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Abstract: In this paper, a design of an adaptive observer for linear parameter varying (LPV) systems is proposed. This observer must be able to estimate simultaneously states and parameters. The estimated parameter vector is used to compute the scheduling functions and thus interpolate the local linear models. Its conditions of existence and stability are given and its design is obtained in terms of a set of linear matrix inequalities (LMI). The efficiency of the proposed approach is illustrated by an academic example.

Keywords: adaptive observer, LPV model,generalized dynamic observer.

1. INTRODUCTION

The estimation problem has been a challenging and open problem since the knowledge of the state variables is important for process supervision and control. Unfortunately, the information of the entire state is not always available due to economic or physical reasons. A solution for these issues is the design of observers which are able to reconstruct signals based on measured variables and a dynamical model which describes the behavior of the system (Luenberger (1964)). The observer design for nonlinear systems is often a difficult task in comparison to linear systems. However, there exist some approaches to approximate the nonlinear behavior of the system as the LPV approach introduced by Shamma and Athans (1991) and Shamma and Cloutier (1993). This kind of systems depends on time-varying parameters which are commonly available. The LPV system can be reformulated in a convex linear combination of linear time-invariant (LTI) systems when the bounds of parameters are known.

Commonly, the observer design assumes that the parameters of the system are known. This fact is not always true in a practical mean, because unknown parameters are encountered frequently in physical systems and these ones can introduce uncertainty in the observer design obtaining an inaccuracy reconstruction of the state variables. In this case, the adaptive observers are an efficient solution to solve this problem due to its characteristic of jointly estimates parameters and states. In the literature, two major approaches have been developed to face the design of adaptive observer. These approaches are essentially based on the following points: the unknown parameter vector is deduced from the stability analysis of a state observer and the convergence property of the parameter error is obtained by the persistence of excitation type constraint, consequently, a parameter adaptation law is proposed. Many contributions deal with this approach as in Zhang (2002); Cho and Rajamani (1997); Alma et al. (2018); and an augmented system for which the adaptive observer design is elaborated. In this case, the system dynamics are augmented with the dynamics of its unknown parameters as in Nagy-Kiss et al. (2015); Bezzaoucha et al. (2013); Srinivasarengan et al. (2018).

More recently, the study of a new dynamic observer, called generalized dynamic observer (GDO) has been introduced for linear time-invariant systems Gao et al. (2016), descriptor systems Osorio-Gordillo et al. (2016) and LPV systems Pérez-Estrada et al. (2018). This observer is based on Park et al. (2002); Marquez (2003), where the principal idea is the introduction of additional dynamics in the observer and the degrees of freedom added to the structure, with the purpose of achieving steady state accuracy and to improve robustness in estimation error against disturbances and parametric uncertainties. The obtained results from Pérez-Estrada et al. (2018); Osorio-Gordillo et al. (2016) generalize the existing results on the PO, the PIO and the dynamic observer (DO) and compare their performances. It is shown that the GDO has better behavior compared to the PIO and PO in the presence of uncertainties in the system.

The contribution of this paper is the design of an adaptive observer for LPV system which is endowed with the GDO properties. An academic example is shown to illustrate the effectiveness of the proposed scheme.

Lemma 1. Consider two matrices X and Y with appropriate dimensions, a time varying matrix $\Delta(t)$ and a positive scalar ϵ. The following inequality is verified

$$X^T\Delta^T(t)Y + Y^T\Delta(t)X \leq \epsilon X^T X + \epsilon^{-1}Y^T Y \quad (1)$$

for $\Delta^T(t)\Delta(t) \leq I$.

2. PROBLEM STATEMENT

Consider the following linear system subject to time varying parametric uncertainty $\theta(t)$

$$\dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t) \tag{2a}$$
$$y(t) = Cx(t) \tag{2b}$$

with $A(\theta(t)) = A_0 + \sum_{j=1}^{n_\theta} \theta_j(t)\overline{A}_j$, $B(\theta(t)) = B_0 + \sum_{j=1}^{n_\theta} \theta_j(t)\overline{B}$, $\theta_j(t) \in [\theta_j^1, \theta_j^2]$ and n_θ is the number of unknown parameters. $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ the input vector, $y(t) \in \mathbb{R}^p$ represents the output vector. $\theta(t) \in \mathbb{R}^{n_\theta}$ is a time varying parameter, non measurable but bounded, hence, it can be written as

$$\theta_j(t) = \mu_j^1(\theta_j(t))\theta_j^1 + \mu_j^2(\theta_j(t))\theta_j^2 \tag{3}$$

where $\mu_j^1 = \frac{\theta_j^2 - \theta_j(t)}{\theta_j^2 - \theta_j^1}$ and $\mu_j^2 = \frac{\theta_j(t) - \theta_j^1}{\theta_j^2 - \theta_j^1}$. The system (2) can be represented in a polytopic form where the parameter $\theta(t)$ evolves inside a polytope represented by $r = 2^{n_\theta}$ vertices.

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\theta(t))(\mathcal{A}_i x(t) + \mathcal{B}_i u(t)) \tag{4a}$$
$$y(t) = Cx(t) \tag{4b}$$

with $\mu_i(\theta(t)) = \prod_{j=1}^{n_\theta} \mu_j^k(\theta_j(t))$ and $\mathcal{A}_i = A_0 + \sum_{j=1}^{n_\theta} \theta_j^k \overline{A}_j$, $\mathcal{B}_i = B_0 + \sum_{j=1}^{n_\theta} \theta_j^k \overline{B}_j$ where k is equal to 1 or 2 depending of the partition of the j^{th} parameter (μ_j^1 or μ_j^2). A_0, B_0, $\overline{A}_j$ and $\overline{B}_j$ are known matrices with suitable dimensions. Now let us consider the following adaptive dynamic observer for system (4)

$$\dot{\zeta}(t) = \sum_{i=1}^{r} \mu_i(\hat{\theta}(t))(N_i\zeta(t) + H_i v(t) + F_i y(t) + J_i u(t)) \tag{5a}$$
$$\dot{v}(t) = \sum_{i=1}^{r} \mu_i(\hat{\theta}(t))(S_i\zeta(t) + L_i v(t) + M_i y(t)) \tag{5b}$$
$$\dot{\hat{\theta}}(t) = \sum_{i=1}^{r} \mu_i(\hat{\theta}(t))(K_{o,i}(C\hat{x}(t) - y(t)) + \alpha_i\hat{\theta}(t)) \tag{5c}$$
$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{5d}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector, $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$, $\hat{\theta}(t) \in \mathbb{R}^{n_\theta}$ is the estimate of $\theta(t)$. The matrices N_i, H_i, F_i, J_i, S_i, L_i, M_i, Q, $K_{o,i}$ and α_i are unknown matrices of appropriate dimensions.

In order to facilitate the comparison between system (4) and adaptive observer (5), the system can be written with scheduling functions depending on the estimated state vector adding and subtracting $\sum_{i=1}^{r} \mu_i(\hat{\theta}(t))(\mathcal{A}_i x(t) + \mathcal{B}_i u(t))$ such that

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\hat{\theta}(t))(\mathcal{A}_i x(t) + \mathcal{B}_i u(t)) + \sum_{i=1}^{r} (\mu_i(\theta(t)) - \mu_i(\hat{\theta}(t)))(\mathcal{A}_i x(t) + \mathcal{B}_i u(t)) \tag{6}$$

Now, let us define:

$$\Delta A(t) = \sum_{i=1}^{r} (\mu_i(\theta(t)) - \mu_i(\hat{\theta}(t)))\mathcal{A}_i = Z_A \Psi_A(t) E_A \tag{7}$$

$$\Delta B(t) = \sum_{i=1}^{r} (\mu_i(\theta(t)) - \mu_i(\hat{\theta}(t)))\mathcal{B}_i = Z_B \Psi_B(t) E_B \tag{8}$$

where $Z_A = [\mathcal{A}_1 \dots \mathcal{A}_r]$, $\Psi_A(t) = \begin{bmatrix} \delta_1(t) I_n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_r(t) I_n \end{bmatrix}$, $Z_B = [\mathcal{B}_1 \dots \mathcal{B}_r]$, $\Psi_B(t) = \begin{bmatrix} \delta_1(t) I_m & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_r I_m \end{bmatrix}$,

$E_A = [I_{n_1} \dots I_{n_r}]^T$ and $E_B = [I_{m_1} \dots I_{m_r}]^T$.

$\delta_i(t) = \mu_i(\theta(t)) - \mu_i(\hat{\theta}(t))$ for $i = 1 \dots r$ and due to the convex property implies that $-1 \leq \delta_i(t) \leq 1$. Therefore the matrices $\Psi_A(t)$ and $\Psi_B(t)$ have the following property

$$\Psi_A(t)^T \Psi_A(t) \leq I, \quad \Psi_B(t)^T \Psi_B(t) \leq I, \tag{9}$$

which it is used to bound the time varying difference between the known and estimated scheduling functions. With the previous conditions, the system (4) is represented as an uncertain system with scheduling functions depending on state estimated vector:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\hat{\theta}(t))((\mathcal{A}_i + \Delta A(t))x(t) + (\mathcal{B}_i + \Delta B(t))u(t)) \tag{10a}$$
$$y(t) = Cx(t) \tag{10b}$$

For the sake of simplicity, the following notation is used $\sum_{i=1}^{r} \hat{\mu}_i = \sum_{i=1}^{r} \mu_i(\hat{\theta}(t))$.

Let $T \in \mathbb{R}^{q_0 \times n}$ be a parameter matrix and considering the transformed error $\varepsilon(t) = \zeta(t) - Tx(t)$, we have its derivative given by

$$\dot{\varepsilon}(t) = \sum_{i=1}^{r} \hat{\mu}_i(N_i\varepsilon(t) + (N_iT + F_iC - T\mathcal{A}_i)x(t) + H_i v(t) + (J_i - T\mathcal{B}_i)u(t) - T\Delta A(t)x(t) - T\Delta B(t)u(t)) \tag{11}$$

By using the definition of $\varepsilon(t)$, equations (5b) and (5d) can be written as

$$\dot{v}(t) = \sum_{i=1}^{r} \hat{\mu}_i(S_i\varepsilon(t) + (S_iT + M_iC)x(t) + L_i v(t)) \tag{12}$$
$$\hat{x}(t) = P\varepsilon(t) + (PT + QC)x(t) \tag{13}$$

If the following conditions are satisfied

$$N_iT + F_iC - T\mathcal{A}_i = 0 \tag{14}$$
$$J_i - T\mathcal{B}_i = 0 \tag{15}$$
$$S_iT + M_iC = 0 \tag{16}$$
$$PT + QC = I \tag{17}$$

the equations (11)-(13) are reduced to the following system

$$\underbrace{\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix}}_{\dot{\varphi}(t)} = \sum_{i=1}^{r} \hat{\mu}_i \left(\underbrace{\begin{bmatrix} N_i & H_i \\ S_i & L_i \end{bmatrix}}_{\mathbb{A}_i} \underbrace{\begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}}_{\varphi(t)} + \underbrace{\begin{bmatrix} -T \\ 0 \end{bmatrix}}_{F_f} \Delta A(t)x(t) + \begin{bmatrix} -T \\ 0 \end{bmatrix} \Delta B(t)u(t) \right) \tag{18}$$

and the estimation error is written as

$$e(t) = \hat{x}(t) - x(t) = P\varepsilon(t). \tag{19}$$

Let us define the parameter estimation error as $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$. The dynamics of this error is given by

$$\dot{\tilde{\theta}}(t) = \sum_{i=1}^{r} \hat{\mu}_i (K_{o,i} CP\varepsilon(t) + \alpha_i \tilde{\theta}(t) + \alpha_i \theta(t) - \dot{\theta}(t)) \quad (20)$$

3. OBSERVER PARAMETERIZATION

In this section, it will give the parameterization of the algebraic constraint (14)-(17). Let $E \in \mathbb{R}^{q_0 \times n}$ be any full row rank matrix such that the matrix $\Sigma = \begin{bmatrix} E \\ C \end{bmatrix}$ is of full column rank and let $\Omega = \begin{bmatrix} I_n \\ C \end{bmatrix}$. Conditions (16) and (17) can be written as:

$$\begin{bmatrix} S_i & M_i \\ P & Q \end{bmatrix} \begin{bmatrix} T \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \quad (21)$$

The necessary and sufficient condition for (21) to be consistent is that $\mathcal{R}\left(\begin{bmatrix} 0 \\ I_n \end{bmatrix}\right) \subset \mathcal{R}\left(\begin{bmatrix} T \\ C \end{bmatrix}\right)$ or equivalently $\text{rank} \begin{bmatrix} T \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} T \\ C \\ 0 \\ I_n \end{bmatrix} = n.$

On the other hand, since $\text{rank} \begin{bmatrix} T \\ C \end{bmatrix} = n$, there always exist matrices $T \in \mathbb{R}^{q_0 \times n}$ and $K \in \mathbb{R}^{q_0 \times p}$ such that $T + KC = E$, which can be written as:

$$[\, T \;\; K \,]\, \Omega = E \quad (22)$$

and since $\text{rank}(\Omega) = \text{rank} \begin{bmatrix} \Omega \\ E \end{bmatrix}$. The general solution to equation (22) is given by:

$$[\, T \;\; K \,] = E\Omega^{+} - Y_1 (I_{n+p} - \Omega\Omega^{+}) \quad (23)$$

From equation equation (23) we deduce that

$$T = T_1 - Y_1 T_2 \quad (24)$$

$$K = K_1 - Y_1 K_2 \quad (25)$$

where $T_1 = E\Omega^{+} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, $T_2 = (I_{n+p} - \Omega\Omega^{+}) \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, $K_1 = E\Omega^{+} \begin{bmatrix} 0 \\ I_p \end{bmatrix}$, $K_2 = (I_{n+p} - \Omega\Omega^{+}) \begin{bmatrix} 0 \\ I_p \end{bmatrix}$ and Y_1 is an arbitrary matrix of appropriate dimensions.

By inserting the value of matrix T given by equation (22) into condition (14) we obtain

$$N_i E + \tilde{K}_i C = T A_i \quad (26)$$

where $\tilde{K}_i = F_i - N_i K$ and (26) can be written as:

$$\left[\, N_i \;\; \tilde{K}_i \,\right] \Sigma = T A_i \quad (27)$$

Since matrix Σ is of full column rank, the general solution to (27) is given by:

$$\left[\, N_i \;\; \tilde{K}_i \,\right] = T A_i \Sigma^{+} - Z_i (I_{n+p} - \Sigma\Sigma^{+}) \quad (28)$$

and by inserting the value of matrix T given in (24) into equation (28) we obtain

$$N_i = N_{1,i} - Y_1 N_{2,i} - Z_i N_3 \quad (29)$$

$$\tilde{K}_i = \tilde{K}_{1,i} - Y_1 \tilde{K}_{2,i} - Z_i \tilde{K}_3 \quad (30)$$

where $N_{1,i} = T_1 A_i \Sigma^{+} \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $N_{2,i} = T_2 A_i \Sigma^{+} \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $N_3 = (I_{q_0+p} - \Sigma\Sigma^{+}) \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $\tilde{K}_{1,i} = T_1 A_i \Sigma^{+} \begin{bmatrix} 0 \\ I_p \end{bmatrix}$, $\tilde{K}_{2,i} = T_2 A_i \Sigma^{+} \begin{bmatrix} 0 \\ I_p \end{bmatrix}$, $\tilde{K}_3 = (I_{qo+p} - \Sigma\Sigma^{+}) \begin{bmatrix} 0 \\ I_p \end{bmatrix}$ and Z_i are arbitrary matrices of appropriate dimension. As matrices N_i, T, K, $\tilde{K}_i$ are known, we can deduce the matrix F_i as:

$$F_i = F_{1,i} - Y_1 F_{2,i} - Z_i F_3 \quad (31)$$

where $F_{1,i} = T_1 A_i \Sigma^{+} \begin{bmatrix} K \\ I_p \end{bmatrix}$, $F_{2,i} = T_2 A_i \Sigma^{+} \begin{bmatrix} K \\ I_p \end{bmatrix}$, $F_3 = (I_{n+p} - \Sigma\Sigma^{+}) \begin{bmatrix} K \\ I_p \end{bmatrix}$.

On the other hand from equation (22) we obtain:

$$\begin{bmatrix} T \\ C \end{bmatrix} = \begin{bmatrix} I_{q_0} & -K \\ 0 & I_p \end{bmatrix} \Sigma \quad (32)$$

replacing equation (32) into the equation (21) we get:

$$\begin{bmatrix} S_i & M_i \\ P & Q \end{bmatrix} \begin{bmatrix} I_{q_0} & -K \\ 0 & I_p \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \quad (33)$$

Since matrix Σ is of full column rank and $\begin{bmatrix} I_{q_0} & -K \\ 0 & I_p \end{bmatrix}^{-1} = \begin{bmatrix} I_{q_o} & K \\ 0 & I_p \end{bmatrix}$ the general solution to equation (33) is given by:

$$\begin{bmatrix} S_i & M_i \\ P & Q \end{bmatrix} = \left(\begin{bmatrix} 0 \\ I_n \end{bmatrix} \Sigma^{+} - \begin{bmatrix} U_{1,i} \\ U_2 \end{bmatrix} (I_{q_0+p} - \Sigma\Sigma^{+}) \right) \begin{bmatrix} I_{q_0} & K \\ 0 & I_p \end{bmatrix}$$

where $U_{1,i}$ and U_2 are arbitrary matrices of appropriate dimensions.

Then matrices S_i, M_i, P and Q can be determined as:

$$S_i = -U_{1,i} N_3 \quad (34)$$

$$M_i = -U_{1,i} F_3 \quad (35)$$

$$P = \Sigma^{+} \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix} - U_2 N_3 \quad (36)$$

$$Q = \Sigma^{+} \begin{bmatrix} K \\ I_p \end{bmatrix} - U_2 F_3 \quad (37)$$

The estimation error (19) shows that $e(t) \to 0$ when $\varepsilon(t) \to 0$, i.e., the error $e(t)$ is independent of the matrix P. Then, without loss of generality, we can take $U_2 = 0$ to obtain $P = \Sigma^{+} \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$ and $Q = \Sigma^{+} \begin{bmatrix} K \\ I_p \end{bmatrix}$.

In order to deal with the uncertain term, by considering the system (10), observer error dynamics (18) and parameter estimation error dynamics (20), we obtain the following augmented system

$$\dot{\beta}(t) = \sum_{i=1}^{r} \hat{\mu}_i \left(\Phi_i(t)\beta(t) + \xi_i(t)\omega(t) \right) \quad (38a)$$

$$e_o(t) = \mathbb{P}_1 \beta(t) \quad (38b)$$

where $\beta(t) = \begin{bmatrix} x(t) \\ \varphi(t) \\ \tilde{\theta}(t) \end{bmatrix}$, $\omega(t) = \begin{bmatrix} u(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$, $e_0(t) = \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix}$,

$\Phi_i(t) = \begin{bmatrix} \mathcal{A}_i + \Delta A(t) & 0 & 0 \\ F_f \Delta A(t) & \mathbb{A}_{1,i} - \mathbb{Y}_i \mathbb{A}_2 & 0 \\ 0 & [\, K_{o,i} CP \;\; 0 \,] & \alpha_i \end{bmatrix}$, $\mathbb{P}_1 = \begin{bmatrix} 0 & \mathbb{P} & 0 \\ 0 & 0 & I \end{bmatrix}$,

$$\xi_i(t) = \begin{bmatrix} \mathcal{B}_i + \Delta B(t) & 0 & 0 \\ F_f \Delta B(t) & 0 & 0 \\ 0 & \alpha_i & -I \end{bmatrix}, \mathbb{A}_{1,i} = \begin{bmatrix} N_{1,i} - Y_1 N_{2,i} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\mathbb{P} = [\, P \;\; 0\,], \mathbb{Y}_i = \begin{bmatrix} Z_i & H_i \\ U_{1,i} & L_i \end{bmatrix} \text{ and } \mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}.$$

4. OBSERVER STABILITY ANALYSIS

For $\omega \neq 0$, we must satisfy $\sup_{w \in \mathcal{L}_2 - \{0\}} \frac{\|e_o(t)\|_2^2}{\|\omega(t)\|_2^2} < \Gamma^2$. Based on the parameterization and the stability analysis, the following Theorem gives the stability conditions which allow the determination of all observer matrices.

Theorem 1. System (38) is asymptotically stable with an attenuation level Γ, such that $\sup_{\omega \in \mathcal{L}_2 - \{0\}} \frac{\|e_o(t)\|_2^2}{\|\omega(t)\|_2^2} < \Gamma^2$ if there exist parameter matrices $\mathbb{Y}_i$, $X_0 = X_0^T > 0$, $X_1 = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{13} \end{bmatrix} > 0$, $X_2 = X_2^T > 0$ and diagonal matrices Γ_1^2, Γ_2^2, Γ_3^2 with appropriate dimension. ϑ, λ_1, λ_2, $\overline{\alpha}_i$ and $\overline{K}_i$ which satisfy the optimization problem (39) under LMIs (41)

$$\min_{X_0, X_1, X_2, \lambda_1, \lambda_2, \Gamma_1^2, \Gamma_2^2, \Gamma_3^2} \vartheta \tag{39}$$

$$\Gamma_{k_1}^2 < \vartheta I, \quad \text{for } k_1 = 1, 2, 3 \tag{40}$$

$$\begin{bmatrix} \mathrm{He}\{X_0 \mathcal{A}_i\} & 0 & 0 & X_0 \mathcal{B}_i \\ 0 & \Pi_{1,i} & N_3^{T\perp} P^T C^T \overline{K}_i^T & 0 \\ 0 & \overline{K}_i C P N_3^{T\perp T} & \overline{\alpha}_i + \overline{\alpha}_i^T & 0 \\ \mathcal{B}^T X_0 & 0 & 0 & -\Gamma_1^2 \\ 0 & 0 & \overline{\alpha}_i^T & 0 \\ 0 & 0 & -X_2^T & 0 \\ 0 & P N_3^{T\perp T} & 0 & 0 \\ 0 & 0 & I & 0 \\ \lambda_1 E_A & 0 & 0 & 0 \\ Z_A^T X_0 & \Pi_2^T & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 E_B \\ Z_B^T X_0 & \Pi_3^T & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \lambda_1 E_A^T & X_0 Z_A & 0 & X_0 Z_B \\ 0 & 0 & N_3^{T\perp} P^T & 0 & 0 & \Pi_2 & 0 & \Pi_3 \\ \overline{\alpha}_i & -X_2 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 E_B^T & 0 \\ -\Gamma_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Gamma_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_1 I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_1 I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 I \end{bmatrix} < 0 \tag{41}$$

where $\Pi_2 = N_3^{T\perp} W_1 T_2 Z_A - N_3^{T\perp} X_{11} T_1 Z_A$,
$\Pi_3 = N_3^{T\perp} W_1 T_2 Z_B - N_3^{T\perp} X_{11} T_1 Z_B$
$\Pi_{1,i} = N_3^{T\perp}(X_{11} N_{1,i} - W_1 N_{2,i} + N_{1,i}^T X_{11} - N_{2,i}^T W_1^T) N_3^{T\perp T}$,

with $Y_1 = X_{11}^{-1} W_1$, $\alpha_i = X_2^{-1} \overline{\alpha}_i$ and $K_{o,i} = X_2^{-1} \overline{K}_i$.

According to the elimination lemma (Skelton et al. (1997)), the matrix $\mathbb{Y}_i$ is parameterized as

$$\mathbb{Y}_i = X_1^{-1}(\mathcal{B}_r^+ \mathcal{K}_i \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+) \tag{42}$$

with

$$\mathcal{K}_i = \mathcal{R}^{-1} \mathcal{B}_l^T \Lambda_i \mathcal{C}_r^T (\mathcal{C}_r \Lambda_i \mathcal{C}_r^T)^{-1} + \mathcal{S}_i^{1/2} \phi (\mathcal{C}_r \Lambda_i \mathcal{C}_r^T)^{-1/2} \tag{43}$$

$$\mathcal{S}_i = \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\Lambda_i - \Lambda_i \mathcal{C}_r^T (\mathcal{C}_r^T \Lambda_i \mathcal{C}_r^T)^{-1} \mathcal{C}_r \Lambda_i] \mathcal{B}_l \mathcal{R}^{-1} \tag{44}$$

$$\Lambda_i = (\mathcal{B}_r \mathcal{R}^{-1} \mathcal{B}_l^T - \mathbb{D}_i)^{-1} > 0 \tag{45}$$

where matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathbb{C} = \mathcal{C}_l \mathcal{C}_r$ and $\mathbb{B} = \mathcal{B}_l \mathcal{B}_r$,

$$\mathbb{D}_i = \begin{bmatrix} \mathrm{He}\{X_0 \mathcal{A}_i\} & 0 & 0 & X_0 \mathcal{B}_i \\ 0 & \mathrm{He}\{X_1 \mathbb{A}_{1,i}\} & \mathbb{Y}_{2.i}^T X_2 & 0 \\ 0 & X_2 \mathbb{Y}_{2,i} & X_2 \alpha_i + \alpha_i^T X_2 & 0 \\ \mathcal{B}^T X_0 & 0 & 0 & -\Gamma_1^2 \\ 0 & 0 & \alpha_i^T X_2 & 0 \\ 0 & 0 & -X_2^T & 0 \\ 0 & \mathbb{P} & 0 & 0 \\ 0 & 0 & I & 0 \\ \lambda_1 E_A & 0 & 0 & 0 \\ Z_A^T X_0 & Z_A^T F_f^T X_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 E_B \\ Z_B^T X_0 & Z_B^T F_f^T X_1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \lambda_1 E_A^T & X_0 Z_A & 0 & X_0 Z_B \\ 0 & 0 & \mathbb{P}^T & 0 & 0 & X_1 F_f Z_A & 0 & X_1 F_f Z_B \\ \overline{\alpha}_i & -X_2 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 E_B^T & 0 \\ -\Gamma_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Gamma_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_1 I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_1 I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 I \end{bmatrix} \tag{46}$$

$\mathbb{Y}_{2,i} = [K_{o.i} C P \; 0]$, $\mathbb{B} = [\, 0 \; -I \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \,]^T$,
$\mathbb{C} = [\, 0 \; \mathbb{A}_2 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \,]$, ϕ is an arbitrary matrix such that $||\phi|| < 1$ and $\mathcal{R} > 0$.

Proof. Consider the following Lyapunov function

$$V(\beta(t)) = \beta(t)^T X \beta(t) > 0 \tag{47}$$

with $X = \begin{bmatrix} X_0 & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & 0 & X_2 \end{bmatrix} > 0$, $X_1 = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{13} \end{bmatrix}$. Its derivative along the trajectory of (38) is given by

$$\dot{V}(\beta(t)) = \sum_{i=1}^{r} \hat{\mu}_i (\beta^T(t)(\Phi_i^T X + X \Phi_i)\beta(t) + \omega^T(t) \xi_i^T(t) X \beta(t) + \beta^T(t) X \xi_i(t) \omega(t)) \tag{48}$$

Now let $\mathcal{S} = \dot{V}(\beta(t)) + e_o^T(t) e_o(t) - \Gamma^2 \omega^T(t) \omega(t)$, then we have

$$\mathcal{S} = \sum_{i=1}^{r} \hat{\mu}_i \begin{bmatrix} \beta(t) \\ \omega(t) \end{bmatrix}^T \underbrace{\begin{bmatrix} \Phi_i^T X + X \Phi_i + \mathbb{P}_1^T \mathbb{P}_1 & X \xi_i \\ (*) & -\Gamma^2 \end{bmatrix}}_{\Theta_i} \begin{bmatrix} \beta(t) \\ \omega(t) \end{bmatrix} \tag{49}$$

with $\Gamma^2 = \mathsf{diag}(\Gamma_1^2, \Gamma_2^2, \Gamma_3^2)$.
We can deduce that if $\Theta_i < 0$ then $\mathcal{S} < 0$. It implies that

$$\dot{V}(\beta(t)) < \Gamma^2 \omega^T(t) \omega(t) - e_o^T(t) e_o(t) \tag{50}$$

By integrating the two sides of this inequality we obtain

$$\int_0^\infty \dot{V}(\beta(t) dt < \int_0^\infty \Gamma^2 \omega^T(t) \omega(t) dt - \int_0^\infty e_o^T(t) e_o(t) dt \tag{51}$$

or equivalently

$$V(\infty) - V(0) < \Gamma^2 \left\|\omega(t)\right\|_2^2 - \left\|e_o(t)\right\|_2^2 \tag{52}$$

For the zero initial condition, it leads to $\frac{\|e_o(t)\|_2^2}{\|\omega(t)\|_2^2} < \Gamma^2$.

By applying the Schur complement to $\Theta_i < 0$, we obtain the following inequality

$$\begin{bmatrix} \Phi_i^T X + X\Phi_i & X\xi_i & \mathbb{P}_1^T \\ (*) & -\Gamma^2 & 0 \\ \mathbb{P}_1 & 0 & -I \end{bmatrix} < 0 \tag{53}$$

which can also be written as

$$\begin{bmatrix} \mathrm{He}\{X_0(\mathcal{A}_i + \Delta A(t))\} & \Delta A^T(t) F_f^T X_1 & 0 \\ (*) & \mathrm{He}\{X_1(\mathbb{A}_{1,i} - \mathbb{Y}_i \mathbb{A}_2)\} & \mathbb{Y}_{2,i}^T X_2 \\ (*) & (*) & \mathrm{He}\{X_2 \alpha_i\} \\ (*) & (*) & (*) \\ (*) & (*) & (*) \\ (*) & (*) & (*) \\ (*) & (*) & (*) \\ (*) & (*) & (*) \end{bmatrix}$$

$$\begin{bmatrix} X_0(\mathcal{B}_i + \Delta B(t)) & 0 & 0 & 0 & 0 \\ X_1 F_f \Delta B(t) & 0 & 0 & \mathbb{P}^T & 0 \\ 0 & X_2\alpha_i & -X_2 & 0 & I \\ -\Gamma_1^2 & 0 & 0 & 0 & 0 \\ 0 & -\Gamma_2^2 & 0 & 0 & 0 \\ 0 & 0 & -\Gamma_3^2 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{54}$$

where $\mathbb{Y}_{2,i} = [K_{o,i} C P \ 0]$.
It is noted that there exists time varying terms in the previous inequality whereby based on Lemma 1, it will obtain the upper bound of each one.

By using the definitions (7) and (8) the inequality (54) can be decomposed into the following terms

$$Q_i + \mathcal{T}(t) + \mathcal{T}^T(t) < 0 \tag{55}$$

where

$$Q_i = \begin{bmatrix} \mathrm{He}\{X_0(\mathcal{A}_i + \Delta A(t))\} & 0 & 0 \\ (*) & \mathrm{He}\{X_1(\mathbb{A}_{1,i} - \mathbb{Y}_i \mathbb{A}_2)\} & \mathbb{Y}_{2,i}^T X_2 \\ (*) & (*) & \mathrm{He}\{X_2\alpha_i\} \\ (*) & (*) & (*) \\ (*) & (*) & (*) \\ (*) & (*) & (*) \\ (*) & (*) & (*) \\ (*) & (*) & (*) \end{bmatrix}$$

$$\begin{bmatrix} X_0\mathcal{B}_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{P}^T & 0 \\ 0 & X_2\alpha_i & -X_2 & 0 & I \\ -\Gamma_1^2 & 0 & 0 & 0 & 0 \\ 0 & -\Gamma_2^2 & 0 & 0 & 0 \\ 0 & 0 & -\Gamma_3^2 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & -I \end{bmatrix}$$

and

$$\mathcal{T}(t) = \begin{bmatrix} X_0 Z_A \\ X_1 F_f Z_A \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Psi_A(t) \begin{bmatrix} E_A^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} X_0 Z_B \\ X_1 F_f Z_B \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Psi_B(t) \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_B^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using Lemma 1 on $\mathcal{T}(t) + \mathcal{T}^T(t)$, there exists positive scalars λ_1 and λ_2 such that

$$\mathcal{T}(t) + \mathcal{T}^T(t) \le \lambda_1^{-1} \begin{bmatrix} X_0 Z_A \\ X_1 F_f Z_A \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} X_0 Z_A \\ X_1 F_f Z_A \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \lambda_1 \begin{bmatrix} E_A^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times$$

$$\begin{bmatrix} E_A^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \lambda_2^{-1} \begin{bmatrix} X_0 Z_B \\ X_1 F_f Z_B \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} X_0 Z_B \\ X_1 F_f Z_B \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_B^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_B^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \tag{56}$$

The quadratic entries of the previous inequality can be handled by Schur's complement. Consequently, the inequality (55) becomes

$$\begin{bmatrix} \mathrm{He}\{X_0\mathcal{A}_i\} & 0 & 0 & X_0\mathcal{B}_i \\ 0 & \mathrm{He}\{X_1(\mathbb{A}_{1,i} - \mathbb{Y}_i\mathbb{A}_2)\} & \mathbb{Y}_{2,i}^T X_2 & 0 \\ 0 & X_2\mathbb{Y}_{2,i} & X_2\alpha_i + \alpha_i^T X_2 & 0 \\ \mathcal{B}^T X_0 & 0 & 0 & -\Gamma_1^2 \\ 0 & 0 & \alpha_i^T X_2 & 0 \\ 0 & 0 & -X_2^T & 0 \\ 0 & \mathbb{P} & 0 & 0 \\ 0 & 0 & I & 0 \\ \lambda_1 E_A & 0 & 0 & 0 \\ Z_A^T X_0 & Z_A^T F_f^T X_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 E_B \\ Z_B^T X_0 & Z_B^T F_f^T X_1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \lambda_1 E_A^T & X_0 Z_A & 0 & X_0 Z_B \\ 0 & 0 & \mathbb{P}^T & 0 & 0 & X_1 F_f Z_A & 0 & X_1 F_f Z_B \\ \overline{\alpha}_i & -X_2 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 E_B^T & 0 \\ -\Gamma_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Gamma_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_1 I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_1 I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 I \end{bmatrix} < 0 \tag{57}$$

which can also be written as

$$\mathbb{B}\mathbb{X}_i\mathbb{C} + (\mathbb{B}\mathbb{X}_i\mathbb{C})^T + \mathbb{D}_i < 0 \tag{58}$$

where $\mathbb{B} = [\,0\ -I\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]^T$,
$\mathbb{C} = [\,0\ \mathbb{A}_2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]$, $\mathbb{X}_i = X_1\mathbb{Y}_i$ and $\mathbb{D}_i$ is represented by (46).

According to the elimination lemma, the solvability conditions of equation (58) is reduced to:

$$\mathbb{C}^{T\perp}\mathbb{D}_i\mathbb{C}^{T\perp T} < 0 \tag{59}$$

5. ILLUSTRATIVE EXAMPLE: DC MOTOR

In order to illustrate the previous results, let us consider the physics-based model of a DC motor which considers an electric equivalent circuit of the armature and the rotor free of charge. The parameters of the DC motor are presented in Liu et al. (2000). Its dynamics is described by the following state-space representation

$$\dot{x}(t) = \begin{bmatrix} -\frac{R(t)}{L} & -\frac{K_a}{L} \\ \frac{k_\tau}{J} & -\frac{K_w}{J} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_e(t) \quad (60)$$

$$y(t) = Cx(t) \quad (61)$$

where $x(t) = \begin{bmatrix} i(t) \\ w(t) \end{bmatrix}$. It is assuming that the electric resistance is an unmeasured time-varying parameter therefore, the DC motor model can be considered as an LPV system where the scheduling function $\theta(t) = R(t) \in [0, 1.2]\Omega$. The system (60) is rewritten as

$$\dot{x}(t) = (A_0 + \theta(t)\overline{A})x(t) + Bv_e(t) \quad (62a)$$

$$y(t) = Cx(t) \quad (62b)$$

$$A_0 = \begin{bmatrix} 0 & -\frac{K_a}{L} \\ \frac{k_\tau}{J} & -\frac{K_w}{J} \end{bmatrix}, \ \overline{A} = \begin{bmatrix} -\frac{1}{L} & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, C = [\,1 \ 0\,]$$

The input variable $v_e = 6\,V$. The initial condition for the system is $x(0) = [0, 0]^T$, the initial condition for the observer is $\hat{x}(0) = [0, 0.4]^T$ and the initial condition of the estimated parameter is $\hat{\theta}(0) = 0$.The observer gains are obtained by solving the LMIs of the Theorem using YALMIP toolbox. The obtained results are depicted in the following figures.

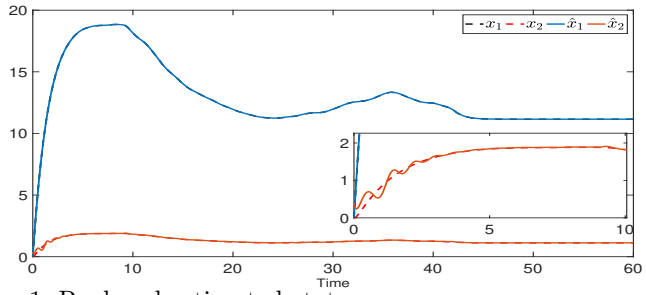

Fig. 1. Real and estimated states

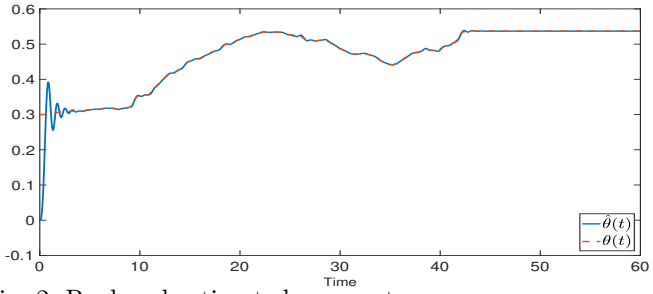

Fig. 2. Real and estimated parameter

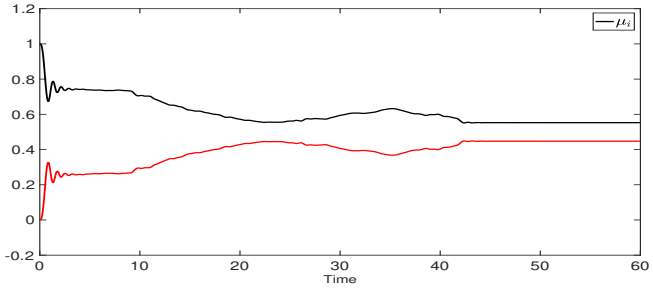

Fig. 3. Scheduling functions

From the depicted figures (1)-(3) we can note that there exist acceptable state and parameter estimations.

6. CONCLUSION

In this paper, we consider the joint estimation of states and parameter for linear parameter varying (LPV) systems. Its conditions of existence and stability are given and its design is obtained in terms of a set of linear matrix inequalities (LMI). In order to illustrate the efficiency of the proposed approach, an academic example was presented.

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