



Collatz conjecture : Geography of the Pascal trihedron.

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Collatz conjecture : Geography of the Pascal trihedron.

Hubert Schaetzel

Abstract The mechanism of descent below the initial flight altitude in Collatz algorithm is clarified by using congruencies modulo powers of 2. We suggest here two three-dimensions' classifications of the integers, emerging from this algorithm, architecture we have called "Pascal trihedron", following here another article where only the headcounts of the planes of the said trihedron were quantified.

Conjecture de Collatz : Géographie du trièdre de Pascal.

Résumé Le mécanisme de descente sous l'altitude initiale dans l'algorithme de Collatz se révèle en utilisant les congruences modulo des puissances de 2. Nous proposons ici deux classements à trois dimensions des nombres entiers dans le cadre de cet algorithme, architecture que nous avons appelée « trièdre de Pascal », faisant suite à un autre article où n'était réalisé que le dénombrement des effectifs des plans dudit trièdre.

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1 Framework.

The Collatz conjecture is also called Syracuse conjecture, Ulam conjecture, Czech conjecture or $3x+1$ problem. We studied this conjecture in another article [1] (writing also a simplified version [2]). The present work follows and completes the said study. The presentation of the conjecture and the results obtained at that time are not repeated here. The reader will refer to the said previous articles if necessary.

We have given at the said time an algorithm to evaluate the size of classes of identical altitude flight times (stopping times). Taking account of its structure, we called this routine "Pascal trihedron".

We contented ourselves of a mere enumeration of classes' populations (of identical altitude flight time) given in this text by theorem 15 page 77. The purpose of this article is to understand the geography of this trihedron and then catch the properties of the members of these classes which draw that geography. This study will in fact lead us towards two geographies, the second deriving from the first one.

2 Brief review and additional definitions.

We maintain all of our initial article notations.

In particular:

- the number v of odd steps of altitude flight time (number of multiplications by 3 (plus 1),
- the number w of even steps of altitude flight time (number of divisions by 2).

For an example, the number of odd and even steps in altitude flight time of 11 is given by the following table :

11	34	17	52	26	13	40	20	10
	I	P	I	P	P	I	P	P

There are 3 odd steps I and 5 even steps P before the first number resulting from the Collatz algorithm becomes smaller than 11.

Theorem 1

Except for finite loops, the number of w pairs and v odd steps are related by :

$$w = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)+1 \quad (1)$$

Proof

For the infinite loops (if they exist), v and w are infinite and the relationship does not apply of course (to the nearest integer).

A multiplication of x by 3 and then adding 1 is equivalent to a multiplication by $3+\varepsilon$, where ε is small in front of 3 when x is large. Performing the Collatz algorithm, we get y, the first number inferior to x, with $x/2 < y < x$ (not including loops where $y = x$) and we have then also $y = (3+\varepsilon_1).(3+\varepsilon_2)...(3+\varepsilon_v)/2^w.x$. The ε_i are all small in front of 3, since the exit of altitude flight by the Collatz algorithm is carried out for $y > x/2$ so for y remaining a large number (same order of magnitude as x). We have therefore $(3+\varepsilon_1).(3+\varepsilon_2)...(3+\varepsilon_v) = (3+\varepsilon)^v$ with ε small in front of 3. We have then, according to previous observations, by substitution $1/2 < (3+\varepsilon)^v/2^w < 1$, which is also $\text{Ln}(3+\varepsilon)/\text{Ln}(2).v < w < \text{Ln}(3+\varepsilon)/\text{Ln}(2).v+1$, thus $w = \text{int}(\text{Ln}(3+\varepsilon)/\text{Ln}(2)).v+1$.

This result is then tested in comparison with $w = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)+1$ for $x = 1$ to $x = 10^8$ (for example). No exception is found to this rule for these x (in particular for pair number x, we have $v = 0$ and $w = 1$). However, as x becomes larger without exception, it is unlikely to see a sudden exception, as ε approximating 0 means that $w = \text{int}(\text{Ln}(3+\varepsilon)/\text{Ln}(2).v)+1$ approaches $w = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)+1$.

Let us consider then x with w pair steps. The altitude flight cycle of $x+k.2^w$ being the same as that of x, let us test then $x_k = x+k.2^w$ for any integer k with regard to $x_k/2 < y_k \leq x_k$, which is also $\text{Ln}(3+\varepsilon_k)/\text{Ln}(2).v \leq w < \text{Ln}(3+\varepsilon_k)/\text{Ln}(2).v+1$. Number x_k is therefore as large as wanted and hence the choice of an ε_k as small as desired can be done. As $\text{Ln}(3)/\text{Ln}(2)$ is transcendental and not in Q, $\text{Ln}(3)/\text{Ln}(2).v$ is not an integer and there is necessarily an ε_k , small enough among our tests, such as $\text{Ln}(3+\varepsilon_k)/\text{Ln}(2)$ is not in Q and $\text{Ln}(3+\varepsilon_k)/\text{Ln}(2).v$ is not an integer and close enough to the precedent one so that $\text{int}(\text{Ln}(3+\varepsilon_k)/\text{Ln}(2).v) = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)$.

The result $w = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)+1$ then applies for this case x_k . Then, the result remains true for all $x_i = x+i.2^w$, i a relative integer (i in Z) completing the proof.

Note that for $x = 1$ which loops on itself, the relationship still applies (because this number belongs to the family $x = 1 \bmod 4$ for which without exception $v = 1$, $w = 2$).

For x small and negative, there are exceptions (by only one unit) found only for the well-known loops :

- For $x = -1$, we have $v = 1$, $w = 1$ and so $w = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)$
- For $x = -5$, we have $v = 2$, $w = 3$ and so $w = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)$
- For $x = -17$, we have $v = 7$, $w = 11$ and so $w = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)$

We then have the populations of modulo 2^w families' associates (term that we adopted in the original article to refer to members of the same modulo 2^w family which all have same altitude flight time) given in the tables below :

We sum up the staffs at the last line (in red font and outside trihedron of staffs) :

$v = 1$, $w = 2$, $\#(v) = 1$, ($2^w = 4$)

Family associates mod 4

5 (or 1)

1
1

$v = 2$, $w = 4$, $\#(v) = 1$, ($2^w = 16$)

Family associates mod 16,

3

1
1

$v = 3, w = 5, \#(v) = 2, (2^w = 32)$

Family associates mod 32,
11, 23

1	1
1	1

$v = 4, w = 7, \#(v) = 3, (2^w = 128)$

Family associates mod 128
7, 15, 59

1	1	1
1	1	1

$v = 5, w = 8, \#(v) = 7, (2^w = 256)$

Family associates mod 256
39, 79, 95, 123, 175, 199, 219

1	1	1	1
0	1	1	1
1	2	2	2

$v = 6, w = 10, \#(v) = 12, (2^w = 1024)$

Family associates mod 1024
287, 347, 367, 423, 507, 575, 583, 735, 815, 923, 975, 999

1	1	1	1	1
0	1	2	2	2
1	2	3	3	3

$v = 7, w = 12, \#(v) = 30, (2^w = 4096)$

Family associates mod 4096
231, 383, 463, 615, 879, 935, 1019, 1087, 1231, 1435, 1647, 1703, 1787, 1823, 1855, 2031, 2203, 2239, 2351, 2587, 2591, 2907, 2975, 3119, 3143, 3295, 3559, 3675, 3911, 4063

1	1	1	1	1	1
0	1	2	3	3	3
0	1	2	3	3	3
1	3	5	7	7	7

Definition 1

We call these successive tables Pascal trihedron staffs' planes.

These planes take their complete form only when three identical columns occur at the end of the table (with a different column in front), that is for $v \geq 5$ ($w \geq 8$).

The number of columns of a plane is $v-1$.

Definition 2

We call the last line (in red) of each of them the simplified decomposition of the Pascal trihedron (of staffs) which we symbolize by $\#TPS(v)$. We have for example $\#TPS(7) = \{1, 3, 5, 7, 7, 7\}$.

Our first goal will be to put all of the set of associates, which we have given the list before each table above, at their proper places in the Pascal trihedron planes.

Doing this, we get another set of tables that we also call Pascal trihedron planes. To distinguish them from the earlier set, we adopt the following denomination :

Definition 3

We call the new entities the Pascal trihedron associates' planes.

Here are a few examples which will ensue from our study, with underneath in red font and off trihedron, a reminder of the staffs :

$v = 1, w = 2, \#(v) = 1$
Family mod 4,

5
1

$v = 2, w = 4, \#(v) = 1$
Family mod 32,

3
1

$v = 3, w = 5, \#(v) = 2$
Family mod 32,

23	11
1	1

$v = 4, w = 7, \#(v) = 3$
Family mod 128

15	7	59
1	1	1

$v = 5, w = 8, \#(v) = 7$
Family mod 256

95	175 79	39 199	219 123
1	2	2	2

$v = 6, w = 10, \#(v) = 12$
Family mod 1024

575	287 735	367 815 975	999 423 583	923 347 507
1	2	3	3	3

$v = 7, w = 12, \#(v) = 30$
Family mod 4096

383	2239 1855 1087	2975 2591 1823 4063 3295	2031 1647 879 3119 2351 1231 463	615 231 3559 1703 935 3911 3143	2587 2203 1435 3675 2907 1787 1019
1	3	5	7	7	7

Definition 4

We identify a number $a_v(i,j)$ on the Pascal trihedron (of associates) by its plane number v and its position in line i and column j (where $i = 1, j = 1$ for the first row and column).

Important note.

We find effectively the simplified decomposition of the Pascal trihedron (of staffs). But, on the other hand, if we consider, for example the $v = 7$ plane, the fine structure of staffs in the last table, we would immediately choose a consolidation of staffs such as :

1	1	1	1	1	1
0	2	2	2	2	2
0	0	2	2	2	2
0	0	0	2	2	2
1	3	5	7	7	7

This is far from the one proposed earlier :

1	1	1	1	1	1
0	1	2	3	3	3
0	1	2	3	3	3
1	3	5	7	7	7

Our previous studies show that the staffs of the v plane v can be deducted additively from the fine structure of the staffs of the $v-1$ plane. A classification in the same mould would then probably allow finding some additive property among the associates' planes to generate more or less the v plane from the $v-1$ plane. Yet this fine model seems set to wrong so far here. The 'geography' of the original staffing table has become totally unreadable. Nevertheless, we will see that anticipations at rank v find place from rank $v-1$ items.

In addition, we will highlight, in the last chapter of this article, a second geography that actually springs up the said additive property.

3 Signature concept.

Let us take the example of the altitude flight of integer 11.

11	34	17	52	26	13	40	20	10
	I	P	I	P	P	I	P	P

A multiplication step (I) is systematically followed by a step of division (P). If we wish so, we can link the two operations into one called (IP). That means for what we had previously :

11	17	26	13	20	10
	IP	IP	P	IP	P

Let us replace the odd steps I by integer 1 and even steps P by integer 0 (and les IP steps by 10). Previous IPIPPIPP becomes the binary number 10100100 that we convert to decimal 164. This one would be all the greater, while considering identical odd-numbered steps (and thus identical even-numbered steps), if the odd steps were more advanced (so IPIPIPPP will give 10101000 or 168).

Definition 5

We will call binary signature the binary number corresponding to the series IP... P... I... PP and decimal signature the result in decimal conversion.

The advantage of decimal signature lies mainly in the ease of a more concise reading.

A combination of bits 0 and 1 does not necessarily correspond to a series IP... P... I... PP generated by the algorithm of Collatz. For example, you can never have two consecutive odd steps I and thus 1 followed by 1 will never appear in a licit binary signature.

Definition 6

We call licit (or permissible, acceptable, legitimate...) signature that one that does not break any rule of the Collatz algorithm within an altitude flight routine.

Let us consider, for example, the binary signature 1000. It is not licit because the number of even steps w is greater than $\text{int}(\ln(3)/\ln(2).v)+1$, where v is the number of odd steps. The altitude flight time is exceeded in this writing. Similarly, writing 1000101010100 is not licit, even if we do have globally $w = \text{int}(\ln(3)/\ln(2).v)+1$, as the flight time was exceeded prematurely writing 1000 at the beginning of the sequence. For some given binary signature, in the same way, one has to check its validity at each new even intermediate step. For example, with signature 1010101010100100001000, the intermediate necessary checks for a licit signature are to be made at $v_1 = 7$, $w_1 = 8$, $v_2 = 8$, $w_2 = 12$ and $v_3 = 9$, $w_3 = 15$. We have $w_1 = 8 < \text{int}(\ln(3)/\ln(2).v_1)+1 = 12$, $w_2 = 12 < \text{int}(\ln(3)/\ln(2).v_2)+1 = 13$, $w_3 = 15 = \text{int}(\ln(3)/\ln(2).v_3)+1$ thus corresponding effectively to a licit signature.

Theorem 2

The binary signatures of a given plane of the Pascal trihedron are all the same size.

Indeed, all the numbers of a plane v have an altitude flight time with v odd steps v and $w = \text{int}(\ln(3)/\ln(2).v)+1$ even steps and so the size of the signatures is identical and equal to $v+w$.

Numerical example

Let us get signatures for all objects a Pascal trihedron plane v , for example, $v = 5$ ($w = 8$, modulo 256) which includes $\#(v) = 7$ elements, namely 39, 79, 95, 123, 175, 199 and 219.

Then, by ordering according to the decreasing values of decimal equivalents, we get the following table :

Objects	Binary signature	Decimal signature
95	1010101010000	5456
175	1010101001000	5448
79	1010101000100	5444
39	1010100101000	5416
199	1010100100100	5412
219	1010010101000	5288
123	1010010100100	5284

4 Construction of signatures' tables.

We intend now to come up with a routine that will enable our geographical initial goal. We know that, for a given plane of the Pascal trihedron, all associates have same size of signatures. We propose therefore to write those exhaustively according to two specific rankings.

The first ranking will help us to establish a set of properties modulo 2^w of the elements of a Pascal trihedron plan constituting the bulk of the so-called geography.

The second ranking will allow counting the staffs of the said plans and establish the corresponding theorem (instead of a conjecture in earlier article).

Let us now start on the first classification. As an example, we choose the plane $v = 7$ and we show how to apply that routine systematically to any plane v .

	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6
Line 1	10101010101000000	1010101010100100000	1010101010010100000	1010101001010100000	1010100101010100000	1010010101010100000
Line 2		1010101010100010000	1010101010010010000	1010101001010010000	1010100101010010000	1010010101010010000
Line 3		1010101010100001000	1010101010010001000	1010101001010001000	1010100101010001000	1010010101010001000
Line 4			1010101010001010000	1010101001001010000	1010100101001010000	1010010101001010000
Line 5			1010101010001001000	1010101001001001000	1010100101001001000	1010010101001001000
Line 6				1010101000101010000	1010100100101010000	1010010100101010000
Line 7				1010101000101001000	1010100100101001000	1010010100101001000

Construction of the first line :

For the v -plane, we have for any element, v odd steps and w pair steps. We will get the first signature by placing the odd steps most on the left side, thus for $v = 7$ (and $w = 12$) we have the following combination 1010101010(10)00000. From this number, the last sequence 10 (on the right side) is then switched giving 1010101010(01)00000 which is retained if it is a licit signature (which is effectively the case here). Then, to the left side of this permutation, we perform a new permutation giving 10101010(01)(01)00000. We follow after that with 10101010(01)(01)(01)00000, then 101010(01)(01)(01)(01)00000, then 1010(01)(01)(01)(01)(01)00000 and 10(01)(01)(01)(01)(01)(01)00000 to finish with. This last step must be rejected because it stops prematurely the altitude flight after sequence 100. It is therefore not licit as explained above.

This kind of construction will be used in the same way regardless of the plane v and will always stop after exactly $v-1$ steps (including the initial step). Hence, a $v-1$ columns table.

Lemma 1

The signatures obtained by permutations of (10) to (01) such as defined above are all licit.

Proof

After a multiplication by 3 (and addition of 1), it takes at least 2 pair steps to go down below the earlier number (and therefore possibly leaving the altitude flight). When two multiplications (multiplication greater than 9) are conducted, it takes 4 pair steps, hence a sequence like (101000) with three 0 at the end of signature to enable to leave the altitude flight. It is the same within a longer signature sequence. This arrangement of three successive zeros never appears in the set of construction described above based on permutations of doublets. The $v-1$ signatures described above are therefore effectively licit.

Construction of columns :

The first line being carried out, the columns' classification is done by gradually pushing to the left in turn the successive 1 beginning with the one most on the left and by the rules of licit signatures.

For column 2, we have thus

1010101010100100000
1010101010100010000
1010101010100001000

The next one in order is not licit because the sequence between parenthesis completes the altitude flight and this signature belongs to another plane (plane $v-1 = 6$) :

(1010101010100000)100

Returning then to the first line, drawing back the second right-side 1, we get :

1010101010010100000

It is none other than the first item in the third column which will then be the subject of continuing the process.

However, let us go focus some more on the second column.

Lemma 2

The number of items in the second column is given by :

$$\#(\text{col2, plane } v) = w(v-1) - v(v) - 1$$

Proof

The number of 0 at the end of the first line signature is $w(v) - v(v)$. In this entry, we mean the number of odd steps $v = v(v)$ and pair steps $w = w(v)$ of the plane v . The size of the illicit sequence (in parentheses above) is given by $v(v-1) + w(v-1)$ as belonging to the $v-1$ plane and we therefore have $v(v) + w(v) - (v(v-1) + w(v-1))$ numbers that follow it, the first being 1 and the others 0. Thus, the number of possible shifts is exactly $(w(v) - v(v)) - (v(v) + w(v) - v(v-1) - w(v-1))$ to which we have to add the solution to the first line, i.e. $v(v-1) + w(v-1) - 2 \cdot v(v) + 1$, and as $v(v) = v(v-1) + 1$, we have a total of $w(v-1) - v(v-1) - 1$ signatures. In the example, the plane $v = 6$ corresponds to $w = 10$ and we got so $10 - 6 - 1 = 3$ signatures in the first column of the plane $v = 7$.

We did this calculation in order to show the importance of the knowledge of the $v-1$ rank for the determination of the staffs of rank v . The object is not here to start again a comprehensive study of the enumeration of the elements of a plane of the Pascal trihedron. To go further, the reader will refer to our other articles.

In the same way that we have developed the second column until you get the first item in the third column, we can obtain the signatures of the other columns successively. The passage from one column to the other is made at the time when the head element of the next column appears as described above.

Lemma 3

To the right of a licit signature always exists a licit signature and this until the $v-1$ column.

Proof

The space between odd steps 1 narrows toward the right with the chosen scheme of construction, which makes impossible an illicit signature.

Theorem 3

The table of signatures is exhaustive at column $v-1$.

Proof

For a given column, there cannot be other elements on a line if the first line fails because a shift of the last odd step 1 to the right makes the signature even 'more' illicit. As there is no first item at column v , which is the case, there cannot be another below.

The lemma and the previous theorem stem the particular left broken shape of the Pascal trihedron planes with completely filled accidentally longer and longer columns.

Theorem 4

Differences in binary signatures (where the two signatures exist) between columns are constants :

$$s_v(i,j)-s_v(i,j-1) = \text{cte}_v(j) \quad (2)$$

Proof

Let us review this by an example as earlier :

Col 3	Col 4
10101010(10)010100000	10101010(01)010100000
10101010(10)010010000	10101010(01)010010000
10101010(10)010001000	10101010(01)010001000
10101010(10)001010000	10101010(01)001010000
10101010(10)001001000	10101010(01)001001000
	10101010(00)101010000
	10101010(00)101001000

By construction, the facing doublets, from a column to the other, are characterised by the same permutation at the same place into the signature, the rest of it, left and right, being identical from one to the other. Their differences are thus identical (here $1000000000 = 2^9$).

Note :

It is of course the same for decimal signatures.

Property 1

The difference between the first two columns is equal to 2^{w-v} and increases column after column by a factor of 4.

v	w	2^{w-v}	2^{w-v+2}	2^{w-v+4}	2^{w-v+6}	2^{w-v+8}	2^{w-v+10}	2^{w-v+12}
3	5	4						
4	7	8	32					
5	8	8	32	128				
6	10	16	64	256	1024			
7	12	32	128	512	2048	8192		
8	13	32	128	512	2048	8192	32768	
9	15	64	256	1024	4096	16384	65536	262144

Proof

Let us go back first to the example of plane $v = 7$ ($w = 12$) signatures.

To go from the left to the right of this table consists simply to swap the 0 with the foregoing 1 at the position $w-v-1+2c$ starting from the right, c being the number of the column in order to find the element of the next column. For example, for the line 5, swapping in 10101010(10)001001000 in column 3 is done at position $12-7-1+6 = 10$ (and position 11 for 1) thus giving 10101010(01)001001000. Then for 101010(10)01001001000 in column 4 swapping is done at position $12-7-1+8 = 12$ to give 101010(01)01001001000. To finish with, for 1010(10)0101001001000 in column 5 swapping is done in position $12-7-1+10 = 14$ to give 1010(01)0101001001000.

These permutations of (10) to (01) are applied in a systematic way for whatever v -plane starting always from a signature like 101010101010...100000...0, then gradually shifting along pairs (10) from those most to the right side towards those on the left side.

The decimal transcription of such writings is of course a gap of 2^{w-v} from column 1 to column 2 (the decimal values get lower), with a gradual increase of the said initial gap by a factor of 4 from a column to the next one.

Theorem 5

Differences in binary signatures (where signatures exist) between lines are constants :

$$s_v(i,j)-s_v(i-1,j) = \text{cte}_v(i) \quad (3)$$

Proof

We give two examples for the plane $v = 7$

	Col 2	Col 3	Col 4	Col 5	Col 6
Line 1	10101010100(10)0000	10101010010(10)0000	10101001010(10)0000	10100101010(10)0000	10100101010(10)0000
Line 2	10101010100(01)0000	10101010010(01)0000	10101001010(01)0000	10100101010(01)0000	10100101010(01)0000

	Col 3	Col 4	Col 5	Col 6
Line 3	101010100(10001)000	101010010(10001)000	101001010(10001)000	101001010(10001)000
Line 4	101010100(01010)000	101010010(01010)000	101001010(01010)000	101001010(01010)000

It is no more a simple permutation here in general, but by construction, identical entities that change at identical positions. The differences thus remain constant.

Note :

It is of course the same for decimal signatures.

Property 2

The difference between the first two lines is equal to 2^{w-v-1} .

The reader can see this on the examples given in the table below. We have 2^{8-5-1} , $8 = 2^{10-6-1}$, $16 = 2^{12-7-1}$ and so on. We do not prove that property here (missing the general case anyway).

The anticipation of the values of these differences between lines would enable us to deduce the associates of any Pascal trihedron plane and is therefore of the utmost importance. We first give some examples of the decimal differences between lines observed before making an analysis, giving here the last column of the tables :

$v = 5$ $w = 8$ $\Delta w = 1$	$v = 6$ $w = 10$ $\Delta w = 2$	$v = 7$ $w = 12$ $\Delta w = 2$	$v = 8$ $w = 13$ $\Delta w = 1$	$v = 9$ $w = 15$ $\Delta w = 2$	$v = 10$ $w = 16$ $\Delta w = 1$		
4	8	16	16	32	32	Continuation1	Continuation2
	32	8	8	16	16	52	4
		56	4	8	8	8	124
		8	52	104	4	4	4
		248	8	16	100	28	852
		8	4	8	16	4	8
			28	56	8	212	4
			4	8	4	8	28
			212	32	52	4	4
			8	392	8	28	124
			4	16	4	4	4
			28	8	28	124	31236
			4	56	4	4	16
			980	8	388	852	8
			8	32	16	8	4
			4	216	8	4	52
			28	8	4	28	8
			4	32	52	4	4
				1672	8	124	28
				16	4	4	4
				8	28	6660	212
				56	4	16	8
				8	212	8	4
				32	8	4	28
				216	4	52	4
				8	28	8	124
				32	4	4	4

v = 5 w = 8 $\Delta w = 1$	v = 6 w = 10 $\Delta w = 2$	v = 7 w = 12 $\Delta w = 2$	v = 8 w = 13 $\Delta w = 1$	v = 9 w = 15 $\Delta w = 2$	v = 10 w = 16 $\Delta w = 1$		
				7816	124	28	852
				16	4	4	8
				8	1540	212	4
				56	16	8	28
				8	8	4	4
				32	4	28	124
					...	...	4

This table offers prospects of anticipation. The principle is to infer the values at rank v from those at rank $v-1$. To this end, we have placed in the header the values of

$$\Delta w_v = \Delta w(v) = w(v) - w(v-1) = w_v - w_{v-1}$$

As $w = \text{int}(\text{Ln}(3)/\text{Ln}(2).v)+1$, Δw at rank v (noted as $\Delta w(v)$ or Δw_v) is equal to 1 or 2. It is easy to see in first line, the differences of decimal signatures $\text{cte}_v(0)$ are multiplied by $\Delta w(v)$ passing from $v-1$ to v . This is deducted immediately also from the ratio of the values previously announced at rank $v-1$ (that is $2^{w(v-1)-(v-1)-1}$) and at rank v (that is $2^{w(v)-v-1}$), hence a ratio of $2^{\Delta w(v)-1} = \text{if}(\Delta w(v) = 1, 1, 2)$. This rule of multiplication at the first line is actually general up to a certain point that we will describe now.

The passage from rank $v-1$ to rank v is done by multiplications by $\Delta w(v)$, by further splits and additions of new items :

v = 5 w = 8 $\Delta w = 1$	v = 6 w = 10 $\Delta w = 2$	
4	8	Multiplication ($8 = 4.\Delta w$).
	32	New item.

v = 6 w = 10 $\Delta w = 2$	v = 7 w = 12 $\Delta w = 2$	
8	16	Multiplication ($16 = 8.\Delta w$).
32	8	Multiplication ($64 = 32.\Delta w$).
	56	Splitting ($64 = 8+56$).
	8	New items
	248	
	8	

v = 7 w = 12 $\Delta w = 2$	v = 8 w = 13 $\Delta w = 1$	
16	16	Multiplication ($16 = 16.\Delta w$).
8	8	Multiplication ($8 = 8.\Delta w$).
56	4	Multiplication ($56 = 56.\Delta w$).
	52	Splitting ($56 = 4+52$).
8	8	Multiplication ($8 = 8.\Delta w$).
248	4	Multiplication ($248 = 248.\Delta w$).
	28	
	4	
	212	
8	8	Multiplication ($8 = 8.\Delta w$).
	4	New items
	28	
	4	
	980	
	8	
	4	
	28	
	4	

v = 8 w = 13 $\Delta w = 1$	v = 9 w = 15 $\Delta w = 2$	
16	32	Multiplication by Δw .
8	16	Multiplication by Δw .
4	8	Multiplication by Δw .
52	104	Multiplication by Δw .
8	16	Multiplication by Δw .
4	8	Multiplication by Δw .
28	56	Multiplication by Δw .
4	8	Multiplication by Δw .
212	32 392	Multiplication $424 = 212 \cdot \Delta w$. Splitting ($424 = 32 + 392$)
8	16	Multiplication by Δw .
4	8	Multiplication by Δw .
28	56	Multiplication by Δw .
4	8	Multiplication by Δw .
980	32 216 8 32 1672	Multiplication $1960 = 980 \cdot \Delta w$. Splitting ($1960 = 32 + 216 + 8 + 32 + 1672$)
8	16	Multiplication by Δw .
4	8	Multiplication by Δw .
28	56	Multiplication by Δw .
4	8	Multiplication by Δw .
	32 216 8 32 7816 16 8 56 8 32	New items

v = 9 w = 15 $\Delta w = 2$	v = 10 w = 16 $\Delta w = 1$	
32	32	Multiplication by Δw .
16	16	Multiplication by Δw .
8	8	Multiplication by Δw .
104	4 100	Multiplication by Δw . Splitting.
16	16	Multiplication by Δw .
8	8	Multiplication by Δw .
56	4 52	Multiplication by Δw . Splitting.
8	8	Multiplication by Δw .
32	4 28	Multiplication by Δw . Splitting.
392	4 388	Multiplication by Δw . Splitting.
16	16	Multiplication by Δw .
8	8	Multiplication by Δw .
56	4 52	Multiplication by Δw . Splitting.
8	8	Multiplication by Δw .
32	4 28	Multiplication by Δw . Splitting.
216	4 212	Multiplication by Δw . Splitting.
8	8	Multiplication by Δw .

v = 9 w = 15 $\Delta w = 2$	v = 10 w = 16 $\Delta w = 1$	
32	4 28	Multiplication by Δw . Splitting.
1672	4 124 4 1540	Multiplication by Δw . Splitting.
16	16	Multiplication by Δw .
8	8	Multiplication by Δw .
56	4 52	Multiplication by Δw . Splitting.
8	8	Multiplication by Δw .
32	4 28	Multiplication by Δw . Splitting.
216	4 212	Multiplication by Δw . Splitting.
8	8	Multiplication by Δw .
32	4 28	Multiplication by Δw . Splitting.
7816	4 124 4 852 8 4 28 4 124 4 6660	Multiplication by Δw . Splitting.
16	16	Multiplication by Δw .
8	8	Multiplication by Δw .
56	4 52	Multiplication by Δw . Splitting.
8	8	Multiplication by Δw .
32	4 28	Multiplication by Δw . Splitting.
	4 212 8 4 28 4 124 4 852 8 4 28 4 124 4 31236 16 8 4 52 8 4 28 4 212 8	New items

v = 9 w = 15 $\Delta w = 2$	v = 10 w = 16 $\Delta w = 1$	
	4 28 4 124 4 852 8 4 28 4 124 4	

The example of the evolution from rank $v = 9$ to rank $v = 10$ shows the rapid complexification of the splitting for larger rankings. The ranks where $\Delta w = 1$ show a much stronger increase in complexity compared to the $\Delta w = 2$ ranks, as well for the splitting (previous table) that for new items (summary below).

v = 5 w = 8 $\Delta w = 1$	v = 6 w = 10 $\Delta w = 2$	v = 7 w = 12 $\Delta w = 2$	v = 8 w = 13 $\Delta w = 1$	v = 9 w = 15 $\Delta w = 2$	v = 10 w = 16 $\Delta w = 1$	v = 11 w = 18 $\Delta w = 2$
/	32	8	4	32	4	32
		248	28	216	124	984
		8	4	8	4	8
			980	32	852	32
			8	7816	8	6792
			4	16	4	16
			28	8	28	8
			4	56	4	56
				8	124	8
				32	4	32
					31236	216
					16	8
					8	32
					4	984
					52	8
					8	32
					4	249832
					28	32
					4	16
					212	8
					8	104
					4	16
					28	8
					4	56
					124	8
					4	32
					852	392
					8	16
					4	8
					28	56
					4	8
					124	32
					4	216
						8
						32

v = 5 w = 8 $\Delta w = 1$	v = 6 w = 10 $\Delta w = 2$	v = 7 w = 12 $\Delta w = 2$	v = 8 w = 13 $\Delta w = 1$	v = 9 w = 15 $\Delta w = 2$	v = 10 w = 16 $\Delta w = 1$	v = 11 w = 18 $\Delta w = 2$
						1672
						16
						8
						56
						8
						32
						216
						8
						32
						984
						8
						32
						6792
						16
						8
						56
						8
						32
						216
						8
						32
						984
						8
						32

In this recapitulation of the additions, we have provided an additional rank data ($v = 11$).
We divide the set of values by $4 \cdot \Delta w(v)$, which gives a new table :

v = 5 w = 8 $\Delta w = 1$	v = 6 w = 10 $\Delta w = 2$	v = 7 w = 12 $\Delta w = 2$	v = 8 w = 13 $\Delta w = 1$	v = 9 w = 15 $\Delta w = 2$	v = 10 w = 16 $\Delta w = 1$	v = 11 w = 18 $\Delta w = 2$
/	4	1 31	1 7	4	1 31	4
		1	1 245 2 1 7	27 1 4 977 2 1 7 1 4	1 213 2 1 7 1 31	123 1 4
			1		1 1665 4 2 1 13 2 1 7 1 53 2 1 7 1 31	849 2 1 7 1 4 27 1 4 123 1 4

v = 5 w = 8 $\Delta w = 1$	v = 6 w = 10 $\Delta w = 2$	v = 7 w = 12 $\Delta w = 2$	v = 8 w = 13 $\Delta w = 1$	v = 9 w = 15 $\Delta w = 2$	v = 10 w = 16 $\Delta w = 1$	v = 11 w = 18 $\Delta w = 2$
					1 213 2 1 7 1 31	
						31229 4 2 1 13 2 1 7 1 4 49 2 1 7 1 4 27 1 4 209 2 1 7 1 4 27 1 4 123 1 4 849 2 1 7 1 4 27 1 4 123 1 4

In this table, where we kept the order of appearance, we have grouped the values so that the sums are systematically equal in powers of 2, with the exception of the first or the last section which has to be added to the others to have the same powers of 2' property.

When $\Delta w(v-1) = 2$, (hence at rank $v-1$.) the first line of each box and each subdivision is equal to 1.

When $\Delta w(v-1) = 1$, then it is the last line of each box and subdivision which is equal to 4.

These few examples show the importance of the powers of 2 and what we will call the "history" (or background) of the process, namely here the previous ranks with the values of $\Delta w(v)$ and $\Delta w(v-1)$. We have compiled tables of differences $cte_v(i)$ up to rank $v = 17$ (the table including then 312455 associates). For a comprehensive study, the presentation of the tables in their entirety is necessary. For a complete locating, we will note the elements of the table at rank v by $ddsl_v(i,j)$, with line index i and column index j ($i = 1$ and $j = 1$ for the first element at the upper left).

Here are the first tables (tables of origin) after divisions by $4.\Delta w(v)$:

$v = 5$

1	1	1
---	---	---

$v = 6$

1	1	1	1
	4	4	4

$v = 7$

2	2	2	2	2
1	1	1	1	1
	7	7	7	7
	1	1	1	1
		31	31	31
		1	1	1

$v = 8$

4	4	4	4	4	4
2	2	2	2	2	2
1	1	1	1	1	1
	13	13	13	13	13
	2	2	2	2	2
	1	1	1	1	1
	7	7	7	7	7
	1	1	1	1	1
		53	53	53	53
		2	2	2	2
		1	1	1	1
		7	7	7	7
		1	1	1	1
			245	245	245
			2	2	2
			1	1	1
			7	7	7
			1	1	1

A set of rules will allow us to move from one table to the next (and vice versa to the previous).

Rule 1. Rule of the staffs of the original table of differences of signatures $ddsl_v(i,j)$.

This rule is trivial but nevertheless necessary.

The number of elements in a column of the table $ddsl_v$ is equal to that of the table of associates minus 1 :

$$\Sigma_i \#ddsl_v(i,j) = \Sigma_i \#a_v(i,j) - 1 \quad (4)$$

The staffs of the table of associates are obtained by referring to the study made in our previous articles [1] and [2]. We thus have for $\Sigma_i \#a_v(i,j)$ and $\Sigma_i \#ddsl_v(i,j)$ respectively (first and second lines) :

$v = 5$

1	2	2	2
0	1	1	1

$v = 6$

1	2	3	3	3
0	1	2	2	2

$v = 7$

1	3	5	7	7	7
0	2	4	6	6	6

$v = 8$

1	4	9	14	19	19	19
0	3	8	13	18	18	18

$v = 9$

1	4	10	19	28	37	37	37
0	3	9	18	27	36	36	36

$v = 10$

1	5	14	30	53	76	99	99	99
0	4	13	29	52	75	98	98	98

$v = 11$

1	5	15	34	65	108	151	194	194	194
0	4	14	33	64	107	150	193	193	193

This rule allows studying only the last column and then coming back on the other elements at the end of the study. The second rule is so committed to this last column.

Rule 2. Rule of the staffs of first generation.

The rule is to take over the second line of the above tables and perform the differences column to column.

$v = 5$

1	1	1
1	/	/

$v = 6$

1	2	2	2
1	1	/	/

$v = 7$

2	4	6	6	6
2	2	2	/	/

$v = 8$

3	8	13	18	18	18
3	5	5	5	/	/

$v = 9$

3	9	18	27	36	36	36
3	6	9	9	9	/	/

$v = 10$

4	13	29	52	75	98	98	98
4	9	16	23	23	23	/	/

$v = 11$

4	14	33	64	107	150	193	193	193
4	10	19	31	43	43	43	/	/

Due to the algorithm that gives the staffs of the Pascal trihedron, we have systematically three identical columns followed two voids in each of the seconds preceding lines (for $v > 7$).

This rule allows to draw up tables of the last column following the said successive staffs and this process step can be done backwards. These tables are called as "first-generation":

v = 5

1

v = 6

1	4
---	---

v = 7

2	7	31
1	1	1

v = 8

4	13	53	245
2	2	2	2
1	1	1	1
	7	7	7
	1	1	1

v = 9

4	13	49	209	977
2	2	2	2	2
1	1	1	1	1
	7	7	7	7
	1	1	1	1
	4	4	4	4
		27	27	27
		1	1	1
		4	4	4

v = 10

8	25	97	385	1665
4	4	4	4	4
2	2	2	2	2
1	1	1	1	1
	13	13	13	13
	2	2	2	2
	1	1	1	1
	7	7	7	7
	1	1	1	1
		53	53	53
		2	2	2
		1	1	1
		7	7	7
		1	1	1
		31	31	31
		1	1	1

v = 11

8	25	93	381	1533	6653	31229
4	4	4	4	4	4	4
2	2	2	2	2	2	2
1	1	1	1	1	1	1
	13	13	13	13	13	13
	2	2	2	2	2	2
	1	1	1	1	1	1
	7	7	7	7	7	7
	1	1	1	1	1	1
	4	4	4	4	4	4
		49	49	49	49	49
		2	2	2	2	2
		1	1	1	1	1
		7	7	7	7	7
		1	1	1	1	1

		4	4	4	4	4
		27	27	27	27	27
		1	1	1	1	1
		4	4	4	4	4
			209	209	209	209
			2	2	2	2
			1	1	1	1
			7	7	7	7
			1	1	1	1
			4	4	4	4
			27	27	27	27
			1	1	1	1
			4	4	4	4
			123	123	123	123
			1	1	1	1
			4	4	4	4
				849	849	849
				2	2	2
				1	1	1
				7	7	7
				1	1	1
				4	4	4
				27	27	27
				1	1	1
				4	4	4
				123	123	123
				1	1	1
				4	4	4

We can also call this rule, the rule of maximum element (of first generation) location, because by construction of the signatures, the difference of the signatures will be here always superior to all those preceding. It is actually the situation that would lead the reader, in absence of knowledge of the algorithm, to change column.

From how signatures are settled, all lines of the table that are thus obtained are identical items, except the voids at the top of lines and except the first line.

Rule 3. Rule of column sums.

Let us note by $sddsl_v(j) = \sum_i dds_l_v(i,j)$ the sum of column j .

Then we have :

$$sddsl_v(j) = 4.\Delta w(v).sddsl_{v-1}(j-1) \quad (5)$$

Examples :

v	$\Delta w(v)$	$sddsl_v(1)$	$sddsl_v(2)$	$sddsl_v(3)$	$sddsl_v(4)$	$sddsl_v(5)$	$sddsl_v(6)$	$sddsl_v(7)$
6	2	1	4					
7	2	3	8	32				
8	1	7	24	64	256			
9	2	7	28	96	256	1024		
10	1	15	56	224	768	2048	8192	
11	2	15	60	224	896	3072	8192	32768

This rule allows, knowing the elements of the last column of the previous table, to deduce all of the elements of the first line, from the knowledge of rank $v-1$.

This then allows focusing only on the last column of the table.

Rule 4. Rule of maxima.

The principle here is to continue the process of maxima's location on the last column of the table already obtained and to repeat it so long maxima are available, the only exception being the first element that is placed in the first column.

For $v = 11$, we will have a table of "second-generation":

31229	4	13	49	209	849
	2	2	2	2	2
	1	1	1	1	1
		7	7	7	7
		1	1	1	1
		4	4	4	4
			27	27	27
			1	1	1
			4	4	4
				123	123
				1	1
				4	4

Then of third generation :

849	2	7	27	123
	1	1	1	1
		4	4	4

Rule 5. Rule of models.

There are 5 final models in the maxima's location process for $v < 18$.

It turns out that the history must be analysed at least up to rank $v-4$ to tell about the origin of these models. Below, we give these models starting with the "history" which gives their rise :

Lists of v	8 13 (20) (25) ...	10 15 (22) (27) ...	5 7 12 17 (19) (24) (29) ...	9 14 (21) (26) ...	6 11 16 (18) (23) (28) (30) ...
$\Delta w(v-1)$	2	2	2	1	1
$\Delta w(v-2)$	2	1	1	2	2
$\Delta w(v-3)$	1	2	2	2	1
$\Delta w(v-4)$	2	2	1	1	2
Models	1	2	3	4	5

Model 1 :

245	2	7
	1	1

Model 2 :

213	2	7	31
	1	1	1

Model 3 :

981	2	7	31
	1	1	1

Model 4 :

977	2	7	27
	1	1	1
		4	4

Model 5 :

849	2	7	27	123
	1	1	1	1
		4	4	4

The v list in parentheses corresponds to the $\Delta w(v-k)$ ' history but we have not checked if the model was good.

It should be noted that for small v , the full model does not appear, due to the absence of sufficient numbers of elements, but only the final part.

Thus, for $v = 6$, model 5 is reduced to :

1
4

For $v = 5$ and $v = 7$, model 3 is reduced to respectively :

1	and	2	7	31
		1	1	1

Property 3

The sum of the last column of a complete model is a power of 2.

The sum of all the elements of a complete model still is a power of 2.

In addition, the sum of all the elements and the sum of the last column of each generation are powers of 2.

Rule 6. Rule of the staffs of i -th generations.

The cuts on maximum elements correspond to the continuation of rule 2 by subtraction of staffs column to column.

For example $v = 11$

Origin		4	14	33	64	107	150	193	193	193
First generation		4	10	19	31	43	43	43	/	/
Second generation	(1)	3	6	9	12	12	/	/	/	/
Third generation	(1)	2	3	3	3	/	/	/	/	/
Fourth generation	(1)	1	1	/	/	/	/	/	/	/

The reader can see in the example given for rule 4 that the staffs of the second and third generations are effectively the ones expected. In addition we extended here count beyond the model 5 adopted earlier, which would result in :

123	1	4
-----	---	---

The staffs of one generation to the next increase by one or two units (in relation with the history of the Δw of course).

Going backwards generations, it may be possible to return to the original table as long as one controls the values of the elements of the first line.

For $v = 11$, let us review these :

First generation		8	25	93	381	1533	6653	31229
Second generation	(31229)	4	13	49	209	849		
Third generation	(849)	2	7	27	123			
Fourth generation	(123)	1	4					

Let us call review table the above table and, for us, the first column will now be the one after the numbers in parentheses.

Rule 7. Rule of the powers of 2.

The first column of the review table are decreasing powers of 2 resulting in 1

Rule 8. Rule of model construction.

The construction of the model requires the last two generations up to $v = 17$.

The last element of the latest generation (last line non-empty element on the right side) is equal 1 or 4 in consistency with the model at work. It will thus locate at first column for 1 to 3 models and in second for models 4 and 5.

Rule 9. Rule of extension of the model.

A model gives birth of the common extensions.

We will return to this extension notion later on with more details.

For now, let give a few examples by adding at the end of tables the relevant ranks and their history $\Delta w(v)$, $\Delta w(v-1)$, $\Delta w(v-2)$, $\Delta w(v-3)$, ..., $\Delta w(6)$.

This history is stopped at the level of $\Delta w(6)$ because this rank is effectively, after checking it, the rank suited to the proposed method.

Extension of Model 1

32	97	361	1449	5801	24233	97961	425641	1998505
16	49	185	761	3065	13305	54265		
8	25	97	417	1697	7841			
4	13	53	245					
2	7							
1								
v = 13	1	2	2	1	2	1	2	2
v = 8	1	2	2					

Extension of Model 2

64	193	713	2793	11241	46057	185321	775145	3134441	13620201	63951849
32	97	361	1417	5769	24201	97929	425609	1736329		
16	49	185	729	3033	13273	54233	250841			
8	25	97	385	1665	7809					
4	13	53	213							
2	7	31								
1										
v = 15	1	2	1	2	2	1	2	1	2	2
v = 10	1	2	1	2	2					

Extension of Model 3

128	385	1417	5545	22185	88745	359081	1473193	5929641	24804009	100301481	435845801	2046458537
64	193	713	2793	11241	45033	184297	774121	3133417	13619177	55562217		
32	97	361	1417	5769	23177	96905	424585	1735305	8026761			
16	49	185	729	3033	12249	53209	249817					
8	25	97	385	1665	6785							
4	13	53	213	981								
2	7	31										
1												
v = 17	1	2	1	2	1	2	2	1	2	1	2	2
v = 12	2	2	1	2	1	2	2					
v = 7	2	2										
v = 5	1											

The 1 of rank 5 in last line is crossed as its history is already out of fields. However, this value is recalled here for the perfect understanding of what will follow later on.

Extension of Model 4

32	97	357	1413	5765	23173	96901	391813	1702533	7993989
16	49	181	725	3029	12245	53205	217045		
8	25	93	381	1661	6781	31357			
4	13	49	209	977					
2	7	27							
1	4								
v = 14	2	1	2	2	1	2	1	2	2
v = 9	2	1	2	2					

Extension of Model 5

64	193	709	2789	11109	44901	184165	741221	3100517	12537701	54480741	255807333
32	97	357	1413	5637	23045	96773	391685	1702405	6945285		
16	49	181	725	2901	12117	53077	216917	1003349			
8	25	93	381	1533	6653	31229					
4	13	49	209	849							
2	7	27	123								
1	4										
v = 16	2	1	2	1	2	2	1	2	1	2	2
v = 11	2	1	2	1	2	2					
v = 6	2										

The reader will notice without any effort that any column with the same number at the end of chain is the identical over all its height regardless of the model.

Rule 10. Rule of the notches.

The histories settle the notches of 1 or 2 units up models.

The histories consist, as we already know, of isolated 1, isolated 2 or 2 in pair.
We have then :

Isolated 1 on column j	Shift a notch on column j
Isolated 2 on column j	No lag, no notch
Pair of 2 on column j and column j + 1	Shift a notch for the pair on column j and j + 1 column

Rule 11. Rule of linearity (affine rule).

The element $r_{m(k)}(i,j)$ of the review table in position (i, j) deduces from that in position (i+1, j) by :

$$r_{m(k)}(i,j) = 2.r_{m(k)}(i+1,j) - \text{cte}_{m(k)}(j) \quad (6)$$

where $\text{cte}_{m(k)}(j)$ are constants for a given model and column.

For each model, we give below the values of $\text{cte}_{m(k)}(j)$.

	$\text{cte}_{m(k)}(1)$	$\text{cte}_{m(k)}(2)$	$\text{cte}_{m(k)}(3)$	$\text{cte}_{m(k)}(4)$	$\text{cte}_{m(k)}(5)$	$\text{cte}_{m(k)}(6)$	$\text{cte}_{m(k)}(7)$	$\text{cte}_{m(k)}(8)$	$\text{cte}_{m(k)}(9)$	$\text{cte}_{m(k)}(10)$
Model 1	0	1	9	73	329	2377	10569	...	...	...
Model 2	0	1	9	41	297	2345	10537	76073	338217	...
Model 3	0	1	9	41	297	1321	9513	75049	337193	2434345
Model 4	0	1	5	37	293	1317	9509	42277	...	...
Model 5	0	1	5	37	165	1189	9381	42149	304293	...

Let us look at the column to column subtractions :

	$\text{cte}_{m(k)}(j) - \text{cte}_{m(k)}(j-1)$									
Model 1		1	8	64	256	2048	8192	...	...	...
Model 2		1	8	32	256	2048	8192	65536	262144	...
Model 3		1	8	32	256	1024	8192	65536	262144	2097152
Model 4		1	4	32	256	1024	8192	32768	...	...
Model 5		1	4	32	128	1024	8192	32768	262144	...

It comes naturally again to powers of 2, with leaps of 2 or 3 from one column to the next:

	Powers of 2 of $\text{cte}_{m(k)}(j) - \text{cte}_{m(k)}(j-1)$									
Model 1		0	3	6	8	11	13	...	...	...
Model 2		0	3	5	8	11	13	16	18	...
Model 3		0	3	5	8	10	13	16	18	21
Model 4		0	2	5	8	10	13	15	...	...
Model 5		0	2	5	7	10	13	15	18	...

By then the rapprochement with the history $\Delta w(v-k)$ of models, we observe a common behaviour that makes us rewrite $\text{cte}_{m(k)}(j)$ as a form which is not dependent on a model but simply of v:

$$\text{cte}_v(1) = 0 \quad (7)$$

$$\text{cte}_v(2) = 1 \quad (8)$$

$$\text{cte}_v(3) = 1 + 2^{\Delta w(v-1)+1} \quad (9)$$

$$\text{cte}_v(j) = 1 + 2^{\Delta w(v-1)+1} + 2^{\Delta w(v-2)+1} + \dots + 2^{\Delta w(v-j+2)+1} \quad (10)$$

The element $r_{m(k)}(i,j)$ of the review table is thus freed in the recursive expression of the model :

$$r_v(i,j) = 2.r_v(i+1,j) - \text{cte}_v(j) \quad (11)$$

We have, to finish with, to find the column's endings values to control all aspects of the routine.
To do this, we will first write down two tables that we juxtapose :

m	2^{m-1}	2^m+1	$(2^{m+1})/2^{m-1}$	2^{m+2}	2^m	s1(m)	s2(m)	st(m)
∞	∞	∞	2	∞	∞	∞	∞	4
...	...	...	...	...	...	...	...	...
9	256	513	2,00390625	2048	512	4104	1025	4,00390244
8	128	257	2,0078125	1024	256	2056	513	4,00779727
7	64	129	2,015625	512	128	1032	257	4,0155642
6	32	65	2,03125	256	64	520	129	4,03100775
5	16	33	2,0625	128	32	264	65	4,06153846
4	8	17	2,125	64	16	136	33	4,12121212
3	4	9	2,25	32	8	72	17	4,23529412
2	2	5	2,5	16	4	40	9	4,44444444
1	1	3	3	8	2	24	5	4,8
0	0,5	2	4	4	1	16	3	5,33333333
-1	0,25	1,5	6	2	0,5	12	2	6
-2	0,125	1,25	10	1	0,25	10	1,5	6,66666667
-3	0,0625	1,125	18	0,5	0,125	9	1,25	7,2
...	...	...	...	...	...	...	...	...
$-\infty$	0	1	∞	0	0	8	1	8

Let us have :

$$\text{sb}(m) = (2^m+1)/2^{m-1} \quad (12)$$

Then :

$$s1(1) = 24 \quad (13)$$

$$s1(m) = s1(m-1) + 2^{m+2} \quad (14)$$

$$s2(1) = 5 \quad (15)$$

$$s2(m) = s2(m-1) + 2^m \quad (16)$$

$$\text{st}(m) = s1(m)/s2(m) \quad (17)$$

Note: The part $m < 0$ of the tables does not serve subsequently and is only here to show the harmony of the lower and upper boundaries : $8 = 4 * 2$.

We then return to the models' extensions and do the differences column to column :

Differences column to column of model extension 1

32	65	264	1088	4352	18432	73728	327680	1572864
16	33	136	576	2304	10240	40960		
8	17	72	320	1280	6144			
4	9	40	192					
2	5							
1								
v = 13	1	2	2	1	2	1	2	2
v = 8	1	2	2					

Differences column to column of model extension 2

64	129	520	2080	8448	34816	139264	589824	2359296	10485760	50331648
32	65	264	1056	4352	18432	73728	327680	1310720		
16	33	136	544	2304	10240	40960	196608			
8	17	72	288	1280	6144					
4	9	40	160							
2	5	24								
1										
v = 15	1	2	1	2	2	1	2	1	2	2
v = 10	1	2	1	2	2					

Differences column to column of model extension 3

128	257	1032	4128	16640	66560	270336	1114112	4456448	18874368	75497472	335544320	1610612736
64	129	520	2080	8448	33792	139264	589824	2359296	10485760	41943040		
32	65	264	1056	4352	17408	73728	327680	1310720	6291456			
16	33	136	544	2304	9216	40960	196608					
8	17	72	288	1280	5120							
4	9	40	160	768								
2	5	24										
1												
v = 17	1	2	1	2	1	2	2	1	2	1	2	2
v = 12	2	2	1	2	1	2	2					
v = 7	2	2										
v = 5	1											

Differences column to column of model extension 4

32	65	260	1056	4352	17408	73728	294912	1310720	6291456
16	33	132	544	2304	9216	40960	163840		
8	17	68	288	1280	5120	24576			
4	9	36	160	768					
2	5	20							
1	3								
v = 14	2	1	2	2	1	2	1	2	2
v = 9	2	1	2	2					

Differences column to column of model extension 5

64	129	516	2080	8320	33792	139264	557056	2359296	9437184	41943040	201326592
32	65	260	1056	4224	17408	73728	294912	1310720	5242880		
16	33	132	544	2176	9216	40960	163840	786432			
8	17	68	288	1152	5120	24576					
4	9	36	160	640							
2	5	20	96								
1	3										
v = 16	2	1	2	1	2	2	1	2	1	2	2
v = 11	2	1	2	1	2	2	1	2	1	2	1
v = 6	2	1	2	1	2	1					

Finally, let us do the ratio of each element in a column to the previous column (same line) :

Ratio of differences column to column of model extension 1

	2,03125	4,061538	4,121212	4	4,235294	4	4,444444	4,8
	2,0625	4,121212	4,235294	4	4,444444	4		
	2,125	4,235294	4,444444	4	4,8			
	2,25	4,444444	4,8					
	2,5							
v = 13	1	2	2	1	2	1	2	2
v = 8	1	2	2					

Ratio of differences column to column of model extension 2

	2,015625	4,031008	4	4,061538	4,121212	4	4,235294	4	4,444444	4,8
	2,03125	4,061538	4	4,121212	4,235294	4	4,444444	4		
	2,0625	4,121212	4	4,235294	4,444444	4	4,8			
	2,125	4,235294	4	4,444444	4,8					
	2,25	4,444444	4							
	2,5	4,8								
v = 15	1	2	1	2	2	1	2	1	2	2
v = 10	1	2	1	2	2					

Ratio of differences column to column of model extension 3

	2,007813	4,015564	4	4,031008	4	4,061538	4,121212	4	4,235294	4	4,444444	4,8
	2,015625	4,031008	4	4,061538	4	4,121212	4,235294	4	4,444444	4		
	2,03125	4,061538	4	4,121212	4	4,235294	4,444444	4	4,8			
	2,0625	4,121212	4	4,235294	4	4,444444	4,8					
	2,125	4,235294	4	4,444444	4							
	2,25	4,444444	4	4,8								
	2,5	4,8										
v = 17	1	2	1	2	1	2	2	1	2	1	2	2
v = 12	2	2	1	2	1	2	2					
v = 7	2	2										
v = 5	4											

Ratio of differences column to column of model extension 4

	2,03125	4	4,061538	4,121212	4	4,235294	4	4,444444	4,8
	2,0625	4	4,121212	4,235294	4	4,444444	4		
	2,125	4	4,235294	4,444444	4	4,8			
	2,25	4	4,444444	4,8					
	2,5	4							
	3								
v = 14	2	1	2	2	1	2	1	2	2
v = 9	2	1	2	2					

Ratio of differences column to column of model extension 5

	2,015625	4	4,031008	4	4,061538	4,121212	4	4,235294	4	4,444444	4,8
	2,03125	4	4,061538	4	4,121212	4,235294	4	4,444444	4		
	2,0625	4	4,121212	4	4,235294	4,444444	4	4,8			
	2,125	4	4,235294	4	4,444444	4,8					
	2,25	4	4,444444	4							
	2,5	4	4,8								
	3										
v = 16	2	1	2	1	2	2	1	2	1	2	2
v = 11	2	1	2	1	2	2					
v = 6	2										

Now, we can give the construction of the endings of column by proper links.

Rule 12. Rule of column endings.

Let us locate by $dr_{m(k)}(i,j)$ an element of a table of differences column to column of the extension of a given model and by $dr_{m(k)}(i_end,j)$, the last non-empty element of a column and by $dr_{m(k)}(i_end+i,j)$ on the same column the elements above offset by i -lines i .

Then, we have :

$\Delta w(v+2-j) = 1$ isolated on column j	$dr_{m(k)}(i_end+i, 1) = 2^i$ $dr_{m(k)}(i_end+i, 2) = sb(i+1)*dr_{m(k)}(i_end+i, 1) = 2^{i+1}+1$ $dr_{m(k)}(i_end+i, j > 2) = 4*dr_{m(k)}(i_end+i, j-1)$
$\Delta w(v+2-j) = 2$ isolated on column j	$dr_{m(k)}(i_end+i, 1) = 2^i$ $dr_{m(k)}(i_end+i, 2) = sb(i+2)*dr_{m(k)}(i_end+i+1, 1) = 2^{i+2}+1$ $dr_{m(k)}(i_end+i, j > 2) = st(i+1)*dr_{m(k)}(i_end+i, j-1)$
$\Delta w(v+2-j) = 2$ and $\Delta w(v+2-j-1) = 2$ in pair on column j and column j+1	$dr_{m(k)}(i_end+i, 1) = 2^i$ $dr_{m(k)}(i_end+i, 2) = sb(i+2)*dr_{m(k)}(i_end+i+1, 1) = 2^{i+2}+1$ $dr_{m(k)}(i_end+i, j > 2) = st(i+2)*dr_{m(k)}(i_end+i, j-1)$ $dr_{m(k)}(i_end+i, j+1 > 3) = st(i+1)*dr_{m(k)}(i_end+i, j)$

Note 1 : $\Delta w(v) = 2$ at the head of the list is considered to be isolated.

Note 2 : When the model (it comes to model 3), displays the possibility at the head list to have either $(\Delta w(v), \Delta w(v-1)) = (1,2)$ or $(\Delta w(v), \Delta w(v-1)) = (1,2)$, then everything happens as if the first choice wins (as a result of the initial $v = 5$ certainly).

The endings of column's rule in fact give the evaluation of the whole table $dr_{m(k)}(i,j)$ and makes obsolete the use of rule 11. We left it as it is interesting matter anyway.

Let us note also that the indexation of $dr_{m(k)}(i,j)$ is not essential which we express in the last rule.

Rule 13. Rule of limitation of model.

A model is unique and applicable only up to a certain rank v.

Due to the history Δw associated to $w = \text{int}(\ln(3)/\ln(2).v)+1$, as rank v increases, the events $\Delta w = 1$ or $\Delta w = 2$ arrive "randomly" and lead necessarily towards bifurcations featuring an infinity of more complex models. The models presented are however useful for priming of the routine.

All of the above rules allow going backwards up to the differences between the lines of the signatures for any v and thus to build a table of signatures. The table of signatures at rank v-1 and those in the previous ranks are not useful. The only necessity is the history $\Delta w(v)$ to $\Delta w(6)$.

Construction of the plane v of associates based on the signatures

Let us proceed by steps.

Property 4

An associate a, in column c ($c = 1$ identifying the first column) is of the form $t.2^{r-1}$, with $r = v+1-c$, r the number of systematic (IP) steps ahead of algorithm and t an odd integer.

But, there is no integer t' such as $a = t'.2^{r+1}-1$.

Proof

Indeed, this number a is odd and the first result of Collatz algorithm (a $3a+1$ operation) is an even number $3t.2^{r-2}$ that can be divided by two providing $3t.2^{r-3}-1$, thus returning to the original form. This number is necessarily odd as long as the exponent of 2 is strictly positive and must therefore undergo a step (IP) after that. We have to conduct r-1 other identical iterations thus leading to $3^r.t.2^0-1 = 3^r.t-1$. This time, this number is necessarily even since t is odd, showing indeed that we cannot have another $r+1^{\text{th}}$ step (IP).

Moreover, the formula $r = v+1-c$ follows trivially from property 1.

Let us now address a fundamental point.

Theorem 6

For any licit signature, there is a number that generates this signature.

Proof

We start with an example for a good understanding of the algorithm that is involved. So let us look for the number m that generates the licit signature 1010101010100100001000.

The Collatz algorithm imposes then the following three pairs of equations :

$$\begin{aligned}
 m &= r_1.2^{v_1}-1 \\
 r_1.3^{v_1}-1 &= m_1.2^{w_1-v_1} \\
 m_1 &= r_2.2^{v_2-v_1}-1
 \end{aligned}$$

$$\begin{aligned} r_2 \cdot 3^{v_2-v_1}-1 &= m_2 \cdot 2^{w_2-w_1-(v_2-v_1)} \\ m_2 &= r_3 \cdot 2^{v_3-v_2}-1 \\ r_3 \cdot 3^{v_3-v_2}-1 &= m_3 \cdot 2^{w_3-w_2-(v_3-v_2)} \end{aligned}$$

Here the couples $(v_1 = 7, w_1 = 8)$, $(v_2 = 8, w_2 = 12)$, $(v_3 = 9, w_3 = 15)$ summarize the signature and allow us to write the operations that take place according to the Collatz algorithm.

For more complex signatures, we will just have to add as many pairs of equations, that will be needed, based on the previous model.

The resolution of this system is then done by addressing in reverse the last equation up to the first one. It alternates diophantine equations of type $ax+by=c$ and simple multiplication giving intermediate values m_i before getting the final m . We have $(a,b) = (2^i, 3^j)$, so a and b are coprime, and the couple $(x,y) = (cx_0, cy_0)$ is solution of equation $ax_0+by_0=1$. By Bézout-Bachet theorem, we are sure of the existence of a solution (s_1, s_2) for this last equation that we complete with the arithmetic series $(s_1+k.y_0, s_2-k.x_0)$.

So, for our example :

$$\begin{aligned} r_3 \cdot 3^{9-8}-1 &= m_3 \cdot 2^{15-12-(9-8)}, \text{ so that } 3r_3-1=4m_3, \text{ so that } r_3=3+4k_1 \text{ and } m_3=2+3k_1 \\ m_2 &= r_3 \cdot 2^{9-8}-1, \text{ so that } m_2=5+8k_1 \\ r_2 \cdot 3^{8-7}-1 &= m_2 \cdot 2^{12-8-(8-7)}, \text{ so that } 3r_2-64k_1=41, \text{ so that } r_2=35+64k_2 \text{ and } k_1=1+3k_2 \\ m_1 &= r_2 \cdot 2^{8-7}-1, \text{ so that } m_1=69+128k_2 \\ r_1 \cdot 3^7-1 &= m_1 \cdot 2^{8-7}, \text{ so that } 2187r_1-256k_2=139, \text{ so that } r_1=1+256k_3 \text{ and } k_2=8+2187k_3 \\ m &= r_1 \cdot 2^7-1, \text{ so that } m=127+32768k_3 \end{aligned}$$

Thus we get finally a set of solution $m = 127+2^{15} \cdot k = 127+2^w \cdot k$, with 127 the associate solution ranging between 0 and 2^w , where k is any integer (negative numbers are solutions also by adapting the flight altitude concept taking absolute values of numbers).

The pairs of equations' system, which we have written above, apply to any signature (including signatures which are not licit, knowing however that there is no need to resolve such systems). An intermediate pair of equations is written whenever appears more than one consecutive 0 in the signature. Each Bézout-Bachet equation leads in its following multiplicative equation to an intermediate m_i of the form $c_i+2^{w_i} \cdot k_j$ and the whole resolution results in a final m of the form $c_f+2^w \cdot k$ where w is the number of 0 in the signature and k any relative integer (in $\mathbb{Z}$), which completes our proof.

In the same time, we have proved also :

Theorem 7

Any odd integer of the arithmetic progression m modulo 2^w has same signature as m .

Definition 7

We call positive sorting the implementation of the classification of a plane v with such signatures as described above.

We can also write immediately (without developing more arguments).

Theorem 8

There is a positive sorting for each Pascal trihedron plane.

5 Pascal trihedron classification by positive sorting.

For plane $v = 7$, using positive sorting, we get the following associates :

383	2239	2975	2031	615	2587
	1855	2591	1647	231	2203
	1087	1823	879	3559	1435
		4063	3119	1703	3675
		3295	2351	935	2907
			1231	3911	1787
			463	3143	1019

The differences from a column to another are as follows ($2^w = 2^{12} = 4096$) :

383-2239 = 2240-4096	2239-2975 = 3360-4096	2975-2031 = 944	2031-615 = 1416	615-2587 = 2124-4096
	1855-2591 = 3360-4096	2591-1647 = 944	1647-231 = 1416	231-2203 = 2124-4096
	1087-1823 = 3360-4096	1823-879 = 944	879-3559 = 1416-4096	3559-1435 = 2124
		4063-3119 = 944	3119-1703 = 1416	1703-3675 = 2124-4096
		3295-2351 = 944	2351-935 = 1416	935-2907 = 2124-4096
			1231-3911 = 1416-4096	3911-1787 = 2124
			463-3143 = 1416-4096	3143-1019 = 2124

The differences from a line to another are as follow :

2239-1855 = 384	2975-2591 = 384	2031-1647 = 384	615-231 = 384	2587-2203 = 384
1855-1087 = 768	2591-1823 = 768	1647-879 = 768	231-3559 = 768-4096	2203-1435 = 768
	1823-4063 = 1856-4096	879-3119 = 1856-4096	3559-1703 = 1856	1435-3675 = 1856-4096
	4063-3295 = 768	3119-2351 = 768	1703-935 = 768	3675-2907 = 768
		2351-1231 = 1120	935-3911 = 1120-4096	2907-1787 = 1120
		1231-463 = 768	3911-3143 = 768	1787-1019 = 768

Theorem 9

The differences from a column to another are constants modulo 2^w .

In the same way, the differences from a line to another :

$$a_v(i,j)-a_v(i-1,j) = ct(i) \bmod 2^w \quad (18)$$

$$a_v(i,j)-a_v(i,j-1) = ct(j) \bmod 2^w \quad (19)$$

Numerical example :

Before we start the theorem proof and in order to easier understand the mechanisms involved, let us look at in details to signatures (Sig) and evolutions by Collatz algorithm (Alg) for a numerical example. We consider the second and third lines of the plane $v = 7$.

1855	2591	1647	231	2203
1087	1823	879	3559	1435

Investigating differences between lines, the number 1855 is associated with 1087, the number 2591 with 1823, and so on, and we get the following table :

Table 1

1855	1087	2591	1823	1647	879	231	3559	2203	1435	1855 mod 2^{12} = -2241	1087	2591 mod 2^{12} = -1505	1823	1647 mod 2^{12} = -2449	879	231	3559	2203 mod 2^{12} = -1893	1435
Sig										Alg									
1	1	1	1	1	1	1	1	1	1	-6722	3262	-4514	5470	-7346	2638	694	10678	-5678	4306
0	0	0	0	0	0	0	0	0	0	-3361	1631	-2257	2735	-3673	1319	347	5339	-2839	2153
1	1	1	1	1	1	1	1	1	1	-10082	4894	-6770	8206	-11018	3958	1042	16018	-8516	6460
0	0	0	0	0	0	0	0	0	0	-5041	2447	-3385	4103	-5509	1979	521	8009	-4258	3230
1	1	1	1	1	1	1	1	0	0	-15122	7342	-10154	12310	-16526	5938	1564	24028	-2129	1615
0	0	0	0	0	0	0	0	1	1	-7561	3671	-5077	6155	-8263	2969	782	12014	-6386	4846
1	1	1	1	1	1	0	0	0	0	-22682	11014	-15230	18466	-24788	8908	391	6007	-3193	2423
0	0	0	0	0	0	1	1	1	1	-11341	5507	-7615	9233	-12394	4454	1174	18022	-9578	7270
1	1	1	1	0	0	0	0	0	0	-34022	16522	-22844	27700	-6197	2227	587	9011	-4789	3635
0	0	0	0	1	1	1	1	1	1	-17011	8261	-11422	13850	-18590	6682	1762	27034	-14366	10906
1	1	0	0	0	0	0	0	0	0	-51032	24784	-5711	6925	-9295	3341	881	13517	-7183	5453
0	0	1	1	1	1	1	1	1	1	-25516	12392	-17132	20776	-27884	10024	2644	40552	-21548	16360
0	0	0	0	0	0	0	0	0	0	-12758	6196	-8566	10388	-13942	5012	1322	20276	-10774	8180
0	0	0	0	0	0	0	0	0	0	-6379	3098	-4283	5194	-6971	2506	661	10138	-5387	4090
1	0	1	0	1	0	1	0	1	0	-19136	1549	-12848	2597	-20912	1253	1984	5069	-16160	2045
0	1	0	1	0	1	0	1	0	1	-9568	4648	-6424	7792	-10456	3760	992	15208	-8080	6136
0	0	0	0	0	0	0	0	0	0	-4784	2324	-3212	3896	-5228	1880	496	7604	-4040	3068
0	0	0	0	0	0	0	0	0	0	-2392	1162	-1606	1948	-2614	940	248	3802	-2020	1534
0	0	0	0	0	0	0	0	0	0	-1196	581	-803	974	-1307	470	124	1901	-1010	767

In the table, on the right side, which focuses on the Collatz algorithm (Alg), the value of the first selected element 1855 has been replaced by $1855 - 4096 = -2241$. Similarly for $2591 - 4096 = -1505$, $1647 - 4096 = -2449$ and $2203 - 4096 = -1893$. On the other hand, we left unchanged 231. The goal here is to have coherence between two associated values, because the Collatz algorithm does not involve modulo operations, thus the differences of a value at head of column to his associated head value must be identical to ensure that the following data have some clarity. Thus, by doing as we do, we get a common first line's difference : $1087 - (-2241) = 1823 - (-1505) = 879 - (-2449) = 3559 - 231 = 1435 - (-1893) = 3328$ and this without involving modulo operations.

The differences column to column are then as follows (table of differences):

Table 2

3328	3328	3328	3328	3328
1792	1792	1792	1792	1792
896	896	896	896	896
2688	2688	2688	2688	2688
3392	3392	3392	3392	3392
1984	1984	1984	1984	3744
3040	3040	3040	3040	3040
928	928	928	1520	1520
464	464	464	464	464
1392	1392	232	232	232
696	696	696	696	696
2088	348	348	348	348
1044	1044	1044	1044	1044
2570	2570	2570	2570	2570
1285	1285	1285	1285	1285
205	3157	1685	3085	1821
1928	1928	1928	1928	1928
3012	3012	3012	3012	3012
3554	3554	3554	3554	3554
1777	1777	1777	1777	1777

So we check in the previous table equality of differences between associated two lines elements in correspondence. We see that this equality is maintained throughout the evolution by the algorithm of Collatz except in some places that we have bold (in black, green, and blue) with a return to identical values as soon as the first following line the so-called evolution.

The reader will also notice that $3744 * 6 = 1984 \bmod 4096$, $1520 * 6 = 928 \bmod 4096$, $232 * 6 = 1392 \bmod 4096$, $348 * 6 = 2088 \bmod 4096$ which is due to a simple shift of an operation of multiplication by 3 (+ 1 falling by difference) and division by 2.

If instead of starting from :

-2241	1087	-1505	1823	-2449	879	231	3559	-1893	1435
-------	------	-------	------	-------	-----	-----	------	-------	------

We choose to start from :

-2241	1087	-1505	1823	-2449	879	231	3559	-1893	1435
+4096k	+4096k	+4096k	+4096k	+4096k	+4096k	+4096k	+4096k	+4096k	+4096k

table 2 will be unchanged except for the line in bold with the following result :

205+488k mod 4096	3157+488k mod 4096	1685+488k mod 4096	3085+488k mod 4096	1821+488k mod 4096
----------------------	-----------------------	-----------------------	-----------------------	-----------------------

If now, we start from :

-2241	1087 +4096k	-1505	1823 +4096k	-2449	879 +4096k	231	3559 +4096k	-1893	1435 +4096k
-------	----------------	-------	----------------	-------	---------------	-----	----------------	-------	----------------

then table 2 evolves from either starting at the third line (for $k = 1$ for example), or at the fifth line (for $k = 6$, for example), ..., or at the second last line (for $k = 2048$ here), that is depending on the choice of k . In particular, varying k from 0 to 4095, we observe that the last line takes all values from 0 to 4095.

Specifically, we have writing df the value of the last line ($3^v = 2187$) :

$$df = 1777 + k \cdot 3^v \bmod 2^w \quad (20)$$

In the case under consideration here, as an example, table 2 will show respectively for $k = 1293$ (where the last line is identical to the first) and $k = 3597$ (where the last line is equal to 0) as follows :

Table 3a et 3b

k = 1293					k = 3597				
3328	3328	3328	3328	3328	3328	3328	3328	3328	3328
1792	1792	1792	1792	1792	1792	1792	1792	1792	1792
2944	2944	2944	2944	2944	2944	2944	2944	2944	2944
640	640	640	640	640	640	640	640	640	640
320	320	320	320	320	320	320	320	320	320
960	960	960	960	2208	960	960	960	2208	2208
2528	2528	2528	2528	2528	2528	2528	2528	2528	2528
3488	3488	3488	1264	1264	3488	3488	3488	1264	1264
3792	3792	3792	3792	3792	3792	3792	3792	3792	3792
3184	3184	3944	3944	3944	3184	3184	3944	3944	3944
3640	3640	3640	3640	3640	3640	3640	3640	3640	3640
2728	1820	1820	1820	1820	2728	1820	1820	1820	1820
1364	1364	1364	1364	1364	1364	1364	1364	1364	1364
2730	2730	2730	2730	2730	2730	2730	2730	2730	2730
1365	1365	1365	1365	1365	1365	1365	1365	1365	1365
245	3197	1725	3125	1861	2293	1149	3773	1077	3909
2048	2048	2048	2048	2048	0	0	0	0	0
1024	1024	1024	1024	1024	0	0	0	0	0
2560	2560	2560	2560	2560	0	0	0	0	0
3328	3328	3328	3328	3328	0	0	0	0	0

If we choose this last table as reference, the last line values are inferred by :

$$df = i \cdot 3^v \bmod 2^w \quad (21)$$

where i is a relative integer ($i = k - 3597$).

We can give, still as an example, all expressions of the numbers written in table 3b :

Table 4

3328	x_0	3328	x_0	3328	x_0	3328	x_0	3328	x_0
1792	$x_1 = 3x_0 \bmod 2^w$	1792	$x_1 = 3x_0 \bmod 2^w$	1792	$x_1 = 3x_0 \bmod 2^w$	1792	$x_1 = 3x_0 \bmod 2^w$	1792	$x_1 = 3x_0 \bmod 2^w$
2944	$x_2 = (x_1 + 2^w)/2 \bmod 2^w$	2944	$x_2 = (x_1 + 2^w)/2 \bmod 2^w$	2944	$x_2 = (x_1 + 2^w)/2 \bmod 2^w$	2944	$x_2 = (x_1 + 2^w)/2 \bmod 2^w$	2944	$x_2 = (x_1 + 2^w)/2 \bmod 2^w$
640	$x_3 = 3x_2 \bmod 2^w$	640	$x_3 = 3x_2 \bmod 2^w$	640	$x_3 = 3x_2 \bmod 2^w$	640	$x_3 = 3x_2 \bmod 2^w$	640	$x_3 = 3x_2 \bmod 2^w$
320	$x_4 = x_3/2 \bmod 2^w$	320	$x_4 = x_3/2 \bmod 2^w$	320	$x_4 = x_3/2 \bmod 2^w$	320	$x_4 = x_3/2 \bmod 2^w$	320	$x_4 = x_3/2 \bmod 2^w$
960	$x_5 = 3x_4 \bmod 2^w$	960	$x_5 = 3x_4 \bmod 2^w$	960	$x_5 = 3x_4 \bmod 2^w$	960	$x_5 = 3x_4 \bmod 2^w$	2208	$x_5 = (x_4 + 2^w)/2 \bmod 2^w$
2528	$x_6 = (x_5 + 2^w)/2 \bmod 2^w$	2528	$x_6 = (x_5 + 2^w)/2 \bmod 2^w$	2528	$x_6 = (x_5 + 2^w)/2 \bmod 2^w$	2528	$x_6 = (x_5 + 2^w)/2 \bmod 2^w$	2528	$x_6 = 3x_5 \bmod 2^w$
3488	$x_7 = 3x_6 \bmod 2^w$	3488	$x_7 = 3x_6 \bmod 2^w$	3488	$x_7 = 3x_6 \bmod 2^w$	1264	$x_7 = x_6/2 \bmod 2^w$	1264	$x_7 = x_6/2 \bmod 2^w$
3792	$x_8 = (x_7 + 2^w)/2 \bmod 2^w$	3792	$x_8 = (x_7 + 2^w)/2 \bmod 2^w$	3792	$x_8 = (x_7 + 2^w)/2 \bmod 2^w$	3792	$x_8 = 3x_7 \bmod 2^w$	3792	$x_8 = 3x_7 \bmod 2^w$
3184	$x_9 = 3x_8 \bmod 2^w$	3184	$x_9 = 3x_8 \bmod 2^w$	3944	$x_9 = (x_8 + 2^w)/2 \bmod 2^w$	3944	$x_9 = (x_8 + 2^w)/2 \bmod 2^w$	3944	$x_9 = (x_8 + 2^w)/2 \bmod 2^w$
3640	$x_{10} = (x_9 + 2^w)/2 \bmod 2^w$	3640	$x_{10} = (x_9 + 2^w)/2 \bmod 2^w$	3640	$x_{10} = 3x_9 \bmod 2^w$	3640	$x_{10} = 3x_9 \bmod 2^w$	3640	$x_{10} = 3x_9 \bmod 2^w$
2728	$x_{11} = 3x_{10} \bmod 2^w$	1820	$x_{11} = x_{10}/2 \bmod 2^w$	1820	$x_{11} = x_{10}/2 \bmod 2^w$	1820	$x_{11} = x_{10}/2 \bmod 2^w$	1820	$x_{11} = x_{10}/2 \bmod 2^w$
1364	$x_{12} = x_{11}/2 \bmod 2^w$	1364	$x_{12} = 3x_{11} \bmod 2^w$	1364	$x_{12} = 3x_{11} \bmod 2^w$	1364	$x_{12} = 3x_{11} \bmod 2^w$	1364	$x_{12} = 3x_{11} \bmod 2^w$
2730	$x_{13} = (x_{12} + 2^w)/2 \bmod 2^w$	2730	$x_{13} = (x_{12} + 2^w)/2 \bmod 2^w$	2730	$x_{13} = (x_{12} + 2^w)/2 \bmod 2^w$	2730	$x_{13} = (x_{12} + 2^w)/2 \bmod 2^w$	2730	$x_{13} = (x_{12} + 2^w)/2 \bmod 2^w$
1365	$x_{14} = x_{13}/2 \bmod 2^w$	1365	$x_{14} = x_{13}/2 \bmod 2^w$	1365	$x_{14} = x_{13}/2 \bmod 2^w$	1365	$x_{14} = x_{13}/2 \bmod 2^w$	1365	$x_{14} = x_{13}/2 \bmod 2^w$
2293	5+8t	1149	5+8t	3773	5+8t	1077	5+8t	3909	5+8t
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

As the initial values are obtained by multiplication by 3 (plus 1) and divisions by 2, the differences of these values are multiplications by 3 and divisions by 2 (sometimes corrected by 2^w before dividing by 2 in operating modulo 2^w) and this as long as the signatures are in correspondence. This is not the case here for the first line in a red box, where the rule is different resulting in another type of evaluation ($5+8t$ assessment is explained later).

Besides, we still can give, as an example, the values to add to the reference table values to pass from table $i = 0$ to $i = 1$, then for table $i = 1$ to $i = 2...$ and then from table i to $i + 1$:

Table 5

0	x_0	0	x_0	0	x_0	0	x_0	0	x_0
0	$x_1 = x_0/2 \bmod 2^w$	0	$x_1 = x_0/2 \bmod 2^w$	0	$x_1 = x_0/2 \bmod 2^w$	0	$x_1 = x_0/2 \bmod 2^w$	0	$x_1 = x_0/2 \bmod 2^w$
2048	$x_2 = (x_1+2^w)/2 \bmod 2^w$	2048	$x_2 = (x_1+2^w)/2 \bmod 2^w$	2048	$x_2 = (x_1+2^w)/2 \bmod 2^w$	2048	$x_2 = (x_1+2^w)/2 \bmod 2^w$	2048	$x_2 = (x_1+2^w)/2 \bmod 2^w$
2048	$x_3 = 3x_2 \bmod 2^w$	2048	$x_3 = 3x_2 \bmod 2^w$	2048	$x_3 = 3x_2 \bmod 2^w$	2048	$x_3 = 3x_2 \bmod 2^w$	2048	$x_3 = 3x_2 \bmod 2^w$
1024	$x_4 = x_3/2 \bmod 2^w$	1024	$x_4 = x_3/2 \bmod 2^w$	1024	$x_4 = x_3/2 \bmod 2^w$	1024	$x_4 = x_3/2 \bmod 2^w$	1024	$x_4 = x_3/2 \bmod 2^w$
3072	$x_5 = 3x_4 \bmod 2^w$	3072	$x_5 = 3x_4 \bmod 2^w$	3072	$x_5 = 3x_4 \bmod 2^w$	3072	$x_5 = 3x_4 \bmod 2^w$	512	$x_5 = x_4/2 \bmod 2^w$
1536	$x_6 = x_5/2 \bmod 2^w$	1536	$x_6 = x_5/2 \bmod 2^w$	1536	$x_6 = x_5/2 \bmod 2^w$	1536	$x_6 = x_5/2 \bmod 2^w$	1536	$x_6 = 3x_5 \bmod 2^w$
512	$x_7 = 3x_6 \bmod 2^w$	512	$x_7 = 3x_6 \bmod 2^w$	2816	$x_7 = (x_6+2^w)/2 \bmod 2^w$	2816	$x_7 = (x_6+2^w)/2 \bmod 2^w$	2816	$x_7 = (x_6+2^w)/2 \bmod 2^w$
256	$x_8 = x_7/2 \bmod 2^w$	256	$x_8 = x_7/2 \bmod 2^w$	256	$x_8 = 3x_7 \bmod 2^w$	256	$x_8 = 3x_7 \bmod 2^w$	256	$x_8 = 3x_7 \bmod 2^w$
768	$x_9 = 3x_8 \bmod 2^w$	768	$x_9 = 3x_8 \bmod 2^w$	2176	$x_9 = (x_8+2^w)/2 \bmod 2^w$	2176	$x_9 = (x_8+2^w)/2 \bmod 2^w$	2176	$x_9 = (x_8+2^w)/2 \bmod 2^w$
2432	$x_{10} = (x_9+2^w)/2 \bmod 2^w$	2432	$x_{10} = (x_9+2^w)/2 \bmod 2^w$	2432	$x_{10} = 3x_9 \bmod 2^w$	2432	$x_{10} = 3x_9 \bmod 2^w$	2432	$x_{10} = 3x_9 \bmod 2^w$
3200	$x_{11} = 3x_{10} \bmod 2^w$	3264	$x_{11} = (x_{10}+2^w)/2 \bmod 2^w$	3264	$x_{11} = (x_{10}+2^w)/2 \bmod 2^w$	3264	$x_{11} = (x_{10}+2^w)/2 \bmod 2^w$	3264	$x_{11} = (x_{10}+2^w)/2 \bmod 2^w$
1600	$x_{12} = x_{11}/2 \bmod 2^w$	1600	$x_{12} = 3x_{11} \bmod 2^w$	1600	$x_{12} = 3x_{11} \bmod 2^w$	1600	$x_{12} = 3x_{11} \bmod 2^w$	1600	$x_{12} = 3x_{11} \bmod 2^w$
2848	$x_{13} = (x_{12}+2^w)/2 \bmod 2^w$	2848	$x_{13} = (x_{12}+2^w)/2 \bmod 2^w$	2848	$x_{13} = (x_{12}+2^w)/2 \bmod 2^w$	2848	$x_{13} = (x_{12}+2^w)/2 \bmod 2^w$	2848	$x_{13} = (x_{12}+2^w)/2 \bmod 2^w$
3472	$x_{14} = (x_{13}+2^w)/2 \bmod 2^w$	3472	$x_{14} = (x_{13}+2^w)/2 \bmod 2^w$	3472	$x_{14} = (x_{13}+2^w)/2 \bmod 2^w$	3472	$x_{14} = (x_{13}+2^w)/2 \bmod 2^w$	3472	$x_{14} = (x_{13}+2^w)/2 \bmod 2^w$
1736	$x_{15} = x_{14}/2 \bmod 2^w$	1736	$x_{15} = x_{14}/2 \bmod 2^w$	1736	$x_{15} = x_{14}/2 \bmod 2^w$	1736	$x_{15} = x_{14}/2 \bmod 2^w$	1736	$x_{15} = x_{14}/2 \bmod 2^w$
1112	$x_{16} = 3x_{15} \bmod 2^w$	1112	$x_{16} = 3x_{15} \bmod 2^w$	1112	$x_{16} = 3x_{15} \bmod 2^w$	1112	$x_{16} = 3x_{15} \bmod 2^w$	1112	$x_{16} = 3x_{15} \bmod 2^w$
556	$x_{17} = x_{16}/2 \bmod 2^w$	556	$x_{17} = x_{16}/2 \bmod 2^w$	556	$x_{17} = x_{16}/2 \bmod 2^w$	556	$x_{17} = x_{16}/2 \bmod 2^w$	556	$x_{17} = x_{16}/2 \bmod 2^w$
278	$x_{18} = x_{17}/2 \bmod 2^w$	278	$x_{18} = x_{17}/2 \bmod 2^w$	278	$x_{18} = x_{17}/2 \bmod 2^w$	278	$x_{18} = x_{17}/2 \bmod 2^w$	278	$x_{18} = x_{17}/2 \bmod 2^w$
2187	$x_{19} = (x_{18}+2^w)/2 \bmod 2^w$	2187	$x_{19} = (x_{18}+2^w)/2 \bmod 2^w$	2187	$x_{19} = (x_{18}+2^w)/2 \bmod 2^w$	2187	$x_{19} = (x_{18}+2^w)/2 \bmod 2^w$	2187	$x_{19} = (x_{18}+2^w)/2 \bmod 2^w$

We observe this time multiplications by 3 or divisions by 2 (corrected sometimes by a 2^w addition before division) starting from the value $x_0 = 0$.

Let us now go back to the fundamentals. All what has been seen so far depends on what follows. The Collatz algorithm is based on equations like those :

$$\begin{aligned}
 m &= r_1 \cdot 2^{v_1} - 1 \\
 r_1 \cdot 3^{v_1} - 1 &= m_1 \cdot 2^{w_1 - v_1} \\
 m_1 &= r_2 \cdot 2^{v_2 - v_1} - 1 \\
 r_2 \cdot 3^{v_2 - v_1} - 1 &= m_2 \cdot 2^{w_2 - w_1 - (v_2 - v_1)} \\
 m_2 &= r_3 \cdot 2^{v_3 - v_2} - 1 \\
 r_3 \cdot 3^{v_3 - v_2} - 1 &= m_3 \cdot 2^{w_3 - w_2 - (v_3 - v_2)} \\
 &\vdots \\
 m_{i-1} &= r_i \cdot 2^{v[i] - v[i-1]} - 1 \\
 r_i \cdot 3^{v[i] - v[i-1]} - 1 &= m_i \cdot 2^{w[i] - w[i-1] - (v[i] - v[i-1])} \\
 &\vdots \\
 m_{j-1} &= r_j \cdot 2^{v[j] - v[j-1]} - 1 \\
 r_j \cdot 3^{v[j] - v[j-1]} - 1 &= m_j \cdot 2^{w[j] - w[j-1] - (v[j] - v[j-1])}
 \end{aligned} \tag{22}$$

The resolution is done starting at the bottom and working up the equations.

In the table below, we visualize changes in correspondence of signatures (in brackets) and differences in the red sidebars, recalling also the previous equations:

Tableau 6

	$v_1 = 6 \ w_1 = 8,$ $v_2 = 7 \ w_2 = 12$	$v_1 = 6 \ w_1 = 9,$ $v_2 = 7 \ w_2 = 12$		$v_1 = 5 \ w_1 = 6,$ $v_2 = 6 \ w_2 = 8,$ $v_3 = 7 \ w_3 = 12$	$v_1 = 5 \ w_1 = 6,$ $v_2 = 6 \ w_2 = 9,$ $v_3 = 7 \ w_3 = 12$		$v_1 = 4 \ w_1 = 5,$ $v_2 = 6 \ w_2 = 8,$ $v_3 = 7 \ w_3 = 12$	$v_1 = 4 \ w_1 = 5,$ $v_2 = 6 \ w_2 = 9,$ $v_3 = 7 \ w_3 = 12$		$v_1 = 3 \ w_1 = 4,$ $v_2 = 6 \ w_2 = 8,$ $v_3 = 7 \ w_3 = 12$	$v_1 = 3 \ w_1 = 4,$ $v_2 = 6 \ w_2 = 9,$ $v_3 = 7 \ w_3 = 12$		$v_1 = 2 \ w_1 = 3,$ $v_2 = 6 \ w_2 = 8,$ $v_3 = 7 \ w_3 = 12$	$v_1 = 2 \ w_1 = 3,$ $v_2 = 6 \ w_2 = 9,$ $v_3 = 7 \ w_3 = 12$
	$r_2 = 3+8k$ $m_1 = 5+16k$ $r_1 = 29+64k$ $m = 1855+4096k$	$r_2 = 3+4k$ $m_1 = 5+8k$ $r_1 = 17+64k$ $m = 1087+4096k$		$r_3 = 3+8k$ $m_2 = 5+16k$ $r_2 = 25+32k$ $m_1 = 49+64k$ $r_1 = 81+128k$ $m = 2591+4096k$	$r_3 = 3+4k$ $m_2 = 5+8k$ $r_2 = 39+32k$ $m_1 = 77+64k$ $r_1 = 57+128k$ $m = 1823+4096k$		$r_3 = 3+8k$ $m_2 = 5+16k$ $r_2 = 19+32k$ $m_1 = 75+128k$ $r_1 = 103+256k$ $m = 1647+4096k$	$r_3 = 3+4k$ $m_2 = 5+8k$ $r_2 = 13+32k$ $m_1 = 51+128k$ $r_1 = 55+256k$ $m = 879+4096k$		$r_3 = 3+8k$ $m_2 = 5+16k$ $r_2 = 17+32k$ $m_1 = 135+256k$ $r_1 = 29+512k$ $m = 231+4096k$	$r_3 = 3+4k$ $m_2 = 5+8k$ $r_2 = 15+32k$ $m_1 = 119+256k$ $r_1 = 445+512k$ $m = 3559+4096k$		$r_3 = 3+8k$ $m_2 = 5+16k$ $r_2 = 27+32k$ $m_1 = 431+512k$ $r_1 = 551+1024k$ $m = 2203+4096k$	$r_3 = 3+4k$ $m_2 = 5+8k$ $r_2 = 5+32k$ $m_1 = 79+512k$ $r_1 = 359+1024k$ $m = 1435+4096k$
3328 3328+4096k	1855 $m = 1855+4096k$ $r_1 = 29+64k$	1087 $m = 1087+4096k$ $r_1 = 17+64k$	3328 3328+4096k	2591 $m = 2591+4096k$ $r_1 = 81+128k$	1823 $m = 1823+4096k$ $r_1 = 57+128k$	3328 3328+4096k	1647 $m = 1647+4096k$ $r_1 = 103+256k$	879 $m = 879+4096k$ $r_1 = 55+256k$	3328 3328+4096k	231 $m = 231+4096k$ $r_1 = 29+512k$	3559 $m = 3559+4096k$ $r_1 = 445+512k$	3328 3328+4096k	2203 $m = 2203+4096k$ $r_1 = 551+1024k$	1435 $m = 1435+4096k$ $r_1 = 359+1024k$
1792 1792+4096k	[1] 1470+4096k	[1] 3262+4096k	1792 1792+4096k	[1] 3678+4096k	[1] 1374+4096k	1792 1792+4096k	[1] 846+4096k	[1] 2638+4096k	1792 1792+4096k	[1] 694+4096k	[1] 2486+4096k	1792 1792+4096k	[1] 2514+4096k	[1] 210+4096k
2944 896+2048k	[0] 735+2048k	[0] 1631+2048k	2944 896+2048k	[0] 1839+2048k	[0] 687+2048k	2944 896+2048k	[0] 423+2048k	[0] 1319+2048k	2944 896+2048k	[0] 347+2048k	[0] 1243+2048k	2944 896+2048k	[0] 1257+2048k	[0] 105+2048k
640 640+2048k	[1] 158+2048k	[1] 798+2048k	640 640+2048k	[1] 1422+2048k	[1] 14+2048k	640 640+2048k	[1] 1270+2048k	[1] 1910+2048k	640 640+2048k	[1] 1042+2048k	[1] 1682+2048k	640+2048k	[1] 1724+2048k	[1] 316+2048k
320 320+1024k	[0] 79+1024k	[0] 399+1024k	320 320+1024k	[0] 711+1024k	[0] 7+1024k	320 320+1024k	[0] 635+1024k	[0] 955+1024k	320 320+1024k	[0] 521+1024k	[0] 841+1024k	320+1024k	[0] 862+1024k	[0] 158+1024k

960 960+1024k	[1] 238+1024k	[1] 174+1024k	960 960+1024k	[1] 86+1024k	[1] 22+1024k	960 960+1024k	[1] 882+1024k	[1] 818+1024k	960 960+1024k	[1] 540+1024k	[1] 476+1024k	2208 160+1024k	[0] m1 = 431+512k r2 = 27+32k	[0] m1 = 79+512k r2 = 5+32k
2528 480+512k	[0] 119+512k	[0] 87+512k	2528 480+512k	[0] 43+512k	[0] 11+512k	2528 480+512k	[0] 441+512k	[0] 409+512k	2528 480+512k	[0] 270+512k	[0] 238+512k	2528 480+512k	[1] 270+512k	[1] 238+512k
3488 416+512k	[1] 358+512k	[1] 262+512k	3488 416+512k	[1] 130+512k	[1] 34+512k	3488 416+512k	[1] 300+512k	[1] 204+512k	1264 240+512k	[0] m1 = 135+256k r2 = 17+32k	[0] m1 = 119+256k r2 = 15+32k	1264 240+512k	[0] 135+256k	[0] 119+256k
3792 208+256k	[0] 179+256k	[0] 131+256k	3792 208+256k	[0] 65+256k	[0] 17+256k	3792 208+256k	[0] 150+256k	[0] 102+256k	3792 208+256k	[1] 150+256k	[1] 102+256k	3792 208+256k	[1] 150+256k	[1] 102+256k
3184 112+256k	[1] 26+256k	[1] 138+256k	3184 112+256k	[1] 196+256k	[1] 52+256k	3944 104+256k	[0] m1 = 75+128k r2 = 19+32k	[0] m1 = 51+128k r2 = 13+32k	3944 104+256k	[0] 75+128k	[0] 51+128k	3944 104+256k	[0] 75+128k	[0] 51+128k
3640 56+128k	[0] 13+128k	[0] 69+128k	3640 56+128k	[0] 98+128k	[0] 26+128k	3640 56+128k	[1] 98+128k	[1] 26+128k	3640 56+128k	[1] 98+128k	[1] 26+128k	3640 56+128k	[1] 98+128k	[1] 26+128k
2728 40+128k	[1] 40+128k	[1] 80+128k	1820 28+128k	[0] m1 = 49+64k r2 = 25+32k	[0] m1 = 13+64k r2 = 7+32k	1820 28+128k	[0] 49+64k	[0] 13+64k	1820 28+128k	[0] 49+64k	[0] 13+64k	1820 28+128k	[0] 49+64k	[0] 13+64k
1364 20+64k	[0] 20+64k	[0] 40+64k	1364 20+64k	[1] 20+64k	[1] 40+64k	1364 20+64k	[1] 20+64k	[1] 40+64k	1364 20+64k	[1] 20+64k	[1] 40+64k	1364 20+64k	[1] 20+64k	[1] 40+64k
2730 10+32k	[0] 10+32k	[0] 20+32k	2730 10+32k	[0] 10+32k	[0] 20+32k	2730 10+32k	[0] 10+32k	[0] 20+32k	2730 10+32k	[0] 10+32k	[0] 20+32k	2730 10+32k	[0] 10+32k	[0] 20+32k
1365 5+16k	[0] m1 = 5+16k r2 = 3+8k	[0] 10+16k	1365 5+16k	[0] m2 = 5+16k r3 = 3+8k	[0] 10+16k	1365 5+16k	[0] m2 = 5+16k r3 = 3+8k	[0] 10+16k	1365 5+16k	[0] m2 = 5+16k r3 = 3+8k	[0] 10+16k	1365 5+16k	[0] m2 = 5+16k r3 = 3+8k	[0] 10+16k
2293 5+8k	[1] 16k	[0] m1 = 5+8k r2 = 3+4k	1149 5+8k	[1] 16k	[0] m2 = 5+8k r3 = 3+4k	3773 5+8k	[1] 16k	[0] m2 = 5+8k r3 = 3+4k	1077 5+8k	[1] 16k	[0] m2 = 5+8k r3 = 3+4k	3909 5+8k	[1] 16k	[0] m2 = 5+8k r3 = 3+4k
0 8k	[0] 8k	[1] 8k	0 8k	[0] 8k	[1] 8k	0 8k	[0] 8k	[1] 8k	0 8k	[0] 8k	[1] 8k	0 8k	[0] 8k	[1] 8k
0 4k	[0] 4k	[0] 4k	0 4k	[0] 4k	[0] 4k	0 4k	[0] 4k	[0] 4k	0 4k	[0] 4k	[0] 4k	0 4k	[0] 4k	[0] 4k
0 2k	[0] 2k	[0] 2k	0 2k	[0] 2k	[0] 2k	0 2k	[0] 2k	[0] 2k	0 2k	[0] 2k	[0] 2k	0 2k	[0] 2k	[0] 2k
0 k	[0] k	[0] k	0 k	[0] k	[0] k	0 k	[0] k	[0] k	0 k	[0] k	[0] k	0 k	[0] k	[0] k

In the first line of the previous table, we mention on the one hand the values v_i and w_i to be taken into account in the system of equations (8) and on the other hand the system of equations which thus results. The previous table is built, after writing the fundamental equations in their places, and then adding all the intermediate expressions that can be evaluated. We choose to this purpose the simplest possible terms. For example, if we have $5+8k = 5 \pmod 8$ followed by signature [1] (reading from top to bottom), we perform the operation $3(5)+1 = 16 = 0 \pmod 8$, or $8k$ (instead of $3(5+8k)+1 = 16+24k = \dots, -32, -8, 16, 40, \dots$ list of numbers that is effectively included in $\dots, -32, -24, -16, -8, 0, 8, 16, 24, 32, 40, \dots$). For the divisions by 2, things are here simpler and $2x+2k.y$ systematically gives $x+k.y$.

We see in particular that the last line shows an arbitrary k : This means that there is effectively a choice for which the last line can display differences systematically equal to 0.

Now, let us see the proof of the theorem (9).

Proof

It comes up to generalize the numerical observations made previously.

To switch from one signature to another, for a given v -plane, means resorting to a certain number of permutations of 10 and 01 or vice versa 10 to 01. (The red boxes in table 6 show that effectively).

Then we have to consider the effect of a first case of shifting, according to the following type, for the numbers on which the signatures act :

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline \end{array}$$

The operation is made generally up line (but down line gives exactly the same conclusion) :

$$\begin{array}{|c|c|c|c|} \hline (2x_1-1)/3 & (2x_2-1)/3 & 2(y_1-1)/3 & 2(y_2-1)/3 \\ \hline 1 & 1 & 0 & 0 \\ \hline 2x_1 & 2x_2 & (y_1-1)/3 & (y_2-1)/3 \\ \hline 0 & 0 & 1 & 1 \\ \hline x_1 & x_2 & y_1 & y_2 \\ \hline \end{array}$$

Then, if the differences are identical $x_2-x_1 = r = y_2-y_1$ before permutation, we will find again identical differences $2(x_2-1)/3 - 2(x_1-1)/3 = 2r/3 = 2(y_2-1)/3 - 2(y_1-1)/3$ after the swapping.

The second case that appeared above is the following :

$$\begin{array}{|c|c|c|c|} \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array}$$

The operation is performed here again up columns :

$(2x_1-1)/3$	$2(x_2-1)/3$	$(2y_1-1)/3$	$2(y_2-1)/3$
1	0	1	0
$2x_1$	$(x_2-1)/3$	$2y_1$	$(y_2-1)/3$
0	1	0	1
x_1	x_2	y_1	y_2

Then, if the differences are identical $x_2-x_1 = r = y_2-y_1$ before permutation, we find the same differences again $2(x_2-1)/3-(2x_1-1)/3 = (2r-1)/3 = 2(y_2-1)/3-(2y_1-1)/3$ when swapping is completed.

The composition of these two cases provides a multitude of configurations that infer all identical differences after the area of permutations.

As for the establishment of a 0 difference at the end of the algorithm, it is just necessary to notice that starting from $x+k.2^w$, the application of v multiplications by 3 (plus 1) and w divisions by 2 leads to $x'+k3^v$, x being transformed into x' and $k.2^w$ into $k.3^v$. Yet $x'+k3^v \bmod 2^w$, x' being fixed (as x is fixed), gives all the numbers from 0 to 2^w-1 (in a specific order), when k described 2^w-1 . Indeed 3^v and 2^w are relative primes (for $v > 0$) and Bézout theorem allows asserting it. Hence it suffices to chosen a proper k for one of a pair of associates to obtain a null difference or any other given number modulo 2^w .

Consider then two pairs of signatures (although not the most general case) :

Line v+w	[1]	[1]	[1]	[1]
	[0]	[0]	[0]	[0]
	...	...	...	...
Line j	[0]	[0]	[1]	[1]
	...	...	...	...
Line i	[1]	[1]	[0]	[0]
	...	...	...	...
Line q	[1]	[0]	[1]	[0]
	...	...	...	...
Line p	[0]	[1]	[0]	[1]
	...	...	...	...
Line 1	[0]	[0]	[0]	[0]

Here, the unfilled lines have four identical elements (either all [0] or all [1]). By a suitable choice of the parameter k , the differences in the last line of each of the two pairs of signatures may be reduced to the same value (for example 0). When the process is then run up columns, line p creates not-identical differences that will be compensated after the q line and in the analogous way, line i will see compensation at line j.

Finally, at line v+w, we will always have differences of equal values.

Therefore, all of the properties observed on the numerical application apply in general way. From a basic operation to another basic operation (permutation), differences turn out the same allowing concluding.

Note :

For a given plane, to a given modulo 2^w offset does correspond a priori a unique (binary or decimal) signatures difference, but the converse is false.

The following examples show exceptions to unity as early as the first decade of planes :

Difference between columns		Difference between lines	
Decimal signatures gaps	Modulo differences	Decimal signatures gaps	Modulo differences
v = 3			
4	12	/	/
v = 4			
8	8	/	/
32	76		
v = 5			
8	176	4	96
32	136		
128	76		

Difference between columns		Difference between lines	
Decimal signatures gaps	Modulo differences	Decimal signatures gaps	Modulo differences
v = 6			
16	288	8	576
64	944	32	864
256	392		
1024	76		
v = 7			
32	2240	8	768
128	3360	16	384
512	944	56	1856
2048	1416	248	1120
8192	2124		
v = 8			
32	4224	4	1024
128	2240	8	512
512	3360	16	256
2048	944	28	7936
8192	1416	52	2944
32768	2124	212	4416
		980	7776
v = 9			
64	19200	8	22528
256	12416	16	11264
1024	18624	32	5632
4096	11552	32	1024
16384	944	56	27136
65536	17800	104	23808
262144	26700	216	18176
		392	18304
		1672	27968
		7816	6752
v = 10			
64	12800	4	8192
256	51968	8	36864
1024	12416	16	51200
4096	51392	28	14336
16384	44320	32	25600
65536	33712	52	31744
262144	17800	100	7680
1048576	26700	124	25600
		212	47616
		388	25856
		852	38656
		1540	20352
		6660	62784
		31236	41568
v = 11			
128	52224	8	49152
512	209408	16	155648
2048	51968	32	204800
8192	77952	32	208896
32768	248000	56	53248
131072	240928	64	104448
524288	99248	104	108544
2097152	148872	200	5120
8388608	223308	216	161792
		392	220160
		744	118272
		984	107520
		1672	170496

Difference between columns		Difference between lines	
Decimal signatures gaps	Modulo differences	Decimal signatures gaps	Modulo differences
		3048	70912
		6792	227072
		12264	262016
		53224	42304
		249832	21088

We did not give completeness of the differences between lines (i.e. all of the redundancies) as in the previous table. We see exceptions to the bijection between values for the $v = 9$ and the $v = 11$ planes.

Property 5

A ratio 1/2 for the differences in decimal signatures, resulting from a shift of the final 1, the rest of the signatures being identical, results in a ratio 2 for the differences between associates modulo 2^w .

Such lag being endemic in the tables of signatures, this ratio 2 (of the differences between associates modulo 2^w) will regularly come out in our study.

We give a simple example here without proof although it would be useful, example that we take in the $v = 15$ plane.

i	Signatures	Decimal differences = dd(i)	Ratios dd(i+1)/dd(i)	Associates = a(i)	Differences d(i) = a(i+1) - a(i) mod 2^w	d(i+1)/d(i) mod 2^w
1	10101010101010101010101010101000000000			2195455		
2	1010101010101010101010101010100100000000	-512		9486335	7290880	
3	1010101010101010101010101010100010000000	-256	1/2	7290879	14581760	2
4	1010101010101010101010101010100001000000	-128	1/2	2899967	12386304	2
5	1010101010101010101010101010100000100000	-64	1/2	10895359	7995392	2
6	1010101010101010101010101010100000010000	-32	1/2	10108927	15990784	2
7	1010101010101010101010101010100000001000	-16	1/2	8536063	15204352	2
8	1010101010101010101010101010100000000100	-8	1/2	5390335	13631488	2
9	10101010101010101010101010101000000000100	-4	1/2	15876095	10485760	2

The proof necessitates, a priori, the resolution of the following three systems (going up equations), resolution that we leave to the ambitious reader :

$m = r_1 \cdot 2^{v1} - 1$ $r_1 \cdot 3^{v1} - 1 = m_1 \cdot 2^{w1-v1}$ $m_1 = r_2 \cdot 2^{v2-v1} - 1$ $r_2 \cdot 3^{v2-v1} - 1 = m_2 \cdot 2^{w2-w1-(v2-v1)}$ $m_2 = r_3 \cdot 2^{v3-v2} - 1$ $r_3 \cdot 3^{v3-v2} - 1 = m_3 \cdot 2^{w3-w2-(v3-v2)}$ $\dots$ $m_{i-1} = r_i \cdot 2^{v[i]-v[i-1]} - 1$ $r_i \cdot 3^{v[i]-v[i-1]} - 1 = m_i \cdot 2^{w[i]-w[i-1]-(v[i]-v[i-1])}$ $\dots$ $m_{j-2} = r_j \cdot 2^{v[j-1]-v[j-2]} - 1$ $r_j \cdot 3^{v[j-1]-v[j-2]} - 1 = m_j \cdot 2^{w[j-1]-w[j-2]-(v[j-1]-v[j-2])}$ $m_{j-1} = r_j \cdot 2^{v[j]-v[j-1]} - 1$ $r_j \cdot 3^{v[j]-v[j-1]} - 1 = m_j \cdot 2^{w[j]-w[j-1]-(v[j]-v[j-1])}$	$m' = r_1 \cdot 2^{v1} - 1$ $r_1 \cdot 3^{v1} - 1 = m_1 \cdot 2^{w1-v1}$ $m_1 = r_2 \cdot 2^{v2-v1} - 1$ $r_2 \cdot 3^{v2-v1} - 1 = m_2 \cdot 2^{w2-w1-(v2-v1)}$ $m_2 = r_3 \cdot 2^{v3-v2} - 1$ $r_3 \cdot 3^{v3-v2} - 1 = m_3 \cdot 2^{w3-w2-(v3-v2)}$ $\dots$ $m_{i-1} = r_i \cdot 2^{v[i]-v[i-1]} - 1$ $r_i \cdot 3^{v[i]-v[i-1]} - 1 = m_i \cdot 2^{w[i]-w[i-1]-(v[i]-v[i-1])}$ $\dots$ $m_{j-2} = r_j \cdot 2^{v[j-1]-v[j-2]} - 1$ $r_j \cdot 3^{v[j-1]-v[j-2]} - 1 = m_j \cdot 2^{w[j-1]-w[j-2]+1-(v[j-1]-v[j-2])}$ $m_{j-1} = r_j \cdot 2^{v[j]-v[j-1]} - 1$ $r_j \cdot 3^{v[j]-v[j-1]} - 1 = m_j \cdot 2^{w[j]-w[j-1]-1-(v[j]-v[j-1])}$	$m'' = r_1 \cdot 2^{v1} - 1$ $r_1 \cdot 3^{v1} - 1 = m_1 \cdot 2^{w1-v1}$ $m_1 = r_2 \cdot 2^{v2-v1} - 1$ $r_2 \cdot 3^{v2-v1} - 1 = m_2 \cdot 2^{w2-w1-(v2-v1)}$ $m_2 = r_3 \cdot 2^{v3-v2} - 1$ $r_3 \cdot 3^{v3-v2} - 1 = m_3 \cdot 2^{w3-w2-(v3-v2)}$ $\dots$ $m_{i-1} = r_i \cdot 2^{v[i]-v[i-1]} - 1$ $r_i \cdot 3^{v[i]-v[i-1]} - 1 = m_i \cdot 2^{w[i]-w[i-1]-(v[i]-v[i-1])}$ $\dots$ $m_{j-2} = r_j \cdot 2^{v[j-1]-v[j-2]} - 1$ $r_j \cdot 3^{v[j-1]-v[j-2]} - 1 = m_j \cdot 2^{w[j-1]-w[j-2]+2-(v[j-1]-v[j-2])}$ $m_{j-1} = r_j \cdot 2^{v[j]-v[j-1]} - 1$ $r_j \cdot 3^{v[j]-v[j-1]} - 1 = m_j \cdot 2^{w[j]-w[j-1]-2-(v[j]-v[j-1])}$
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Now, let us say that positive sorting of a large staff's plane can be a long task.

Thus, it may be interesting to find additional properties relating associates, hence what follows.

6 Intra-column properties.

Theorem 10

For $v > 1$, the number $a_v(1,1)$ in the first column of a Pascal trihedron associates' plane is the unique number of this plane, which has a cycle of length 2^{w-v} (modulo 2^w), that is such as

$$(a_v(1,1))^{2^{w-v}} = 1 \mod 2^w \quad (23)$$

and

$$(a_v(1,1))^i \neq 1 \mod 2^w \text{ if } 0 < i < 2^{w-v} \quad (24)$$

For example in the preceding tables :

$$3^{2^{4-2}} = 1 \mod 2^4, 23^{2^{5-3}} = 1 \mod 2^5, 15^{2^{7-4}} = 1 \mod 2^7, 95^{2^{8-5}} = 1 \mod 2^8, 575^{2^{10-6}} = 1 \mod 2^{10}, 383^{2^{12-7}} = 1 \mod 2^{12}$$

being the first occurrence of $1 \mod 2^w$ when proceeding to successive iterations of m .

We have also, for $v = 1$, $5^{2^{2-1}} = 1 \mod 2^2$, but the length of the cycle is shorter than 2 (since $5^1 = 1 \mod 2^2$).

Proof

The first associate is the unique number of its ranked v family with multiplication steps executed prior to division steps in the Collatz algorithm : A multiplication step (I) is systematically followed by a phase of division (P) which is concatenated as an IP-step. The process is illustrated below and it is clear that there is only a single number that verifies this routine for a given v (since w is related to v in a unique way).

$a_v(1,1) = pr$	$(3pr+1)/2$	...	...	...	...	...	res < pr
	IP	...	IP	P	...	P	P

From property 4 , the said number, at rank v , is necessarily of the form $a_v(1,1) = t \cdot 2^v - 1$, this corresponding to systematic IP-type sequences. Let us notice also that we have necessarily $a_v(1,1) \neq t' \cdot 2^{v+1} - 1$, which is ending the previous routine. The binomial development of $(a_v(1,1))^{2^{w-v}} = (-1+t \cdot 2^v)^{2^{w-v}} = 1 + 2^{w-v} \cdot (t \cdot 2^v + x_1 \cdot 2^{2v} + \dots + x_i \cdot 2^{i \cdot v}) = 1 + k \cdot 2^w$ is then equivalent to the first part of what we want to prove. As, per elsewhere, $a_v(1,1) \neq t' \cdot 2^{v+1} - 1$, there is no equality $a_v(1,1))^{2^{w-v}} = 1 + k \cdot 2^{w-c}$ with $c > 0$.

Note: All of the numbers $(a_v(1,1) + \text{or}(0,2)) + k \cdot 2^{v+1}$, k an integer, have exactly same cycle length (modulo 2^w) than $a_v(1,1)$ because of the form $t \cdot 2^v - 1$. There are exactly 2^{w-v} such numbers in the range 1 to $2^w - 1$ ($t = 1, 3, \dots, 2^{w-v+1} - 1$) and they are the only ones to have this property.

Among these numbers, $a_v(1,1)$ is the only one belonging to the Pascal trihedron plane v (as proved above).

None of the numbers $a_v(1,1) + k \cdot 2^{v+1}$ is in the Pascal trihedron plane $v=1$. (because $3 \mod 4$).

All of the numbers $a_v(1,1) + 2 + k \cdot 2^{v+1}$ are in Pascal trihedron plane $v=1$ (because $1 \mod 4$).

The number $a_v(1,1)$ is thus always ahead of his fellow pair $a_v(1,1) + 2$.

The table below gives a sample with the number $a_v(1,1)$ in bold font.

v	3	4	5	6
w	5	7	8	10
2^{w-v}	4	8	8	16
2^{v+1}	16	32	64	128

7	15	31	63
9	17	33	65
23	47	95	191
25	49	97	193
	79	159	319
	81	161	321
	111	223	447
	113	225	449
			575
			577
			703
			705
			831
			833
			959
			961

Theorem 11

The cycles' length of the numbers $a_v(1,c)$ is doubled passing from one column to the next, that is

$$(a_v(1,c))^{2^{w-v+c-1}} = 1 \mod 2^w \quad (25)$$

and

$$(a_v(1,c))^i \neq 1 \mod 2^w \text{ if } 0 < i < 2^{w-v+c-1} \quad (26)$$

where c is the c^{th} column.

We have thus as examples $11^{2^{5-3+2-1}} = 1 \mod 2^5$ and $7^{2^{7-4+2-1}} = 1 \mod 2^7$.

Proof

We only have to repeat word for word the demonstration used previously and replace $a_v(1,1)$ by $a_v(1,c)$, $t.2^v-1$ by $t.2^{v+c}-1$ and $t.2^{v+1}-1$ by $t.2^{v+c+1}-1$.

The lengths of the cycles are illustrated below at the last line of each table :

$v = 2, w = 4, \#(v) = 1$
Family mod 16,

3
4

$v = 3, w = 5, \#(v) = 2$
Family mod 32,

23	11
4	8

$v = 4, w = 7, \#(v) = 3$
Family mod 128

15	7	59
8	16	32

$v = 5, w = 8, \#(v) = 7$
Family mod 256

95	175	39	219
	79	199	123
8	16	32	64

$v = 6, w = 10, \#(v) = 12$
Family mod 1024

575	287	367	999	923
	735	815	423	347
		975	583	507
16	32	64	128	256

$v = 7, w = 12, \#(v) = 30$
Family mod 4096

383	2239	2975	2031	615	2587
	1855	2591	1647	231	2203
	1087	1823	879	3559	1435
		4063	3119	1703	3675
		3295	2351	935	2907
			1231	3911	1787
			463	3143	1019
32	64	128	256	512	1024

Evaluation of the number of planes including a cycle of given length

This evaluation is not crucial and may be skipped.

From theorem 11, a plane v includes associates with different cycles of lengths 2^i where integer i ranges from $w-v$ up to $w-2$. So, we get using the relationship between v and w :

$$\text{int}((\ln(3)/\ln(2)).v)-v+1 \leq i \leq \text{int}((\ln(3)/\ln(2)).v)-1 \quad (27)$$

This relationship allows case by case to find precisely the number of planes with associates of a given cycle length. However we look here simply for an approximate value when v is increasing.

Thus we have :

$$((\ln(3)/\ln(2)-1).v \approx i \approx (\ln(3)/\ln(2)).v \quad (28)$$

Hence

$$(\ln(2)/\ln(3)).i \approx v \approx (\ln(2)/(\ln(3)-\ln(2))).i \quad (29)$$

Considering the interval defined in this way, the number of planes v for large i is around :

$$v \approx i.\ln^2(2)/(\ln(3).(\ln(3)-\ln(2))) \approx 1,0786.i \quad (30)$$

meaning i and v are always of the same order of magnitude.

Theorem 12

Associates a_r and a_s of a same column are linked by a relationship of the following type (i an odd integer) :

$$a_s = a_r^i \bmod 2^w \quad (31)$$

Proof

Let us consider column c . According to property 4, a_r is of the form $t.2^s-1$ with t an odd integer. Integer s is the exact number of sequences of IP-type at the beginning of the Collatz algorithm for the integer a_r , the last sequence being followed by at least one extra P sequence. Let us take the term to an odd exponent i , we get then by binomial development $(t.2^s-1)^i = x.t.2^s-1$ where x is an odd number also. When i is incremented ($= 0, 1, 2, \dots$), the term $(t.2^s-1)^i \bmod 2^w$ takes (for any odd integer t) exactly 2^{w-s} different values. When i is now incremented by steps of 2 units ($= 1, 3, 5, \dots$), the expression takes exactly half of these values, 2^{w-s-1} different values. The odd integer t being fixed, the term $x.t.2^s-1 \bmod 2^w$ takes then exactly $2^{w-s}/2$ different values when x is incremented according to odd values, that is the same as $x.2^s-1 \bmod 2^w$ to within a permutation. This are therefore the same numbers as those taken by $(t.2^s-1)^i \bmod 2^w$. Thus $(t.2^s-1)^i \bmod 2^w$ describes all the integers set that can be taken by an expression such as $x.t.2^s-1 \bmod 2^w$. So a_s is necessarily of the form $a_r^i \bmod 2^w$ for some odd integer i .

Note: For even values of i and x , there is a difference of 2 between the terms of the set $(t.2^s-1)^i \bmod 2^w$ and those of the set $x.2^s-1 \bmod 2^w$ classified in increasing values.

The reader can check by samples this relationship thanks to the values given below where the power i is given (the reference for each column being the first number of the previous subfamilies and recalling also here that the first non-trivial table starts at $v = 5$) :

$v = 5, w = 8, \#(v)=7$

Family mod 256

1	1	1	1
	15	21	9

(as $79 = 175^{15} \bmod 2^8$, $199 = 39^{21} \bmod 2^8$, $123 = 219^9 \bmod 2^8$.)

$v = 6, w = 10, \#(v)=12$

Family mod 1024

1	1	1	1	1
	31	37	121	49
		27	109	121

$v = 7, w = 12, \#(v)=30$

Family mod 4096

1	1	1	1	1	1
	63	69	89	337	545
	59	77	9	497	609
		107	93	329	849
		115	13	489	913
			3	253	153
			179	413	217

Theorem 13

There is no iterative relationship between associates a_r and a_s of separate columns :

$$a_s \neq a_r^i \bmod 2^w \quad (32)$$

Proof

Otherwise, we would have $a_s = a_r^i \bmod 2^w$ with a_r of the form $t.2^u - 1$ for some odd integer t , a_s of the form $t'.2^{u'} - 1$ for some odd integer t' (u and u' permissible maximum values) and $u \neq u'$. The role of a_s and a_r is interchangeable, namely that if $a_s = a_r^i \bmod 2^w$, there are also some j such as $a_r = a_s^j \bmod 2^w$. This also means that we can assume $u' > u$. By binomial development, we get $(t.2^u - 1)^i = x.t.2^u - 1 = t'.2^{u'} - 1 \bmod 2^w$ when i is odd (and x will also be), which is impossible, because $x.t.2^{u-\min(u,u')} = t'.2^{u'-\min(u,u')} \bmod 2^{w-\min(u,u')}$ has a parity contradiction and we have $(t.2^u - 1)^i = x.t.2^{u+1} + 1 = t'.2^{u'} - 1 \bmod 2^w$ when i is even (and x will still be odd), that is also $x.t.2^u + 1 = t'.2^{u'-1} \bmod 2^{w-1}$. To satisfy the parity, we have then $u'-1 = 0$, and as $u' > u$, we conclude $u = 0$, that means a_r is even, number that does not interest us (all numbers in the Pascal trihedron are odd).

For example, for $v = 7$ ($w = 12$),

383	2239	2975	2031	615	2587
	1855	2591	1647	231	2203
	1087	1823	879	3559	1435
		4063	3119	1703	3675
		3295	2351	935	2907
			1231	3911	1787
			463	3143	1019

383 gives by successive iterations modulo 2^{12} , neither 2239 nor 2975, 2031, 615, 2587 (or other numbers respectively of each of the columns), similarly 2239 gives neither 383 nor 2975, 2031, 615, 2587, etc.

7 Intra-line properties.

There are no property specific to a single line based on the previous method as we shall see. The initial difficulty is, as we have seen, that there is no iterative link such as $a_s = a_r^i \bmod 2^w$ between two associates a_r and a_s of separate columns. Nevertheless, we can write links such as $a_s^j = a_r^i \bmod 2^w$, but these have a little pronounced differentiation character, only allowing the grouping of associates of identical staffs' lines.

Property 6

The strength of the relationship between two associates is expressed by :

$$(a_v(i,j+1))^{2^{k(w-k)}} = (a_v(i,j))^{2^{k(w-k)}} \bmod 2^w \quad (33)$$

where k is the largest possible.

This is actually more a technique of classification than a property to be proven.

For example, let us take the last column of the Pascal trihedron associates' plane for $v = 7$ ($w = 12$, modulo 4096) and let us focus on the two last columns.

By theorem 11, we have $2587^{1024} = 1 \bmod 4096$ and similarly $2203^{1024} = 1 \bmod 4096$, $1435^{1024} = 1 \bmod 4096$, $3675^{1024} = 1 \bmod 4096$, $2907^{1024} = 1 \bmod 4096$, $1787^{1024} = 1 \bmod 4096$ and $1019^{1024} = 1 \bmod 4096$.

Let us divide then the exponent 1024 by 2, then by 4, then by 8 and so on. We get modulo 4096 a table which allows, thanks to the results on the columns of 64, 32 and 16 iterations, the following classification :

Numbers of iterations of	1024	512	256	128	64	32	16	8	4	2	1
2587	1	2049	3073	3585	1793	2945	1473	2273	1009	3801	2587
2203	1	2049	3073	3585	1793	2945	3521	1249	497	3545	2203
1435	1	2049	3073	3585	1793	2945	3521	3297	3569	3033	1435
3675	1	2049	3073	3585	1793	897	449	3809	1777	1113	3675
2907	1	2049	3073	3585	1793	897	449	1761	753	601	2907
1787	1	2049	3073	3585	3841	3969	4033	3553	1649	2585	1787
1019	1	2049	3073	3585	3841	3969	4033	1505	625	2073	1019

This table gives us lines' aggregate score {1,2,2,2} as expected by the simplified decomposition #TPS (7) = {1,3,5,7,7,7} of the studied plane. In particular, this technique allows always to find the correct first line, here through the column of 16 iterations.

Then let us do the same with the second last column :

Numbers of iterations of	1024	512	256	128	64	32	16	8	4	2	1
615	1	1	2049	3073	3585	3841	1921	3009	3041	1393	615
231	1	1	2049	3073	3585	3841	3969	1985	481	113	231
3559	1	1	2049	3073	3585	3841	3969	4033	3553	1649	3559
1703	1	1	2049	3073	3585	1793	897	2497	737	241	1703
935	1	1	2049	3073	3585	1793	897	449	3809	1777	935
3911	1	1	2049	3073	1537	769	2433	3265	1121	1457	3911
3143	1	1	2049	3073	1537	769	2433	1217	97	2993	3143

Differentiation is still done by the columns of 64, 32 and 16 iterations.

We can also realize differences modulo 4096 of these two tables and we get :

Iterations Matchings	1024	512	256	128	64	32	16	8	4	2	1
2587/615	0	2048	1024	512	2304	3200	3648	3360	2064	2408	1972
2203/231	0	2048	1024	512	2304	3200	3648	3360	16	3432	1972
1435/3559	0	2048	1024	512	2304	3200	3648	3360	16	1384	1972
3675/1703	0	2048	1024	512	2304	3200	3648	1312	1024	872	1972
2907/935	0	2048	1024	512	2304	3200	3648	1312	1024	2920	1972
1787/3911	0	2048	1024	512	2304	3200	1600	288	528	1128	1972
1019/3143	0	2048	1024	512	2304	3200	1600	288	528	3176	1972

The differentiation is done this time thanks to the columns of 16, 8 and 4 iterations.

However, this method only links associates belonging to identical staff lines. We wrote, for example, pairs 2203/231 and 1435/3559, but 2203/3559 and 1435/231 matchings would give a table with the same characteristics when these associates are not on the same lines of the plane $v = 7$.

To go further, we adopt a different strategy.

8 Values of the first element of the Pascal trihedron planes.

We seek to deduce the value of $a_{v+1}(1,1)$ from $a_v(1,1)$. To do this, we compare the current value to 6 times the precedent one plus 5 modulo 2^w and check the differences :

v	w	Δw	$a_v(1,1)$	$6.a_{v-1}(1,1)+5$ mod 2^w	differences diff	td = diff/ 2^{v+3}	t	t-td
2	4	2	3					
3	5	1	23	23	0	0	3	3
4	7	2	15	15	0	0	1	1
5	8	1	95	95	0	0	3	3
6	10	2	575	575	0	0	9	9
7	12	2	383	3455	3072	3	3	0
8	13	1	255	2303	2048	1	1	0
9	15	2	5631	1535	-4096	-1	11	12
10	16	1	25599	33791	8192	1	25	24
11	18	2	104447	153599	49152	3	51	48
12	20	2	69631	626687	557056	17	17	0
13	21	1	745471	417791	-327680	-5	91	96
14	23	2	3293183	4472831	1179648	9	201	192
15	24	1	2195455	2981887	786432	3	67	64
16	26	2	12648447	13172735	524288	1	193	192
17	27	1	97910783	75890687	-22020096	-21	747	768
18	29	2	65273855	50593791	-14680064	-7	249	256
19	31	2	43515903	391643135	348127232	83	83	0

Conjecture 1

The first associate at rank v is derived from the first associate at rank $v-1$ by multiplying by 6 and 5 with an additive correction $td.2^{v+3}$ modulo 2^w , td a relative integer.

We will come back to the values of td later on.

Theorem 14

Let us have $a_v(1,1)$ the first element of the Pascal trihedron v -plane.

Let us have $a_1(1,1) = 1$ (for $v = 1$, $w = 2$ and thus $5 > 2^w$, and $a_1(1,1) = 5$ is unsuitable to the present case where instead 1 is).

Let us have $m(v)$ defined by the recursive relation :

$$m(v+1) = (m(v) + r \cdot 3^v) / 2^{\Delta w(v+1)-1} \quad (34)$$

starting with $m(1) = 1$.

Then :

$$a_{v+1}(1,1) = (2 \cdot (a_v(1,1) + r \cdot 2^w) - 1) / 3 \quad (35)$$

for the smallest r such that $m(v+1)$ and $a_{v+1}(1,1)$ are simultaneous integers.

The integer $m(v)$ is precisely the result of the Collatz algorithm on $a_v(1,1)$ when the altitude flight comes to an end (except for 1 which forms a loop with a result equal to the initial value).

Moreover, the solution is necessarily reached for r in the set $\{0, 1, 2, 3, 4, 5\}$.

For memory

$$w = w(v) = \text{int}((\text{Ln}(3)/\text{Ln}(2)) \cdot v) + 1$$

and thus

$$\Delta w(v+1) = w(v+1) - w(v) = \text{int}((\text{Ln}(3)/\text{Ln}(2)) \cdot (v+1)) - \text{int}((\text{Ln}(3)/\text{Ln}(2)) \cdot v)$$

Proof

We focus first on the parameter t .

Let us recall (property 4), that $a_v(1,1)$, ranked v , is necessarily of the form :

$$a_v(1,1) = t \cdot 2^v - 1 \quad (36)$$

where t is a positive integer.

It is the only acceptable form, otherwise the intermediate results would provide a premature even step. Then to finish Collatz algorithm with the objective to have only even steps, it is necessary and suffices to impose

$$3^v \cdot t - 1 = m \cdot 2^{w-v} \quad (37)$$

where $w-v$ is the remaining number of steps (which are even). The Bézout theorem assures us of the existence of a solution (t, m) of integers (here positive) to $3^v \cdot t - m \cdot 2^{w-v} = 1$, as 2 and 3 are relative primes, from which we derive then :

$$t = (1 + m \cdot 2^{w-v}) / (3^v) \quad (38)$$

Hence :

$$a_v(1,1) = (1 + m \cdot 2^{w-v}) \cdot (2/3)^v - 1 \quad (39)$$

In practice, the unique number $a_v(1,1)$ is obtained by searching the smallest m such as $(1 + m \cdot 2^{w-v}) \cdot (2/3)^v$ is an integer. This is what we have done below :

v	w	Δw	$a_v(1,1)$	m	t	r
1	2		5	1	1	1
2	4	2	3	2	1	2
3	5	1	23	20	3	0
4	7	2	15	10	1	1
5	8	1	95	91	3	3
6	10	2	575	410	9	0
7	12	2	383	205	3	0
8	13	1	255	205	1	1
9	15	2	5631	3383	11	1
10	16	1	25599	23066	25	2
11	18	2	104447	70582	51	0
12	20	2	69631	35291	17	1
13	21	1	745471	566732	91	2

v	w	Δw	$a_v(1,1)$	m	t	r
14	23	2	3293183	1877689	201	0
15	24	1	2195455	1877689	67	1
16	26	2	12648447	8113298	193	2
17	27	1	97910783	94206740	747	0
18	29	2	65273855	47103370	249	0
19	31	2	43515903	23551685	83	1
20	32	1	1460666367	1185813152	1393	2
21	34	2	6700400639	4079690977	3195	0
22	35	1	4466933759	4079690977	1065	3
23	37	2	71697432575	49111434902	8547	0
24	39	2	47798288383	24555717451	2849	2
25	40	1	764873277439	589414790413	22795	1
26	42	2	1242923270143	718351699928	18521	1
27	43	1	3760646520831	3260217528257	28019	1
28	45	2	8371159695359	5442907506622	31185	0
29	46	1	5580773130239	5442907506622	10395	0
30	48	2	3720515420159	2721453753311	3465	3
31	50	2	565430297034751	310197425018629	263299	1

The attentive reader noted a few values of m staying identical when we go from rank v-1 to rank v and other m values in a 1/2 ratio. This is related to the resolution of the Bézout equation for these two ranks (we note below $w = w(v)$ at ranked v and $w(v-1)$ at rank v - 1 and Δw their difference) :

$$3^{v-1} \cdot t_1 \cdot m_1 \cdot 2^{w(v-1)-v+1} = 1 \quad (40)$$

$$3^v \cdot t_2 \cdot m_2 \cdot 2^{w(v)-v} = 1 \quad (41)$$

If $w(v)-w(v-1) = \Delta w = 1$, one can have $m_2 = m_1$ (with $t_2 = 3t_1$), but you cannot have $m_2 = m_1/2$. Case of $v = 8, 15, 22$ and 29 . If $\Delta w = 2$, can have $m_2 = m_1/2$ (with $t_2 = 3t_1$), but you cannot have $m_2 = m_1$. Case of $v = 4, 7, 12, 18, 19$ and 30 .

The anticipation of m seems simpler than that of $a_v(1,1)$ because of these cases.

More generally, parameter m is inferred from the v-plane to the v+1-plane, thanks to the twin equations (40) and (41), by the following relation :

$$m(v+1) = (m(v) + r(v) \cdot 3^v) / 2^{\Delta w(v+1)-1} \quad (42)$$

where $m(v)$ is the value of m for the v-plane and $m(v+1)$ and $\Delta w(v+1)$ are respectively the values of m and Δw for the v+1-plane and $r(v)$ an integer.

A similar anticipation allows the evaluation of t :

$$t(v+1) = (t(v) + r(v) \cdot 2^{w-v}) / 3 \quad (43)$$

where $r(v)$ is the same parameter as that used previously.

These relationships enable completing the table above incrementing r until what t, m are integers.

Parameter m alternates integer and integer plus 1/2 values when r is incremented. Parameter t, meanwhile, alternates the integer, integer plus 1/3 and integer plus 2/3 when r is incremented. Thus, test r from 0 to 5 will provide certainly the adequate choice of r among them, knowing that we seek the minimum value of $a_v(1,1)$.

The frequency of the values $r = 0$, $r = 1$ and $r = 2$ giving the good $a_v(1,1)$ solution seems roughly the same order of magnitude. Similarly, the frequency of the values $r = 3$, $r = 4$ and $r = 5$ giving an $a_v(1,1)$ seems of the same order of magnitude, but this time half of the previous.

An alternative to the use of the relationship (39) is to take the recursive formula :

$$a_{v+1}(1,1) = (2 \cdot (a_v(1,1) + r \cdot 2^w) - 1) / 3 \quad (44)$$

As previously, the solution is reached when $a_{v+1}(1,1)$ is an odd integer, $r = r(v)$ taking one of the integer values between 0 and 5. However, and as previously, a single equation is not sufficient to identify the right solution. Indeed, when r describes 0 to 5, $a_{v+1}(1,1)$ takes 2 odd integer's values and here the smallest of them is not necessarily the good solution. It is therefore always necessary to use parallel evaluations of $m(v)$ in the absence of a simpler discriminative relation for r. If this relationship is related to the history of the Δw 's, it seems relatively complex (little discrimination is observed on analysis of 6 generations of Δw).

Let us note that when we replace $r(v)$ by $r(v)+6$, we have $a'_{v+1}(1,1) = a_{v+1}(1,1)+2^{w+2}$ which is indeed outside the domain $[1,2^{w+\Delta w}]$ where all of the numbers of the $v+1$ plane are supposed to be (knowing that $a_{v+1}(1,1)$ is positive).

In addition, equation (37) means precisely that $m = m(v)$ is the first number obtained after Collatz algorithm altitude flight on $a_v(1,1)$ is executed.

We have conducted this test until the value $v = 17000$ (with systematic surveys up to $v = 3000$).

A record for a short time.

It is easy to find numbers with flight time above an arbitrary value.

For example, the Mersenne numbers $a = 2^n - 1$ have n odd steps at the beginning of Collatz algorithm and hence an altitude flight time superior to $n + \text{int}(\text{Ln}(3)/\text{Ln}(2) \cdot n) + 1$ (which is in the order of magnitude of $2,585 n$) steps. Assuming a total flight time 5 times greater than the altitude flight time, the order of magnitude of the total flight time would be $13n$. Thus $a = 2^{\text{int}(m/13)} - 1$ would have a total flight time about m .

For our part, we are looking somewhat little different numbers, because of the factor t in front of 2^n , but the results are quite similar.

We stopped at $a_{v=17000}(1,1)$, as computation time is being increasingly longer for a poor flight time gain.

v	TFA	TF	v	TFA	TF	v	TFA	TF
100	259	1215	5800	14993	84139	11500	29728	158984
200	517	2812	5900	15252	82909	11600	29986	159772
300	776	3983	6000	15510	86161	11700	30245	162785
400	1034	5418	6100	15769	84223	11800	30503	164126
500	1293	6668	6200	16027	88313	11900	30762	166503
600	1551	8323	6300	16286	89405	12000	31020	166668
700	1810	10162	6400	16544	89383	12100	31279	166975
800	2068	10711	6500	16803	89218	12200	31537	169271
900	2327	12657	6600	17061	93575	12300	31796	176762
1000	2585	13998	6700	17320	96603	12400	32054	172298
1100	2844	15452	6800	17578	94944	12500	32313	175574
1200	3102	17386	6900	17837	97498	12600	32571	176130
1300	3361	17826	7000	18095	101173	12700	32830	183660
1400	3619	19908	7100	18354	100015	12800	33088	182153
1500	3878	21109	7200	18612	101195	12900	33347	179320
1600	4136	23357	7300	18871	101300	13000	33605	180025
1700	4395	24861	7400	19129	101086	13100	33864	186272
1800	4653	24984	7500	19388	104763	13200	34122	186729
1900	4912	26817	7600	19646	109416	13300	34381	186927
2000	5170	27562	7700	19905	106843	13400	34639	186449
2100	5429	29085	7800	20163	110578	13500	34897	187840
2200	5687	31864	7900	20422	110996	13600	35156	194120
2300	5946	32587	8000	20680	112596	13700	35414	192448
2400	6204	33610	8100	20939	116283	13800	35673	194468
2500	6463	34607	8200	21197	114642	13900	35931	192263
2600	6721	36146	8300	21456	116322	14000	36190	193957
2700	6980	37449	8400	21714	119419	14100	36448	196543
2800	7238	38207	8500	21973	118913	14200	36707	200805
2900	7497	42412	8600	22231	118334	14300	36965	199377
3000	7755	42909	8700	22490	122742	14400	37224	201365
3100	8014	43628	8800	22748	123448	14500	37482	199770
3200	8272	45090	8900	23007	122749	14600	37741	206488
3300	8531	45391	9000	23265	127217	14700	37999	205924

v	TFA	TF	v	TFA	TF	v	TFA	TF
3400	8789	46154	9100	23524	126968	14800	38258	211375
3500	9048	47942	9200	23782	128288	14900	38516	210150
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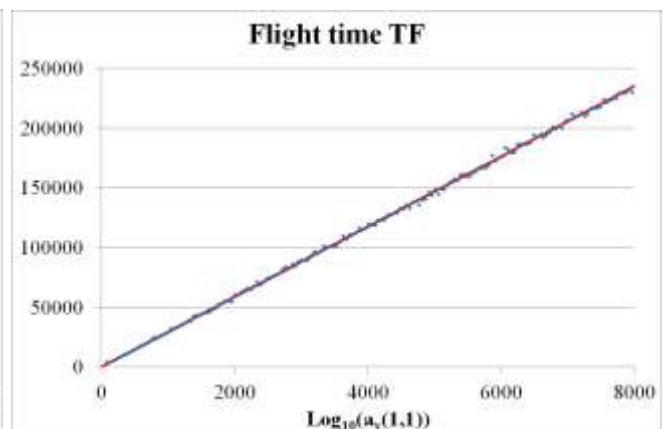
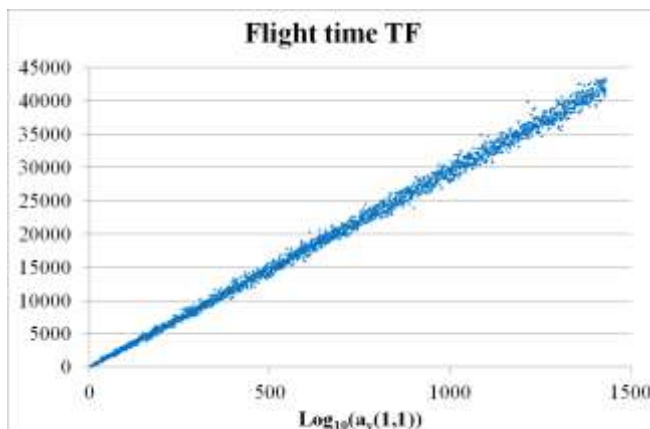
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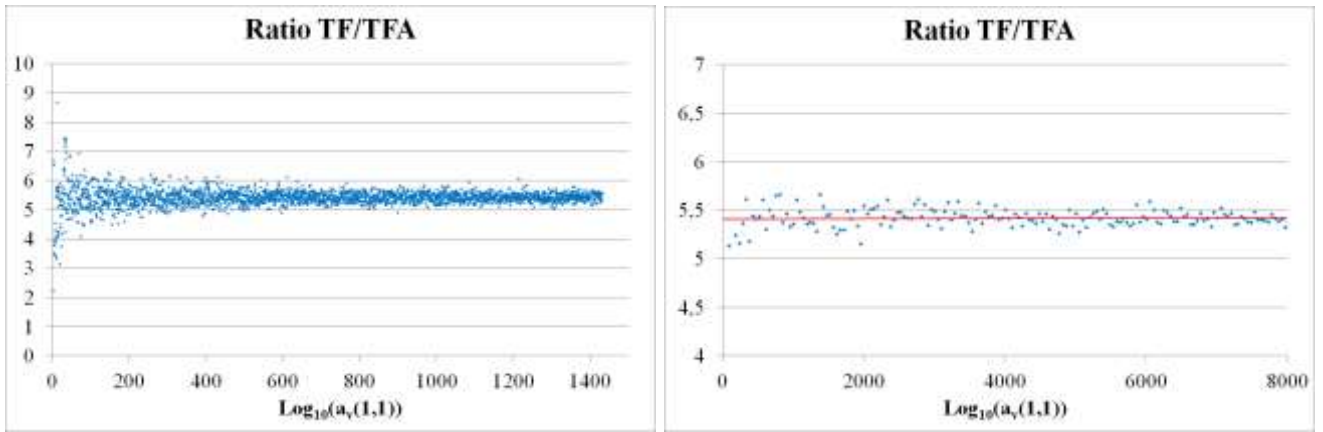
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233816667313105945551725135711620882891969967598181777341552669042202998270304087784840064868663384
556121582549011174074851460397633010000697505901398378527711643050733028022892596733395477878530011
494856389105516543967355520219123060799227402577947588827376848678480451462777726467901291444986703
886464152990536646272616428899188054895573958089628618810777609225642233223488238172496219670778921
642570298745111873380966378587694787507474538633931934614586033497279656140694259120733462940958392
982086657364494955357101224651834491778702053214754203483061413993158398117879447906474272703682138
749682324670043667029812556086029116262652317931842471130684538054233781159568256886551838246273918
931559810325955758187802972851495934936769357490210866263100027354303214976517277068139544277906287
017049233376618333160398103965497708389879318129364064423762630276613905393955713676774176056138797
198893020569795289758813442996751207380071646200903625106363605913930019136318514551110511223188645
612382129564469635506967546312054210126408993721857941637789186870688506838637511713526853334533762
833189240641363924938589387153771447029331556246318877905550495867257129358169572382245147817069837
718150845538587798699091778516365755580228362046944181886675731777338674423333560219233038236077278
153461932588059376443012058298974250329458077095643956465879652604340152612661861981492297633406339
101489689402907087341145627221667475940829208973201031407546453700895120001372772256438737012393604
030160450546924661230220085591774533828268572943742099639679720189352557395843553647273282164515381
502941702643415867177511622311123127445560631818743970621227359496628042564610426226405762512077548
912724615788918415355781304131507018663430803119851898858295682891786785037507045293959704325456411
060438781288403794105810014615454169260275169847631022959652134445544422807314951258063526116089211
767777809780031021475098537075142422251948835925223949806497921695880536331078676961529398523036108
699892936656947835890357764199017910406560107638562644706690196436677971673871231195208339774827227
334023228954801977836688487048528856084158241560328434420178098787342846194273403517061213290765469
009359256012196977573506710503058001543727204455700350437578113707135685125220452533498714173415445
858345853018644513921908635680498132670833183137906143494360308967945137497555683070169077396261107
630797917625926410261170229112884891117909553078317803639820435371254901517389248036051453056356659
300257449060420375554678951765274709018050930118897371621317211102556128913530921594553028231772859
078764955974089819066365482648894319050715271564743109051069008134593836512817094191511728953717254
409747115232607469731781206847142254276803908636335204293799637525434271714085339957420611295396666
117138190268578513837148761573986930576062476436481027024337133243501494250199018033998867845618637
131511368203745962845774625595243856432519651298913882144054307385325331779555206120863910565389859
847959498046643651571772475837383504369062197465316402857428695881430808412844176563322809142052500
916835104470624641117347747706399215538786472449677341245107642827532043961309426771477132261665301
73309731144084186532336329803023925411121544152684569805312886171645215772433789146240122879.

The number of digits of the last number is 8111.

Intermediary values of the flight times and TF/TFA ratios are given on charts below. Graphics on the right are only on v-hundreds (v = 100, v = 200,... , v = 17000). The regression line of the total flight time is :

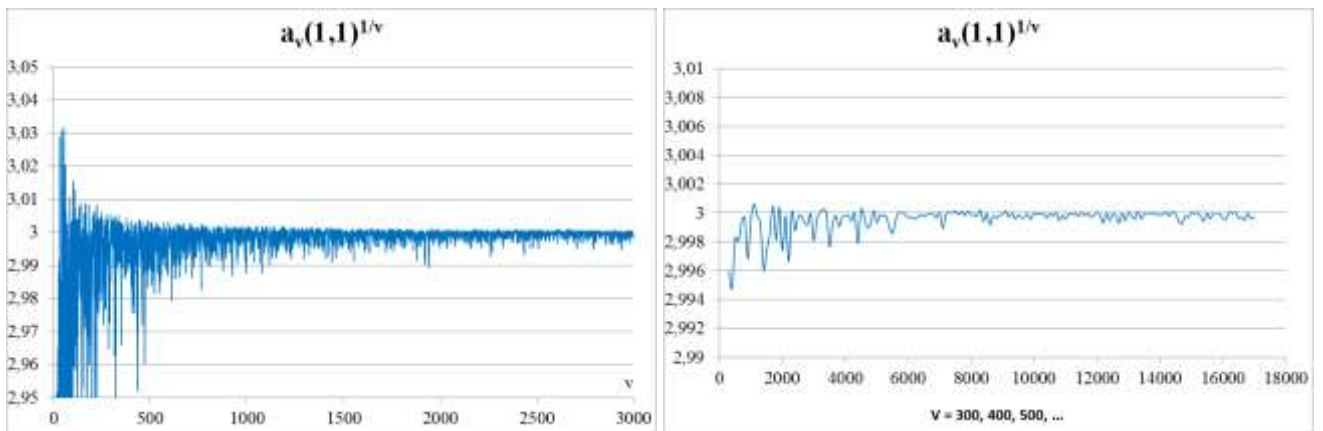
$$TF(v) \approx 29,4.Log_{10}(a_v(1,1))$$





From results on t and m at rank $v = 17000$ given above and from the recursive algorithm equations (42), (43) and (36), the reader can find new $a_v(1,1)$ numbers with larger flight times (TFA and TF).

As for the value of $a_v(1,1)$, it increases haltingly with v in a mean ratio of 3. The graphs below give an outline of it expressing $a_v(1,1)^{1/v}$ according to v .



The increase in $a_v(1,1)$ is not regular also including reductions in values. Their frequency is 23% for the planes v from 1 to 3000. All these decreases correspond, a priori, to the general form $a_{v+1}(1,1) = (2 \cdot a_v(1,1) - 1)/3$. Increases from one plan to the next are sometimes significant to catch up “backlogs”. So for $v = 1945$, we have $a_v(1,1) \approx 2394 \cdot a_{v-1}(1,1)$.

The evolution with an average ratio of 3 is not an exclusive property to $a_v(1,1)$.

It is a statistical property of the associates of a plane v . Indeed, let us suppose that we choose integers at random first among the numbers 1 to $2^{w(v-1)}$, then among 1 to $2^{w(v)}$, where $w(v-1)$ and $w(v)$ are large enough. For a large sample, the average of values in the first case would be of the order of magnitude of $2^{w(v-1)}/2$ and for the second case of $2^{w(v)}/2$, hence a ratio from the last to the first of $2^{w(v)}/2^{w(v-1)} = 2^{\Delta w(v)} = 2^{\Delta w}$.

We checked on a small sample the averages of the values of v -planes' associates. These do increase within an order of magnitude of 2 if $\Delta w = 1$ and 4 if $\Delta w = 2$. Due to the fact that $w = \text{int}(\ln(3)/\ln(2) \cdot v) + 1$, the frequency of the events $\Delta w = 1$ is $2 - \ln(3)/\ln(2)$ and that of the events $\Delta w = 2$ is $\ln(3)/\ln(2) - 1$. The average increase of the associates' values on incrementing v is hence $2^{2 - \ln(3)/\ln(2)} \cdot 4^{\ln(3)/\ln(2) - 1} = 2^{\ln(3)/\ln(2)} = 3$.

More on parameter t

The study of t has allowed us to deepen the knowledge of the first element of a Pascal trihedron plane. Let us see this in a more general way. Therefore, let us give first the t -values for the first planes of the Pascal trihedron.

v	w	w-v-3	t-td	t-td	t = t(v)	t/3	(t-td)/2 ^{w-v-4}
3	5	-1	3	3 ¹	3		12
4	7	0	1	3 ⁰	1	1	
5	8	0	3	3 ¹	3		6
6	10	1	9	3 ²	9		9
7	12	2	0	0	3	3	
8	13	2	0	0	1	1	
9	15	3	12	3 ¹ .2 ²	11		3
10	16	3	24	3 ¹ .2 ³	25		6
11	18	4	48	3 ¹ .2 ⁴	51		6
12	20	5	0	0	17	17	
13	21	5	96	3 ¹ .2 ⁵	91		6
14	23	6	192	3 ¹ .2 ⁶	201		6
15	24	6	64	2 ⁶	67	67	
16	26	7	192	3 ¹ .2 ⁶	193		3
17	27	7	768	3 ¹ .2 ⁸	747		12
18	29	8	256	2 ⁸	249	249	
19	31	9	0	0	83	83	
20	32	9	1536	3 ¹ .2 ⁹	1393		6
21	34	10	3072	3 ¹ .2 ¹⁰	3195		6
22	35	10	1024	2 ¹⁰	1065	1065	
23	37	11	9216	3 ² .2 ¹⁰	8547		9
24	39	12	0	0	2849	2849	
25	40	12	24576	3 ¹ .2 ¹³	22795		12
26	42	13	20480	5 ¹ .2 ¹¹	18521		
27	43	13	24576	3 ¹ .2 ¹³	28019		6
28	45	14	24576	3 ¹ .2 ¹³	31185		3
29	46	14	0	0	10395	10395	
30	48	15	0	0	3465	3465	
31	50	16	294912	3 ¹ .2 ¹⁵	263299		9

The differences t-td are loosely correlated to 2^{w-v-3} with sometimes straight returns to 0. If t(v)/t(v-1) = 1/3, then t-td at rank v is either equal to 0 or equal to 2^{w-v-3} in the preceding table. For v = 26, we observe a factor 5, which contrasts with other differences. We can attribute, a priori, this characteristic to the fact that t(v)/t(v-1) < 1 without being equal to the 1/3 ratio. In all other cases, which are characterized by t(v) / t(v-1) > 1, we have here tdw = (t-td)/ 2^{w-v-4} = 3, 6, 9 or 12.

The overall situation is more complex. The table below gives a wider range of values. To date, a general rule of anticipation of these values seems difficult to find.

c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw
32	8	82	8	132	9	182	2	232	2	282	16	332	6	382	2	432	4	482	12
33	6	83	6	133	8	183	15	233	15	283	8	333	8	383	3	433	3	483	11
34	14	84	12	134	15	184	4	234	4	284	0	334	9	384	12	434	0	484	15
35	17	85	4	135	10	185	3	235	9	285	0	335	8	385	5	435	0	485	10
36	2	86	0	136	6	186	0	236	6	286	6	336	12	386	6	436	0	486	9
37	0	87	0	137	9	187	9	237	14	287	2	337	17	387	15	437	0	487	8
38	3	88	6	138	8	188	8	238	11	288	9	338	4	388	4	438	3	488	15
39	12	89	9	139	15	189	0	239	2	289	3	339	3	389	15	439	12	489	2
40	5	90	14	140	10	190	6	240	15	290	0	340	6	390	11	440	11	490	0
41	0	91	5	141	3	191	14	241	10	291	3	341	6	391	2	441	2	491	9
42	12	92	0	142	3	192	11	242	9	292	0	342	12	392	3	442	9	492	8
43	14	93	0	143	12	193	8	243	0	293	12	343	16	393	12	443	15	493	6
44	16	94	0	144	8	194	6	244	6	294	4	344	5	394	5	444	16	494	8
45	17	95	15	145	8	195	3	245	15	295	12	345	6	395	3	445	14	495	12
46	16	96	5	146	0	196	12	246	16	296	17	346	15	396	12	446	16	496	2
47	5	97	0	147	0	197	5	247	11	297	10	347	10	397	11	447	5	497	12
48	6	98	3	148	9	198	0	248	6	298	6	348	3	398	14	448	15	498	8
49	14	99	0	149	9	199	0	249	14	299	14	349	0	399	8	449	10	499	2
50	8	100	6	150	8	200	0	250	17	300	8	350	0	400	8	450	6	500	9
51	8	101	15	151	3	201	3	251	10	301	12	351	6	401	0	451	14	501	15
52	3	102	4	152	0	202	9	252	9	302	10	352	14	402	15	452	14	502	16
53	15	103	12	153	12	203	2	253	2	303	9	353	17	403	4	453	4	503	8
54	10	104	4	154	8	204	15	254	6	304	14	354	5	404	6	454	12	504	2

c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw	c	tdw
55	15	105	12	155	8	205	16	255	3	305	14	355	6	405	2	455	8	505	15
56	4	106	2	156	6	206	11	256	12	306	16	356	12	406	9	456	8	506	4
57	6	107	6	157	8	207	6	257	2	307	17	357	4	407	0	457	12	507	0
58	14	108	15	158	15	208	8	258	6	308	5	358	15	408	0	458	4	508	9
59	14	109	16	159	8	209	15	259	0	309	12	359	11	409	3	459	0	509	14
60	14	110	11	160	2	210	10	260	12	310	17	360	14	410	12	460	0	510	17
61	4	111	8	161	9	211	6	261	16	311	16	361	5	411	5	461	12	511	4
62	12	112	15	162	2	212	9	262	11	312	17	362	12	412	3	462	17	512	15
63	16	113	11	163	12	213	2	263	2	313	2	363	8	413	12	463	10	513	5
64	8	114	2	164	4	214	12	264	3	314	6	364	8	414	2	464	12	514	0
65	9	115	9	165	3	215	4	265	12	315	3	365	6	415	0	465	5	515	12
66	2	116	2	166	15	216	3	266	16	316	0	366	6	416	15	466	0	516	16
67	0	117	9	167	16	217	0	267	5	317	0	367	14	417	10	467	9	517	14
68	6	118	9	168	11	218	0	268	0	318	15	368	8	418	0	468	14	518	8
69	6	119	2	169	2	219	15	269	6	319	4	369	2	419	0	469	8	519	2
70	14	120	3	170	3	220	4	270	2	320	3	370	12	420	6	470	2	520	3
71	8	121	6	171	0	221	0	271	15	321	0	371	8	421	0	471	0	521	0
72	9	122	0	172	0	222	0	272	8	322	0	372	8	422	12	472	0	522	9
73	8	123	0	173	3	223	12	273	14	323	0	373	3	423	2	473	0	523	2
74	6	124	15	174	12	224	17	274	17	324	3	374	6	424	12	474	9	524	0
75	2	125	11	175	11	225	10	275	10	325	3	375	9	425	16	475	14	525	12
76	0	126	2	176	2	226	6	276	3	326	12	376	2	426	11	476	14	526	16
77	0	127	3	177	3	227	2	277	9	327	14	377	9	427	2	477	11	527	11
78	6	128	6	178	6	228	15	278	8	328	10	378	15	428	9	478	8	528	8
79	3	129	6	179	2	229	16	279	12	329	6	379	4	429	8	479	12	529	9
80	12	130	3	180	6	230	2	280	10	330	12	380	12	430	15	480	10	530	6
81	11	131	6	181	14	231	6	281	15	331	4	381	16	431	14	481	0	531	14

For the report $tdw = (t-td)/2^{w-v-4}$, other values appear not present in the first list and we have stock of the occurrences (Nb_app) between $v = 3$ and $v = 531$:

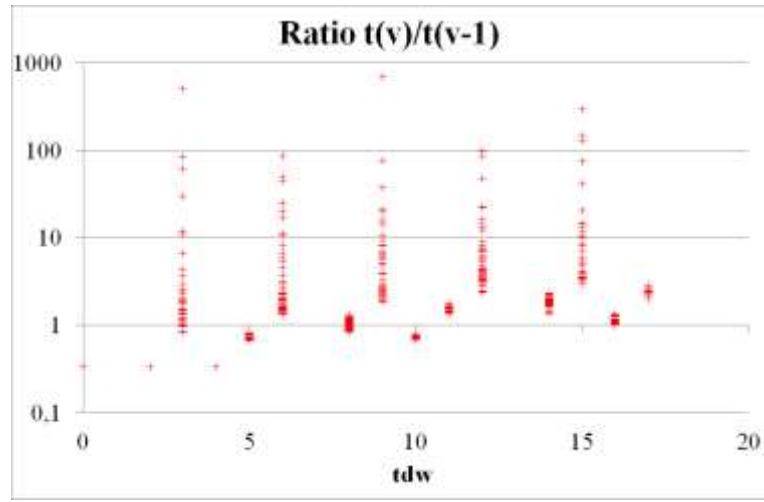
tdw	0	6	12	8	2	3	9	15	14	4	16	11	10	5	17
Nb_app	77	58	51	45	42	41	37	37	31	23	21	19	18	16	13
% occurrences	14,6%	11,0%	9,6%	8,5%	7,9%	7,8%	7,0%	7,0%	5,9%	4,3%	4,0%	3,6%	3,4%	3,0%	2,5%

Returns to 0 are the most frequent, then masterfully 6 and 12.

If $t(v)/t(v-1) = 1/3$, then tdw at rank v is either 0, 2 or 4. Specifically, $v_{tdw=0} = 12, 19, 24, 29, 30, 37, 41, 67, 76, 77, 86, 87, 92, 93, 94, 97, 99, 122, 123, 146, 147, 152, 171, 172, 186, 189, 198, 199, 200, 217, 218, 221, 222, 243, 259, 268, 284, 285, 290, 292, 316, 317, 321, 322, 323, 349, 350, 401, 407, 408, 415, 418, 419, 421, 434, 435, 436, 437, 459, 460, 466, 471, 472, 473, 481, 490, 507, 514, 521, 524, v_{tdw=2} = 15, 18, 22, 36, 66, 75, 106, 114, 116, 119, 126, 160, 162, 169, 176, 179, 182, 203, 213, 227, 230, 232, 239, 253, 257, 263, 270, 287, 313, 369, 376, 382, 391, 405, 414, 423, 427, 441, 470, 489, 496, 499, 504, 519, 523$ and $v_{tdw=4} = 56, 61, 85, 102, 104, 164, 184, 215, 220, 234, 294, 319, 331, 338, 357, 379, 388, 403, 432, 453, 458, 506, 511, 533$.

We find this data in the form of three points lined up at the bottom left of the chart below.

For $tdw = 3, 6, 9, 12, 15$, the ratio $t(v)/t(v-1)$ seems not possessing an upper boundary (when v tends to infinity). The $tdw = 3$ value is special because the points cross axis $t(v)/t(v-1) = 1$, which seems not the case for other values modulo 3. Values $tdw = 5, 8, 10, 11, 14, 16$ and 17 seem to have lower and upper boundaries, 5 and 10 being under the said axis, 8 crossing it, and 11, 14, 16 and 17 being above it.



9 Values of the second element of the Pascal trihedron planes.

We are talking about the number $a_v(1,2)$ lying on line 1 and column 2 of the Pascal trihedron.
We have $2 \cdot a_v(1,2) + 1 = a_v(1,1) \bmod 2^w$, or:

Conjecture 2

$$a_v(1,2) = (a_v(1,1) - 1 + \text{or}(0,1) \cdot 2^w) / 2 \quad (47)$$

The table below gives the values of $\text{or}(0,1)$ for $v = 3$ to $v = 19$.

v	w	Δw	2^w	$a_v(1,1)$	$a_v(1,2)$	$\text{or}(0,1)$	m	n
3	5	1	32	23	11	0	20	10
4	7	2	128	15	7	0	10	5
5	8	1	256	95	175	1	91	167
6	10	2	1024	575	287	0	410	205
7	12	2	4096	383	2239	1	205	1196
8	13	1	8192	255	4223	1	205	3383
9	15	2	32768	5631	19199	1	3383	11533
10	16	1	65536	25599	12799	0	23066	11533
11	18	2	262144	104447	52223	0	70582	35291
12	20	2	1048576	69631	559103	1	35291	283366
13	21	1	2097152	745471	372735	0	566732	283366
14	23	2	8388608	3293183	5840895	1	1877689	3330329
15	24	1	16777216	2195455	9486335	1	1877689	8113298
16	26	2	67108864	12648447	6324223	0	8113298	4056649
17	27	1	134217728	97910783	48955391	0	94206740	47103370
18	29	2	536870912	65273855	32636927	0	47103370	23551685
19	31	2	2147483648	43515903	1095499775	1	23551685	592906576

The anticipation of the value of $\text{or}(0,1)$ seems a difficult problem. In any case, it seems not related to Δw , as Δw never has three successive identical values while we observe this for $\text{or}(0,1)$. It should be noted however that the term $\text{or}(0,1)$ is directly related to the parity of m given sooner. If m is even then the value of $\text{or}(0,1)$ is 0 otherwise it is 1 :

$$\text{or}(0,1) = \text{if}(m = 0 \bmod 2, 0, 1) \quad (48)$$

It is easy to find this result through an analysis of the Collatz algorithm as we have done in paragraph 8, knowing that the second element has just an odd step offset of one unit versus the first element.
All calculations performed, we find:

$$a_v(1,2) = (1 + n \cdot 2^{w-v+1}) \cdot (2^{v-1} / 3^v) - 1 \quad (49)$$

with n an integer

$$n = (m + \text{or}(0,1) \cdot 3^v) / 2 \quad (50)$$

and if m is even, the value $\text{or}(0,1) = 0$ is required to get an integer n , and if m is odd, it is $\text{or}(0,1) = 1$ which we have to choose.

The reader will check that $n = n(v)$ is the first resulting number of execution of the Collatz algorithm on $a_v(1,2)$ just after the

altitude flight. We will address this point in a general framework later on.

10 Values of the elements of the first line of the Pascal trihedron planes.

The values of the first line $a_v(1,j+1)$, $j \geq 2$, infer the preceding by recurrence :

Conjecture 3

$$a_v(1,j+1) = a_v(1,j) - d_v(2,2) + C(j-1,1).d_v(3,3) - C(j-1,2).d_v(4,4) + \dots + (-1)^j.C(j-1,j-1).d_v(j+1,j+1) \mod 2^w \quad (51)$$

where $C(i,j)$ are the binomial coefficients

$$C(i,j) = i!/(j!(i-j)!)$$

and $d_v(i,j)$ meets

$$d_v(i+1,j+1) = (\text{or}(1,2).2^w - d_v(i,j))/2 \quad (52)$$

where

$$\text{or}(1,2) = 1 \text{ or } 2 \quad (53)$$

and

$$d_v(2,2) = a_v(1,1) - a_v(1,2) \mod 2^w \quad (54)$$

Let us take an example ($v = 6$) for which the Pascal trihedron plane has been sorted according to lines and columns, the first line is as follows :

575	287	367	999	923
-----	-----	-----	-----	-----

We proceed to differences modulo 2^w (here modulo 1024) column-by-column between the number left on the diagonal and the one above on the column, that is

$$d_v(i+1,j+1) = d_v(i,j) - d_v(i,j+1) \mod 2^w \quad (55)$$

This gives the table :

$i \backslash j$	1	2	3	4	5
1	575	287	367	999	923
2		288	944	392	76
3			368	552	316
4				840	236
5					604

For example $288 = 575 - 287$, $944 = 287 - 367 + 1024$, $368 = 288 - 944 + 1024$, $392 = 367 - 999 + 1024$, $552 = 944 - 392$, etc. The elements on the diagonal (288, 368, 840, 604) infer each other by :

$$d_v(i+1,j+1) = (\text{or}(1,2).2^w - d_v(i,j))/2 \quad (56)$$

where $\text{or}(1,2)$ takes either the value 1 or the value 2.

Here $368 = (1024 - 288)/2$, $840 = (2.1024 - 368)/2$, $604 = (2.1024 - 840)/2$.

Having found these values, we go back to the elements of the first line with reverse operations :

$$d_v(i,j+1) = d_v(i,j) - d_v(i+1,j+1) \mod 2^w \quad (57)$$

Let us have $a_v(1,1)$, $d_v(2,2)$, $d_v(3,3)$, $d_v(4,4)$, $d_v(5,5)$,... the numbers appearing on the main diagonal. The elements of the first line are then given by :

$$\begin{aligned} a_v(1,3) &= a_v(1,2) - d_v(2,2) + d_v(3,3) \mod 2^w \\ a_v(1,4) &= a_v(1,3) - d_v(2,2) + 2d_v(3,3) - d_v(4,4) \mod 2^w \\ a_v(1,5) &= a_v(1,4) - d_v(2,2) + 3d_v(3,3) - 3d_v(4,4) + d_v(5,5) \mod 2^w \\ &\dots \\ a_v(1,j+1) &= a_v(1,j) - d_v(2,2) + C(j-1,1).d_v(3,3) - C(j-1,2).d_v(4,4) + \dots + (-1)^j.C(j-1,j-1).d_v(j+1,j+1) \mod 2^w \end{aligned}$$

The coefficients in front of $d_v(i,i)$ are thus growing according to a Pascal triangle with alternating signs, $C(i,j)$ being the binomial coefficients.

Alternative approach

Instead of making the differences according to the relation (12), let us proceed by the opposite :

$$d_v(i+1,j+1) = -(d_v(i,j) - d_v(i,j+1)) \mod 2^w \quad (58)$$

We get the following table (for $v=6$) :

$i \backslash j$	1	2	3	4	5
1	575	287	367	999	923
2		736	80	632	948
3			368	552	316
4				184	788
5					604

The elements of the main diagonal (excluding 575 and 736) infer each other by

$$(or(0,1).2^w + d_v(i,j))/d_v(i+1,j+1) = 2 \quad (59)$$

meaning that the elements on the main diagonal are divided by two, with from time to time, the addition of 2^{w-1} .

The recurrence process is simple, but rests the problem around the anticipation of values $or(1,2)$ (or else $or(0,1)$ if we proceed by the alternative). Of course $or(1,2)$ and $or(0,1)$ are closely linked here.

This method applies similarly to all of the lines of the Pascal trihedron planes.

For example, for the tenth line of the plane corresponding to $v = 12$ ($w = 20$, $2^w = 1048576$), we have the following table :

Line 10	713727	1028607	452351	112255	650687	934047	834799	161639	200475
		314880	472320 157440	708480 236160 78720	538432 878528 642368 563648	283360 793504 963552 321184 806112	949328 665968 921040 1006064 684880 927344	375416 474664 857272 984808 1027320 342440 463672	38836 711996 237332 428636 492404 513660 171220 756124
$d_{12}(i,i)$		314880	157440	78720	563648	806112	927344	463672	756124
$d_{12}(i,i)/d_{12}(i+1,i+1)$			2	2				2	
$(2^w + d_{12}(i,i))/d_{12}(i+1,i+1)$					2	2	2		2

The anticipation of the values $or(1,2)$ (or $or(0,1)$) would solve largely, with conjectures on the first and second elements of a plane, the problem at hand.

The rest of the hereby text settles the problem a little differently but remains largely related to what has already been stated.

11 Pascal trihedron whole set anticipation.

We proceed by an example (which is not yet conveniently sorted at this stage) for clarity.

We choose $v = 9$ (and $w(at v) = 15$), that is also $v-1 = 8$ (and $w(at v-1) = 13$).

The values of Pascal trihedron plane for $v-1$ are as follows:

255	4223	1983	6815	5871	4455	2331
	3967	1727	6559	5615	4199	2075
	3455	1215	6047	5103	3687	1563
	2431	191	5023	4079	2663	539
		5439	2079	1135	7911	5787
		4927	1567	623	7399	5275
		3903	543	7791	6375	4251
		4159	799	8047	6631	4507
		3135	7967	7023	5607	3483
			3551	2607	1191	7259
			3039	2095	679	6747
			2015	1071	7847	5723
			2271	1327	8103	5979
			1247	303	7079	4955
				719	7495	5371

207	6983	4859
7375	5959	3835
7631	6215	4091
6607	5191	3067

The values for v are the followings :

5631	19199	6783	20927	9375	8431	23399	29467
	13567	1151	15295	3743	2799	17767	23835
	2303	22655	4031	25247	24303	6503	12571
	12543	127	14271	2719	1775	16743	22811
		9087	23231	11679	10735	25703	31771
		30591	11967	415	32239	14439	20507
		8063	22207	10655	9711	24679	30747
		13695	27839	16287	15343	30311	3611
		23935	5311	26527	25583	7783	13851
		22911	4287	25503	24559	6759	12827
			18751	7199	6255	21223	27291
			7487	28703	27759	9959	16027
			17727	6175	5231	20199	26267
			23359	11807	10863	25831	31899
			831	22047	21103	3303	9371
			32575	21023	20079	2279	8347
			14399	2847	1903	16871	22939
			24639	13087	12143	27111	411
			23615	12063	11119	26087	32155
				16863	15919	30887	4187
				5599	4655	19623	25691
				15839	14895	29863	3163
				21471	20527	2727	8795
				31711	30767	12967	19035
				30687	29743	11943	18011
				12511	11567	26535	32603
				22751	21807	4007	10075
				21727	20783	2983	9051
					14031	28999	2299
					2767	17735	23803
					13007	27975	1275
					18639	839	6907
					28879	11079	17147
					27855	10055	16123
					9679	24647	30715
					19919	2119	8187
					18895	1095	7163

In both tables, we already ordered values following columns and rows. Thus, the differences between lines are as reported earlier constants, modulo 8192 for the first table and modulo 32768 for the second.

We gather then lines of equal staffs.

Thus, we get the following associations :

19199	6783	20927	9375	8431	23399	29467
13567	1151	15295	3743	2799	17767	23835
2303	22655	4031	25247	24303	6503	12571
12543	127	14271	2719	1775	16743	22811

and

255	4223	1983	6815	5871	4455	2331
-----	------	------	------	------	------	------

then,

9087	23231	11679	10735	25703	31771
30591	11967	415	32239	14439	20507
8063	22207	10655	9711	24679	30747
13695	27839	16287	15343	30311	3611
23935	5311	26527	25583	7783	13851
22911	4287	25503	24559	6759	12827

and

3967	1727	6559	5615	4199	2075
3455	1215	6047	5103	3687	1563
2431	191	5023	4079	2663	539

then,

18751	7199	6255	21223	27291
7487	28703	27759	9959	16027
17727	6175	5231	20199	26267
23359	11807	10863	25831	31899
831	22047	21103	3303	9371
32575	21023	20079	2279	8347
14399	2847	1903	16871	22939
24639	13087	12143	27111	411
23615	12063	11119	26087	32155

and

5439	2079	1135	7911	5787
4927	1567	623	7399	5275
3903	543	7791	6375	4251
4159	799	8047	6631	4507
3135	7967	7023	5607	3483

then,

16863	15919	30887	4187
5599	4655	19623	25691
15839	14895	29863	3163
21471	20527	2727	8795
31711	30767	12967	19035
30687	29743	11943	18011
12511	11567	26535	32603
22751	21807	4007	10075
21727	20783	2983	9051

and

3551	2607	1191	7259
3039	2095	679	6747
2015	1071	7847	5723
2271	1327	8103	5979
1247	303	7079	4955

then to end with,

14031	28999	2299
2767	17735	23803
13007	27975	1275
18639	839	6907
28879	11079	17147
27855	10055	16123
9679	24647	30715
19919	2119	8187
18895	1095	7163

and

719	7495	5371
207	6983	4859
7375	5959	3835
7631	6215	4091
6607	5191	3067

We then consider the differences modulo $2^{w(\text{at } v-1)}$ (here $2^{13} = 8192$) between lines of equal staffs. The choice of the differences is multiple, and to limit ourselves to one only, we subtract the first tables (ranking v) to the first line only of the following table underneath it (ranking $v-1$).

We then get the following table :

Dif1 col 1	Dif1 col 2	Dif1 col 3	Dif1 col 4	Dif1 col 5	Dif1 col 6	Dif1 col 7	Divisors	Ratios
2560	2560	2560	2560	2560	2560	2560	$2^{10} = 1024$	2,5
5120	5120	5120	5120	5120	5120	5120		5
2048	2048	2048	2048	2048	2048	2048		2
4096	4096	4096	4096	4096	4096	4096		4
	5120	5120	5120	5120	5120	5120	$2^9 = 512$	10
	2048	2048	2048	2048	2048	2048		4
	4096	4096	4096	4096	4096	4096		8
	1536	1536	1536	1536	1536	1536		3
	3584	3584	3584	3584	3584	3584		7
	2560	2560	2560	2560	2560	2560		5
		5120	5120	5120	5120	5120	$2^8 = 256$	20
		2048	2048	2048	2048	2048		8
		4096	4096	4096	4096	4096		16
		1536	1536	1536	1536	1536		6
		3584	3584	3584	3584	3584		14
		2560	2560	2560	2560	2560		10
		768	768	768	768	768		3
		2816	2816	2816	2816	2816		11
		1792	1792	1792	1792	1792		7
			5120	5120	5120	5120	$2^8 = 256$	20
			2048	2048	2048	2048		8
			4096	4096	4096	4096		16
			1536	1536	1536	1536		6
			3584	3584	3584	3584		14
			2560	2560	2560	2560		10
			768	768	768	768		3
			2816	2816	2816	2816		11
			1792	1792	1792	1792	$2^8 = 256$	7
				5120	5120	5120		20
				2048	2048	2048		8
				4096	4096	4096		16
				1536	1536	1536		6
				3584	3584	3584		14
				2560	2560	2560		10
				768	768	768		3
				2816	2816	2816		11
				1792	1792	1792		7

Each row has constant values (cf. theorem 9). We divide then the value by the factors set out in column “divisors” to get the column “ratio”.

Conjecture 4

Let us take the divisors (in the column divisors) as follows :

- for the first set of lines (with v-2 columns): 2^{v+1}
- for the second set of lines (with v-3 columns): 2^v (if $v \geq 6$)
- for the third set of lines (with v-4 columns): 2^{v-1} (if $v \geq 7$)
- ...
- for the i-1th set of lines (with v-i columns): 2^{v-i+3} (if $v \geq i+3$)
- ...
- for the last three sets of lines (with 5, 4 and 3 columns): 2^8

The last rule does not apply for $v < 6$, it does partially for $v = 6$ (the last column) and $v = 7$ (last two columns) and completely for $v \geq 8$.

Then, the elements of the “ratios” column are all integers, except for one which is an integer + 1/2.

The half-integer corresponding line will be selected to take his place at the first line of the current plane (what we anticipated for our example). In each set of rows, we classify the values in the column “ratios” by increasing values (what we have also done from the beginning).

This process gives a unique classification of the lines of each Pascal trihedron plane. It is not the best one and we will get to this a little later.

Meanwhile, let us rewrite the column “ratios” in a way that seems us most appropriate, taking into account the grouping of lines of identical staffs, thus highlighting the last three sets of data:

2,5	5	10	20	20	20
	2	4	8	8	8
	4	8	16	16	16
		3	6	6	6
		7	14	14	14
		5	10	10	10
			3	3	3
			11	11	11
			7	7	7

Somehow, we find a kind of Pascal trihedron plane (with the last three identical columns). We will call these planes by the term (Pascal) secondary planes. However, everything remains to discover about their anticipation.

Working so far modulo $2^{w(at\ v-1)}$ (here $2^{13} = 8192$), the previous table cannot map the Pascal trihedron plane ranked v from that ranked $v-1$.

We need resume the same work, but this time by making differences modulo $2^{w(at\ v)}$ (here $2^{15} = 32768$) between lines of equal lengths.

Dif2 col 1	Dif2 col 2	Dif2 col 3	Dif2 col 4	Dif2 col 5	Dif2 col 6	Dif2 col 7
18944	2560	18944	2560	2560	18944	27136
13312	29696	13312	29696	29696	13312	21504
2048	18432	2048	18432	18432	2048	10240
12288	28672	12288	28672	28672	12288	20480
	5120	21504	5120	5120	21504	29696
	26624	10240	26624	26624	10240	18432
	4096	20480	4096	4096	20480	28672
	9728	26112	9728	9728	26112	1536
	19968	3584	19968	19968	3584	11776
	18944	2560	18944	18944	2560	10752
		13312	5120	5120	13312	21504
		2048	26624	26624	2048	10240
		12288	4096	4096	12288	20480
		17920	9728	9728	17920	26112
		28160	19968	19968	28160	3584
		27136	18944	18944	27136	2560
		8960	768	768	8960	17152
		19200	11008	11008	19200	27392
		18176	9984	9984	18176	26368
			13312	13312	29696	29696
			2048	2048	18432	18432
			12288	12288	28672	28672
			17920	17920	1536	1536
			28160	28160	11776	11776
			27136	27136	10752	10752
			8960	8960	25344	25344
			19200	19200	2816	2816
			18176	18176	1792	1792
				13312	21504	29696
				2048	10240	18432
				12288	20480	28672
				17920	26112	1536
				28160	3584	11776
				27136	2560	10752
				8960	17152	25344
				19200	27392	2816
				18176	26368	1792

Then we perform the subtractions Dif2-Dif1(modulo 2^w) and we divide by $2^{w(at\ v-1)}$ (so here $2^{13} = 8192$). This gives:

Dif2-Dif1 col 1	Dif2-Dif1 col 2	Dif2-Dif1 col 3	Dif2-Dif1 col 4	Dif2-Dif1 col 5	Dif2-Dif1 col 6	Dif2-Dif1 col 7
2	0	2	0	0	2	3
1	3	1	3	3	1	2
0	2	0	2	2	0	1
1	3	1	3	3	1	2
	0	2	0	0	2	3
	3	1	3	3	1	2
	0	2	0	0	2	3
	1	3	1	1	3	0
	2	0	2	2	0	1
	2	0	2	2	0	1
		1	0	0	1	2
		0	3	3	0	1
		1	0	0	1	2
		2	1	1	2	3
		3	2	2	3	0
		3	2	2	3	0
		1	0	0	1	2
		2	1	1	2	3
		2	1	1	2	3
			1	1	3	3
			0	0	2	2
			1	1	3	3
			2	2	0	0
			3	3	1	1
			3	3	1	1
			1	1	3	3
			2	2	0	0
			2	2	0	0
				1	2	3
				0	1	2
				1	2	3
				2	3	0
				3	0	1
				3	0	1
				1	2	3
				2	3	0
				2	3	0

If the multiplicative factor between $2^{w(at\ v-1)}$ and $2^{w(at\ v)}$ is 2, we will get in these tables only 0 and 1, if it is instead 4, we will get 0, 1, 2 or 3.

We will call these planes by the term tertiary planes.

If we can anticipate secondary and tertiary planes, we can build by successive iterations all values constituting the Pascal trihedron.

Conjecture 5

The values of secondary planes, ordered by positive sorting, can be deducted from one column to the next one by multiplication by 2 except the first column (trivially) on the one hand and the last two columns, which are identical to the previous one, on the other hand.

The reader can check this, for example, for the secondary plane of plane $v = 9$ given above, as well as the secondary plane of the planes $v = 4$ to $v = 11$ given in Chapter 12.

Knowledge of the first element in each row means knowing all the other objects.

Here, the rule applies from the first number (in parentheses), but this is not a general case.

Indeed, let us note $s_v(i,j)$ an element of secondary plane v , i the line number line and j the column number, we have then:

Conjecture 6

For any secondary plane, the difference between the first element $s_v(1,1)$ and half of the next $s_v(1,2)$ is either zero, or a power of 2.

v	$s_v(1,1)$	$s_v(1,2)$	$s_v(1,1)-s_v(1,2)/2$
5	0,5	1	0
6	1,5	1	2^0
7	2,5	1	2^1
8	7,5	7	2^2
9	2,5	5	0
10	3,5	7	0
11	6,5	13	0
12	23,5	15	2^4

Conjecture 7

The associates of the first line at rank v are drawn from the associates next ranked v-1 by multiplying by 6 and adding 5 and after iterations of the result :

$$a_v(1,j) = (6.a_{v-1}(1,j)+5)^k \bmod 2^w \quad (60)$$

We will show this by examples as we do usually in this article and therefore we start by giving the first lines of some associates' planes :

v	w	2^w	2^{w-v}	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9	Col 10
1	2	4	2	5									
2	4	16	4	3									
3	5	32	4	23	11								
4	7	128	8	15	7	59							
5	8	256	8	95	175	39	219						
6	10	1024	16	575	287	367	999	923					
7	12	4096	32	383	2239	2975	2031	615	2587				
8	13	8192	32	255	4223	1983	6815	5871	4455	2331			
9	15	32768	64	5631	19199	6783	20927	9375	8431	23399	29467		
10	16	65536	64	25599	12799	26367	13951	28095	49311	15599	63335	36635	
11	18	262144	128	104447	52223	104959	52991	237183	251327	10399	173295	24423	63259

Each item of columns "Col i" is drawn from that which is above it, unless the box is empty, in which case the value 0 does not apply.

As examples:

(5).6+5 = 3 mod 16 and $3^1 = 3 \bmod 16$.

(3).6+5 = 23 mod 32 and $23^1 = 23 \bmod 32$.
but (0).6+5 = 5 mod 32 and $5^i \neq 11 \bmod 32$ for any i .

(23).6+5 = 15 mod 128 and $15^1 = 15 \bmod 128$.
(11).6+5 = 71 mod 128 and $71^9 = 7 \bmod 128$.
but (0).6+5 = 5 mod 128 and $5^i \neq 59 \bmod 128$ for any i .

(15).6+5 = 95 mod 256 and $95^1 = 95 \bmod 256$.
(7).6+5 = 47 mod 256 and $47^9 = 175 \bmod 256$.
(59).6+5 = 103 mod 256 and $103^{25} = 39 \bmod 256$.
but (0).6+5 = 5 mod 256 and $5^i \neq 219 \bmod 256$ for any i .

(95).6+5 = 575 mod 1024 and $575^1 = 575 \bmod 1024$.
(175).6+5 = 31 mod 1024 and $31^9 = 287 \bmod 1024$.
(39).6+5 = 239 mod 1024 and $239^{57} = 367 \bmod 1024$.
(219).6+5 = 295 mod 1024 and $295^{89} = 999 \bmod 1024$.
but (0).6+5 = 5 mod 1024 and $5^i \neq 923 \bmod 1024$ for any i .

Continuing that way, we get the following table. The "Exp" columns are the exponents to apply to 6m+5, to get the value under the number m :

v	w	2^w	2^{w-v}	Exp 1	Exp 2	Exp 3	Exp 4	Exp 5	Exp 6	Exp 7	Exp 8	Exp 9	Exp 10
1	2	4	2										
2	4	16	4	1									
3	5	32	4	1									
4	7	128	8	1	9								
5	8	256	8	1	9	25							
6	10	1024	16	1	9	57	89						
7	12	4096	32	25	25	57	121	25					
8	13	8192	32	25	57	57	185	121	25				
9	15	32768	64	25	121	185	57	441	1657	2073			
10	16	65536	64	57	57	57	57	825	1977	2681	2073		
11	18	262144	128	57	185	57	57	57	1849	1977	2681	2073	

The exponents are always inferior to $2^{w-v+c-1}$, c being the c^{th} column.

This is analogous to what has been said in the statement of theorem 11.

The exponents have remarkably linked values, such $9 = 1+8$, $25 = 9+16$, $57 = 25+32$, $89 = 25+64$, $121 = 57+64$, $185 = 121+64$, $441 = 185+256$, $825 = 185+128+512$, $1657 = 121+512+1024$, $1849 = 825+1024$, $1977 = 441+512+1024$, $2681 = 121+512+2048$, $2073 = 25+2048$.

However, the plain anticipation of these values remains a mystery.

12 Sorted planes.

The reader will find below the first Pascal trihedron planes (of associates) classified by positive sorting and also related secondary and tertiary planes.

$v = 1$, $w = 2$, $\#(v) = 1$

Family mod 4,

1
5

$v = 2$, $w = 4$, $\#(v) = 1$

Family mod 32,

1
3

$v = 3$, $w = 5$, $\#(v) = 2$

Family mod 32,

1	1
23	11

$v = 4$, $w = 7$, $\#(v) = 3$

Family mod 128

1	1	1
15	7	59

$v = 5$, $w = 8$, $\#(v) = 7$

Family mod 256

1	2	2	2
95	175	39	219
	79	199	123

$v = 6$, $w = 10$, $\#(v) = 12$

Family mod 1024

1	2	3	3	3
575	287	367	999	923
	735	815	423	347
		975	583	507

$v = 7, w = 12, \#(v) = 30$
Family mod 4096

1	3	5	7	7	7
383	2239	2975	2031	615	2587
	1855	2591	1647	231	2203
	1087	1823	879	3559	1435
		4063	3119	1703	3675
		3295	2351	935	2907
			1231	3911	1787
			463	3143	1019

$v = 8, w = 13, \#(v) = 85$
Family mod 8192

1	4	9	14	19	19	19
255	4223	1983	6815	5871	4455	2331
	3967	1727	6559	5615	4199	2075
	3455	1215	6047	5103	3687	1563
	2431	191	5023	4079	2663	539
		5439	2079	1135	7911	5787
		4927	1567	623	7399	5275
		3903	543	7791	6375	4251
		4159	799	8047	6631	4507
		3135	7967	7023	5607	3483
			3551	2607	1191	7259
			3039	2095	679	6747
			2015	1071	7847	5723
			2271	1327	8103	5979
			1247	303	7079	4955
				719	7495	5371
				207	6983	4859
				7375	5959	3835
				7631	6215	4091
				6607	5191	3067

$v = 9, w = 15, \#(v) = 173$
Family mod 32768

1	4	10	19	28	37	37	37
5631	19199	6783	20927	9375	8431	23399	29467
	13567	1151	15295	3743	2799	17767	23835
	2303	22655	4031	25247	24303	6503	12571
	12543	127	14271	2719	1775	16743	22811
		9087	23231	11679	10735	25703	31771
		30591	11967	415	32239	14439	20507
		8063	22207	10655	9711	24679	30747
		13695	27839	16287	15343	30311	3611
		23935	5311	26527	25583	7783	13851
		22911	4287	25503	24559	6759	12827
			18751	7199	6255	21223	27291
			7487	28703	27759	9959	16027
			17727	6175	5231	20199	26267
			23359	11807	10863	25831	31899
			831	22047	21103	3303	9371
			32575	21023	20079	2279	8347
			14399	2847	1903	16871	22939
			24639	13087	12143	27111	411
			23615	12063	11119	26087	32155
				16863	15919	30887	4187
				5599	4655	19623	25691
				15839	14895	29863	3163
				21471	20527	2727	8795
				31711	30767	12967	19035
				30687	29743	11943	18011
				12511	11567	26535	32603
				22751	21807	4007	10075
				21727	20783	2983	9051

					14031	28999	2299
					2767	17735	23803
					13007	27975	1275
					18639	839	6907
					28879	11079	17147
					27855	10055	16123
					9679	24647	30715
					19919	2119	8187
					18895	1095	7163

v = 10, w = 16, #(v) = 476
Family mod 65536

1	5	14	30	53	76	99	99	99
25599	12799	26367	13951	28095	49311	15599	63335	36635
	52735	767	53887	2495	23711	55535	37735	11035
	1535	15103	2687	16831	38047	4335	52071	25371
	30207	43775	31359	45503	1183	33007	15207	54043
	22015	35583	23167	37311	58527	24815	7015	45851
		27903	15487	29631	50847	17135	64871	38171
		42239	29823	43967	65183	31471	13671	52507
		5375	58495	7103	28319	60143	42343	15643
		62719	50303	64447	20127	51951	34151	7451
		30975	18559	32703	53919	20207	2407	41243
		59647	47231	61375	17055	48879	31079	4379
		51455	39039	53183	8863	40687	22887	61723
		37119	24703	38847	60063	26351	8551	47387
		28927	16511	30655	51871	18159	359	39195
			56191	4799	26015	57839	40039	13339
			4991	19135	40351	6639	54375	27675
			33663	47807	3487	35311	17511	56347
			25471	39615	60831	27119	9319	48155
			59263	7871	29087	60911	43111	16411
			22399	36543	57759	24047	6247	45083
			14207	28351	49567	15855	63591	36891
			65407	14015	35231	1519	49255	22555
			57215	5823	27039	58863	41063	14363
			9599	23743	44959	11247	58983	32283
			38271	52415	8095	39919	22119	60955
			30079	44223	65439	31727	13927	52763
			15743	29887	51103	17391	65127	38427
			7551	21695	42911	9199	56935	30235
			47487	61631	17311	49135	31335	4635
			39295	53439	9119	40943	23143	61979
				33087	54303	20591	2791	41627
				47423	3103	34927	17127	55963
				10559	31775	63599	45799	19099
				2367	23583	55407	37607	10907
				36159	57375	23663	5863	44699
				64831	20511	52335	34535	7835
				56639	12319	44143	26343	65179
				42303	63519	29807	12007	50843
				34111	55327	21615	3815	42651
				52031	7711	39535	21735	60571
				15167	36383	2671	50407	23707
				6975	28191	60015	42215	15515
				58175	13855	45679	27879	1179
				49983	5663	37487	19687	58523
				24383	45599	11887	59623	32923
				16191	37407	3695	51431	24731
				43071	64287	30575	12775	51611
				6207	27423	59247	41447	14747
				63551	19231	51055	33255	6555
				49215	4895	36719	18919	57755
				41023	62239	28527	10727	49563
				15423	36639	2927	50663	23963
				7231	28447	60271	42471	15771

					31199	63023	45223	18523
					45535	11823	59559	32859
					8671	40495	22695	61531
					479	32303	14503	53339
					34271	559	48295	21595
					62943	29231	11431	50267
					54751	21039	3239	42075
					40415	6703	54439	27739
					32223	64047	46247	19547
					50143	16431	64167	37467
					13279	45103	27303	603
					5087	36911	19111	57947
					56287	22575	4775	43611
					48095	14383	62119	35419
					22495	54319	36519	9819
					14303	46127	28327	1627
					41183	7471	55207	28507
					4319	36143	18343	57179
					61663	27951	10151	48987
					47327	13615	61351	34651
					39135	5423	53159	26459
					13535	45359	27559	859
					5343	37167	19367	58203
						61135	43335	16635
						9935	57671	30971
						38607	20807	59643
						30415	12615	51451
						64207	46407	19707
						27343	9543	48379
						19151	1351	40187
						4815	52551	25851
						62159	44359	17659
						14543	62279	35579
						43215	25415	64251
						35023	17223	56059
						20687	2887	41723
						12495	60231	33531
						52431	34631	7931
						44239	26439	65275
						5583	53319	26619
						34255	16455	55291
						26063	8263	47099
						11727	59463	32763
						3535	51271	24571
						43471	25671	64507
						35279	17479	56315

v = 11, w = 18, #(v)= 961
Family mod 262144

1	5	15	34	65	108	151	194	194	194
104447	52223	104959	52991	237183	251327	10399	173295	24423	63259
	209919	511	210687	132735	146879	168095	68847	182119	220955
	1023	53759	1791	185983	200127	221343	122095	235367	12059
	107519	160255	108287	30335	44479	65695	228591	79719	118555
	58367	111103	59135	243327	257471	16543	179439	30567	69403
		105983	54015	238207	252351	11423	174319	25447	64283
		159231	107263	29311	43455	64671	227567	78695	117531
		3583	213759	135807	149951	171167	71919	185191	224027
		216575	164607	86655	100799	122015	22767	136039	174875
		108031	56063	240255	254399	13471	176367	27495	66331
		214527	162559	84607	98751	119967	20719	133991	172827
		165375	113407	35455	49599	70815	233711	84839	123675
		112127	60159	244351	258495	17567	180463	31591	70427
		62975	11007	195199	209343	230559	131311	244583	21275
		120319	68351	252543	4543	25759	188655	39783	78619

212223	134271	148415	169631	70383	183655	222491
3327	187519	201663	222879	123631	236903	13595
109823	31871	46015	67231	230127	81255	120091
60671	244863	259007	18079	180975	32103	70939
214271	136319	150463	171679	72431	185703	224539
58623	242815	256959	16031	178927	30055	68891
9471	193663	207807	229023	129775	243047	19739
218367	140415	154559	175775	76527	189799	228635
169215	91263	105407	126623	27375	140647	179483
226559	148607	162751	183967	84719	197991	236827
6399	190591	204735	225951	126703	239975	16667
112895	34943	49087	70303	233199	84327	123163
63743	247935	262079	21151	184047	35175	74011
10495	194687	208831	230047	130799	244071	20763
223487	145535	159679	180895	81647	194919	233755
18687	202879	217023	238239	138991	252263	28955
119039	41087	55231	76447	239343	90471	129307
69887	254079	6079	27295	190191	41319	80155
127231	49279	63423	84639	247535	98663	137499
	240511	254655	13727	176623	27751	66587
	31615	45759	66975	229871	80999	119835
	138111	152255	173471	74223	187495	226331
	88959	103103	124319	25071	138343	177179
	242559	256703	15775	178671	29799	68635
	86911	101055	122271	23023	136295	175131
	37759	51903	73119	236015	87143	125979
	246655	260799	19871	182767	33895	72731
	197503	211647	232863	133615	246887	23579
	254847	6847	28063	190959	42087	80923
	34687	48831	70047	232943	84071	122907
	141183	155327	176543	77295	190567	229403
	92031	106175	127391	28143	141415	180251
	38783	52927	74143	237039	88167	127003
	251775	3775	24991	187887	39015	77851
	46975	61119	82335	245231	96359	135195
	147327	161471	182687	83439	196711	235547
	98175	112319	133535	34287	147559	186395
	155519	169663	190879	91631	204903	243739
	247167	261311	20383	183279	34407	73243
	91519	105663	126879	27631	140903	179739
	42367	56511	77727	240623	91751	130587
	251263	3263	24479	187375	38503	77339
	202111	216255	237471	138223	251495	28187
	259455	11455	32671	195567	46695	85531
	97663	111807	133023	33775	147047	185883
	48511	62655	83871	246767	97895	136731
	105855	119999	141215	41967	155239	194075
	260479	12479	33695	196591	47719	86555
	211327	225471	246687	147439	260711	37403
	6527	20671	41887	204783	55911	94747
		20799	42015	204911	56039	94875
		74047	95263	258159	109287	148123
		180543	201759	102511	215783	254619
		131391	152607	53359	166631	205467
		22847	44063	206959	58087	96923
		129343	150559	51311	164583	203419
		80191	101407	2159	115431	154267
		26943	48159	211055	62183	101019
		239935	261151	161903	13031	51867
		35135	56351	219247	70375	109211
		77119	98335	261231	112359	151195
		183615	204831	105583	218855	257691
		134463	155679	56431	169703	208539
		81215	102431	3183	116455	155291
		32063	53279	216175	67303	106139
		89407	110623	11375	124647	163483

					189759	210975	111727	224999	1691
					140607	161823	62575	175847	214683
					197951	219167	119919	233191	9883
					27455	48671	211567	62695	101531
					133951	155167	55919	169191	208027
					84799	106015	6767	120039	158875
					31551	52767	215663	66791	105627
					244543	3615	166511	17639	56475
					39743	60959	223855	74983	113819
					140095	161311	62063	175335	214171
					90943	112159	12911	126183	165019
					148287	169503	70255	183527	222363
					40767	61983	224879	76007	114843
					253759	12831	175727	26855	65691
					48959	70175	233071	84199	123035
					84031	105247	5999	119271	158107
					190527	211743	112495	225767	2459
					141375	162591	63343	176615	215451
					88127	109343	10095	123367	162203
					38975	60191	223087	74215	113051
					96319	117535	18287	131559	170395
					196671	217887	118639	231911	8603
					147519	168735	69487	182759	221595
					204863	226079	126831	240103	16795
					97343	118559	19311	132583	171419
					48191	69407	232303	83431	122267
					105535	126751	27503	140775	179611
						84447	247343	98471	137307
						137695	38447	151719	190555
						244191	144943	258215	34907
						195039	95791	209063	247899
						86495	249391	100519	139355
						192991	93743	207015	245851
						143839	44591	157863	196699
						90591	253487	104615	143451
						41439	204335	55463	94299
						98783	261679	112807	151643
						140767	41519	154791	193627
						247263	148015	261287	37979
						198111	98863	212135	250971
						144863	45615	158887	197723
						95711	258607	109735	148571
						153055	53807	167079	205915
						253407	154159	5287	44123
						204255	105007	218279	257115
						261599	162351	13479	52315
						91103	253999	105127	143963
						197599	98351	211623	250459
						148447	49199	162471	201307
						95199	258095	109223	148059
						46047	208943	60071	98907
						103391	4143	117415	156251
						203743	104495	217767	256603
						154591	55343	168615	207451
						211935	112687	225959	2651
						104415	5167	118439	157275
						55263	218159	69287	108123
						112607	13359	126631	165467
						147679	48431	161703	200539
						254175	154927	6055	44891
						205023	105775	219047	257883
						151775	52527	165799	204635
						102623	3375	116647	155483
						159967	60719	173991	212827
						260319	161071	12199	51035
						211167	111919	225191	1883

						6367	169263	20391	59227
						160991	61743	175015	213851
						111839	12591	125863	164699
						169183	69935	183207	222043
							48847	162119	200955
							102095	215367	254203
							208591	59719	98555
							159439	10567	49403
							50895	164167	203003
							157391	8519	47355
							108239	221511	260347
							54991	168263	207099
							5839	119111	157947
							63183	176455	215291
							105167	218439	257275
							211663	62791	101627
							162511	13639	52475
							109263	222535	261371
							60111	173383	212219
							117455	230727	7419
							217807	68935	107771
							168655	19783	58619
							225999	77127	115963
							55503	168775	207611
							161999	13127	51963
							112847	226119	2811
							59599	172871	211707
							10447	123719	162555
							67791	181063	219899
							168143	19271	58107
							118991	232263	8955
							176335	27463	66299
							68815	182087	220923
							19663	132935	171771
							77007	190279	229115
							112079	225351	2043
							218575	69703	108539
							169423	20551	59387
							116175	229447	6139
							67023	180295	219131
							124367	237639	14331
							224719	75847	114683
							175567	26695	65531
							232911	84039	122875
							125391	238663	15355
							76239	189511	228347
							133583	246855	23547

This classification is based on the secondary planes :

$v = 4$, $w = 7$, $\#(v) = 3$

Family mod 128

0,5

$v = 5$, $w = 8$, $\#(v) = 7$

Family mod 256

0,5	1
-----	---

$v = 6$, $w = 10$, $\#(v) = 12$

Family mod 1024

1,5	1	2
-----	---	---

$v = 7, w = 12, \#(v) = 30$
Family mod 4096

2,5	1	2	2
	2	4	4

$v = 8, w = 13, \#(v) = 85$
Family mod 8192

7,5	7	14	14	14
	6	12	12	12
	4	8	8	8
		9	9	9
		5	5	5

$v = 9, w = 15, \#(v) = 173$
Family mod 32768

2,5	5	10	20	20	20
	2	4	8	8	8
	4	8	16	16	16
		3	6	6	6
		7	14	14	14
		5	10	10	10
			3	3	3
			11	11	11
			7	7	7

$v = 10, w = 16, \#(v) = 476$
Family mod 65536

3,5	7	14	28	56	56	56
	14	28	56	112	112	112
	12	24	48	96	96	96
	8	16	32	64	64	64
		17	34	68	68	68
		13	26	52	52	52
		5	10	20	20	20
		23	46	92	92	92
		15	30	60	60	60
			1	2	2	2
			57	114	114	114
			41	82	82	82
			13	26	26	26
			61	122	122	122
			11	22	22	22
			59	118	118	118
				95	95	95
				79	79	79
				47	47	47
				119	119	119
				87	87	87
				115	115	115
				83	83	83

$v = 11, w = 18, \#(v) = 961$
Family mod 262144

6,5	13	26	52	104	208	208	208
	10	20	40	80	160	160	160
	4	8	16	32	64	64	64
	8	16	32	64	128	128	128
		27	54	108	216	216	216
		15	30	60	120	120	120
		23	46	92	184	184	184
		29	58	116	232	232	232
		5	10	20	40	40	40
		1	2	4	8	8	8

			43	86	172	172	172
			19	38	76	76	76
			35	70	140	140	140
			47	94	188	188	188
			63	126	252	252	252
			55	110	220	220	220
			25	50	100	100	100
			41	82	164	164	164
			33	66	132	132	132
				117	234	234	234
				69	138	138	138
				101	202	202	202
				125	250	250	250
				29	58	58	58
				13	26	26	26
				81	162	162	162
				113	226	226	226
				97	194	194	194
				15	30	30	30
				47	94	94	94
				31	62	62	62
					199	199	199
					103	103	103
					167	167	167
					215	215	215
					23	23	23
					247	247	247
					127	127	127
					191	191	191
					159	159	159
					251	251	251
					59	59	59
					27	27	27

The tertiary planes are :

$v = 5, w = 8, \#(v) = 7$
Family mod 256

1	0	1
0	1	0

$v = 6, w = 10, \#(v) = 12$
Family mod 1024

0	0	3	2
2	2	1	0
	3	1	1

$v = 7, w = 12, \#(v) = 30$
Family mod 4096

1	2	1	3	1
1	2	1	3	1
0	1	0	2	0
	3	2	1	3
	2	1	0	2
		0	3	1
		3	2	0

$v = 8, w = 13, \#(v) = 85$
Family mod 8192

0	1	0	0	0	1
0	1	0	0	0	1
0	1	0	0	0	1
0	1	0	0	0	1
	0	1	1	1	0

0	1	1	1	0
0	1	1	1	0
0	1	1	1	0
0	1	1	1	0
	1	1	1	0
	1	1	1	0
	1	1	1	0
	1	1	1	0
	1	1	1	0
		1	0	0
		1	0	0
		1	0	0
		1	0	0
		1	0	0

v = 9, w = 15, #(v)= 173
Family mod 32768

2	0	2	0	0	2	3
1	3	1	3	3	1	2
0	2	0	2	2	0	1
1	3	1	3	3	1	2
	0	2	0	0	2	3
	3	1	3	3	1	2
	0	2	0	0	2	3
	1	3	1	1	3	0
	2	0	2	2	0	1
	2	0	2	2	0	1
		1	0	0	1	2
		0	3	3	0	1
		1	0	0	1	2
		2	1	1	2	3
		3	2	2	3	0
		3	2	2	3	0
		1	0	0	1	2
		2	1	1	2	3
		2	1	1	2	3
			1	1	3	3
			0	0	2	2
			1	1	3	3
			2	2	0	0
			3	3	1	1
			3	3	1	1
			1	1	3	3
			2	2	0	0
			2	2	0	0
				1	2	3
				0	1	2
				1	2	3
				2	3	0
				3	0	1
				3	0	1
				1	2	3
				2	3	0
				2	3	0

v = 10, w = 16, #(v)= 476
Family mod 65536

0	0	0	0	1	0	1	0
1	1	1	1	0	1	0	1
1	1	1	1	0	1	0	1
0	0	0	0	1	0	1	0
0	0	0	0	1	0	1	0
	0	0	0	1	0	1	0
	0	0	0	1	0	1	0
	1	1	1	0	1	0	1
	1	1	1	0	1	0	1

	1	0	1	1
		1	0	0
		1	0	0
		0	1	1
		0	1	1
		1	0	0
		0	1	1
		0	1	1
		1	0	0
		1	0	0
		0	1	1
		0	1	1
		0	1	1
		1	0	0
		1	0	0
		0	1	1
		1	0	0
		0	1	1
		0	1	1
		1	0	0
		1	0	0
		0	1	1
		1	0	0
		1	0	0
		0	1	1
		0	1	1

v = 11, w = 18, #(v)= 961
Family mod 262144

0	1	0	3	3	3	2	3	0
2	3	2	1	1	1	0	1	2
3	0	3	2	2	2	1	2	3
1	2	1	0	0	0	3	0	1
0	1	0	3	3	3	2	3	0
	0	0	2	3	3	1	3	0
	1	1	3	0	0	2	0	1
	3	3	1	2	2	0	2	3
	2	2	0	1	1	3	1	2
	0	0	2	3	3	1	3	0
	2	2	0	1	1	3	1	2
	1	1	3	0	0	2	0	1
	0	0	2	3	3	1	3	0
	0	0	2	3	3	1	3	0
	1	1	3	0	0	2	0	1
		2	1	1	1	0	1	2
		3	2	2	2	1	2	3
		1	0	0	0	3	0	1
		0	3	3	3	2	3	0
		2	1	1	1	0	1	2
		0	3	3	3	2	3	0
		3	2	2	2	1	2	3
		2	1	1	1	0	1	2
		3	2	2	2	1	2	3
		3	2	2	2	1	2	3
		1	0	0	0	3	0	1
		0	3	3	3	2	3	0
		3	2	2	2	1	2	3
		2	1	1	1	0	1	2
		3	2	2	2	1	2	3
		1	0	0	0	3	0	1
		0	3	3	3	2	3	0
		1	0	0	0	3	0	1
			2	3	3	1	3	0
			3	0	0	2	0	1
			1	2	2	0	2	3
			0	1	1	3	1	2

2	3	3	1	3	0
0	1	1	3	1	2
3	0	0	2	0	1
2	3	3	1	3	0
2	3	3	1	3	0
3	0	0	2	0	1
3	0	0	2	0	1
1	2	2	0	2	3
0	1	1	3	1	2
3	0	0	2	0	1
2	3	3	1	3	0
3	0	0	2	0	1
1	2	2	0	2	3
0	1	1	3	1	2
1	2	2	0	2	3
2	3	3	1	3	0
0	1	1	3	1	2
3	0	0	2	0	1
2	3	3	1	3	0
2	3	3	1	3	0
3	0	0	2	0	1
0	1	1	3	1	2
3	0	0	2	0	1
0	1	1	3	1	2
3	0	0	2	0	1
2	3	3	1	3	0
3	0	0	2	0	1
	3	3	2	0	0
	0	0	3	1	1
	2	2	1	3	3
	1	1	0	2	2
	3	3	2	0	0
	1	1	0	2	2
	0	0	3	1	1
	3	3	2	0	0
	3	3	2	0	0
	0	0	3	1	1
	0	0	3	1	1
	2	2	1	3	3
	1	1	0	2	2
	0	0	3	1	1
	3	3	2	0	0
	1	1	0	2	2
	0	0	3	1	1
	3	3	2	0	0
	0	0	3	1	1
	0	0	3	1	1
	2	2	1	3	3
	1	1	0	2	2
	0	0	3	1	1
	0	0	3	1	1
	0	0	3	1	1
	2	2	1	3	3
	1	1	0	2	2
	0	0	3	1	1
	0	0	3	1	1
	2	2	1	3	3
	1	1	0	2	2

2	2	1	3	3
0	0	3	1	1
0	0	3	1	1
1	1	0	2	2
	0	2	0	1
	1	3	1	2
	3	1	3	0
	2	0	2	3
	0	2	0	1
	2	0	2	3
	1	3	1	2
	0	2	0	1
	0	2	0	1
	1	3	1	2
	1	3	1	2
	3	1	3	0
	2	0	2	3
	1	3	1	2
	0	2	0	1
	1	3	1	2
	3	1	3	0
	2	0	2	3
	3	1	3	0
	0	2	0	1
	2	0	2	3
	1	3	1	2
	0	2	0	1
	0	2	0	1
	1	3	1	2
	2	0	2	3
	1	3	1	2
	2	0	2	3
	1	3	1	2
	0	2	0	1
	1	3	1	2
	1	3	1	2
	3	1	3	0
	2	0	2	3
	1	3	1	2
	1	3	1	2
	1	3	1	2
	3	1	3	0
	2	0	2	3
	3	1	3	0
	1	3	1	2
	1	3	1	2
	2	0	2	3
		3	1	2
		0	2	3
		2	0	1
		1	3	0
		3	1	2
		1	3	0
		0	2	3
		3	1	2
		3	1	2
		0	2	3
		0	2	3
		2	0	1
		1	3	0
		0	2	3
		3	1	2
		0	2	3
		2	0	1
		1	3	0
		2	0	1
		1	3	0
		2	0	1

	3	1	2
	1	3	0
	0	2	3
	3	1	2
	3	1	2
	0	2	3
	1	3	0
	0	2	3
	1	3	0
	0	2	3
	3	1	2
	0	2	3
	0	2	3
	2	0	1
	1	3	0
	0	2	3
	0	2	3
	0	2	3
	2	0	1
	1	3	0
	2	0	1
	0	2	3
	0	2	3
	1	3	0

13 A classification for enumeration.

The objective is here to count the staffs of numbers of the Pascal trihedron planes. This has already been achieved in a previous article. However in the absence of a "geography" by signatures, we were limited to a conjecture on the said count instead of the theorem that we will establish now.

In fact, we will create a second geography. This new classification does not give Pascal trihedron planes in a compact "triangular" form, but in a form more indented on the left side. For an easy understanding, let us start by examples according to our regular practice.

We choose two plans for reasons explained further, namely $v = 8$ ($w = 13$) and $v = 9$ ($w = 15$).

We will not explain here again the notion of licit signatures. The tables are made up only of this kind.

1010101010101000000	101010101010100100000	101010101010010100000	101010101001010100000	101010100101010100000	101010010101010100000	101001010101010100000
	1010101010100010000	101010101010010010000	101010101001010010000	101010100101010010000	101010010101010010000	101001010101010010000
		101010101010001010000	101010101000101010000	101010100101001010000	101010010101001010000	101001010101001010000
			101010101001001010000	101010100101001010000	101010010100101010000	101001010100101010000
				101010100010101010000	101010010010101010000	101001010010101010000
	101010101010100001000	101010101010010001000	101010101001010001000	101010100101010001000	101010010101010001000	101001010101010001000
		101010101010001001000	101010101001001001000	101010100101001001000	101010010101001001000	101001010101001001000
		101010101010000101000	101010101001000101000	101010100101000101000	101010010101000101000	101001010101000101000
			101010101000101001000	101010100100101001000	101010010100101001000	101001010100101001000
			101010101000100101000	101010100100100101000	101010010100100101000	101001010100100101000
				1010101000100101001000	101010010010101001000	101001010010101001000
				101010100010100101000	101010010010100101000	101001010010100101000
	101010101010100000100	101010101010010000100	101010101001010000100	101010100101010000100	101010010101010000100	101001010101010000100
		101010101010001000100	101010101001001000100	101010100101001000100	101010010101001000100	101001010101001000100
		101010101010000100100	101010101001000100100	101010100101000100100	101010010101000100100	101001010101000100100
			101010101000101000100	10101010010100101000100	10101001010100101000100	10100101010100101000100
			101010101000100100100	10101010010100100100100	10101001010100100100100	10100101010100100100100
				101010100010101000100	101010010010101000100	101001010010101000100
				101010100010100100100	101010010010100100100	101001010010100100100
				101010100010100100100	101010010010100100100	101001010010100100100
				101010100010100100100	101010010010100100100	101001010010100100100

The principle of formation of this plane is the following :

We start from the first classification of the chosen plane and store items in the same columns as in this one.

The first line is identical to that of this first classification.

The first element of the plane is still at the top left and contains v sequences 10 at the beginning of signatures (reading from left to right) and $w-v$ sequences 0 at the end, hence $w-v+1$ sequences 0 follow the most on the right 1 (called final 1 of signature subsequently).

The elements right on this line were then all $w-v$ sequences 0 after the final 1.

The other elements are then classified according to a positive sorting of signatures with the additional constraint of a classification of the final part (which starts at the final 1), gradually decreasing the number of final 0.

$$v = 8, \Delta w_{\text{prec1}} = 2, \#(v) = 85$$

1	1	1	1	1	1	1
0	1	2	3	4	4	4
0	1	3	5	7	7	7
0	1	3	5	7	7	7
1	4	9	14	19	19	19

For the second table, we have :

[illegible]

The corresponding table of staffs is the following :

$$v = 9, \Delta W_{\text{prec1}} = 1, \#(v) = 173$$

1	1	1	1	1	1	1	1
0	1	2	3	4	5	5	5
0	1	3	6	9	12	12	12
0	1	4	9	14	19	19	19
1	4	10	19	28	37	37	37

Each line in the previous table, offline summation, is matching with a "triangular subblock" (that we can number 1, 2, 3, ..., from the top down) of the Pascal trihedron plane v.

Each triangular subblock, so numbered, by construction, has the same number of final 0 off the first subblock whose first item has an extra 0 at the signature end (after the final 1) comparing to the other elements of the subblock.

We found previously (see ref [1] and [2]) and the relationships linking the staffing tables of rank v to those of rank $v-1$. Thus:

Theorem 15

Let us have $\#(v,i,j)$ an element of the staffing table (excluding the final red line). The number v is the number of the plane, the number i is the index of line starting at 1 (downwards), j is the index of the column starting at 1 (to the right).

Then we have :

$\#(v,1,j) = 1, j = 1 \text{ à } v-1$	(61)
$\#(v,i,1) = 0, i > 1$	(62)
$\#(v,i,j) = \#(v,i-1,j) + \#(v-1,i,j-1), i > 1, j > 1$	(63)
The last line is doubled if $\Delta w_{\text{prec1}}(v) = 2$, then incrementing the number of lines	(64)

Proof :

It is recalled that any licit signature matches a number that generates it (see theorem 6) and that there is therefore exhaustivity in a list of licit signatures.

Subsequently, we assimilate the $\#(v,i,j)$ count and the elements that it counts.

The first two lines of the theorem table are trivial (as a result of the construction procedure of the trihedron page 6).

Switch from one signature to another is done with a gradual shift of the 1 on the most right to the right, then the second, etc. If there are matching signatures (bijection) on the first two subblocks to be treated, this correspondence will necessarily pass on a column until the final subblock.

Matching works on the first pair of subblocks by adopting $\#(v-1,1,j-1)$ and $\#(v,1,j)$ which means simply align the subblocks of the $v-1$ plane and the v plane on the right side, the first element of the plan v being thus isolated.

We give the following example :

v-1 = 8, column 4		v = 9, column 5	
111	101010101001010100000	211	101010101001010101000000
121	101010101001010010000	221	101010101001010100100000
122	101010101001001010000	222	101010101001010010100000
123	101010101000101010000	223	101010101001001010100000
		224	101010101000101010100000
131	101010101001010001000	231	101010101001010100010000
132	101010101001001001000	232	101010101001010010010000
133	101010101001000101000	233	101010101001010001010000
134	101010101000101001000	234	101010101001001010010000
135	101010101000100101000	235	101010101001001001010000
		236	101010101001000101010000
		237	101010101000101010010000
		238	101010101000101001010000
		239	101010101000100101010000
141	101010101001010000100	241	101010101001010100001000
142	101010101001001000100	242	101010101001010010001000
143	101010101001000100100	243	101010101001010001001000
144	101010101000101000100	244	101010101001010000101000
145	101010101000100100100	245	101010101001001010001000
		246	101010101001001001001000
		247	101010101001001000101000
		248	101010101001000101001000
		249	101010101001000100101000
		250	101010101000101010001000
		251	101010101000101001001000
		252	101010101000101000101000
		253	101010101000100101001000
		254	101010101000100100101000

The lines have been numbered for easier location. There are two types of matches, namely those of $\#(v-1,i,j-1)$ with $\#(v,i,j)$ in blue and green fonts and those of $\#(v,i-1,j)$ with $\#(v,i,j)$ in red and green fonts.

The first match is performed on the first two items (the first line of the table)

111	101010101001010100000
211	101010101001010101000000

then on the two following subblocks

211	1010101001010101000000
221	101010101001010100100000
121	101010101001010010000
222	101010101001010010100000
122	101010101001001010000
223	101010101001001010100000
123	101010101000101010000
224	101010101000101010100000

and so on

221	101010101001010100100000
231	101010101001010100010000
222	101010101001010010100000
232	101010101001010010010000
223	101010101001001010100000
234	101010101001001010010000
224	101010101000101010100000
237	101010101000101010010000
131	101010101001010001000
233	101010101001010001010000
132	101010101001001001000
235	101010101001001001010000
133	101010101001000101000
236	101010101001000101010000
134	101010101000101001000
238	101010101000101001010000
135	101010101000100101000
239	101010101000100101010000

231	101010101001010100010000
241	101010101001010100001000
232	101010101001010010010000
242	101010101001010010001000
233	101010101001010001010000
243	101010101001010001001000
234	101010101001001010010000
245	101010101001001010001000
235	101010101001001001010000
246	101010101001001001001000
236	101010101001000101010000
248	101010101001000101001000
237	101010101000101010010000
250	101010101000101010001000
238	101010101000101001010000
251	101010101000101001001000
239	101010101000100101010000
253	101010101000100101001000
141	101010101001010000100
244	101010101001010000101000
142	101010101001001000100
247	101010101001001000101000
143	101010101001000100100
249	101010101001000100101000
144	101010101000101000100
252	101010101000101000101000
145	101010101000100100100
254	101010101000100100101000

The scheme is totally reproducible for each of the two categories of matchings.

For the first category (red and green fonts), it is obviously a simple shift of the last 1, since signatures are stored with the constraint of an identical final block 10... 0 with a decreasing number of 0. By the method of signatures' construction with constraint, which is exhaustive to the left of the last 1, any signature of subblock n of type x... x0010...0 has necessarily a counterpart in the subblock n-1 (in his column) of type x...x0100...0 (the x...x being identical).

For the second category (blue and green fonts), where the offset of the 1's extends to the right without finding a counterpart in the subblock above it, there is an extra 01 after the last 1, then 00 or 000 at the end of signatures as $\Delta w = 1$ or $\Delta w = 2$. The offset being progressive and comprehensive in any subblock, a bijection results necessarily if there is an initial matching of the last 1 of the plan v-1 with the second last 1 of the plan v for subblocks in question. For this, it is necessary to check the matching between the first elements of the first subblock of the v-1 plans and v.

We start with the first element of the plan v-1 and the second component of the plan v :

v	Signatures
4	10101010000
5	1010101001000
5	1010101010000
6	1010101010010000
6	1010101010100000
7	1010101010100100000
7	1010101010101000000
8	101010101010100100000
8	101010101010101000000
9	101010101010101001000000
9	101010101010101010000000
10	10101010101010101001000000

The last 1 examined of the plane v-1 item is effectively aligned with the second last of the other element to be considered in the plane v. When v is incremented, by construction, the signatures will grow up with simply 01 addition after the last 1 on the first list (green police components) and addition (if $\Delta w = 2$) or not (if $\Delta w = 1$) of a 0 at the signature end and 01 addition after the second last 1 of the second list (and same rule for the final 0's).

Having checked these first elements, it should be then check the setting of the other columns, noting the evolution from one to the other column. The case v-1 = 6 and v = 7 is representative :

1010101010100000	1010101010010000	1010101001010000	1010100101010000	1010010101010000
1010101010100100000	1010101010010100000	1010101001010100000	1010100101010100000	1010010101010100000

Indeed, starting from the first pair of elements, subject of the previous discussion, by construction, the other signatures will have the same final sequence of 0's (that is one less for the first line and the same number in second line because the first signature of the first subblock of a plan has an additional 0 as noted earlier). Thus, the last 1 of the elements of the first line remains aligned with the second last 1 of the second line and by completeness (exhaustivity) all the preceding 1's (to the said last and second last) are aligned from one line to the other.

By completeness also, the two lists in the presence will stop at the same time. For skeptics, as we still have to prove assertion (64), the underneath study will be reassuring as it is back on these final elements of lists.

When $\Delta w_{\text{prec1}}(v) = 2$, we have a twinning of the last two triangular subblocks by moving the last 1 a step to the right (or what amounts to swapping 10 in 01)

For example, for v = 7, $\Delta w = 2$, $\Delta w_{\text{prec1}} = 2$, the last two subblocks are as follows :

1010101010100010000	1010101010010010000	1010101001010010000	1010100101010010000	1010010101010010000
	1010101010001010000	1010101001001010000	1010100101001010000	1010010101001010000
		1010101000101010000	1010100100101010000	1010010100101010000
1010101010100001000	1010101010010001000	1010101001010001000	1010100101010001000	1010010101010001000
	1010101010001001000	1010101001001001000	1010100101001001000	1010010101001001000
		1010101000101001000	1010100100101001000	1010010100101001000

It suffices to have two 0 between the two last 1 of the last triangular subblock for this twinning to appear. Indeed, all items above or left have by construction a spacing between the latter two 1 equal or superior to this one, which allows to systematically switch 01 into 10 in the second last subblock.

On the other hand, if a single 0 is there instead, then the twinning (in its entirety) is impossible.

Yet, this last element does increase as follows as v increases (we have not represented the cases $v = 1$ to $v = 3$ because this "last element" does not yet exist in these planes in a way):

v	w	Δw	Last signature
4	7	2	10100101000
5	8	1	1010010100100
6	10	2	1010010100101000
7	12	2	1010010100101001000
8	13	1	101001010010100100100
9	15	2	101001010010100100101000
10	16	1	10100101001010010010100100
11	18	2	10100101001010010010100101000
12	20	2	10100101001010010010100101001000
13	21	1	1010010100101001001010010100100100
14	23	2	10100101001010010010100101001001001000
15	24	1	1010010100101001001010010100100100100100

This arises from the fact that the last element is the one for which all of the 1's are most to the right. So, from $v-1$ to v , the latest addition which is 10 or 100 (depending on whether $\Delta w = 1$ or $\Delta w = 2$) will be done with an offset of 1 of a single notch to the left (because otherwise the signature would be the previous + the add and stop prematurely). Thus, the latter stores the whole history of the Δw in the 10 and 100 sequences of which it is formed, and this in order. In particular, the second last sequence is 10 if $\Delta w_{\text{prec1}} = 1$ and 100 if $\Delta w_{\text{prec1}} = 2$ which allows to conclude the proof.

On the other hand, the table also shows us that the elements of the last triangular subblock of a plan do end with 100 if $\Delta w = 1$ and 1000 if $\Delta w = 2$.

As the first subblock ends with $w-v+1-1$ sequences 0 and by construction the number of final 0's decrease from one subblock to another by one unit, we can easily count the number of subblocks at rank v , namely $w-v-\Delta w$.

We will finish our study by two properties and recall a third one. To facilitate the writing of these, we identify the constituting elements of subblocks by $sb(v, n, i, j)$, where v is the plan, n the n th subblock (starting at 1), i the line index (starting at 1) in the subblock, j the index column (starting at 1) in the subblock. The first element of the plan v is spotted apart by $sb(v, 1, 1, 0)$.

For the first property, we focus on the first line of each subblock of a plane v .

Property 7

$$sb(v, n, 1, j) = sb(v, n-1, 1, j) + cte(n) \bmod 2^w \quad j > 0 \quad (65)$$

where $cte(n)$ depends only on n (but not on j).

Property 8 generalizes property 7.

Property 8

There is a pair of two lists of length equal such as

$$sb(v, n, 1, j) = sb(v, n-1, 1, j') + cte1(n) \bmod 2^w \quad j \text{ et } j' > 0 \quad (66)$$

and

$$sb(v, n, 1, j) = sb(v-1, n, 1, j') + cte2(n) \bmod 2^{w(v-1)} \quad j \text{ et } j' > 0 \quad (67)$$

where $cte1(n)$ and $cte2(n)$ depend only on n and $w(v-1)$ means w at rank $v-1$.

We find in these formulas the additive properties that we had foreseen in the last paragraph of chapter 2 before tackling the notion of signature.

Proof

These properties are derived from theorem 9, which does indicate that equal differences of signatures means equal differences of associates modulo 2^w . The two pairs of lists are those discussed earlier as we compared signatures in red and green fonts and signatures in blue and green fonts. When signatures concern planes $v-1$ and v (blue and green fonts), the value modulo must obviously be taken on rank $v-1$, that is modulo $2^{w(v-1)}$.

Example :

Numbering	Signatures	Associates	Diff_ mod $2^{w(v)}$	Diff_ mod $2^{w(v-1)}$
111	1010101001010100000	6815		
211	10101010010101000000	9375		2560

211	10101010010101000000	9375		
221	1010101001010100100000	3743	27136	
121	1010101001010010000	6559		
222	1010101001010010100000	11679		5120
122	1010101001001010000	2079		
223	1010101001001010100000	7199		5120
123	1010101000101010000	3551		
224	1010101000101010100000	16863		5120

221	1010101001010100100000	3743		
231	1010101001010100010000	25247	21504	
222	1010101001010010100000	11679		
232	1010101001010010010000	415	21504	
223	1010101001001010100000	7199		
234	1010101001001010010000	28703	21504	
224	1010101000101010100000	16863		
237	1010101000101010010000	5599	21504	
131	1010101001010001000	6047		
233	1010101001010001010000	16287		2048
132	1010101001001001001000	1567		
235	1010101001001001010000	11807		2048
133	1010101001000101000	799		
236	1010101001000101010000	2847		2048
134	1010101000101001000	3039		
238	1010101000101001010000	21471		2048
135	1010101000100101000	2271		
239	1010101000100101010000	12511		2048

231	1010101001010100010000	25247		
241	101010100101010001000	2719	10240	
232	1010101001010010010000	415		
242	1010101001010010001000	10655	10240	
233	1010101001010001010000	16287		
243	1010101001010001001000	26527	10240	
234	1010101001001010010000	28703		
245	1010101001001010001000	6175	10240	
235	1010101001001001010000	11807		
246	1010101001001001001000	22047	10240	
236	1010101001000101010000	2847		
248	1010101001000101001000	13087	10240	

237	101010101000101010010000	5599		
250	101010101000101010001000	15839	10240	
238	101010101000101001010000	21471		
251	101010101000101001001000	31711	10240	
239	101010101000100101010000	12511		
253	101010101000100101001000	22751	10240	
141	101010101001010000100	5023		
244	101010101001010000101000	25503		4096
142	101010101001001000100	543		
247	101010101001001000101000	21023		4096
143	101010101001000100100	7967		
249	101010101001000100101000	12063	!	4096
144	101010101000101000100	2015		
252	101010101000101000101000	30687		4096
145	101010101000100100100	1247		
254	101010101000100100101000	21727		4096

As we perform offsets of the final 1's between subblocks, property 5 springs up with ratios of 2 from one to the other. Thus $21504 = 2 \cdot 27136 \bmod 2^{15}$, $10240 = 2 \cdot 21504 \bmod 2^{15}$ and $5120 = 2 \cdot 2560 \bmod 2^{13}$, $2048 = 2 \cdot 5120 \bmod 2^{13}$, $4096 = 2 \cdot 2048 \bmod 2^{13}$.

Finally, when $\Delta w_{\text{prec1}}(v) = 2$, we have a twinning of the last two triangular subblocks and the differences between twin associates of these two blocks are thus equal.

The plan $v = 7$ ($w = 12$, $2^w = 4096$) provides a synthesis of all these properties ($768 = 2 \cdot 384$) :

Lines		Associates				
1	383	2239	2975	2031	615	2587
2		1855	2591	1647	231	2203
3			4063	3119	1703	3675
4				1231	3911	1787
5		1087	1823	879	3559	1435
6			3295	2351	935	2907
7				463	3143	1019
Differences modulo 2^w						
1-2		384	384	384	384	384
2-5		768	768	768	768	768
3-6			768	768	768	768
4-7				768	768	768

14 Recursive construction of the Pascal trihedron set of values.

This construction applies to this second geography. The calculation efficiency stems from its simplicity.

Theorem 16

Let us have $a_v(r,s)$ the element on line r and column s of the Pascal trihedron v -plane.

Let us have $m_v(r,s)$ the result of the Collatz algorithm on $a_v(r,s)$ at the end of altitude flight.

We have then

$$a_{v+1}(r,s) = (2 \cdot (a_v(r,s) + r \cdot 2^w) - 1) / 3 \quad (68)$$

and

$$m_{v+1}(r,s) = (m_v(r,s) + r \cdot 3^v) / 2^{\Delta w(v+1)-1} \quad (69)$$

for the smallest integer r such as $a_{v+1}(r,s)$ and $m_{v+1}(r,s)$ are simultaneous integers.

Moreover, the solution is necessarily obtained for r in the set $\{0,1,2,3,4,5\}$.

Note

Passing from plane v to plane $v + 1$, the size of the plane increases and new integers $a_{v+1}(r,s)$ where $a_v(r,s)$ is empty are to be evaluated directly.

References

- [1] <https://sites.google.com/site/schaetzelhubertdiophantien/> Altitude flight with Collatz numbers : the genesis of a Pascal trihedron.
- [2] <https://sites.google.com/site/schaetzelhubertdiophantien/> Pascal trihedron and Collatz algorithm.
- [3] Rihó Terras. A stopping time problem on the positive integers. *Acta Arithmetica* 30 (1976), 241–252

Appendix 1 Construction of the v-plane.

The plane $v = 1$ includes a single item : the integer 1.

The plane $v = 2$ includes a single item : the integer 3.

The choice of v , greater than or equal to 3, below enables to calculate the v -plan v line by line, each element being in its exact position in the columns, the empty positions being viewed by the number 0. The number of elements increasing exponentially with v , there is a rapid saturation of the system of calculation for comprehensive planes. Hence, the need for alternative code programming given afterwards.

Program 1

```

/*Choose v >= 3*/
/*Recommending to limit to v = 14 max with already 8045 elements*/
v = 6;

lm = log(3)/log(2); w = 1+floor(lm*v); infini = 10^1000;
qt1 = qt2 = vector(v); cdplt=2; for(i=3, v, qt1[1] = qt = 1; for(j=2, cdplt, qt1[j] = qt1[j-1]+qt2[j]; qt = qt+qt1[j]); for(j=1,
cdplt, qt2[j] = qt1[j]); dw = floor(lm*i)-floor(lm*(i-1)); if(dw ==2, cdplt++));
print("The \"v\"-plane of the trihedron contains \"qt\" integers (0 is void):");
sgorig = 0;
for(i = 1, v, sgorig = concat(sgorig,1));
for(i = 1, w-v-1, sgorig = concat(sgorig,0));
sgorig = sgorig[^1];
sgref = 0;
for(t = 1,v, wp = 1+floor(lm*(t-1)); wc = 1+floor(lm*t); dw = wc-wp;
if(dw == 1,sgref = concat(sgref,1), sgref = concat(sgref,1); sgref = concat(sgref,0));sgref = sgref[^1];
invtaborig = 0;
for(i = 1, v, invtaborig = concat(invtaborig,i));
invtaborig = invtaborig[^1];
invtabref = 0;tt= 0;
for(t = 1,v, wp = 1+floor(lm*(t-1)); wc = 1+floor(lm*t); dw = wc-wp; tt++; invtabref = concat(invtabref,tt);
if(dw == 2,tt++));
invtabref = invtabref[^1];
taborig = vector(v); tabref = vector(v);
for(i = 1, v, j= v-i+1; taborig[j] = invtaborig[i]; tabref[j] = invtabref[i]);
tabact = tabdpl = vector(qt); tabact[1] = taborig; tabdpl = taborig;
i = 0; r = 1;
for(t = 1,infini, i++; if(i > v, break);
if(tabdpl[i] < tabref[i],
tabdpl[i]++; r++;
if(i > 1,for(j = 1, i-1, x = i-j; tabdpl[x] = tabdpl[x+1]+1)); i = 0;
tabact[r] = tabdpl;
if(tabdpl[1] == v+1, lig = 1, lig = lig+1);
if(tabdpl == tabref, break));
ligm = lig;
col = 1; lig = 0; tabplan = matrix(ligm,v-1);
for(t = 1,qt, tabdpl = tabact[t];
res = 0; long = 2;
for(i = 1, v, ii = v-i+1;
if(tabdpl[ii]<>v-ii+1, res = concat(res, tabdpl[ii+1]); long = v-i+3; break));
for(j = v+1-ii, v, iii = v-j+1; res = concat(res, tabdpl[iii]-if(long == 2,0,1)));
res = res[^1];
res = concat(res, w);
resp = res;
resp[2] = res[2]-res[1];
for(i = 3, long, resp[i] = res[i]-res[i-1]-1);
sec = resp[long]; add = sec;
for(i = 1, long-2, ii = long-i; add = add + resp[ii]+1; sec = concat(add,sec));
sec = concat(res[1],sec);
nb1 = 0; nb2 = 0;
if(long > 2, for(i = 1, long-2, ii = long-i+1;
for(rech = 1, infini, nb1 = (rech*(2^sec[ii])+nb2*(2^resp[ii])+1)/3;

```

```

if(nb1 == floor(nb1), nb2 = 2*nb1-1;break));
for(rech = 1, infini, nb1 = (rech*(2^sec[2])+nb2*(2^resp[2]+1)/(3^resp[1]));
if(nb1 == floor(nb1), nb2 = nb1*(2^resp[1])-1;break));
if(tabdpl[1] == v+1, col = col+1;lig = 1, lig = lig+1);
tabplan[lig,col] = nb2);
for(i = 1,ligm, print(tabplan[i,]));}

```

Example 1

The plane $v = 6$:

The 6-plane of the trihedron contains 12 integers (0 is void):
[575, 287, 367, 999, 923]
[0, 735, 815, 423, 347]
[0, 0, 975, 583, 507]

The plane $v = 10$:

The 10-plane of the trihedron contains 476 integers (0 is void):
[25599, 12799, 26367, 13951, 28095, 49311, 15599, 63335, 36635]
[0, 52735, 767, 53887, 2495, 23711, 55535, 37735, 11035]
[0, 1535, 15103, 2687, 16831, 38047, 4335, 52071, 25371]
[0, 30207, 43775, 31359, 45503, 1183, 33007, 15207, 54043]
[0, 22015, 35583, 23167, 37311, 58527, 24815, 7015, 45851]
[0, 0, 27903, 15487, 29631, 50847, 17135, 64871, 38171]
[0, 0, 42239, 29823, 43967, 65183, 31471, 13671, 52507]
[0, 0, 5375, 58495, 7103, 28319, 60143, 42343, 15643]
[0, 0, 62719, 50303, 64447, 20127, 51951, 34151, 72987]
[0, 0, 30975, 18559, 32703, 53919, 20207, 67943, 41243]
[0, 0, 59647, 47231, 61375, 17055, 48879, 31079, 69915]
[0, 0, 51455, 39039, 53183, 8863, 40687, 22887, 61723]
[0, 0, 37119, 24703, 38847, 60063, 26351, 8551, 47387]
[0, 0, 28927, 16511, 30655, 51871, 18159, 65895, 39195]
[0, 0, 0, 56191, 4799, 26015, 57839, 40039, 13339]
[0, 0, 0, 4991, 19135, 40351, 6639, 54375, 27675]
[0, 0, 0, 33663, 47807, 3487, 35311, 17511, 56347]
[0, 0, 0, 25471, 39615, 60831, 27119, 9319, 48155]
[0, 0, 0, 59263, 7871, 29087, 60911, 43111, 16411]
[0, 0, 0, 22399, 36543, 57759, 24047, 6247, 45083]
[0, 0, 0, 14207, 28351, 49567, 15855, 63591, 36891]
[0, 0, 0, 65407, 14015, 35231, 1519, 49255, 22555]
[0, 0, 0, 57215, 5823, 27039, 58863, 41063, 14363]
[0, 0, 0, 9599, 23743, 44959, 11247, 58983, 32283]
[0, 0, 0, 38271, 52415, 8095, 39919, 22119, 60955]
[0, 0, 0, 30079, 44223, 65439, 31727, 13927, 52763]
[0, 0, 0, 15743, 29887, 51103, 17391, 65127, 38427]
[0, 0, 0, 7551, 21695, 42911, 9199, 56935, 30235]
[0, 0, 0, 47487, 61631, 17311, 49135, 31335, 70171]
[0, 0, 0, 39295, 53439, 9119, 40943, 23143, 61979]
[0, 0, 0, 0, 33087, 54303, 20591, 68327, 41627]
[0, 0, 0, 0, 47423, 3103, 34927, 17127, 55963]
[0, 0, 0, 0, 10559, 31775, 63599, 45799, 19099]
[0, 0, 0, 0, 2367, 23583, 55407, 37607, 76443]
[0, 0, 0, 0, 36159, 57375, 23663, 5863, 44699]
[0, 0, 0, 0, 64831, 20511, 52335, 34535, 73371]
[0, 0, 0, 0, 56639, 12319, 44143, 26343, 65179]
[0, 0, 0, 0, 42303, 63519, 29807, 12007, 50843]
[0, 0, 0, 0, 34111, 55327, 21615, 3815, 42651]
[0, 0, 0, 0, 52031, 7711, 39535, 21735, 60571]
[0, 0, 0, 0, 15167, 36383, 2671, 50407, 23707]
[0, 0, 0, 0, 6975, 28191, 60015, 42215, 15515]
[0, 0, 0, 0, 58175, 13855, 45679, 27879, 66715]
[0, 0, 0, 0, 49983, 5663, 37487, 19687, 58523]
[0, 0, 0, 0, 24383, 45599, 11887, 59623, 32923]
[0, 0, 0, 0, 16191, 37407, 3695, 51431, 24731]

[0, 0, 0, 0, 43071, 64287, 30575, 12775, 51611]
[0, 0, 0, 0, 6207, 27423, 59247, 41447, 14747]
[0, 0, 0, 0, 63551, 19231, 51055, 33255, 72091]
[0, 0, 0, 0, 49215, 4895, 36719, 18919, 57755]
[0, 0, 0, 0, 41023, 62239, 28527, 10727, 49563]
[0, 0, 0, 0, 15423, 36639, 2927, 50663, 23963]
[0, 0, 0, 0, 7231, 28447, 60271, 42471, 15771]
[0, 0, 0, 0, 0, 31199, 63023, 45223, 18523]
[0, 0, 0, 0, 0, 45535, 11823, 59559, 32859]
[0, 0, 0, 0, 0, 8671, 40495, 22695, 61531]
[0, 0, 0, 0, 0, 479, 32303, 14503, 53339]
[0, 0, 0, 0, 0, 34271, 66095, 48295, 21595]
[0, 0, 0, 0, 0, 62943, 29231, 11431, 50267]
[0, 0, 0, 0, 0, 54751, 21039, 68775, 42075]
[0, 0, 0, 0, 0, 40415, 6703, 54439, 27739]
[0, 0, 0, 0, 0, 32223, 64047, 46247, 19547]
[0, 0, 0, 0, 0, 50143, 16431, 64167, 37467]
[0, 0, 0, 0, 0, 13279, 45103, 27303, 66139]
[0, 0, 0, 0, 0, 5087, 36911, 19111, 57947]
[0, 0, 0, 0, 0, 56287, 22575, 4775, 43611]
[0, 0, 0, 0, 0, 48095, 14383, 62119, 35419]
[0, 0, 0, 0, 0, 22495, 54319, 36519, 75355]
[0, 0, 0, 0, 0, 14303, 46127, 28327, 67163]
[0, 0, 0, 0, 0, 41183, 7471, 55207, 28507]
[0, 0, 0, 0, 0, 4319, 36143, 18343, 57179]
[0, 0, 0, 0, 0, 61663, 27951, 10151, 48987]
[0, 0, 0, 0, 0, 47327, 13615, 61351, 34651]
[0, 0, 0, 0, 0, 39135, 5423, 53159, 26459]
[0, 0, 0, 0, 0, 13535, 45359, 27559, 66395]
[0, 0, 0, 0, 0, 5343, 37167, 19367, 58203]
[0, 0, 0, 0, 0, 61135, 43335, 16635]
[0, 0, 0, 0, 0, 9935, 57671, 30971]
[0, 0, 0, 0, 0, 38607, 20807, 59643]
[0, 0, 0, 0, 0, 30415, 12615, 51451]
[0, 0, 0, 0, 0, 64207, 46407, 19707]
[0, 0, 0, 0, 0, 27343, 9543, 48379]
[0, 0, 0, 0, 0, 19151, 66887, 40187]
[0, 0, 0, 0, 0, 4815, 52551, 25851]
[0, 0, 0, 0, 0, 62159, 44359, 17659]
[0, 0, 0, 0, 0, 14543, 62279, 35579]
[0, 0, 0, 0, 0, 43215, 25415, 64251]
[0, 0, 0, 0, 0, 35023, 17223, 56059]
[0, 0, 0, 0, 0, 20687, 68423, 41723]
[0, 0, 0, 0, 0, 12495, 60231, 33531]
[0, 0, 0, 0, 0, 52431, 34631, 73467]
[0, 0, 0, 0, 0, 44239, 26439, 65275]
[0, 0, 0, 0, 0, 5583, 53319, 26619]
[0, 0, 0, 0, 0, 34255, 16455, 55291]
[0, 0, 0, 0, 0, 26063, 8263, 47099]
[0, 0, 0, 0, 0, 11727, 59463, 32763]
[0, 0, 0, 0, 0, 3535, 51271, 24571]
[0, 0, 0, 0, 0, 43471, 25671, 64507]
[0, 0, 0, 0, 0, 35279, 17479, 56315]

Appendix 2

Search for the position of an integer in the Pascal trihedron from its value.

The choice of the number nb gives its modulo 2^w representative within the Pascal trihedron, possibly different from nb, and coordinates (plane, column, row).

Program 2

```

/* Give integer : nb */
nb = 40160091701359;

infini = 10^10000; nbt = nb; sg = 0; v = 0; lm = log(3)/log(2);
for(i = 1, infiniti,
if(nbt/2 == floor(nbt/2), nbt = nbt/2; sg = concat(sg,0); if(nbt <= nb, break), nbt = (3*nbt+1)/2; v = v+1; sg = concat(sg,1)));
w = 1+floor(lm*v);
sg = sg[^1]; sg = sg[^w];
for(i = 1, w-1, if(sg[i] == 0, rep = i-1 ;break));col = v+1-rep;
vv = 0; tabrech = 0;
for(i = 1, w-1, vv = vv+1;if(sg[i] == 1, tabrech = concat(vv,tabrech)));
tabrech = tabrech[^v+1];
sgorig = 0;
for(i = 1, rep, sgorig = concat(sgorig,1)); sgorig = concat(sgorig,0);
for(i = 1, v-rep, sgorig = concat(sgorig,1));
for(i = 1, w-v-1, sgorig = concat(sgorig,0));
sgorig = sgorig[^1];
sgref = 0;
for(t = 1,v, wp = 1+floor(lm*(t-1)); wc = 1+floor(lm*t); dww = wc-wp;
if(dww == 1,sgref = concat(sgref,1), sgref = concat(sgref,1); sgref = concat(sgref,0));sgref = sgref[^1];
invtaborig = 0;j = 1;
for(i = 1, w-1,
if(sgorig[i]== 1,invtaborig = concat(invtaborig,i);j = j+1));
invtaborig = invtaborig[^1];
invtabref = 0;tt = 0;
for(t = 1,v, wp = 1+floor(lm*(t-1)); wc = 1+floor(lm*t); dww = wc-wp; tt++; invtabref = concat(invtabref,tt);
if(dww == 2,tt++));
invtabref = invtabref[^1];
taborig = vector(v); tabref = vector(v);
for(i = 1, v, j = v-i+1; taborig[j] = invtaborig[i]; tabref[j] = invtabref[i]);
tabdpl = taborig; tabdpl[1] = tabdpl[1]-1;
i = 0; r = 1;
for(t = 1,infiniti, i++; if(i > v, break);
if(tabdpl[i] < tabref[i],
tabdpl[i]++; r++;
if(i > 1,for(j = 1, i-1, x = i-j; tabdpl[x] = tabdpl[x+1]+1)); i = 0;
if(tabdpl == tabrech, break));
nbpl = nb-(2^w)*floor(nb/(2^w));
if(nbpl <> nb, print("Integer "nb" is represented by "nbpl" in the trihedron."));
print("The location of integer "nbpl" in the trihedron is plane "v", column "col", line "r-1".")

```

Examples 2

Integer nb = 40160091701359 :

Integer 40160091701359 is represented by 111 in the trihedron.
The location of integer 111 in the trihedron is plane 19, column 16, line 150386.

Integer nb = 110420130004991 :

The location of integer 110420130004991 in the trihedron is plane 30, column 10, line 5.

Appendix 3

Search for the value of an integer from its position in the Pascal trihedron.

The choice of the position (plane v, column, row) gives the modulo 2^w representative within the trihedron.

Program 3

```

/* Give plan rank : v */
v = 7;
/* Give column : column (<= v-1) */
column = 2;
/* Give line : line (must be not void)*/
line = 2;

infini = 10^10000; lm = log(3)/log(2); w = 1+floor(lm*v); alert = 0;
sgorig = 0; rep = v+1-column;
for(i = 1, rep, sgorig = concat(sgorig,1)); sgorig = concat(sgorig,0);
for(i = 1, v-rep, sgorig = concat(sgorig,1));
for(i = 1, w-v-1, sgorig = concat(sgorig,0));
sgorig = sgorig[^1];
sgref = 0;
for(t = 1,v, wp = 1+floor(lm*(t-1)); wc = 1+floor(lm*t); dww = wc-wp;
if(dww == 1, sgref = concat(sgref,1), sgref = concat(sgref,1); sgref = concat(sgref,0)); sgref = sgref[^1];
invtaborig = 0; j = 1;
for(i = 1, w-1,
if(sgorig[i] == 1, invtaborig = concat(invtaborig,i); j = j+1));
invtaborig = invtaborig[^1];
invtabref = 0; tt = 0;
for(t = 1,v, wp = 1+floor(lm*(t-1)); wc = 1+floor(lm*t); dww = wc-wp; tt++; invtabref = concat(invtabref,tt);
if(dww == 2, tt++));
invtabref = invtabref[^1];
taborig = tabref = tabdpl = vector(v);
for(i = 1, v, j = v-i+1; taborig[j] = invtaborig[i]; tabref[j] = invtabref[i]);
tabdpl = taborig; tabdpl[1] = tabdpl[1]-1;
i = 0; r = 1;
for(t = 1, infini, i++; if(i > v, break);
if(tabdpl[i] < tabref[i],
tabdpl[i]++; r++;
if(tabdpl[column] == archive+1, if(column >= 1, alert = 1; print("Wrong choice : Line position outside the trihedron.")));
if(i > 1, for(j = 1, i-1, x = i-j; tabdpl[x] = tabdpl[x+1]+1)); i = 0;
archive = tabdpl[column];
if(r-1 == line, break));
if(alert == 1, break,
parv = 0; absc = 0;
for(i = 1, w-1, if(tabdpl[v-absc] == i, parv = concat(parv,1); absc = absc+1; if(absc > v-1, absc = v-1), parv =
concat(parv,0));
parv = parv[^1];
res = 0; long = 2;
for(i = 1, v, ii = v-i+1;
if(tabdpl[ii] < v-ii+1, res = concat(res, tabdpl[ii+1]); long = v-i+3; break));
for(j = v+1-ii, v, iii = v-j+1; res = concat(res, tabdpl[iii]-if(long == 2, 0, 1)));
res = res[^1];
res = concat(res, w);
resp = res;
resp[2] = res[2]-res[1];
for(i = 3, long, resp[i] = res[i]-res[i-1]-1);
sec = resp[long]; add = sec;
for(i = 1, long-2, ii = long-i; add = add + resp[ii]+1; sec = concat(add, sec));
sec = concat(res[1], sec);
nb1 = 0; nb2 = 0;
if(long > 2, for(i = 1, long-2, ii = long-i+1;
for(rech = 1, infini, nb1 = (rech*(2^sec[ii])+nb2*(2^resp[ii])+1)/3;
if(nb1 == floor(nb1), nb2 = 2*nb1-1; break)))));
for(rech = 1, infini, nb1 = (rech*(2^sec[2])+nb2*(2^resp[2])+1)/(3^resp[1]);
if(nb1 == floor(nb1), nb2 = nb1*(2^resp[1])-1; break));
print("The element of the "v"-plane in line "line" and column "column" is equal to "nb2".");

```

Exemple 3

The integer of plan $v = 19$ with coordinates (341651,16) is :

The element of the 19-plane in line 341651 and column 16 is equal to 1507847631

Integer of plane $v = 14$ with coordinates (3388,12) doesn't exist :

Wrong choice : Line position outside the trihedron.

Appendix 4

Getting a sequence of numbers answering successive additions of 1 at the beginning of the parity vector.

The choice of an integer aa , whose parity vector is vp , gives by the algorithm below a sequence of numbers having parity vectors $[1, vp]$, then $[1,1,vp,0]$, then $[1,1,1,vp,0]$,... the number of final 0 being adjusted to the relationship linking v the number of multiplication steps to w the number of division steps, that is $w = \text{int}(\ln(3)/\ln(2) \cdot v) + 1$ (in altitude flight).

The calculations are done without any recourse to the parity vectors (aiming a very high speed of execution). This program allows for example to recursively calculate the rest of the first elements of planes $a_v(1,1)$.

It also allows to recursively calculate the series of the second elements of planes $a_v(1,2)$.

Similarly, each element of a first line $a_v(1,i)$ gives an analog series $a_{v+n}(1,i)$.

This continues throughout the second block of a Pascal trihedron plane : $a_v(i$ in second block, $j)$ gives $a_{v+n}(i,j)$.

In addition, calculation gives jointly for $a_v(i,j)$ its flight altitude output value $m_v(i,j)$.

Program 4

```
/*Choose starting integer */
aa = 5;
/*Choose number of steps */
steps = 100;

infini = 10000000; lm = log(3)/log(2);
mm = aa; m = a = vector(steps); id = 0;
for(t = 1,infini,
if(mm/2 == floor(mm/2), mm = mm/2, id = id+1; mm = 3*mm+1);
if(aa >= mm, break));
wid = 1+floor(lm*id); ret1 = 2^wid; acal= aa-ret1*floor(aa/ret1); ret2 = 3^id; mcal= mm-ret2*floor(mm/ret2);
if(acal <> aa, print("Integer "aa" is represented by "acal" in the trihedron."));
a[1] = acal; m[1] = mcal;
print("v   m(v)   a(v)");
print(id"   "m[1]"   "a[1]);
limit = steps+id-1;
for(v = id, limit-1, w = 1+floor(lm*v) ; wp = 1+floor(lm*(v+1)); dw = wp-w;
for(r = 0, 5, vv = v-id+1; mm = (m[vv]+r*(3^v))/(2^(dw-1)) ; aa = (2*(a[vv]+r*(2^w))-1)/3;
if(aa == floor(aa), if(mm == floor(mm), t = v+1; tt = t-id+1; m[tt] = mm; a[tt] = aa; ret = 2^wp;
m[tt]= m[tt]-ret*floor(m[tt]/ret);
a[tt]= a[tt]-ret*floor(a[tt]/ret);
print(t"   "m[tt]"   "a[tt]);break))))}
```

Example 4

The series of the 50 first elements of each plane (starting with 5) :

Integer 5 is represented by 1 in the trihedron.

v	m(v)	a(v)
1	1	1
2	2	3
3	20	23
4	10	15
5	91	95
6	410	575
7	205	383
8	205	255
9	3383	5631
10	23066	25599
11	70582	104447
12	35291	69631
13	566732	745471
14	1877689	3293183
15	1877689	2195455
16	8113298	12648447
17	94206740	97910783
18	47103370	65273855
19	23551685	43515903
20	1185813152	1460666367
21	4079690977	6700400639
22	4079690977	4466933759
23	49111434902	71697432575
24	24555717451	47798288383

25 589414790413 764873277439
 26 718351699928 1242923270143
 27 3260217528257 3760646520831
 28 5442907506622 8371159695359
 29 5442907506622 5580773130239
 30 2721453753311 3720515420159
 31 310197425018629 565430297034751
 32 927870821302576 1127553469251583
 33 2316955599503129 3754102064414719
 34 13435076732614175 14512333715931135
 35 48410492615473510 69722884175560703
 36 24205246307736755 46481922783707135
 37 24205246307736755 30987948522471423
 38 237244576099367059 404965800550596607
 39 2938948011445351237 3344434545985323007
 40 3495751582232163752 5304080376275140607
 41 3495751582232163752 3536053584183427071
 42 74693868545457654682 100740004115906559999
 43 256184912535753545759 460690542983074807807
 44 912698847324827701013 1094188109133657407487
 45 2918276679121441932709 4664764141813809283071
 46 8826902092223109329995 9406331405035399806975
 47 8844920105937805212962 12567376247183126822911
 48 31011274411926405894268 58750159982064966828031
 49 190544160565671425620990 240654409923814840336383
 50 334571409513453242400578 563411546487419618459647

Number 1000th of the series

1000 110277737662239392555996382436853839831230580402291185298951946358498956333614674111382402477750726619658
 15015703553603845291360312966963410184891820532930751507365891913029425327815060977692438340696297884506443216
 65401676961180452393456153975938038355963675587842309211959117160252848025523124344888826836411307889067702769
 55086625775371298447569894063609760659751963882667459405174174888685722485109676483971192668155890160044032425
 2134496987219467406007352136396783848666645 113181711197902800523896881066218160102709667045393956950389619432
 94809801395772938409808943142435536912466639492491375516278778672506461456066473815462993585162043989461638387
 68783595410569772717210118527009685388805997434584268801949287629806470254334768295217540415359597797337275682
 09962344081540465463827583787623071447065173716333421030611964595317030946266729588187240878123884668706939368
 0025440093827021268188760756952836484710583975703867188025783792797611927080534015

Number 10000th of the series

10000 13558044774597476769556430862363748607046843647477779449081961354110954815445264866427449002556336388372
 67643410057077511354707949277740682238193617463134922632594673616132161885228906360109753208876527602232200913
 06409595254659790830847536846559584490142528619755613896103001949718252404393971748244533592429470898267674550
 24906431057922213261017636087691882429751636849272672342666169317533557433048794949093694654671636722715291477
 95641936207010184672370823718059478996768397366047614788952621584153539081882887167096960278626972606653659877
 86376199816082415678409091216040475535421744944158855174666052331443956021266752594155897878103412503818039129
 82567955059729692835890240188229308585650934225513518874654887002639126440329324782195088425200942530232320245
 50458757959855874732998196989145257758224781891631108741556199948089626632883969507655564160187174690125882845
 39630243557499624084795215003068581408993494724023031818939390906027675439621906981948139143879954025460746314
 2636789843078384319230846295354025210125907454190555680826502593947397207543728059268252784668523876824678067
 03626757133097803177221402713232767971478948576298978054318022279446894238959274675285348240050418508148753878
 59102344885863355677070455333052059857753649148177948722483776555181707993116091866520197648701043235394252349
 76112007334968187513521752203927651140732761791970739941930334272323353869979822617840376755082567058636354287
 96362885058451176964084167644049992028771756306236323370777731715822525158629026506004610147846767547187698323
 29929209950233271987503018924721071493838417870146126532628582000290280671190712729598783382820514220941066159
 23812483100232790669197486686085086430488889611917155999071847939267668574801309992978294745512453029002072604
 41972193998449663792988391299920425036507414712095771719704950660140839649411539058034377990178433278505136910
 90223383443584510801231752223269527861834731783430011188928000035591956262726059424618385249866897481880604254
 52063612434132930646335738078774763905997061386143573187714919025090322500269984659629049035042696369624626151
 62910071765780489429237917694827779240424456951138540213728969595416307240835061932795582485510236939495385496
 96237590512537883918841878393731261532349869544283524046346596110953226703962134588889665769200540729823787524
 10451962896611863929896168935634693105014223950923629383665611847543934547155818842264266714176646047791159898
 78375643658872580574192346623376570557092626950621252293930051827094054427363476320391242949742416104755419493
 42520476576404942741533078340690793631994287824578637914986876509775530174192398096268398204209075203529790864
 35267540783930388237385570780693930734805029804079253946325787529248547240103264346716531803818946250424917858
 61031513654195568828240480840831798497060896630026127668740338747051210720733993004002555195212826640700744011
 29704011149592801379530216753123077150447046555659329577909175126448433964743855136515955602927790897953967518
 5863184072888222911043498220633046554710768148258718062968552686105896995314241610901554496637291480109665548
 43244635065175132943105102508568079598146304020889812899853085110301460706251370931703076245250225499088725611

01255131496321969511236709532431483359037881405236107715855474649996324096403302742835783687247760190901337024
85637035598931831510599973227218275203418329130764498608973244251164005061630706769322453412598796168361316510
77878103257550671584907143105656754920765987311775367625075039824969935513797516631241485953263608647512813095
99170820338837250100980068159374121240920981986085709064463786789195146381768134789569125203687496924162132092
52393881909178394485081948739100188316039816254659988678454604369182424605740711751304839296753760355402565203
41806279491829853851319374346707274183821383878191332256882409993483396958078742097154469236840395372505165590
61106354913502151879260423574361940887490764157511532995069960974946428792489868674684141624228726349726353086
94878378377036842333586447847566839962255247249445650064961802548815153835947729340398501259866025351205293010
77434217481236671886909136250159886875145997810626471857319960474783099744341266356817066110003599434732654832
48039851829696915407452488555393709422127458737105589054387424290458372098514425815103384967716110419033497097
95343607374014062889443067845595926749404036785245420524335889073943390306027857825927952750432711657836786204
9004362088347854754169420690083926722405773055032430968002982827549908049123593016243202403790309247774890098
44169320148142962799385718471426738127837579478620955880997551254681169965585841238891876012622941149851024859
53603032602017133827979796260333313310752457310047473131970127828369830340407525699078593523860558026363040476
17192685622308909718247483202941467282086072349 1758252085792249411761416308381793239188059313865951574886800
06112650104863512630706923277544752250122259570381477821182637950775129877728250659243489036352093416116539323
04996754387661362184278182789969743983839415156980600519775698541546738264881643280879727211391375719349131874
31574523555297451379031260984038494866019150392657329150763352172743067954213864922512064910473388998938364322
45327321816691546173437261957338491777637982437927967276409208177652614493204580181446640712033539559441472419
58319447866118694684882111474316211035882865660638555047209275588151404938766096316566238518915302737028548056
4931011668627850972489584341004007589958529077454373784574262035530938878958410970268641412091500668561370597
1493559510885395737057655071377938294056230446887900717369618779746346545957061995934340047313865951574886800
67770231827087730610071082748147427274297585349862450569045656614623345700009479891328607321518911699798642476
40982328864575945875830964190756442660719087093701975414725901269028483772944927993499313643680036653001542938
23379386273953134657098115412209953549488248338955230071061768463535963047605859638602286252769783729717389210
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62009147743074222401108801470794036295170556070784365526050606051931969747356066928979695611634012774449402140
07452174357771272730514898941918790865284642332556328210770002377482893680402078268822121937632468333971063856
83244294654434095982039559319430942041495900469787476277890644609696264869729480208979890151767862846483916185
83085787839356488500664992237045799137672220910955625434198909001747232986604052175851814350383910346395471188
43944055969852614748363343564344273780073813566406288334379393693389109063803735061464008769495910971700646181
48844700464052132132409434363686909528818219825593130541195049026782624484148440573901802268879579112792244507
02918403469236683260024270484742058158663565764803426687374840450231242201738208807292727372137097607624110548
796772314723559679522294987436584491658522719137476751064751656469933447009069059733204875629402830403783
23457189971955582358910577200467920042896720985394818867754476885797185179536765320653627450112129596533340256
98838972454963988569621407037981098677592223664379838973579022768909505994798839655392413392855817397372394544
59745115233001602478682786174531081990173077364952948327202789691603241122922796997608126088341307035795231326
39543005302626441204493037294663202124489716454944083207576075868909582036153856688276819600799641290222489794
32734616759128052543205669262548179173829645878637249427238225004205627937764454258817233861308949416713246650
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19608329964941879015839808582270194928562410100973927766058942480093484976905500829253944968114918267143981121
72505107258674493499124502006251311553693881694717013723862356190498687883474052110456961362984843225648119432
89532033410576353474468365306296985006283553381972197838723947956846843097918293818689291673132662505114563505
38297584654996165215896564082631746680654633130265148412667703353184819163807054090830915453647715377945876348
36860943811252379075013593898153705911085805978424707660927338632085556416043839227747203989860616268816429811
38913238055865699285530447465350181494029393979572345931733040123227236389136678524367589475968770688849701522
86453914978016001214283328764302428613593730643678158206624628459429175228516389415445646792823483451542534817
80476733809423343689258753451641890024289403785008804142560384432926750651928324112383508811841444102195957465
38545956321885327998734852780383973327671186226783365580400218882539682138252858046142165016972201277222325583
71329999678657202150616669413198647027301404067959144854222989257083239022219478336435548278569648998943868339
56295951797946417514961167144648280216819548499101174493963991484006542057536522653316415632677000758215347801
35122585634124752554563158217488421151118627242577642053173974609421509793823406567722663462876366258008821734
10339589364529979492368234855778118163941298931848978463727236078231234276169920992705641660427416372217262120
46812588312643159261165023842154427947413849334070764270590594705531457663897386549284579910644589115598118604
00941424183257151661633858867934850481279313674420595913917819781743497891205647122715383351984074124773202966
0393693594003373121148614264784410150308906500928051117167789621724158511499573201733681151

Search for the value of an integer from its parity vector.

The choice of licit parity vector “vectpar” gives the integer nb2 that generates it.

The choice of a parity vector without precaution is often wrong. The number w of divisions by 2 and v of multiplications by 3 plus 1 are linked by $w = \text{int}(\ln(3)/\ln(2).v)+1$. Moreover, a bad choice can also come from the fact that too many divisions by 2 are selected at the beginning of the vector. A bad choice is detected (but not corrected).

Program 5

```

{ /* Choose a valid parity vector */
vectpar = [1,0,1,1,1,0,1,1,1,0,0,0];

lg = #vectpar; lm = log(3)/log(2); nbi = 0; nbp = 0;
for(i = 1, lg, if(vectpar[i] == 1, nbi = nbi+1, nbp = nbp+1));
wred = 1+floor(lm*nbi)-nbi;
alert = 0;
if(wred > nbp, print("Missing "wred-nbp" zero(s) in the parity vector "vectpar" . Retype valid vector."); alert = 1);
if(wred < nbp, print(nbp-wred" zero(s) in excess in parity vector "vectpar" . Retype valid vector."); alert = 1);
nbi = 0; nbp = 0;
if(alert == 0, for(i = 1, lg-1, if(vectpar[i] == 1, nbi = nbi+1, nbp = nbp+1); wred = 1+floor(lm*nbi)-nbi;
if(wred <= nbp, print("Premature zero(s) in the parity vector "vectpar" . Retype valid vector."); alert = 1;break)););
if(alert == 1, break, vectpar = vectpar[^lg];
v = nbi; w = 1+floor(lm*v); tabdpl = 0; infini = 10^1000;
for(i = 1, w-1, if(vectpar[i] == 1, tabdpl = concat(i, tabdpl)));
tabdpl = tabdpl[^v+1];
res = 0; long = 2;
for(i = 1, v, ii = v-i+1;
if(tabdpl[ii] <> v-ii+1, res = concat(res, tabdpl[ii+1]); long = v-i+3; break));
for(j = v+1-ii, v, iii = v-j+1; res = concat(res, tabdpl[iii]-if(long == 2,0,1)));
res = res[^1]; res = concat(res, w); resp = res; resp[2] = res[2]-res[1];
for(i = 3, long, resp[i] = res[i]-res[i-1]-1);
sec = resp[long]; add = sec;
for(i = 1, long-2, ii = long-i; add = add + resp[ii]+1; sec = concat(add,sec));
sec = concat(res[1],sec);
nb1 = 0; nb2 = 0;
if(long > 2, for(i = 1, long-2, ii = long-i+1;
for(rech = 1, infini, nb1 = (rech*(2^sec[ii])+nb2*(2^resp[ii])+1)/3;
if(nb1 == floor(nb1), nb2 = 2*nb1-1;break)))));
for(rech = 1, infini, nb1 = (rech*(2^sec[2])+nb2*(2^resp[2])+1)/(3^resp[1]);
if(nb1 == floor(nb1), nb2 = nb1*(2^resp[1])-1;break));
print("The integer corresponding to the parity vector "vectpar" is "nb2".".);) }

```

Examples 5

[illegible]

is

Appendix 6

Getting the parity vector from the integer value.

The use of the Collatz algorithm gives the parity vector without difficulty.

Program 6

```
{/* Choose a number */
nb = 27+2^59;
nbr = nb; infini = 10^1000; vectpar = 0; v = 1; nbr = (3*nbr+1)/2; vectpar = concat(vectpar, 1);
for(i = 1, infini,
if(nbr > nb,
if(nbr/2 == floor(nbr/2), nbr = nbr/2; vectpar = concat(vectpar, 0), v = v+1; nbr = (3*nbr+1)/2; vectpar = concat(vectpar, 1)),
break));
vectpar = vectpar[^1];
lm = log(3)/log(2); w = 1+floor(lm*v); ret = 2^w;
nbpl = nb-ret*floor(nb/ret);
if(nbpl <> nb, print("Integer \"nb\" is represented by \"nbpl\" within the trihedron."));
print("Integer \"nbpl\" is located in plane \"v\" and as parity vector \"vectpar\".")}
```

Examples 6

Integer 576460752303423515 is represented by 27 within the trihedron.
Integer 27 is located in plane 37 and has parity vector [1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0].

Integer 71 is located in plane 32 and as parity vector [1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0]