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Robust Semiparametric Efficient Estimators in Complex Elliptically Symmetric Distributions

Stefano Fortunati, *Member, IEEE*, Alexandre Renaux, *Member, IEEE*, Frédéric Pascal, *Senior Member, IEEE*

Abstract—Covariance matrices play a major role in statistics, signal processing and machine learning applications. This paper focuses on the *semiparametric covariance/scatter matrix estimation problem in elliptical distributions*. The class of elliptical distributions can be seen as a semiparametric model where the finite-dimensional vector of interest is given by the location vector and by the (vectorized) covariance/scatter matrix, while the density generator represents an infinite-dimensional nuisance function. The main aim of this work is then to provide possible estimators of the finite-dimensional parameter vector able to reconcile the two dichotomic concepts of *robustness* and (semiparametric) *efficiency*. An R -estimator satisfying these requirements has been recently proposed by Hallin, Oja and Paindaveine for real-valued elliptical data by exploiting the Le Cam’s theory of *one-step efficient estimators* and the *rank-based statistics*. In this paper, we firstly recall the building blocks underlying the derivation of such real-valued R -estimator, then its extension to complex-valued data is proposed. Moreover, through numerical simulations, its estimation performance and robustness to outliers are investigated in a finite-sample regime.

Index Terms—Semiparametric models, robust estimation, elliptically symmetric distributions, scatter matrix estimation, Le Cam’s one-step estimator, ranks.

I. INTRODUCTION

Semiparametric inference is the branch of theoretical and applied statistics dealing with point estimation or testing in semiparametric model. In short, a semiparametric model is a family of probability density functions (pdfs) parameterized by a finite-dimensional parameter vector of interest, say $\phi \in \Omega \subseteq \mathbb{R}^q$ (or \mathbb{C}^q), and by an infinite-dimensional parameter, say $g \in \mathcal{G}$, where \mathcal{G} is a suitable set of functions [1]. In the vast majority of applications where semiparametric models are used, the infinite-dimensional parameter g plays the role of a *nuisance* function.

Despite of their generality and practical relevance, the use of semiparametric models in Signal Processing (SP) applications is still limited to very few cases. To name some examples, we refer to [2] for a semiparametric approach to blind source separation, to [3] for robust non-linear regression and to [4] for empirical likelihood methods applied to covariance estimation. More recently, in [5,6], the class of the Real and Complex Elliptically Symmetric (RES and CES) distributions [7] has been revised from a semiparametric standpoint (see also [8]–[12] in the statistical literature). The family of Elliptically Symmetric (ES) distributions is in fact a typical example of

semiparametric model where the finite-dimensional parameter vector of interest is given by the location vector μ and by the (vectorized version of) the covariance/scatter matrix Σ , while the density generator g can be considered as a nuisance function. In particular, in [5] the RES class has been framed in the context of *semiparametric group models*, then a Semiparametric Cramér-Rao Bound (SCRB) for the joint estimation of μ and Σ in the presence of the nuisance density generator g has been derived. The second work [6] extended the previously obtained SCRБ to semiparametric estimation of complex parameters in CES distributed data. A semiparametric version of the celebrated Slepian-Bangs formula has been also proposed. However, the following fundamental question has not been addressed in [5,6] which were focused on lower bounds: *is it possible to derive a robust and semiparametric efficient estimator of the covariance/scatter matrix Σ of a set of ES distributed observations?* As we will see ahead, a first positive answer to this question has been provided in [10] for the RES case while its extension to CES distributions will be given in this paper.

To start, let us take a closer look to the two main features that this estimator should have. Firstly, it should be *semiparametric efficient*, at least asymptotically. In other words, we require that the error covariance matrix of this estimator should be equal to the SCRБ given in [5,6] as the number of observations goes to infinity. The second desirable feature is the *distributional robustness*. As said before, a semiparametric model allows for the presence of a nuisance function that, in the case of ES distributed observations, is the unknown density generator g characterizing the shape of their actual distribution. So, a distributionally robust estimator is basically an estimator of Σ whose statistical properties do not rely on $g \in \mathcal{G}$, and consequently on the actual ES distribution of the data. It is worth to underline that, even if robust estimators of covariance matrices are already available in the statistics and SP literature ([7,13]–[17], [18, Ch. 4] and references therein), they fail to be semiparametric efficient as shown in [5,6].

A good candidate for the estimator that we are looking for is the one proposed by Hallin, Oja and Paindaveine in their seminal paper [10]. Building upon their previous work [9], in [10] the Authors propose an estimator of the constrained, real-valued scatter matrix Σ in RES distributed data that meets the two requirements of *nearly* semiparametric efficiency and distributional robustness. To achieve the semiparametric efficiency, the Le Cam’s theory of *one-step efficient estimators* [19], [20, Ch. 6] has been exploited. In fact, as showed by Le Cam, it is possible to derive asymptotically efficient estimators that, unlike the Maximum Likelihood (ML) one, do not search for the maxima of the log-likelihood function. This

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is of great importance in practical applications, where the ML estimator can present computational difficulties in the resulting optimization problem or even existence/uniqueness issues [21, Ch. 6]. The second requirement of distributional robustness has been addressed in [10] using a rank-based approach [22], [23, Ch. 13]. Originally developed in the context of order statistics, rank-based methods have been used in robust statistics to derive distributionally robust estimators and tests that are usually referred to as R -estimators and R -tests [24, Ch. 3].

After a semiparametric formalization of the shape matrix estimation problem given in Section II, the subsequent Section III provides a review of the methodology used in [10] to derive a semiparametric efficient R -estimator of the constrained, real-valued, scatter matrix Σ in RES distributed data. This first part has the twofold goal of *i*) introducing two statistical procedures (i.e. semiparametric one-step estimators and rank-based robustification) that are not yet widespread among the SP community and then *ii*) showing how they can be applied to derive original estimators of scatter matrices. To this end, additional in-depth supporting material will be provided separately from the main body of the paper. In addition, the code containing our Matlab and Python implementation of both real- and complex-valued R -estimator can be found at [25]. Section IV focuses on the extension of the previously derived outcomes to the complex-valued parameter case with Complex ES distributed data. In Section V the Mean Squared Error (MSE) performance and the robustness properties of the proposed semiparametric efficient R -estimator will be investigated through numerical simulations in a “finite-sample” regime. The theoretical analysis, in fact, can only provide us with asymptotic guarantees on the good behavior of an estimator but, since in practice the number of available observation is always finite, a “finite-sample” performance characterization is necessary as well. To this end, the error covariance matrix of the proposed R -estimator (evaluated using independent Monte Carlo runs) will be compared with the SCRBS in [5,6] in different scenarios. The second feature that is going to be assessed in Section V is the robustness to the presence of *outliers* in the observations. In the present context, an outlier can be represented by an observation vector whose distribution does not belong to the ES family.

Algebraic notation: Throughout this paper, italics indicates scalar quantities (a), lower case and upper case boldface indicate column vectors (\mathbf{a}) and matrices (\mathbf{A}), respectively. Each entry of a matrix \mathbf{A} is indicated as $a_{ij} \triangleq [\mathbf{A}]_{i,j}$. \mathbf{I}_N defines the $N \times N$ identity matrix. The superscripts $*$, \top and \mathbf{H} indicate the complex conjugation, the transpose and the Hermitian operators respectively, then $\mathbf{A}^{\mathbf{H}} = (\mathbf{A}^*)^{\top}$. Moreover, $\mathbf{A}^{-\top} \triangleq (\mathbf{A}^{-1})^{\top} = (\mathbf{A}^{\top})^{-1}$, $\mathbf{A}^{-*} \triangleq (\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1}$ and $\mathbf{A}^{-\mathbf{H}} \triangleq (\mathbf{A}^{-1})^{\mathbf{H}} = (\mathbf{A}^{\mathbf{H}})^{-1}$. The Euclidean norm of a vector \mathbf{a} is indicated as $\|\mathbf{a}\|$. The determinant and the Frobenius norm of a matrix \mathbf{A} are indicated as $|\mathbf{A}|$ and $\|\mathbf{A}\|_F$, respectively. The symbol vec indicates the standard vectorization operator that maps column-wise the entry of an $N \times N$ matrix \mathbf{A} in an N^2 -dimensional column vector $\text{vec}(\mathbf{A})$. The operator $\text{vec}(\mathbf{A})$ defines the $N^2 - 1$ -dimensional vector obtained from $\text{vec}(\mathbf{A})$ by deleting its first element, i.e. $\text{vec}(\mathbf{A}) \triangleq [a_{11}, \text{vec}(\mathbf{A})^{\top}]^{\top}$. A matrix \mathbf{A} whose first top-left entry is constrained to be equal

to 1, i.e. $a_{11} \triangleq 1$, is indicated as \mathbf{A}_1 .

For any $N \times N$ symmetric matrix \mathbf{A} :

- $\text{vecs}(\mathbf{A})$ indicates the $N(N+1)/2$ -dimensional vector of the entries of the lower (or upper) triangular part of \mathbf{A} .
- According to the notation previously introduced, $\text{vecs}(\mathbf{A}) \triangleq [a_{11}, \text{vecs}(\mathbf{A})^{\top}]^{\top}$.
- If $a_{11} = 0$, then \mathbf{M}_N is the $N(N+1)/2 - 1 \times N^2$ matrix such that (s.t.) $\mathbf{M}_N^{\top} \text{vecs}(\mathbf{A}) = \text{vec}(\mathbf{A})$. Note that \mathbf{M}_N^{\top} can be obtained from the *duplication matrix* \mathbf{D}_N [26,27] by removing its first column. Note that \mathbf{D}_N is implicitly defined as the unique $N^2 \times N(N+1)/2$ matrix satisfying $\mathbf{D}_N \text{vecs}(\mathbf{A}) = \text{vec}(\mathbf{A})$ for any symmetric matrix \mathbf{A} .

Statistical notation: Let x_l be a sequence of random variables in the same probability space. We write:

- $x_l = o_P(1)$ if $\lim_{l \rightarrow \infty} \Pr\{|x_l| \geq \epsilon\} = 0, \forall \epsilon > 0$ (*convergence in probability to 0*),
- $x_l = O_P(1)$ if for any $\epsilon > 0$, there exists a finite $M > 0$ and a finite $L > 0$, s.t. $\Pr\{|x_l| > M\} < \epsilon, \forall l > L$ (*stochastic boundedness*).

The cumulative distribution function (cdf) and the related probability density function (pdf) of a random variable x or a random vector \mathbf{x} are indicated as P_X and p_X , respectively. For random variables and vectors, $\stackrel{d}{=}$ stands for “has the same distribution as”. The symbol $\underset{L \rightarrow \infty}{\sim}$ indicates the convergence in distribution. According to the notation introduced in [5,6,28], we indicate the *true* pdf as $p_0(\mathbf{x}) \triangleq p_X(\mathbf{x}|\phi_0, g_0)$, where ϕ_0 and g_0 indicate the true parameter vector to be estimated and the true nuisance function, respectively. We define as $E_{\phi, g}\{f(\mathbf{x})\} = \int f(\mathbf{x})p_X(\mathbf{x}|\phi, g)d\mathbf{x}$ the expectation operator of a *measurable* function f of a random vector \mathbf{x} . Moreover, we simply indicate as $E_0\{\cdot\}$ the expectation with respect to (w.r.t.) the true pdf $p_0(\mathbf{x})$. The superscript $*$ indicates a \sqrt{L} -consistent, *preliminary*, estimator $\hat{\phi}^*$ of ϕ_0 , s.t. $\sqrt{L}(\hat{\phi}^* - \phi_0) = O_P(1)$. The dependence of \mathbf{x} of a function $f(\mathbf{x})$ is often dropped for notation simplicity: $f \equiv f(\mathbf{x})$.

II. THE SEMIPARAMETRIC SHAPE MATRIX ESTIMATION

Let $\{\mathbf{x}_l\}_{l=1}^L$ be a set of N -dimensional, real-valued, *independent* and *identically distributed* (i.i.d.) observation vectors. Each observation is assumed to be sampled from a real elliptical pdf [7,29,30] of the form:

$$p_X(\mathbf{x}_l|\boldsymbol{\mu}, \boldsymbol{\Sigma}, g) = 2^{-N/2} |\boldsymbol{\Sigma}|^{-1/2} g((\mathbf{x}_l - \boldsymbol{\mu})^{\top} \boldsymbol{\Sigma}^{-1} (\mathbf{x}_l - \boldsymbol{\mu})), \quad (1)$$

where $\boldsymbol{\mu} \in \mathbb{R}^N$ is a location vector, $\boldsymbol{\Sigma} \in \mathcal{M}_N^{\mathbb{R}}$ is a $N \times N$ *scatter* matrix in the set $\mathcal{M}_N^{\mathbb{R}}$ of the symmetric, positive definite, real matrices. The function $g \in \mathcal{G}$ is the *density generator*, an infinite-dimensional parameter that characterizes the specific distribution in the RES family. In order to guarantee the integrability of the pdf in Eq. (1), the set of all the possible density generators is defined as $\mathcal{G} = \{g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \mid \int_0^{\infty} t^{N/2-1} g(t) dt < \infty, \int p_X d\mathbf{x} = 1\}$ [29]. Each random vector whose pdf is given by Eq. (1), say $\mathbf{x} \sim RES_N(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, admits the following *stochastic representation* [7,29]:

$$\mathbf{x} \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R} \boldsymbol{\Sigma}^{1/2} \mathbf{u}, \quad (2)$$

where $\mathbf{u} \sim \mathcal{U}(S_{\mathbb{R}^N})$ is uniformly distributed on the unit sphere $S_{\mathbb{R}^N} \triangleq \{\mathbf{u} \in \mathbb{R}^N \mid \|\mathbf{u}\| = 1\}$, $\mathcal{R} \triangleq \sqrt{\mathcal{Q}}$ is called *modular variate* while \mathcal{Q} , usually referred to as *2nd-order modular variate*, is such that (s.t.)

$$\mathcal{Q} \stackrel{d}{=} (\mathbf{x}_l - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_l - \boldsymbol{\mu}) \triangleq Q_l, \forall l. \quad (3)$$

Moreover, \mathcal{Q} has pdf given by:

$$p_{\mathcal{Q}}(q) = (\pi/2)^{N/2} \Gamma(N/2)^{-1} q^{N/2-1} g(q), \quad (4)$$

where $\Gamma(\cdot)$ stands for the Gamma function.

The expression of the elliptical pdf in Eq. (1) and the stochastic representation in Eq. (2) are not uniquely defined due to the well-know scale ambiguity between the scatter matrix $\boldsymbol{\Sigma}$ and the density generator g . Specifically, from Eq. (1), it is immediate to verify that $RES_N(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g(t)) \equiv RES_N(\boldsymbol{\mu}, c\boldsymbol{\Sigma}, g(ct)), \forall c > 0$. In an equivalent way, from Eq. (2), we have that $\mathbf{x} \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R}\boldsymbol{\Sigma}^{1/2}\mathbf{u} \stackrel{d}{=} \boldsymbol{\mu} + (c^{-1}\mathcal{R})(c\boldsymbol{\Sigma}^{1/2})\mathbf{u}, \forall c > 0$. This readily implies that $\boldsymbol{\Sigma}$ is identifiable only up to a scale factor and consequently only a *scaled* version of $\boldsymbol{\Sigma}$ can be estimated. To avoid this identifiability problem, following [7,11,12], let us define the symmetric and positive definite *shape* matrix \mathbf{V} as:

$$\mathbf{V} = \boldsymbol{\Sigma}/s(\boldsymbol{\Sigma}), \quad (5)$$

where $s : \mathcal{M}_N^{\mathbb{R}} \rightarrow \mathbb{R}^+$ is a scalar functional on $\mathcal{M}_N^{\mathbb{R}}$ satisfying the following assumptions [11,12]:

- A1 Homogeneity: $s(c \cdot \boldsymbol{\Sigma}) = c \cdot s(\boldsymbol{\Sigma}), \forall c > 0$,
- A2 Differentiability over $\mathcal{M}_N^{\mathbb{R}}$ with $\frac{\partial s(\boldsymbol{\Sigma})}{\partial [\boldsymbol{\Sigma}]_{1,1}} \neq 0$,
- A3 $s(\mathbf{I}_N) = 1$.

Typical examples of this class of scale functional are $s(\boldsymbol{\Sigma}) = [\boldsymbol{\Sigma}]_{1,1}$, $s(\boldsymbol{\Sigma}) = \text{tr}(\boldsymbol{\Sigma})/N$ and $s(\boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|^{1/N}$. Each scale functional s corresponds to a differentiable constraint on the shape matrix \mathbf{V} . As an example, the constraints induced by the three above-mentioned scale functionals are $v_{11} = 1$, $\text{tr}(\mathbf{V}) = N$ and $|\mathbf{V}|^{1/N} = 1$. It is easy to verify that, under A1, A2 and A3, the first top-left entry of \mathbf{V} , i.e. v_{11} , can always be expressed as function of the other entries. This consideration, along with the fact that \mathbf{V} is symmetric by definition, suggests us that, to avoid the identifiability problem, in the semiparametric estimation problem, we just need to consider the vector $\text{vecs}(\mathbf{V})$ as unknown. Moreover, as discussed in [11] and verified here in Sec. V, the optimality properties of the proposed semiparametric estimator of the shape matrix do not depend on the particular scale functional. Consequently, in order to avoid tedious matrix calculation that may confuse the derivation of the algorithm, we choose the simple scale functional $s(\boldsymbol{\Sigma}) = [\boldsymbol{\Sigma}]_{1,1}$, i.e. the one that constrains the shape matrix \mathbf{V} to have its first top-left entry equal to 1. In the rest of the paper, a generic shape matrix satisfying this constraint is indicated as \mathbf{V}_1 according with the notation previously introduced.

Having said that, we can formally state the semiparametric estimation problem that we are going to analyze in the following sections. Let $\Omega \subseteq \mathbb{R}^q$ be a parameter space of dimension $q = N(N+3)/2 - 1 (= N + N(N+1)/2 - 1$ where

the “-1” term is due to the 1-dimensional scale constraint). Each element of Ω is a vector ϕ of the form:

$$\phi = (\boldsymbol{\mu}^\top, \text{vecs}(\mathbf{V}_1)^\top)^\top, \quad (6)$$

where $\boldsymbol{\mu} \in \mathbb{R}^N$ and $\mathbf{V}_1 \in \mathcal{M}_N^{\mathbb{R}}$. Let us define the RES semiparametric model as the following set of (uniquely defined) pdfs:

$$\mathcal{P}_{\phi,g} = \left\{ p_X | p_X(\mathbf{x} | \phi, g) = 2^{-N/2} |\mathbf{V}_1|^{-1/2} \times g((\mathbf{x}_l - \boldsymbol{\mu})^\top \mathbf{V}_1^{-1} (\mathbf{x}_l - \boldsymbol{\mu})) ; \phi \in \Omega, g \in \mathcal{G} \right\}. \quad (7)$$

The semiparametric estimation problem that we want to address is then to find a robust and semiparametric efficient estimator of a true parameter vector $\phi_0 \in \Omega$ in the presence of a nuisance function $g_0 \in \mathcal{G}$.

III. AN R -ESTIMATOR FOR SHAPE MATRICES IN RES DATA

The aim of this section is to trace the procedure adopted in [10] to derive the R -estimator of real-valued scatter matrices in RES data. In particular, the concepts of Le Cam’s one-step estimators and ranks-based robustification will be firstly introduced and their application to the particular semiparametric estimation problem at hand discussed. Finally, a ready-to-use expression of the resulting R -estimator is provided, while the related Matlab and Python implementation is given in [25].

A. Semiparametric efficient one-step estimators

The main ingredient for the derivation of a one-step estimator for the parametric part (location vector and scatter matrix) of the semiparametric RES model $\mathcal{P}_{\phi,g}$ in Eq. (7) is the notion of *efficient* score vector. Specifically, the efficient score vector $\bar{\mathbf{s}}_{\phi,g_0}$ for the estimation of $\phi \in \Omega$ in the presence of a nuisance density generator $g_0 \in \mathcal{G}$ is given by [28], [5, Th. IV.1]:

$$\bar{\mathbf{s}}_{\phi,g_0}(\mathbf{x}_l) \equiv \bar{\mathbf{s}}_{\phi,g_0} \triangleq \mathbf{s}_{\phi,g_0} - \Pi(\mathbf{s}_{\phi,g_0} | \mathcal{T}_{g_0}), \quad (8)$$

where $\mathbf{s}_{\phi,g_0}(\mathbf{x}_l)$ is the usual score vector defined as:

$$\mathbf{s}_{\phi,g_0}(\mathbf{x}_l) \triangleq \nabla_{\phi} \ln p_X(\mathbf{x}_l | \phi, g_0) = \begin{pmatrix} \mathbf{s}_{\boldsymbol{\mu},g_0}(\mathbf{x}_l) \\ \mathbf{s}_{\text{vecs}(\mathbf{V}_1),g_0}(\mathbf{x}_l) \end{pmatrix}, \quad (9)$$

and $\Pi(\mathbf{s}_{\phi,g_0} | \mathcal{T}_{g_0})$ is the orthogonal projection of the score vector \mathbf{s}_{ϕ,g_0} in Eq. (9) on the semiparametric nuisance tangent space \mathcal{T}_{g_0} [5,31]. Then, the semiparametric counterpart of the Fisher Information Matrix (FIM) is the *efficient* semiparametric FIM (SFIM) [28],[5, Th. IV.1]:

$$\bar{\mathbf{I}}(\phi | g_0) \triangleq E_{\phi,g_0} \{ \bar{\mathbf{s}}_{\phi,g_0}(\mathbf{x}) \bar{\mathbf{s}}_{\phi,g_0}(\mathbf{x})^\top \}. \quad (10)$$

Finally, we introduce the *efficient central sequence* as:

$$\bar{\Delta}_{\phi,g_0}(\mathbf{x}_1, \dots, \mathbf{x}_L) \equiv \bar{\Delta}_{\phi,g_0} \triangleq L^{-1/2} \sum_{l=1}^L \bar{\mathbf{s}}_{\phi,g_0}(\mathbf{x}_l). \quad (11)$$

Note that the previous three quantities depend on the true, and generally unknown, density generator g_0 .

The next Theorem provides us with the expression of the one-step estimator of ϕ together with its asymptotic properties.

Theorem 1. *Let $\{\mathbf{x}_l\}_{l=1}^L$ be a set of i.i.d. observations sampled from a RES distribution whose pdf $p_0(\mathbf{x}) \in \mathcal{P}_{\phi,g}$ in Eq. (7).*

Let $\hat{\phi}^*$ be any preliminary \sqrt{L} -consistent estimator of the true parameter vector $\phi_0 \triangleq (\boldsymbol{\mu}_0^\top, \text{vecs}(\mathbf{V}_{1,0})^\top)^\top$. Then, the semiparametric one-step estimator

$$\hat{\phi}_s = \hat{\phi}^* + L^{-1/2} \bar{\mathbf{I}}(\hat{\phi}^* | g_0)^{-1} \bar{\boldsymbol{\Delta}}_{\hat{\phi}^*, g_0}, \quad (12)$$

has the following properties:

PS1 \sqrt{L} -consistency

$$\sqrt{L} (\hat{\phi}_s - \phi_0) = O_P(1), \quad (13)$$

PS2 Asymptotic normality and efficiency

$$\sqrt{L} (\hat{\phi}_s - \phi_0) \underset{L \rightarrow \infty}{\sim} \mathcal{N}(\mathbf{0}, \bar{\mathbf{I}}(\phi_0 | g_0)^{-1}), \quad (14)$$

where $\bar{\mathbf{I}}(\phi_0 | g_0)^{-1} = \text{CSCRB}(\boldsymbol{\mu}_0, \mathbf{V}_{1,0} | g_0)$ and the constrained semiparametric CRB (CSCRB) [5] is evaluated for the constraint $[\mathbf{V}_{1,0}]_{11} = 1$.

Remark: The proof of Theorem 1 is given in [10] (see the proof of the Proposition 2.1). In addition, we refer the interested reader to our supporting material for a tutorial introduction of the Le Cam's theory underlying it.

Even if semiparametric efficient, the ‘‘clairvoyant’’ estimator $\hat{\phi}_s$ in Eq. (12) relies on the true, and generally unknown, density generator g_0 , so it is not useful for practical inference problems. Consequently, a distributionally robust alternative to $\hat{\phi}_s$ has to be derived, at the price of a possible loss in efficiency. Before addressing the crucial issue of robustness, we provide a ‘‘tangible’’ expression of the clairvoyant estimator of \mathbf{V}_1 that will be useful ahead.

B. Semiparametric clairvoyant estimator of shape matrices

To construct $\hat{\phi}_s$ in Eq. (12) we need explicit expressions of the efficient score vector $\bar{\mathbf{s}}_{\phi, g_0} = (\bar{\mathbf{s}}_{\boldsymbol{\mu}, g_0}^\top, \bar{\mathbf{s}}_{\text{vecs}(\mathbf{V}_1), g_0}^\top)^\top$, the efficient SFIM $\bar{\mathbf{I}}(\phi | g_0)$ and a preliminary \sqrt{L} -consistent estimator $\hat{\phi}^*$ of ϕ_0 . Building upon the results in our previous work [5], $\bar{\mathbf{s}}_{\boldsymbol{\mu}}$ and $\bar{\mathbf{s}}_{\text{vecs}(\mathbf{V}_1)}$ can be expressed as [5, Eq. (53)]:

$$\bar{\mathbf{s}}_{\boldsymbol{\mu}, g_0} = \mathbf{s}_{\boldsymbol{\mu}, g_0} = -2\sqrt{Q_l} \psi_0(Q_l) \mathbf{V}_1^{-1/2} \mathbf{u}_l, \quad (15)$$

$$\bar{\mathbf{s}}_{\text{vecs}(\mathbf{V}_1), g_0} = -Q_l \psi_0(Q_l) \mathbf{K}_{\mathbf{V}_1} \text{vec}(\mathbf{u}_l \mathbf{u}_l^\top), \quad (16)$$

where Q_l is defined in Eq. (3) and

$$\mathbf{K}_{\mathbf{V}_1} = \mathbf{M}_N \left(\mathbf{V}_1^{-1/2} \otimes \mathbf{V}_1^{-1/2} \right) \Pi_{\text{vecs}(\mathbf{I}_N)}^\perp, \quad (17)$$

$$\mathbf{u}_l = (Q_l \mathbf{V}_1)^{-1/2} (\mathbf{x}_l - \boldsymbol{\mu}), \quad (18)$$

$$\psi_0(t) = d \ln g_0(t) / dt, \quad (19)$$

$$\Pi_{\text{vecs}(\mathbf{I}_N)}^\perp = \mathbf{I}_{N^2} - N^{-1} \text{vec}(\mathbf{I}_N) \text{vec}(\mathbf{I}_N)^\top, \quad (20)$$

where \mathbf{M}_N is defined in the notation section. Before moving forward, some comments are in order. As already proved in [5], the efficient score vector $\bar{\mathbf{s}}_{\boldsymbol{\mu}, g_0}$ in Eq. (15) of the mean vector is equal to the score vector $\mathbf{s}_{\boldsymbol{\mu}, g_0}$, or in other words, $\bar{\mathbf{s}}_{\boldsymbol{\mu}, g_0}$ is orthogonal to the nuisance tangent space \mathcal{T}_{g_0} . This implies that, knowing or not knowing the true density generator g_0 does not have any impact on the asymptotic performance of an estimator of $\boldsymbol{\mu}$. The expression of the efficient score vector for the shape matrix in Eq. (16) of this

paper comes directly from Eq. (53) of [5]. Even if clearly related, the main difference between these two expressions is in the fact that, while in Eq. (53) of [5] the gradient is taken w.r.t. $\text{vecs}(\boldsymbol{\Sigma}_0)$ where $\boldsymbol{\Sigma}_0$ is the *unconstrained* scatter matrix, in this paper the gradient is taken w.r.t. $\text{vecs}(\mathbf{V}_1)$ where \mathbf{V}_1 is the *constrained* shape matrix s.t. $[\mathbf{V}_1]_{11} = 1$. This is the reason why we have the matrix \mathbf{M}_N instead of the duplication matrix \mathbf{D}_N as in Eq. (53) of [5]. Moreover, Eq. (16) follows from Eq. (53) of [5] through basic matrix algebra and the fact that $\text{tr}(\mathbf{u}_l \mathbf{u}_l^\top) = \|\mathbf{u}_l\|^2 = 1, \forall l$ and allows us to write a more compact expression for $\mathbf{s}_{\text{vecs}(\mathbf{V}_1), g_0}$.

The efficient SFIM $\bar{\mathbf{I}}(\phi | g_0)$ in Eq. (10) can be immediately obtained from the results in Eq. (15) and Eq. (16) and from the expression given in [5, Eq. (54)] as:

$$\begin{aligned} \bar{\mathbf{I}}(\phi | g_0) &\triangleq E_{\phi, g_0} \{ \bar{\mathbf{s}}_{\phi, g_0}(\mathbf{x}) \bar{\mathbf{s}}_{\phi, g_0}(\mathbf{x})^\top \} \\ &= \begin{pmatrix} \bar{\mathbf{I}}(\boldsymbol{\mu} | g_0) & \mathbf{0} \\ \mathbf{0}^\top & \bar{\mathbf{I}}(\text{vecs}(\mathbf{V}_1) | g_0) \end{pmatrix}, \end{aligned} \quad (21)$$

The block-diagonal structure of $\bar{\mathbf{I}}(\phi | g_0)$ in Eq. (21) implies that a lack of *a priori* knowledge about the mean vector $\boldsymbol{\mu}$ does not have any impact on the asymptotic performance of an estimator of the shape matrix \mathbf{V}_1 . In other words, the estimate of $\boldsymbol{\mu}$ and the one of \mathbf{V}_1 are asymptotically decorrelated. This and the above-mentioned fact that $\bar{\mathbf{s}}_{\boldsymbol{\mu}, g_0} \perp \mathcal{T}_{g_0}$ allow us to consider the estimation of $\boldsymbol{\mu}$ and the one of \mathbf{V}_1 as two separate problems. For this reason, from now on, we will focus our attention only on the estimation of \mathbf{V}_1 .

From Eq. (16) and building upon the expression already derived in Eq. (56) of [5], we have that:

$$\bar{\mathbf{I}}(\text{vecs}(\mathbf{V}_1) | g_0) = \alpha_0 \mathbf{K}_{\mathbf{V}_1} \mathbf{K}_{\mathbf{V}_1}^\top, \quad \text{where} \quad (22)$$

$$\alpha_0 \triangleq 2E\{\mathcal{Q}^2 \psi_0(\mathcal{Q})^2\} / N(N+2). \quad (23)$$

By substituting the expression of $\bar{\mathbf{s}}_{\text{vecs}(\mathbf{V}_1), g_0}$ given in Eq. (16) in the definition of the efficient central sequence in Eq. (11), we get:

$$\bar{\boldsymbol{\Delta}}_{\mathbf{V}_1, g_0} = -L^{-1/2} \mathbf{K}_{\mathbf{V}_1} \sum_{l=1}^L Q_l \psi_0(Q_l) \text{vec}(\mathbf{u}_l \mathbf{u}_l^\top). \quad (24)$$

Finally, we just need to put Eq. (24) and the expression of $\bar{\mathbf{I}}(\text{vecs}(\mathbf{V}_1) | g_0)$, given in Eq. (22), in the definition of one-step estimator in Eq. (12). This yields the following estimator:

$$\text{vecs}(\hat{\mathbf{V}}_{1,s}) = \text{vecs}(\hat{\mathbf{V}}_1^*) - \frac{1}{L\alpha_0} \left[\mathbf{K}_{\hat{\mathbf{V}}_1^*} \mathbf{K}_{\hat{\mathbf{V}}_1^*}^\top \right]^{-1} \times \quad (25)$$

$$\mathbf{K}_{\hat{\mathbf{V}}_1^*} \sum_{l=1}^L \hat{Q}_l^* \psi_0(\hat{Q}_l^*) \text{vec}(\hat{\mathbf{u}}_l^* (\hat{\mathbf{u}}_l^*)^\top),$$

where:

$$\hat{Q}_l^* \triangleq (\mathbf{x}_l - \hat{\boldsymbol{\mu}}^*)^\top [\hat{\mathbf{V}}_1^*]^{-1} (\mathbf{x}_l - \hat{\boldsymbol{\mu}}^*), \quad (26)$$

$$\hat{\mathbf{u}}_l^* \triangleq (\hat{Q}_l^*)^{-1/2} [\hat{\mathbf{V}}_1^*]^{-1/2} (\mathbf{x}_l - \hat{\boldsymbol{\mu}}^*), \quad (27)$$

while, as the notation suggests, the matrix $\mathbf{K}_{\hat{\mathbf{V}}_1^*}$ is obtained from $\mathbf{K}_{\mathbf{V}_1}$ in Eq. (17) by substituting \mathbf{V}_1 with its preliminary estimator $\hat{\mathbf{V}}_1^*$.

The last thing to do is to choose preliminary estimators for the mean vector and for the shape matrix. To this end, we can use the joint Tyler's shape and mean vector estimator [32, Eq. (6)], i.e. $\hat{\boldsymbol{\mu}}^* = \hat{\boldsymbol{\mu}}_{Ty}$ and $\hat{\mathbf{V}}_1^* = \hat{\mathbf{V}}_{1, Ty}$ with the constraint

$[\widehat{\mathbf{V}}_{1, Ty}]_{11} = 1$. This is a good choice since such $\hat{\phi}^*$ is \sqrt{L} -consistent under any possible density generator $g \in \mathcal{G}$.

As previously said, the clairvoyant estimators provided in Eq. (25) cannot be directly exploited for semiparametric inference since it still depends on the true density generator g_0 from two different standpoints:

- i) *Statistical dependence*: The estimator $\widehat{\mathbf{V}}_{1,s}$ in Eq. (25) relies on the random variables $\{\hat{Q}_l^*\}_{l=1}^L$ whose pdf depends on g_0 through the one of the data $\{\mathbf{x}_l\}_{l=1}^L$ (see Eq. (26)).
- ii) *Functional dependence*: The scalar α_0 in Eq. (23) is function of $E\{Q^2\psi_0(Q)^2\}$ that depends on g_0 through the function ψ_0 in Eq. (19) and the pdf of Q in Eq. (4).

In [10], Hallin, Oja and Paindaveine showed that rank-based statistics can be exploited to overcome the above-mentioned dependences and obtain a distributionally robust estimator of the shape matrix able to dispense with the knowledge of g_0 . However, to fully understand the theory underlying the outcomes of [10], a strong knowledge of the Le Cam theory and of its *invariance-based* extension to semiparametric framework [33] is required. The aim of the following subsections is then to supply any SP practitioner with a “ready-to-use” formulation of the resulting R -estimator. Anyway, the interested reader can find additional tutorial-style discussions about the semiparametric extension of the Le Cam’s theory in the supporting material of this paper.

C. Preliminaries on rank-based statistics

Let $\{x_l\}_{l=1}^L$ be a set of L continuous i.i.d. random variables s.t. $x_l \sim p_X, \forall l$. We define the vector of the *order statistics* as $\mathbf{v}_X \triangleq [x_{L(1)}, x_{L(2)}, \dots, x_{L(L)}]^\top$ whose entries $x_{L(1)} < x_{L(2)} < \dots < x_{L(L)}$ are the values of $\{x_l\}_{l=1}^L$ ordered in an ascending way.¹ Then, the rank $r_l \in \mathbb{N}/\{0\}$ of x_l is the position index of x_l in \mathbf{v}_X . Finally, we define $\mathbf{r}_X \triangleq [r_1, \dots, r_L]^\top \in \mathbb{N}^L$ as the vector collecting the ranks.

Lemma 1. *Let \mathcal{K} be the family of score functions ² $K : (0, 1) \rightarrow \mathbb{R}^+$ that are continuous, square integrable and that can be expressed as the difference of two monotone increasing functions. Then, we have:*

- 1) *The vectors \mathbf{v}_X and \mathbf{r}_X are independent,*
- 2) *Regardless the actual pdf p_X , the rank vector \mathbf{r}_X is uniformly distributed on the set of all $L!$ permutations on $\{1, 2, \dots, L\}$ and $!$ stands for the factorial notation,*
- 3) *For each $l = 1, \dots, L$, we have that $K\left(\frac{r_l}{L+1}\right) = K(u_l) + o_P(1)$ where $K \in \mathcal{K}$ and $u_l \sim \mathcal{U}[0, 1]$ is a random variable uniformly distributed in $(0, 1)$.*

Remark: The proof can be found in [22], [23, Ch. 13].

To understand why Lemma 1 is useful to derive a distributionally robust and semiparametric efficient estimator of the shape matrix we should take a step back.

¹Note that, since $x_l, \forall l$ are continuous random variable the equality occurs with probability 0.

²Even if this can create some ambiguity, we decide to indicate the elements in \mathcal{K} as “score functions” in order to maintain the consistency with the terminology used in classical references about ranks.

D. Robust approximations of $\overline{\Delta}_{\mathbf{v}_1, g_0}$ and $\bar{\mathbf{I}}(\text{vecs}(\mathbf{V}_1)|g_0)$

From the stochastic representation in Eq. (2), there is a one-to-one correspondence between a RES distributed observation vector $\mathbf{x}_l \sim RES_N(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_0)$ and the couple (Q_l, \mathbf{u}_l) , where $Q_l \stackrel{d}{=} Q$ is defined in Eq. (3) and whose pdf p_Q is given in Eq. (4), while $\mathbf{u} \sim \mathcal{U}(S_{\mathbb{R}^N})$. Then, Point 2) in the Lemma 1 tells us that the distribution of \mathbf{r}_Q is invariant w.r.t. the pdf p_Q in Eq. (4) that depends on the actual, and generally unknown, density generator $g_0 \in \mathcal{G}$. This feature is very attractive for robust inference since it allows us to derive rank-based (or R -) estimators and tests that are distributionally robust. Point 3) of Lemma 1 provides us with the missing piece to obtain a distributionally robust approximation of the efficient central sequence $\overline{\Delta}_{\mathbf{v}_1, g_0}$. Specifically, let

$$P_{Q,0}(q) = (\pi/2)^{N/2} \Gamma(N/2)^{-1} \int_0^q t^{N/2-1} g_0(t) dt \quad (28)$$

be the true, and generally unknown, cdf of 2nd-order modular variates whose pdf is given in Eq. (4). Let us now recall the basic fact that (see e.g. [34, Th. 2.1.10])

$$P_{Q,0}^{-1}(u_l) = Q_l, \quad u_l \sim \mathcal{U}[0, 1], \quad Q_l \sim P_{Q,0} \quad \forall l \quad (29)$$

where $P_{Q,0}^{-1}$ indicates the inverse function of the cdf. Finally, we have to introduce the “true” score function

$$K_0(u) = -P_{Q,0}^{-1}(u) \psi_0(P_{Q,0}^{-1}(u)), \quad u \in (0, 1), \quad (30)$$

that can be shown to belong to the set \mathcal{K} [35]. Note that K_0 depends on the true density generator g_0 through ψ_0 in Eq. (19) and $P_{Q,0}$ in Eq. (28). From Point 3) of Lemma 1 and by using the relation Eq. (29) we have

$$K_0\left(\frac{r_l}{L+1}\right) = -Q_l \psi_0(Q_l) + o_P(1). \quad (31)$$

Consequently, substituting Eq. (31) in Eq. (16) yields to the following approximation of the efficient central sequence in Eq. (24):

$$\overline{\Delta}_{\mathbf{v}_1, g_0} = \frac{1}{\sqrt{L}} \mathbf{K}_{\mathbf{v}_1} \sum_{l=1}^L K_0\left(\frac{r_l}{L+1}\right) \text{vec}(\mathbf{u}_l \mathbf{u}_l^\top) + o_P(1). \quad (32)$$

The expression in Eq. (32) depends “statistically” only on the ranks r_l and on the random vectors \mathbf{u}_l whose distributions are invariant w.r.t. the actual RES distribution of the data. However, we still have a functional dependence from g_0 due to the score function K_0 . To get rid of this dependence, we may adopt a “misspecified approach” [36]: *since we do not know which is the true density generator g_0 , let us build the score function K_g by substituting in Eq. (30) a, possibly misspecified, $g \in \mathcal{G}$ instead of the unknown g_0 .* Consequently, by substituting \mathbf{V}_1 with a consistent preliminary estimator $\widehat{\mathbf{V}}_1^*$, a distributionally robust approximation of the efficient central sequence $\overline{\Delta}_{\mathbf{v}_1}$ in Eq. (24) can be obtained as:

$$\widetilde{\Delta}_{\widehat{\mathbf{v}}_1^*} \triangleq \frac{1}{\sqrt{L}} \mathbf{K}_{\widehat{\mathbf{v}}_1^*} \sum_{l=1}^L K_g\left(\frac{r_l^*}{L+1}\right) \text{vec}(\hat{\mathbf{u}}_l^* (\hat{\mathbf{u}}_l^*)^\top), \quad (33)$$

where r_l^* is the rank of \hat{Q}_l^* already defined in Eq. (26) and $\hat{\mathbf{u}}_l^*$ is given in Eq. (27). As a useful example of score function

K_g , we may cite the *van der Waerden* score function K_{vdW} . Specifically, K_{vdW} is obtained by assuming a, possibly misspecified, Gaussian distribution for the acquired data. Since, under Gaussianity, the density generator is $g_G(t) = \exp(-t/2)$ and \mathcal{Q} in Eq. (3) is distributed as a χ -squared random variable with N degrees of freedom, i.e. $\mathcal{Q} \sim \chi^2(N)$, from Eq. (30) we have:

$$K_{vdW}(u) = \Psi^{-1}(u)/2, \quad u \in (0, 1), \quad (34)$$

where $\Psi(u)$ indicates the cdf of $\chi^2(N)$. On the same line, if we assume a t -distribution for the collected data, we obtain the score function:

$$K_{t_\nu}(u) = \frac{N(N + \nu)F_{N,\nu}^{-1}(u)}{2(\nu + NF_{N,\nu}^{-1}(u))}, \quad u \in (0, 1), \quad (35)$$

where $F_{N,\nu}(u)$ stands for the cdf of a Fisher random variable with N and $\nu \in (0, \infty)$ degrees of freedom, i.e. $F_{N,\nu}$. In particular, the expression of K_{t_ν} comes from the fact that, under an assumed t -distribution, the density generator is $g_{t_\nu}(t) = (1 + t/\nu)^{-(\nu+N)/2}$ while $\mathcal{Q}/N \sim F_{N,\nu}$ [30, Ex. 2.5]. Note that, from the properties of the F -distribution [37, Ch. 27], it follows that $\lim_{\nu \rightarrow \infty} K_{t_\nu}(u) = K_{vdW}(u)$. This is not surprising since it is well known that the t -distribution collapses into the Gaussian one as $\nu \rightarrow \infty$. We note, that other possible score function may be built upon the loss functions discussed in [38].

As expected, a misspecification of the density generator will bring to a loss in semiparametric efficiency. Remarkably, as we will see in Sec. V, such performance loss are small, especially if the Gaussian *van der Waerden* score is adopted. A theoretical justification of this surprisingly small loss of efficiency may be related to the so-called ‘‘Chernoff-Savage result’’ for non-parametric R -tests [39]. Some preliminary investigation towards this direction have been provided in [35], but a comprehensive and in-depth analysis of this phenomenon is still missing. Even if of crucial importance, this aspect falls outside the aims of this paper and it is left to future works.

Let us now focus on the efficient SFIM in Eq. (22). In [10], it is proved that $\bar{\mathbf{I}}(\text{vecs}(\mathbf{V}_1)|g_0)$ can be approximated as:

$$\bar{\mathbf{I}}(\text{vecs}(\mathbf{V}_1)|g_0) = \hat{\alpha} \mathbf{K}_{\hat{\mathbf{V}}_1^*} \mathbf{K}_{\hat{\mathbf{V}}_1^*}^\top + o_P(1), \quad (36)$$

where $\hat{\alpha}$ is a *consistent* estimator of α_0 in Eq. (23). In particular, in [10, Sec. 4] it is shown that a possible candidate for $\hat{\alpha}$ is:

$$\hat{\alpha} = \|\bar{\Delta}_{\hat{\mathbf{V}}_1^* + L^{-1/2}\mathbf{H}^0} - \bar{\Delta}_{\hat{\mathbf{V}}_1^*}\| / \|\mathbf{K}_{\hat{\mathbf{V}}_1^*} \mathbf{K}_{\hat{\mathbf{V}}_1^*}^\top \text{vecs}(\mathbf{H}^0)\|, \quad (37)$$

where \mathbf{H}^0 may be any symmetric matrix whose first top-left entry is equal to 0, i.e. $[\mathbf{H}^0]_{1,1} = 0$. Therefore, the consistent estimator $\hat{\alpha}$ depends on this ‘‘small perturbation’’ matrix \mathbf{H}^0 that can be considered as an hyper-parameter to be defined by the user. Some consideration on the choice of \mathbf{H}^0 will be provided in Sec. V-C where a numerical analysis of the performance of the proposed shape matrix estimator is presented. Note that the estimator $\hat{\alpha}$ in Eq. (37) is only an example of a possible estimator for α_0 , but other procedures may be adopted as well. In [10, Sec. 4.2] for example, an ML-based approach is implemented to derive

a consistent and efficient estimator for α_0 . However, such ML-based estimator requires the solution of an optimization problem that may become computationally heavy as the matrix dimension increases.

We conclude this subsection with an important remark on the distributional robustness of $\bar{\Delta}_{\hat{\mathbf{V}}_1^*}$ in Eq. (33) and of the approximation of the SFIM given in Eq. (36). These two terms, needed to build a robust version of the R -estimator in Eq. (25), depend on four random quantities: the preliminary estimator $\hat{\mathbf{V}}_1^*$, the ranks r_l^* , the vectors $\hat{\mathbf{u}}_l^*$ and $\hat{\alpha}$. If, as consistent preliminary estimator, we use a distribution-free estimator as the Tyler’s one, it can be easily shown that r_l^* and $\hat{\mathbf{u}}_l^*$ are distribution-free as well. This implies that the ‘‘approximated’’ central sequence $\bar{\Delta}_{\hat{\mathbf{V}}_1^*}$ is itself distribution-free [10, Prop. 2.1]. This is not the case for the estimator $\hat{\alpha}$ in Eq. (37). In fact, even if $\bar{\Delta}_{\hat{\mathbf{V}}_1^*}$ is distribution-free, this is not true for its ‘‘perturbed’’ version $\bar{\Delta}_{\hat{\mathbf{V}}_1^* + L^{-1/2}\mathbf{H}^0}$ as proved in [10, Prop. 2.1, Point (iv)]. Consequently, the resulting R -estimator will not be fully distribution-free. However, it still remain distributionally robust, since $\hat{\alpha}$ is proven to be a consistent estimator of α_0 for every possible density generator $g \in \mathcal{G}$ [10, Sec. 4].

E. The final expression for the real-valued R -estimator

The desired R -estimator of real-valued shape matrices in RES distributed data can then be obtained from the the expression of the semiparametric one-step estimator in Theorem 1 by replacing the efficient central sequence $\bar{\Delta}_{\hat{\phi}^*, g_0}$ and the efficient SFIM $\bar{\mathbf{I}}(\text{vecs}(\mathbf{V}_1)|g_0)$ with their approximations provided in Eqs. (33) and (36), respectively. In particular, a distributionally robust, one-step estimator of \mathbf{V}_1 is given by:

$$\text{vecs}(\hat{\mathbf{V}}_{1,R}) = \text{vecs}(\hat{\mathbf{V}}_1^*) + \frac{1}{L\hat{\alpha}} \left[\mathbf{K}_{\hat{\mathbf{V}}_1^*} \mathbf{K}_{\hat{\mathbf{V}}_1^*}^\top \right]^{-1} \times \mathbf{K}_{\hat{\mathbf{V}}_1^*} \sum_{l=1}^L K_g \left(\frac{r_l^*}{L+1} \right) \text{vec}(\hat{\mathbf{u}}_l^* (\hat{\mathbf{u}}_l^*)^\top), \quad (38)$$

where $\{r_l^*\}_{l=1}^L$ are the ranks of the random variables $\{\hat{Q}_l^*\}_{l=1}^L$ defined in Eq. (26), while $\hat{\mathbf{u}}_l^*$ is defined in Eq. (27). Again, as preliminary estimator of the (constrained) shape matrix we may use the Tyler’s estimator $\hat{\mathbf{V}}_1^* = \hat{\mathbf{V}}_{1,Ty}$.

Before moving on, one last comment is in order. It is immediate to verify from the expressions of $\hat{\mathbf{V}}_{1,R}$ and $\hat{\alpha}$, given in Eqs. (38) and (37) respectively, that the R -estimator, as function of the score K_g , satisfies the following homogeneity property: $\hat{\mathbf{V}}_{1,R}(cK_g) = \hat{\mathbf{V}}_{1,R}(K_g)$ for every positive scalar $c > 0$. However, if a different estimator of α_0 is adopted, this may not be the case and the score should be normalized, e.g. as $\int_0^1 K_g(u) = N$ [40, Assumption S3].

IV. EXTENSION TO COMPLEX ES DISTRIBUTIONS

Building upon the previously obtained results, this section aims at providing an extension of the R -estimator in Eq. (38) to the complex-valued shape matrix estimation problem in CES-distributed data. As already shown in [7], [18, Ch. 4] and [6, Def. II.1], there exists a one-to-one mapping between the set of the CES distributions and a subset of the RES ones. This implies that the theory already developed for the real-valued

case can be applied straight to complex-valued data. However, the use of a real representation of complex quantities usually leads to a loss in the clarity and even in the “interpretability” of the results. This is because the entries of the complex parameter vector are “scrambled” by the $\mathbb{C} \rightarrow \mathbb{R}^2$ mapping and the analysis of the statistical properties of the resulting real version of the estimator may be quite cumbersome. This problem is even more serious when we have to estimate a complex matrix where, in addition to the “scrambling” of the real and imaginary parts due to the $\mathbb{C} \rightarrow \mathbb{R}^2$ mapping, we must take care of the row-column ordering. Having a mathematical tool that allows us to operate directly in the complex field enables us to represent the entries of the parameter vector/matrix in a compact way gaining a lot in terms of both interpretability and feasibility of the obtained estimator. Best practice is then to use the Wirtinger calculus [41]–[44]. Basically, the Wirtinger calculus generalizes the concept of complex derivative to non-holomorphic, real-valued functions of complex variables. In our recent paper [6], the Wirtinger calculus has been exploited to derive the SCRb for the joint estimation of the complex-valued location vector and scatter matrix of a set of CES distributed data. In particular, the complex-valued counterparts of the efficient score vector and of the SFIM for shape matrices in CES data have been evaluated in [6]. As for the real-valued case, these two quantities are the basic ingredients to derive a complex version of the R -estimator in Eq. (38). Note that, due to the strong similarity between the properties of the CES and RES distributed random vectors, in the following we will mostly reuse the same notation introduced in Section II for the corresponding entities.

A. CES distributed data: a recall

Let $\{\mathbf{z}_l\}_{l=1}^L \in \mathbb{C}^N$ be a set of complex i.i.d. observation vectors. Let $\mathcal{G}_{\mathbb{C}}$ be the following set of functions $\mathcal{G}_{\mathbb{C}} = \{h : \mathbb{R}^+ \rightarrow \mathbb{R}^+ | \int_0^\infty t^{N-1} h(t) dt < \infty, \int p_Z d\mathbf{z} = 1\}$ [7]. Moreover, we indicate with $\mathcal{M}_{\mathbb{C}}^N$ the set of the Hermitian, positive definite, $N \times N$ complex matrices.

Any CES-distributed random vector $\mathbf{z}_l = \mathbf{x}_{R,l} + j\mathbf{x}_{I,l} \sim CES(\boldsymbol{\mu}, \boldsymbol{\Sigma}, h)$ satisfies the properties [7],[6, Sec. II]:

- $\mathbf{z}_l \in \mathbb{C}^N$ is CES distributed iff $[\mathbf{x}_{R,l}^\top, \mathbf{x}_{I,l}^\top]^\top \in \mathbb{R}^{2N}$ has a $2N$ -variate RES distribution,
- Its pdf p_Z is fully specified by the location vector $\boldsymbol{\mu} \in \mathbb{C}^N$, by the scatter matrix $\boldsymbol{\Sigma} \in \mathcal{M}_{\mathbb{C}}^N$ and by the density generator $h \in \mathcal{G}_{\mathbb{C}}$ and it can be expressed as:

$$p_Z(\mathbf{z}_l | \boldsymbol{\mu}, \boldsymbol{\Sigma}, h) = |\boldsymbol{\Sigma}|^{-1} h((\mathbf{z}_l - \boldsymbol{\mu})^H \boldsymbol{\Sigma}^{-1} (\mathbf{z}_l - \boldsymbol{\mu})). \quad (39)$$

- *Stochastic representation*: $\mathbf{z}_l \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R}\boldsymbol{\Sigma}^{1/2}\mathbf{u}$, where \mathcal{R} is the *modular variate* and $\mathbf{u} \sim \mathcal{U}(S_{\mathbb{C}^N})$ is uniformly distributed on $S_{\mathbb{C}^N} \triangleq \{\mathbf{u} \in \mathbb{C}^N | \|\mathbf{u}\| = 1\}$.
- The *2nd-order modular variate* $\mathcal{Q} \triangleq \mathcal{R}^2$ is s.t.

$$\mathcal{Q} \stackrel{d}{=} (\mathbf{z}_l - \boldsymbol{\mu})^H \boldsymbol{\Sigma}^{-1} (\mathbf{z}_l - \boldsymbol{\mu}) \triangleq \mathcal{Q}_l, \forall l, \quad (40)$$

and it admits a pdf $p_{\mathcal{Q}}$ of the form:

$$p_{\mathcal{Q}}(q) = \pi^N \Gamma(N)^{-1} q^{N-1} h(q). \quad (41)$$

Exactly as for the real-valued case, the complex scatter matrix $\boldsymbol{\Sigma}$ is not identifiable and only a scaled version of it

can be estimated. Then, the shape matrix $\mathbf{V} \triangleq \boldsymbol{\Sigma}/s(\boldsymbol{\Sigma})$ has to be introduced, where $s(\cdot)$ is a scalar functional on $\mathcal{M}_{\mathbb{C}}^N$ satisfying conditions A1, A2 and A3 given in Sec. II. As for the real case, among all the possible scale functionals, we choose $s(\boldsymbol{\Sigma}) = [\boldsymbol{\Sigma}]_{1,1}$ for simplicity.

At first, we need to define the unknown complex-valued parameter vector $\boldsymbol{\phi}$ to be estimated. As shown in [6] and in analogy with the real-valued case, the estimation of the mean vector and of the shape matrix are asymptotically decorrelated. Consequently, we focus only of the shape matrix estimation from the “centered” data set $\{\mathbf{z}_l - \hat{\boldsymbol{\mu}}^*\}_{l=1}^L$, where $\hat{\boldsymbol{\mu}}^*$ is any \sqrt{L} -consistent estimator of $\boldsymbol{\mu} \in \mathbb{C}^N$. The interested reader may find additional considerations of the joint estimation of $\boldsymbol{\mu}$ and \mathbf{V}_1 in [45]. According to the basics of the Wirtinger calculus, $\boldsymbol{\phi}$ has to be constructed stacking in a single vector the unknown parameters and their complex conjugate [41,44]. Then, according to the detailed discussion provided in [6, Sec. III.A], we have that $\boldsymbol{\phi} = \text{vec}(\mathbf{V}_1)$.

As shown in Theorem 1, the basic building blocks for a semiparametric efficient estimators are the semiparametric efficient score vector $\bar{\mathbf{s}}_{\boldsymbol{\phi}, h_0} \equiv \bar{\mathbf{s}}_{\text{vec}(\mathbf{V}_1), h_0}$ and the efficient SFIM $\bar{\mathbf{I}}(\text{vec}(\mathbf{V}_1)|h_0)$. Both $\bar{\mathbf{s}}_{\text{vec}(\mathbf{V}_1), h_0}$ and $\bar{\mathbf{I}}(\text{vec}(\mathbf{V}_1)|h_0)$ have been already introduced in full details in our previous work [6] and their expressions are recalled here for clarity. Let us start by defining the following matrices:

$$\mathbf{P} = [\mathbf{e}_2 | \mathbf{e}_3 | \cdots | \mathbf{e}_{N^2}], \quad (42)$$

where \mathbf{e}_i is the i -th vector of the canonical basis of \mathbb{R}^{N^2} ,

$$\mathbf{L}_{\mathbf{V}_1} = \mathbf{P} \left(\mathbf{V}_1^{-T/2} \otimes \mathbf{V}_1^{-1/2} \right) \Pi_{\text{vec}(\mathbf{I}_N)}^\perp, \quad (43)$$

and $\Pi_{\text{vec}(\mathbf{I}_N)}^\perp$ has already been defined in Eq. (20). Then, from the calculation in [6, Sec. III.B],³ using some matrix algebra, we obtain the following expression for the complex efficient semiparametric score vector

$$\bar{\mathbf{s}}_{\text{vec}(\mathbf{V}_1), h_0} = -\mathcal{Q}_l \psi_0(\mathcal{Q}_l) \mathbf{L}_{\mathbf{V}_1} \text{vec}(\mathbf{u}_l \mathbf{u}_l^H), \quad (44)$$

where $\psi_0(t) = d \ln h_0(t) / dt$, $\mathbf{u}_l \triangleq (\mathcal{Q}_l \mathbf{V}_1)^{-1/2} (\mathbf{z}_l - \boldsymbol{\mu})$ and \mathcal{Q}_l has been defined in Eq. (40). Note that the function ψ_0 here is defined by means of the true density generator h_0 related to the CES pdf in Eq. (39). Moreover, from [6, Eq. (29)]:

$$\bar{\mathbf{I}}(\text{vec}(\mathbf{V}_1)|h_0) = \alpha_{\mathbb{C},0} \mathbf{L}_{\mathbf{V}_1} \mathbf{L}_{\mathbf{V}_1}^H, \quad \text{where} \quad (45)$$

$$\alpha_{\mathbb{C},0} \triangleq E\{\mathcal{Q}^2 \psi_0(\mathcal{Q})^2\} / N(N+1). \quad (46)$$

It is worth to underline that the matrix \mathbf{P} in Eq. (42) has been introduced in order to take into account the fact that the first top-left entry of \mathbf{V}_1 is equal to 1, i.e. $[\mathbf{V}_1]_{1,1} = 1$, and it does not have to be estimated.

B. An R -estimator for shape matrices in CES data

The derivation of the complex-valued R -estimator mimics the one proposed in Section III for the real case. In particu-

³Note that in [6, Eq. (25)] there is a typo. In fact, a minus “-” is missing in front of the right-hand side.

lar, an approximation of the complex-valued efficient central sequence can be obtained as:

$$\tilde{\Delta}_{\hat{\mathbf{V}}_1^*}^{\mathbb{C}} \triangleq \frac{1}{\sqrt{L}} \mathbf{L}_{\hat{\mathbf{V}}_1^*} \sum_{l=1}^L K_h \left(\frac{r_l^*}{L+1} \right) \text{vec}(\hat{\mathbf{u}}_l^* (\hat{\mathbf{u}}_l^*)^H), \quad (47)$$

where $\hat{\mathbf{V}}_1^*$ is any \sqrt{L} -consistent estimator of the (complex-valued) shape matrix and r_l^* is the rank of \hat{Q}_l^* defined, in analogy with Eq. (26), as

$$\hat{Q}_l^* \triangleq (\mathbf{z}_l - \hat{\boldsymbol{\mu}}^*)^H [\hat{\mathbf{V}}_1^*]^{-1} (\mathbf{z}_l - \hat{\boldsymbol{\mu}}^*), \quad (48)$$

$$\hat{\mathbf{u}}_l^* \triangleq (\hat{Q}_l^*)^{-1/2} [\hat{\mathbf{V}}_1^*]^{-1/2} (\mathbf{z}_l - \hat{\boldsymbol{\mu}}^*). \quad (49)$$

Moreover, the score function $K_h(\cdot)$ is the ‘‘complex’’ counterpart of the one defined in Eq. (30). Specifically, $K_h(\cdot)$ can be obtained from the expression Eq. (30) by evaluating $P_{\mathcal{Q}}^{-1}$ and ψ_0 by means of an assumed, and possibly misspecified, $h \in \mathcal{G}_{\mathbb{C}}$ instead of its real counterpart $g \in \mathcal{G}$. For example, the ‘‘complex version’’ of the *van der Waerden* score function in Eq. (34) can be obtained from Eq. (30) by noticing that the complex circular Gaussian distribution has a density generator given by $h_{\mathbb{C}G}(t) = \exp(-t)$ while $\mathcal{Q} \sim \text{Gamma}(N, 1)$ [7]. Then, the ‘‘complex’’ *van der Waerden* score function is:

$$K_{\mathbb{C}vdW}(u) = \Phi_G^{-1}(u), \quad u \in (0, 1), \quad (50)$$

where Φ_G indicates the cdf of a Gamma-distributed random variable with parameters $(N, 1)$. Similarly, the ‘‘complex version’’ of the t_{ν} -score in Eq. (35) is given by:

$$K_{\mathbb{C}t_{\nu}}(u) = \frac{N(2N + \nu)F_{2N, \nu}^{-1}(u)}{\nu + 2NF_{2N, \nu}^{-1}(u)}, \quad u \in (0, 1), \quad (51)$$

where, as in Eq. (35), $F_{2N, \nu}(u)$ stands for the Fisher cdf with $2N$ and $\nu \in (0, \infty)$ degrees of freedom, where we used the fact that $h_{\mathbb{C}G}(t) = (1 + 2t/\nu)^{-(2N+\nu)/2}$ and $\mathcal{Q}/N \sim F_{2N, \nu}$ [7]. We note that, as for the real case previously discussed, we have that $\lim_{\nu \rightarrow \infty} K_{\mathbb{C}t_{\nu}}(u) = K_{\mathbb{C}vdW}(u)$. The complex-valued approximation of the efficient SFIM in Eq. (45) can be obtained as:

$$\bar{\mathbf{I}}(\text{vec}(\mathbf{V}_1)|h_0) = \hat{\alpha}_{\mathbb{C}} \mathbf{L}_{\hat{\mathbf{V}}_1^*} \mathbf{L}_{\hat{\mathbf{V}}_1^*}^H + o_P(1), \quad \text{where} \quad (52)$$

$$\hat{\alpha}_{\mathbb{C}} = \|\tilde{\Delta}_{\hat{\mathbf{V}}_1^* + L^{-1/2} \mathbf{H}_{\mathbb{C}}^0} - \tilde{\Delta}_{\hat{\mathbf{V}}_1^*}^{\mathbb{C}}\| / \|\mathbf{L}_{\hat{\mathbf{V}}_1^*} \mathbf{L}_{\hat{\mathbf{V}}_1^*}^H \text{vec}(\mathbf{H}_{\mathbb{C}}^0)\|, \quad (53)$$

and $\mathbf{H}_{\mathbb{C}}^0$ is a ‘‘small perturbation’’, Hermitian, matrix s. t. $[\mathbf{H}_{\mathbb{C}}^0]_{1,1} = 0$. Finally, putting together the previous results, the complex extension of the distributionally robust, one-step estimator in Eq. (38) can be obtained as:

$$\boxed{\text{vec}(\hat{\mathbf{V}}_{1,R}) = \text{vec}(\hat{\mathbf{V}}_1^*) + \frac{1}{L\hat{\alpha}_{\mathbb{C}}} \left[\mathbf{L}_{\hat{\mathbf{V}}_1^*} \mathbf{L}_{\hat{\mathbf{V}}_1^*}^H \right]^{-1} \times \mathbf{L}_{\hat{\mathbf{V}}_1^*} \sum_{l=1}^L K_h \left(\frac{r_l^*}{L+1} \right) \text{vec}(\hat{\mathbf{u}}_l^* (\hat{\mathbf{u}}_l^*)^H)}. \quad (54)}$$

In the following, the pseudocode to implement the proposed R -estimator is reported, while its related Matlab and Python code can be found at [25]. A good preliminary estimator of the constrained, complex-valued shape matrix, may be Tyler’s estimator $\hat{\mathbf{V}}_1^* = \hat{\mathbf{V}}_{1,Ty}$.

Algorithm 1 Semiparametric efficient R -estimator for \mathbf{V}_1

Input: $\mathbf{z}_1, \dots, \mathbf{z}_L; \hat{\boldsymbol{\mu}}^*; \hat{\mathbf{V}}_1^*; K_h(\cdot); \mathbf{H}_{\mathbb{C}}^0$.

Output: $\hat{\mathbf{V}}_{1,R}$.

- 1: **for** $l = 1$ to L **do**
 - 2: $\hat{Q}_l^* \leftarrow (\mathbf{z}_l - \hat{\boldsymbol{\mu}}^*)^H [\hat{\mathbf{V}}_1^*]^{-1} (\mathbf{z}_l - \hat{\boldsymbol{\mu}}^*)$,
 - 3: $\hat{\mathbf{u}}_l^* \leftarrow (\hat{Q}_l^*)^{-1/2} [\hat{\mathbf{V}}_1^*]^{-1/2} (\mathbf{z}_l - \hat{\boldsymbol{\mu}}^*)$,
 - 4: **end for**
 - 5: Evaluate the ranks $\{r_1^*, \dots, r_L^*\}$ of $\{\hat{Q}_1^*, \dots, \hat{Q}_L^*\}$,
 - 6: $\mathbf{L}_{\hat{\mathbf{V}}_1^*} \leftarrow \mathbf{P}([\hat{\mathbf{V}}_1^*]^{-T/2} \otimes [\hat{\mathbf{V}}_1^*]^{-1/2}) \Pi_{\text{vec}(\mathbf{I}_N)}^{\perp}$,
 - 7: $\tilde{\Delta}_{\hat{\mathbf{V}}_1^*}^{\mathbb{C}} \leftarrow L^{-1/2} \mathbf{L}_{\hat{\mathbf{V}}_1^*} \sum_{l=1}^L K_h \left(\frac{r_l^*}{L+1} \right) \text{vec}(\hat{\mathbf{u}}_l^* (\hat{\mathbf{u}}_l^*)^H)$,
 - 8: Evaluate $\tilde{\Delta}_{\hat{\mathbf{V}}_1^* + L^{-1/2} \mathbf{H}_{\mathbb{C}}^0}^{\mathbb{C}}$ following step 7 with $\hat{\mathbf{V}}_1^* \leftarrow \hat{\mathbf{V}}_1^* + L^{-1/2} \mathbf{H}_{\mathbb{C}}^0$,
 - 9: Evaluate $\hat{\alpha}_{\mathbb{C}}$ as in Eq. (53).
 - 10: $\text{vec}(\hat{\mathbf{V}}_{1,R}) \leftarrow \text{vec}(\hat{\mathbf{V}}_1^*) + L^{-1/2} [\hat{\alpha}_{\mathbb{C}} \mathbf{L}_{\hat{\mathbf{V}}_1^*} \mathbf{L}_{\hat{\mathbf{V}}_1^*}^H]^{-1} \tilde{\Delta}_{\hat{\mathbf{V}}_1^*}^{\mathbb{C}}$,
 - 11: Reshape $\text{vec}(\hat{\mathbf{V}}_{1,R})$ in a $N \times N$ matrix with $[\hat{\mathbf{V}}_{1,R}]_{1,1} = 1$.
 - 12: **return** $\hat{\mathbf{V}}_{1,R}$
-

V. NUMERICAL ANALYSIS

In this section, through numerical simulations, we investigate three different aspects of the considered R -estimator of shape matrices: *i*) its semiparametric efficiency, *ii*) its robustness to outliers and *iii*) its algorithmic properties. In the following, we limit ourselves to report the results related to the complex-valued R -estimator proposed in Sec. IV, while the corresponding analysis of the real-valued case is provided in the supporting material.

In order to distinguish different estimators, each of them will be indicated as $\hat{\mathbf{V}}_{1,\gamma}^{\varphi}$ where γ and φ specify the estimator at hand as we will see below. For the sake of consistency with the SP literature on scatter matrix estimation, in the figures, we re-normalized $\hat{\mathbf{V}}_{1,\gamma}^{\varphi}$ in order to have $\text{tr}(\hat{\mathbf{V}}_{1,\gamma}^{\varphi}) = N$. According to the discussion on Sec. II, we can define the re-scaled estimator as:

$$\hat{\mathbf{V}}_{\gamma}^{\varphi} = N \hat{\mathbf{V}}_{1,\gamma}^{\varphi} / \text{tr}(\hat{\mathbf{V}}_{1,\gamma}^{\varphi}). \quad (55)$$

Plotting the MSE of this re-scaled estimator will allow us to underline the fact that the semiparametric efficiency property of the derived R -estimator does not depend on the particular scale functional adopted. As a reference, in the figures we also report the Constrained Semiparametric CRB (CSCRB) derived in closed form in [6]. As performance index for the shape matrix estimators, we use

$$\varepsilon_{\gamma}^{\varphi} = \|E\{\text{vec}(\hat{\mathbf{V}}_{\gamma}^{\varphi} - \mathbf{V}_0) \text{vec}(\hat{\mathbf{V}}_{\gamma}^{\varphi} - \mathbf{V}_0)^H\}\|_F, \quad (56)$$

Similarly, as performance bound, we adopt the index:

$$\varepsilon_{\text{CSCRB}} = \|[\text{CSCRB}(\boldsymbol{\Sigma}_0, g_0)]\|_F. \quad (57)$$

Note that the CSCRB in [6] is evaluated for a generic scatter matrix, then we have to choose the constraint accordingly to the definition of the shape matrix at hand (see Sec. II).

We generate the data according to a (true but unknown to the estimators) complex Generalized Gaussian (GG) distribution. The interested reader may find additional simulation related to the complex t -distribution in [46]. The data power is chosen

to be $\sigma_X^2 = E_{\mathcal{Q}}\{\mathcal{Q}\}/N = 4$. Finally, all the numerical indices have been evaluated through 10^6 Monte Carlo runs. The density generator of the complex Generalized Gaussian (GG) distribution is [7]:

$$h_0(t) = \frac{s\Gamma(N)b^{-N/s}}{\pi^N\Gamma(N/s)} \exp\left(-\frac{t^s}{b}\right), t \in \mathbb{R}^+ \quad (58)$$

and, according to the value of the shape parameter $s > 0$, it can model a distribution with both heavier tails ($0 < s < 1$) and lighter tails ($s > 1$) compared to the Gaussian distribution ($s = 1$). The versatility of the GG distribution is useful to assess the distributional robustness of the proposed R -estimator since its properties can be checked in Gaussian, super-Gaussian and sub-Gaussian scenarios. The setting used in our simulation is as follows:

- Σ_0 is a Toeplitz Hermitian matrix whose first column is given by $[1, \rho, \dots, \rho^{N-1}]^\top$; $\rho = 0.8e^{j2\pi/5}$ and $N = 8$.
- The ‘‘small perturbation’’ matrix \mathbf{H}_C^0 is chosen to be a symmetric random matrix s.t. $\mathbf{H}_C^0 = (\mathbf{G}_C + \mathbf{G}_C^H)/2$ where $[\mathbf{G}_C]_{i,j} \sim \mathcal{CN}(0, v^2)$, $[\mathbf{G}_C]_{1,1} = 0$ and $v = 0.01$. Note that v has to be small enough to guarantee that $\widehat{\mathbf{V}}_1^* + L^{-1/2}\mathbf{H}_C^0 \in \mathcal{M}_N^C$. A more exhaustive discussion on the choice of v will be given in Sec. V-C.

As previously discussed, the R -estimator in Eq. (54) depends on two ‘‘user-defined’’ quantities: 1) the preliminary estimator $\widehat{\mathbf{V}}_1^*$ and 2) the score function K_h . In order to assess the impact of their choice on the performance of the R -estimator, we perform our simulations by using the Tyler’s and the Huber’s estimators as preliminary estimators. Moreover, for the Huber’s estimator, three different values of the tuning parameter q (i.e. $q = 0.9, 0.5, 0.1$) have been adopted [7, Sec. V.C]. Note that the Sample Covariance Matrix (SCM) and Tyler’s estimators can be obtained from the Huber’s one when $q \rightarrow 1$ and $q \rightarrow 0$, respectively. As score functions, we exploit the *van der Waerden* one given in Eq. (50) and the t_ν -score in Eq. (51) for three different values of ν ($\nu = 0.1, 1, 5$).

A. Semiparametric efficiency

In Figs. 1 and 2, MSE indices of the R -estimator in Eq. (54) are plotted as function of the number L of observations and then compared with the CSCRb for a shape parameter of the GG distribution equal to 0.5, i.e. for a heavy-tailed scenario. Specifically, in Fig. 1 the asymptotic efficiency of the R -estimator, exploiting a *van der Waerden* score, is investigated for the two considered preliminary estimators, i.e. Tyler’s and Huber’s one. As we can see, the impact of the choice of the preliminary estimator on the asymptotic efficiency of the R -estimator is negligible. Similar consideration can be done for the choice of the particular score function. As shown in Fig. 2 in fact, the MSE curves of the R -estimator are very similar to each other and close to the CSCRb as $L \rightarrow \infty$. These simulations confirm the *nearly* semiparametric efficiency of the proposed R -estimator. We said ‘‘nearly’’ because, as anticipated in Sec. III-D, the choice of the score function does have an impact on the finite-sample performance and on the robustness to outliers. To see this, in Fig. 3, we report the MSE indices obtained for the *van der Waerden* and t_ν -scores

as function of the shape parameter s in a non-asymptotic regime, i.e. for $L = 5N$. The results in Fig. 3 seems to suggest that the *van der Waerden* score provide the lowest MSE index for $0.3 < s < 2$ while it presents small loss in highly heavy-tailed scenarios ($0.1 < s < 0.3$). Note that *van der Waerden* score is perfectly specified for $s = 1$, i.e. when the data are Gaussian distributed. As anticipated in Sec. III-D, this surprisingly good performance of the *van der Waerden* score is related to the so-called ‘‘Chernoff-Savage’’ result for rank-based statistics [35,39].

The t_ν -scores are more flexible since the additional parameter ν can be used to tune the desired trade-off between semiparametric efficiency and robustness to outliers, as we will see ahead. In particular, t_ν -scores characterized by a small value of ν improves the robustness of the resulting R -estimator at the price of a loss of efficiency. On the other hand, larger values of ν will provide a better efficiency, in particular in sub-Gaussian scenario, sacrificing the robustness as addressed in the next section. However, it is important to stress here that the MSE index of the resulting R -estimator is lower than the one of Tyler’s estimator for all the (non-degenerating) score functions. Moreover, due to the semiparametric nature of the R -estimator this conclusion holds true regardless the actual density generator characterizing the data distribution. While the choice of the score function has an impact on the properties of the resulting R -estimator, simulation results have highlighted that the impact of the preliminary estimator is negligible, as long as it is \sqrt{L} -consistent and robust (see also [46] for additional discussions). For this reason and for the sake of brevity, in the following we will only report the results obtained by adopting the preliminary Tyler’s estimator.

B. Robustness to outliers

Along with the semiparametric efficiency and distributional robustness, another fundamental property of a shape matrix estimator is the robustness to outliers. In the present context, an outlier is defined as an observation vector that does not share the same statistical behavior of the main data set, i.e. it is not CES distributed or/and it hasn’t the same shape matrix or location parameter. The two main tools used to quantify the robustness to outliers of an estimator are the *breakdown point* (BP) and the *influence function* (IF) [24, Ch. 11 and 12]. Roughly speaking, the BP indicates the percentage of ‘‘arbitrarily large’’ outliers that an estimator can tolerate before providing unreliable ‘‘arbitrarily large’’ estimates. On the other hand, the IF gives us a measure of the impact that an infinitesimal perturbation (at a given point) of the samples distribution may have on the estimation performance. Unfortunately, the evaluation of the BP and IF may be involved and difficult to obtain in closed form. Anyway, their ‘‘finite-sample’’ counterparts, called *finite-sample BP* [47] and *empirical IF* (EIF) [48], or *sensitivity curve*, can be easily evaluated through numerical simulations.

To evaluate the finite-sample BP for the proposed R -estimator, we follow the approach discussed in [49]. Let us start by indicating with $Z = \{\mathbf{z}_l\}_{l=1}^L \sim CES(\mathbf{0}, \mathbf{V}_1, h_0)$ the

“pure” GG data set whose h_0 is given in Eq. (58) and with $Z_\varepsilon = \{\mathbf{z}_l\}_{l=1}^L \sim f_{Z_\varepsilon}$ the ε -contaminated data set s.t.:

$$f_{Z_\varepsilon}(\mathbf{z}|\mathbf{V}_1, h_0, \varrho) = (1 - \varepsilon)CES(\mathbf{0}, \mathbf{V}_1, h_0) + \varepsilon q_Z(\varrho), \quad (59)$$

where $\varepsilon \in [0, 1/2]$ is a contamination parameter. The function $q_Z(\varrho)$ represents the pdf of an outlier $\tilde{\mathbf{z}}$ that we arbitrarily choose to be as $\tilde{\mathbf{z}} = \tau^{-1}\mathbf{u}$ where, as before, $\mathbf{u} \sim \mathcal{U}(S_{\mathbb{C}^N})$ while $\tau \sim \text{Gam}(\varrho, 1/\varrho)$ and Gam indicates the Gamma distribution. Consequently, $\tilde{\mathbf{z}}|\tau$ is uniformly distributed on the N sphere of ray τ^{-1} , i.e. $S_{\mathbb{C}^N}^\tau \triangleq \{\tilde{\mathbf{z}} \in \mathbb{C}^N \mid \|\tilde{\mathbf{z}}\| = \tau^{-1}\}$. This implies that we can obtain “arbitrarily large” outlier by generating arbitrarily small values of $\tau \sim \text{Gam}(\varrho, 1/\varrho)$. This can be achieved by choosing arbitrarily small values of the shape parameter $\varrho > 0$ in the Gamma distribution. Let $\widehat{\mathbf{V}}_\gamma^\varphi(Z)$ and $\widehat{\mathbf{V}}_\gamma^\varphi(Z_\varepsilon)$ be two shape matrix estimators evaluated from the pure and the ε -contaminated data sets, respectively. Then the finite-sample BP curves can be evaluated as [49]:

$$BP_\gamma^\varphi(\varepsilon) \triangleq \max \left\{ \lambda_{\gamma,1}^\varphi(\varepsilon), 1/\lambda_{\gamma,N}^\varphi(\varepsilon) \right\}, \quad (60)$$

where $\lambda_{\gamma,i}^\varphi(\varepsilon)$ is the i -th ordered eigenvalue of the matrix $[\widehat{\mathbf{V}}_\gamma^\varphi(Z)]^{-1}\widehat{\mathbf{V}}_\gamma^\varphi(Z_\varepsilon)$, s.t. $\lambda_{\gamma,1}^\varphi(\varepsilon) \geq \dots \geq \lambda_{\gamma,N}^\varphi(\varepsilon)$. Clearly, when there is no contamination ($\varepsilon = 0$), we have that $BP_\gamma^\varphi(0) = 1$. Any robust estimator should then have a BP value close to 1 for every value of ε , while it may be arbitrarily large for a non-robust estimator. Fig. 4 shows the BP curves of the proposed R -estimator exploiting the *van der Waerden* and three t_ν -scores ($\nu = 0.1, 1, 5$). Since $BP_\gamma^\varphi(\varepsilon)$ depends on Z and Z_ε , we plot its averaged value over 10^4 realizations of these data sets. For the sake of comparison, we report also the BP value of Tyler’s estimator. All the BP curves, related to the resulting R -estimator, remain close to the Tyler’s one for every value of ε . On the other hand, the BP of the non-robust Sample Covariance Matrix (SCM) estimator explodes to 10^{17} as soon as $\varepsilon \neq 0$, so we do not include it in the plot. A visual inspection of Fig. 4 confirms us what already said in Sec. V-A: t_ν -scores with a small value of ν lead to more robust estimators. In particular, it can be noted that the BP curves of the R -estimator with $t_{0.1}$ - and t_1 -score functions coincide with the one of Tyler’s estimator.

Let us now focus on the EIF [48]. For the shape matrix estimation at hand, it can be defined as:

$$EIF_\gamma^\varphi \triangleq (L + 1) \|\widehat{\mathbf{V}}_\gamma^\varphi(Z) - \widehat{\mathbf{V}}_\gamma^\varphi(Z, \tilde{\mathbf{z}})\|_F, \quad (61)$$

where $\tilde{\mathbf{z}}$ is an outlier distributed according to the pdf $q_Z(\varrho)$ defined in Eq. (59). As Eq. (61) suggests, the EIF_γ^φ gives us a measure of the impact that a single outlier $\tilde{\mathbf{z}}$ has on the shape matrix estimator $\widehat{\mathbf{V}}_\gamma^\varphi$ when it is added to the “pure” data set Z . Moreover, if L is sufficiently large, the expression in Eq. (61) is a good approximation of the theoretical IF [48]. For this reason, in our simulation we use $L = 1000$. Since EIF_γ^φ depends on Z and $\tilde{\mathbf{z}}$, we plot its averaged value over 10^4 realizations of the data set and the outlier. As for the IF, the most important property that the EIF of a robust estimator should have is the boundedness. In fact, this indicates that the impact of a single outlier on the estimation performance is limited. In Fig. 5, we report the EIF of the proposed R -estimator exploiting the *van der Waerden* and three t_ν -scores

($\nu = 0.1, 1, 5$). As benchmark, the EIF of the Tyler’s estimator is adopted since it is known that the relevant IF is continuous and bounded [7]. On the other hand, the EIF of the non-robust SCM grows rapidly to 10^4 as the norm of the outlier $\tilde{\mathbf{z}}$ increases (i.e. when $\varrho \rightarrow 0$), so we do not include it in the plot. As we can see from Fig. 5, the EIFs of the proposed R -estimator remain bounded and close to the one of the Tyler’s estimator for arbitrarily large values of $\|\tilde{\mathbf{z}}\|$ ($\varrho \rightarrow 0$).

C. Algorithmic considerations

This last subsection collects some observations on the algorithmic implementation of the proposed R -estimator. As can be seen from the pseudo-code in Sec. IV, the R -estimator is obtained by applying a linear “one-step” correction $L^{-1/2}[\hat{\alpha}_C \mathbf{L}_{\widehat{\mathbf{V}}_1^*} \mathbf{L}_{\widehat{\mathbf{V}}_1^*}^H]^{-1} \Delta_{\widehat{\mathbf{V}}_1^*}^C$ to a preliminary estimator $\widehat{\mathbf{V}}_1^*$ (see step 10 in Algo. 1). In particular, unlike M -estimators that are obtained as implicit solution of a fixed point equation, it does not require any iterative implementation. Consequently, leaving aside the computation of $\widehat{\mathbf{V}}_1^*$, the computational load of the proposed R -estimator is roughly given by the amount of calculation needed to *i*) obtain the L ranks r_i^* and vectors $\hat{\mathbf{u}}_i^*$ (see steps 2 and 3 in Algo. 1) and *ii*) deal with the $(N^2 - 1) \times (N^2 - 1)$ matrices $\mathbf{L}_{\mathbf{V}_1}$, $[\mathbf{L}_{\mathbf{V}_1} \mathbf{L}_{\mathbf{V}_1}^H]$ and $[\mathbf{L}_{\mathbf{V}_1} \mathbf{L}_{\mathbf{V}_1}^H]^{-1}$. Clearly, this represents a problem as the dimension N of the observations increases. A possible way out would be to exploit the structure of $\mathbf{L}_{\widehat{\mathbf{V}}_1^*}$, given in Eq. (43), to reduce the global computational load but this point falls outside the scope of the present paper.

The second algorithmic consideration is related the choice of the “small perturbation” matrix \mathbf{H}_C^0 . The theory does not provide us with any hint about the optimal selection of this hyper-parameter, so we decided to define it as a random matrix $\mathbf{H}_C^0 = (\mathbf{G}_C + \mathbf{G}_C^H)/2$ where $[\mathbf{G}_C]_{i,j} \sim \mathcal{CN}(0, v^2)$, $[\mathbf{G}_C]_{1,1} = 0$. The problem then is reduced to the simpler choice of the scalar perturbation parameter v . Fortunately, simulation results seem to suggest that the R -estimator is quite robust w.r.t. the choice of v for various density generators and various levels of non-Gaussianity. On the other hand, the choice of v is sensitive to the data dimension N and to the number of observations L . As an example, Fig. 6 shows the MSE index of the $\widehat{\mathbf{V}}_{R,vdW}^{Ty}$ as function of v for different data dimension N . As we can see, the MSE index remains stable for a sufficiently large range of values for v allowing us for its safe selection.

VI. CONCLUSIONS

In this paper, a distributionally robust and nearly semiparametric efficient R -estimator of the shape matrix in Real and Complex ES distributions has been discussed and analyzed. This estimator has been firstly proposed by Hallin, Oja and Paindaveine in their seminal paper [10] where the Le Cam’s theory of one-step efficient estimators and the properties of rank-based statistics have been exploited as basic building blocks for its derivation. In the first part of this paper, a survey of the main statistical concepts underlying such R -estimator has been provided for the case of RES-distributed data. Then, its extension to CES distributions has been derived by means of the Wirtinger calculus. Finally, the finite-sample

performance of the R -estimator has been investigated in different scenarios in terms of MSE and robustness to outliers. However, a number of fundamental issues still remain to be fully addressed. In our opinion, the most important one is related to the estimation of $\alpha_{\mathbb{C},0}$ in Eq. (46) (or, for the real-valued case, α_0 in Eq. (23)). The estimator in Eq. (53) in fact is consistent under any possible density generator $h \in \mathcal{G}_{\mathbb{C}}$ but it does not satisfy any optimality property. Moreover, it depends on an hyper-parameter, i.e. the “small perturbation” matrix $\mathbf{H}_{\mathbb{C}}^0$ (or \mathbf{H}^0 in the real-valued case), that has to be defined by the user in an heuristic way and, currently, without any theoretical guidelines. A possible improvement w.r.t. the estimator in Eq. (53) is discussed in [10, Sec. 4.2] and it will be the subject of future works. Other important open questions are related to the evaluation of the theoretical BP point and IF. Closed form expressions of these two quantities will help to fully understand the robustness properties of the proposed R -estimator with respect to classical M -estimators.

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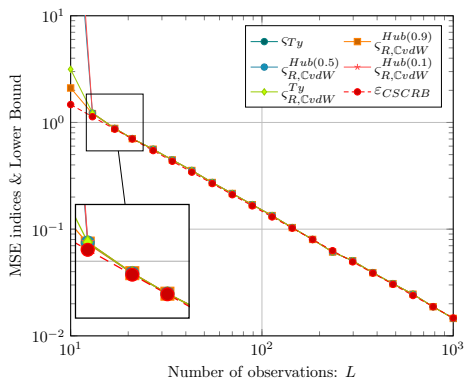


Fig. 1: MSE indices vs preliminary Tyler's and Huber's estimators as function of L ($s = 0.5$).

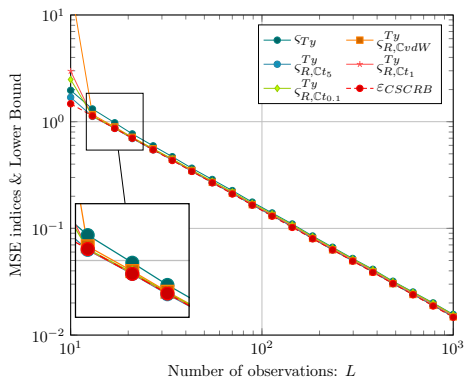


Fig. 2: MSE indices vs different score functions K_h as function of L ($s = 0.5$).

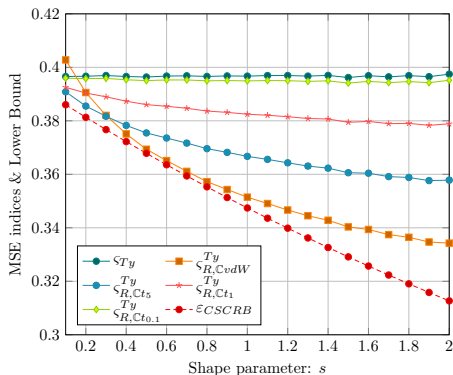


Fig. 3: MSE indices vs different score functions K_h as function of s ($L = 5N$).

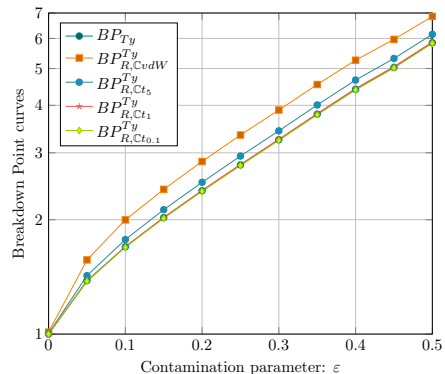


Fig. 4: BP vs different score functions K_h as function of ϵ ($L = 5N$, $\rho = 0.1$, $s = 0.1$).

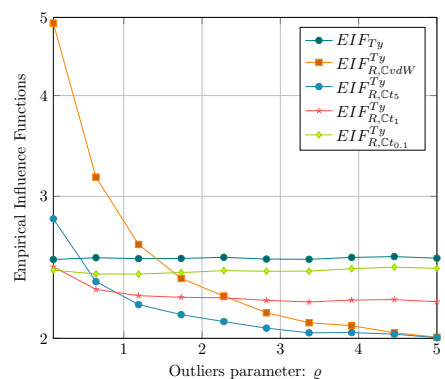


Fig. 5: EIF vs different score functions K_h as function of ρ ($L = 1000$, $s = 0.1$).

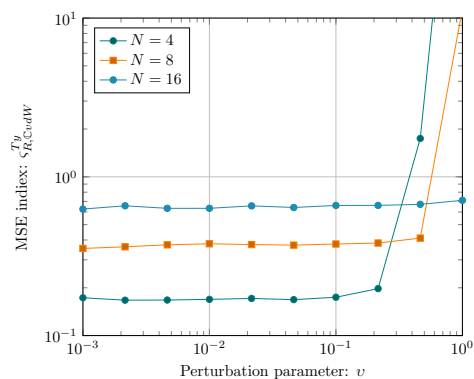


Fig. 6: MSE index $S_{R,CvdW}^{Ty}$ as function of the perturbation parameter v ($L = 5N$, $s = 0.5$).