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# A HARMONIC MEAN INEQUALITY FOR THE MODIFIED DEGENERATE GAMMA FUNCTION

KWARA NANTOMAH

ABSTRACT. In this work, we prove that for  $z > 0$ , the harmonic mean of  $\Gamma_\lambda(z)$  and  $\Gamma_\lambda(1/z)$  is always greater than or equal to  $\lambda/\ln(\lambda + 1)$ . Where  $\Gamma_\lambda(z)$  is the modified degenerate gamma function and  $\lambda \in (0, 1)$ . At the end, we pose a conjecture concerning a harmonic mean inequality for the modified degenerate digamma function,  $\psi_\lambda(z)$ .

## 1. INTRODUCTION

Degenerate special functions and polynomials have been a subject of intense discussion in recent times. For instance, see [13], [14], [16], [18], [20] and the related references therein.

In 2017, Kim and Kim [15] introduced the degenerate gamma function as

$$\Gamma_\lambda(z) = \int_0^\infty t^{z-1} (1 + \lambda t)^{-\frac{1}{\lambda}} dt$$

where  $\lambda \in (0, \infty)$  and  $0 < \Re(z) < \frac{1}{\lambda}$ . This was motivated by the degenerate exponential function which is defined as [15]

$$e_\lambda^t = (1 + \lambda t)^{\frac{1}{\lambda}}$$

where  $\lambda \in (0, \infty)$ . It is clear that  $\lim_{\lambda \rightarrow 0} e_\lambda^t = e^t$  and  $\lim_{\lambda \rightarrow 0} \Gamma_\lambda(z) = \Gamma(z)$ , where  $\Gamma(z)$  is the classical gamma function. For some interesting properties of the degenerate gamma function, one may refer to the recent article [17].

In 2018, Kim et al. [19] introduced the modified degenerate gamma function which is defined as

$$\Gamma_\lambda(z) = \int_0^\infty t^{z-1} (1 + \lambda)^{-\frac{t}{\lambda}} dt \quad (1)$$

where  $\lambda \in (0, 1)$  and  $\Re(z) > 0$ . This definition is equivalent to

$$\Gamma_k(z) = \int_0^\infty t^{z-1} (1 + 1/k)^{-kt} dt$$

where  $1 < k < \infty$  and  $\Re(z) > 0$ . Here,  $\lim_{k \rightarrow \infty} \Gamma_k(z) = \Gamma(z)$ . The modified degenerate gamma function (1) satisfies the following properties [19].

$$\Gamma_\lambda(1) = \frac{\lambda}{\ln(1 + \lambda)}, \quad (2)$$

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$$\Gamma_\lambda(z+1) = \frac{\lambda z}{\ln(1+\lambda)} \Gamma_\lambda(x), \quad (3)$$

$$\Gamma_\lambda(m+1) = \frac{\lambda^{m+1} m!}{(\ln(1+\lambda))^{m+1}}, \quad m \in \mathbb{N}. \quad (4)$$

In a recent work, He et al. [11] introduced the modified degenerate digamma function which is defined as

$$\psi_\lambda(z) = \frac{d}{dz} \ln \Gamma_\lambda(z) = \frac{\Gamma'_\lambda(z)}{\Gamma_\lambda(z)} \quad (5)$$

and has the following representations among others.

$$\begin{aligned} \psi_\lambda(z) &= -\gamma + \ln \left( \frac{\lambda}{\ln(1+\lambda)} \right) + \sum_{k=0}^{\infty} \frac{z-1}{(k+1)(k+z)}, \\ &= -\gamma + \ln \left( \frac{\lambda}{\ln(1+\lambda)} \right) - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(k+z)}, \\ &= -\gamma + \ln \left( \frac{\lambda}{\ln(1+\lambda)} \right) + \int_0^1 \frac{1-t^{z-1}}{1-t} dt, \\ &= -\gamma + \ln \left( \frac{\lambda}{\ln(1+\lambda)} \right) + \int_0^\infty \frac{e^{-t} - e^{-zt}}{1-e^{-t}} dt, \end{aligned} \quad (6)$$

where  $\gamma$  is the Euler-Mascheroni constant. It also satisfies the following basic properties

$$\psi_\lambda(1) = -\gamma + \ln \left( \frac{\lambda}{\ln(1+\lambda)} \right), \quad (7)$$

$$\psi_\lambda(z+1) = \psi_\lambda(z) + \frac{1}{z}, \quad (8)$$

and similarly, it is clear that  $\lim_{\lambda \rightarrow 0} \psi_\lambda(z) = \psi(z)$  where  $\psi(z)$  is the classical digamma function. For additional properties of the function  $\psi_\lambda(z)$ , one may refer to [11].

In 1974, Gautschi [9] proved that for  $z > 0$ , the harmonic mean of  $\Gamma(z)$  and  $\Gamma(1/z)$  is always greater than or equal to 1. That is,

$$\frac{2\Gamma(z)\Gamma(1/z)}{\Gamma(z) + \Gamma(1/z)} \geq 1, \quad z > 0. \quad (9)$$

Due to the importance of this elegant inequality, some refinements and extensions have been studied by other researchers. See for example [1], [2], [3], [4], [5], [6], [10] and [12].

In 2017, Alzer and Jameson [7] established analogous results for the digamma function which is given as

$$\frac{2\psi(z)\psi(1/z)}{\psi(z) + \psi(1/z)} \geq -\gamma, \quad z > 0. \quad (10)$$

In 2018, Yin et al. [26] extended (10) to the  $k$ -digamma function by establishing that

$$\frac{2\psi_k(z)\psi_k(1/z)}{\psi_k(z) + \psi_k(1/z)} \geq \frac{\ln^2 k + \gamma^2 - 2(\gamma + 1) \ln k}{k [\ln k + \psi(1/k)]}, \quad z > 0, \frac{1}{\sqrt[3]{3}} \leq k \leq 1. \quad (11)$$

In relation to other special functions, in 2019, Nantomah [23] posed as a conjecture the inequality

$$\frac{2\beta(z)\beta(1/z)}{\beta(z) + \beta(1/z)} \leq \ln 2, \quad z > 0 \quad (12)$$

where  $\beta(z)$  is the Nielsen's beta function [22]. Subsequently, Matejicka [21] provided a concrete proof of (12).

Recently, Yildirim [25] improved inequality (11) by relaxing the condition on  $k$  and proving that

$$\frac{2\psi_k(z)\psi_k(1/z)}{\psi_k(z) + \psi_k(1/z)} \geq \psi_k(1), \quad z > 0, k > 0. \quad (13)$$

When  $k = 1$ , both (11) and (13) reduce to (10).

Also, in a recent work, Bouali [8] extended (9) and (10) to the  $q$ -analogues of the gamma and digamma functions by proving that

$$\frac{2\Gamma_q(z)\Gamma_q(1/z)}{\Gamma_q(z) + \Gamma_q(1/z)} \geq 1, \quad z > 0 \quad (14)$$

and

$$\frac{2\psi_q(z)\psi_q(1/z)}{\psi_q(z) + \psi_q(1/z)} \geq \psi_q(1), \quad z > 0, q \in (0, p_0) \quad (15)$$

where  $p_0 \simeq 3.239945$  and by letting  $q \rightarrow 1$ , the inequalities (14) and (15) respectively reduce to (9) and (10).

Inspired by the above works, the main purpose of this paper is to extend Gautschi's inequality (9) to the modified degenerate gamma function defined in (1). At the end, we pose a conjecture concerning a harmonic mean inequality for the modified degenerate digamma function.

## 2. MAIN RESULTS

We begin with the following lemma which is well known as the convolution theorem for Laplace transforms.

**Lemma 2.1.** *Let  $f(t)$  and  $g(t)$  be any two functions with convolution  $f * g = \int_0^t f(s)g(t-s)ds$ . Then the Laplace transform of the convolution is given as*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \mathcal{L}\{g\}.$$

*That is*

$$\int_0^\infty \left[ \int_0^t f(s)g(t-s)ds \right] e^{-zt} dt = \int_0^\infty f(t)e^{-zt} dt \int_0^\infty g(t)e^{-zt} dt. \quad (16)$$

**Lemma 2.2.** *The function  $A(z) = z\psi'_\lambda(z)$  is decreasing on  $(0, \infty)$ .*

*Proof.* By direct computations and by applying Lemma 2.1, we have

$$\begin{aligned}
\frac{A'(z)}{z} &= \frac{1}{z} \psi'_\lambda(z) + \psi''_\lambda(z) \\
&= \int_0^\infty e^{-zt} dt \int_0^\infty \frac{te^{-zt}}{1-e^{-t}} dt - \int_0^\infty \frac{t^2 e^{-zt}}{1-e^{-t}} dt \\
&= \int_0^\infty \left[ \int_0^t \frac{s}{1-e^{-s}} ds \right] e^{-zt} dt - \int_0^\infty \frac{t^2 e^{-zt}}{1-e^{-t}} dt \\
&= \int_0^\infty Q(t) e^{-zt} dt
\end{aligned}$$

where

$$Q(t) = \int_0^t \frac{s}{1-e^{-s}} ds - \frac{t^2}{1-e^{-t}}.$$

Then

$$Q'(t) = \frac{t}{(1-e^{-t})^2} (e^{-t} + te^{-t} - 1) < 0$$

which shows that  $Q(t)$  is decreasing. In this way,

$$Q(t) < Q(0) = \lim_{t \rightarrow 0} Q(t) = 0.$$

Therefore  $A'(z) < 0$  which completes the proof.  $\square$

**Lemma 2.3.** *The inequality*

$$\psi_\lambda(z) + \psi_\lambda(1/z) \leq 2 \left[ -\gamma + \ln \left( \frac{\lambda}{\ln(\lambda+1)} \right) \right] \quad (17)$$

*holds for  $z > 0$  and with equality when  $z = 1$ .*

*Proof.* Let  $B(z) = \psi_\lambda(z) + \psi_\lambda(1/z)$  for  $z > 0$ . Then

$$\begin{aligned}
zB'(z) &= z\psi'_\lambda(z) - \frac{1}{z}\psi'_\lambda(1/z) \\
&= T(z)
\end{aligned}$$

and by applying Lemma 2.2, we conclude that  $T(z) < 0$  if  $z \in (1, \infty)$  and  $T(z) > 0$  if  $z \in (0, 1)$ . Thus,  $B(z)$  is decreasing on  $(1, \infty)$  and increasing on  $(0, 1)$ . In either case, we have

$$B(z) < B(1) = 2 \left[ -\gamma + \ln \left( \frac{\lambda}{\ln(\lambda+1)} \right) \right]$$

which completes the proof.  $\square$

**Lemma 2.4.** *The inequality*

$$z - \psi_\lambda(1/z) > \psi_\lambda(z) \quad (18)$$

*holds for  $z > 0$ .*

*Proof.* It suffices to show that  $\psi_\lambda(z) + \psi_\lambda(1/z) < 0$ . Let  $h(r) = -\gamma + \ln(r/\ln(r+1))$  where  $r \in (0, 1)$ . Then  $h'(r) = \frac{1}{r} - \frac{1}{(r+1)\ln(r+1)} > 0$  which shows that  $h(r)$  is increasing. Hence  $h(r) < h(1) = -\gamma - \ln(\ln 2) < 0$  and the conclusion follows from Lemma 2.3.  $\square$

**Lemma 2.5.** *The function  $K(z) = z\Gamma_\lambda(1/z)\psi_\lambda(z)$  is increasing on  $(0, \infty)$ .*

*Proof.* By direct computations and by utilizing Lemma 2.4, we obtain

$$\begin{aligned} K'(z) &= \Gamma_\lambda(1/z)\psi_\lambda(z) - \frac{1}{z}\Gamma'_\lambda(1/z)\psi_\lambda(z) + z\Gamma_\lambda(1/z)\psi'_\lambda(z), \\ \frac{zK'(z)}{\Gamma_\lambda(1/z)} &= [z - \psi_\lambda(1/z)]\psi_\lambda(z) + z^2\psi'_\lambda(z) \\ &> [\psi_\lambda(z)]^2 + x^2\psi'_\lambda(z) \\ &> 0. \end{aligned}$$

Hence  $K'(z) > 0$  which gives the desired results.  $\square$

**Theorem 2.6.** *The inequality*

$$\frac{2\Gamma_\lambda(z)\Gamma_\lambda(1/z)}{\Gamma_\lambda(z) + \Gamma_\lambda(1/z)} \geq \frac{\lambda}{\ln(\lambda + 1)} \quad (19)$$

*holds for  $z > 0$  and with equality when  $z = 1$ .*

*Proof.* For  $z > 0$ , let  $H(z) = \frac{2\Gamma_\lambda(z)\Gamma_\lambda(1/z)}{\Gamma_\lambda(z) + \Gamma_\lambda(1/z)}$  and  $\phi(z) = \ln H(z)$ . Then

$$\begin{aligned} &z[\Gamma_\lambda(z) + \Gamma_\lambda(1/z)]\phi'(z) \\ &= z\Gamma_\lambda(z)\psi_\lambda(z) - \frac{1}{z}\Gamma_\lambda(z)\psi_\lambda(1/z) + z\Gamma_\lambda(1/z)\psi_\lambda(z) - \frac{1}{z}\Gamma_\lambda(1/z)\psi_\lambda(1/z) \\ &\quad - z\Gamma'_\lambda(z) + \frac{1}{z}\Gamma'_\lambda(1/z) \\ &= z\Gamma_\lambda(1/z)\psi_\lambda(z) - \frac{1}{z}\Gamma_\lambda(z)\psi_\lambda(1/z) \\ &= \beta(z). \end{aligned}$$

It follows from Lemma 2.5 that,  $\beta(z) > 0$  if  $z \in (1, \infty)$  and  $\beta(z) < 0$  if  $z \in (0, 1)$ . Hence  $\phi(z)$  is increasing on  $(1, \infty)$  and decreasing on  $(0, 1)$ . Consequently,  $H(z)$  is also increasing on  $(1, \infty)$  and decreasing on  $(0, 1)$ . In either case, we have

$$H(z) > H(1) = \frac{\lambda}{\ln(\lambda + 1)}$$

which completes the proof.  $\square$

**Remark 2.7.** As a consequence of Theorem 2.6, we recover the inequalities

$$\begin{aligned} \Gamma_\lambda(z)\Gamma_\lambda(1/z) &\geq \left(\frac{\lambda}{\ln(\lambda + 1)}\right)^2, \quad z > 0 \\ \Gamma_\lambda(z) + \Gamma_\lambda(1/z) &\geq \frac{2\lambda}{\ln(\lambda + 1)}, \quad z > 0 \end{aligned}$$

which were obtained in [24].

In relation to (10), (11), (13) and (15), we pose the following conjecture concerning the modified degenerate digamma function.

**Conjecture 2.8.** The inequality

$$\frac{2\psi_\lambda(z)\psi_\lambda(1/z)}{\psi_\lambda(z) + \psi_\lambda(1/z)} \geq -\gamma + \ln\left(\frac{\lambda}{\ln(\lambda + 1)}\right) \quad (20)$$

holds for  $z > 0$  and with equality when  $z = 1$ .

### 3. CONCLUSION

We have established that for  $z > 0$ , the harmonic mean of  $\Gamma_\lambda(z)$  and  $\Gamma_\lambda(1/z)$  is always greater than or equal to  $\lambda/\ln(\lambda + 1)$ , where  $\Gamma_\lambda(z)$  is the modified degenerate gamma function and  $\lambda \in (0, 1)$ . By letting  $\lambda \rightarrow 0$  in the established results, we obtain the famous results by Gautschi.

### 4. DATA AVAILABILITY

No data were used to support this study.

### 5. CONFLICTS OF INTEREST

The author declares that there is no conflict of interests regarding the publication of this paper.

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