \lambda_i(\Omega)$ as $k \rightarrow +\infty$ have the same value $\frac{2}{N}$, which is independent of Ω ;

(b) The limit (1.13) is proved by showing the inequalities

$$\ln k + \ln \frac{2}{eNd_N|\Omega|} - \ln \ln \frac{2k}{eNd_N|\Omega|} \leq \frac{N}{2} \lambda_k(\Omega) \leq \ln k + 2 + \tilde{c}_0 (\ln \ln(k+1))^2 \quad (1.15)$$

for $k \geq \frac{e^{e+1}Nd_N}{2}|\Omega|$, where $\tilde{c}_0 > 0$ is independent of k , but depends of $|\Omega|$. Here the first and second inequalities follow by (1.10) and (1.11) respectively, along with the monotonicity of the sequence of eigenvalues.

Compared with (1.15), the inequalities (1.14) in Corollary 1.4 provide a sharper lower bound for $\lambda_k(\Omega)$, and a uniform upper bound c_0 , thanks to related estimates of \mathcal{N} from [23].

The rest of this paper is organized as follows. Section 2 is devoted to proving the lower bound by developing Li-Yau's method, and then we prove Theorem 1.2. In Section 3, we show the upper bounds for the first k -eigenvalues in Theorem 1.3. Finally, we prove the Weyl's limit of eigenvalues in Corollary 1.4 and obtain the sharper bounds of $\lambda_k(\Omega)$ obtained in Section 4.

2 Lower bounds

Let g be the auxiliary function defined by

$$g(r) = r \ln r \quad \text{for } r > 0.$$

Then $g(e) = e$, $g(1) = 0$ and $g(\frac{1}{e}) = -\frac{1}{e}$ and the following properties hold.

Lemma 2.1. *For $c \geq -\frac{1}{e}$, there exists a unique real number $r_c \geq \frac{1}{e}$ such that*

$$g(r_c) = c,$$

and we have that $r_c \leq 1 + c$. Furthermore,

(i) for $-\frac{1}{e} \leq c \leq 0$,

$$r_c \geq 1 + (e - 1)c \geq \frac{1}{e};$$

(ii) for $0 \leq c \leq e$,

$$r_c \geq 1 + \frac{e - 1}{e}c;$$

(iii) for $c \geq e$,

$$r_c \geq 1 + \frac{e - 1}{e}c$$

and

$$\frac{c}{\ln c} \leq r_c \leq \frac{c}{\ln c - \ln \ln c}. \quad (2.1)$$

Proof. The function g is increasing and g is convex from $[\frac{1}{e}, +\infty)$ onto $[-\frac{1}{e}, +\infty)$. Hence r_c is uniquely determined if $c \geq -\frac{1}{e}$ and $c \mapsto r_c$ is increasing from $[-\frac{1}{e}, +\infty)$ onto $[\frac{1}{e}, +\infty)$.

For $a > 0$, we define $\psi_a(x) = (1 + ax) \ln(1 + ax) - x$ for $x > -\frac{1}{a}$. Then $\psi_a(x) > 0$ (resp. $\psi_a(x) < 0$) is equivalent to $1 + ax > r_x$ (resp. $1 + ax < r_x$). Note that $\psi'_a(x) = a(1 + \ln(1 + ax)) - 1$ and $\psi''_a(x) = \frac{a^2}{1 + ax} > 0$. Since $\psi'_a(-\frac{1}{a}) = -\infty$ and ψ'_a is increasing, $\psi'_a(0) = a - 1$ is the maximum (resp. minimum) of ψ'_a on $(-\frac{1}{a}, 0]$ (resp. on $[0, \infty)$). Therefore, if $a > 1$, ψ_a is positive on $(-\frac{1}{a}, r_a^*)$ for some $r_a^* \in (-\frac{1}{a}, 0)$, negative on $(r_a^*, 0)$ and positive on $(0, \infty)$. If $0 < a < 1$, ψ_a is positive on $(-\frac{1}{a}, 0)$, negative on $(0, r_a^*)$ for some $r_a^* > 0$ and positive on (r_a^*, ∞) . If $a = 1$, ψ_1 is positive on $[-\frac{1}{e}, 0) \cup (0, \infty)$ and vanishes only at 0. Then $\psi_1 \geq 0$ which implies the first assertion.

Since $e - 1 > 1$ and $\psi_{e-1}(-\frac{1}{e}) = 0$, $\psi_{e-1}(x) < 0$ for $x \in (-\frac{1}{e}, 0)$. This gives (i).

Since $0 < \frac{e-1}{e} < 1$, $\psi_{\frac{e-1}{e}}$ is negative on $(0, r_{\frac{e-1}{e}}^*)$ and positive on $(r_{\frac{e-1}{e}}^*, \infty)$. Since $\psi_{\frac{e-1}{e}}(e) = 0$, $r_{\frac{e-1}{e}}^* = e$ and we get (ii) and (iii).

Since g is increasing on $[e, \infty)$, (2.1) is equivalent to

$$c - \frac{\ln c}{\ln \ln c} \leq c \leq \frac{c}{\ln c - \ln \ln c} \ln \left(\frac{c}{\ln c - \ln \ln c} \right) = c \frac{\ln c - \ln(\ln c - \ln \ln c)}{\ln c - \ln \ln c}.$$

Set $C = \ln c$, then

$$\frac{\ln c - \ln(\ln c - \ln \ln c)}{\ln c - \ln \ln c} = \frac{C - \ln(C - \ln C)}{C - \ln C} > 1 \quad \text{for } C > 1$$

and (2.1) follows. □

Lemma 2.2. *Let f be a real measurable function defined in \mathbb{R}^N with $0 \leq f \leq M_1$ a.e. and*

$$2 \int_{\mathbb{R}^N} \ln |z| f(z) dz = M_2.$$

Then

$$(i) \quad M_2 \geq -\frac{2\omega_{N-1}}{N^2}M_1,$$

and

$$(ii) \quad \int_{\mathbb{R}^N} f(z)dz \leq \frac{M_1\omega_{N-1}}{N} \left(e + \frac{N^2M_2}{2M_1\omega_{N-1}} \right) = \frac{e\omega_{N-1}}{N}M_1 + \frac{N}{2}M_2.$$

(iii) If we assume furthermore that $\frac{M_2}{M_1} \geq \frac{2e^2\omega_{N-1}}{N^2}$, there holds

$$\int_{\mathbb{R}^N} f(z)dz \leq \frac{NM_2}{2} \left(\ln \left(\frac{N^2M_2}{2eM_1\omega_{N-1}} \right) - \ln \ln \left(\frac{N^2M_2}{2eM_1\omega_{N-1}} \right) \right)^{-1}.$$

Proof. We have

$$\begin{aligned} \frac{M_2}{2} &= \int_{B_1} \ln |z|f(z)dz + \int_{B_1^c} \ln |z|f(z)dz \\ &\geq M_1 \int_{B_1} \ln |z|dz + \int_{B_1^c} \ln |z|f(z)dz \geq -\frac{\omega_{N-1}}{N^2}M_1. \end{aligned}$$

Hence (i) holds.

Let $R > 0$, we have that

$$(\ln |z| - \ln R)(f(z) - M_1\mathbf{1}_{B_R}) \geq 0.$$

By integration over \mathbb{R}^N we get

$$\frac{M_2}{2} + \frac{M_1\omega_{N-1}R^N}{N^2} \geq \ln R \int_{\mathbb{R}^N} f(z)dz.$$

Since R is arbitrary, $\int_{\mathbb{R}^N} f(z)dz$ satisfies the following upper estimate

$$\int_{\mathbb{R}^N} f(z)dz \leq \inf \left\{ A > 0 \text{ s.t. } \frac{M_2}{2} + \frac{M_1\omega_{N-1}R^N}{N^2} - A \ln R \geq 0 \text{ for all } R > 0 \right\}. \quad (2.2)$$

Set

$$\Theta_A(R) = \frac{M_2}{2} + \frac{M_1\omega_{N-1}R^N}{N^2} - A \ln R,$$

then Θ_A achieves the minimum if

$$\frac{M_1\omega_{N-1}R^N}{N} = A \iff R = R_A := \left(\frac{NA}{\omega_{N-1}M_1} \right)^{\frac{1}{N}}.$$

Hence

$$\Theta_A(R_A) = \frac{M_2}{2} + \frac{A}{N} - \frac{A}{N} \ln \left(\frac{NA}{\omega_{N-1}M_1} \right). \quad (2.3)$$

Put $r = \frac{NA}{M_1\omega_{N-1}}$, then

$$\Theta_A(R_A) \geq 0 \iff r \ln r - r \leq \frac{N^2M_2}{2M_1\omega_{N-1}} \iff g\left(\frac{r}{e}\right) \leq \frac{N^2M_2}{2eM_1\omega_{N-1}}. \quad (2.4)$$

Then $\frac{r}{e} \leq r_c$ with $c = \frac{N^2M_2}{2eM_1\omega_{N-1}}$, inequality $r_c \leq 1 + c$ in Lemma 2.1 yields

$$r = \frac{NA}{M_1\omega_{N-1}} \leq e + \frac{N^2M_2}{2M_1\omega_{N-1}} \implies \int_{\mathbb{R}^N} f(z)dz \leq \frac{M_1\omega_{N-1}}{N} \left(e + \frac{N^2M_2}{2M_1\omega_{N-1}} \right),$$

which is (ii).

Assuming now that $\frac{M_2}{M_1} \geq \frac{2e^2\omega_{N-1}}{N^2}$, we can apply Lemma 2.1-(iii) and get

$$\int_{\mathbb{R}^N} f(z)dz \leq \frac{NM_2}{2} \left(\ln \left(\frac{N^2M_2}{2eM_1\omega_{N-1}} \right) - \ln \ln \left(\frac{N^2M_2}{2eM_1\omega_{N-1}} \right) \right)^{-1},$$

which is (iii) and ends the proof. \square

Lemma 2.3. *Let*

$$\tilde{g}(r) = \frac{r}{\ln r - \ln \ln r} \quad \text{for } r > e.$$

Then for $t > \frac{e^e}{e-1}$, there exists a unique point $r_t > e$ such that $\tilde{g}(r_t) = t$. Furthermore,

$$t(\ln t - \ln \ln t) \leq r_t < t \ln t. \quad (2.5)$$

Proof. Since

$$\begin{aligned} \tilde{g}'(r) &= \frac{1}{\ln r - \ln \ln r} - \frac{1 - (\ln r)^{-1}}{(\ln r - \ln \ln r)^2} \\ &\geq \frac{1}{\ln r - \ln \ln r} \left(1 - \frac{1}{\ln r - \ln \ln r} \right) > 0, \end{aligned}$$

the function \tilde{g} is increasing from $(e, +\infty)$ onto $(\frac{e^e}{e-1}, +\infty)$. Setting $r_t^* = t(\ln t - \ln \ln t)$, then

$$\begin{aligned} \tilde{g}(r_t^*) &= \frac{t(\ln t - \ln \ln t)}{\ln t + \ln(\ln t - \ln \ln t) - \ln \ln(t(\ln t - \ln \ln t))} \\ &\leq \frac{t(\ln t - \ln \ln t)}{\ln t + \ln(\ln t - \ln \ln t) - \ln \ln(t \ln t)} \\ &= \frac{\ln t - \ln \ln t}{\ln t - \ln \frac{\ln(t \ln t)}{\ln t - \ln \ln t}} t \\ &\leq t, \end{aligned}$$

where the last inequality holds if

$$\frac{\ln(t \ln t)}{\ln t - \ln \ln t} \leq \ln t,$$

and this inequality is equivalent to

$$\tilde{h}(\tau) := \tau^2 - (\ln \tau + 1)\tau - \ln \tau \geq 0, \quad \tau = \ln t.$$

Freezing the coefficient $\ln \tau$, $\tilde{h}(\tau) = (\tau - \tau_1)(\tau - \tau_2)$, where the τ_1, τ_2 depend of τ , but $\tau_1 < 0 < \tau_2$, since $\tau_1\tau_2 = -\ln \tau < 0$. Because $\tilde{h}(e) = e^2 - 2e - 1 = 0.9584 \pm 10^{-4}$, we have $e > \tau_1$. Hence $\tau > e$ implies $\tau > \tau_1$ which in turn implies $\tilde{h}(\tau) > 0$. Hence $r_t^* \leq r_t$ using the monotonicity of \tilde{g} .

Let $s_t = t \ln t$, then

$$\tilde{g}(s_t) = \frac{t \ln t}{\ln t + \ln \ln t - \ln \ln(t \ln t)} < t$$

using the fact that

$$\ln \ln t - \ln \ln(t \ln t) < 0 \quad \text{for } t > e.$$

Hence $s_t \geq r_t$, which ends the proof. \square

Proof of Theorem 1.2. Denote

$$\Phi_k(x, y) = \sum_{j=1}^k \phi_j(x)\phi_j(y), \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

and

$$\widehat{\Phi}_k(z, y) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \Phi_k(x, y) e^{-ix \cdot z} dx,$$

where $\widehat{\Phi}_k$ is the Fourier transform with respect to x . Since the $\{\phi_j\}_{j \in \mathbb{N}}$ are orthogonal in $L^2(\Omega)$, we have the identity

$$\int_{\mathbb{R}^N} \int_{\Omega} |\widehat{\Phi}_k(z, y)|^2 dz dy = \int_{\Omega} \int_{\Omega} |\Phi_k(x, y)|^2 dx dy = k.$$

Furthermore, we note that

$$\begin{aligned} \int_{\Omega} |\widehat{\Phi}_k(z, y)|^2 dy &= \int_{\Omega} \left(\sum_{j=1}^k \widehat{\phi}_j(z) \phi_j(y) \right) \overline{\left(\sum_{j=1}^k \widehat{\phi}_j(z) \phi_j(y) \right)} dy \\ &= \int_{\Omega} \left(\sum_{j, \ell=1}^k \widehat{\phi}_j(z) \overline{\widehat{\phi}_\ell(z)} \phi_j(y) \phi_\ell(y) \right) dy \\ &= \sum_{j=1}^k |\widehat{\phi}_j(z)|^2. \end{aligned} \quad (2.6)$$

Using again the orthonormality of the $\{\phi_j\}_{j \in \mathbb{N}}$ in $L^2(\Omega)$, we infer by the k-dim Pythagore theorem,

$$\begin{aligned} \int_{\Omega} |\widehat{\Phi}_k(z, y)|^2 dy &= (2\pi)^{-N} \int_{\Omega} \left| \sum_{j=1}^k \left(\int_{\Omega} e^{-ix \cdot z} \phi_j(x) dx \right) \phi_j(y) \right|^2 dy \\ &\leq (2\pi)^{-N} |\Omega|. \end{aligned} \quad (2.7)$$

We have, from the Fourier expression of L_{Δ} applied in the variable x ,

$$\begin{aligned} \sum_{j=1}^k \lambda_j(\Omega) &= \int_{\Omega} \int_{\Omega} \Phi_k(x, y) L_{\Delta} \Phi_k(x, y) dy dx \\ &= 2 \sum_{j=1}^k \int_{\mathbb{R}^N} |\widehat{\phi}_j(z)|^2 \ln |z| dz \\ &= 2 \int_{\mathbb{R}^N} \left(\int_{\Omega} |\widehat{\Phi}_k(z, y)|^2 dy \right) \ln |z| dz. \end{aligned}$$

Now we apply Lemma 2.2 to the function

$$f(z) = \int_{\Omega} |\widehat{\Phi}_k(z, y)|^2 dy$$

with

$$M_1 = (2\pi)^{-N} |\Omega| \quad \text{and} \quad M_2 = \sum_{j=1}^k \lambda_j(\Omega).$$

Part (i): By Lemma 2.2 (i),

$$\sum_{j=1}^k \lambda_j(\Omega) \geq -\frac{2\omega_{N-1}}{N^2(2\pi)^N} |\Omega| = -d_N |\Omega|,$$

where d_N is the constant defined in (1.9).

Part (ii):

$$k = \int_{\mathbb{R}^N} f(z) dz \leq \frac{e\omega_{N-1}|\Omega|}{N(2\pi)^N} + \frac{N}{2} \sum_{j=1}^k \lambda_j(\Omega),$$

which implies that

$$\sum_{j=1}^k \lambda_j(\Omega) \geq \frac{2k}{N} - \frac{2e\omega_{N-1}|\Omega|}{N^2(2\pi)^N}.$$

Part (iii): for $k \in \mathbb{N}$, if

$$\sum_{j=1}^k \lambda_j(\Omega) \geq \frac{2e^2\omega_{N-1}|\Omega|}{N^2},$$

then

$$k \leq \frac{NM_2}{2} \left(\ln \left(\frac{N^2M_2}{2eM_1\omega_{N-1}} \right) - \ln \ln \left(\frac{N^2M_2}{2eM_1\omega_{N-1}} \right) \right)^{-1}.$$

Setting

$$r = \frac{N^2M_2}{2eM_1\omega_{N-1}} \quad \text{and} \quad t = \frac{Nk}{eM_1\omega_{N-1}} = \frac{(2\pi)^N Nk}{e\omega_{N-1}|\Omega|},$$

we have from (2.5) that

$$r \geq r_t \geq t(\ln t - \ln \ln t) \tag{2.8}$$

for any $t > e^e$, that is

$$k > \frac{e^{e+1}\omega_{N-1}|\Omega|}{(2\pi)^N N} = \frac{e^{e+1}Nd_N|\Omega|}{2}.$$

This implies

$$\frac{N^2M_2}{2eM_1\omega_{N-1}} \geq \frac{(2\pi)^N Nk}{e\omega_{N-1}|\Omega|} \left(\ln \left(\frac{(2\pi)^N Nk}{e\omega_{N-1}|\Omega|} \right) - \ln \ln \left(\frac{(2\pi)^N Nk}{e\omega_{N-1}|\Omega|} \right) \right),$$

from what we infer

$$\sum_{j=1}^k \lambda_j(\Omega) \geq \frac{2k}{N} \left(\ln \left(\frac{2k}{eNd_N|\Omega|} \right) - \ln \ln \left(\frac{2k}{eNd_N|\Omega|} \right) \right), \tag{2.9}$$

which completes the proof. \square

Remark 2.4. *Since the mapping $k \mapsto \lambda_k(\Omega)$ is nondecreasing, we have that*

$$\lambda_k(\Omega) \geq \frac{1}{k} \sum_{i=1}^k \lambda_i(\Omega) > 0 \quad \text{if} \quad k \geq \frac{eNd_N|\Omega|}{2}.$$

It is worth noticing that some similar conditions assuring the positivity of λ_k could be derived from [23, Corollary 3.2], which asserts that

$$\mathcal{N}(\lambda) \leq e^{\lambda N+1} \frac{1}{(2\pi)^N} |\Omega| |B_1|,$$

which implies

$$\lambda_k(\Omega) \geq \frac{1}{N} \left(\ln \left(k \frac{(2\pi)^N}{|\Omega| |B_1|} \right) - 1 \right) > 0 \quad \text{if} \quad k > \frac{e|\Omega| |B_1|}{(2\pi)^N}.$$

3 Upper bound

For any bounded complex valued functions u, v defined on Ω , there holds

$$L_\Delta(uv)(x) = u(x)L_\Delta v(x) + c_N \int_{B_1(x)} \frac{u(x) - u(\zeta)}{|x - \zeta|^N} v(\zeta) d\zeta. \quad (3.1)$$

Lemma 3.1. For $z \in \mathbb{R}^N \setminus \{0\}$, we denote

$$\mu_z(x) = e^{ix \cdot z}, \quad \forall x \in \mathbb{R}^N,$$

then

$$L_\Delta \mu_z(x) = (2 \ln |z|) \mu_z(x), \quad \forall x \in \mathbb{R}^N. \quad (3.2)$$

Proof. From Lemma 5.1 in the appendix, for any $z \in \mathbb{R}^N \setminus \{0\}$ the function $x \mapsto L_\Delta \mu_z(x)$ is continuous and bounded in \mathbb{R}^N . We first prove that for $s \in (0, 1)$,

$$(-\Delta)^s \mu_z(x) = |z|^{2s} \mu_z(x), \quad \forall x \in \mathbb{R}^N. \quad (3.3)$$

Since μ_z is a bounded, we can consider the distribution T_{μ_z} in \mathbb{R}^N with support in Ω defined by

$$\langle T_{\mu_z}, \zeta \rangle := \int_{\mathbb{R}^N} \mu_z(x) \zeta(x) dx \quad \text{for all } \zeta \in C_c^\infty(\Omega).$$

Then $\langle (-\Delta)^s T_{\mu_z}, \zeta \rangle := \langle T_{\mu_z}, (-\Delta)^s \zeta \rangle$. Since

$$\frac{d}{ds} (-\Delta)^s \zeta \Big|_{s=0} = L_\Delta \zeta$$

there holds in the sense of distributions

$$\frac{d}{ds} (-\Delta)^s T_{\mu_z} \Big|_{s=0} = L_\Delta T_{\mu_z},$$

in the sense that

$$\langle L_\Delta T_{\mu_z}, \zeta \rangle = \langle T_{\mu_z}, L_\Delta \zeta \rangle = \int_{\mathbb{R}^N} \mu_z L_\Delta \zeta dx.$$

By definition of the Fourier transform of distributions \mathcal{F} in the class $\mathcal{S}'(\mathbb{R}^N)$ (see [32]), there holds

$$\langle L_\Delta T_{\mu_z}, \zeta \rangle = \int_{\mathbb{R}^N} e^{iz \cdot x} L_\Delta \zeta(x) dx = \mathcal{F}(L_\Delta \zeta)(z) = 2 \ln |z| \mathcal{F}(\zeta)(z) = 2 \ln |z| \int_{\mathbb{R}^N} \mu_z(x) \zeta(x) dx$$

Since $x \mapsto L_\Delta \mu_z(x)$ is locally integrable, $L_\Delta T_{\mu_z} = T_{L_\Delta \mu_z}$, hence

$$\langle T_{L_\Delta \mu_z}, \zeta \rangle = \int_{\mathbb{R}^N} L_\Delta \mu_z \zeta dx = 2 \ln |z| \int_{\mathbb{R}^N} \mu_z \zeta dx.$$

Because ζ is arbitrary this implies $T_{L_\Delta \mu_z} = 2 \ln |z| \mu_z$, a.e. in \mathbb{R}^N and finally everywhere by continuity, which is the claim. \square

Next, let $\eta_0 \in C^1(\mathbb{R})$ be a nondecreasing real valued function such that $\|\eta_0'\|_{L^\infty} \leq 2$ satisfying

$$\eta_0(t) = 1 \quad \text{if } t \geq 1, \quad \eta_0(t) = 0 \quad \text{if } t \leq 0.$$

Since Ω is a bounded domain, there exists a C^1 domain $\mathcal{O} \subset \Omega$ such that $|\mathcal{O}| \geq \frac{3}{4}|\Omega|$. For $\sigma > 0$, we set again

$$w_\sigma(x) = \eta_0(\sigma^{-1} \bar{\rho}(x)), \quad \forall x \in \mathbb{R}^N. \quad (3.4)$$

where $\bar{\rho}(x) = \text{dist}(x, \partial\mathcal{O})$. Observe that $w_\sigma \in \mathbb{H}_0(\Omega)$ and

$$w_\sigma \rightarrow 1 \quad \text{in } \mathcal{O} \quad \text{as } \sigma \rightarrow 0^+.$$

Thus, there exists $\sigma_1 > 0$ such that for $\sigma \in (0, \sigma_1]$,

$$|\Omega| > \int_{\Omega} w_\sigma dx \geq \int_{\Omega} w_\sigma^2 dx > \frac{|\Omega|}{2}.$$

Lemma 3.2. *Let*

$$\mathcal{L}_z w_\sigma(x) = \int_{B_1(x)} \frac{w_\sigma(x) - w_\sigma(\zeta)}{|x - \zeta|^N} e^{-i\zeta \cdot z} d\zeta,$$

then there holds

$$\left| \mathcal{L}_z w_\sigma(x) \right| \leq \frac{2\omega_{N-1}}{\sigma} \quad \text{for } x \in \Omega.$$

Proof. Actually, if $x \in \Omega$, we have that

$$|w_\sigma(x) - w_\sigma(\zeta)| \leq \|Dw_\sigma\|_{L^\infty} |x - \zeta| \leq \sigma^{-1} \|\eta'_0\|_{L^\infty} |x - \zeta|,$$

then

$$\left| \int_{B_1(x)} \frac{w_\sigma(x) - w_\sigma(\zeta)}{|x - \zeta|^N} e^{-i\zeta \cdot z} d\zeta \right| \leq \frac{\|\eta'_0\|_{L^\infty}}{\sigma} \int_{B_1(x)} \frac{d\zeta}{|\zeta - x|^{N-1}} \leq \frac{2\omega_{N-1}}{\sigma},$$

since $\|\eta'_0\|_{L^\infty} \leq 2$. This ends the proof. \square

Proof of Theorem 1.3. We recall that $\Phi_k(x, y)$ and $\widehat{\Phi}_k(z, y)$ have been defined in the proof of Theorem 1.2. If we denote

$$\tilde{v}_{\sigma, z}(x) := v_\sigma(x, z) = w_\sigma(x) e^{-ix \cdot z},$$

the projection of v_σ onto the subspace of $L^2(\Omega)$ spanned by the ϕ_j for $1 \leq j \leq k$ can be written in terms of the Fourier transform of $w_\sigma \Phi_k$ with respect to the x -variable:

$$\int_{\Omega} v_\sigma(x, z) \Phi_k(x, y) dx = (2\pi)^{N/2} \mathcal{F}_x(w_\sigma \Phi_k)(z, y).$$

Put

$$v_{\sigma, k}(z, y) = v_\sigma(z, y) - (2\pi)^{N/2} \mathcal{F}_x(w_\sigma \Phi_k)(z, y)$$

and the Rayleigh-Ritz formula shows that

$$\lambda_{k+1}(\Omega) \int_{\Omega} |v_{\sigma, k}(z, y)|^2 dy \leq \int_{\Omega} \overline{v_{\sigma, k}(z, y)} L_{\Delta, y} v_{\sigma, k}(z, y) dy$$

for any $z \in \mathbb{R}^N$ and $\sigma > 0$, where the right hand side is a real value

$$\int_{\Omega} \overline{v_{\sigma, k}(z, y)} L_{\Delta, y} v_{\sigma, k}(z, y) dy = \int_{\mathbb{R}^N} \overline{v_{\sigma, k}(z, y)} L_{\Delta, y} v_{\sigma, k}(z, y) dy = \int_{\mathbb{R}^N} 2 \ln |\xi| \left| \mathcal{F}(v_{\sigma, k})(z, \xi) \right|^2 d\xi,$$

although $v_{\sigma, k}$ is a complex valued function. Then, integrating this last inequality with respect to z in $B_r \setminus B_1$, for $r > 1$, we obtain

$$\lambda_{k+1}(\Omega) \leq \inf_{\sigma > 0} \frac{\int_{B_r \setminus B_1} \int_{\Omega} \overline{v_{\sigma, k}(z, y)} L_{\Delta, y} v_{\sigma, k}(z, y) dy dz}{\int_{B_r \setminus B_1} \int_{\Omega} |v_{\sigma, k}(z, y)|^2 dy dz}.$$

By Pythagore's theorem, we have that

$$\int_{\Omega} |v_{\sigma, k}(z, y)|^2 dy = \int_{\Omega} |v_\sigma(z, y)|^2 dy - (2\pi)^N \int_{\Omega} \sum_{j=1}^k |\mathcal{F}_x(w_\sigma \phi_j)(z)|^2 \phi_j(y)^2 dy.$$

Integrating over $B_r \setminus B_1$ we obtain that

$$\int_{B_r \setminus B_1} \int_{\Omega} |v_{\sigma, k}(z, y)|^2 dy dz \geq \frac{\omega_{N-1} r^N}{N} \int_{\Omega} w_\sigma^2(y) dy - (2\pi)^N \sum_{j=1}^k \int_{B_r \setminus B_1} |\mathcal{F}_x(w_\sigma \phi_j)(z)|^2 dz.$$

On the other hand,

$$\begin{aligned} \int_{B_r \setminus B_1} \int_{\Omega} \overline{v_{\sigma,k}(z,y)} L_{\Delta,y} v_{\sigma,k}(z,y) dy dz &= \int_{B_r \setminus B_1} \int_{\Omega} \overline{v_{\sigma}(z,y)} L_{\Delta,y} v_{\sigma}(z,y) dy dz \\ &\quad - (2\pi)^N \int_{B_r \setminus B_1} \int_{\Omega} \overline{\mathcal{F}_x(w_{\sigma}\Phi_k)(z,y)} L_{\Delta,y} \mathcal{F}_x(w_{\sigma}\Phi_k)(z,y) dy dz, \end{aligned}$$

where

$$\int_{B_r \setminus B_1} \int_{\Omega} \overline{\mathcal{F}_x(w_{\sigma}\Phi_k)(z,y)} L_{\Delta,y} \mathcal{F}_x(w_{\sigma}\Phi_k)(z,y) dy dz = \sum_{j=1}^k \lambda_j(\Omega) \int_{B_r \setminus B_1} |\mathcal{F}_x(w_{\sigma}\phi_j)(z)|^2 dz,$$

and

$$\begin{aligned} \int_{B_r \setminus B_1} \int_{\Omega} \overline{v_{\sigma}(z,y)} L_{\Delta,y} v_{\sigma}(z,y) dy dz &\leq \int_{B_r \setminus B_1} \int_{\Omega} w_{\sigma}^2(y) |L_{\Delta,y} e^{-iy \cdot z}| dy dz + \int_{B_r \setminus B_1} \int_{\Omega} w_{\sigma}(y) |\mathcal{L}_z w_{\sigma}(y)| dy dz \\ &\leq \int_{B_r \setminus B_1} \int_{\Omega} w_{\sigma}^2(y) \ln |z| dy dz + \frac{2\omega_{N-1}}{\sigma} \int_{B_r \setminus B_1} \int_{\Omega} w_{\sigma}(y) dy dz \\ &= \frac{\omega_{N-1}}{N} \varrho_{2,\sigma} \left(r^N \ln r - \frac{1}{N} (r^N - 1) \right) + \frac{\omega_{N-1}^2}{N\sigma} \varrho_{1,\sigma} (r^N - 1) \\ &\leq \frac{\omega_{N-1}}{N} \varrho_{2,\sigma} r^N \ln r + \frac{\omega_{N-1}^2}{N\sigma} \varrho_{1,\sigma} r^N \end{aligned}$$

with

$$\varrho_{1,\sigma} = \int_{\Omega} w_{\sigma}(y) dy \quad \text{and} \quad \varrho_{2,\sigma} = \int_{\Omega} w_{\sigma}^2(y) dy.$$

Because of Parseval's identity, there holds

$$\int_{B_r \setminus B_1} |\mathcal{F}_x(w_{\sigma}\phi_i)(z)|^2 dz \leq \int_{\Omega} (w_{\sigma}\phi_i)^2 dx \leq 1.$$

If k_0 is the smallest positive integer such that $k_0 \geq \frac{\epsilon N d_N}{2} |\Omega|$, then $\lambda_{k_0}(\Omega) \geq 0$.

For $k \geq k_0$, we choose $r > 1$ such that

$$\frac{2\omega_{N-1} r^N \ln r}{N^2} \geq \frac{\omega_{N-1} r^N}{N} \iff r \geq e^{\frac{N}{2}} \quad \text{and} \quad \frac{\omega_{N-1} r^N}{N} |\Omega| > 2(2\pi)^N k,$$

then we have that

$$\lambda_{k+1}(\Omega) \leq \frac{\frac{\omega_{N-1} r^N}{N} \left(\varrho_{2,\sigma} \frac{2}{N} \ln \frac{r^N}{e} + \varrho_{1,\sigma} \frac{\omega_{N-1}}{\sigma} \right) - (2\pi)^N \sum_{j=1}^k \lambda_j(\Omega) \int_{B_r} |\mathcal{F}_x(w_{\sigma}\phi_j)(z)|^2 dz}{\frac{\omega_{N-1} r^N}{N} \varrho_{2,\sigma} - (2\pi)^N \sum_{j=1}^k \int_{B_r} |\mathcal{F}_x(w_{\sigma}\phi_j)(z)|^2 dz}$$

Denote

$$A_1 = \frac{\omega_{N-1} r^N}{N} \left(\varrho_{2,\sigma} \frac{2}{N} \ln \frac{r^N}{e} + \varrho_{1,\sigma} \frac{\omega_{N-1}}{\sigma} \right) \quad \text{and} \quad A_2 = \frac{\omega_{N-1} r^N}{N} \varrho_{2,\sigma},$$

then

$$\begin{aligned} 0 &\leq \frac{A_1 - (2\pi)^N \sum_{j=1}^k \lambda_j(\Omega) \int_{B_r} |\mathcal{F}_x(w_{\sigma}\phi_j)(z)|^2 dz}{A_2 - (2\pi)^N \sum_{j=1}^k \int_{B_r} |\mathcal{F}_x(w_{\sigma}\phi_j)(z)|^2 dz} - \lambda_{k+1}(\Omega) \\ &= \frac{(A_1 - A_2 \lambda_{k+1}(\Omega)) + (2\pi)^N \sum_{j=1}^k \left(\lambda_{k+1}(\Omega) - \lambda_j(\Omega) \right) \int_{B_r} |\mathcal{F}_x(w_{\sigma}\phi_j)(z)|^2 dz}{A_2 - (2\pi)^N \sum_{j=1}^k \int_{B_r} |\mathcal{F}_x(w_{\sigma}\phi_j)(z)|^2 dz} \\ &\leq \frac{(A_1 - A_2 \lambda_{k+1}(\Omega)) + (2\pi)^N \sum_{j=1}^k \left(\lambda_{k+1}(\Omega) - \lambda_j(\Omega) \right)}{A_2 - (2\pi)^N k}, \end{aligned}$$

since $\lambda_{k+1}(\Omega) \geq \lambda_j(\Omega)$ for $j < k + 1$ and $\int_{B_r} |\mathcal{F}_x(w_\sigma \phi_j)(z)|^2 dz \in (0, 1)$. As a consequence, we obtain that

$$\lambda_{k+1}(\Omega) \leq \frac{\frac{\omega_{N-1} r^N}{N} \left(\varrho_{2,\sigma} \frac{2}{N} \ln \frac{r^N}{e} + \varrho_{1,\sigma} \frac{\omega_{N-1}}{\sigma} \right) - (2\pi)^N \sum_{j=1}^k \lambda_j(\Omega)}{\frac{\omega_{N-1} r^N}{N} \varrho_{2,\sigma} - (2\pi)^N k}, \quad (3.5)$$

where

$$\frac{\omega_{N-1} r^N}{N} \varrho_{2,\sigma} - (2\pi)^N k > \frac{\omega_{N-1} r^N}{N} \frac{|\Omega|}{2} - (2\pi)^N k > 0.$$

We fix $\sigma = \sigma_1$ and first impose $k, r > 1$ such that

$$\frac{\omega_{N-1} r^N}{N} \varrho_{2,\sigma} = (2\pi)^N (k + 1),$$

and take $r = k^{\frac{1}{N}}$ for $k \geq k_0$, then we recall that

$$\lambda_{k_0} \geq 0$$

and

$$\begin{aligned} (2\pi)^N \sum_{j=1}^{k+1} \lambda_j(\Omega) &\leq \frac{2\omega_{N-1}}{N^2} \varrho_{2,\sigma} r^N \ln \frac{r^N}{e} + \frac{\omega_{N-1}^2}{N} \sigma^{-1} |\Omega|^{\frac{1}{2}} \sqrt{\varrho_{2,\sigma}} r^N \\ &\leq (2\pi)^N \frac{2(k+1)}{N} \left(\ln(k+1) + \ln \frac{p_N}{\varrho_{2,\sigma}} + \frac{\omega_{N-1}}{\varrho_{2,\sigma}} \ln \ln \frac{p_N(k+1)}{\varrho_{2,\sigma}} \right) \\ &\leq (2\pi)^N \frac{2(k+1)}{N} \left(\ln(k+1) + \ln \frac{2p_N}{|\Omega|} + \frac{2\omega_{N-1}}{\sqrt{|\Omega|}} \ln \ln \frac{2p_N(k+1)}{|\Omega|} \right), \end{aligned}$$

where

$$p_N = \frac{2(2\pi)^N N}{\omega_{N-1}} \quad \text{and} \quad \frac{|\Omega|}{2} \leq \varrho_{2,\sigma} \leq |\Omega|.$$

Moreover, (1.12) follows by the lower bound (1.10) and the upper bound (1.11) directly. \square

4 Further discussion

Proof of Corollary 1.4 (i). On the one hand, it follows by the nondecreasing monotonicity of $k \mapsto \lambda_k(\Omega)$ and (1.10) that

$$\lambda_k(\Omega) \geq \frac{1}{k} \sum_{i=1}^k \lambda_i(\Omega) \geq \frac{2}{N} \left(\ln k + \ln \left(\frac{2}{eNd_N|\Omega|} \right) - \ln \ln \left(\frac{2k}{eNd_N|\Omega|} \right) \right). \quad (4.1)$$

On the other hand, we take $m = \lceil \frac{k}{\ln \ln k} \rceil + 1$ and obtain that

$$\begin{aligned} \lambda_{k+1}(\Omega) &\leq \frac{1}{m} \left(\sum_{j=1}^{k+m} \lambda_j(\Omega) - \sum_{j=1}^k \lambda_j(\Omega) \right) \\ &\leq \frac{2}{Nm} (k+m) \left(\ln(k+m) + 1 + \ln \left(\frac{p_N}{|\Omega|} \right) + \frac{\omega_{N-1}}{\sqrt{|\Omega|}} \ln \ln \left(\frac{p_N(k+1+m)}{|\Omega|} \right) \right) \\ &\quad - \frac{2}{Nm} k \left(\ln k + \ln \left(\frac{2}{eNd_N|\Omega|} \right) - \ln \ln \left(\frac{2k}{eNd_N|\Omega|} \right) \right) \\ &\leq \frac{2}{N} \ln(k+m) + \frac{2}{N} (\ln \ln k) \ln \left(1 + \frac{1}{\ln \ln k} \right) + c_1 \ln \ln k + c_2 (\ln \ln k)^2 \\ &\leq \frac{2}{N} \ln(k+1) + \frac{4}{N} + c_3 (\ln \ln(k+1))^2, \end{aligned} \quad (4.2)$$

where $c_1, c_2, c_3 > 0$ is independent of k . Thus, (1.15) holds true and we have that

$$\lim_{k \rightarrow +\infty} \frac{\lambda_k(\Omega)}{\ln k} = \frac{2}{N}.$$

We complete the proof. □

4.1 Proof of Corollary 1.4 (ii)

With the help of some estimates in [23], we can prove the bounds (1.14) for $\lambda_k(\Omega)$.

Proposition 4.1. *Let Ω be a bounded domain in \mathbb{R}^N and $\lambda_k(\Omega)$ be the k -th eigenvalue of Dirichlet problem (1.3). Then we have that for $k \geq 1$*

$$\lambda_k(\Omega) \leq \frac{2}{N} \ln k + c_3 (\ln \ln(k + e))^2 + \frac{2}{N} \ln \frac{1}{R},$$

where $R = \frac{|\Omega|}{|B_1|}$. In particular,

$$\lambda_1(\Omega) \leq 2 \ln \frac{1}{R} + c_3,$$

where $c_3 \in \mathbb{R}$ is independent of Ω .

Proof. Note that the constant c_0 in the second inequality (1.15) could be dependent of $|\Omega|$, but independent of the shape of Ω . Let

$$\Omega_R = \{R^{-\frac{1}{N}}x : x \in \Omega\}.$$

From (1.15) we obtain that

$$\lambda_k(\Omega_R) \leq \frac{2}{N} \ln k + c_4 (\ln \ln(k + e))^2, \quad k \geq \frac{e^{e+1} N d_N}{2} |B_1|,$$

where c_4 is independent of k and Ω_R . The above inequality holds for $k \geq 1$ only by adjusting the constant c_4 . It is shown a scaling property for logarithmic Laplacian [23, Lemma 2.5] that

$$\lambda_k(\Omega) = \lambda_k(t\Omega) - 2 \ln \frac{1}{t},$$

where $t\Omega = \{tx : x \in \Omega\}$. From the assumption that $\Omega_R = R^{-\frac{1}{N}}\Omega$, we have that

$$\lambda_k(\Omega) = \lambda_k(\Omega_R) + \frac{2}{N} \ln \frac{1}{R}.$$

We omit the left proof. □

Proposition 4.2. *Let Ω be a bounded domain in \mathbb{R}^N and $\lambda_k(\Omega)$ be the k -th eigenvalue of Dirichlet problem (1.3). Then we have that for $k \geq 1$*

$$\lambda_k(\Omega) \geq \frac{2}{N} \left(\ln k + \ln \frac{2}{e N d_N |\Omega|} \right).$$

In particular, $\lambda_1(\Omega) \geq \frac{2}{N} \left(-\ln |\Omega| + c_5 \right)$, where $c_5 = \ln \frac{2}{e N d_N}$.

Proof. It follows by [23, Lemma 2.5] that

$$\mathcal{N}(\lambda) \leq \frac{|\Omega| \omega_{N-1}}{N (2\pi)^N} e^{\frac{\lambda}{2} N + 1}.$$

Note that

$$\lambda_k(\Omega) \geq \lambda \text{ is equivalent to } \mathcal{N}(\lambda) \leq k,$$

and then

$$\lambda_k(\Omega) \geq \frac{2}{N} \ln \frac{N (2\pi)^N k}{|\Omega| \omega_{N-1}} = \frac{2}{N} \ln \frac{2k}{e N d_N |\Omega|}.$$

Then all inequalities follow. □

5 Appendix: definedness of $L_{\Delta}\mu_z(x)$

Lemma 5.1. *Let $z \in \mathbb{R}^N \setminus \{0\}$ and $\mu_z(x) = e^{ix \cdot z}$ for all $x \in \mathbb{R}^N$. Then $L_{\Delta}\mu_z(x)$ is well defined for any $x \in \mathbb{R}^N$, and $x \mapsto L_{\Delta}\mu_z(x)$ is continuous and bounded in \mathbb{R}^N .*

Proof. Since the function μ_z is bounded and smooth in \mathbb{R}^N , $L_{\Delta}\mu_z(x)$ is well defined if the following limit exists

$$\lim_{r \rightarrow +\infty} \int_{B_r(x) \setminus B_1(x)} \frac{\mu_z(y)}{|x-y|^N} dy.$$

If it holds, we have,

$$L_{\Delta}\mu_z(x) = c_N \int_{B_1(x)} \frac{\mu_z(x) - \mu_z(y)}{|x-y|^N} dy - c_N \lim_{r \rightarrow +\infty} \int_{B_r(x) \setminus B_1(x)} \frac{\mu_z(y)}{|x-y|^N} dy + \rho_N \mu_z(x).$$

We introduce the laplacian of μ_z ,

$$\Delta\mu_z(t) = \Delta e^{i \sum_j t_j z_j} = \Delta \left(\prod_j e^{it_j z_j} \right) = - \left(\sum_j z_j^2 \right) \prod_j e^{it_j z_j} = -|z|^2 \mu_z(t).$$

There holds

$$\int_{B_r(x) \setminus B_1(x)} \frac{\mu_z(y)}{|x-y|^N} dy = \mu_z(x) \int_{B_r \setminus B_1} \frac{\mu_z(t)}{|t|^N} dt = -\frac{\mu_z(x)}{|z|^2} \int_{B_r \setminus B_1} \frac{\Delta\mu_z(t)}{|t|^N} dt$$

By Green's formula

$$\int_{B_r \setminus B_1} |t|^{-N} \Delta\mu_z(t) dt = \int_{B_r \setminus B_1} \mu_z(t) \Delta |t|^{-N} dt + \int_{\partial(B_r \setminus B_1)} \left(|t|^{-N} \frac{\partial\mu_z(t)}{\partial n} - \mu_z(t) \frac{\partial |t|^{-N}}{\partial n} \right) dS(t)$$

We have

$$\begin{aligned} \Delta |t|^{-N} &= 2N |t|^{-N-2}, \\ \frac{\partial |t|^{-N}}{\partial n} &= \begin{cases} -Nr^{-N-1} & \text{on } \partial B_r \\ N & \text{on } \partial B_1 \end{cases} \end{aligned}$$

and

$$\frac{\partial\mu_z(t)}{\partial n} = \begin{cases} \frac{it \cdot z}{r} \mu_z(t) & \text{on } \partial B_r \\ -it \cdot z \mu_z(t) & \text{on } \partial B_1 \end{cases}$$

since n is the unit outward normal vector either on ∂B_r or on ∂B_1 . Letting $r \rightarrow \infty$, we obtain

$$\lim_{r \rightarrow +\infty} \int_{B_r(x) \setminus B_1(x)} \frac{\mu_z(y)}{|x-y|^N} dy = -\frac{\mu_z(x)}{|z|^2} \left(2N \int_{\mathbb{R}^N \setminus B_1} \frac{\mu_z(t)}{|t|^{N+2}} dt - \int_{\partial B_1} (N + it \cdot z) \mu_z(t) dS(t) \right).$$

Finally we obtain

$$\begin{aligned} L_{\Delta}\mu_z(x) &= c_N \int_{B_1(x)} \frac{\mu_z(x) - \mu_z(y)}{|x-y|^N} dy + \rho_N \mu_z(x) \\ &\quad + \frac{c_N \mu_z(x)}{|z|^2} \left(2N \int_{\mathbb{R}^N \setminus B_1} \frac{\mu_z(t)}{|t|^{N+2}} dt - \int_{\partial B_1} (N + it \cdot z) \mu_z(t) dS(t) \right). \end{aligned} \tag{5.1}$$

The boundedness and the continuity of $x \mapsto L_{\Delta}\mu_z(x)$ follow from this formula. \square

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