



**HAL**  
open science

# Consistent Risk Measure on $L^0$ : NA Condition, Pricing and Dual Representation

Emmanuel Lépinette, Duc Think Vu

► **To cite this version:**

Emmanuel Lépinette, Duc Think Vu. Consistent Risk Measure on  $L^0$ : NA Condition, Pricing and Dual Representation. *International Journal of Theoretical and Applied Finance*, 2021, 24 (06n07), 10.1142/S0219024921500370 . hal-02976860

**HAL Id: hal-02976860**

**<https://hal.science/hal-02976860>**

Submitted on 23 Oct 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Consistent Risk Measure on $L^0$ : NA Condition, Pricing and Dual Representation

Emmanuel LEPINETTE,<sup>1</sup> Duc Think VU<sup>1</sup>

<sup>1</sup> *Ceremade, UMR CNRS 7534, Paris Dauphine University, PSL National Research, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France and GosaeF, Faculty of Sciences of Tunis, Tunisia. Email: emmanuel.lepinette@ceremade.dauphine.fr*

**Abstract:** The NA condition is one of the pillars supporting the classical theory of financial mathematics. We revisit this condition for financial market models where a dynamic risk-measure defined on  $L^0$  is fixed to characterize the family of acceptable wealths that play the role of non negative financial positions. We provide in this setting a new version of the fundamental theorem of asset pricing and we deduce a dual characterization of the super-hedging prices of an European option. Moreover we provide an example where it is possible to obtain a dual representation of the risk-measure on  $L^0$ .

**Keywords and phrases:** NA condition, Risk-hedging prices, Dynamic risk-measures, Dual representation, No-arbitrage

Mathematics Subject Classification (2010): 49J53, 60D05, 91G20, 91G80.  
JEL Classification: C02, C61, G13.

## 1. Introduction

The NA condition originates from the work of Black and Scholes [5] and Merton [31]. In these articles, the risky asset is modeled by a geometric Brownian motion. The NA condition means the absence of arbitrage opportunities, i.e. a terminal portfolio value starting from the zero initial endowment can not be acceptable. A financial position in the classical arbitrage theory is acceptable if it is non negative almost surely. In our work, the new contribution is that we consider a larger class of acceptable positions which are defined from a risk-measure.

The NA condition is characterized through the famous Fundamental The-

orem of Asset Pricing (FTAP), for a variety of financial models, essentially as equivalent to the existence of a so-called risk-neutral probability measure, under which the price process is a martingale. There are several versions of the FTAP and, in discrete-time, the most popular is certainly the one formulated by Dalang, Morton and Willinger [8], see also [37], [36], [22], [25], [26]. In continuous time, the formulation of the FTAP theorem is only possible once continuous-time self-financing portfolios are defined, see the seminal work of Black and Scholes [5]. This gave rise to an extensive development of stochastic calculus, e.g. for semi-martingales [20], making possible formulation of several versions of the FTAP theorem as given in [9], [11], [10], [12], [19].

The main contribution of the FTAP theorems is the link between the concept of arbitrage and the pricing technique which is deduced. It is now very well known that the super-hedging prices of an European claim are dually identified through the risk-neutral probability measures characterizing the NA condition. We may observe that the NA condition has been suitably chosen in the models of consideration in such a way that the set of all attainable claims is closed, see [27, Theorem 2.1.1]. This allows one to apply the Hahn-Banach separation theorem, see [13], and obtain dual elements that characterize the super-hedging prices. This is also the case for financial models with proportional transaction costs, see [27, Section 3] and the references mentioned therein.

The increasing use of risk-measures in the context of the Basel banking supervision naturally calls into question the definition of the super-hedging condition which is commonly accepted in the usual literature. Recall that a portfolio process  $(V_t)_{t \in [0, T]}$  super-replicates a contingent claim  $h_T$  at the horizon date  $T > 0$  means that  $V_T \geq h_T$  a.s.. In practice, this inequality remains difficult to achieve and practitioners accept to take a moderate risk, choosing for example  $\alpha \in (0, 1)$  so that  $P(V_T - h_T \geq 0) \geq 1 - \alpha$  is close to 1. This is the case when considering the Value At Risk measure, see [23], and we say that  $V_T - h_T$  is acceptable. More generally,  $V_T - h_T$  is said acceptable for a risk-measure  $\rho$  if  $\rho(V_T - h_T) \leq 0$ , see [14], [15], [16], [18], [28], [1] for frictionless markets and [24], [17], [4], [2] for conic models. The acceptable positions play the role of the almost surely non negative random variables and allows one to take risk controlled by the risk measure we choose. Moreover, by considering a larger family of acceptable positions, the hedging prices may be lowered as shown in [34] for the Black and Scholes models with proportional transaction

costs, see also the discussion in [33].

Pricing with a coherent risk-measure has been explored and developed by Cherny in two major papers [6] and [7] for coherent risk-measures defined on the space of bounded random variables. Cherny supposes that the risk-measure  $\rho$  (or equivalently the utility measure  $u = -\rho$ ) is defined by a weakly compact determining set  $\mathcal{D}$  of equivalent probability measures, i.e. such that  $\rho(X) = \sup_{Q \in \mathcal{D}} E_Q(-X)$  for any  $X \in L^\infty$ . This representation automatically holds for coherent risk-measures defined on  $L^\infty$ . This motivates the choice of Cherny to suppose such a representation for the risk-measures he considers on  $L^0$  as he claims that *it is hopeless to axiomatize the notion of a risk measure on  $L^0$  and then to obtain the corresponding representation theorem*, see [7].

Actually, the recent paper [30] proposes an axiomatic construction of a dynamic coherent risk-measure on  $L^0$  from the set of all acceptable sets. We consider such a dynamic risk-measure and we define the discrete-time portfolio processes as the processes  $(V_t)_{t \leq T}$  adapted to a filtration  $(\mathcal{F}_t)_{t \leq T}$  such that  $V_t + \theta_t \Delta S_{t+1} - V_{t+1}$  is acceptable at time  $t$  for some  $\mathcal{F}_t$ -measurable strategy  $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ . This is a generalization of the classical definition where, usually, acceptable means non negative so that  $V_t + \theta_t \Delta S_{t+1} \geq V_{t+1}$  almost surely. We then introduce a no-arbitrage condition we call NA as in the classical literature and we show that it coincides with the usual NA condition if the acceptable positions are the non negative random variables. This NA condition allows one to dually characterize the super-hedging prices, at least when  $\rho$  is time consistent.

Similarly, Cherny proposes in his papers [6] and [7] a no-arbitrage condition *No Good Deal* (NGD) which is the key point to define the super-hedging prices. The approach is a priori slightly different: The NGD condition holds if there is no bounded claim  $X$  attainable from the zero initial capital such that  $\rho(X) < 0$ . In our setting, the NA condition is formulated from the minimal price super-hedging the zero claim, which is required to be non negative. Clearly, there is a link between the NA and the NGD condition as  $\rho(X)$  appears to be a possible super-hedging price for the zero claim. In particular, the NA condition implies that  $\rho(X) \geq 0$  so that *NGD* holds. Reciprocally, the NGD condition implies that the NA condition holds by [7, Theorem 3.4] and the FTAP theorem we formulate in this paper. Indeed, by [7, Theorem 3.4], NGD implies the existence of a risk-neutral probability measure among the probability measures representing the risk measure. Therefore, the NGD and the NA conditions are equivalent in the setting of Cherny. Although, in

our paper we do not need to suppose the existence of a priori given probability measures representing the risk-measure. This is why the proof of the FTAP theorem we formulate is more challenging as we cannot directly use an immediate compactness argument as done in [7] to obtain a risk-neutral probability measure. We then deduce a dual representation of the superhedging prices in the case where the risk-measure is time-consistent. At last, we propose a discussion about dual representation of a risk-measure defined on  $L^0$ . We propose conditions under which a dual representation exists.

## 2. Model

In discrete-time, we consider a stochastic basis  $(\Omega, \mathcal{F} := (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$  where the complete <sup>1</sup>  $\sigma$ -algebra  $\mathcal{F}_t$  represents the information of the market available at time  $t$ . For any  $t \leq T$ ,  $L^0(\mathbf{R}^d, \mathcal{F}_t)$ ,  $d \geq 1$ , is the space of all  $\mathbf{R}^d$ -valued random variables which are  $\mathcal{F}_t$ -measurable, and endowed with the topology of convergence in probability. Similarly,  $L^p(\mathbf{R}^d, \mathcal{F}_t)$ ,  $p \in [1, \infty)$  (resp.  $p = \infty$ ), is the normed space of all  $\mathbf{R}^d$ -valued random variables which are  $\mathcal{F}_t$ -measurable and admit a moment of order  $p$  under the probability measure  $\mathbb{P}$  (resp. bounded). In particular,  $L^p(\mathbf{R}_+, \mathcal{F}_t) = \{X \in L^p(\mathbf{R}, \mathcal{F}_t) | X \geq 0\}$  and  $L^p(\mathbf{R}_-, \mathcal{F}_t) = -L^p(\mathbf{R}_+, \mathcal{F}_t)$  when  $p = 0$  or  $p \in [1, \infty]$ . All equalities and inequalities between random variables are understood to hold everywhere on  $\Omega$  up to a negligible set. If  $A_t$  is a set-valued mapping (i.e. a random set of  $\mathbf{R}^d$ ), we denote by  $L^0(A_t, \mathcal{F}_t)$  the set of all  $\mathcal{F}_t$ -measurable random variables  $X_t$  such that  $X_t \in A_t$  a.s.. The topology in  $L^0$  is defined from the convergence in probability. We say that  $X_t \in L^0(A_t, \mathcal{F}_t)$  is a measurable selection of  $A_t$ . In our paper, a random set  $A_t$  is said  $\mathcal{F}_t$ -measurable if it is graph-measurable, see [32], i.e.

$$\text{Gr } A_t = \{(\omega, x) \in \Omega \times \mathbf{R}^d : x \in A_t(\omega)\} \in \mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d).$$

It is well known that  $L^0(A_t, \mathcal{F}_t) \neq \emptyset$  if and only if  $A_t \neq \emptyset$  a.s., see [21, Th. 4.4]. When referring to this property, we shall say that we use a "measurable selection argument" as it is usual to say.

We consider a dynamic consistent risk-measure  $X \mapsto (\rho_t(X))_{t \leq T}$  defined on the space  $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ ,  $\overline{\mathbf{R}} = [-\infty, \infty]$ . Precisely, we consider the risk-measure

---

<sup>1</sup>This means that the  $\sigma$ -algebra contains the negligible sets so that an equality between two random variables is understood up to a negligible set.

of [30], where an extension to the whole space  $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$  is proposed. Recall that, in this paper, the risk-measure is constructed from its closed acceptance sets  $(\mathcal{A}_t)_{t \leq T}$  of acceptable financial positions  $\mathcal{A}_t$  at time  $t \leq T$ . We suppose that  $\mathcal{A}_t$  is a closed convex cone. In the following, we use the conventions:

$$\begin{aligned} 0 \times (\pm\infty) &= 0, & (0, \infty) \times (\pm\infty) &= \{\pm\infty\}, \\ \mathbf{R} + (\pm\infty) &= \pm\infty, & \infty - \infty &= -\infty + \infty = +\infty. \end{aligned}$$

For  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ ,  $\rho_t(X)$  may be infinite and  $\rho_t(X) \in \mathbf{R}$  a.s. if and only if  $X \in \text{Dom } \mathcal{A}_t$  where

$$\begin{aligned} \text{Dom } \mathcal{A}_t &:= \{X \in L^0(\mathbf{R}, \mathcal{F}_T) : \mathcal{A}_t^X \neq \emptyset\}, \\ \mathcal{A}_t^X &:= \{C_t \in L^0(\mathbf{R}, \mathcal{F}_t) | X + C_t \in \mathcal{A}_t\}. \end{aligned}$$

Actually, we have  $\rho_t(X) = \text{ess inf}_{\mathcal{F}_t} \mathcal{A}_t^X$  if  $X \in \text{Dom } \mathcal{A}_t$ . Recall that the following properties hold (see [30]):

**Proposition 2.1.** *The risk-measure  $\rho_t$  satisfies the following properties:*

*Normalization:*  $\rho_t(0) = 0$ ;

*Monotonicity:*  $\rho_t(X) \geq \rho_t(X')$  whatever  $X, X' \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$  s.t.  $X \leq X'$ ;

*Cash invariance:*  $\rho_t(X + m_t) = \rho_t(X) - m_t$  if  $m_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ , and  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ ;

*Subadditivity:*  $\rho_t(X + X') \leq \rho_t(X) + \rho_t(X')$  if  $X, X' \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$  ;

*Positive homogeneity:*  $\rho_t(k_t X) = k_t \rho_t(X)$  if  $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ ,  $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ .

Moreover,  $\rho_t$  is lower semi-continuous i.e., if  $X_n \rightarrow X$  a.s., then  $\rho_t(X) \leq \liminf_n \rho_t(X_n)$  a.s., and we have

$$\mathcal{A}_t = \{X \in \text{Dom } \mathcal{A}_t | \rho_t(X) \leq 0\}. \quad (2.1)$$

We generalize the definition of acceptable set  $\mathcal{A}_t$  to  $\mathcal{A}_{t,u} = \mathcal{A}_t \cap L^0(\overline{\mathbf{R}}, \mathcal{F}_u)$  for dates between  $t \leq T$  and  $u \in [t, T]$ . The corresponding acceptable sets and risk measures can be represented as

$$\mathcal{A}_{t,u} = \{X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_u) : \rho_{t,u}(X) \leq 0\}.$$

Let  $(S_t)_{t \leq T}$  be a process describing the discounted prices of  $d$  risky assets such that  $S_t \in L^0(\mathbf{R}_+^d, \mathcal{F}_t)$  for any  $t \geq 0$ . A contingent claim with maturity date  $t + 1$  is defined by a real-valued  $\mathcal{F}_{t+1}$ -measurable random variable  $h_{t+1}$ . In the paper [30], the super-hedging problem for the payoff  $h_{t+1}$  is solved with respect to the dynamic risk-measure  $(\rho_t)_{t \leq T}$ . Precisely:

**Definition 2.2.** A payoff  $h_{t+1} \in L^0(\mathbf{R}, \mathcal{F}_T)$  is said to be risk-hedged at time  $t$  if there exists a risk-hedging price  $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  and a strategy  $\theta_t$  in  $L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that  $P_t + \theta_t \Delta S_{t+1} - h_{t+1}$  is acceptable at time  $t$ .

Let  $\mathcal{P}_t(h_{t+1})$  be the set of all risk-hedging prices  $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  at time  $t$  as in Definition 2.2. In the following, we suppose that  $h_{t+1}$  is non negative and  $\mathcal{P}_t(h_{t+1}) \neq \emptyset$ . This is the case if there exist  $a_t, b_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that  $h_{t+1} \leq a_t S_{t+1} + b_t$ . This inequality trivially holds for European call and put options.

**Definition 2.3.** The minimal risk-hedging price of the contingent claim  $h_{t+1}$  at time  $t$  is defined as

$$P_t^* := \operatorname{ess\,inf}_{\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} \mathcal{P}_t(h_{t+1}). \quad (2.2)$$

Note that the minimal risk-hedging price  $P_t^*$  of  $h_{t+1}$  is not necessarily a price, i.e. it is not necessarily an element of  $\mathcal{P}_t(h_{t+1})$  if this set is not closed. The aim of our paper is to study a no-arbitrage condition under which  $P_t^* \in \mathcal{P}_t(h_{t+1})$ .

Starting from the contingent claim  $h_T$  at time  $T$ , we recursively define

$$P_T^* := h_T, \quad P_t^* := \operatorname{ess\,inf}_{\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*),$$

where  $P_{t+1}^*$  may be interpreted as a contingent claim  $h_{t+1}$ . The interesting question is whether  $P_t^*$  is actually a price, i.e. an element of  $\mathcal{P}_t(P_{t+1}^*)$ , or equivalently whether  $\mathcal{P}_t(P_{t+1}^*)$  is closed. In the classical setting, recall that closedness is obtained under the NA condition.

**Definition 2.4.** A stochastic process  $(V_t)_{t \leq T}$  adapted to  $(\mathcal{F}_t)_{t \leq T}$ , starting from an initial endowment  $V_0$  is a portfolio process if, for all  $t \leq T - 1$ , there exists  $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that  $V_t + \theta_t \Delta S_{t+1} - V_{t+1}$  is acceptable at time  $t$ . Moreover, we say that it super-hedges the payoff  $h_T \in L^0([-\infty, \infty), \mathcal{F}_T)$  if  $V_T \geq h_T$  a.s..

Note that  $V_{T-1} + \theta_{T-1} \Delta S_T - V_T$  is supposed to be acceptable at time  $T-1$ . Therefore,  $V_T \geq h_T$  implies that  $V_{T-1} + \theta_{T-1} \Delta S_T - h_T$  is acceptable at time  $T-1$ . In the following, we actually set  $V_T = h_T$  where  $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$  is a European claim. Notice that, if  $P_{T-1}^* = -\infty$  on some non null set, then, the one step pricing procedure of [30] may be applied as the risk-measure is defined on  $L^0([-\infty, \infty], \mathcal{F}_T)$ . Actually, this is trivial to super hedges  $P_{T-1}^* = -\infty$  by  $P_{T-2}^* = -\infty$ . This means that the backward procedure of [30] may be applied without any no-arbitrage condition. Let us recall this procedure.

We define  $P_T^* = h_T =: h$  and let us consider the set  $\mathcal{P}_t(P_{t+1}^*)$  of all prices  $p_t$  at time  $t$  allowing one to start a portfolio strategy  $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that  $p_t + \theta_t \Delta S_{t+1} = P_{t+1}^* + a_{t,t+1}$  where  $a_{t,t+1} \in L^0(\mathbf{R}, \mathcal{F}_t)$  is an acceptable position at time  $t$ . This is a generalization of the classical super-hedging inequality  $p_t + \theta_t \Delta S_{t+1} \geq P_{t+1}^*$ . We have

$$\mathcal{P}_t(P_{t+1}^*) = \{\theta_t S_t + \rho_t(\theta_t S_{t+1} - P_{t+1}^*) : \theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)\} + L^0(\mathbf{R}_+, \mathcal{F}_t),$$

and, recursively, we define:

$$P_t^* = \operatorname{ess\,inf}_{\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*).$$

In [30], a jointly measurable version of the random function  $g_t$  that appears above in the characterization of  $\mathcal{P}_t(P_{t+1}^*)$ , i.e.

$$g_t^h(\omega, x) := x S_t + \rho_t(x S_{t+1} - P_{t+1}^*), \quad (2.3)$$

is constructed in the one-dimensional case. With the same arguments, we may obtain a jointly measurable version of  $g_t^h(\omega, x) := x S_t + \rho_t(x S_{t+1} - P_{t+1}^*)$  if  $x \in \mathbf{R}^d$ . Moreover, by similar arguments, we also show that  $P_t^* = \inf_{x \in \mathbf{R}^d} g_t^h(x)$ .

Let  $V$  be a portfolio process with  $V_T = h_T = h$ . By definition, we have that  $\rho_{T-1}(V_{T-1} + \theta_{T-1} \Delta S_T - h_T) \leq 0$ . We deduce that  $V_{T-1} \geq P_{T-1}^*$  and, by induction, we get that  $V_t \geq P_t^*$  for all  $t \leq T$ , since  $V_t$  is a risk-hedging price for  $V_{t+1} \geq P_{t+1}^*$  at time  $t+1$ . In particular,  $V_t \in \mathcal{P}_t(P_{t+1}^*) \neq \emptyset$  for all  $t \in T-1$ .

### 2.1. No-arbitrage conditions

An immediate profit is the possibility to super-replicate the zero contingent claim at a negative price, see [3].



**Definition 2.5.** *Absence of Immediate Profit (AIP) holds if, for any  $t \leq T$ ,*

$$\mathcal{P}_t(0) \cap L^0(\mathbf{R}_-, \mathcal{F}_t) = \{0\}. \quad (2.4)$$

It is clear that AIP holds at time  $T$  since  $\mathcal{P}_T(0) = L^0(\mathbf{R}_+, \mathcal{F}_T)$ . We now formulate characterizations of the AIP condition in the multi-dimensional setting. We denote by  $S(0, 1)$  the set of all  $z \in \mathbf{R}^d$  such that  $|z| = 1$ .

**Theorem 2.6.** *The following statements are equivalent:*

1. *AIP holds between time  $t - 1$  and  $t$ .*
2.  *$\rho_{t-1}(x\Delta S_t) \geq 0$ , for any  $x \in \mathbf{R}^d$ , a.s..*
3.  *$\rho_{t-1}(z\Delta S_t) \geq 0$ , for any  $z \in S(0, 1)$ , a.s..*
4. *Let  $x_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$ . If  $x_{t-1}\Delta S_t$  is acceptable on some non null set  $F_{t-1} \in \mathcal{F}_{t-1}$ , then  $\rho_{t-1}(x_{t-1}\Delta S_t) = 0$  on  $F_{t-1}$ .*

*Proof.*  $1 \iff 2$ . Consider  $h_t = 0$  under AIP. As  $P_t^* = \inf_{x \in \mathbf{R}^d} g_t^0(x) \geq 0$ , we deduce that, for all  $x \in \mathbf{R}^d$ ,  $g_t^0(x) = xS_{t-1} + \rho_{t-1}(xS_t) = \rho_{t-1}(x\Delta S_t) \geq 0$ . The equivalence  $2 \iff 3$  is clear by homogeneity. Let us show that  $2 \implies 4$ . Suppose that  $x_{t-1}\Delta S_t$  is acceptable on  $F_{t-1}$ , i.e.  $\rho_{t-1}(x_{t-1}\Delta S_t) \leq 0$  on  $F_{t-1}$ . Then, by  $2$ , we have  $\rho_{t-1}(x_{t-1}\Delta S_t) = 0$  on  $F_{t-1}$ . Let us show that  $4$  implies  $2$ . Consider the set  $F_{t-1} = \{\rho_{t-1}(x_{t-1}\Delta S_t) < 0\} \in \mathcal{F}_{t-1}$ . Then,  $x_{t-1}\Delta S_t$  is acceptable on  $F_{t-1}$  hence by  $4$ ,  $\rho_{t-1}(x_{t-1}\Delta S_t) = 0$ , which implies that  $P(F_{t-1}) = 0$ . Therefore,  $\rho_{t-1}(x_{t-1}\Delta S_t) \geq 0$  a.s..  $\square$

In the following, we consider a contingent claim  $h_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  and a jointly measurable version (see [30]) of the random function

$$g_{t-1}(\omega, x) := xS_{t-1} + \rho_{t-1}(xS_t - h_t) \quad (2.5)$$

which is associated to  $h_t$ .

**Theorem 2.7.** *Suppose that AIP holds and consider  $z_{t-1} \in L^0(S(0, 1), \mathcal{F}_{t-1})$ . Then, on the set  $F_{t-1} = \{\rho_{t-1}(z_{t-1}\Delta S_t) = 0\} \cap \{\rho_{t-1}(-z_{t-1}\Delta S_t) = 0\}$ , the random function  $g_{t-1}$  given by (2.5) is constant on the line  $\mathbf{R}z_{t-1}$ .*

*Proof.* Consider  $x_{t-1}, y_{t-1} \in L^0(\mathbf{R}z_{t-1}, \mathcal{F}_{t-1})$  such that  $x_{t-1} \neq y_{t-1}$ . Without loss of generality, we assume that  $y_{t-1} = r_2 z_{t-1}$  and  $x_{t-1} = r_1 z_{t-1}$  with  $r_2 > r_1$  and  $r_1, r_2 \in L^0(\mathbf{R}, \mathcal{F}_{t-1})$ . We have:

$$\begin{aligned} \rho_{t-1}(y_{t-1}S_t - h_t) &= \rho_{t-1}(x_{t-1}S_t - h_t + (y_{t-1} - x_{t-1})S_t), \\ &\leq \rho_{t-1}(x_{t-1}S_t - h_t) + \rho_{t-1}((y_{t-1} - x_{t-1})S_t). \end{aligned}$$

We deduce that on the set  $F_{t-1}$  we have:

$$\begin{aligned} g_{t-1}(y_{t-1}) &= y_{t-1}S_{t-1} + \rho_{t-1}(y_{t-1}S_t - h_t), \\ &\leq y_{t-1}S_{t-1} + \rho_{t-1}(x_{t-1}S_t - h_t) + \rho_{t-1}((y_{t-1} - x_{t-1})S_t), \\ &= g_{t-1}(x_{t-1}) + (y_{t-1} - x_{t-1})S_{t-1} + \rho_{t-1}((y_{t-1} - x_{t-1})S_t). \end{aligned}$$

This is equivalent to:

$$\begin{aligned} g_{t-1}(y_{t-1}) - g_{t-1}(x_{t-1}) &\leq (y_{t-1} - x_{t-1})S_{t-1} + \rho_{t-1}((y_{t-1} - x_{t-1})S_t), \\ &\leq (r_2 - r_1)\rho_{t-1}(z_{t-1}\Delta S_t) = 0. \end{aligned}$$

By symmetry, we also have:

$$g_{t-1}(x_{t-1}) - g_{t-1}(y_{t-1}) \leq (r_2 - r_1)\rho_{t-1}(-z_{t-1}\Delta S_t) = 0.$$

This implies that  $g_{t-1}$  is a constant on the line  $\mathbf{R}z_{t-1}$ . Indeed, on the contrary case, the  $\mathcal{F}_{t-1}$ -measurable set  $\Gamma_{t-1}(\omega) = \{\alpha \in \mathbf{R} : g_{t-1}(\alpha z_{t-1}) \neq g_{t-1}(z_{t-1})\}$  is non empty on the non null set  $G_{t-1} = \{\omega \in \Omega : \Gamma_{t-1}(\omega) \neq \emptyset\} \in \mathcal{F}_{t-1}$ . We then deduce a measurable selection  $\tilde{z}_t \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$  such that  $\tilde{z}_t = \alpha_t z_t$  and  $\alpha_t \in \Gamma_{t-1}$  on the set  $G_{t-1}$  and we put  $\tilde{z}_t = z_t$  on the complimentary set  $\Omega \setminus G_{t-1}$ . By the first part above, we deduce that  $g_{t-1}(\tilde{z}_t) = g_{t-1}(z_t)$  a.s., which contradicts the fact that  $\alpha_t \in \Gamma_{t-1}$  on  $F_{t-1}$ .  $\square$

**Definition 2.8.** *We say that the symmetric risk neutral condition SRN holds at time  $t$  if, almost surely, for any  $z_t \in L^0(S(0, 1), \mathcal{F}_t)$ ,  $\rho_t(z_t \Delta S_{t+1}) = 0$  if and only if  $\rho_t(-z_t \Delta S_{t+1}) = 0$ . We say that SRN holds if it holds at any time.*

Observe that the SRN condition means that a zero cost position  $z_t$  is risk-neutral if and only if  $-z_t$  is risk neutral.

**Definition 2.9.** *We say that the no-arbitrage NA condition holds at time  $t$  when both conditions AIP and SRN holds at time  $t$ . We say that NA holds if it holds at any time.*

We recall that a function  $f : \Omega \times \mathbf{R}^d \rightarrow \overline{\mathbf{R}}$  is an  $\mathcal{F}_t$ -normal integrand, if its epigraph is  $\mathcal{F}_t$ -measurable and closed. Since the probability space is complete, we know by [35, Corollary 14.34] that it is equivalent to suppose that  $f(\omega, x)$  is  $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable and lower semi-continuous (lsc) in  $x$ . Moreover, by [35, Theorem 14.37], we have:

**Proposition 2.10.** *If  $f$  is an  $\mathcal{F}_t$ -normal integrand,  $\inf_{y \in \mathbf{R}^d} f(\omega, y)$  is  $\mathcal{F}_t$ -measurable and  $\{(\omega, x) \in \Omega \times \mathbf{R}^d : f(\omega, x) = \inf_{y \in \mathbf{R}^d} f(\omega, y)\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$  is a measurable closed set.*

As we may choose a jointly measurable version of  $g_t(\omega, x)$  when the payoff is  $h_{t+1} = 0$ , we consider a jointly measurable version of  $\rho_t(\omega, x) := \rho_t(x\Delta S_{t+1})$  i.e.  $\rho_t(\omega, x)$  is  $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable. Then,  $\rho_t$  is an  $\mathcal{F}_t$ -normal integrand. By Proposition 2.10, the set  $\Gamma_t = \{z : \rho_t(z\Delta S_{t+1}) = \inf_{y \in S(0,1)} \rho_t(y\Delta S_{t+1})\}$  is  $\mathcal{F}_t$ -measurable. Moreover, each  $\omega$ -section of  $\Gamma_t$  is non empty since  $\rho_t$  is lsc and  $S(0, 1)$  is compact. Therefore, by a measurable selection argument, we may select  $z_t \in L^0(S(0, 1), \mathcal{F}_t)$  such that  $\rho(z_t\Delta S_{t+1}) = \inf_{z \in S(0,1)} \rho_t(z\Delta S_{t+1})$  a.s..

**Theorem 2.11.** *Let  $h_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  be a payoff such that  $\rho_{t-1}(h_t) < \infty$  a.s.. Consider the random function  $g_{t-1}$  associated to  $h_t$  given by (2.5). For any  $z_{t-1} \in L^0(S(0, 1), \mathcal{F}_{t-1})$ , consider the random set*

$$F_{t-1} = \{\rho_{t-1}(z_{t-1}\Delta S_t) > 0\} \cap \{\rho_{t-1}(-z_{t-1}\Delta S_t) > 0\}.$$

We have:

$$\lim_{|r| \rightarrow \infty} g_{t-1}(\omega, rz_{t-1}) = +\infty, \quad \forall \omega \in F_{t-1}.$$

hence  $g_{t-1}$  admits a minimum on the line  $\mathbf{R}z_{t-1}$  when  $\omega \in F_{t-1}$ .

*Proof.* Consider  $r_n \in L^0((0, \infty), \mathcal{F}_t)$ , for all  $n \geq 1$ , such that  $r_n \rightarrow \infty$  a.s.. We have:

$$\begin{aligned} g_{t-1}(r_n z_{t-1}) &= r_n z_{t-1} S_{t-1} + \rho_{t-1}(r_n z_{t-1} S_t - h_t), \\ &= r_n \left( z_{t-1} S_{t-1} + \rho_{t-1} \left( z_{t-1} S_t - \frac{h_t}{r_n} \right) \right), \\ &= r_n \left( \rho_{t-1}(z_{t-1}\Delta S_t) + \rho_{t-1} \left( z_{t-1} S_t - \frac{h_t}{r_n} \right) - \rho_{t-1}(z_{t-1} S_t) \right). \end{aligned}$$

Observe that:

$$\rho_{t-1} \left( z_{t-1} S_t - \frac{h_t}{r_n} \right) - \rho_{t-1}(z_{t-1} S_t) \geq -\frac{1}{r_n} \rho_{t-1}(h_t).$$

As  $\rho_{t-1}(h_t) < \infty$  a.s., we deduce that  $\lim_{n \rightarrow +\infty} g_{t-1}(r_n z_{t-1}) = +\infty$  on  $F_{t-1}$ . Now, let us suppose that there is a non null set  $G_{t-1}$  of  $\mathcal{F}_{t-1}$  such that  $g_{t-1}(\omega, rz_{t-1})$  does not converge to  $+\infty$  if  $r \rightarrow \infty$  when  $\omega \in G_{t-1}$ . Note that  $\omega \in G_{t-1}$  if and only if there exists  $m(\omega) \in \mathbf{R}$  such that, for all  $n \geq 1$ , there exists  $r_n(\omega) \geq n$  such that  $g_{t-1}(\omega, r_n(\omega)) \leq m(\omega)$ . Consider the following set

$$\Gamma_{t-1}(\omega) = \{(m, (r_n)_{n=1}^\infty) \in \mathbf{R} \times \mathbf{R}^{\mathbf{N}} : r_n \geq n \text{ and } g_t(\omega, r_n) \leq m, \forall n \geq 1\}.$$

The Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbf{R}^{\mathbb{N}})$  is defined as the smallest topology on  $\mathbf{R}^{\mathbb{N}}$  such that the projection mappings  $P^n : (r_j)_{j=1}^{\infty} \mapsto r_n$ ,  $n \geq 1$ , are continuous. Therefore, we deduce that  $\Gamma_{t-1}$  is  $\mathcal{F}_{t-1}$ -measurable. As  $\Gamma_{t-1}$  is non empty on  $G_{t-1}$ , we deduce a  $\mathcal{F}_{t-1}$ -measurable selection  $(m, (r_n)_{n=1}^{\infty})$  of  $\Gamma_{t-1}$  on  $G_{t-1}$  that we extend to the whole space  $\Omega$  by  $m(\omega) = +\infty$  and  $r_n(\omega) = n$ , if  $\omega \in \Omega \setminus G_{t-1}$ . Since the  $\mathcal{F}_{t-1}$ -measurable sequence  $(r_n)_{n=1}^{\infty}$  converges a.s. to  $+\infty$ , we deduce that  $\lim_{n \rightarrow +\infty} g_{t-1}(r_n z_{t-1}) = +\infty$  on  $G_{t-1}$  by the first part of the proof. This is in contradiction with the property  $g_t(\omega, r_n(\omega)) \leq m(\omega)$ , for all  $n \geq 1$ , if  $\omega \in G_{t-1}$ .

Similarly, by symmetry, we may also prove that  $\lim_{r \rightarrow -\infty} g_{t-1}(r z_{t-1}) = +\infty$  on  $F_t$ . As  $g_{t-1}$  is lsc, we finally deduce that  $g_{t-1}$  achieves a minimum on  $\mathbf{R}z_{t-1}$ .  $\square$

**Proposition 2.12.** *Let  $h_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  be such that  $\rho_{t-1}(h_t) < \infty$  and  $\rho_{t-1}(-h_t) < \infty$  a.s.. Consider the function  $g_{t-1}$  associated to  $h_t$  given by (2.5). Consider  $z_{t-1} \in L^0(S(0, 1), \mathcal{F}_{t-1})$  such that we have  $\rho_{t-1}(z_{t-1} \Delta S_t) = \inf_{z \in S(0, 1)} \rho_t(z \Delta S_t)$ . Then, on the set*

$$F_{t-1} = \{\rho_{t-1}(z_{t-1} \Delta S_t) > 0\} \cap \{\rho_{t-1}(-z_{t-1} \Delta S_t) > 0\},$$

the random function  $g_{t-1}$  admits a minimum.

*Proof.* For any  $z \in S(0, 1)$ , we have  $\rho_{t-1}(z \Delta S_t) > 0$  and  $\rho_{t-1}(-z \Delta S_t) > 0$  by definition of  $F_{t-1}$  and  $z_{t-1}$ . By theorem 2.11, there exists  $r_{t-1} \in L^0(\mathbf{R}_+, \mathcal{F}_{t-1})$  such that  $\inf_{r \in \mathbf{R}} g_{t-1}(r z_{t-1}) = g_{t-1}(r_{t-1} z_{t-1})$ . Notice that, by definition, we have  $g_{t-1}(r_{t-1} z_{t-1}) \leq g_{t-1}(0) = \rho_{t-1}(-h_t)$ . On the set  $\{r_{t-1} > 0\}$ , this is equivalent to

$$\begin{aligned} r_{t-1} \left( z_{t-1} S_{t-1} + \rho_{t-1} \left( z_{t-1} S_t - \frac{h_t}{r_{t-1}} \right) \right) &\leq \rho_{t-1}(-h_t), \\ r_{t-1} \left( z_{t-1} S_{t-1} + \rho_{t-1}(z_{t-1} S_t) + \rho_{t-1} \left( z_{t-1} S_t - \frac{h_t}{r_{t-1}} \right) - \rho_{t-1}(z_{t-1} S_t) \right) &\leq \rho_{t-1}(-h_t). \end{aligned}$$

We observe that:

$$\rho_{t-1} \left( z_{t-1} S_t - \frac{h_t}{r_{t-1}} \right) - \rho_{t-1}(z_{t-1} S_t) \geq -\frac{1}{r_{t-1}} \rho_{t-1}(h_t).$$

Therefore,  $r_{t-1}(z_{t-1} S_{t-1} + \rho_{t-1}(z_{t-1} S_t)) \leq \rho_{t-1}(-h_t) + \rho_{t-1}(h_t)$ , i.e.

$$r_{t-1} \leq \frac{\rho_{t-1}(-h_t) + \rho_{t-1}(h_t)}{\rho_{t-1}(z_{t-1} \Delta S_t)}.$$

Similarly, on the set  $\{r_{t-1} < 0\}$ , we deduce that:

$$-r_{t-1} \leq \frac{\rho_{t-1}(-h_t) + \rho_{t-1}(h_t)}{\rho_{t-1}(-z_{t-1}\Delta S_t)}.$$

We finally deduce that, in any case, we have:

$$|r_{t-1}| \leq \max\left(\frac{|\rho_{t-1}(-h_t) + \rho_{t-1}(h_t)|}{\rho_{t-1}(z_{t-1}\Delta S_t)}, \frac{|\rho_{t-1}(-h_t) + \rho_{t-1}(h_t)|}{\rho_{t-1}(-z_{t-1}\Delta S_t)}\right) = M_{t-1} < \infty.$$

on  $F_{t-1}$ . At last, we deduce that for each  $\omega \in F_{t-1}$ :

$$\inf_{x \in \mathbf{R}^d} g_{t-1}(x) = \inf_{r \in [-M_{t-1}, M_{t-1}]} \inf_{z \in S(0,1)} g_{t-1}(rz) = \inf_{x \in \bar{B}(0, M_{t-1})} g_{t-1}(x),$$

where  $\bar{B}(0, M_{t-1})$  is the closed ball of radius  $M_{t-1}$  and centered at the origin. Since  $\bar{B}(0, M_{t-1})$  is compact and  $g_{t-1}$  is lsc, we deduce that  $g_{t-1}$  admits a minimum on  $\bar{B}(0, M_{t-1})$ . By Proposition 2.10, observe that there exists a measurable version of an argmin, using a measurable selection argument.  $\square$

We now prove that, under NA, infimum super-hedging prices are minimal prices.

**Theorem 2.13.** *Suppose that NA holds at time  $t \leq T$  and consider a payoff  $h_{t+1} \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that  $|\rho_t(h_{t+1})| + |\rho_t(-h_{t+1})| < \infty$  a.s.. Then, the minimal risk-hedging price  $P_t^*$  for the payoff  $h_{t+1}$  is a price.*

*Proof.* Suppose first that  $d = 2$ . Since  $\rho_t$  is lsc, there exists  $z_t \in L^0(S(0, 1), \mathcal{F}_t)$  such that  $\inf_{z \in S(0,1)} \rho_t(z\Delta S_{t+1}) = \rho_t(z_t\Delta S_{t+1})$ . By Proposition 2.12 and under SRN,  $g_t$  attains a minimum on  $\mathbf{R}^2$  when  $\omega \in F_t = \{\rho_t(z_t\Delta S_{t+1}) > 0\} \in \mathcal{F}_t$ .

Let us now suppose that  $\omega \in F_t^c = \{\rho_t(z_t\Delta S_{t+1}) = \rho_t(-z_t\Delta S_{t+1}) = 0\}$ . We consider a line that is parallel to the line  $\mathbf{R}z_t$ . For any  $z_1, z_2 \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  on that line such that  $z_1 - z_2 = r_t z_t \in \mathbf{R}z_t$ ,  $r_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ , we have:

$$\begin{aligned} g_t(z_1) &= \rho_t((z_2 + r_t z_t)\Delta S_{t+1} - h_{t+1}) \\ &\leq \rho_t(z_2\Delta S_{t+1} - h_{t+1}) + \rho_t(r_t z_t\Delta S_{t+1}) = g_t(z_2) \end{aligned}$$

By symmetry, we also have:  $g_t(z_2) \leq g_t(z_1)$ , hence  $g_t(z_1) = g_t(z_2)$ . Therefore,  $g_t$  is constant on any line which is parallel to  $\mathbf{R}z_t$ . Moreover,

$$\{(\omega, z_t^\perp) \in \Omega \times \mathbf{R}^2 : z_t^\perp z_t(\omega) = 0\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^2).$$

By measurable selection argument, we may choose  $z_t^\perp \in L^0(S(0, 1), \mathcal{F}_t)$  such that the line  $\mathbf{R}z_t^\perp$  is orthogonal to  $\mathbf{R}z_t$ . Since  $d = 2$ , for any  $x \in \mathbf{R}^2$ , there exist  $\lambda \in \mathbf{R}$  such that  $x - \lambda z_t^\perp \in \mathbf{R}z_t$ . We then have:

$$\inf_{x \in \mathbf{R}^2} g_t(x) = \inf_{\lambda \in \mathbf{R}} g_t(\lambda z_t^\perp).$$

On the set  $\{\rho_t(z_t^\perp \Delta S_{t+1}) = 0\}$ , we get that  $\inf_{\lambda \in \mathbf{R}} g_t(\lambda z_t^\perp) = g_t(0)$  by Proposition 2.7. On the other hand, on the set  $\{\rho_t(z_t^\perp \Delta S_{t+1}) > 0\}$ , we get that  $\lim_{|\lambda| \rightarrow \infty} g_t(\lambda z_t^\perp) = +\infty$  by Proposition 2.11 and SRN, hence  $g_t$  achieves a minimum on the line  $\mathbf{R}z_t^\perp$ .

Let us now prove the  $d$ -dimensional case by induction. Recall that there exists  $z_t \in L^0(S(0, 1), \mathcal{F}_t)$  such that  $\rho_t(z_t \Delta S_{t+1}) = \inf_{z \in S(0, 1)} \rho_t(z \Delta S_{t+1})$ . On  $F_t = \{\rho_t(z_t \Delta S_{t+1}) > 0\}$ , by Proposition 2.12 and SRN,  $g_t$  attains a minimum on  $\mathbf{R}^d$ . On  $F_t^c = \{\rho_t(z_t \Delta S_{t+1}) = 0\}$ , consider an hyperplane  $I_{d-1}$  which is orthogonal to  $\mathbf{R}z_t$  and admits an orthonormal basis  $(z_1, z_2, \dots, z_{d-1})$  such that for each  $\omega \in \Omega$ ,  $\hat{z} = (z_t, z_1, \dots, z_{d-1})$  is an orthonormal basis for  $\mathbf{R}^d$ . Note that each  $z_i$  can be chosen in  $L^0(S(0, 1), \mathcal{F}_t)$ . Indeed, similarly to the case  $d = 2$ , we first choose  $z_1 \in L^0(S(0, 1), \mathcal{F}_t)$  orthogonal to  $z_t$ . Recursively, for  $i \in \{2, \dots, d-1\}$ , we have:

$$\{(\omega, z_i) \in \Omega \times \mathbf{R}^d : z_i z_j(\omega) = 0 \text{ for all } j = 0, \dots, i-1\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d).$$

By measurable selection argument, we then choose  $z_i \in L^0(S(0, 1), \mathcal{F}_t)$ . We denote by  $M_t$  the matrix such that  $z_{i-1} = M_t e_i$ , for every  $i \geq 1$ , where  $e_i = (0, \dots, 1, \dots, 0) \in \mathbf{R}^d$ . We recall the change of variable  $x = M_t \tilde{x}$  where  $x$  and  $\tilde{x}$  are the coordinates of an arbitrary vector of  $\mathbf{R}^d$  in the basis  $(e_i)_{i \geq 1}$  and  $(z_i)_{i \geq 0}$  respectively. The  $i$  column vector of  $M_t$  coincides with  $z_i$  expressed in the basis  $(e_i)_{i \geq 1}$ , hence each entry of  $M_t$  belongs to  $L^0(\mathbf{R}, \mathcal{F}_t)$  and so do the components of  $M_t^{-1}$ . We then define the adapted processes  $\tilde{S}_u = M_t^{-1} S_u = M'_t S_u$ , for  $u = t, t+1$ . We have:

$$g_t(x) = \rho_t(x \Delta S_{t+1} - h_{t+1}) = \rho_t(\tilde{x} \Delta \tilde{S}_{t+1} - h_{t+1}).$$

We observe that  $\tilde{S}_{u=t, t+1}$  forms a new market model which also satisfies the NA condition between  $t$  and  $t+1$ . Indeed, for any  $z \in S(0, 1)$ , we have:

$$\rho_t(z \Delta \tilde{S}_{t+1}) = \rho_t(z M'_t \Delta S_{t+1}),$$

hence  $\rho_t(z\Delta\tilde{S}_{t+1}) = 0$  implies that  $\rho_t(-zM'_t\Delta S_{t+1}) = 0$  by the NA condition satisfied in the market formed by  $S$  which, in turn, implies  $\rho_t(-z\Delta\tilde{S}_{t+1}) = 0$ .

Fix  $\omega$  and, for any  $x \in \mathbf{R}^d$ , consider the orthogonal projection  $\bar{x}$  of  $x$  onto  $I_{d-1}$ . We then have  $g_t(x) = g_t(\bar{x})$ . For  $\bar{x} \in I_{d-1}$ , we denote  $\hat{x} = M_t^{-1}\bar{x}$ , we have:

$$\bar{x}\Delta S_{t+1} = \hat{x}\Delta\tilde{S}_{t+1} := \sum_{i=1}^d \hat{x}^i \Delta\tilde{S}_{t+1}^i = \sum_{i=2}^d \hat{x}^i \Delta\tilde{S}_{t+1}^i,$$

since the first coordinate of  $\hat{x}$  equals 0 in the new basis. We deduce that:

$$\inf_{x \in \mathbf{R}^d} g_t(x) = \inf_{x \in I_{d-1}} \rho_t(x\Delta S_{t+1} - h_{t+1}) = \inf_{\hat{x} \in \mathbf{R}^{d-1}} \rho_t \left( \sum_{i=2}^d \hat{x}^i \Delta\tilde{S}_{t+1}^i - h_{t+1} \right)$$

This means that we have reduced the optimization problem to a market with only  $d - 1$  assets defined by  $(\tilde{S}^2, \dots, \tilde{S}^d)$ . As it satisfies the NA condition, we deduce that  $\inf_{x \in \mathbf{R}^d} g_t(x)$  is attained by induction.  $\square$

Observe that the theorem above provides the existence of an optimal hedging strategy  $\theta_t^* \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that

$$P_t^* = g_t(\theta_t^*) = \theta_t^* S_t + \rho_t(\theta_t^* S_{t+1} - h_{t+1}) \in \mathcal{P}_t(h_{t+1}).$$

In the following, we say that a payoff  $h_{t+1}$  is not freely attainable at time  $t$  if it satisfies  $\rho_t(-h_{t+1}) > 0$  a.s. and  $|\rho_t(h_{t+1})| + |\rho_t(-h_{t+1})| < \infty$  a.s.. Note that if  $\rho_t(-h_{t+1}) > 0$ , then it is not possible to get the payoff  $h_{t+1}$  from nothing when writing  $0 = h_{t+1} + (-h_{t+1})$  and letting aside  $(-h_{t+1})$  since the latter is not acceptable. Notice that, in the usual case where  $\rho_t(X) = -\text{ess inf}_{\mathcal{F}_t}(X)$ ,  $\rho_t(-h_{t+1}) > 0$  means that  $\text{ess sup}_{\mathcal{F}_t}(h_{t+1}) > 0$  and recall that  $h_{t+1}$  is acceptable if  $h_{t+1} \geq 0$  a.s.. The following theorem gives an interpretation of the NA condition. Precisely, NA means that the price of any no freely attainable and acceptable payoff is strictly positive. In the usual case, a no freely attainable and acceptable payoff is a non negative payoff which does not vanish on a non null  $\mathcal{F}_t$ -measurable set.

**Theorem 2.14.** *The NA condition holds at time  $t \leq T$  if and only if the infimum risk-hedging price  $P_t^*$  of any no freely attainable and acceptable payoff  $h_{t+1}$  at time  $t$  is strictly positive. Moreover, under NA, the infimum risk-hedging price  $P_t^*$  of any contingent claim  $h_{t+1}$  satisfies*

$$\rho_t(-h_{t+1}) \geq P_t^* \geq -\rho_t(h_{t+1}).$$

*Proof.* Suppose that NA holds. By Theorem 2.13, there is  $z_t \in L^0(S(0, 1), \mathcal{F}_t)$  and  $r_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  such that  $P_t^* = \rho_t(r_t z_t \Delta S_{t+1} - h_{t+1})$ . Suppose that  $\rho_t(z_t \Delta S_{t+1})$  and  $\rho_t(-z_t \Delta S_{t+1})$  are both equal to 0. Then, the function  $g_t$  associated to  $h_{t+1}$ , see (2.5), is constant on the line  $\mathbf{R}z_t$  by Theorem 2.7. Therefore,  $P_t^* = g_t(0) = \rho_t(-h_{t+1}) > 0$ . Otherwise, under NA,  $\rho_t(z_t \Delta S_{t+1}) > 0$  and  $\rho_t(-z_t \Delta S_{t+1}) > 0$ . Using triangular inequalities, and the assumption  $\rho_t(h_{t+1}) \leq 0$ , we then deduce that:

$$\begin{aligned} P_t^* &= r_t z_t S_t + \rho_t(r_t z_t S_{t+1} - h_{t+1}), \\ &= \rho_t(-h_{t+1}) 1_{r_t=0} + r_t \rho_t \left( z_t \Delta S_{t+1} - \frac{h_{t+1}}{r_t} \right) 1_{r_t>0} \\ &\quad - r_t \rho_t \left( -z_t \Delta S_{t+1} + \frac{h_{t+1}}{r_t} \right) 1_{r_t<0}, \\ &\geq \rho_t(-h_{t+1}) 1_{r_t=0} + r_t \rho_t(z_t \Delta S_{t+1}) 1_{r_t>0} - r_t \rho_t(-z_t \Delta S_{t+1}) 1_{r_t<0}, \\ &> 0. \end{aligned}$$

For the reverse implication, let us prove first that AIP holds. We fix  $h_{t+1}$  such that  $\rho_t(-h_{t+1}) > 0$  and  $\rho_t(h_{t+1}) \leq 0$ . So, with the function  $g_t$  associated to  $h_{t+1}$ , see (2.5), we have  $P_t^* = P_t^*(h_{t+1}) = \inf_{x \in \mathbf{R}} g_t(x) > 0$  by assumption and  $g_t(rz) > 0$  for all  $r \in \mathbf{R}$  and  $z \in S(0, 1)$ . Let us show that the set  $\{z S_t + \rho_t(z S_{t+1}) < 0\}$  is empty for all  $z \in S(0, 1)$  a.s.. In the contrary case, by measurable selection, we may construct  $z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that we have  $P(z_t S_t + \rho_t(z_t S_{t+1}) < 0) > 0$ . We then define

$$r_t := -\frac{\rho_t(-h_{t+1})}{\rho_t(z_t \Delta S_{t+1})} 1_{\{\rho_t(z_t \Delta S_{t+1}) < 0\}} \geq 0.$$

We have

$$\begin{aligned} g_t(r_t z_t) &= r_t z_t S_t + \rho_t(r_t z_t S_{t+1} - h_{t+1}), \\ &\leq r_t z_t S_t + \rho_t(r_t z_t S_{t+1}) + \rho_t(-h_{t+1}), \\ &\leq r_t \rho_t(z_t \Delta S_{t+1}) + \rho_t(-h_{t+1}), \\ &\leq \rho_t(-h_{t+1}) 1_{\{\rho_t(z_t \Delta S_{t+1}) \geq 0\}}. \end{aligned}$$

Therefore,  $P_t^* \leq 0$  on the set  $\{\rho_t(z_t \Delta S_{t+1}) < 0\}$  in contradiction with  $P_t^* > 0$ .



Let us show that  $\rho_t(-z\Delta S_{t+1}) = 0$  if  $\rho_t(z\Delta S_{t+1}) = 0$  for any  $z \in S(0, 1)$ . Otherwise, by measurable selection argument, there exists  $z_t \in L^0(S(0, 1), \mathcal{F}_t)$  such that  $\Lambda_t := \{\rho_t(z_t\Delta S_{t+1}) = 0\} \cap \{\rho_t(-z_t\Delta S_{t+1}) > 0\}$  satisfies  $P(\Lambda_t) > 0$ . If  $h_{t+1} = z_t\Delta S_{t+1}$ , then  $\rho_t(-h_{t+1}) = \rho_t(-z_t\Delta S_{t+1}) > 0$  on  $\Lambda_t$ . On the complementary set, we fix  $h_{t+1} = \gamma_t > 0$ ,  $\gamma_t \in L^0((0, \infty), \mathcal{F}_t)$ . It follows that  $\rho_t(-h_{t+1}) > 0$ . Moreover,  $\rho_t(h_{t+1}) = \rho_t(z_t\Delta S_{t+1}) = 0$  on  $\Lambda_t$  and, otherwise,  $\rho_t(h_{t+1}) = -\gamma_t < 0$ . Therefore,  $\rho_t(h_{t+1}) \leq 0$ . We deduce that  $P_t^*(h_{t+1}) > 0$ , by assumption. On the other hand, if  $r \geq 1$ , and  $\omega \in \Lambda_t$ ,

$$P_t^*(h_{t+1}) \leq \rho_t(rz_t\Delta S_{t+1} - z_t\Delta S_{t+1}) = (r - 1)\rho_t(z_t\Delta S_{t+1}) = 0.$$

It follows that  $P_t^*(h_{t+1}) \leq 0$  on  $\Lambda_t$ , i.e. a contradiction. We conclude that  $\rho_t(z\Delta S_{t+1}) = 0$  if and only if  $\rho_t(-z\Delta S_{t+1}) = 0$  for any  $z \in S(0, 1)$ .

At last, it is clear that  $P_t^*(h_{t+1}) \leq g_t(0) = \rho_t(-h_{t+1})$ . Moreover, for all  $x \in \mathbf{R}^d$ ,  $0 \leq \rho_t(x\Delta S_{t+1}) \leq \rho_t(x\Delta S_{t+1} - h_{t+1}) + \rho_t(h_{t+1})$ . Taking the infimum in the r.h.s. of this inequality, we get that  $0 \leq P_t^*(h_{t+1}) + \rho_t(h_{t+1})$  and we may conclude.  $\square$

**Proposition 2.15.** *Suppose that the risk-measure is  $\rho_t(X) = -\text{ess inf}_{\mathcal{F}_t} X$ . Then, the NA condition coincides with the classical NA condition of frictionless models, i.e. it is equivalent to the existence of a risk-neutral probability measure.*

*Proof.* We know that the existence of a risk-neutral probability measure  $Q \sim P$  implies AIP. Moreover, suppose that  $\rho_t(z\Delta S_{t+1}) = 0$  on  $F_t \in \mathcal{F}_t$  where  $z \in S(0, 1)$ . Then, by definition of  $\rho_t$ ,  $1_{F_t}z\Delta S_{t+1} \geq 0$ . As  $E_Q(1_{F_t}z\Delta S_{t+1}) = 0$ , we deduce that  $1_{F_t}z\Delta S_{t+1} = 0$  hence  $\rho_t(-z\Delta S_{t+1}) = 0$  on  $F_t$ . By symmetry, we deduce that SRN holds.

Reciprocally, suppose that AIP and SRN conditions hold. Let  $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$  such that  $\theta_t\Delta S_{t+1} \geq 0$  a.s.. Let us write  $\theta_t = r_t z_t$  where  $r_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  and  $z_t \in L^0(S(0, 1), \mathcal{F}_t)$ . On the set  $F_t = \{r_t > 0\}$ ,  $z_t\Delta S_{t+1} \geq 0$  hence  $\text{ess inf}_{\mathcal{F}_t}(z_t\Delta S_{t+1}) \geq 0$ . By the AIP condition,  $\rho_t(z_t\Delta S_{t+1}) \geq 0$ . We deduce that  $\text{ess inf}_{\mathcal{F}_t}(z_t\Delta S_{t+1}) = 0 = \rho_t(z\Delta S_{t+1})$ . Under SRN, we deduce that  $\rho_t(-z\Delta S_{t+1}) = 0$  hence  $z\Delta S_{t+1} \geq 0$  so that  $z_t\Delta S_{t+1} = 0$ . By a similar reasoning on the set  $F_t = \{r_t < 0\}$ , we also get that  $z_t\Delta S_{t+1} = 0$  hence  $\theta_t\Delta S_{t+1} = 0$ . We then conclude by [27, Condition (g), p. 73, Section 2.1.1].  $\square$

## 2.2. No arbitrage and pricing under consistency in time

**Definition 2.16.** A dynamic risk-measure  $(\rho_t)_{t \leq T}$  is said time-consistent if  $\rho_{t+1}(X) = \rho_{t+1}(Y)$  implies  $\rho_t(X) = \rho_t(Y)$  for  $X, Y \in L^0(\mathbf{R}, \mathcal{F}_T)$  and  $t \leq T - 1$  (see Section 5 in [16]).

The following result is very well known, see [1].

**Lemma 2.17.** A dynamic risk-measure  $(\rho_t)_{t \leq T}$  is time-consistent if and only if its family of acceptable sets  $(\mathcal{A}_t)_{t \leq T}$  satisfies

$$\mathcal{A}_{t,T} = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1,T} \quad (2.6)$$

for any  $t \leq T - 1$ .

Observe that, if  $(\rho_t)_{t \leq T}$  is time-consistent, we may show by induction that  $\rho_t(-\rho_{t+s}(\cdot)) = \rho_t(\cdot)$  for any  $t \leq T$  and  $s \geq 0$  such that  $s \leq T - t$ . In the following, we introduce another possible definition for the risk-hedging prices in the multi-period model, where the risk is only measured at time  $t$ .

**Definition 2.18.** The contingent claim  $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$  is said directly risk-hedged at time  $t \leq T - 1$  if there exists a (direct) price  $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$  and a strategy  $(\theta_u)_{u=t}^{T-1}$  such that that  $P_t + \sum_{t \leq u \leq T-1} \theta_u \Delta S_{u+1} - h_T$  is acceptable at time  $t$ .

The set of all direct risk-hedging prices at time  $t$  is then given by

$$\bar{\mathcal{P}}_t(h_T) = \left\{ \rho_t \left( \sum_{t \leq u \leq T-1} \theta_u \Delta S_{u+1} - h_T \right) : \theta_u \in L^0(\mathbf{R}^d, \mathcal{F}_u) \right\} + L^0(\mathbf{R}, \mathcal{F}_t).$$

and the infimum direct risk-hedging price is

$$\bar{P}_t^*(h_T) := \operatorname{ess\,inf}_{(\theta_u)_{u=t}^{T-1}} \bar{\mathcal{P}}_t(h_T).$$

The following result from [30] proves that the direct infimum risk-hedging prices may coincide with the infimum prices derived from the step by step backward procedure developed before, i.e. such that

$$P_t^*(h_T) = \operatorname{ess\,inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*(h_T)),$$

where  $P_T^*(h_T) = h_T$ .

**Theorem 2.19.** *Suppose that the dynamic risk-measure  $(\rho_t)_{t \leq T}$  is time consistent. Then,  $\bar{P}_t^*(h_T) = P_t^*(h_T)$  for any  $t \leq T - 1$ . Moreover, the direct infimum risk-hedging prices are direct prices if and only if the infimum prices of the backward procedure are prices.*

**Corollary 2.20.** *Suppose that the dynamic risk-measure  $(\rho_t)_{t \leq T}$  is time consistent. Then,  $\bar{\mathcal{P}}_t(h_T) = \mathcal{P}_t(h_T)$  for all  $t \leq T$ .*

In the following, we consider the set of all attainable claims  $\mathcal{R}_t^T$  between  $t$  and  $T$ , when starting from the zero initial endowment, i.e.

$$\mathcal{R}_{t,T} := \left\{ \sum_{u=t+1}^T \theta_{u-1} \Delta S_u : \theta_u \in L^0(\mathbf{R}^d, \mathcal{F}_u), u \geq t \right\}.$$

We observe that  $\bar{\mathcal{P}}_t(0) = (\mathcal{A}_{t,T} - \mathcal{R}_{t,T}) \cap L^0(\mathbf{R}, \mathcal{F}_t)$ . In the following, we consider the sets  $\mathcal{Z}_{t,T} := \mathcal{R}_{t,T} - \mathcal{A}_{t,T}$  and the sets

$$\mathcal{A}_{t,T}^0 = \{X \in L^0(\mathbf{R}, \mathcal{F}_T) : \rho_t(X) = \rho_t(-X) = 0\}.$$

**Remark 2.21.** *Note that  $\mathcal{A}_{t,T}^0 = \mathcal{A}_{t,T} \cap (-\mathcal{A}_{t,T})$ . Indeed, first observe that  $\mathcal{A}_{t,T}^0 \subseteq \mathcal{A}_{t,T} \cap (-\mathcal{A}_{t,T})$ . Let  $x_{t,T} \in \mathcal{A}_{t,T} \cap (-\mathcal{A}_{t,T})$ . We have:*

$$0 = \rho_t(x_{t,T} - x_{t,T}) \leq \rho_t(x_{t,T}) + \rho_t(-x_{t,T}) \leq 0.$$

*This implies  $\rho_t(x_{t,T}) = \rho_t(-x_{t,T}) = 0$  hence  $x_{t,T} \in \mathcal{A}_{t,T}^0$ .*

*The set  $\mathcal{Z}_{t,T}$  is the family of all claims that are attainable up to an acceptable position at time  $t$  since every attainable claim  $r_{t,T} \in \mathcal{R}_t^T$  may be written as  $r_{t,T} = (r_{t,T} - a_{t,T}) + a_{t,T}$  where  $a_{t,T} \in \mathcal{A}_{t,T}$  is let aside and  $r_{t,T} - a_{t,T} \in \mathcal{Z}_{t,T}$ .*

**Theorem 2.22.** *Assume that the risk measure is time-consistent. Suppose that  $\mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}_{t,T}^0$ . Then, AIP holds and  $\mathcal{Z}_{t,T}$  is closed in  $L^0$  for every  $t \leq T - 1$ .*

*Proof.* Consider  $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ . By Theorem 2.6, it suffices to show that  $\rho_t(\theta_t \Delta S_{t+1}) \geq 0$  a.s.. Otherwise, the set  $\Lambda_t = \{\rho_t(\theta_t \Delta S_{t+1}) < 0\}$  admits a positive probability and  $\theta_t \Delta S_{t+1} 1_{\Lambda_t} \in \mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}_{t,T}^0$ . It follows that  $\rho_t(\theta_t \Delta S_{t+1} 1_{\Lambda_t}) = 0$  hence a contradiction. Therefore, AIP holds.

Let us show that  $\bar{\mathcal{Z}}_{t,T} \subseteq \mathcal{Z}_{t,T}$ . In the one step model, let us suppose that  $\gamma^n = \theta_{T-1}^n \Delta S_T - \epsilon_{T-1,T}^n \in \mathcal{Z}_{T-1,T}$  converges to  $\gamma^\infty \in L^0(\mathbf{R}, \mathcal{F}_T)$  in probability. We suppose that  $\epsilon_{T-1,T}^n \in \mathcal{A}_{T-1,T}$ . We need to show that  $\gamma^\infty \in \mathcal{Z}_{T-1,T}$ .

On the  $\mathcal{F}_{T-1}$ -measurable set  $\Lambda_{T-1} := \{\liminf_n |\theta_{T-1}^n| < \infty\}$ , by [27, Lemma 2.1.2], we may assume w.l.o.g. that  $\theta_{T-1}^n$  is convergent to some  $\theta_{T-1}^\infty$  hence  $\epsilon_{T-1,T}^n$  is also convergent hence  $\gamma^\infty 1_{\Lambda_{T-1}} \in \mathcal{Z}_{T-1,T}$ .

Otherwise, on  $\Omega \setminus \Lambda_{T-1}$ , we use the normalized sequences,

$$\tilde{\theta}_{T-1}^n := \theta_{T-1}^n / (|\theta_{T-1}^n| + 1), \quad \tilde{\epsilon}_{T-1,T}^n := \epsilon_{T-1,T}^n / (|\theta_{T-1}^n| + 1).$$

By [27, Lemma 2.1.2], we may assume that a.s.  $\tilde{\theta}_{T-1}^n \rightarrow \tilde{\theta}_{T-1}^\infty$ ,  $\tilde{\epsilon}_{T-1,T}^n \rightarrow \tilde{\epsilon}_{T-1,T}^\infty$  and  $\tilde{\theta}_{T-1}^\infty \Delta S_T - \tilde{\epsilon}_{T-1,T}^\infty = 0$  a.s.. Note that  $|\tilde{\theta}_{T-1}^\infty| = 1$  a.s.. As  $\tilde{\theta}_{T-1}^\infty \Delta S_T$  is acceptable ( $\epsilon_{T-1,T}^\infty \in \mathcal{A}_{T-1,T}$ ) then  $\tilde{\theta}_{T-1}^\infty \Delta S_T \in \mathcal{A}_{t,T}^0$  by assumption. We follow the recursive arguments on the dimension of [26]. Since  $|\tilde{\theta}_{T-1}^\infty| = 1$ , there exists a partition of  $\Omega \setminus \Lambda_{T-1}$  into  $d$  disjoint subsets  $G_{T-1}^i \in \mathcal{F}_{T-1}$  such that  $\tilde{\theta}_{T-1}^{\infty,i} \neq 0$  on  $G_{T-1}^i$ . Define on  $G_{T-1}^i$ ,  $\tilde{\theta}_{T-1}^n := \theta_{T-1}^n - \beta_{T-1}^n \tilde{\theta}_{T-1}^\infty$  where  $\beta_{T-1}^n := \theta_{T-1}^{n,i} / \tilde{\theta}_{T-1}^{\infty,i}$ . Observe that  $\gamma^n = \tilde{\theta}_{T-1}^n \Delta S_T - \tilde{\epsilon}_{T-1,T}^n$  where the position  $\tilde{\epsilon}_{T-1,T}^n = \epsilon_{T-1,T}^n - \beta_{T-1}^n \tilde{\theta}_{T-1}^\infty \Delta S_T$  is acceptable since  $\pm \tilde{\theta}_{T-1}^\infty \Delta S_T$  are acceptable. As  $\tilde{\theta}_{T-1}^n = 0$  on  $G_{T-1}^i$ , we repeat the entire procedure on each  $G_{T-1}^i$  with the new expression  $\gamma^n = \tilde{\theta}_{T-1}^n \Delta S_T - \tilde{\epsilon}_{T-1,T}^n$  such that the number of components of  $\tilde{\theta}_{T-1}^n$  is reduced by one. We then conclude by recursion on the number of non-zero components since the conclusion is trivial if all the coordinates vanish.

We now show the result in multi-step models by induction. Fix some  $s \in \{t, \dots, T-1\}$ . We show that  $\overline{\mathcal{Z}_{s+1}^T} \subseteq \mathcal{Z}_{s+1}^T$  implies the same property for  $s$  instead of  $s+1$ .

Since AIP holds, we get that  $\mathcal{Z}_{s+1}^T \cap L^0(\mathbf{R}_+, \mathcal{F}_{s+1}) = \{0\}$  hence  $\overline{\mathcal{Z}_{s+1}^T} \subseteq \mathcal{Z}_{s+1}^T$  implies that  $\overline{\mathcal{Z}_{s+1}^T} \cap L^1(\mathbf{R}_+, \mathcal{F}_{s+1}) = \{0\}$ . Using the Hahn-Banach separation theorem in  $L^1$ , we deduce  $Q^{(s+1)} \ll P$  with  $\frac{dQ^{(s+1)}}{dP} \in L^\infty$  such that  $\rho_{s+1} := E_P(\frac{dQ^{(s+1)}}{dP} | \mathcal{F}_{s+1}) = 1$  a.s.,  $(S_u)_{u \geq s+1}$  is a martingale under  $Q^{(s+1)}$  and  $E_Q(a_{s+1,T} | \mathcal{F}_{s+1}) \geq 0$  for all  $a_{s+1,T} \in \mathcal{A}_{s+1,T}$  such that  $E_Q(|a_{s+1,T}| | \mathcal{F}_{s+1}) < \infty$  a.s.. Suppose that

$$\gamma^n = \sum_{u=s+1}^T \theta_{u-1}^n \Delta S_u - \epsilon_{s,T}^n \in \mathcal{Z}_{s,T} \text{ converges to } \gamma^\infty \in L^0(\mathbf{R}, \mathcal{F}_T).$$

We suppose that  $\epsilon_{s,T}^n \in \mathcal{A}_{s,T}$ . By Lemma 2.17,  $\epsilon_{s,T}^n = \epsilon_{s,s+1}^n + \epsilon_{s+1,T}^n$ , where  $\epsilon_{s,s+1}^n \in \mathcal{A}_{s,s+1}$  and  $\epsilon_{s+1,T}^n \in \mathcal{A}_{s+1,T}$ . As before, on the  $\mathcal{F}_s$ -measurable set

$\Lambda_s := \{\liminf_n |\theta_s^n| < \infty\}$ , we may assume w.l.o.g. that  $\theta_s^n$  converges to  $\theta_s^\infty$ . Therefore, on  $\Lambda_s$ ,

$$\sum_{u=s+2}^T \theta_{u-1}^n \Delta S_u - \epsilon_{s,T}^n = \gamma^n - \theta_s^n \Delta S_{s+1} \rightarrow \gamma^\infty - \theta_s^\infty \Delta S_{s+1}.$$

On the subset  $\hat{\Lambda}_{s+1} := \{\liminf_n |\epsilon_{s,s+1}^n| = \infty\} \cap \Lambda_s \in \mathcal{F}_{s+1}$ , we use the normalization procedure as previously, i.e. we divide by  $|\epsilon_{s,s+1}^n|$ , up to a subsequence, and, by the induction hypothesis, we obtain that

$$\sum_{u=s+2}^T \tilde{\theta}_{u-1}^n \Delta S_u - \tilde{\epsilon}_{s+1,T} = \tilde{\epsilon}_{s,s+1},$$

where  $\tilde{\epsilon}_{s+1,T} \in \mathcal{A}_{s+1,T}$  and  $\tilde{\epsilon}_{s,s+1} \in \mathcal{A}_{s,s+1}$  satisfies  $|\tilde{\epsilon}_{s,s+1}| = 1$  a.s.. Moreover, by assumption, we may show that

$$E_{Q^{(s+1)}} \left( \sum_{u=s+2}^T \tilde{\theta}_{u-1}^n \Delta S_u | \mathcal{F}_{s+1} \right) = 0.$$

Moreover, still by assumption,  $E_{Q^{(s+1)}}(\tilde{\epsilon}_{s+1,T} | \mathcal{F}_{s+1}) \geq 0$ . We deduce that  $\tilde{\epsilon}_{s,s+1} = E_{Q^{(s+1)}}(\tilde{\epsilon}_{s,s+1} | \mathcal{F}_{s+1}) \leq 0$ . Therefore,  $\tilde{\epsilon}_{s,s+1} = -1$  hence  $\rho_s(\tilde{\epsilon}_{s,s+1}) = \rho_s(-1) = 1$ , which is in contradiction with  $\rho_s(\tilde{\epsilon}_{s,s+1}) \leq 0$ . Therefore, we may suppose, on  $\Lambda_s$ , that  $\epsilon_{s,s+1}^n$  converges a.s. to some  $\epsilon_{s,s+1} \in \mathcal{A}_{s,s+1}$ . By the induction hypothesis, we then deduce that  $\sum_{u=s+2}^T \theta_{u-1}^n \Delta S_u - \epsilon_{s+1,T}^n$  also converges to an element of  $\mathcal{Z}_{s+1}^T$  and we conclude that  $\gamma^\infty 1_{\Lambda_s} \in \mathcal{Z}_s^T$ .

On  $\Omega \setminus \Lambda_s$ , we use the normalisation procedure as before, and deduce the equality

$$\sum_{u=s+1}^T \tilde{\theta}_{u-1}^\infty \Delta S_u - \tilde{\epsilon}_{s,T}^\infty = 0 \text{ a.s.}$$

for some  $\tilde{\theta}_u^\infty \in L^0(\mathbf{R}, \mathcal{F}_u)$ ,  $u \in \{s, \dots, T-1\}$  and  $\tilde{\epsilon}_{s,T}^\infty \in \mathcal{A}_{s,T}$ . By Lemma 2.17, we write  $\tilde{\epsilon}_{s,T}^\infty = \tilde{\epsilon}_{s,s+1}^\infty + \tilde{\epsilon}_{s+1,T}^\infty$  where  $\tilde{\epsilon}_{s,s+1}^\infty \in \mathcal{A}_{s,s+1}$  and  $\tilde{\epsilon}_{s+1,T}^\infty \in \mathcal{A}_{s+1,T}$ . Moreover,  $|\tilde{\theta}_s^\infty| = 1$  a.s.. We deduce that:

$$\tilde{\theta}_s^\infty \Delta S_{s+1} + \sum_{u=s+2}^T \tilde{\theta}_{u-1}^\infty \Delta S_u - \tilde{\epsilon}_{s+1,T}^\infty = \tilde{\epsilon}_{s,s+1}^\infty \text{ a.s..}$$

Taking the conditional expectation knowing  $\mathcal{F}_{s+1}$  under  $Q^{(t+1)}$ , we deduce that  $\tilde{\epsilon}_{s,s+1}^\infty \leq \tilde{\theta}_s^\infty \Delta S_{s+1}$ . It follows that  $\rho_s(\tilde{\theta}_s^\infty \Delta S_{s+1}) \leq \rho_s(\tilde{\epsilon}_{s,s+1}^\infty) \leq 0$  hence  $\tilde{\theta}_s^\infty \Delta S_{s+1} \in \mathcal{A}_{s,T}^0$  by the assumption. Using the one step arguments based on the elimination of non-zero components of the sequence  $\theta_s^n$ , we may replace  $\theta_s^n$  by  $\tilde{\theta}_s^n$  such that  $\tilde{\theta}_s^n$  converges. We then repeat the same arguments on the set  $\Lambda_s$  to conclude that  $\gamma^\infty 1_{\Omega \setminus \Lambda_s} \in \mathcal{Z}_s^T$ .  $\square$

**Theorem 2.23.** *Suppose that the risk-measure is time-consistent. Suppose that NA holds and  $\mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$ , for every  $t \leq T$ . Then, we have  $\mathcal{Z}_{t,T} \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$  and  $\mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}_{t,T}^0$  for every  $t$ .*

*Proof.* Let us consider  $W_{t,T} \in \mathcal{R}_{t,T} \cap \mathcal{A}_{t,T}$ . Then,  $W_{t,T}$  is of the form:

$$W_{t,T} = \sum_{s=t+1}^T \theta_{s-1} \Delta S_s = \sum_{s=t+1}^T a_{s-1,s},$$

where  $\theta_{s-1} \in L^0(\mathbf{R}, \mathcal{F}_{s-1})$  and  $a_{s-1,s} \in \mathcal{A}_{s-1,s}$ , for all  $s = t+1, \dots, T$ . It follows that:

$$\theta_t \Delta S_{t+1} - a_{t,t+1} + \sum_{s=t+2}^T (\theta_{s-1} \Delta S_s - a_{s-1,s}) = 0. \quad (2.7)$$

Therefore,  $p_t = \theta_t \Delta S_{t+1} - a_{t,t+1}$  is a (direct) price at time  $s = t+1$  for the zero claim. Under AIP condition, we get that  $\theta_t \Delta S_{t+1} \geq a_{t,t+1}$  hence  $\rho_t(\theta_t \Delta S_{t+1}) \leq 0$ . As  $\rho_t(\theta_t \Delta S_{t+1}) \geq 0$  by AIP,  $\rho_t(\theta_t \Delta S_{t+1}) = 0$  and, by RN, we get that  $\rho_t(\theta_t \Delta S_{t+1}) = \rho_t(-\theta_t \Delta S_{t+1}) = 0$ . We then deduce that  $-p_t \in \mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$  hence  $p_t = 0$  and  $\theta_t \Delta S_{t+1} = a_{t,t+1} \in \mathcal{A}_{t,T}^0$ . The equality (2.7) may be rewritten as:

$$\theta_{t+1} \Delta S_{t+2} - a_{t+1,t+2} + \sum_{s=t+3}^T (\theta_{s-1} \Delta S_s - a_{s-1,s}) = 0. \quad (2.8)$$

By induction, we finally deduce that  $\theta_s \Delta S_{t+1} = a_{s,s+1} \in \mathcal{A}_{s,s+1}^0$  for all  $s \geq t$ . By Remark 2.21, we have  $W_{t,T} \in \mathcal{A}_{t,T}^0$ .

Consider now  $\epsilon_T^+ \in \mathcal{Z}_{t,T} \cap L^0(\mathbf{R}_+, \mathcal{F}_T)$ . We may write  $\epsilon_T^+ = r_{t,T} - a_{t,T}$  where  $r_{t,T} \in \mathcal{R}_{t,T}$  and  $a_{t,T} \in \mathcal{A}_{t,T}$ . We get that  $r_{t,T} = a_{t,T} + \epsilon_T^+ \in \mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}_{t,T}^0$  hence  $-r_{t,T} \in \mathcal{A}_{t,T}$ . It follows that  $-\epsilon_T^+ \in \mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$ .  $\square$

**Theorem 2.24** (FTAP). *Suppose that the risk-measure is time-consistent and  $\mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$  for every  $t \leq T$ . Then, the following statements are equivalent:*

- 1) NA
- 2)  $\mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}_{t,T}^0$ , for every  $t \leq T$ .
- 3)  $\mathcal{R}_{0,T} \cap \mathcal{A}_{0,T} = \mathcal{A}_{0,T}^0$ .
- 4)  $\overline{\mathcal{Z}_{t,T}} \cap \mathcal{A}_{t,T} = \mathcal{A}_{t,T}^0$ , for every  $t \leq T$ .
- 5)  $\overline{\mathcal{Z}_{0,T}} \cap \mathcal{A}_{0,T} = \mathcal{A}_{0,T}^0$ .
- 6)  $\mathcal{Z}_{0,T} \cap \mathcal{A}_{0,T} = \mathcal{A}_{0,T}^0$  and  $\mathcal{Z}_{0,T}$  is closed in  $L^0$ .
- 7) For all  $t \leq T-1$ , there exists  $Q = Q^t \sim P$  with  $dQ/dP \in L^\infty((0, \infty), \mathcal{F}_T)$  such that  $(S_u)_{u=t}^T$  is a  $Q$ -martingale and, for all  $t \leq T-1$ , for all  $X$  such that  $E_Q(X^- | \mathcal{F}_t) < \infty$  a.s.,  $\rho_t(X) \geq -E_Q(X | \mathcal{F}_t)$ .

Moreover, for all  $x \in \mathcal{A}_{t,T} \setminus \mathcal{A}_{t,T}^0$ , there exists such a  $Q = Q_x^t$  such that  $P(E_Q(x | \mathcal{F}_t) \neq 0) > 0$ .

*Proof.* Suppose that 1) holds. By Theorem 2.23, we deduce that 3) holds. Note that 2) and 3) are equivalent since the risk measure is time consistent. Suppose that 3) holds. Since  $-\mathcal{A}_{t,T} \subseteq \mathcal{Z}_{t,T}$ , it follows that  $\mathcal{A}_{t,T}^0 \subseteq \mathcal{Z}_{t,T} \cap \mathcal{A}_{t,T}$ . Reciprocally, consider  $x_{t,T} = W_{t,T} - a_{t,T} \in \mathcal{Z}_{t,T} \cap \mathcal{A}_{t,T}$ , where  $W_{t,T} \in \mathcal{R}_{t,T}$  and  $a_{t,T} \in \mathcal{A}_{t,T}$ , then  $W_{t,T} \in \mathcal{A}_{t,T}$  hence  $W_{t,T} \in \mathcal{A}_{t,T}^0$  by 2). It follows that  $x_{t,T} \in (-\mathcal{A}_{t,T})$  and we conclude that  $\mathcal{Z}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}_{t,T}^0$ . Moreover, by Theorem 2.22,  $\mathcal{Z}_{t,T}$  is closed in probability hence 4) holds. Note that 4) and 5) are equivalent since the risk measure is time consistent.

Assume that 4) holds. The existence of  $Q$  in 7) holds by standard arguments based on the Hahn-Banach separation theorem. In particular, NA holds under  $P'$  such that  $P' \sim P$ . We suppose w.l.o.g that  $S_t$  is integrable under  $P$  for every  $t$ . If  $x \in L^1(\mathbf{R}, \mathcal{F}_T) \cap (\mathcal{A}_{t,T} \setminus \mathcal{A}_{t,T}^0)$ ,  $x \notin \mathcal{Z}_{t,T} \cap L^1(\mathbf{R}, \mathcal{F}_T)$ . By the Hahn-Banach separation theorem, we deduce the existence of  $p_x \in L^\infty(\mathbf{R}, \mathcal{F}_T)$  and  $c \in \mathbf{R}$  such that:

$$E(p_x X) < c < E(x p_x), \quad \forall X \in \mathcal{Z}_{t,T}.$$

As  $\mathcal{Z}_{t,T}$  is a cone, we get that  $E(p_x X) \leq 0$  for all  $X \in \mathcal{Z}_{t,T}$  and since  $-L^0(\mathbf{R}_+, \mathcal{F}_T) \subseteq \mathcal{Z}_{t,T}$ , we deduce that  $p_x \geq 0$  a.s.. With  $X = 0$ , we get that  $E(x p_x) > 0$  and, as  $\mathcal{R}_{t,T}$  is a vector space,  $E(p_x X) = 0$  for all  $X \in \mathcal{R}_{t,T}$ .

As  $P(p_x > 0) > 0$ , we may renormalize and suppose that  $\|p_x\|_\infty = 1$ . Let us consider the family  $G = (\Gamma_x)_{x \in I}$  where  $I = L^1(\mathbf{R}, \mathcal{F}_T) \cap (\mathcal{A}_{t,T} \setminus \mathcal{A}_{t,T}^0)$  and  $\Gamma_x = \{p_x > 0\}$ . For any  $\Gamma \in \mathcal{F}_T$  such that  $P(\Gamma) > 0$ ,  $x = 1_\Gamma \in I$  since  $\mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$ . Therefore,  $E(xp_x) = E(1_\Gamma p_x) > 0$  implies that  $P(\Gamma_x \cap \Gamma) > 0$ . By Lemma 2.1.3 in [27], we deduce a countable family  $(x_i)_{i=1}^\infty$  of  $I$  such that  $\Omega = \bigcup_{i=1}^\infty \Gamma_{x_i}$ . We define  $p = \sum_{i=1}^\infty 2^{-i} p_{x_i}$ . We have  $p > 0$  a.s and we renormalize  $p$  such that  $p \in L^\infty(\mathbf{R}_+, \mathcal{F}_T)$  and  $E_P(p) = 1$ . We define  $Q \sim P$  such that  $dQ/dP = p$ . We have  $E(pX) = 0$  for all  $X \in \mathcal{R}_{t,T}$ . Therefore, with  $F_{u-1} \in \mathcal{F}_{u-1}$ ,  $1_{F_{u-1}} \Delta S_u \in \mathcal{R}_{t,T}$  if  $u \geq t+1$ , so  $E_Q(1_{F_{u-1}} \Delta S_u) = 0$ . This implies that  $E_Q(\Delta S_u | \mathcal{F}_{u-1}) = 0$ , i.e.  $(S_u)_{u=t}^T$  is a  $Q$ -martingale.

Moreover, by the construction of  $Q$  above, for all  $x \in \mathcal{A}_{t,T} \cap L^1(\mathbf{R}, \mathcal{F}_T)$ , we have  $E_Q(x | \mathcal{F}_t) \geq 0$ . By truncature and homogeneity, we may extend this property to every  $x$  such that  $E(|x| | \mathcal{F}_t) < \infty$  a.s. since  $x/(1 + E(|x| | \mathcal{F}_t))$  is integrable. Finally, this also holds if  $E_Q(x^- | \mathcal{F}_t) < \infty$  a.s.. At last, since  $\rho_t(X) + X \in \mathcal{A}_{t,T}$ , we may conclude that  $\rho_t(X) \geq -E_Q(X | \mathcal{F}_t)$ , for all  $X$  such that  $E_Q(X^- | \mathcal{F}_t) < \infty$  a.s.. If  $x \in \mathcal{A}_{t,T} \setminus \mathcal{A}_{t,T}^0$ , it suffices to consider the probability measure  $Q_x = \frac{1}{2}(Q + \tilde{Q})$  where  $\tilde{Q}$  is defined by its density  $d\tilde{Q}/dP = p_x$ . Indeed, since  $E_{\tilde{Q}}(x) > 0$  and  $E_Q(x) \geq 0$ , this implies that  $E_{Q_x}(x) > 0$  hence  $P(E_{Q_x}(x | \mathcal{F}_t) \neq 0) > 0$ .

Assume that 7) holds. For some martingale measure  $Q \sim P$  we have  $\rho_t(\theta_t \Delta S_{t+1}) \geq -E_Q(\theta_t \Delta S_{t+1} | \mathcal{F}_t) = 0$ , hence AIP holds. If  $\rho_t(\theta_t \Delta S_{t+1}) = 0$  on some non null set  $\Lambda_t$ , we have  $\rho_t(\theta_t \Delta S_{t+1} 1_{\Lambda_t}) = 0$ . This implies  $\theta_t \Delta S_{t+1} 1_{\Lambda_t}$  is acceptable. Moreover, if  $\theta_t \Delta S_{t+1} 1_{\Lambda_t} \notin \mathcal{A}_{t,T}^0$ ,  $E_{Q_x}(\theta_t \Delta S_{t+1} 1_{\Lambda_t} | \mathcal{F}_t) \neq 0$  by 7), which yields contradiction. Therefore,  $\rho_t(\theta_t \Delta S_{t+1}) = \rho_t(-\theta_t \Delta S_{t+1}) = 0$  on  $\Lambda_t$ , i.e. RN holds, and we deduce that 1) holds. Note that 5) and 6) are equivalent by Theorem 2.22.  $\square$

We recall that the No Good Deal condition (NGD) of Cherny [7] may be rephrased in our setting as follows:

**Definition 2.25.** *The NGD condition holds at any time  $t \leq T$  if there is no  $X_{t,T} \in \mathcal{R}_{t,T}$  such that  $\rho_t(X_{t,T}) < 0$  on a non null set.*

In the setting of Cherny, we suppose that

$$\rho_t(X) = \text{ess sup}_{Q^t \in \mathcal{D}_t} E_{Q^t}(-X), \quad (2.9)$$

where  $\mathcal{D}_t$  is a weakly compact subset of  $L^1$  with respect to the  $\sigma(L^1, L^\infty)$  topology and we use the definition  $E_{Q^t}(-X) = E_{Q^t}(X^-) - E_{Q^t}(X^+)$  with



the convention  $\infty - \infty = +\infty$ . Adapting [7, Theorem 3.4], we immediately get the following:

**Corollary 2.26.** *Suppose that the risk-measure is given by (2.9). Then, the NA condition and the NGD condition are equivalent to the existence of a probability measure  $Q^t \in \mathcal{D}_t$  such that the price process  $(S_u)_{u=t}^T$  is a  $Q^t$ -martingale for all  $t \leq T - 1$ .*

The following step is to provide a dual description of the payoffs that can be super-hedged. To do so, we denote by  $\mathcal{Q}_t^e$  (and  $\mathcal{Q}^e = \mathcal{Q}_0^e$ ) the set of equivalent martingale measures  $Q$  that satisfies  $\rho_t(X) \geq -E_Q(X|\mathcal{F}_t)$ , for all  $X$  such that  $E_Q(X^-|\mathcal{F}_t) < \infty$  a.s.. We have  $\mathcal{Q}_t^e \neq \emptyset$  under NA. We restrict the payoffs to the class  $L_S(\mathbf{R}, \mathcal{F}_T)$  of random variables  $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$  satisfying:

$$|h_T| \leq c^0 + \sum_{i=1}^d c^i S_T^i, \quad P - a.s.$$

for some constants  $c^0, \dots, c^d$  that may depend on  $h_T$ .

**Theorem 2.27.** *Suppose that the risk-measure is time-consistent and we have  $\mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$  for every  $t \leq T$ . Consider the following sets:*

$$\begin{aligned} \Gamma_{0,T} &:= \mathcal{Z}_{0,T} \cap L_S(\mathbf{R}, \mathcal{F}_T), \\ \Theta_{0,T} &:= \left\{ h_T \in L_S(\mathbf{R}, \mathcal{F}_T), \sup_{Q \in \mathcal{Q}^e} E_Q(h_T) \leq 0 \right\}. \end{aligned}$$

*Then, under the NA condition,  $\Gamma_{0,T} = \Theta_{0,T}$  and the minimal risk-hedging price  $P_0^*(h_T)$  of any contingent claim  $h_T \in L_S(\mathbf{R}, \mathcal{F}_T)$  is given by*

$$P_0^*(h_T) = \sup_{Q \in \mathcal{Q}^e} E_Q(h_T).$$

*Proof.* By Theorems 2.22 and 2.24, we know that  $\Gamma_{0,T}$  is closed in probability. For any  $h_T \in \Gamma_{0,T}$ , there exists  $\sum_{t=0}^T \theta_{t-1} \Delta S_t \in \mathcal{R}_{0,T}$  such that  $\rho_0 \left( \sum_{t=0}^T \theta_{t-1} \Delta S_t - h_T \right) \leq 0$ . Since,  $h_T \in L_S$ , we suppose w.l.o.g that  $S_T$  and  $h_T$  are integrable under  $P$ .

Set  $\gamma_t := \sum_{t=0}^t \theta_{t-1} \Delta S_t - h_T$  for every  $t \leq T$ . For any  $Q \in \mathcal{Q}^e \neq \emptyset$ , we have:

$$|\gamma_T| \leq \left| \sum_{t=0}^{T-1} \theta_{t-1} \Delta S_t \right| + |\theta_{T-1}| |\Delta S_T| + |h_T|,$$

hence:

$$E_Q(|\gamma_T| | \mathcal{F}_{T-1}) \leq \left| \sum_{t=0}^{T-1} \theta_{t-1} \Delta S_t \right| + |\theta_{T-1}| E_Q(|\Delta S_T| | \mathcal{F}_{T-1}) + E_Q(|h_T| | \mathcal{F}_{T-1}) < \infty \text{ a.s..}$$

By Statement 7) of Theorem 2.24 and the martingale property, we deduce that:

$$\rho_{T-1}(\gamma_T) \geq -E_Q(\gamma_{T-1} | \mathcal{F}_{T-1}). \quad (2.10)$$

At time  $T - 2$ , by time-consistency of the risk measure and (2.10), we get that

$$\rho_{T-2}(\gamma_T) = \rho_{T-2}(-\rho_{T-1}(\gamma_T)) \geq \rho_{T-2}(E_Q(\gamma_{T-1} | \mathcal{F}_{T-1})).$$

Moreover,  $E_Q(|E_Q(\gamma_{T-1} | \mathcal{F}_{T-1})| | \mathcal{F}_{T-2}) \leq E_Q(|\gamma_{T-1}| | \mathcal{F}_{T-2})$  and

$$\begin{aligned} E_Q(|\gamma_{T-1}| | \mathcal{F}_{T-2}) &\leq \left| \sum_{t=0}^{T-2} \theta_{t-1} \Delta S_t \right| + |\theta_{T-2}| E_Q(|\Delta S_{T-1}| | \mathcal{F}_{T-2}) \\ &\quad + E_Q(|h_T| | \mathcal{F}_{T-2}) < \infty \text{ a.s..} \end{aligned}$$

We deduce by Statement 7) of Theorem 2.24 that

$$\rho_{T-2}(E_Q(\gamma_{T-1} | \mathcal{F}_{T-1})) \geq -E_Q(\gamma_{T-1} | \mathcal{F}_{T-2}).$$

By the martingale property, we finally deduce that  $\rho_{T-2}(\gamma_T) \geq -E_Q(\gamma_{T-2} | \mathcal{F}_{T-2})$ . Recursively, we finally obtain:

$$0 \geq \rho_0 \left( \sum_{t=0}^T \theta_{t-1} \Delta S_t - h_T \right) \geq -E_Q(\gamma_1 | \mathcal{F}_0) \geq -E_Q(\theta_0 \Delta S_1 - h_T) \geq E_Q(h_T). \quad (2.11)$$

This implies  $\Gamma_{0,T} \subset \Theta_{0,T}$ .

Reciprocally, assume that there is  $\hat{h}_T \in \Theta_{0,T} \setminus \Gamma_{0,T}$ . Since  $\hat{h}_T \in L_S(\mathbf{R}, \mathcal{F}_T)$ ,  $\hat{h}_T$  is integrable under  $Q \in \mathcal{Q}^e$ . Moreover, since  $\Gamma_{0,T}$  is closed in probability,  $\tilde{\Gamma}_{0,T} := \Gamma_{0,T} \cap L_Q^1(\mathbf{R}, \mathcal{F}_T)$  is closed in  $L^1$ . By the Hahn-Banach separation theorem, as  $\hat{h}_T \notin \tilde{\Gamma}_{0,T}$ , we deduce the existence of  $Y \in L^\infty(\mathbf{R}, \mathcal{F}_T)$  such that:

$$\sup_{X \in \tilde{\Gamma}_{0,T}} E_Q(YX) < E_Q(Y\hat{h}_T).$$

Let  $H$  be the density  $Q$  w.r.t  $P$ , i.e.  $H = dQ/dP$ . We have:

$$\sup_{X \in \tilde{\Gamma}_{0,T}} E(HYX) < E(HY\hat{h}_T).$$

Since  $\tilde{\Gamma}_{0,T}$  is a cone, we deduce that  $E(HYX) \leq 0$  for all  $X \in \tilde{\Gamma}_{0,T}$ . Moreover,  $E(HY\hat{h}_T) > 0$ ,  $HY \geq 0$  a.s. and  $E(HY) > 0$ . Therefore, we deduce that  $\hat{H} := HY/E(HY)$  defines the density of a probability measure  $\hat{Q} \in \mathcal{Q}^a$ .

We define  $H^\epsilon := \epsilon H + (1 - \epsilon)\hat{H}$ . Since  $E(\hat{H}\hat{h}_T) > 0$ , we may choose  $\epsilon \in (0, 1)$  small enough so that  $E(H^\epsilon\hat{h}_T) > 0$ . Since  $H^\epsilon$  defines the density of a probability measure  $Q^\epsilon \in \mathcal{Q}^e$ , we should have  $E_{Q^\epsilon}\hat{h}_T = E(H^\epsilon\hat{h}_T) \leq 0$ , as  $\hat{h}_T \in \Theta_{0,T}$ . This yields a contradiction. We conclude that  $\Gamma_{0,T} = \Theta_{0,T}$ .

At last,  $P_0$  is a super-hedging price for  $h_T$  if and only if  $h_T - P_0 \in \Gamma_{0,T}$ . By the first part, we deduce that  $P_0^* \geq \sup_{Q \in \mathcal{Q}^e} E_Q(h_T)$ . Suppose there exists  $\epsilon > 0$  such that  $P_0^* - \epsilon \geq \sup_{Q \in \mathcal{Q}^e} E_Q(h_T)$ . Then,  $(h_T - P_0^* + \epsilon) \in \Theta_{0,T}$ . Since  $\Theta_{0,T} = \Gamma_{0,T}$ , there exists  $W_{0,T} \in \mathcal{R}_{0,T}$  such that  $\rho_0(W_{0,T} - h_T + P_0^* - \epsilon) \leq 0$ . This implies that  $P_0^* - \epsilon \geq \rho_0(W_{0,T} - h_T)$ . Since  $\rho_0(W_{0,T} - h_T)$  is a super-hedging price for  $h_T$ , we also deduce that  $\rho_0(W_{0,T} - h_T) \geq P_0^*$  which yields a contradiction. We conclude that  $P_0^* = \sup_{Q \in \mathcal{Q}^e} E_Q(h_T)$ .  $\square$

### 3. An example of dual representation for a consistent risk-measure

As mentioned by Cherny [7, Theorem 2.2] and shown in [15], any consistent risk-measure  $\rho_t$  restricted to the set of all bounded random variables is characterized by a family  $\mathcal{D}_t$  of absolutely continuous probability measures such that  $\rho_t(X) = \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t)$ . In the following, we consider the risk-measure  $\rho_t$  on  $L^0$  as defined in this paper. For  $X \in L^0$ , we define  $E_Q(-X|\mathcal{F}_t)$  as  $E_Q(-X|\mathcal{F}_t) = E_Q(X^-|\mathcal{F}_t) - E_Q(X^+|\mathcal{F}_t)$  with the convention  $\infty - \infty = \infty$ . We say that a random variable  $X$  is  $\mathcal{F}_t$ -bounded from above if  $X \leq c_t$  a.s. for some  $c_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ .

**Proposition 3.1.** *Let  $(\rho_t)_{t=0, \dots, T}$  be the coherent risk-measure as defined in Section 2. Then, there exists a family  $\mathcal{D}_t$  of absolutely continuous probability measures such that, for every  $\mathcal{F}_t$ -bounded from above random variable  $X$ , we have:*

$$\rho_t(X) = \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t). \quad (3.12)$$

*Proof.* By [1], [15], there exists  $\mathcal{D}_t$  such that (3.12) holds if  $X \in L^\infty$ . By homogeneity, it is clear that (3.12) still holds if  $X$  is  $\mathcal{F}_t$ -bounded, i.e.  $|X| \leq c_t$  where  $c_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ . Let us show that (3.12) still holds for any random variable  $X$  such that  $X \leq c_t$  a.s. for some  $c_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$  and  $X$  is acceptable. Let us define  $X^M = X1_{X \geq -M}$  for any  $M > 0$ . Then,  $X^M$  is bounded a.s.. As  $X^M = X - X1_{X < -M}$  and  $-X1_{X < -M} \geq 0$ , then  $X^M$  is acceptable i.e.  $\rho_t(X^M) \leq 0$ . By 3.12, we deduce that  $E_Q(X^M|\mathcal{F}_t) \geq 0$  for all  $Q \in \mathcal{D}_t$ . This implies that  $E_Q((X^M)^+|\mathcal{F}_t) \geq E_Q((X^M)^-|\mathcal{F}_t)$  and, as  $M \rightarrow \infty$ , we get that  $c_t \geq E_Q(X^+|\mathcal{F}_t) \geq E_Q(X^-|\mathcal{F}_t)$  hence  $\infty > E_Q(X|\mathcal{F}_t) \geq 0$ . More generally, for any  $X$  such that  $X \leq c_t$  for some  $c_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ ,  $\rho_t(X) + X$  is acceptable hence  $\rho_t(X) \geq E_Q(-X|\mathcal{F}_t)$  for any  $Q \in \mathcal{D}_t$ . We deduce that the inequality  $\rho_t(X) \geq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t)$  holds. Reciprocally, if  $\text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t) = +\infty$ , then we deduce that 3.12 holds. Otherwise, for any  $Q \in \mathcal{D}_t$ ,  $E_Q(X^-|\mathcal{F}_t) < \infty$ . Moreover, for any  $\epsilon > 0$ ,

$$\text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t) + E_Q(X^M|\mathcal{F}_t) + \epsilon \geq E_Q(-X|\mathcal{F}_t) + E_Q(X^M|\mathcal{F}_t) + \epsilon.$$

As  $E_Q(X^+|\mathcal{F}_t) \leq c_t$ , we may apply the dominated convergence theorem as  $\lim_{M \rightarrow \infty} X^M = X$  a.s. and we deduce that for  $M \geq M_0(t)$ , the expectation of  $\gamma_t^n = \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t) + X^{M_0(t)+n} + \epsilon$ ,  $n \geq 0$ , is non negative. As  $\gamma_t^n$  is  $\mathcal{F}_t$ -bounded, we get that  $\rho_t(\gamma_t^n) \leq 0$  by (3.12) hence  $\gamma_t^n$  is acceptable for any  $n \geq 0$ . Since the set of acceptable positions is closed, we deduce that  $\text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t) + X + \epsilon$  is acceptable. Finally, as  $\epsilon \rightarrow 0$ ,  $\text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t) + X$  is acceptable. As  $\rho_t(X) = \text{ess inf}_{\mathcal{F}_t} \mathcal{A}_t^X$ , we deduce that  $\rho_t(X) \leq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X|\mathcal{F}_t)$  so that the equality (3.12) holds for any random variable that are  $\mathcal{F}_t$ -bounded form above.  $\square$

Unfortunately, it is unrealistic to expect that (3.12) may be extended in general from  $L^\infty$  to  $L^0$ , as mentioned by Cherny, [7]. The main problem is about the non negatives random variables as we shall see in the proof of the next proposition. Before, let us see a trivial example where we may meet some difficulties for non negative random variables.

We consider  $\Omega = [0, 1]$  equipped with the Borel  $\sigma$ -algebra and the Lebesgue measure  $P$ . The random variable  $X(\omega) = \omega^{-1}1_{(0,1]}(\omega)$  is non negative hence acceptable. Suppose that the set of all acceptable positions is the closure in  $L^0$  of the random variables  $X$  such that  $E_P(X) = E_P(X^+) - E_P(X^-) \geq 0$ . We then define  $\rho$  on  $L^0$  as in Section 2, see [30]. As  $E_P(X) = \infty$ , we deduce that  $Z_\alpha := X - \alpha$  is acceptable for all  $\alpha > 0$  if (3.12) holds. On the other

hand,  $P(Z_\alpha < 0) = 1 - \alpha^{-1}$  tends to 1 as  $\alpha \rightarrow \infty$ , which is unrealistic if  $Z_\alpha$  is acceptable.

Consider  $Q \in \mathcal{D}_0$  and  $Z = dQ/dP$ . Suppose that  $P(Z > 1) > 0$ . We then choose  $\alpha < 0$  and  $\beta > 0$  such that  $\alpha P(Z > 1) + \beta P(Z \leq 1) = 0$ . Then,  $X = \alpha 1_{Z>1} + \beta 1_{Z \leq 1}$  is acceptable as  $E_P(X) = 0$ . Therefore, by (3.12),  $E_Q(X) \geq 0$ . Actually,

$$E_Q(X) = E_P(XZ) = E_P(\alpha Z 1_{Z>1} + \beta Z 1_{Z \leq 1}) \leq E_P(X) = 0$$

and  $E_Q(X) = 0$  if and only if  $\alpha Z 1_{Z>1} + \beta Z 1_{Z \leq 1} = X$ . In that case  $Z = 1$  on  $\{Z > 1\}$  hence a contradiction. We deduce that  $Z \leq 1$  a.s.. At last, since  $Z \leq 1$  a.s., we deduce that  $Z = 1$  a.s.. We then deduce that  $\mathcal{D}_0 = \{P\}$ .

Also, as another example, consider  $X(\omega) = \omega^{-1} 1_{(0,\alpha)}(\omega) + (\omega-1)^{-1} 1_{(\alpha,1)}(\omega)$ ,  $\omega \in \Omega$ . Since  $E_P(X) = \infty - \infty = \infty$ , we deduce that  $X$  is acceptable. Nevertheless,  $P(X < 0) = 1 - \alpha$  tends to 1 as  $\alpha \rightarrow 0$ , which is clearly unrealistic.

In the following, we denote by  $\mathcal{A}_t^{\infty,+}$  the set of all acceptable positions at time  $t$  which are  $\mathcal{F}_t$ -bounded from above.

**Proposition 3.2.** *Suppose that  $\mathcal{A}_t$  is the closure of  $\mathcal{A}_t^{\infty,+} + L^0(\mathbf{R}^+, \mathcal{F}_T)$  in  $L^0$  and assume that, for some fixed  $\varepsilon > 0$ ,  $\mathcal{A}_t^{\infty,+}$  contains all the random variables  $Z$  which are  $\mathcal{F}_t$ -bounded from above and satisfy  $P(Z < 0) \leq \varepsilon$ . Let  $(\rho_t)_{t=0,\dots,T}$  be the coherent risk-measure as defined in Section 2. Then, there exists a family  $\mathcal{D}_t$  of absolutely continuous probability measures such that we have.*

$$\rho_t(X) = \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-X | \mathcal{F}_t), \quad \forall X \in L^0. \quad (3.13)$$

*Proof.* Suppose that  $Z = X + \epsilon^+$  where  $X$  is  $\mathcal{F}_t$ -bounded from above and acceptable and  $\epsilon^+ \geq 0$  a.s.. Then, for all  $Q \in \mathcal{D}_t$ ,  $E_Q(Z | \mathcal{F}_t) \geq E_Q(X | \mathcal{F}_t) \geq 0$  by Proposition 3.1. As  $\rho_t(Z) + Z$ , admits the same form than  $Z$ , we deduce that  $\rho_t(Z) + Z$  admits non negative conditional expectations under  $Q \in \mathcal{D}_t$ . Therefore,  $\rho_t(Z) \geq E_Q(-Z | \mathcal{F}_t)$  for all  $Z \in \mathcal{D}_t$  hence  $\rho_t(Z) \geq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z | \mathcal{F}_t)$ , at least when  $\rho_t(Z) > -\infty$ . Otherwise, when  $\rho_t(Z) = -\infty$ ,  $Z_\alpha = -\alpha + Z$  is acceptable for all  $\alpha > 0$ , hence  $E_Q(Z_\alpha | \mathcal{F}_t) \geq 0$ , i.e.  $E_Q(Z | \mathcal{F}_t) \geq \alpha$  for all  $\alpha > 0$ . It follows that  $E_Q(Z^- | \mathcal{F}_t) - E_Q(Z^+ | \mathcal{F}_t) \leq -\alpha$  and finally  $\rho_t(Z) = \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z | \mathcal{F}_t) = -\infty$  as  $\alpha \rightarrow \infty$ .

Consider an acceptable position  $Z$ . Then, by assumption,  $Z = \limsup_n Z^n$  where  $Z^n$  is of the form  $Z^n = X^n + \epsilon_n^+$  with  $\epsilon_n^+ \geq 0$  a.s. and  $X^n$  is  $\mathcal{F}_t$ -bounded from above. Note that  $\sup_{n \leq k} X_n$  is still  $\mathcal{F}_t$ -bounded from above for

all  $k \geq n$ . Since  $\sup_{n \geq k} Z_n \geq \sup_{k \leq n \leq m} Z_n$ , for all  $m \geq k$ , we deduce that  $\sup_{n \geq k} Z_n$  is of the form  $X_k + \epsilon_k^+$  where  $X_k$  is  $\mathcal{F}_t$ -bounded from above and acceptable while  $\epsilon_k^+ \geq 0$  a.s.. It follows that any acceptable position is of the form  $Z = \lim \downarrow Z_n$  where  $Z_n$  is of the form  $Z_n = X_n + \epsilon_n^+$  where  $X_n$  is  $\mathcal{F}_t$ -bounded from above and acceptable while  $\epsilon_n^+ \geq 0$  a.s.. As  $Z \leq Z_n$ , we deduce that  $\rho_t(Z) \geq \rho_t(Z_n) \geq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z_n | \mathcal{F}_t)$  by virtue of the inequality we have shown in the first part. As  $(-Z_n)$  is non decreasing we finally deduce that  $\rho_t(Z) \geq E_Q(-Z | \mathcal{F}_t)$  for any  $Q \in \mathcal{D}_t$ , when  $n \rightarrow \infty$ . It follows that  $\rho_t(Z) \geq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z | \mathcal{F}_t)$ .

Moreover, suppose that (3.13) holds for any acceptable position  $Z_n$  of the form  $Z_n = X_n + \epsilon_n^+$  where  $X_n$  is  $\mathcal{F}_t$ -bounded from above and acceptable and  $\epsilon_n^+ \geq 0$  a.s.. By lower semi-continuity,

$$\rho_t(Z) \leq \liminf_n \rho_t(Z_n) = \liminf_n \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z_n | \mathcal{F}_t).$$

As  $Z \leq Z_n$ ,  $E_Q(-Z_n | \mathcal{F}_t) \leq E_Q(-Z | \mathcal{F}_t)$ , and we deduce the inequality  $\rho_t(Z) \leq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z | \mathcal{F}_t)$ . We then conclude that (3.13) holds for every acceptable position  $Z$  and, finally, for every  $X \in L^0$  as  $\rho_t(X) + X$  is acceptable.

It remains to show that (3.13) holds for  $Z = X + \epsilon^+ \in \mathcal{A}_t^{\infty,+} + L^0(\mathbf{R}^+, \mathcal{F}_T)$ . To get it, it is sufficient to prove that  $\rho_t(Z) \leq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z | \mathcal{F}_t)$ . Let us define  $Z^n = X + \epsilon^+ 1_{\epsilon^+ \leq n} + \alpha_n 1_{\epsilon^+ > n} \in \mathcal{A}_t^{\infty,+}$  where  $\alpha_n > 0$  is chosen large enough in such a way that  $P(\alpha_n < \epsilon^+) < \varepsilon$ . Then,  $(\alpha_n - \epsilon^+) 1_{\epsilon^+ > n}$  is acceptable by hypothesis for  $P((\alpha_n - \epsilon^+) 1_{\epsilon^+ > n} < 0) \leq P(\alpha_n < \epsilon^+) < \varepsilon$ .

Since  $Z^n \rightarrow Z$  a.s., we deduce that  $\rho_t(Z) \leq \liminf_n \rho_t(Z^n)$ . Recall that  $\rho_t(Z^n) = \sup_{Q \in \mathcal{D}_t} E_Q(-Z^n | \mathcal{F}_t)$  by Proposition 3.1. Hence,

$$\rho_t(Z^n) \leq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z | \mathcal{F}_t) + \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(Z - Z^n | \mathcal{F}_t).$$

Moreover, since  $Z^n - Z$  is  $\mathcal{F}_t$ -bounded from below, we have

$$\text{ess sup}_{Q \in \mathcal{D}_t} E_Q(Z - Z^n | \mathcal{F}_t) = \rho_t(Z^n - Z) = \rho_t((\alpha_n - \epsilon^+) 1_{\epsilon^+ > n}) \leq 0.$$

We then deduce that  $\rho_t(Z) \leq \text{ess sup}_{Q \in \mathcal{D}_t} E_Q(-Z | \mathcal{F}_t)$  and the conclusion follows.  $\square$

The proof of the proposition above shows that (3.13) holds as soon as it holds for any acceptable position which is the sum of an  $\mathcal{F}_t$ -bounded position plus a non negative one. By Proposition 3.1, (3.13) holds for any  $\mathcal{F}_t$ -bounded position. Therefore, the difficulty in proving (3.13) stems from the non negative random variables.

## References

- [1] Acciaio B. and Penner I. Dynamics convex risk measures. *Advanced Mathematical Methods for Finance*, 2010,1, Ed. Giulia Di Nunno and Bent Øksendal, Springer Heidelberg Dordrecht London New-York.
- [2] Ararat C., Hamel A. and Rudloff B. Set-valued shortfall and divergence risk measures. *International Journal of Theoretical and Applied Finance*, 2017, 20 (5), 1750026.
- [3] Baptiste J., Carassus L. and Lépinette E. Pricing without martingale measures. Preprint. <https://hal.archives-ouvertes.fr/hal-01774150>.
- [4] Ben Tahar I. and Lépinette E. Vector valued coherent risk measure processes. *IJTAF*, 17, 02 (2014).
- [5] Black F. and Scholes M. The pricing of options and corporate liabilities. *Journal of Political Economy*, 1973, 81, 637-659.
- [6] Cherny A. Pricing and hedging European options with discrete-time coherent risk. *Finance and Stochastics*, 2007, 13, 537-569.
- [7] Cherny A. Pricing with coherent risk. *Theory of Probability and Its Applications*, 2007, 52(3), 389-415.
- [8] Dalang E.C., Morton A. and Willinger W. Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics and Stochastic Reports*, 1990, 29, 185-201.
- [9] Delbaen F. and Schachermayer W. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 1994, 300, 463-520.
- [10] Delbaen F. and Schachermayer W. The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen*, 1996, 312, 215-250.
- [11] Delbaen F. and Schachermayer W. The no-arbitrage condition under a change of numéraire. *Stochastics and Stochastic Reports*, 1995, 53, 213-226.
- [12] Delbaen F. and Schachermayer W. *The mathematics of arbitrage*. Springer Finance, 2006, 371.
- [13] Schachermayer W. A Hilbert space proof of the fundamental theorem of asset pricing in finite discrete time. *Insurance: Mathematics and Economics*, 1992 (4), 11, 249-257.
- [14] Delbaen F. Coherent risk measures. *Lecture Notes*, Cattedra Galileiana, Scuola Normale Superiore di Pisa, 2000.

- [15] Delbaen F. Coherent risk measures on general probability spaces. *Advances in Finance and Stochastics: essays in honor of Dieter Sondermann*, Springer, Heidelberg, 2002, 1-37.
- [16] Detlefsen K. and Scandolo G. Conditional and dynamic convex risk measures. *Finance and Stochastics*, 2005, 9, 539-561.
- [17] Feinstein Z. and Rudloff B. 2013. Time consistency of dynamic risk measures in markets with transaction costs. *Quantitative Finance*, 2013; 13 (9), 1473-1489.
- [18] Fölmer H. and Penner I. Convex risk measures and the dynamics of their penalty functions. *Statistics and Decisions*, 2006, 2, 24.
- [19] Guasoni P, Rásonyi M. and Schachermayer W. The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Annals of Finance*, 2010, 6, 157-191.
- [20] Harrison J.M. and Pliska S.R. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 1981, 11, 215-260.
- [21] Hess C. Set-valued integration and set-valued probability theory: An overview. In: E. Pap (ed.) *Handbook of Measure Theory*, Elsevier (2002), 14, 617–673.
- [22] Jacod J. and Shiryaev A.N. Local martingales and the fundamental asset pricing theorems in the discrete-time case. *Finance and Stochastics*, 1998 (3), 2, 259-273.
- [23] Jorion P. *Value at Risk: the new benchmark for managing financial risk*. McGraw-Hill Professional, 2006 (3).
- [24] Jouini E., Meddeb M. and Touzi N. Vector-valued measure of risk. *Finance and Stochastics*, 8, 531-552.
- [25] Kabanov Y.M. and Kramkov D. No arbitrage and equivalent martingale measures: an elementary proof of the Harrison-Pliska theorem. *Theory of Probability and its Applications*, 2001 (3) 39, 523-527.
- [26] Kabanov Y. and Stricker C. A Teachers' note on no-arbitrage criteria. In *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, Springer Berlin, 2001, 149-152.
- [27] Kabanov Y. and Safarian, M. *Markets with transaction costs*. *Mathematical Theory*. Springer-Verlag, 2009.
- [28] Kaina M. and Ruschendorf L. On convex risk measures on  $L^p$ -spaces. *Mathematical Methods of Operations Research*. 2009, 69(3), 475-495.
- [29] Lépinette E. and Molchanov I. Risk arbitrage and hedging to acceptability. *Finance and Stochastics*, to appear.



- [30] Lépinette E. and Zhao J. Super-hedging a European option with a coherent risk-measure and without no-arbitrage condition. Preprint.
- [31] Merton R.C. The theory of rational option pricing. *Bell J. Econ. Manag. Sci.*,1973, 4, 141-183.
- [32] Molchanov I. *Theory of Random Sets*. 2nd edition. Springer, London, 2017.
- [33] Lépinette Risk arbitrage and hedging to acceptability (with Molchanov I.). *Finance and Stochastics*. To appear.
- [34] Pergamenshchikov S. Limit theorem for Leland's strategy. *The Annals of Applied Probability*, 2003, 13, 3, 1099-1118.
- [35] Rockafellar, R Tyrrell and Wets, Roger J-B. *Variational analysis*. Springer Science & Business Media, 2009.
- [36] Rogers L.C.G. Equivalent martingale measures and no-arbitrage. *Stochastics and Stochastic Reports*, 1994, 51, 41-49.
- [37] Ross S. The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 1976, 13, 341-360.