Bounds for eigenvalues of the Dirichlet problem for the logarithmic Laplacian

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Abstract

We provide bounds for the sequence of eigenvalues $\{\lambda_i(\Omega)\}_i$ of the Dirichlet problem

$$L_\Delta u = \lambda u \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ in } \ \mathbb{R}^N \setminus \Omega,$$

where L_Δ is the logarithmic Laplacian operator with Fourier transform symbol $2\ln|\zeta|$. The logarithmic Laplacian operator is not positively definitive if the volume of the domain is large enough, hence the principle eigenvalue is no longer always positive. We also give asymptotic estimates of the sum of the first k eigenvalues. To study the principle eigenvalue, we construct lower and upper bounds by a Li-Yau type method and calculate the Rayleigh quotient for some particular functions respectively. Our results point out the role of the volume of the domain in the bound of the principle eigenvalue. For the asymptotic of sum of eigenvalues, lower and upper bounds are built by a duality argument and by Kröger's method respectively. Finally, we obtain the limit of eigenvalues and prove that the limit is independent of the volume of the domain.

Keywords: Dirichlet eigenvalues; Logarithmic Laplacian.

MSC2010: 35P15; 35R09.

1 Introduction and main results

Let L_Δ be the logarthmic Laplacian in $\mathbb{R}^N$, $N \geq 1$, defined by

$$L_\Delta u(x) = c_N \int_{\mathbb{R}^N} \frac{u(x)1_{B_1(x)}(y) - u(y)}{|x-y|^N} dy + \rho_N u(x) \tag{1.1}$$

where

$$c_N := \pi^{-N/2}\Gamma(N/2) = \frac{2}{\omega_{N-1}}, \qquad \rho_N := 2\ln 2 + \psi(\tfrac{N}{2}) - \gamma, \tag{1.2}$$

$\omega_{N-1} := H^{N-1}(S^{N-1}) = \int_{S^{N-1}} dS$, $\gamma = -\Gamma'(1)$ is the Euler Mascheroni constant and $\psi = \frac{\Gamma'}{\Gamma}$ is the Digamma function.

The aim of this article is to provide estimates of the eigenvalues of the operator L_Δ in a bounded domain $\Omega \subset \mathbb{R}^N$ which are the real numbers λ such that there exists a solution to the Dirichlet problem

$$\begin{aligned} L_\Delta u &= \lambda u \quad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{1.3}$$

In recent years, there has been a renewed and increasing interest in the study of boundary value problems involving linear and nonlinear integro-differential operators. This growing interest

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is justified both seminal advances in the understanding of nonlocal phenomena from a PDE or a probabilistic point of view, see e.g. [3–6, 14, 15, 21, 32, 35, 36] and the references therein, and by important applications. Among nonlocal differential order operators, the simplest and most studied examples, are the fractional powers of the Laplacian which exhibit many phenomenological properties. Recall that, for $s \in (0,1)$, the fractional Laplacian of a function $u \in C_c^\infty(\mathbb{R}^N)$ is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\widehat{u}(\xi) \qquad \text{for all } \xi \in \mathbb{R}^N,$$

where and in the sequel both $\mathcal{F}$ and $\widehat{\cdot}$ denote the Fourier transform. Equivalently, $(-\Delta)^s$ can be written as a singular integral operator under the following form,

$$(-\Delta)^s u(x) = c_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \tag{1.4}$$

where $c_{N,s} = 2^{2s}\pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$ and Γ is the Gamma function, see e.g. [36].

The fractional Laplacian has the following limiting properties when s approaches the values 0 and 1:

$$\lim_{s\to 1^-} (-\Delta)^s u(x) = -\Delta u(x) \quad \text{and} \quad \lim_{s\to 0^+} (-\Delta)^s u(x) = u(x) \qquad \text{for } u \in C_c^2(\mathbb{R}^N),$$

see e.g. [14]. Recently, [8] shows a further expansion at $s = 0$ that

$$(-\Delta)^s u(x) = u(x) + sL_\Delta u(x) + o(s) \quad \text{as} \quad s \to 0^+$$

for $u \in C_c^2(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, where, formally, the operator

$$L_\Delta := \frac{d}{ds}\Big|_{s=0} (-\Delta)^s \tag{1.5}$$

is given as a *logarithmic Laplacian*; indeed,

(i) for $1 < p \le \infty$, we have $L_\Delta u \in L^p(\mathbb{R}^N)$ and $\frac{(-\Delta)^s u - u}{s} \to L_\Delta u$ in $L^p(\mathbb{R}^N)$ as $s \to 0^+$;

(ii) $\mathcal{F}(L_\Delta u)(\xi) = 2\ln|\xi|\,\widehat{u}(\xi)$ for a.e. $\xi \in \mathbb{R}^N$.

Note that the problems with integral-differential operators given by kernels with a singularity of order $-N$ have received growing interest recently, as they give rise to interesting limiting regularity properties and Harnack inequalities without scaling invariance, see e.g. [25]. Another important domain of study consists in understanding the eigenvalues of the Dirichlet problem with zero exterior value [8]. We refer to [19, 24] for more topics related to the logarithmic Laplacian and also [16, 23] for general nonlocal operator and related embedding results. Let $\mathbb{H}(\Omega)$ denote the space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ with $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and

$$\iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\le 1}} \frac{(u(x)-u(y))^2}{|x-y|^N} dxdy < +\infty.$$

As we will see it, $\mathbb{H}(\Omega)$ is a Hilbert space under the inner product

$$\mathcal{E}(u,w) = \frac{c_N}{2} \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\le 1}} \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^N} dxdy,$$

where c_N is given in (1.2), with associated norm $\|u\|_{\mathbb{H}(\Omega)} = \sqrt{\mathcal{E}(u,u)}$. By [13, Theorem 2.1], the embedding $\mathbb{H}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Throughout this article we identify $L^2(\Omega)$ with the space of functions in $L^2(\mathbb{R}^N)$ which vanish a.e. in $\mathbb{R}^N \setminus \Omega$. The quadratic form associated with L_Δ is well-defined on $\mathbb{H}(\Omega)$ by

$$\mathcal{E}_L : \mathbb{H}(\Omega)\times\mathbb{H}(\Omega) \to \mathbb{R}, \quad \mathcal{E}_L(u,w) = \mathcal{E}(u,w) - c_N \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\ge 1}} \frac{u(x)w(y)}{|x-y|^N} dxdy + \rho_N \int_{\mathbb{R}^N} uw\,dx,$$

where ρ_N is defined in (1.2). A function $u \in \mathbb{H}(\Omega)$ is an eigenfunction of (1.3) corresponding to the eigenvalue λ if

$$\mathcal{E}_L(u,\phi) = \lambda \int_\Omega u\phi\, dx \qquad \text{for all } \phi \in \mathbb{H}(\Omega).$$

Proposition 1.1. *[8, Theorem 1.4] Let Ω be a bounded domain in $\mathbb{R}^N$. Then problem (1.3) admits a sequence of eigenvalues*

$$\lambda_1(\Omega) < \lambda_2(\Omega) \le \cdots \le \lambda_i(\Omega) \le \lambda_{i+1}(\Omega) \le \cdots$$

and corresponding eigenfunctions ϕ_i, $i \in \mathbb{N}$ such that the following holds:

(a) $\lambda_i(\Omega) = \min\{\mathcal{E}_L(u,u) \,:\, u \in \mathbb{H}_i(\Omega) \,:\, \|u\|_{L^2(\Omega)} = 1\}$, *where*

$$\mathbb{H}_1(\Omega) := \mathbb{H}(\Omega) \quad \text{and} \quad \mathbb{H}_i(\Omega) := \{u \in \mathbb{H}(\Omega) \,:\, \int_\Omega u\phi_i\, dx = 0 \text{ for } i = 1, \dots i-1\} \quad \text{for } i > 1;$$

(b) $\{\phi_i \,:\, i \in \mathbb{N}\}$ *is an orthonormal basis of* $L^2(\Omega)$;

(c) ϕ_1 *is positive in* Ω. *Moreover,* $\lambda_1(\Omega)$ *is simple, i.e., if* $u \in \mathbb{H}(\Omega)$ *satisfies (1.3) in weak sense with* $\lambda = \lambda_1(\Omega)$, *then* $u = t\phi_1$ *for some* $t \in \mathbb{R}$;

(d) $\lim_{i\to\infty} \lambda_i(\Omega) = +\infty$.

Due to lack of the homogenous property for the logarithmic Laplacian operator, the effect of the domain for the principle eigenvalue can't be expected as the Laplacian or fractional Laplacian, just by scaling the domain by their homogeneous property of such operators. Secondly, the logarithmic Laplacian operator is no longer positively definitive if $|\Omega|$ is to large, since it is proved in [8] that the positivity of the principle eigenvalue is equivalent to the comparison principle, which does not hold for balls with large radius. These properties of the logarithmic Laplacian operator enrich the asymptotics of the Dirichlet eigenvalues as we will see below, but also make more difficult the obtention of bounds for eigenvalues.

Observe that the inclusion $\mathbb{H}(O_1) \subset \mathbb{H}(O_2)$ implies that the mapping $O \mapsto \lambda_1(O)$ is nonincreasing, i.e. $\lambda_1(O_1) \ge \lambda_1(O_2)$ if $O_1 \subset O_2$. Our first results deal with upper and lower bounds on the the principle eigenvalue and they are connected both with the measure and the distortion of the domain. *We denote by H^k the k-dimensional Hausdorff measure in $\mathbb{R}^N$ and for simplicity $H^N(E) = |E|$ for any Borel set $E \subset \mathbb{R}^N$.* We also define the signed distance function to $\partial\Omega$ by

$$\rho(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \overline{\Omega} \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \in \overline{\Omega}^c \end{cases} \tag{1.6}$$

and, for $\nu > 0$, the internal and external foliations of $\partial\Omega$ by

$$T_\nu^+ = \Big\{x \in \mathbb{R}^N : \rho(x) = \nu\Big\} \subset \Omega \tag{1.7}$$

and

$$T_\nu^- = \Big\{x \in \mathbb{R}^N : \rho(x) = -\nu\Big\}, \tag{1.8}$$

respectively, and $T_\nu = T_\nu^+ \cup T_\nu^-$.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ and $\lambda_1(\Omega)$ be the principle eigenvalue of Dirichlet problem (1.3) obtained in Proposition 1.1.*

(i) For $R > 2$, if we assume that $B_R \subset \Omega \subset B_{2R}$, and that there exists $c_0 > 1$ depending only on N such that for any $\nu \in [0, \frac{1}{2})$, there holds

$$\frac{1}{c_0} R^{N-1} \le H^{N-1}(T_\nu^+) \le c_0 R^{N-1} \quad \text{if } N \ge 2. \tag{1.9}$$

Then for $R \geq \max\left\{2, \frac{Nc_0}{2\omega_{N-1}}\right\}$, *we have that*

$$\lambda_1(\Omega) \leq \omega_{N-1} \ln \frac{1}{R} + z_1(R), \tag{1.10}$$

where

$$z_1(R) = \rho_N + \omega_{N-1} \ln 2 + \frac{4c_0}{R}\Big(1 + \frac{c_0}{2\omega_{N-1}R}\Big).$$

(ii) For $R \in (0, \frac{1}{4})$, *if we assume again that* $B_R \subset \Omega \subset B_{2R}$, *and that there exists* $c_0 > 1$ *such that for any* $\nu \in [0, \frac{R}{4}]$, *there holds*

$$\frac{1}{c_0} R^{N-1} \leq H^{N-1}(T_\nu) \leq c_0 R^{N-1} \quad \text{if } N \geq 2. \tag{1.11}$$

Then

$$\lambda_1(\Omega) \leq 4 \ln \frac{1}{R} + c_1, \tag{1.12}$$

where $c_1 > 0$ *independent of* R.

Since the function z_1 is decreasing, estimate (1.10) indicates that there exists R^* such that $\lambda_1(\Omega) < 0$ when $R > R^*$. Furthermore,

$$\lambda_1(\Omega) \leq \omega_{N-1} \ln \frac{2}{R} + \rho_N + O(R^{-1}) \quad \text{as } R \to \infty.$$

Our upper bounds are obtained by considering the Rayleigh quotient $\lambda_1(\Omega) \leq \frac{\mathcal{E}(u,u)}{\int_\Omega u^2 dx}$ with the particular function $u = w_\sigma(x) = \min\{\max\{\sigma\rho(x), 0\}, 1\}$, where ρ is defined in (1.6) and $\sigma > 0$. However, the dominating term of upper bounds arises from $c_N \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\geq 1}} \frac{w_\sigma(x) w_\sigma(y)}{|x-y|^N} dx$ when R is large, while it does from $\mathcal{E}(w_\sigma, w_\sigma)$ for $R > 0$ small enough.

Next we prove lower bounds of the principle eigenvalue.

Theorem 1.3. *Assume that* Ω *is a bounded domain in* $\mathbb{R}^N$ *and* $\lambda_1(\Omega)$ *is the principle eigenvalue of Dirichlet problem (1.3) obtained in Proposition 1.1. Let*

$$d_N = \frac{2\omega_{N-1}}{N^2(2\pi)^N}. \tag{1.13}$$

Then we have that
(i)

$$\lambda_1(\Omega) \geq -d_N|\Omega|;$$

(ii) if $|\Omega| < \frac{2}{eNd_N}$,

$$\lambda_1(\Omega) > 0;$$

(iii) if $|\Omega| \leq \frac{2}{e^{e+1}Nd_N}$,

$$\lambda_1(\Omega) \geq \frac{2}{N}\left(\ln\Big(\frac{2}{eNd_N|\Omega|}\Big) - \ln\ln\Big(\frac{2}{eNd_N|\Omega|}\Big)\right),$$

where e *is the Euler number.*

We summarize our results by the following table of the main asymptotic term of principle eigenvalue with respect to the volume of domain Ω from the upper bound in Theorem 1.2 and the lower bound in Theorem 1.3 in the particular where $\Omega = B_R$ where $0 < r_0 < 1 < R_0 < +\infty$:

R	$(0, r_0)$	$(R_0, +\infty)$
Upper bound of $\lambda_1(\Omega)$	$2\Big(1+\frac{c_0}{2\omega_{N-1}}\Big)\ln\frac{1}{R}$	$\omega_{N-1}\ln\frac{1}{R}$
Lower bound of $\lambda_1(\Omega)$	$\frac{2}{N}\ln\frac{1}{R}$	$-\frac{2e\omega_{N-1}}{N^2 d_N}R^N$

The mian order asymptotic of principle eigenvalue in a ball.

From above table we note that the lower bound of principle eigenvalue for large value R is rather unprecise.

The Hilbert-Pólya conjecture is to associate the zero of the Riemann Zeta function with the eigenvalue of a Hermitian operator. This quest initiated the mathematical interest for estimating the sum of Dirichlet eigenvalues of the Laplacian while in physics the question is related to count the number of bound states of a one body Schrödinger operator and to study their asymptotic distribution. In 1912, Weyl in [37] shows that the k-th eigenvalue $\mu_k(\Omega)$ of Dirichlet problem with the Laplacian operator has the asymptotic behavior $\mu_k(\Omega) \sim C_N(k|\Omega|)^{\frac{2}{N}}$ as $k \to +\infty$, where $C_N = (2\pi)^2|B_1|^{-\frac{2}{N}}$. Later, Pólya [33] (in 1960) proved that

$$\mu_k(\Omega) \geq C\left(\frac{k}{|\Omega|}\right)^{\frac{2}{N}} \tag{1.14}$$

holds for $C = C_N$ and any "plane-covering domain" D in $\mathbb{R}^2$, (his proof also works in dimension $N \geq 3$) and he also conjectured that (1.14) holds with $C = C_N$ for any bounded domain in $\mathbb{R}^N$. Rozenbljium [34] and independently Lieb [29] proved (1.14) with a positive constant C for general bounded domain. Li-Yau [28] improved the constant $C = \frac{N}{N+2}C_N$, and with that constant (1.14) is also called Berezin-Li-Yau inequality because this constant is achieved with the help of Legendre transform as in the Berezin's earlier paper [2]. The Berezin-Li-Yau inequality then is generalized in [11–13,26,29,31], for degenerate elliptic operators in [9,22,38] for the fractional Laplacian $(-\Delta)^s$ defined in (1.4) and the inequality reads

$$\mu_{s,k}(\Omega) \geq \frac{N}{N+2s}C_N\left(\frac{k}{|\Omega|}\right)^{\frac{2s}{N}}. \tag{1.15}$$

Due to the expression of the Fourier symbol of L_Δ, Berezin-Li-Yau method can not be applied to our problem (1.3). Our results are based on the appropriate estimates for the solutions of equations:

$$r\ln r = c \quad \text{and} \quad \frac{r}{\ln r - \ln\ln r} = t.$$

The estimates that we obtain provide a uniform lower bound of the sum of the first k-eigenvalues, independently of k, an estimate which has a particular interest when these eigenvalues are negative. More precisely, we have the following inequalities:

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\{\lambda_i(\Omega)\}_{i\in\mathbb{N}}$ be the sequence of eigenvalues of problem (1.3) obtained in Proposition 1.1 and d_N be given in (1.13). Then there holds*

(i) for any $k \in \mathbb{N}^*$,

$$\sum_{i=1}^{k}\lambda_i(\Omega) \geq -d_N|\Omega|;$$

(ii) if $k > \frac{eNd_N}{2}|\Omega|$,

$$\sum_{i=1}^{k}\lambda_i(\Omega) > 0;$$

(iii) if $k \geq \frac{e^{e+1}Nd_N}{2}|\Omega|$,

$$\sum_{i=1}^{k}\lambda_i(\Omega) \geq \frac{2}{N}k\left(\ln k + \ln\frac{2}{eNd_N|\Omega|} - \ln\ln\Big(\frac{2k}{eNd_N|\Omega|}\Big)\right). \tag{1.16}$$

Using the monotonicity of the sequence of eigenvalues, we deduce the following lower bound for $\lambda_k(\Omega)$ from Theorem 1.4 part *(iii)*.

Corollary 1.5. *Under the assumption of Theorem 1.4, for* $k \geq \frac{e^{e+1}Nd_N}{2}|\Omega|$, *we have that*

$$\lambda_k(\Omega) > 0$$

and

$$\lambda_k(\Omega) \geq \frac{2}{N}\left(\ln k + \ln\Big(\frac{2}{eNd_N|\Omega|}\Big) - \ln\ln\Big(\frac{2k}{eNd_N|\Omega|}\Big)\right).$$

Our goal is to provide an upper bound for the sum of eigenvalues. Motivated by Kröger's result for the Laplacian [26], we prove the following upper bound.

Theorem 1.6. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and* $\{\lambda_i(\Omega)\}_{i\in\mathbb{N}}$ *be the sequence of eigenvalues of problem (1.3). Then for* $k > k_0 := ed_N|\Omega|$,

$$\sum_{i=1}^{k}\lambda_i(\Omega) \leq \frac{2}{N}k\Big(\ln k + 1 + \ln\frac{p_N}{|\Omega|} + \frac{\omega_{N-1}}{\sqrt{|\Omega|}}\ln\ln\frac{p_N(k+1)}{|\Omega|}\Big) \tag{1.17}$$

and

$$\lim_{k\to+\infty}(k\ln k)^{-1}\sum_{i=1}^{k}\lambda_i(\Omega) = \frac{2}{N}, \tag{1.18}$$

where $p_N = \frac{2(2\pi)^N N}{\omega_{N-1}}$.

Note that from (1.18) the limit of the sum of the first k-eigenvalues does not depend on the volume of Ω. Finally, we build the Wely's formula for the logarithmic Laplacian and indeed we have the following asymptotic estimate.

Theorem 1.7. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and* $\{\lambda_i(\Omega)\}_{i\in\mathbb{N}}$ *be the sequence of eigenvalues of problem (1.3). Then*

$$\lim_{k\to+\infty}\frac{\lambda_k(\Omega)}{\ln k} = \frac{2}{N}. \tag{1.19}$$

The rest of this paper is organized as follows. In Section 2, we build the upper bound of the principle eigenvalue of Theorem 1.2 by considering particular test functions in the Rayleigh quotient. Section 3 is devoted to proving the lower bound by developing Li-Yau's method, and then we prove Theorem 1.3, Theorem 1.4 and Corollary 1.5. In Section 4, we show the upper bounds for the first k-eigenvalues in Theorem 1.6. In Section 5, we give the Wely's limit of eigenvalues in Theorem 1.7. Finally, we annex a proof of identities for the fractional Laplacian.

2 Upper bounds for the principle eigenvalue

2.1 Large domain: proof of Theorem 1.2 -(i)

Set

$$\eta(t) = \min\{\max\{0,t\},1\} \quad \text{for all } t\in\mathbb{R},$$

and, for $\sigma > 0$,

$$w_\sigma(x) = \eta(\sigma^{-1}\rho(x)) \quad \text{for all } x\in\mathbb{R}^N. \tag{2.1}$$

Note that $w_\sigma \in \mathbb{H}(\Omega)$, $w_\sigma \to 1$ in Ω as $\sigma \to 0^+$, and for $\sigma \in (0, 2R]$,

$$w_\sigma(x) = \frac{1}{\sigma}\rho(x) \quad \text{for all } x \in \Omega.$$

Since $|\eta'| \le 1$ and the signed distance function ρ is a contraction mapping, there always hols

$$|w_\sigma(x) - w_\sigma(y)| \le \frac{1}{\sigma}|x - y| \quad \text{for all } x, y \in \mathbb{R}^N. \tag{2.2}$$

By definition of $\lambda_1(\Omega)$,

$$\lambda_1(\Omega) \le \inf_{\sigma>0} \frac{\mathcal{E}_L(w_\sigma, w_\sigma)}{\int_\Omega w_\sigma^2(x)dx}. \tag{2.3}$$

If $\sigma \ge 2R$, we have

$$\int_\Omega w_\sigma^2(x)dx = \sigma^{-2}\int_\Omega \rho^2 dx,$$

while if $\sigma < 2R$, there holds

$$\begin{aligned}\int_\Omega w_\sigma^2(x)dx &= \sigma^{-2}\int_{\Omega_\sigma} \rho^2 dx + \int_{\Omega\setminus\Omega_\sigma} dx \\ &= \sigma^{-2}\int_{\Omega_\sigma} \rho^2 dx + |\Omega| - |\Omega_\sigma|,\end{aligned}$$

where

$$\Omega_\sigma = \Big\{x \in \mathbb{R}^N : 0 < \rho(x) < \sigma\Big\} \subset \Omega. \tag{2.4}$$

Then

$$\begin{aligned}\int_\Omega w_\sigma^2(x)dx &= \sigma^{-2}\int_{\Omega_\sigma} \rho^2 dx + |\Omega| - |\Omega_\sigma| \\ &\ge |\Omega| - |\Omega_\sigma|\end{aligned}$$

Taking $\sigma \le \frac{1}{2}$, using (1.9) and the co-area formula since $|\nabla\rho(x)| = 1$, we have that

$$\begin{aligned}|\Omega| - |\Omega_\sigma| &= \int_{\Omega\setminus\Omega\sigma} |\nabla\rho(x)|dx = |\Omega| - \int_0^\sigma H^{N-1}(T_t^+)dt \\ &\ge |\Omega| - c_0\sigma R^{N-1} = |\Omega| - \frac{Nc_0\sigma}{\omega_{_{N-1}}R}|B_R|,\end{aligned}$$

since $N|B_1| = \omega_{_{N-1}}$. Hence, under the assumption of Theorem 1.2-(i), we have that

$$\int_\Omega w_\sigma^2(x)dx \ge \left(1 - \frac{Nc_0\sigma}{\omega_{_{N-1}}R}\right)|\Omega| \quad \text{for all } \sigma \in (0, \tfrac{1}{2}). \tag{2.5}$$

Concerning the term $\mathcal{E}_L(w_\sigma, w_\sigma)$, we have the following upper bound.

Lemma 2.1. *Under the assumption of Theorem 1.2-(i), let $\sigma = \frac{1}{4}$, then we have that*

$$\mathcal{E}_L(w_\sigma, w_\sigma) \le \frac{4c_0}{R}|\Omega| + \Big(\rho_N - \omega_{_{N-1}}\ln\frac{R}{2}\Big)\int_\Omega w_\sigma^2 dx. \tag{2.6}$$

Proof. We recall that

$$\mathcal{E}_L(w_\sigma, w_\sigma) = \mathcal{E}(w_\sigma, w_\sigma) - c_N \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\ge 1}} \frac{w_\sigma(x)w_\sigma(y)}{|x-y|^N}dxdy + \rho_N\int_\Omega w_\sigma^2\,dx$$

and our proof is divided into two parts.

Step 1 : Note that for $x, y \in \mathbb{R}^N$ such that $|x-y| \leq 1$, we have that

$$|w_\sigma(x) - w_\sigma(y)| = 0, \quad \forall\, (x,y) \in (\Omega \setminus \Omega_\sigma)^2 \cup (\Omega^c)^2.$$

Using (2.2), we have that

$$\begin{aligned}
\mathcal{E}(w_\sigma, w_\sigma) &= \frac{c_N}{2} \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\leq 1}} \frac{(w_\sigma(x) - w_\sigma(y))^2}{|x-y|^N} dxdy \\
&\leq c_N \sigma^{-2} \int_{\Omega_\sigma} \int_{B_1(x)} |x-y|^{2-N} dydx \\
&= \frac{c_N \omega_{_{N-1}}}{2\sigma^2} |\Omega_\sigma| \\
&\leq \frac{c_0}{R\sigma} |\Omega| = \frac{4c_0}{R} |\Omega|,
\end{aligned}$$

thanks to the identity $c_N \omega_{_{N-1}} = 2$.
Step 2 : We have that

$$c_N \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\geq 1}} \frac{w_\sigma(x) w_\sigma(y)}{|x-y|^N} dydx = c_N \int_\Omega \int_{\Omega \cap B_1^c(x)} \frac{w_\sigma(x) w_\sigma(y)}{|x-y|^N} dydx.$$

Note that for $\sigma = \frac{1}{4}$, we have that

$$\inf_{x\in\Omega} |\Omega \setminus (\Omega_\sigma \cup B_1(x))| \geq |\Omega| - \frac{1}{4} c_0 R^{N-1} - \frac{\omega_{_{N-1}}}{N}.$$

Set

$$\begin{aligned}
\mathcal{D}_1(\sigma, x) &= (\Omega \setminus B_1(x)) \cap \Omega_\sigma = \Omega_\sigma \cap B_1^c(x) \\
\mathcal{D}_2(\sigma, x) &= (\Omega \setminus B_1(x)) \cap (\Omega \setminus \Omega_\sigma) = \Omega \cap \Omega_\sigma^c \cap B_1^c(x).
\end{aligned}$$

Then $\mathcal{D}_1(\sigma,x) \cap \mathcal{D}_2(\sigma,x) = \emptyset$, $\mathcal{D}_1(\sigma,x) \cup \mathcal{D}_2(\sigma,x) = \Omega \cap B_1^c(x)$. If $x \in B_{\frac{R}{2}}$, then $B_{\frac{R}{2}}(x) \cap B_1^c(x) \subset \mathcal{D}_2(\sigma,x)$, which implies

$$\int_{\mathcal{D}_2(\sigma,x)} \frac{1}{|x-y|^N} dy \geq \int_{B_{\frac{R}{2}}(x) \cap B_1^c(x)} \frac{1}{|x-y|^N} dy = \omega_{_{N-1}} \ln \frac{R}{2}$$

and

$$\int_{\Omega \cap B_1^c(x)} \frac{w_\sigma(y)}{|x-y|^N} dy = \int_{\mathcal{D}_1(\sigma,x)} \frac{\sigma\rho(y)}{|x-y|^N} dy + \int_{\mathcal{D}_2(\sigma,x)} \frac{1}{|x-y|^N} dy > \omega_{_{N-1}} \ln \frac{R}{2},$$

since the first term of the right-hand side is positive. Thus, we obtain

$$\begin{aligned}
\int_\Omega \int_{\Omega \cap B_1^c(x)} \frac{w_\sigma(x) w_\sigma(y)}{|x-y|^N} dydx &\geq \omega_{_{N-1}} \ln \frac{R}{2} \int_\Omega w_\sigma(x) dx \\
&\geq \omega_{_{N-1}} \ln \frac{R}{2} \int_\Omega w_\sigma^2(x) dx.
\end{aligned}$$

As a consequence and since $\sigma = \frac{1}{4}$, we infer that

$$\mathcal{E}_L(w_\sigma, w_\sigma) \leq \frac{4c_0}{R} |\Omega| + \Big(\rho_N - \omega_{_{N-1}} \ln \frac{R}{2}\Big) \int_\Omega w_\sigma^2 dx.$$

We complete the proof. □

Proof of Theorem 1.2-(i). Since $R > \frac{Nc_0}{2\omega_{_{N-1}}}$, we have that

$$\Big(1 - \frac{Nc_0}{4\omega_{_{N-1}} R}\Big)^{-1} \leq 1 + \frac{Nc_0}{2\omega_{_{N-1}} R}$$

and

$$\lambda_1(\Omega) \le \frac{\mathcal{E}_L(w_\sigma, w_\sigma)}{\int_\Omega w_\sigma^2(x)dx} \le \frac{4Nc_0}{R}\Big(1 - \frac{c_0}{4\omega_{_{N-1}}R}\Big)^{-1} + \rho_N - \omega_{_{N-1}} \ln\frac{R}{2}$$
$$\le -\omega_{_{N-1}} \ln R + \rho_N + \omega_{_{N-1}} \ln 2 + \frac{4c_0}{R}\Big(1 + \frac{Nc_0}{2\omega_{_{N-1}}R}\Big),$$

which is the result. □

2.2 Small domain: proof of Theorem 1.2-(ii)

The following upper estimate of $\mathcal{E}_L(w_\sigma, w_\sigma)$ holds.

Lemma 2.2. *Under the assumptions of Theorem 1.2-(ii), there exists $c_2 > 0$ independent of R such that for $\sigma = \min\Big\{\frac{R}{4}, \frac{2R\omega_{N-1}}{Nc_0}\Big\}$,*

$$\mathcal{E}_L(w_\sigma, w_\sigma) \le \Big(2\ln\frac{1}{R} + c_2\Big)|\Omega| + \rho_N \int_\Omega w_\sigma^2\, dx. \tag{2.7}$$

Proof. Since $\Omega \subset B_{2R}$ and $R < \frac{1}{4}$, there holds

$$\mathcal{E}_L(w_\sigma, w_\sigma) = \mathcal{E}(w_\sigma, w_\sigma) + \rho_N \int_\Omega w_\sigma^2\, dx.$$

For $r > 0$, we denote

$$\Omega^{1,r} = \{x \in \mathbb{R}^N : |\rho(x)| < r\} \text{ and } \Omega^{2,r} = \{x \in \mathbb{R}^N : \rho(x) > -r\} = \Omega \cup \Omega^{1,r},$$

and for $r_1 > r_2 > 0$ set

$$\mathcal{A}^{r_1,r_2} = \Omega^{2,r_1} \setminus \Omega^{2,r_2} = \{x \in \mathbb{R}^N : -r_2 \ge \rho(x) > -r_1\}.$$

Note that for $r > 2R$, $\Omega \subset \Omega^{1,r} = \Omega^{2,r}$ since $\Omega \subset B_{2R}$. When $\sigma \in (0, \frac{R}{4}]$, we set

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ s.t. } |x-y| < 1 \text{ and } |w_\sigma(x) - w_\sigma(y)| > 0\}.$$

Then $\mathcal{D} \subset \mathcal{D}_1 \cup \mathcal{D}_2$, where

$$\mathcal{D}_1 = (\Omega^{1,\sigma} \times \Omega^{2,\sigma}) \cup (\Omega^{2,\sigma} \times \Omega^{1,\sigma}) \text{ and } \mathcal{D}_2 = (\mathcal{A}^{1,\sigma} \times \Omega) \cup (\Omega \times \mathcal{A}^{1,\sigma}).$$

Using (2.2), $|\Omega^{2,\sigma}| \le |B_{2R+\sigma}(x)|$, we obtain from the definition of $\mathcal{D}_1$ and since $c_N\omega_{_{N-1}} = 2$,

$$\begin{aligned}
\frac{c_N}{2}\iint_{\mathcal{D}_1} \frac{(w_\sigma(x) - w_\sigma(y))^2}{|x-y|^N}dxdy &\le c_N\sigma^{-2}\int_{\Omega^{1,\sigma}}\int_{\Omega^{2,\sigma}} |x-y|^{2-N}dydx \\
&\le c_N\sigma^{-2}\int_{\Omega^{1,\sigma}}\int_{B_{2R+\sigma}(x)} |x-y|^{2-N}dydx \\
&= \frac{c_N\omega_{_{N-1}}}{2}(\frac{2R}{\sigma} + 1)^2|\Omega^{1,\sigma}| \\
&\le 2c_0(\frac{2R}{\sigma} + 1)^2R^{N-1}\sigma \\
&\le \frac{Nc_Nc_0\sigma}{R}(\frac{2R}{\sigma} + 1)^2|\Omega|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{c_N}{2}\iint_{\mathcal{D}_2}\frac{(w_\sigma(x)-w_\sigma(y))^2}{|x-y|^N}dxdy &\le c_N\int_{\mathcal{A}^{1,\sigma}}\int_\Omega |x-y|^{-N}dydx\\ &= c_N\int_\Omega\int_{\mathcal{A}^{1,\sigma}}|x-y|^{-N}dxdy\\ &\le 2|\Omega|\ln\frac{2}{\sigma},\end{aligned}$$

since for any $y\in\Omega$, $x\in\mathcal{A}^{1,\sigma}\Longrightarrow x\in B_2(y)\setminus B_\sigma(y)$, which implies

$$\int_{\mathcal{A}^{1,\sigma}}|x-y|^{-N}dx\le\int_{B_2(y)\setminus B_\sigma(y)}|x-y|^{-N}dx=\omega_{N-1}\ln\frac{2}{\sigma}.$$

Taking $\sigma=\varrho_0 R$ with

$$\varrho_0=\min\left\{\frac{1}{4},\frac{\omega_{N-1}}{2Nc_0}\right\},$$

we obtain that

$$\begin{aligned}\mathcal{E}(w_\sigma,w_\sigma)=\frac{c_N}{2}\iint_{\substack{x,y\in\mathbb{R}^N\\|x-y|\le 1}}\frac{(w_\sigma(x)-w_\sigma(y))^2}{|x-y|^N}dxdy&\\ \le\Big(\frac{Nc_Nc_0\sigma}{R}(\frac{2R}{\sigma}+1)^2+2\ln\frac{2}{\sigma}\Big)|\Omega|&\\ <\Big(2\ln\frac{1}{R}+c_2\Big)|\Omega|,&\end{aligned}$$

where $c_2=\frac{81Nc_0}{2\omega_N}+4\ln 2$, which yields (2.7). □

End of the proof of Theorem 1.2-(ii). Estimate (2.5) is valid, hence, if $\sigma=\varrho_0 R$, we derive that $1-\frac{Nc_0\sigma}{R\omega_{N-1}}\le\frac{1}{2}$ and

$$\int_\Omega w_\sigma^2(x)dx\ge\frac{1}{2}|\Omega|.$$

Therefore,

$$\lambda_1(\Omega)\le\frac{\mathcal{E}_L(w_\sigma,w_\sigma)}{\int_\Omega w_\sigma^2(x)dx}\le 2\Big(2\ln\frac{1}{R}+c_2\Big)+\rho_N=4\ln\frac{1}{R}+2c_2+\rho_N,$$

which ends the proof. □

3 Lower bounds

Let

$$g(r)=r\ln r\quad\text{for } r>0,$$

then $g(e)=e$, $g(1)=0$ and $g(\frac{1}{e})=-\frac{1}{e}$.

Lemma 3.1. *For $c\ge-\frac{1}{e}$, there exists a unique point $r_c\ge\frac{1}{e}$ such that*

$$g(r_c)=c,$$

and we have that $r_c\le 1+c$. Furthermore,
(i) for $-\frac{1}{e}\le c\le 0$,

$$r_c\ge 1+(e-1)c\ge\frac{1}{e};$$

(ii) for $0 \le c \le e$,

$$r_c \ge 1 + \frac{e-1}{e}c;$$

(iii) for $c \ge e$,

$$r_c \ge 1 + \frac{e-1}{e}c$$

and

$$\frac{c}{\ln c} \le r_c \le \frac{c}{\ln c - \ln\ln c}. \tag{3.1}$$

Proof. The function g is increasing in $[\frac{1}{e}, +\infty)$ with value in $[-\frac{1}{e}, +\infty)$. Hence r_c is uniquely determined if $c \ge -\frac{1}{e}$, $c \mapsto r_c$ is increasing from $[-\frac{1}{e}, +\infty)$ onto $[\frac{1}{e}, +\infty)$, and g is convex.

For $a > 0$, we define $\psi_a(x) = (1+ax)\ln(1+ax) - x$ for $x > -\frac{1}{a}$. Then $\psi_a(x) > 0$ (resp. $\psi_a(x) < 0$) is equivalent to $1+ax > r_x$ (resp. $1+ax < r_x$). Note that $\psi_a'(x) = a(1+\ln(1+ax)) - 1$. Since $\psi_a'(-\frac{1}{a}) = -\infty$ and ψ_a' is increasing, $\psi_a'(0) = a-1$ is the maximal (resp. minimal) value of ψ_a' on $(-\frac{1}{a}, 0]$ (resp. on $[0, \infty)$). Therefore, if $a > 1$, ψ_a is positive on $(-\frac{1}{a}, r_a^*)$ for some $r_a^* \in (-\frac{1}{a}, 0)$, negative on $(r_a^*, 0)$ and positive on $(0, \infty)$. If $0 < a < 1$, ψ_a is positive on $(-\frac{1}{a}, 0)$, negative on $(0, r_a^*)$ for some $r_a^* > 0$ and positive on (r_a^*, ∞). If $a = 1$, ψ_1 is positive on $[-\frac{1}{e}, 0) \cup (0, \infty)$ and vanishes only at 0. Then $\psi_1 \ge 0$ implies the first assertion.

Since $e - 1 > 1$ and $\psi_{e-1}(-\frac{1}{e}) = 0$, $\psi_{e-1}(x) < 0$ for $x \in (-\frac{1}{e}, 0)$. This gives (i).

Since $0 < \frac{e-1}{e} < 1$, $\psi_{\frac{e-1}{e}}$ is negative on $(0, r^*_{\frac{e-1}{e}})$ and positive on $(r^*_{\frac{e-1}{e}}, \infty)$. Since $\psi_{\frac{e-1}{e}}(e) = 0$, $r^*_{\frac{e-1}{e}} = e$ and we get (ii) and (iii).

Since g is increasing on $[e, \infty)$, (3.1) is equivalent to

$$c - \frac{\ln c}{\ln\ln c} \le c \le \frac{c}{\ln c - \ln\ln c}\ln\left(\frac{c}{\ln c - \ln\ln c}\right) = c\frac{\ln c - \ln(\ln c - \ln\ln c)}{\ln c - \ln\ln c}.$$

Set $C = \ln c$, then

$$\frac{\ln c - \ln(\ln c - \ln\ln c)}{\ln c - \ln\ln c} = \frac{C - \ln C}{C - \ln C} > 1 \quad \text{for } C > 1$$

and (3.1) follows. □

Lemma 3.2. *Let* f *be a real-valued function defined in* $\mathbb{R}^N$ *with* $0 \le f \le M_1$ *and*

$$2\int_{\mathbb{R}^N} \ln|z|\, f(z)dx = M_2.$$

Then (i)

$$M_2 \ge -\frac{2\omega_{N-1}}{N^2}M_1;$$

(ii)

$$\int_{\mathbb{R}^N} f(z)dz \le \frac{M_1\omega_{N-1}}{N}\left(e + \frac{N^2M_2}{2M_1\omega_{N-1}}\right) = \frac{e\omega_{N-1}}{N}M_1 + \frac{N}{2}M_2;$$

(iii) Furthermore, if $\frac{M_2}{M_1} \ge \frac{2e^2\omega_{N-1}}{N^2}$, *there holds*

$$\int_{\mathbb{R}^N} f(z)dz \le \frac{NM_2}{2}\left(\ln\left(\frac{N^2M_2}{2eM_1\omega_{N-1}}\right) - \ln\ln\left(\frac{N^2M_2}{2eM_1\omega_{N-1}}\right)\right)^{-1}.$$

Proof. We have

$$\begin{aligned}\frac{M_2}{2} &= \int_{B_1}\ln|z|f(z)dz + \int_{B_1^c}\ln|z|f(z)dz \\ &\ge M_1\int_{B_1}\ln|z|dz + \int_{B_1^c}\ln|z|f(z)dz \ \ge -\frac{\omega_{N-1}}{N^2}M_1.\end{aligned}$$

Hence (i) holds.

For $R > 0$ we have that

$$(\ln|z| - \ln R)(f(z) - M_1 \mathbf{1}_{B_R}) \geq 0.$$

By integration over $\mathbb{R}^N$ we get

$$\frac{M_2}{2} + \frac{M_1 \omega_{N-1} R^N}{N^2} \geq \ln R \int_{\mathbb{R}^N} f(z)dz.$$

The estimate from above of $\int_{\mathbb{R}^N} f(z)dz$ is obtained by

$$\int_{\mathbb{R}^N} f(z)dz \leq \inf\Big\{A > 0 \text{ s.t. } \frac{M_2}{2} + \frac{M_1 \omega_{N-1} R^N}{N^2} - A \ln R \geq 0 \text{ for all } R > 0\Big\}. \tag{3.2}$$

Set

$$\Theta_A(R) = \frac{M_2}{2} + \frac{M_1 \omega_{N-1} R^N}{N^2} - A \ln R,$$

then Θ_A achieves the minimum if

$$\frac{M_1 \omega_{N-1} R^N}{N} = A \Longleftrightarrow R = R_A := \left(\frac{NA}{\omega_{N-1} M_1}\right)^{\frac{1}{N}}.$$

Hence

$$\Theta_A(R_A) = \frac{M_2}{2} + \frac{A}{N} - \frac{A}{N} \ln\left(\frac{NA}{\omega_{N-1} M_1}\right). \tag{3.3}$$

Put $r = \frac{NA}{M_1 \omega_{N-1}}$, then

$$\Theta_A(R_A) \geq 0 \Longleftrightarrow r \ln r - r \leq \frac{N^2 M_2}{2 M_1 \omega_{N-1}} \Longleftrightarrow g\left(\frac{r}{e}\right) \leq \frac{N^2 M_2}{2e M_1 \omega_{N-1}}. \tag{3.4}$$

Then $\frac{r}{e} \leq r_c$ with $c = \frac{N^2 M_2}{2e M_1 \omega_{N-1}}$, inequality $r_c \leq 1 + c$ in Lemma 3.1 yields

$$r = \frac{NA}{M_1 \omega_{N-1}} \leq e + \frac{N^2 M_2}{2 M_1 \omega_{N-1}} \Longrightarrow \int_{\mathbb{R}^N} f(z)dz \leq \frac{M_1 \omega_{N-1}}{N}\left(e + \frac{N^2 M_2}{2 M_1 \omega_{N-1}}\right),$$

which is (ii).

Assuming now $\frac{M_2}{M_1} \geq \frac{2e^2 \omega_{N-1}}{N^2}$, we can apply Lemma 3.1-(iii) and get

$$\int_{\mathbb{R}^N} f(z)dz \leq \frac{N M_2}{2}\left(\ln\left(\frac{N^2 M_2}{2e M_1 \omega_{N-1}}\right) - \ln\ln\left(\frac{N^2 M_2}{2e M_1 \omega_{N-1}}\right)\right)^{-1},$$

which is (iii) and ends the proof. □

Lemma 3.3. *Let*

$$\tilde{g}(r) = \frac{r}{\ln r - \ln\ln r} \quad \text{for } r > e.$$

Then for $t > \frac{e^e}{e-1}$, there exists a unique point $r_t > e$ such that $\tilde{g}(r_t) = t$. Furthermore,

$$t(\ln t - \ln\ln t) \leq r_t < t \ln t. \tag{3.5}$$

Proof. Since

$$\begin{aligned}\tilde{g}'(r) &= \frac{1}{\ln r - \ln\ln r} - \frac{1-(\ln r)^{-1}}{(\ln r - \ln\ln r)^2} \\ &\geq \frac{1}{\ln r - \ln\ln r}\Big(1 - \frac{1}{\ln r - \ln\ln r}\Big) > 0,\end{aligned}$$

the function $\tilde{g}$ is increasing from $(e, +\infty)$ onto $(\frac{e^e}{e-1}, +\infty)$. Setting $r_t^* = t(\ln t - \ln\ln t)$, then

$$\begin{aligned}\tilde{g}(r_t^*) &= \frac{t(\ln t - \ln\ln t)}{\ln t + \ln(\ln t - \ln\ln t) - \ln\ln(t(\ln t - \ln\ln t))} \\ &\leq \frac{t(\ln t - \ln\ln t)}{\ln t + \ln(\ln t - \ln\ln t) - \ln\ln(t\ln t)} \\ &= \frac{\ln t - \ln\ln t}{\ln t - \ln\frac{\ln(t\ln t)}{\ln t - \ln\ln t}} t \\ &\leq t,\end{aligned}$$

where the last inequality holds if

$$\frac{\ln(t\ln t)}{\ln t - \ln\ln t} \leq \ln t,$$

which is equivalent to

$$\tilde{h}(\tau) := \tau^2 - (\ln\tau + 1)\tau - \ln\tau \geq 0, \quad \tau = \ln t.$$

Freezing the coefficient $\ln\tau$, $\tilde{h}(\tau) = (\tau - \tau_1)(\tau - \tau_2)$, where the τ_1, τ_2 depend of τ, but $\tau_1 < 0 < \tau_2$, since $\tau_1\tau_2 = -\ln\tau < 0$. Because $\tilde{h}(e) = e^2 - 2e - 1 = 0.9584 \pm 10^{-4}$, we have $e > \tau_1$. Hence $\tau > e$ implies $\tau > \tau_1$ which in turn implies $\tilde{h}(\tau) > 0$. Hence $r_t^* \leq r_t$ using the monotonicity of $\tilde{g}$.

Let $s_t = t\ln t$, then

$$\tilde{g}(s_t) = \frac{t\ln t}{\ln t + \ln\ln t - \ln\ln(t\ln t)} < t$$

by the fact that

$$\ln\ln t - \ln\ln(t\ln t) < 0 \quad \text{for } t > e.$$

Hence $s_t \geq r_t$, which ends the proof. □

Proof of Theorem 1.4. Denote

$$\Phi_k(x,y) = \sum_{j=1}^{k} \phi_j(x)\phi_j(y), \qquad (x,y) \in \Omega\times\Omega,$$

and

$$\widehat{\Phi}_k(z,y) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \Phi_k(x,y) e^{ix\cdot z} dx,$$

where $\widehat{\Phi}_k$ is the Fourier transform with respect to x. Hence we have that

$$\int_{\mathbb{R}^N}\int_{\Omega} |\widehat{\Phi}_k(z,y)|^2 dzdy = \int_{\Omega}\int_{\Omega} |\Phi_k(x,y)|^2 dxdy = k$$

by the orthonormality of the $\{\phi_j\}_{i\in\mathbb{N}}$ in $L^2(\Omega)$. Furthermore, we note that

$$\begin{aligned}\int_{\Omega} |\widehat{\Phi}_k(z,y)|^2 dy &= \int_{\Omega} \left(\sum_{j=1}^{k} \widehat{\phi}_j(z)\phi_j(y)\right)\left(\sum_{j=1}^{k} \overline{\widehat{\phi}_j(z)}\phi_j(y)\right) dy \\ &= \int_{\Omega} \left(\sum_{j,\ell=1}^{k} \widehat{\phi}_j(z)\overline{\widehat{\phi}_\ell(z)}\phi_j(y)\phi_\ell(y)\right) dy \qquad (3.6) \\ &= \sum_{j=1}^{k} |\widehat{\phi}_j(z)|^2.\end{aligned}$$

Using again the orthonormality of the $\{\phi_j\}_{i\in\mathbb{N}}$ in $L^2(\Omega)$, we infer by the k-dim Pythagore theorem,

$$\begin{aligned}\int_\Omega |\widehat{\Phi}_k(z,y)|^2 dy &= (2\pi)^{-N}\int_\Omega \left|\sum_{j=1}^k \left(\int_\Omega e^{ix.z}\phi_j(x)dx\right)\phi_j(y)\right|^2 dy \\ &= (2\pi)^{-N}\sum_{j=1}^k \left|\int_\Omega e^{ix.z}\phi_j(x)dx\right|^2 \\ &\leq (2\pi)^{-N}|\Omega|.\end{aligned} \tag{3.7}$$

We have, from the Fourier expression of L_Δ,

$$\begin{aligned}\sum_{j=1}^k \lambda_j(\Omega) &= \int_\Omega\int_\Omega \Phi_k(x,y)L_\Delta\Phi_k(x,y)dydx \\ &= 2\sum_{j=1}^k \int_{\mathbb{R}^N} |\widehat{\phi}_j(z)|^2 \ln|z|dz \\ &= 2\int_{\mathbb{R}^N}\left(\int_\Omega |\widehat{\Phi}_k(z,y)|^2 dy\right)\ln|z|dz.\end{aligned}$$

Now we apply Lemma 3.2 to the function

$$f(z) = \int_\Omega |\widehat{\Phi}_k(z,y)|^2 dy$$

with

$$M_1 = (2\pi)^{-N}|\Omega| \quad \text{and} \quad M_2 = \sum_{j=1}^k \lambda_j(\Omega).$$

Part (i): By Lemma 3.2 (i),

$$\sum_{j=1}^k \lambda_j(\Omega) \geq -\frac{2\omega_{_{N-1}}}{N^2(2\pi)^N}|\Omega| = -d_N|\Omega|,$$

where d_N is constant defined in (1.13).
Part (ii):

$$k = \int_{\mathbb{R}^N} f(z)dz \leq \frac{e\omega_{_{N-1}}|\Omega|}{N(2\pi)^N} + \frac{N}{2}\sum_{j=1}^k \lambda_j(\Omega),$$

which implies that

$$\sum_{j=1}^k \lambda_j(\Omega) \geq \frac{2k}{N} - \frac{2e\omega_{_{N-1}}|\Omega|}{N^2(2\pi)^N}.$$

Part (iii): for $k\in\mathbb{N}$, if

$$\sum_{j=1}^k \lambda_j(\Omega) \geq \frac{2e^2\omega_{_{N-1}}}{N^2}|\Omega|,$$

then

$$k \leq \frac{NM_2}{2}\left(\ln\left(\frac{N^2M_2}{2eM_1\omega_{_{N-1}}}\right) - \ln\ln\left(\frac{N^2M_2}{2eM_1\omega_{_{N-1}}}\right)\right)^{-1}.$$

Setting

$$r = \frac{N^2 M_2}{2eM_1\omega_{_{N-1}}} \quad \text{and} \quad t = \frac{Nk}{eM_1\omega_{_{N-1}}} = \frac{(2\pi)^N Nk}{e\omega_{_{N-1}}|\Omega|},$$

we have from (3.5) that

$$r \geq r_t \geq t\,(\ln t - \ln\ln t)\,, \tag{3.8}$$

for any $t > e^e$, i.e.

$$k > \frac{e^{e+1}\omega_{_{N-1}}|\Omega|}{(2\pi)^N N} = \frac{e^{e+1}Nd_N}{2}|\Omega|.$$

This implies

$$\frac{N^2 M_2}{2eM_1\omega_{_{N-1}}} \geq \frac{(2\pi)^N Nk}{e\omega_{_{N-1}}|\Omega|}\left(\ln\Big(\frac{(2\pi)^N Nk}{e\omega_{_{N-1}}|\Omega|}\Big) - \ln\ln\Big(\frac{(2\pi)^N Nk}{e\omega_{_{N-1}}|\Omega|}\Big)\right),$$

from what we infer

$$\sum_{j=1}^{k}\lambda_j(\Omega) \geq \frac{2k}{N}\left(\ln\Big(\frac{2k}{eNd_N|\Omega|}\Big) - \ln\ln\Big(\frac{2k}{eNd_N|\Omega|}\Big)\right), \tag{3.9}$$

which completes the proof. □

Proof of Theorem 1.3 and Corollary 1.5. It is clear that Theorem 1.4 with $k = 1$ implies Theorem 1.3. From the inequality (3.9) we derive

$$\tilde{\lambda}_k(\Omega) := \frac{1}{k}\sum_{i=1}^{k}\lambda_i(\Omega) \geq \frac{2}{N}\left(\ln\left(\frac{2k}{eNd_N|\Omega|}\right) - \ln\ln\left(\frac{2k}{eNd_N|\Omega|}\right)\right).$$

Since $k \mapsto \lambda_k(\Omega)$ is nondecreasing, we conclude Corollary 1.5. □

4 Upper bounds for the sum of eigenvalues

For any bounded complex valued functions u, v defined on Ω, there holds

$$L_\Delta(uv)(x) = u(x)L_\Delta v(x) + c_N\int_{B_1(x)}\frac{u(x)-u(\zeta)}{|x-\zeta|^N}v(\zeta)d\zeta. \tag{4.1}$$

Lemma 4.1. *For $z \in \mathbb{R}^N$, we denote*

$$\mu_z(x) = e^{\mathrm{i}x\cdot z}, \quad \forall\, x \in \mathbb{R}^N,$$

then

$$L_\Delta\mu_z(x) = (2\ln|z|)\mu_z(x), \quad \forall\, x \in \mathbb{R}^N. \tag{4.2}$$

Proof. From Lemma A.1, we have that

$$(-\Delta)^s\mu_z(x) = |z|^{2s}\mu_z(x), \quad \forall\, x \in \mathbb{R}^N,$$

and from the property (1.5) of L_Δ since μ_z is bounded,

$$\begin{aligned} 0 &= \frac{(-\Delta)^s\mu_z(x) - |z|^{2s}\mu_z(x)}{s} \\ &= \frac{(-\Delta)^s\mu_z(x) - \mu_z(x)}{s} - \frac{|z|^{2s}-1}{s}\mu_z(x) \\ &\to L_\Delta\mu_z(x) - (2\ln|z|)\mu_z(x) \qquad \text{as} \quad s \to 0^+, \end{aligned}$$

hence,

$$L_\Delta\mu_z(x) = (2\ln|z|)\mu_z(x), \quad \forall\, x \in \mathbb{R}^N,$$

which is the claim. □

Next, let $\eta_0 \in C^1(\mathbb{R})$ be a nondecreasing real value function such that $\|\eta_0'\|_{L^\infty} \le 2$ satisfying

$$\eta_0(t) = 1 \ \text{ if } \ t \ge 1, \qquad \eta_0(t) = 0 \ \text{ if } \ t \le 0.$$

Since Ω is a bounded domain, there exists a C^1 domain $\mathcal{O} \subset \Omega$ such that $|\mathcal{O}| \ge \frac{3}{4}|\Omega|$. For $\sigma > 0$, we reset

$$w_\sigma(x) = \eta_0(\sigma^{-1}\bar{\rho}(x)), \quad \forall\, x \in \mathbb{R}^N. \tag{4.3}$$

where $\bar{\rho}(x) = \mathrm{dist}(x, \partial\mathcal{O})$. Observe that $w_\sigma \in \mathbb{H}_0(\Omega)$ and

$$w_\sigma \to 1 \quad \text{in } \ \mathcal{O} \ \text{ as } \ \sigma \to 0^+.$$

Thus, there exists $\sigma_1 > 0$ such that for $\sigma \in (0, \sigma_1]$,

$$|\Omega| > \int_\Omega w_\sigma \, dx \ge \int_\Omega w_\sigma^2 \, dx > \frac{|\Omega|}{2}.$$

Lemma 4.2. *Let*

$$\mathcal{L}_z w_\sigma(x) = \int_{B_1(x)} \frac{w_\sigma(x) - w_\sigma(\zeta)}{|x-\zeta|^N} e^{i\zeta\cdot z} d\zeta,$$

then there holds

$$\Big|\mathcal{L}_z w_\sigma(x)\Big| \le \frac{2\omega_{N-1}}{\sigma} \quad \text{for } \ x \in \Omega.$$

Proof. Actually, if $x \in \Omega$, we have that

$$|w_\sigma(x) - w_\sigma(\zeta)| \le \|Dw_\sigma\|_{L^\infty}|x-\zeta| \le \sigma^{-1}\|\eta_0'\|_{L^\infty}|x-\zeta|,$$

then

$$\Big|\int_{B_1(x)} \frac{w_\sigma(x) - w_\sigma(\zeta)}{|x-\zeta|^N} e^{i\zeta\cdot z} d\zeta\Big| \le \frac{\|\eta_0'\|_{L^\infty}}{\sigma} \int_{B_1(x)} \frac{d\zeta}{|\zeta - x|^{N-1}} \le \frac{2\omega_{N-1}}{\sigma},$$

since $\|\eta_0'\|_{L^\infty} \le 2$. This ends the proof. □

Proof of Theorem 1.6. We recall that $\Phi_k(x,y)$ and $\widehat{\Phi}_k(z,y)$ have been defined in the proof of Theorem 1.4. If we set

$$\tilde{v}_{\sigma,z}(x) := v_\sigma(x,z) = w_\sigma(x) e^{\mathrm{i}x\cdot z},$$

the projection of v_σ onto the subspace of $L^2(\Omega)$ spanned by the ϕ_j for $1 \le j \le k$ can be written in terms of the Fourier transform of $w_\sigma\Phi_k$ with respect to the x-variable:

$$\int_\Omega v_\sigma(x,z)\Phi_k(x,y)dx = (2\pi)^{N/2}\mathcal{F}_x(w_\sigma\Phi_k)(z,y).$$

Set

$$v_{\sigma,k}(z,y) = v_\sigma(z,y) - (2\pi)^{N/2}\mathcal{F}_x(w_\sigma\Phi_k)(z,y)$$

and the Rayleigh-Ritz formula shows that

$$\lambda_{k+1}(\Omega)\int_\Omega |v_{\sigma,k}(z,y)|^2 dy \le \int_\Omega \overline{v_{\sigma,k}(z,y)} L_{\Delta,y} v_{\sigma,k}(z,y) dy$$

for any $z \in \mathbb{R}^N$ and $\sigma > 0$, where the right hand side is a real value

$$\int_\Omega \overline{v_{\sigma,k}(z,y)} L_{\Delta,y} v_{\sigma,k}(z,y) dy = \int_{\mathbb{R}^N} \overline{v_{\sigma,k}(z,y)} L_{\Delta,y} v_{\sigma,k}(z,y) dy = \int_{\mathbb{R}^N} 2\ln|\xi| \Big|\mathcal{F}(v_{\sigma,k})(z,\xi)\Big|^2 d\xi,$$

although $v_{\sigma,k}$ is complex valued function. Then, integrating this last inequality with respect to z in $B_r \setminus B_1$, for $r > 1$, we obtain

$$\lambda_{k+1}(\Omega) \leq \inf_{\sigma>0} \frac{\displaystyle\int_{B_r\setminus B_1}\int_\Omega \overline{v_{\sigma,k}(z,y)} L_{\Delta,y} v_{\sigma,k}(z,y)dydz}{\displaystyle\int_{B_r\setminus B_1}\int_\Omega |v_{\sigma,k}(z,y)|^2 dydz}.$$

By Pythagore's theorem, we have that

$$\int_\Omega |v_{\sigma,k}(z,y)|^2 dy = \int_\Omega |v_\sigma(z,y)|^2 dy - (2\pi)^N \int_\Omega \sum_{j=1}^k |\mathcal{F}_x(w_\sigma\phi_i)(z)|^2 \phi_i(y)^2 dy,$$

integrating over $B_r \setminus B_1$ implies that

$$\int_{B_r\setminus B_1}\int_\Omega |v_{\sigma,k}(z,y)|^2 dydz \geq \frac{\omega_{N-1} r^N}{N} \int_\Omega w_\sigma^2(y)dy - (2\pi)^N \sum_{j=1}^k \int_{B_r\setminus B_1} |\mathcal{F}_x(w_\sigma\phi_i)(z)|^2 dz.$$

On the other hand,

$$\begin{aligned}\int_{B_r\setminus B_1}\int_\Omega \overline{v_{\sigma,k}(z,y)} L_{\Delta,y} v_{\sigma,k}(z,y)dydz &= \int_{B_r\setminus B_1}\int_\Omega \overline{v_\sigma(z,y)} L_{\Delta,y} v_\sigma(z,y)dydz \\ &\quad - (2\pi)^N \int_{B_r\setminus B_1}\int_\Omega \overline{\mathcal{F}_x(w_\sigma\Phi_k)(z,y)} L_{\Delta,y} \mathcal{F}_x(w_\sigma\Phi_k)(z,y)dydz,\end{aligned}$$

where

$$\int_{B_r\setminus B_1}\int_\Omega \overline{\mathcal{F}_x(w_\sigma\Phi_k)(z,y)} L_{\Delta,y} \mathcal{F}_x(w_\sigma\Phi_k)(z,y)dydz = \sum_{j=1}^k \lambda_j(\Omega) \int_{B_r\setminus B_1} |\mathcal{F}_x(w_\sigma\phi_j)(z)|^2 dz$$

and

$$\begin{aligned}&\int_{B_r\setminus B_1}\int_\Omega \overline{v_\sigma(z,y)} L_{\Delta,y} v_\sigma(z,y)dydz \\ &\qquad \leq \int_{B_r\setminus B_1}\int_\Omega w_\sigma^2(y) |L_{\Delta,y} e^{\mathrm{i}y.z}| dydz + \int_{B_r\setminus B_1}\int_\Omega w_\sigma(y) |\mathcal{L}_z w_\sigma(y)| dydz \\ &\qquad \leq \int_{B_r\setminus B_1}\int_\Omega w_\sigma^2(y) \ln|z| dydz + \frac{2\omega_{N-1}}{\sigma} \int_{B_r\setminus B_1}\int_\Omega w_\sigma(y) dydz \\ &\qquad = \frac{\omega_{N-1}}{N} \varrho_{2,\sigma} \Big(r^N \ln r - \frac{1}{N}(r^N - 1)\Big) + \frac{\omega_{N-1}^2}{N\sigma} \varrho_{1,\sigma} \Big(r^N - 1\Big) \\ &\qquad \leq \frac{\omega_{N-1}}{N} \varrho_{2,\sigma}\, r^N \ln r + \frac{\omega_{N-1}^2}{N\sigma} \varrho_{1,\sigma}\, r^N\end{aligned}$$

with

$$\varrho_{1,\sigma} = \int_\Omega w_\sigma(y)dy \quad \text{and} \quad \varrho_{2,\sigma} = \int_\Omega w_\sigma^2(y)dy.$$

Because of Parseval's identity, there holds

$$\int_{B_r\setminus B_1} |\mathcal{F}_x(w_\sigma\phi_i)(z)|^2 dz \leq \int_\Omega (w_\sigma\phi_i)^2 dx \leq 1.$$

If we choose $r > 1$ such that

$$\frac{2\omega_{N-1}}{N^2} r^N \ln r \geq \frac{\omega_{N-1} r^N}{N} \Longleftrightarrow r \geq e^{\frac{N}{2}} \quad \text{and} \quad \frac{\omega_{N-1} r^N}{N} |\Omega| > 2(2\pi)^N k,$$

we have that

$$\begin{aligned}\lambda_{k+1}(\Omega) &\le \frac{\frac{\omega_{N-1}r^N}{N}\Big(\varrho_{2,\sigma}\frac{2}{N}\ln\frac{r^N}{e}+\varrho_{1,\sigma}\frac{\omega_{N-1}}{\sigma}\Big)-(2\pi)^N\sum_{j=1}^k\lambda_j(\Omega)\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_j)(z)|^2dz}{\frac{\omega_{N-1}r^N}{N}\varrho_{2,\sigma}-(2\pi)^N\sum_{j=1}^k\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_j)(z)|^2dz}\\ &\le \frac{\frac{\omega_{N-1}r^N}{N}\Big(\varrho_{2,\sigma}\frac{2}{N}\ln\frac{r^N}{e}+\varrho_{1,\sigma}\frac{\omega_{N-1}}{\sigma}\Big)-(2\pi)^N\sum_{j=1}^k\lambda_j(\Omega)}{\frac{\omega_{N-1}r^N}{N}\varrho_{2,\sigma}-(2\pi)^Nk},\end{aligned} \tag{4.4}$$

where

$$\frac{\omega_{N-1}r^N}{N}\varrho_{2,\sigma}-(2\pi)^Nk>\frac{\omega_{N-1}r^N}{N}\frac{|\Omega|}{2}-(2\pi)^Nk>0.$$

We fix $\sigma=\sigma_1$ and choose k, $r>1$ such that

$$\frac{\omega_{N-1}r^N}{N}\varrho_{2,\sigma}=(2\pi)^N(k+1),$$

and take $r=k^{\frac{1}{N}}$ for $k\ge k_0:=ed_N|\Omega|$, then we have that

$$\lambda_{k_0}\ge 0$$

and

$$\begin{aligned}(2\pi)^N\sum_{i=1}^k\lambda_i(\Omega)&\le\frac{2\omega_{N-1}}{N^2}\varrho_{2,\sigma}\,r^N\ln\frac{r^N}{e}+\frac{\omega_{N-1}^2}{N}\sigma^{-1}|\Omega|^{\frac12}\sqrt{\varrho_{2,\sigma}}\,r^N\\ &\le(2\pi)^N\frac{2}{N}(k+1)\Big(\ln(k+1)+\ln\frac{p_N}{\varrho_{2,\sigma}}+\frac{\omega_{N-1}}{\varrho_{2,\sigma}}\ln\ln\frac{p_N(k+1)}{\varrho_{2,\sigma}}\Big)\\ &\le(2\pi)^N\frac{2}{N}(k+1)\Big(\ln(k+1)+\ln\frac{2p_N}{|\Omega|}+\frac{2\omega_{N-1}}{\sqrt{|\Omega|}}\ln\ln\frac{2p_N(k+1)}{|\Omega|}\Big),\end{aligned}$$

where

$$p_N=\frac{2(2\pi)^NN}{\omega_{N-1}}\quad\text{and}\quad\frac{|\Omega|}{2}\le\varrho_{2,\sigma}\le|\Omega|.$$

Moreover, (1.18) follows by the lower bound (1.16) and the upper bound (1.17) directly. □

5 Weyl's limit of eigenvalues

To study the asymptotic distribution we introduce the counting function $\mathcal{N}(\Lambda)$ that counts the number of eigenvalues below $\Lambda>0$:

$$\mathcal{N}(\Lambda)=\sum_{k\in\mathbb{N}}sgn_+(\Lambda-\lambda_k(\Omega))=\sum_{k\in\mathbb{N}}(\Lambda-\lambda_k(\Omega))_+^0$$

Here $sgn_+(r)=1$ if $r>0$, $sgn_+(r)=0$ if $r\le0$ and $r_\pm=(|r|\pm r)/2$ denotes the positive and negative part of $x\in\mathbb{R}$. The counting function can also be expressed by introducing the trace of an operator

$$\mathcal{N}(\Lambda)=\mathrm{tr}(L_\Delta-\Lambda)_-^0,$$

where we use the same convention for self-adjoint operators.

Theorem 5.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N$, $\{\lambda_i(\Omega)\}_{i\in\mathbb{N}}$ is the increasing sequence of eigenvalues of problem (1.3). Then we have the asymptotic formula*

$$\lim_{\Lambda\to+\infty}\Lambda^{-1}e^{-\frac{N}{2}\Lambda}\mathcal{N}(\Lambda)=\frac{\omega_{N-1}}{2(2\pi)^NN}\,|\Omega|. \tag{5.1}$$

To prove Theorem 5.1, we need following preliminary lemmas.

Lemma 5.2. *[20, Theorem 2] Let $\mathcal{L}_T$ be a differential operator with symbol $T : \mathbb{R}^N \to \mathbb{R}$. For $\Omega \subset \mathbb{R}^d$ with finite volume and all $\Lambda > 0$, there holds*

$$\operatorname{tr}(\mathcal{L}_T - \Lambda)_- \leq \frac{|\Omega|}{(2\pi)^N} \int_{\mathbb{R}^N} (T(p) - \Lambda)_- \, dp \,.$$

Also see the proof in [18] for the Laplacian and in [17, 20] for the fractional Laplacian and in our case: $\mathcal{L}_T = L_\Delta$ and $T(p) = 2\ln|p|$.

Lemma 5.3. *Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a non-decreasing sequence of positive numbers that tends to infinity. Assume that there are finite constants $A > 0$ and $N > 0$ such that*

$$\lim_{\Lambda\to+\infty} \Lambda^{-1} e^{-\frac{N}{2}\Lambda} \sum_{k\in\mathbb{N}} (\Lambda - \xi_k)_+ = A.$$

Then

$$\lim_{\Lambda\to+\infty} \Lambda^{-1} e^{-\frac{N}{2}\Lambda} \sum_{k\in\mathbb{N}} (\Lambda - \xi_k)_+^0 = \frac{N}{2} A.$$

Proof. Let us introduce the notation $\mathcal{S}(\Lambda) = \sum_{k\in\mathbb{N}} (\Lambda - \xi_k)_+$ and $\mathcal{N}(\Lambda) = \sum_{k\in\mathbb{N}} (\Lambda - \xi_k)_+^0$. For any $h > 0$ and $\Lambda > 0$, $k \in \mathbb{N}$, we have that

$$(\Lambda + h - \xi_k)_+ - (\Lambda - \xi_k)_+ \geq h(\Lambda - \xi_k)_+^0,$$

and then

$$\mathcal{S}(\Lambda + h) - \mathcal{S}(\Lambda) \geq h\mathcal{N}(\Lambda). \tag{5.2}$$

For any $\epsilon \in (0, 1)$, there exists $\Lambda_0 = \Lambda_0(\epsilon) > 1$ such that for $\Lambda > \Lambda_0$,

$$|\Lambda^{-1} e^{-\frac{N}{2}\Lambda} \mathcal{S}(\Lambda) - A| \leq \epsilon.$$

Hence, for $0 < h \leq \Lambda^{-1}$ relation (5.2) implies

$$\begin{aligned}\mathcal{N}(\Lambda) &\leq \frac{1}{h}\left(A\Big((\Lambda + h)e^{\frac{N}{2}(\Lambda+h)} - \Lambda e^{\frac{N}{2}\Lambda}\Big) + \epsilon\Big((\Lambda + h)e^{\frac{N}{2}(\Lambda+h)} + \Lambda e^{\frac{N}{2}\Lambda}\Big)\right) \\ &\leq Ae^{\frac{N}{2}\Lambda}(\Lambda + 1) + N^2\Lambda^{-1}e^{\frac{N}{2}(\Lambda+1)} + 2e^{\frac{N}{2}}\epsilon\,(\Lambda + 1)e^{\frac{N}{2}\Lambda},\end{aligned}$$

thus we have that

$$\Lambda^{-1} e^{-\frac{N}{2}\Lambda} \mathcal{N}(\Lambda) \leq \frac{N}{2} A(1 + o(1)) \quad \text{as } \Lambda \to +\infty.$$

The lower bound follows similarly using the fact that $(\Lambda - \xi_k)_+ - (\Lambda - h - \xi_k)_+ \leq h(\Lambda - \xi_k)_+^0$ for suitable $h > 0$. □

Now we are in a position to prove Theorem 5.1:

Proof of Theorem 5.1. We first show that

$$\sum_k (\Lambda - \lambda_k(\Omega))_+ = \frac{\omega_{N-1}}{(2\pi)^N N^2} |\Omega| \Lambda e^{\frac{N}{2}\Lambda} (1 + o(1)) \quad \text{as } \Lambda \to \infty \,. \tag{5.3}$$

By Lemma 5.2 and the fact that $\operatorname{tr}(L_\Delta - \Lambda)_- = \sum_k (\Lambda - \lambda_k(\Omega))_+$, it suffices to prove the asymptotics

$$\operatorname{tr}(L_\Delta - \Lambda)_- = \frac{\omega_{N-1}}{(2\pi)^N N^2} |\Omega| \Lambda e^{\frac{N}{2}\Lambda} (1 + o(1)) \quad \text{as } \Lambda \to \infty \,. \tag{5.4}$$

The upper bound: From Lemma 5.2, we have that

$$\begin{aligned}\operatorname{tr}(L_\Delta-\Lambda)_- &\le \frac{|\Omega|}{(2\pi)^N}\int_{\mathbb{R}^N}(2\ln|x|-\Lambda)_-\,dx\\ &= \frac{|\Omega|}{(2\pi)^N}\int_{|x|\le e^{\frac{\Lambda}{2}}}(\Lambda-2\ln|x|)dx = \frac{2\omega_{N-1}}{N^2(2\pi)^N}e^{\frac{N}{2}\Lambda}|\Omega|.\end{aligned}$$

The lower bound: Fix $\delta>0$ and put $\Omega_\delta=\{x\in\Omega:\operatorname{dist}(x,\Omega^c)>\delta\}$. By the dominated convergence theorem, $|\Omega_\delta|\to|\Omega|$ as $\delta\to 0$, hence it suffices to show the lower bound in (5.4) with Ω replaced by Ω_δ.

Let $g\in C_0^\infty(\mathbb{R}^N)$ be a real-valued with support in $\{x\in\mathbb{R}^N:|x|\le\delta/2\}$ and such that $\|g\|_{L^2(\mathbb{R}^N)}=1$. For $p\in\mathbb{R}^N$ and $q\in\Omega_\delta$ we introduce the coherent states

$$F_{p,q}(x)=e^{ip\cdot x}g(x-q).$$

Then the properties of coherent states (see, e.g, [30, Thm. 12.8]) imply that

$$\operatorname{tr}(L_\Delta-\Lambda)_-\ge\frac{1}{(2\pi)^N}\iint_{\mathbb{R}^N\times\Omega_\delta}\left\langle F_{p,q},(L_\Delta-\Lambda)_-\,F_{p,q}\right\rangle dp\,dq\,.$$

Note that the mapping: $t\mapsto(t-\Lambda)_-$ is convex and that $\|F_{p,q}\|_{L^2(\mathbb{R}^N)}=1$. Thus we can apply Jensen's inequality to the spectral measure of L_Δ to obtain that

$$\operatorname{tr}(L_\Delta-\Lambda)_-\ge\frac{1}{(2\pi)^N}\iint_{\mathbb{R}^N\times\Omega_\delta}(\langle F_{p,q},L_\Delta F_{p,q}\rangle-\Lambda)_-\,dp\,dq\,.$$

Recall that $T(p)=2\ln|p|$, then there is a constant $C>0$ (independent of p, q, and Λ) such that

$$\langle F_{p,q},L_\Delta F_{p,q}\rangle\le 2\ln|p|+C, \tag{5.5}$$

where

$$\langle F_{p,q},L_\Delta F_{p,q}\rangle=\frac{1}{(2\pi)^N}\iiint e^{\mathrm{i}(p-\xi)\cdot(x-y)}g_q(x)g_q(y)T(\xi)\,dx\,dy\,d\xi.$$

The proof could see [20] and we addressed the proof in Appendix B for the reader's convenience.

Inserting (5.5) into the bound above, letting $r=e^{\frac{N}{2}(\Lambda+C)}$, we have that for $\Lambda>2C+1$,

$$\begin{aligned}\operatorname{tr}(\mathcal{L}_T-\Lambda)_- &\ge \frac{|\Omega_\delta|}{(2\pi)^N}\int_{\mathbb{R}^N}(2\ln|p|+C-\Lambda)_-\,dp\\ &= \frac{|\Omega_\delta|}{(2\pi)^N}\int_{B_r}(\Lambda-C-2\ln|x|)dx = \frac{|\Omega_\delta|}{(2\pi)^N}\frac{\omega_{N-1}}{N^2}(\Lambda-C)e^{\frac{N}{2}(\Lambda-C)}.\end{aligned}$$

Thus we conclude that

$$\liminf_{\Lambda\to\infty}\Lambda^{-1}e^{-\frac{N}{2}\Lambda}\operatorname{tr}(\mathcal{L}_T-\Lambda)_-\ge\frac{\omega_{N-1}}{(2\pi)^N N^2}|\Omega_\delta|\,.$$

By the arbitrary of $\delta>0$, we obtain the required lower bound of (5.4).

Finally, (5.1) follows (5.3) and Lemma 5.3 directly. □

Proof of Theorem 1.7. From Theorem 5.1 with $\Lambda=\lambda_k(\Omega)$, we have that

$$\lim_{k\to+\infty}\lambda_k(\Omega)^{-1}e^{-\frac{N}{2}\lambda_k(\Omega)}k=\frac{\omega_{N-1}}{2(2\pi)^N N}\,|\Omega|=:m_0.$$

Thus, there exists $\bar{k}>1$ such that for $k\ge\bar{k}$,

$$\frac{m_0}{2}\le\lambda_k(\Omega)^{-1}e^{-\frac{N}{2}\lambda_k(\Omega)}k\le 2m_0.$$

Letting

$$F(\mu) = \mu^{-1} e^{-\frac{N}{2}\mu}$$

we note that for k large enough

$$F(\frac{2}{N}\ln k) = \frac{N}{2\ln k}\frac{1}{k} \quad \text{and} \quad F(\frac{2}{N}(\ln k - \ln(\ln^2 k))) \geq \frac{N\ln k}{2}\frac{1}{k},$$

which imply that for such k

$$\frac{2}{N}(\ln k - \ln(\ln^2 k)) \leq \lambda_k(\Omega) \leq \frac{2}{N}\ln k.$$

Thus, we have that

$$\lim_{k\to+\infty} \frac{\lambda_k(\Omega)}{\ln k} = \frac{2}{N}.$$

We complete the proof. □

A Identities for the fractional Laplacian

Lemma A.1. *For fixed $z \in \mathbb{R}^N \setminus \{0\}$, denote*

$$v_z(x) = e^{\mathrm{i}x\cdot z}, \quad \forall\, x \in \mathbb{R}^N,$$

then

$$(-\Delta)^s v_z(x) = |z|^{2s} v_z(x), \quad \forall\, x \in \mathbb{R}^N. \tag{A.1}$$

Proof. Without loss of generality, it is enough to prove (A.1) with $z = te_1$, where $t > 0$ and $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$. For this, we write

$$v_t(x) = v_t(x_1) = e^{\mathrm{i}tx_1}, \quad x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{N-1}.$$

For $N \geq 2$ it implies by [7, Lemma 3.1] that

$$(-\Delta)^s v_t(x) = (-\Delta)^s_{\mathbb{R}} v_t(x_1).$$

Now we claim that

$$(-\Delta)^s_{\mathbb{R}} v_t(x_1) = t^{2s} v_t(x_1), \quad \forall\, x_1 \in \mathbb{R}. \tag{A.2}$$

Indeed, observe that $-\Delta_{\mathbb{R}} := -(v_t)_{x_1x_1} = t^2 v_t$ in $\mathbb{R}$ and then

$$(|\xi_1|^2 - t^2)\widehat{v_t}(\xi_1) = \mathcal{F}\left(-\Delta_{\mathbb{R}} v_t - t^2 v_t\right)(\xi_1) = 0,$$

which implies that

$$\mathrm{supp}(\widehat{v_t}) \subset \{\pm t\},$$

which in turn implies

$$(|\xi_1|^{2s} - t^{2s})\widehat{v_t}(\xi_1) = 0 = \mathcal{F}\left((-\Delta)^s_{\mathbb{R}} v_t - t^{2s} v_t\right)(\xi_1).$$

and finally

$$\left((-\Delta)^s_{\mathbb{R}} v_t - t^{2s} v_t\right)(\xi_1) = 0 \quad \text{in } \mathbb{R},$$

which yields

$$(-\Delta)^s v_t(x) = (-\Delta)^s_{\mathbb{R}} v_t = t^{2s} v_t(x), \quad \forall\, x \in \mathbb{R}^N,$$

and complete the proof. □

B Proof of (5.5)

By the definition of $\mathcal{L}_T$ and $F_{p,q}$, we have that

$$\langle F_{p,q}, L_\Delta F_{p,q}\rangle = \frac{1}{(2\pi)^N}\iiint e^{\mathrm{i}(p-\xi)\cdot(x-y)} g_q(x)g_q(y)T(\xi)\,dx\,dy\,d\xi\,,$$

where $T(p) = 2\log|p|$ in our setting, we write $g_q(x) = g(x-q)$ and all integrals are taken over $\mathbb{R}^N$. We insert the identity $g_q(x)g_q(y) = (g_q(x)^2 + g_q(y)^2 - (g_q(x)-g_q(y))^2)/2$ and use the symmetry of the integral above to obtain

$$\begin{aligned}\langle F_{p,q}, L_\Delta F_{p,q}\rangle &= \iiint e^{\mathrm{i}(p-\xi)\cdot(x-y)}\left(g_q(x)^2 - \frac{1}{2}\left(g_q(x)-g_q(y)\right)^2\right)T(\xi)\frac{dxdyd\xi}{(2\pi)^N}\\ &= T(p) - \iiint e^{\mathrm{i}(p-\xi)\cdot(x-y)}\left(g_q(x)-g_q(y)\right)^2 T(\xi)\frac{dxdyd\xi}{2(2\pi)^N}\,,\end{aligned} \tag{B.1}$$

where in the second equality we used the fact that $\int g(x-q)^2dx = 1$ for all $q\in\mathbb{R}^N$ and

$$T(p) = \mathcal{F}^{-1}(\hat{T})(p) = \frac{1}{(2\pi)^N}\iint e^{\mathrm{i}(p-\xi)z}T(\xi)d\xi dz.$$

To estimate the contribution of the second summand we substitute $y = x - z$ and apply Plancherel's Theorem to get

$$\iiint e^{\mathrm{i}(p-\xi)\cdot z}\left(g_q(x)-g_q(x-z)\right)^2 T(\xi)\frac{dxdzd\xi}{2(2\pi)^N} = \iiint e^{\mathrm{i}(p-\xi)\cdot z}|\hat{g}_q(\eta)|^2\left|1-e^{-\mathrm{i}z\cdot\eta}\right|^2 T(\xi)\frac{d\eta dzd\xi}{2(2\pi)^N}.$$

We note that $|\hat{g}_q(\eta)|^2 = |\hat{g}(\eta)|^2$ is independent of q. Moreover, we write $|1-e^{-\mathrm{i}z\cdot\eta}|^2 = 2-e^{\mathrm{i}z\cdot\eta}-e^{-\mathrm{i}z\cdot\eta}$ and perform the integration in z and ξ. We find

$$\iiint e^{\mathrm{i}(p-\xi)\cdot(x-y)}\left(g_q(x)-g_q(y)\right)^2 T(\xi)\frac{dxdyd\xi}{2(2\pi)^N} = \int\left(T(p) - \frac{1}{2}\left(T(p+\eta)+T(p-\eta)\right)\right)|\hat{g}(\eta)|^2 d\eta.$$

the right-hand side is bounded below by $-C_0\int\log(e+|\eta|)|\hat{g}(\eta)|^2d\eta \geq -C$ due to the fact that $T(p) = 2\log|p|$, here C_0 and C are positive constants depending on $|p|$.

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