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Constant Along Primal Rays Conjugacies and Generalized Convexity for Functions of the Support

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Abstract

The support of a vector in \mathbb{R}^d is the set of indices with nonzero entries. Functions of the support have the property to be 0-homogeneous and, because of that, the Fenchel conjugacy fails to provide relevant analysis. In this paper, we define the coupling Capra between \mathbb{R}^d and itself by dividing the classic Fenchel scalar product coupling by a given (source) norm on \mathbb{R}^d . Our main result is that, when both the source norm and its dual norm are orthant-strictly monotonic, any nondecreasing finite-valued function of the support mapping is Capra-convex, that is, is equal to its Capra-biconjugate (generalized convexity). We also establish that any such function is the composition of a proper convex lower semi continuous function on \mathbb{R}^d with the normalization mapping on the unit sphere (hidden convexity), and that, when normalized, it admits a variational formulation, which involves a family of generalized local- K -support dual norms.

Key words: support of a vector, hidden convexity, Fenchel-Moreau conjugacy, Capra conjugacy, generalized convexity, orthant-strictly monotonic norms, coordinate- K norms, top- K norms, K -support norms.

1 Introduction

The support of a vector in \mathbb{R}^d , where $d \geq 1$ is a fixed integer, is the set of indices in $V = \{1, \dots, d\}$ with nonzero entries. We consider the support mapping that maps vectors in \mathbb{R}^d to subsets of indices in 2^V , and set functions that go from 2^V to $\overline{\mathbb{R}}$. By composing any set function with the support mapping, we obtain *functions of the support mapping* (in short, FSM). An example of FSM is the so-called ℓ_0 pseudonorm (also called counting function, or cardinality function) which counts the number of nonzero components of a vector. The ℓ_0 pseudonorm measures the sparsity of a vector, and it is mentioned in an abundant literature in sparse optimization. However, because of its combinatorial nature,

the problems of minimizing the ℓ_0 pseudonorm under constraints or of minimizing a criterion under k -sparsity constraint (ℓ_0 pseudonorm less than a given integer k) are usually not tackled as such. Most of the literature in sparse optimization studies surrogate problems where the ℓ_0 pseudonorm either enters a penalization term or is replaced by a regularizing term. We refer the reader to [12] that provides a brief tour of the literature dealing with least squares minimization constrained by k -sparsity, and to [7] for a survey of the rank function of a matrix, that shares many properties with the ℓ_0 pseudonorm.

Conjugacies, and more generally dualities, are a powerful tool to tackle suitable classes of optimization problems. For instance, the Fenchel conjugacy plays a central role in analyzing solutions of convex problems (and beyond) [14]. Unfortunately, FSM have the property to be 0-homogeneous and, because of that, the Fenchel conjugacy fails to provide relevant analysis. As an illustration, the Fenchel biconjugate of the characteristic function of the level sets of the ℓ_0 pseudonorm and the Fenchel biconjugate of the ℓ_0 pseudonorm both are zero. However, the field of generalized convexity goes beyond the Fenchel conjugacy and convex functions, and provides conjugacies that are adapted to analyze classes of functions such as increasing positive homogeneous, difference of convex, quasi-convex, increasing and convex-along-rays. For more details on the theory, and more examples, we refer the reader to the books [16, 15] and to the nice introduction paper [10]. To our knowledge, none of the conjugacies in the literature is adapted to FSM. In this paper, we study FSM as such (and not surrogate functions) and we display a class of conjugacies that are suitable for FSM; thus equipped, we obtain results for nondecreasing finite-valued FSM.

The paper is organized as follows. In Sect. 2, we define functions of the support mapping (FSM), we provide background on couplings and conjugacies, and we introduce a source norm $\|\cdot\|$ on \mathbb{R}^d and the induced constant along primal rays coupling \mathfrak{c} (Capra). Then, we state our main result: if both the source norm $\|\cdot\|$ and the dual norm $\|\cdot\|_*$ are orthant-strictly monotonic, any nondecreasing finite-valued FSM is equal to its Capra-biconjugate, that is, is a Capra-convex function in the sense of generalized convexity. In Sect. 3, we introduce families of local norms — (dual) local-coordinate- K norms, generalized top- K and local- K -support dual norms — based on the source norm $\|\cdot\|$. Then, we provide formulas for Capra-conjugates, Capra-subdifferentials and Capra-biconjugates of FSM. In Sect. 4, we still suppose that both the source norm $\|\cdot\|$ and the dual norm $\|\cdot\|_*$ are orthant-strictly monotonic, and we arrive at unexpected results. Indeed, we show that any nondecreasing finite-valued FSM coincides, on the unit sphere $\mathbb{S} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$, with a proper convex lower semi continuous function on \mathbb{R}^d , and we deduce a variational formula for nondecreasing finite-valued FSM taking the value 0 on the null vector. Sect. 5 concludes. In the Appendix, Sect. A provides background on the constant along primal rays coupling (Capra), and Sect. B gathers material on local-coordinate- K and generalized local-top- K norms, and their dual norms.

2 Capra-convexity of nondecreasing functions of the support mapping (FSM)

In §2.1, we define the support mapping, set functions and functions of the support mapping (FSM). In §2.2, we gather background on Fenchel-Moreau conjugacies with respect to a coupling, and we introduce the constant along primal rays coupling \dagger (Capra) induced by a source norm $\|\cdot\|$. In §2.3, we state our main result on Capra-convexity of nondecreasing functions of FSM.

We work on the Euclidian space \mathbb{R}^d (with $d \in \mathbb{N}^*$), equipped with the scalar product $\langle \cdot, \cdot \rangle$ (but not necessarily with the Euclidian norm). We use the notation $\overline{\mathbb{R}} = [-\infty, +\infty]$.

2.1 The support mapping

Let $d \geq 1$ be a fixed integer. We denote by V the following set of indices:

$$V = \{1, \dots, d\} . \quad (1)$$

We consider the so-called \mathbb{R}^d -*support mapping*, denoted by supp and defined by

$$\begin{aligned} \text{supp} : \mathbb{R}^d &\rightarrow 2^V \\ x &\mapsto \{j \in V \mid x_j \neq 0\} . \end{aligned} \quad (2)$$

We consider *set functions*

$$F : 2^V \rightarrow \overline{\mathbb{R}} . \quad (3)$$

A set function F is said to be normalized if $F(\emptyset) = 0$.

Definition 1 *A function of the support mapping (in short, FSM) is a function of the form $F \circ \text{supp}$, where F is a set function as in (3). A FSM is said to be normalized if it takes the value 0 on the null vector.*

An example of function of the support mapping is given by $\ell_0 : \mathbb{R}^d \rightarrow V \subset \overline{\mathbb{R}}$, defined by $\ell_0(x) = |\text{supp}(x)| = \text{number of nonzero components of } x, \forall x \in \mathbb{R}^d$, where $|K|$ denotes the cardinal of a subset $K \subset V$. The function ℓ_0 falls in our scope with $F = |\cdot|$, as $\ell_0 = |\cdot| \circ \text{supp}$.

It is relevant to observe that FSM are 0-homogeneous, since the mapping supp itself is 0-homogeneous in the sense that:

$$\text{supp}(\rho x) = \text{supp}(x) , \quad \forall \rho \in \mathbb{R} \setminus \{0\} , \quad \forall x \in \mathbb{R}^d . \quad (4)$$

We introduce the *level sets* of the mapping supp

$$\text{supp}^{\subset K} = \{x \in \mathbb{R}^d \mid \text{supp}(x) \subset K\} , \quad \forall K \subset V , \quad (5a)$$

and the *level curves* of the mapping supp

$$\text{supp}^{\perp K} = \{x \in \mathbb{R}^d \mid \text{supp}(x) = K\}, \quad \forall K \subset V. \quad (5b)$$

For any subset $K \subset V$, we denote the subspace of \mathbb{R}^d made of vectors whose components vanish outside of K by¹

$$\mathcal{R}_K = \mathbb{R}^K \times \{0\}^{-K} = \{x \in \mathbb{R}^d \mid x_j = 0, \quad \forall j \notin K\} \subset \mathbb{R}^d, \quad (6)$$

where $\mathcal{R}_\emptyset = \{0\}$. We denote by $\pi_K : \mathbb{R}^d \rightarrow \mathcal{R}_K$ the *orthogonal projection mapping* and, for any vector $x \in \mathbb{R}^d$, by $x_K = \pi_K(x) \in \mathcal{R}_K$ the vector which coincides with x , except for the components outside of K that are zero. It is easily seen that the orthogonal projection mapping π_K is self-dual, giving

$$\langle x_K, y_K \rangle = \langle x_K, y \rangle = \langle \pi_K(x), y \rangle = \langle x, \pi_K(y) \rangle = \langle x, y_K \rangle, \quad \forall x \in \mathbb{R}^d, \quad \forall y \in \mathbb{R}^d. \quad (7)$$

The level sets of the supp mapping in (5a) are easily related to the subspaces \mathcal{R}_K of \mathbb{R}^d , as defined in (6), by

$$\text{supp}^{\perp K} = \{x \in \mathbb{R}^d \mid \text{supp}(x) \subset K\} = \mathcal{R}_K, \quad \forall K \subset V. \quad (8)$$

As we manipulate functions with values in $\overline{\mathbb{R}} = [-\infty, +\infty]$, we adopt the Moreau *lower* ($\dot{+}$) *and upper* ($\dot{+}$) *additions* [11], which extend the usual addition ($+$) with $(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = -\infty$ and $(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty$. Let \mathbb{W} be a set. For any function $h : \mathbb{W} \rightarrow \overline{\mathbb{R}}$, its *epigraph* is $\text{epih} = \{(w, t) \in \mathbb{W} \times \mathbb{R} \mid h(w) \leq t\}$, its *effective domain* is $\text{dom}h = \{w \in \mathbb{W} \mid h(w) < +\infty\}$. A function $h : \mathbb{W} \rightarrow \overline{\mathbb{R}}$ is said to be *proper* if it never takes the value $-\infty$ and if $\text{dom}h \neq \emptyset$. When \mathbb{W} is equipped with a topology, the function $h : \mathbb{W} \rightarrow \overline{\mathbb{R}}$ is said to be *lower semi continuous (lsc)* if its epigraph is closed. For any subset $W \subset \mathbb{W}$, $\delta_W : \mathbb{W} \rightarrow \overline{\mathbb{R}}$ denotes the *characteristic function* of the set W :

$$\delta_W(w) = 0 \text{ if } w \in W, \quad \delta_W(w) = +\infty \text{ if } w \notin W. \quad (9)$$

2.2 Constant along primal rays coupling (Capra)

In §2.2.1, we gather background on Fenchel-Moreau conjugacies with respect to a coupling c and we give definition and characterization of c -convex functions. Then, in §2.2.2, we show how to define a constant along primal rays coupling \dot{c} (Capra) by means of a norm.

2.2.1 Background on Fenchel-Moreau conjugacies

We review concepts and notations related to the Fenchel conjugacy (we refer the reader to [14]), and then we present how they are extended to general conjugacies [16, 15, 10].

¹Here, following notation from Game Theory, we have denoted by $-K$ the complementary subset of K in V : $K \cup (-K) = V$ and $K \cap (-K) = \emptyset$.

The Fenchel conjugacy on \mathbb{R}^d . The classic *Fenchel conjugacy* \star is defined, for any functions $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ and $g : \mathbb{Y} \rightarrow \overline{\mathbb{R}}$, by²

$$f^\star(y) = \sup_{x \in \mathbb{R}^d} \left(\langle x, y \rangle \dot{+} (-f(x)) \right), \quad \forall y \in \mathbb{R}^d, \quad (10a)$$

$$g^{\star'}(x) = \sup_{y \in \mathbb{Y}} \left(\langle x, y \rangle \dot{+} (-g(y)) \right), \quad \forall x \in \mathbb{R}^d, \quad (10b)$$

$$f^{\star\star'}(x) = \sup_{y \in \mathbb{R}^d} \left(\langle x, y \rangle \dot{+} (-f^\star(y)) \right), \quad \forall x \in \mathbb{R}^d. \quad (10c)$$

Recall that a function is said to be *convex* if its epigraph is a convex subset of $\mathbb{R}^d \times \mathbb{R}$, and is said to be *closed* if it is either lsc and nowhere having the value $-\infty$, or is the constant function $-\infty$ [14, p. 15]. It is proved that the Fenchel conjugacy induces a one-to-one correspondence between the closed convex functions on \mathbb{R}^d and themselves [14, Theorem 5]. Closed convex functions are the two constant functions $-\infty$ and $+\infty$ united with all proper convex lsc functions.³

The general case [16, 15, 10]. We consider two sets \mathbb{X} (“primal”), \mathbb{Y} (“dual”), not necessarily vector spaces, together with a *coupling* function

$$c : \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}. \quad (11)$$

With any coupling, one associates *conjugacies* from $\overline{\mathbb{R}}^{\mathbb{X}}$ to $\overline{\mathbb{R}}^{\mathbb{Y}}$ and from $\overline{\mathbb{R}}^{\mathbb{Y}}$ to $\overline{\mathbb{R}}^{\mathbb{X}}$ as follows.

Definition 2 *The c -Fenchel-Moreau conjugate of a function $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$, with respect to the coupling c , is the function $f^c : \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ defined by*

$$f^c(y) = \sup_{x \in \mathbb{X}} \left(c(x, y) \dot{+} (-f(x)) \right), \quad \forall y \in \mathbb{Y}. \quad (12a)$$

With the coupling c , we associate the reverse coupling c' defined by

$$c' : \mathbb{Y} \times \mathbb{X} \rightarrow \overline{\mathbb{R}}, \quad c'(y, x) = c(x, y), \quad \forall (y, x) \in \mathbb{Y} \times \mathbb{X}. \quad (12b)$$

The c' -Fenchel-Moreau conjugate of a function $g : \mathbb{Y} \rightarrow \overline{\mathbb{R}}$, with respect to the coupling c' , is the function $g^{c'} : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ defined by

$$g^{c'}(x) = \sup_{y \in \mathbb{Y}} \left(c(x, y) \dot{+} (-g(y)) \right), \quad \forall x \in \mathbb{X}. \quad (12c)$$

The c -Fenchel-Moreau biconjugate of a function $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$, with respect to the coupling c , is the function $f^{cc'} : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ defined by

$$f^{cc'}(x) = (f^c)^{c'}(x) = \sup_{y \in \mathbb{Y}} \left(c(x, y) \dot{+} (-f^c(y)) \right), \quad \forall x \in \mathbb{X}. \quad (12d)$$

²In convex analysis, one does not use the notation \star' in (10b) and (10c), but simply \star . We use \star' and $\star\star'$ to be consistent with the notation (12c) and (12d) for general conjugacies.

³In particular, any closed convex function that takes at least one finite value is necessarily proper convex lsc.

The biconjugate of a function $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ satisfies

$$f^{cc'}(x) \leq f(x) , \quad \forall x \in \mathbb{X} . \quad (13)$$

With the notion of c -biconjugate, the classic notion of convex function is generalized as follows.

Definition 3 *A function $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is said to be c -convex if it is equal to its c -biconjugate:*

$$f \text{ is } c\text{-convex} \iff f^{cc'} = f . \quad (14)$$

In generalized convexity, it is established that c -convex functions are all functions of the form $g^{c'}$, for all $g : \mathbb{Y} \rightarrow \overline{\mathbb{R}}$, or, equivalently, all functions of the form $f^{cc'}$, for all $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$. As an illustration, the \star -convex functions are the closed convex functions since, as recalled above, the Fenchel conjugacy induces a one-to-one correspondence between the closed convex functions on \mathbb{R}^d and themselves.

2.2.2 Constant along primal rays coupling (Capra)

Let $\|\cdot\|$ be a norm on \mathbb{R}^d , called the *source norm*. We denote the unit sphere \mathbb{S} and the unit ball \mathbb{B} of the norm $\|\cdot\|$ by

$$\mathbb{S} = \{x \in \mathbb{R}^d \mid \|x\| = 1\} , \quad \mathbb{B} = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\} . \quad (15)$$

Following [4, 3], we introduce the *constant along primal rays coupling* \mathfrak{c} (Capra).

Definition 4 ([3, Definition 8]) *We define the coupling \mathfrak{c} , or Capra, between \mathbb{R}^d and \mathbb{R}^d by*

$$\forall y \in \mathbb{R}^d , \quad \begin{cases} \mathfrak{c}(x, y) = \frac{\langle x, y \rangle}{\|x\|} , & \forall x \in \mathbb{R}^d \setminus \{0\} , \\ \mathfrak{c}(0, y) = 0. \end{cases} \quad (16)$$

We stress the point that, in (16), the Euclidian scalar product $\langle x, y \rangle$ and the norm term $\|x\|$ need not be related, that is, the norm $\|\cdot\|$ is not necessarily Euclidian.

The coupling Capra has the property of being *constant along primal rays*, hence the acronym Capra (Constant Along Primal RAYs). We introduce the primal *normalization mapping* n , from \mathbb{R}^d towards the unit sphere \mathbb{S} united with $\{0\}$, as follows:

$$n : \mathbb{R}^d \rightarrow \mathbb{S} \cup \{0\} , \quad n(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } x \neq 0 , \\ 0 & \text{if } x = 0 . \end{cases} \quad (17)$$

2.3 Capra-convexity of nondecreasing FSM

We recall definitions of orthant-monotonic and orthant-strictly monotonic norms. For any $x \in \mathbb{R}^d$, we denote by $|x|$ the vector of \mathbb{R}^d with components $|x_i|$, $i = 1, \dots, d$:

$$x = (x_1, \dots, x_d) \implies |x| = (|x_1|, \dots, |x_d|). \quad (18)$$

Definition 5 A norm $\|\cdot\|$ on the space \mathbb{R}^d is called

- orthant-monotonic [6] if, for all x, x' in \mathbb{R}^d , we have ($|x| \leq |x'|$ and $x \circ x' \geq 0 \implies \|x\| \leq \|x'\|$), where $|x| \leq |x'|$ means $|x_i| \leq |x'_i|$ for all $i = 1, \dots, d$, and where $x \circ x' = (x_1 x'_1, \dots, x_d x'_d)$ is the Hadamard (entrywise) product,
- orthant-strictly monotonic [5, Definition 3] if, for all x, x' in \mathbb{R}^d , we have ($|x| < |x'|$ and $x \circ x' \geq 0 \implies \|x\| < \|x'\|$), where $|x| < |x'|$ means that $|x_i| \leq |x'_i|$ for all $i = 1, \dots, d$, and there exists $j \in \{1, \dots, d\}$ such that $|x_j| < |x'_j|$.

We now establish that, when both the source norm and its dual norm are orthant-strictly monotonic, any nondecreasing finite-valued FSM is equal to its Capra-biconjugate, that is, is a Capra-convex function.

Theorem 6 Let $\|\cdot\|$ be the source norm with associated coupling ζ , as in Definition 4. Suppose that both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_*$ are orthant-strictly monotonic. Then, for any nondecreasing finite-valued set function $F : 2^V \rightarrow \mathbb{R}$, we have

$$(F \circ \text{supp})^{\zeta\zeta'} = F \circ \text{supp}, \quad (19)$$

that is, the function $F \circ \text{supp}$ is a Capra-convex function (see Definition 3).

The proof of Theorem 6 is relegated in §3.5, after we analyze functions of the support mapping by means of the Capra-conjugacy in Sect. 3.

3 The Capra-conjugacy and functions of the support mapping (FSM)

In §3.1, we introduce families of local norms — local-coordinate- K norms and their dual norms, as well as generalized local-top- K and local- K -support dual norms. Then, we provide formulas for Capra-conjugates of functions of the support mapping in §3.2, for Capra-subdifferentials in §3.3, and for Capra-biconjugates in §3.4. Finally, we prove Theorem 6 in §3.5.

3.1 Local- K norms

To analyze the support mapping by means of the Capra conjugacy, we introduce families of local norms — on the subspaces $\mathcal{R}_K \subset \mathbb{R}^d$ in (6), for any subset $K \subset V$ — as follows.

Restriction norms. We introduce the so-called restriction norms.

Definition 7 For any norm $\|\cdot\|$ on \mathbb{R}^d and any subset $K \subset \llbracket 1, d \rrbracket$, we define

- the K -restriction norm $\|\cdot\|_K$ on the subspace \mathcal{R}_K of \mathbb{R}^d , as defined in (6), by

$$\|x\|_K = \|x\|, \quad \forall x \in \mathcal{R}_K. \quad (20)$$

- the (K, \star) -norm $\|\cdot\|_{K, \star}$, on the subspace \mathcal{R}_K of \mathbb{R}^d , which is the norm $(\|\cdot\|_K)_\star$, given by the dual norm (on the subspace \mathcal{R}_K) of the restriction norm $\|\cdot\|_K$ to the subspace \mathcal{R}_K (first restriction, then dual),
- the (\star, K) -norm $\|\cdot\|_{\star, K}$ is the norm $(\|\cdot\|_\star)_K$, given by the restriction to the subspace \mathcal{R}_K of the dual norm $\|\cdot\|_\star$ (first dual, then restriction).

We have that [5, Equation (14b)]

$$\|y\|_{K, \star} = \sigma_{\mathcal{R}_K \cap \mathbb{B}}(y) = \sigma_{\mathcal{R}_K \cap \mathbb{S}}(y), \quad \forall y \in \mathcal{R}_K. \quad (21)$$

Local-coordinate- K norms and their dual norms.

Definition 8 For any subset $K \subset V$, we call local-coordinate- K norm (associated with the source norm $\|\cdot\|$) the norm $\|\cdot\|_{(K)}^\mathcal{R}$ (on the subspace \mathcal{R}_K in (6)) given by

$$\|\cdot\|_{(K)}^\mathcal{R} = (\|\cdot\|_{(K), \star}^\mathcal{R})_\star, \quad (22)$$

that is, whose dual norm (on the subspace \mathcal{R}_K) is the dual local-coordinate- K norm, denoted by $\|\cdot\|_{(K), \star}^\mathcal{R}$, with expression

$$\|y\|_{(K), \star}^\mathcal{R} = \sup_{J \subset K} \|y_J\|_{J, \star}, \quad \forall y \in \mathcal{R}_K, \quad (23)$$

where the (K, \star) -norm $\|\cdot\|_{K, \star}$ is given in Definition 7.

We adopt the convention $\|\cdot\|_{(\emptyset), \star}^\mathcal{R} = 0$. We denote the unit sphere and the unit ball of the local-coordinate- K norm $\|\cdot\|_{(K)}^\mathcal{R}$ in (22) by

$$\mathbb{S}_{(K)}^\mathcal{R} = \{x \in \mathcal{R}_K \mid \|x\|_{(K)}^\mathcal{R} = 1\} \subset \mathcal{R}_K, \quad \forall K \subset V, \quad (24a)$$

$$\mathbb{B}_{(K)}^\mathcal{R} = \{x \in \mathcal{R}_K \mid \|x\|_{(K)}^\mathcal{R} \leq 1\} \subset \mathcal{R}_K, \quad \forall K \subset V. \quad (24b)$$

We denote the unit sphere and the unit ball of the dual local-coordinate- K norm $\|\cdot\|_{(K), \star}^\mathcal{R}$ in (23) by

$$\mathbb{S}_{(K), \star}^\mathcal{R} = \{y \in \mathcal{R}_K \mid \|y\|_{(K), \star}^\mathcal{R} = 1\} \subset \mathcal{R}_K, \quad \forall K \subset V, \quad (25a)$$

$$\mathbb{B}_{(K), \star}^\mathcal{R} = \{y \in \mathcal{R}_K \mid \|y\|_{(K), \star}^\mathcal{R} \leq 1\} \subset \mathcal{R}_K, \quad \forall K \subset V. \quad (25b)$$

Generalized local-top- K and local- K -support dual norms.

Definition 9 For any subset $K \subset V$, we call generalized local-top- K dual norm (associated with the source norm $\|\cdot\|$) the local norm (on the subspace \mathcal{R}_K in (6)) defined by

$$\|y\|_{\star, (K)}^{\text{tn}} = \sup_{J \subset K} \|y_J\|_{\star} = \sup_{J \subset K} \|y_J\|_{\star, J}, \quad \forall y \in \mathcal{R}_K, \quad (26)$$

where we adopt the convention $\|\cdot\|_{\star, (\emptyset)}^{\text{tn}} = 0$. We call generalized local- K -support dual norm the local dual norm (on the subspace \mathcal{R}_K) of the generalized local-top- K dual norm, denoted by⁴ $\|\cdot\|_{\star, (K)}^{\text{sn}}$:

$$\|\cdot\|_{\star, (K)}^{\text{sn}} = \left(\|\cdot\|_{\star, (K)}^{\text{tn}} \right)_{\star}. \quad (27)$$

3.2 Capra-conjugates of FSM

With the Fenchel conjugacy, we calculate that $\delta_{\text{supp} \subset K}^{\star} = \delta_{\{0\}}$ — where $K \neq \emptyset$ and $\delta_{\text{supp} \subset K}$ is the characteristic function of the level sets (5a) as defined in (9) — and, more generally, that $(F \circ \text{supp})^{\star} = \sup \{ -F(\emptyset), \delta_{\{0\}} + (-\inf_{K \neq \emptyset} F(K)) \}$, for any set function F as in (3). Hence, the Fenchel conjugacy is not suitable to handle FSM. By contrast, we will now show that we obtain more interesting formulas with the Capra-conjugacy.

Proposition 10 Let $\|\cdot\|$ be the source norm with associated coupling \mathfrak{C} , as in Definition 4. Let $F : 2^V \rightarrow \overline{\mathbb{R}}$ be a set function.

We have

$$(F \circ \text{supp})^{\mathfrak{C}}(y) = \sup_{K \subset V} \left(\|y_K\|_{(K), \star}^{\mathcal{R}} - F(K) \right), \quad \forall y \in \mathbb{R}^d, \quad (28)$$

where the family $\left\{ \|\cdot\|_{(K), \star}^{\mathcal{R}} \right\}_{K \subset V}$ of dual local-coordinate- K norms is as in Definition 8, and with the convention $\|y_{\emptyset}\|_{(\emptyset), \star}^{\mathcal{R}} = 0$.

If, in addition, the norm $\|\cdot\|$ is orthant-monotonic, then we have that

$$(F \circ \text{supp})^{\mathfrak{C}} = \sup_{K \subset V} \left[\|\cdot\|_{\star, (K)}^{\text{tn}} - F(K) \right], \quad (29)$$

where the family $\left\{ \|\cdot\|_{\star, (K)}^{\text{tn}} \right\}_{K \subset V}$ of generalized local-top- K dual norms is as in Definition 9, and with the convention that $\|\cdot\|_{\star, (\emptyset)}^{\text{tn}} = 0$.

⁴We use the symbol \star in the superscript to indicate that the generalized local- K -support dual norm $\|\cdot\|_{\star, (K)}^{\text{sn}}$ is a dual norm.

Proof. To prove (28), we apply the postponed Lemma 11, as well as results about the dual local-coordinate- K norms $\|\cdot\|_{(K),\star}^{\mathcal{R}}$ gathered in Sect. B. For any $y \in \mathbb{R}^d$, we have

$$\begin{aligned} (F \circ \text{supp})^{\dot{\zeta}}(y) &= \sup_{K \subset V} \left(\sigma_{\mathbb{S} \cap (\text{supp}=K)}(y) \dot{+} (-F(K)) \right) && \text{(Lemma 11 with } \Gamma = \mathbb{R}^d) \\ &= \sup_{K \subset V} \left(\|y_K\|_{(K),\star}^{\mathcal{R}} - F(K) \right). \\ &\quad \text{(as } \sigma_{\text{supp}=K \cap \mathbb{S}}(y) = \sigma_{\text{supp}=K \cap \mathbb{S}}(y_K) = \|y_K\|_{(K),\star}^{\mathcal{R}} \text{ by (60a))} \end{aligned}$$

If the norm $\|\cdot\|$ is orthant-monotonic, we have that $\|\cdot\|_{(K),\star}^{\mathcal{R}} = \|\cdot\|_{\star,(K)}^{\text{tn}}$ by Proposition 25.

This ends the proof. \square

Lemma 11 *Let $\|\cdot\|$ be the source norm with associated coupling $\dot{\zeta}$, as in Definition 4.*

For any set function $F : 2^V \rightarrow \overline{\mathbb{R}}$ and any $\Gamma \subset \mathbb{R}^d$ such that $0 \in \Gamma$ and $n(\Gamma \cap \text{supp}^=K) = \Gamma \cap n(\text{supp}^=K)$ for all $K \subset V$, where the normalization mapping n has been defined in (17), we have

$$(F \circ \text{supp} \dot{+} \delta_{\Gamma})^{\dot{\zeta}}(y) = \sup_{K \subset V} \left(\sigma_{\mathbb{S} \cap \Gamma \cap (\text{supp}=K)}(y) \dot{+} (-F(K)) \right), \quad \forall y \in \mathbb{R}^d. \quad (30)$$

Proof. Using the level curves of the supp mapping in (5b), we obtain a representation of the function $F \circ \text{supp}$ as

$$F \circ \text{supp} = \inf_{K \subset V} (\delta_{\text{supp}=K} \dot{+} F(K)). \quad (31)$$

For all $y \in \mathbb{R}^d$, we have

$$\begin{aligned} (F \circ \text{supp} \dot{+} \delta_{\Gamma})^{\dot{\zeta}}(y) &= \left(\inf_{K \subset V} (\delta_{(\text{supp}=K)} \dot{+} F(K)) \dot{+} \delta_{\Gamma} \right)^{\dot{\zeta}}(y) && \text{(using Equation (31))} \\ &= \left(\inf_{K \subset V} (\delta_{\Gamma} \dot{+} \delta_{(\text{supp}=K)} \dot{+} F(K)) \right)^{\dot{\zeta}}(y) \end{aligned}$$

as $\inf_K \varphi(K) \dot{+} r = \inf_K (\varphi(K) \dot{+} r)$, for any set function $\varphi : 2^V \rightarrow \overline{\mathbb{R}}$ and any $r \in \overline{\mathbb{R}}$ [11]

$$\begin{aligned} &= \left(\inf_{K \subset V} (\delta_{\Gamma \cap (\text{supp}=K)} \dot{+} F(K)) \right)^{\dot{\zeta}}(y) && \text{(as } \delta_{\Gamma} \dot{+} \delta_{(\text{supp}=K)} = \delta_{\Gamma \cap (\text{supp}=K)}) \\ &= \sup_{K \subset V} \left(\delta_{\Gamma \cap (\text{supp}=K)} \dot{+} (-F(K)) \right)^{\dot{\zeta}}(y) \end{aligned}$$

as conjugacies, being dualities, turn infima into suprema

$$\begin{aligned}
&= \sup_{K \subset V} \left(\delta_{\Gamma \cap (\text{supp}^=K)}^{\dot{\mathcal{C}}}(y) \dot{+} (-F(K)) \right) && \text{(by property of conjugacies)} \\
&= \sup_{K \subset V} \left(\sigma_{n(\Gamma \cap \text{supp}^=K)}(y) \dot{+} (-F(K)) \right) \\
&\quad \text{(as } \delta_S^{\dot{\mathcal{C}}} = \sigma_{n(S)} \text{ for any subset } S \subset \mathbb{R}^d, \text{ by (53))} \\
&= \sup_{K \subset V} \left(\sigma_{\Gamma \cap n(\text{supp}^=K)}(y) \dot{+} (-F(K)) \right) \\
&\quad \text{(because } n(\Gamma \cap \text{supp}^=K) = \Gamma \cap n(\text{supp}^=K) \text{ by assumption)} \\
&= \sup_{K \subset V} \left(\sigma_{\Gamma \cap (\{0\} \cup (\mathbb{S} \cap \text{supp}^=K))}(y) \dot{+} (-F(K)) \right) \\
&\quad \text{(as } n(\text{supp}^=K) = \{0\} \cup (\mathbb{S} \cap \text{supp}^=K) \text{ by (17))} \\
&= \sup_{K \subset V} \left(\sup (0, \sigma_{\Gamma \cap \mathbb{S} \cap \text{supp}^=K})(y) \dot{+} (-F(K)) \right)
\end{aligned}$$

as $\Gamma \cap (\{0\} \cup (\mathbb{S} \cap \text{supp}^=K)) = \{0\} \cup (\Gamma \cap \mathbb{S} \cap \text{supp}^=K)$ since $0 \in \Gamma$ by assumption

$$= \sup_{K \subset V} \left(\sigma_{\mathbb{S} \cap \Gamma \cap \text{supp}^=K}(y) \dot{+} (-F(K)) \right). \quad \text{(as } \sigma_{\mathbb{S} \cap \text{supp}^=K} \geq 0 \text{)}$$

This ends the proof. \square

3.3 Capra-subdifferentials of FSM

We now show that FSM display Capra-subdifferentials, as in (55b), that are related to the family of dual local-coordinate- K norms, in Definition 8, as follows. For this purpose, we recall that the *normal cone* $N_C(x)$ to the (nonempty) closed convex subset $C \subset \mathbb{R}^d$ at $x \in C$ is the closed convex cone defined by [8, p.136]

$$N_C(x) = \{y \in \mathbb{R}^d \mid \langle x' - x, y \rangle \leq 0, \forall x' \in C\}. \quad (32)$$

Proposition 12 *Let $\|\cdot\|$ be the source norm with associated coupling $\dot{\mathcal{C}}$, as in Definition 4, and with associated families $\left\{ \|\cdot\|_{(K)}^{\mathcal{R}} \right\}_{K \subset V}$ of local-coordinate- K norms and $\left\{ \|\cdot\|_{(K),\star}^{\mathcal{R}} \right\}_{K \subset V}$ of dual local-coordinate- K norms, as in Definition 8.*

We consider a set function $F : 2^V \rightarrow \overline{\mathbb{R}}$ and a vector $x \in \mathbb{R}^d$.

- *The Capra-subdifferential, as in (55d), of the function $F \circ \text{supp}$ at $x = 0$ is given by*

$$\partial_{\dot{\mathcal{C}}}(F \circ \text{supp})(0) = \bigcap_{K \subset V} [F(K) \dot{+} (-F(\emptyset))] \mathbb{B}_{(K),\star}^{\mathcal{R}}, \quad (33a)$$

where, by convention, $\lambda \mathbb{B}_{(K),\star}^{\mathcal{R}} = \emptyset$, for any $\lambda \in [-\infty, 0[$, and $+\infty \mathbb{B}_{(K),\star}^{\mathcal{R}} = \mathbb{R}^d$.

- The Capra-subdifferential, as in (55e), of the function $F \circ \text{supp}$ at $x \neq 0$ is given by the following cases

- if $L = \text{supp}(x) \neq \emptyset$ and either $F(L) = -\infty$ or $F \equiv +\infty$, then $\partial_{\dot{\zeta}}(F \circ \text{supp})(x) = \mathbb{R}^d$,
- if $L = \text{supp}(x) \neq \emptyset$ and $F(L) = +\infty$ and there exists $K \subset V$ such that $F(K) \neq +\infty$, then $\partial_{\dot{\zeta}}(F \circ \text{supp})(x) = \emptyset$,
- if $L = \text{supp}(x) \neq \emptyset$ and $-\infty < F(L) < +\infty$, then

$$y \in \partial_{\dot{\zeta}}(F \circ \text{supp})(x) \iff \begin{cases} y \in N_{\mathbb{B}_{(L)}^{\mathcal{R}}}\left(\frac{x}{\|x\|_{(L)}^{\mathcal{R}}}\right) \text{ and} \\ L \in \arg \max_{K \subset V} [\|y\|_{(K),*}^{\mathcal{R}} - F(K)] . \end{cases} \quad (33b)$$

Proof. We have

$$\begin{aligned} y \in \partial_{\dot{\zeta}}(F \circ \text{supp})(x) &\iff (F \circ \text{supp})^{\dot{\zeta}}(y) = \dot{\zeta}(x, y) \dot{+} (-(F \circ \text{supp})(x)) \\ &\quad \text{(by definition (55b) of the Capra-subdifferential)} \\ &\iff \sup_{K \subset V} [\|y\|_{(K),*}^{\mathcal{R}} - F(K)] = \dot{\zeta}(x, y) \dot{+} (-(F \circ \text{supp})(x)) \\ &\quad \text{(as } (F \circ \text{supp})^{\dot{\zeta}}(y) = \sup_{K \subset V} [\|y\|_{(K),*}^{\mathcal{R}} - F(K)] \text{ by (28))} \\ &\iff \left(x = 0 \text{ and } \sup_{K \subset V} [\|y\|_{(K),*}^{\mathcal{R}} - F(K)] = -F(\emptyset) \right) \\ &\quad \text{(by definition (16) of } \dot{\zeta}(x, y) \text{ when } x = 0) \\ \text{or } &\left(x \neq 0 \text{ and } \sup_{K \subset V} [\|y\|_{(K),*}^{\mathcal{R}} - F(K)] = \frac{\langle x, y \rangle}{\|x\|} - F(\text{supp}(x)) \right) . \\ &\quad \text{(by definition (16) of } \dot{\zeta}(x, y) \text{ when } x \neq 0) \end{aligned}$$

Therefore, on the one hand, we obtain that

$$\begin{aligned} y \in \partial_{\dot{\zeta}}(F \circ \text{supp})(0) &\iff \|y\|_{(K),*}^{\mathcal{R}} - F(K) \leq -F(\emptyset), \quad \forall K \subset V \quad (\text{as } \|y\|_{(\emptyset),*}^{\mathcal{R}} = 0 \text{ by convention}) \\ &\iff \|y\|_{(K),*}^{\mathcal{R}} \leq F(K) \dot{+} (-F(\emptyset)), \quad \forall K \subset V \\ &\quad \text{(because } u \dot{+} (-v) \leq w \iff u \leq v \dot{+} w \text{ for all } u, v, w \text{ in } \overline{\mathbb{R}} \text{ [11])} \\ &\iff y \in \bigcap_{K \subset V} [F(K) \dot{+} (-F(\emptyset))] \mathbb{B}_{(K),*}^{\mathcal{R}}, \end{aligned}$$

where, by convention $\lambda \mathbb{B}_{(K),*}^{\mathcal{R}} = \emptyset$, for any $\lambda \in [-\infty, 0[$, and $+\infty \mathbb{B}_{(K),*}^{\mathcal{R}} = \mathbb{R}^d$.

On the other hand, when $x \neq 0$, we get

$$y \in \partial_{\dot{\zeta}}(F \circ \text{supp})(x) \iff \sup_{K \subset V} [\|y\|_{(K),*}^{\mathcal{R}} - F(K)] = \frac{\langle x, y \rangle}{\|x\|} - F(\text{supp}(x)) . \quad (34)$$

We now establish necessary and sufficient conditions for y to belong to $\partial_{\dot{\zeta}}(F \circ \text{supp})(x)$ when $x \neq 0$.

For this purpose, we consider $x \in \mathbb{R}^d \setminus \{0\}$, and we denote $L = \text{supp}(x)$. We have

$$y \in \partial_{\dot{C}}(F \circ \text{supp})(x)$$

$$\iff \sup_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] = \frac{\langle x, y \rangle}{\|x\|} - F(L) \quad (\text{by (34) with } \text{supp}(x) = L)$$

$$\iff \|y\|_{(L),\star}^{\mathcal{R}} - F(L) \leq \sup_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] = \frac{\langle x, y \rangle}{\|x\|} - F(L)$$

$$\iff \|y_L\|_{L,\star} - F(L) \leq \|y\|_{(L),\star}^{\mathcal{R}} - F(L) \leq \sup_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] = \frac{\langle x, y \rangle}{\|x\|} - F(L)$$

as $\|y_L\|_{L,\star} \leq \|y\|_{(L),\star}^{\mathcal{R}}$ by the expression (23) of the dual local-coordinate- L norm $\|y\|_{(L),\star}^{\mathcal{R}}$

$$\iff \|y_L\|_{L,\star} - F(L) \leq \|y\|_{(L),\star}^{\mathcal{R}} - F(L) \leq \sup_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] = \frac{\langle x, y \rangle}{\|x\|} - F(L) \leq \|y_L\|_{L,\star} - F(L)$$

as we have $\frac{\langle x, y \rangle}{\|x\|} = \frac{\langle x_L, y_L \rangle}{\|x_L\|} = \frac{\langle x_L, y_L \rangle}{\|x_L\|_L} \leq \|y_L\|_{L,\star}$ since $x = x_L$ and by property (56) of the dual norm

$$\iff \|y_L\|_{L,\star} - F(L) = \|y\|_{(L),\star}^{\mathcal{R}} - F(L) = \sup_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] = \frac{\langle x, y \rangle}{\|x\|} - F(L)$$

(as all terms in the inequalities are necessarily equal)

$$\iff \begin{cases} \text{either } F(L) = -\infty \\ \text{or } (F(L) = +\infty \text{ and } F(K) = +\infty, \forall K \subset V) \\ \text{or } (-\infty < F(L) < +\infty \text{ and} \\ \quad \|y_L\|_{L,\star} = \|y\|_{(L),\star}^{\mathcal{R}} = \frac{\langle x, y \rangle}{\|x\|} \text{ and } \|y\|_{(L),\star}^{\mathcal{R}} - F(L) = \sup_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)]) \end{cases}$$

Let us make a brief insert and notice that

$$\begin{aligned} x = x_L, \text{ sup}(x) = L \neq \emptyset, \langle x, y \rangle &= \|x\| \times \|y\|_{(L),\star}^{\mathcal{R}} \\ \implies \text{sup}(x) = L \neq \emptyset, \langle x_L, y_L \rangle &= \|x_L\|_L \times \|y\|_{(L),\star}^{\mathcal{R}} && (\text{by (7)}) \\ \implies \text{sup}(x) = L \neq \emptyset, \|x_L\|_L \times \|y\|_{(L),\star}^{\mathcal{R}} &\leq \|x_L\|_L \times \|y_L\|_{L,\star} && (\text{by (56)}) \\ \implies \|y\|_{(L),\star}^{\mathcal{R}} &\leq \|y_L\|_{L,\star} && (\text{since } \|x_L\|_L = \|x_L\| \neq 0 \text{ because } \text{sup}(x) = L \neq \emptyset) \\ \implies \|y\|_{(L),\star}^{\mathcal{R}} &= \|y_L\|_{L,\star} \end{aligned}$$

as $\|y_L\|_{L,\star} \leq \|y\|_{(L),\star}^{\mathcal{R}}$ by the expression (23) of the dual local-coordinate- L norm $\|y\|_{(L),\star}^{\mathcal{R}}$.

Now, let us go back to the equivalences regarding $y \in \partial_{\dot{C}}(F \circ \text{supp})(x)$. Focusing on the case where $\text{supp}(x) = L \neq \emptyset$ and $-\infty < F(L) < +\infty$, we have

$$\begin{aligned} y \in \partial_{\dot{C}}(F \circ \text{supp})(x) &\Leftrightarrow \|y_L\|_{L,\star} = \|y\|_{(L),\star}^{\mathcal{R}} = \frac{\langle x, y \rangle}{\|x\|} \text{ and } L \in \arg \max_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] \\ &\Leftrightarrow \|y_L\|_{L,\star} = \|y\|_{(L),\star}^{\mathcal{R}} \text{ and } \langle x, y \rangle = \|x\| \times \|y\|_{(L),\star}^{\mathcal{R}} \text{ and } L \in \arg \max_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] \\ &\Leftrightarrow \langle x, y \rangle = \|x\| \times \|y\|_{(L),\star}^{\mathcal{R}} \text{ and } L \in \arg \max_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] \end{aligned}$$

as just established in the insert right above

$$\begin{aligned} \Leftrightarrow \langle x, y \rangle &= \|x\|_{(L)}^{\mathcal{R}} \times \|y\|_{(L),\star}^{\mathcal{R}} \text{ and } L \in \arg \max_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] \\ &\quad (\text{as } \text{supp}(x) = L \implies x \in \mathcal{R}_L \implies \|x\| = \|x\|_{(L)}^{\mathcal{R}} \text{ by (61e)}) \\ \Leftrightarrow y &\in N_{\mathbb{B}_{(L)}^{\mathcal{R}}} \left(\frac{x}{\|x\|_{(L)}^{\mathcal{R}}} \right) \text{ and } L \in \arg \max_{K \subset V} [\|y\|_{(K),\star}^{\mathcal{R}} - F(K)] \end{aligned}$$

by the equivalence $\langle x, y \rangle = \|x\|_{(l)}^{\mathcal{R}} \times \|y\|_{(l),\star}^{\mathcal{R}} \iff y \in N_{\mathbb{B}_{(l)}^{\mathcal{R}}} \left(\frac{x}{\|x\|_{(l)}^{\mathcal{R}}} \right)$.

This ends the proof. \square

We now show that, when both the source norm and its dual norm are orthant-strictly monotonic, the Capra-subdifferential of a nondecreasing finite-valued FSM is nonempty.

Proposition 13 *Let $\|\cdot\|$ be the source norm with associated coupling \mathfrak{c} , as in Definition 4, and with associated families $\left\{ \|\cdot\|_{(K)}^{\mathcal{R}} \right\}_{K \subset V}$ of local-coordinate- K norms and $\left\{ \|\cdot\|_{(K),\star}^{\mathcal{R}} \right\}_{K \subset V}$ of dual local-coordinate- K norms, as in Definition 8.*

Suppose that both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_{\star}$ are orthant-strictly monotonic. Let $F : 2^V \rightarrow \mathbb{R}$ be a nondecreasing finite-valued set function. Then, we have

$$\partial_{\mathfrak{c}}(F \circ \text{supp})(x) \neq \emptyset, \quad \forall x \in \mathbb{R}^d.$$

More precisely, $\partial_{\mathfrak{c}}(F \circ \text{supp})(0) = \bigcap_{K \subset V} [F(K) \dot{+} (-F(\emptyset))] \mathbb{B}_{(K),\star}^{\mathcal{R}} \neq \emptyset$ and, when $x \neq 0$, for any $y \in \mathbb{R}^d$ such that $\text{supp}(y) = \text{supp}(x)$, and that $\langle x, y \rangle = \|x\| \times \|y\|_{\star}$, we have that $\lambda y \in \partial_{\mathfrak{c}}(F \circ \text{supp})(x)$ for $\lambda > 0$ large enough.

Proof. When $x = 0$, we have, by (33a), that $\partial_{\mathfrak{c}}(F \circ \text{supp})(0) = \bigcap_{K \subset V} [F(K) \dot{+} (-F(\emptyset))] \mathbb{B}_{(K),\star}^{\mathcal{R}}$ because $F(K) \dot{+} (-F(\emptyset)) = F(K) - F(\emptyset)$ since the function F takes finite values. The set $\bigcap_{K \subset V} [F(K) \dot{+} (-F(\emptyset))] \mathbb{B}_{(K),\star}^{\mathcal{R}}$ is nonempty (it contains 0), because $F(K) - F(\emptyset) \geq 0$ for $K \subset V$ since $F : 2^V \rightarrow \mathbb{R}$ is nondecreasing.

From now on, we consider $x \in \mathbb{R}^d \setminus \{0\}$ such that $\text{supp}(x) = L \subset V$, where $L \neq \emptyset$ since $x \neq 0$. We will use the characterization (33b) of the subdifferential $\partial_{\mathfrak{c}}(F \circ \text{supp})(x)$.

Since the norm $\|\cdot\|$ is orthant-strictly monotonic, by Proposition 21 (equivalence between Item 1 and Item 3), there exists a vector $y \in \mathbb{R}^d$ such that

$$L = \text{supp}(x) = \text{supp}(y) \neq \emptyset, \tag{35a}$$

$$\text{and } \langle x, y \rangle = \|x\| \times \|y\|_{\star}. \tag{35b}$$

Since both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_{\star}$ are orthant-strictly monotonic, using Proposition 26 we have that⁵

$$K \subsetneq L \implies \|y\|_{(K),\star}^{\mathcal{R}} < \|y\|_{(L),\star}^{\mathcal{R}} = \|y\|_{(V),\star}^{\mathcal{R}} = \|y\|_{\star}. \tag{36}$$

⁵ $K \subsetneq L$ stands for $K \subset L$ and $K \neq L$.

• First, we are going to establish that we have $y \in N_{\mathbb{B}_{(L)}^{\mathcal{R}}}(\frac{x}{\|x\|_{(L)}^{\mathcal{R}}})$, that is, the first of the two conditions in the characterization (33b) of the subdifferential $\partial_{\dot{C}}(F \circ \text{supp})(x)$.

On the one hand, by (36), we have that $\|y\|_{\star} = \|y\|_{(L),\star}^{\mathcal{R}}$. On the other hand, we have $\|x\| = \|x\|_{(L)}^{\mathcal{R}}$ by (61e) since $\text{supp}(x) = L$. Hence, from (35b), we get that $\langle x, y \rangle = \|x\|_{(L)}^{\mathcal{R}} \times \|y\|_{(L),\star}^{\mathcal{R}}$, from which we obtain $y \in N_{\mathbb{B}_{(L)}^{\mathcal{R}}}(\frac{x}{\|x\|_{(L)}^{\mathcal{R}}})$ by the equivalence $\langle x, y \rangle = \|x\|_{(L)}^{\mathcal{R}} \times \|y\|_{(L),\star}^{\mathcal{R}} \iff y \in N_{\mathbb{B}_{(L)}^{\mathcal{R}}}(\frac{x}{\|x\|_{(L)}^{\mathcal{R}}})$ as $x \neq 0$. To close this part, notice that, for all $\lambda > 0$, we have that $\lambda y \in N_{\mathbb{B}_{(L)}^{\mathcal{R}}}(\frac{x}{\|x\|_{(L)}^{\mathcal{R}}})$, because this last set is a (normal) cone.

• Second, we turn to proving the second of the two conditions in the characterization (33b) of the subdifferential $\partial_{\dot{C}}(F \circ \text{supp})(x)$. More precisely, we are going to show that, for λ large enough, $\|\lambda y\|_{(L),\star}^{\mathcal{R}} - F(L) = \sup_{K \subset V} [\|\lambda y\|_{(K),\star}^{\mathcal{R}} - F(K)]$. For this purpose, we consider the mapping $\psi :]0, +\infty[\rightarrow \mathbb{R}$ defined by

$$\psi(\lambda) = \|\lambda y\|_{(L),\star}^{\mathcal{R}} - F(L) - \sup_{K \subset V} [\|\lambda y\|_{(K),\star}^{\mathcal{R}} - F(K)] , \quad \forall \lambda > 0 ,$$

and we show that $\psi(\lambda) = 0$ for λ large enough. We have

$$\psi(\lambda) = \inf_{K \subset V} \phi_K(\lambda) \text{ with } \phi_K(\lambda) = \left(\lambda (\|y\|_{(L),\star}^{\mathcal{R}} - \|y\|_{(K),\star}^{\mathcal{R}}) + F(K) - F(L) \right) .$$

Using Proposition 26, we get the following.

- If $K \cap L = L$ (which is equivalent to $L \subset K$), we get that $(\|y\|_{(L),\star}^{\mathcal{R}} - \|y\|_{(K),\star}^{\mathcal{R}}) = 0$, which implies that $\phi_K(\lambda) = F(K) - F(L) \geq 0$ since F is assumed nondecreasing.
- If $K \cap L \subsetneq L$ (which is equivalent to $L \not\subset K$), we get that $(\|y\|_{(L),\star}^{\mathcal{R}} - \|y\|_{(K),\star}^{\mathcal{R}}) > 0$, which implies that $\phi_K(\lambda)$ goes to infinity with λ .

Wrapping up the above results, we have shown that, for any vector $y \in \mathbb{R}^d$ such that $\text{supp}(y) = \text{supp}(x)$, and that $\langle x, y \rangle = \|x\| \times \|y\|_{\star}$, then, for $\lambda > 0$ large enough, λy satisfies the two conditions in the characterization (33b) of the subdifferential $\partial_{\dot{C}}(F \circ \text{supp})(x)$.

This ends the proof. □

3.4 Capra-biconjugates of FSM

With the Fenchel conjugacy, we calculate that $\delta_{\text{supp} \subset K}^{\star\star'} = 0$ — where $K \neq \emptyset$, and $\delta_{\text{supp} \subset K}$ is the characteristic function of the level sets (5a) as defined in (9) — and, more generally, that $(F \circ \text{supp})^{\star\star'} = \inf \{ \delta_{\{0\}} + F(\emptyset), \inf_{K \neq \emptyset} F(K) \}$, with any set function F as in (3). Hence, the Fenchel conjugacy is not suitable to handle FSM.

By contrast, we will now show that FSM are related to the families of local-coordinate- K norms and dual local-coordinate- K norms, in Definition 8, by the following Capra-biconjugacy formulas.

In the sequel we will use the notation \mathbb{R}_V to refer to the subset $\prod_{K \subset V} \mathcal{R}_K$ of $(\mathbb{R}^d)^{2^V}$, and the notation Δ_V to refer to the simplex of \mathbb{R}_V (which can be identified with the simplex of \mathbb{R}^v , where $v = |2^V|$). Hence, an element $z \in \mathbb{R}_V$ is a family $\{z_{(K)}\}_{K \subset V}$ where $z_{(K)} \in \mathcal{R}_K$ for $K \subset V$ (with the convention $z_{(\emptyset)} = 0$), and an element $\lambda \in \Delta_V$ is a family $\{\lambda_K\}_{K \subset V}$ of nonnegative real numbers such that $\sum_{K \subset V} \lambda_K = 1$. We will also use $\mathbb{R}_V^{\setminus \emptyset}$ to denote the projection of \mathbb{R}_V when we get rid of the element λ_{\emptyset} , and we define $\Delta_V^{\setminus \emptyset}$ in the same way, that is,

$$\Delta_V^{\setminus \emptyset} = \left\{ \left\{ \lambda_K \in \mathbb{R}_V^{\setminus \emptyset} \right\}_{K \subset V, K \neq \emptyset} \mid \lambda_K \geq 0 \text{ and } \sum_{K \subset V, K \neq \emptyset} \lambda_K \leq 1 \right\}. \quad (37)$$

We denote by $\mathbb{B}_{(\cdot)}^{\mathcal{R}}$ (resp. $\mathbb{S}_{(\cdot)}^{\mathcal{R}}$) the family $\{\mathbb{B}_{(K)}^{\mathcal{R}}\}_{K \subset V}$ of unit local balls in (24b) (resp. $\{\mathbb{S}_{(K)}^{\mathcal{R}}\}_{K \subset V}$ of unit local spheres in (24a)) with the convention $\mathbb{B}_{(\emptyset)}^{\mathcal{R}} = \mathbb{S}_{(\emptyset)}^{\mathcal{R}} = \{0\}$. Then, $z \in \mathbb{B}_{(\cdot)}^{\mathcal{R}}$ will denote a family $\{z_{(K)}\}_{K \subset V, K \neq \emptyset}$, such that $z_{(K)} \in \mathbb{B}_{(K)}^{\mathcal{R}}$ for all $K \subset V$, with $K \neq \emptyset$.

Proposition 14 *Let $\|\cdot\|$ be the source norm with associated coupling \mathfrak{c} , as in Definition 4, and with associated families $\{\|\cdot\|_{(K)}^{\mathcal{R}}\}_{K \subset V}$ of local-coordinate- K norms and $\{\|\cdot\|_{(K),*}^{\mathcal{R}}\}_{K \subset V}$ of dual local-coordinate- K norms, as in Definition 8.*

Let $\{\Gamma_K\}_{K \subset V}$ be a family of subsets of \mathbb{R}^d such that the closed convex hull $\overline{\text{co}}(\Gamma_K) = \mathbb{B}_{(K)}^{\mathcal{R}}$ for all $K \subset V$ (with the convention that $\Gamma_{\emptyset} = \{0\}$).

- *For any set function $F : 2^V \rightarrow \overline{\mathbb{R}}$, we have*

$$(F \circ \text{supp})^{\mathfrak{c}\mathfrak{c}'}(x) = ((F \circ \text{supp})^{\mathfrak{c}})^{\star'}\left(\frac{x}{\|\cdot\|}\right), \quad \forall x \in \mathbb{R}^d \setminus \{0\}, \quad (38a)$$

where the function $((F \circ \text{supp})^{\mathfrak{c}})^{\star'}$ is closed convex and has the following expression as a Fenchel conjugate

$$((F \circ \text{supp})^{\mathfrak{c}})^{\star'} = \left(\sup_{K \subset V} \left(\|\pi_K(\cdot)\|_{(K),*}^{\mathcal{R}} - F(K) \right) \right)^{\star'}, \quad (38b)$$

and also has the following two equivalent expressions as a Fenchel biconjugate

$$= \left(\inf_{K \subset V} (\delta_{\Gamma_K} \dot{+} F(K)) \right)^{\star\star'}, \quad (38c)$$

$$= \left(x \mapsto \inf \{ F(K) \mid x \in \Gamma_K \} \right)^{\star\star'}. \quad (38d)$$

- *For any finite-valued set function $F : 2^V \rightarrow \mathbb{R}$, the function $((F \circ \text{supp})^{\mathfrak{c}})^{\star'}$ is proper convex lsc and has the following variational expression*

$$((F \circ \text{supp})^{\mathfrak{c}})^{\star'}(x) = \min_{\substack{\lambda \in \Delta_V \\ x \in \sum_{K \subset V} \lambda_K \Gamma_K}} \sum_{K \subset V} \lambda_K F(K), \quad \forall x \in \mathbb{R}^d, \quad (38e)$$

- For any finite-valued set function $F : 2^V \rightarrow \mathbb{R}$ such that $F(\emptyset) = 0$, the function $((F \circ \text{supp})^{\dot{\zeta}})^{\star'}$ is proper convex lsc and has the following variational expression

$$((F \circ \text{supp})^{\dot{\zeta}})^{\star'}(x) = \min_{\substack{z \in \mathbb{R}_V^{\setminus \emptyset} \\ \sum_{\emptyset \subsetneq K \subset V} \|z(K)\|_{(K)}^{\mathcal{R}} \leq 1 \\ \sum_{\emptyset \subsetneq K \subset V} z(K) = x}} \sum_{\emptyset \subsetneq K \subset V} \|z(K)\|_{(K)}^{\mathcal{R}} F(K), \quad \forall x \in \mathbb{R}^d, \quad (38f)$$

and the $\dot{\zeta}$ -biconjugate function $(F \circ \text{supp})^{\dot{\zeta}\dot{\zeta}'}$ has the following variational expression

$$(F \circ \text{supp})^{\dot{\zeta}\dot{\zeta}'}(x) = \frac{1}{\|x\|} \min_{\substack{z \in \mathbb{R}_V^{\setminus \emptyset} \\ \sum_{\emptyset \subsetneq K \subset V} \|z(K)\|_{(K)}^{\mathcal{R}} \leq \|x\| \\ \sum_{\emptyset \subsetneq K \subset V} z(K) = x}} \sum_{\emptyset \subsetneq K \subset V} \|z(K)\|_{(K)}^{\mathcal{R}} F(K), \quad \forall x \in \mathbb{R}^d \setminus \{0\}. \quad (38g)$$

Proof.

- We consider the set function $F : 2^V \rightarrow \overline{\mathbb{R}}$. By (52) and the definition (17), of the normalization mapping n , we immediately get (38a).
- We consider the set function $F : 2^V \rightarrow \overline{\mathbb{R}}$. We have

$$((F \circ \text{supp})^{\dot{\zeta}})^{\star'} = \left(\sup_{K \subset V} (\| \pi_K(\cdot) \|_{(K), \star}^{\mathcal{R}} - F(K)) \right)^{\star'} \quad (\text{by (28)})$$

so that we have proved (38b)

$$\begin{aligned} &= \left(\sup_{K \subset V} (\sigma_{\mathbb{B}_{(K)}^{\mathcal{R}}} - F(K)) \right)^{\star'} \\ (\text{by (58) as } \mathbb{B}_{(K)}^{\mathcal{R}} \text{ is the unit ball of the norm } \|\cdot\|_{(K)}^{\mathcal{R}} \text{ by (24b), and with the convention } \mathbb{B}_{(\emptyset)}^{\mathcal{R}} = \{0\}) \\ &= \left(\sup_{K \subset V} (\delta_{\Gamma_K}^{\star} - F(K)) \right)^{\star'} \end{aligned}$$

because $\delta_{\Gamma_K}^{\star} = \sigma_{\Gamma_K} = \sigma_{\overline{\text{co}}(\Gamma_K)} = \sigma_{\mathbb{B}_{(K)}^{\mathcal{R}}}$, where the last equality comes from the assumption $\overline{\text{co}}(\Gamma_K) = \mathbb{B}_{(K)}^{\mathcal{R}}$

$$\begin{aligned} &= \left(\sup_{K \subset V} ((\delta_{\Gamma_K} \dot{+} F(K))^{\star}) \right)^{\star'} \quad (\text{by property of conjugacies}) \\ &= \left(\left(\inf_{K \subset V} (\delta_{\Gamma_K} \dot{+} F(K)) \right)^{\star} \right)^{\star'} \end{aligned}$$

as conjugacies, being dualities, turn infima into suprema

$$= \left(\inf_{K \subset V} (\delta_{\Gamma_K} \dot{+} F(K)) \right)^{\star\star'}. \quad (\text{by definition (10c) of the Fenchel biconjugate})$$

Thus, we have obtained (38c) and (38d).

- We consider the set function $F : 2^V \rightarrow \mathbb{R}$. For the remaining expressions for $((F \circ \text{supp})^\zeta)^{\star'}$, we use a general formula [18, Corollary 2.8.11] for the Fenchel conjugate of the supremum of proper convex functions $f_K : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$, $K \subset V$:

$$\bigcap_{K \subset V} \text{dom} f_K \neq \emptyset \implies \left(\sup_{K \subset V} f_K \right)^\star = \min_{(\lambda_K)_{K \subset V} \in \Delta_V} \left(\sum_{K \subset V} \lambda_K f_K \right)^\star. \quad (39)$$

Thus, we obtain

$$\begin{aligned} \left((F \circ \text{supp})^\zeta \right)^{\star'} &= \left(\sup_{K \subset V} (\|\cdot\|_{(K),\star}^\mathcal{R} - F(K)) \right)^{\star'} && \text{(by (28))} \\ &= \left(\sup_{K \subset V} (\sigma_{\mathbb{B}_{(K)}^\mathcal{R}} - F(K)) \right)^{\star'} \\ \text{(by (58) as } \mathbb{B}_{(K)}^\mathcal{R} \text{ is the unit ball of the norm } \|\cdot\|_{(K)}^\mathcal{R} \text{ by (24b), and with the convention } \mathbb{B}_{(\emptyset)}^\mathcal{R} = \{0\}) \\ &= \left(\sup_{K \subset V} (\sigma_{\Gamma_K} - F(K)) \right)^{\star'} && \text{(as } \overline{\text{co}}(\Gamma_K) = \mathbb{B}_{(K)}^\mathcal{R} \text{ by assumption)} \\ &= \min_{(\lambda_K)_{K \subset V} \in \Delta_V} \left(\sum_{K \subset V} \lambda_K (\sigma_{\Gamma_K} - F(K)) \right)^{\star'} && \text{(by (39))} \end{aligned}$$

as the functions $f_K = \sigma_{\Gamma_K} - F(K)$ are proper convex (they even take finite values) for $K \subset V$

$$= \min_{(\lambda_K)_{K \subset V} \in \Delta_V} \left(\sigma_{\sum_{K \subset V} \lambda_K \Gamma_K} - \sum_{K \subset V} \lambda_K F(K) \right)^{\star'}$$

as, for all $K \subset V$, $\lambda_K \sigma_{\Gamma_K} = \sigma_{\lambda_K \Gamma_K}$ since $\lambda_K \geq 0$, and then using the well-known property that the support function of a Minkowski sum of subsets is the sum of the support functions of the individual subsets [8, p. 226]

$$\begin{aligned} &= \min_{(\lambda_K)_{K \subset V} \in \Delta_V} \left((\sigma_{\sum_{K \subset V} \lambda_K \Gamma_K})^{\star'} + \sum_{K \subset V} \lambda_K F(K) \right) \\ &\quad \text{(by property of conjugacies)} \\ &= \min_{(\lambda_K)_{K \subset V} \in \Delta_V} \left(\delta_{\sum_{K \subset V} \lambda_K \Gamma_K} + \sum_{K \subset V} \lambda_K F(K) \right). \\ &\quad \text{(because } \sum_{K \subset V} \lambda_K \Gamma_K \text{ is a closed convex set)} \end{aligned}$$

Therefore, for all $x \in \mathbb{R}^d$, we have

$$\left((F \circ \text{supp})^\zeta \right)^{\star'}(x) = \min_{\substack{(\lambda_K)_{K \subset V} \in \Delta_V \\ x \in \sum_{K \subset V} \lambda_K \Gamma_K}} \sum_{K \subset V} \lambda_K F(K),$$

which is (38e).

- We consider the set function $F : 2^V \rightarrow \mathbb{R}$, such that $F(\emptyset) = 0$. We are going to prove (38f). For this purpose, we start from the already proven Equation (38e) where we choose $\Gamma_K = \mathbb{S}_{(K)}^\mathcal{R}$ for all

$K \subset V$ (except $K = \emptyset$ for which $\Gamma_K = \{0\}$), which is licit since $\overline{\text{co}}(\mathbb{S}_{(K)}^{\mathcal{R}}) = \mathbb{B}_{(K)}^{\mathcal{R}}$. We obtain

$$\begin{aligned}
((F \circ \text{supp})^{\dot{\zeta}})^{\star'}(x) &= \min_{\substack{\lambda \in \Delta_V \\ x \in \sum_{K \subset V} \lambda_K \Gamma_K}} \sum_{K \subset V} \lambda_K F(K) && \text{(by (38e))} \\
&= \min_{\substack{\lambda \in \Delta_V \setminus \emptyset \\ x \in \sum_{\emptyset \subsetneq K \subset V} \lambda_K \Gamma_K}} \sum_{\emptyset \subsetneq K \subset V} \lambda_K F(K) && \text{(since } F(\emptyset) = 0 \text{ and } \Gamma_{\emptyset} = \{0\}) \\
&= \min_{\substack{\lambda \in \Delta_V \setminus \emptyset \\ x \in \sum_{\emptyset \subsetneq K \subset V} \lambda_K \mathbb{S}_{(K)}^{\mathcal{R}}}} \sum_{\emptyset \subsetneq K \subset V} \lambda_K F(K) && \text{(using } \Gamma_K = \mathbb{S}_{(K)}^{\mathcal{R}}) \\
&= \min_{\substack{z' \in \mathbb{S}_{(\cdot)}^{\mathcal{R}}, \lambda \in \Delta_V \setminus \emptyset \\ \sum_{\emptyset \subsetneq K \subset V} \lambda_K z'_{(K)} = x}} \sum_{\emptyset \subsetneq K \subset V} \lambda_K F(K) && \text{(getting rid of } \lambda_{\emptyset} \text{ since } F(\emptyset) = 0) \\
&= \min_{\substack{z \in \mathbb{R}_V \setminus \emptyset \\ \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{(K)}^{\mathcal{R}} \leq 1 \\ \sum_{\emptyset \subsetneq K \subset V} z_{(K)} = x}} \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{(K)}^{\mathcal{R}} F(K),
\end{aligned}$$

by putting $z_{(K)} = \lambda_K z'_{(K)}$, for all $\emptyset \subsetneq K \subset V$. Thus, we have obtained (38f).

Finally, from $(F \circ \text{supp})^{\dot{\zeta}\dot{\zeta}'} = ((F \circ \text{supp})^{\dot{\zeta}})^{\star'} \circ n$, by (52), we get that

$$(F \circ \text{supp})^{\dot{\zeta}\dot{\zeta}'}(x) = \frac{1}{\|x\|} \min_{\substack{z \in \mathbb{R}_V \setminus \emptyset \\ \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{(K)}^{\mathcal{R}} \leq \|x\| \\ \sum_{\emptyset \subsetneq K \subset V} z_{(K)} = x}} \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{(K)}^{\mathcal{R}} F(K), \quad \forall x \in \mathbb{R}^d \setminus \{0\}.$$

Therefore, we have proved (38g).

This ends the proof. \square

3.5 Proof of Theorem 6

Proof. By (28), we have that $(F \circ \text{supp})^{\dot{\zeta}} = \sup_{K \subset V} \left(\|\cdot\|_{(K),\star}^{\mathcal{R}} - F(K) \right)$, where $\|\cdot\|_{(K),\star}^{\mathcal{R}} = \|\cdot\|_{\star,(K)}^{\text{tn}}$ by Equation (69) in Proposition 25 since the norm $\|\cdot\|$ is orthant-monotonic. Hence, we obtain (29).

As, by assumption, both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_{\star}$ are orthant-strictly monotonic, Proposition 13 applies. Therefore, for any vector $x \in \mathbb{R}^d$ and any $y \in \partial_{\dot{\zeta}}(F \circ \text{supp})(x) \neq \emptyset$, we obtain

$$\begin{aligned}
(F \circ \text{supp})^{\dot{\zeta}\dot{\zeta}'}(x) &\geq \dot{\zeta}(x, y) \dot{+} (-(F \circ \text{supp})^{\dot{\zeta}}(y)) && \text{(by definition (12d) of the biconjugate)} \\
&= \dot{\zeta}(x, y) - (F \circ \text{supp})^{\dot{\zeta}}(y) && \text{(because } -\infty < \dot{\zeta}(x, y) < +\infty \text{ by (16))} \\
&= \dot{\zeta}(x, y) - (\dot{\zeta}(x, y) - (F \circ \text{supp})(x)) \\
&\quad \text{(by definition (55b) of the Capra-subdifferential } \partial_{\dot{\zeta}}(F \circ \text{supp})(x)) \\
&= (F \circ \text{supp})(x).
\end{aligned}$$

On the other hand, we have that $(F \circ \text{supp})^{\dot{\zeta}\dot{\zeta}'}(x) \leq (F \circ \text{supp})(x)$ by (13). We conclude that $(F \circ \text{supp})^{\dot{\zeta}\dot{\zeta}'}(x) = (F \circ \text{supp})(x)$, which is (19).

This ends the proof. \square

4 Hidden convexity and variational formulations for nondecreasing FSM

In this Sect. 4, we suppose that both the source norm $\|\cdot\|$ and the dual norm $\|\cdot\|_*$ are orthant-strictly monotonic. From our main result obtained in Sect. 2 — namely, Theorem 6 which provides conditions under which a function of the support mapping (FSM) is a Capra-convex function — we will derive the following results. In §4.1, we show that any nondecreasing finite-valued FSM coincides, on the unit sphere $\mathbb{S} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$, with a proper convex lsc function on \mathbb{R}^d . In §4.2, we deduce that a nondecreasing finite-valued FSM taking the value 0 on the null vector admits a variational formula which involves the whole family of generalized local- K -support dual norms in Definition 9. Then, in §4.3, we show that the variational formulation obtained yields a family of lower and upper bounds for suitable nondecreasing FSM, as a ratio between two norms.

4.1 Hidden convexity in nondecreasing FSM

We now present a (rather unexpected) consequence of the just established Theorem 6.

Proposition 15 *Let $\|\cdot\|$ be the source norm with associated coupling $\dot{\zeta}$, as in Definition 4, and with associated families $\left\{\|\cdot\|_{*,(K)}^{\text{tn}}\right\}_{K \subset V}$ of generalized local-top- K dual norms and $\left\{\|\cdot\|_{*,(K)}^{\text{sn}}\right\}_{K \subset V}$ of generalized local- K -support dual norms, as in Definition 9.*

Suppose that both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_$ are orthant-strictly monotonic. For any nondecreasing finite-valued set function $F : 2^V \rightarrow \mathbb{R}$, the following statements hold true.*

(i) *The following function $\mathcal{L}_0^F : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$, defined by*

$$\mathcal{L}_0^F = ((F \circ \text{supp})^{\dot{\zeta}})^{\star'} , \quad (40a)$$

is proper convex lsc.

(ii) *The function $F \circ \text{supp}$ coincides, on the unit sphere $\mathbb{S} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$, with the function \mathcal{L}_0^F in (40a):*

$$(F \circ \text{supp})(x) = \mathcal{L}_0^F(x) , \quad \forall x \in \mathbb{S} . \quad (40b)$$

(iii) The function $F \circ \text{supp}$ can be expressed as the composition of the proper convex lsc function \mathcal{L}_0^F in (40a) with the normalization mapping n in (17):

$$(F \circ \text{supp})(x) = \mathcal{L}_0^F\left(\frac{x}{\|x\|}\right), \quad \forall x \in \mathbb{R}^d \setminus \{0\}. \quad (40c)$$

(iv) The proper convex lsc function \mathcal{L}_0^F in (40a) is given by

$$\mathcal{L}_0^F = \left(\sup_{K \subset V} \left[\|\cdot\|_{\star, (K)}^{\text{tn}} - F(K) \right] \right)^{\star'}. \quad (40d)$$

(v) The function \mathcal{L}_0^F in (40a) is the largest proper convex lsc function below the function

$$\inf_{K \subset V} \left[\delta_{\mathbb{B}_{\star, (K)}^{\text{sn}}} \dot{+} F(K) \right], \quad (40e)$$

that is, below the function $x \in \mathbb{R}^d \mapsto \inf \{F(K) \mid K \subset V, x \in \mathbb{B}_{\star, (K)}^{\text{sn}}\}$, with the convention that $\mathbb{B}_{\star, (\emptyset)}^{\text{sn}} = \{0\}$ and that $\inf \emptyset = +\infty$.

(vi) The function \mathcal{L}_0^F in (40a) is the largest proper convex lsc function below the function

$$\inf_{K \subset V} \left[\delta_{\mathbb{S}_{\star, (K)}^{\text{sn}}} \dot{+} F(K) \right], \quad (40f)$$

that is, below the function $x \in \mathbb{R}^d \mapsto \inf \{F(K) \mid K \subset V, x \in \mathbb{S}_{\star, (K)}^{\text{sn}}\}$, with the convention that $\mathbb{S}_{\star, (\emptyset)}^{\text{sn}} = \{0\}$ and that $\inf \emptyset = +\infty$.

(vii) The function \mathcal{L}_0^F in (40a) also has the following variational expressions

$$\mathcal{L}_0^F(x) = \min_{\substack{\lambda \in \Delta_V \\ x \in \sum_{K \subset V} \lambda_K \mathbb{B}_{\star, (K)}^{\text{sn}}}} \sum_{K \subset V} \lambda_K F(K), \quad \forall x \in \mathbb{R}^d \quad (40g)$$

$$= \min_{\substack{\lambda \in \Delta_V \\ x \in \sum_{K \subset V} \lambda_K \mathbb{S}_{\star, (K)}^{\text{sn}}}} \sum_{K \subset V} \lambda_K F(K), \quad \forall x \in \mathbb{R}^d \quad (40h)$$

and, if $F(\emptyset) = 0$,

$$\mathcal{L}_0^F(x) = \min_{\substack{z \in \mathbb{R}^d \setminus \emptyset \\ \sum_{\emptyset \subsetneq K \subset V} \|z(K)\|_{\star, (K)}^{\text{sn}} \leq 1 \\ \sum_{\emptyset \subsetneq K \subset V} z(K) = x}} \sum_{\emptyset \subsetneq K \subset V} \|z(K)\|_{\star, (K)}^{\text{sn}} F(K), \quad \forall x \in \mathbb{R}^d. \quad (40i)$$

Proof. Since the norm $\|\cdot\|$ is orthant-strictly monotonic, it is orthant-monotonic, so that we have $\|\cdot\|_{(K)}^{\mathcal{R}} = \|\cdot\|_{\star, (K)}^{\text{sn}}$ and $\|\cdot\|_{(K), \star}^{\mathcal{R}} = \|\cdot\|_{\star, (K)}^{\text{tn}}$, for any $K \subset V$ by Equation (69) in Proposition 25 (with the proper conventions that they are all zero in the case $K = \emptyset$).

•(i) The function \mathcal{L}_0^F in (40a) is a Fenchel conjugate, hence is a closed convex function since the Fenchel conjugacy induces a one-to-one correspondence between the closed convex functions on \mathbb{R}^d and themselves [14, Theorem 5]. By (40b), proven below, the function \mathcal{L}_0^F takes finite values on the unit sphere. Thanks to Footnote 3, we conclude that the function \mathcal{L}_0^F is proper convex lsc.

•(ii) The assumptions make it possible to conclude that $(F \circ \text{supp})^{\dot{\mathcal{C}}'} = F \circ \text{supp}$, thanks to Theorem 6. We deduce from (38a) that

$$(F \circ \text{supp})(x) = (F \circ \text{supp})^{\dot{\mathcal{C}}'}(x) = ((F \circ \text{supp})^{\dot{\mathcal{C}}})^{\star'}\left(\frac{x}{\|x\|}\right), \quad \forall x \in \mathbb{R}^d \setminus \{0\}.$$

Thus, the function $F \circ \text{supp}$ coincides, on the unit sphere \mathbb{S} , with the closed convex function $\mathcal{L}_0^F : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ given by (52), namely $\mathcal{L}_0^F = ((F \circ \text{supp})^{\dot{\mathcal{C}}})^{\star'}$. Thus, we have proved (40b).

•(iii) The equality (40c) is an easy consequence of the property (4), implying that the function $F \circ \text{supp}$ is invariant along any open ray of \mathbb{R}^d .

•(iv) As $\mathcal{L}_0^F = ((F \circ \text{supp})^{\dot{\mathcal{C}}})^{\star'}$ by definition (40a), and as $((F \circ \text{supp})^{\dot{\mathcal{C}}})^{\star'} = \left(\sup_{K \subset V} \left[\|\cdot\|_{\star, (K)}^{\text{tn}} - F(K)\right]\right)^{\star'}$ by (29), we get (40d).

•(v) We use (38c) with $\Gamma_K = \mathbb{B}_{\star, (K)}^{\text{sn}}$ to obtain (40e). Indeed, since $\mathbb{B}_{\star, (K)}^{\text{sn}} = \mathbb{B}_{(K)}^{\mathcal{R}}$ (because $\|\cdot\|_{(K)}^{\mathcal{R}} = \|\cdot\|_{\star, (K)}^{\text{sn}}$), we have that $\overline{\text{co}}(\Gamma_K) = \mathbb{B}_{\star, (K)}^{\text{sn}} = \mathbb{B}_{(K)}^{\mathcal{R}}$ and thus $\mathbb{B}_{\star, (K)}^{\text{sn}}$ can be used as Γ_K .

•(vi) In the same way, we use (38c) with $\Gamma_K = \mathbb{S}_{\star, (K)}^{\text{sn}}$ to obtain (40f).

•(vii) We use (38e) with $\Gamma_K = \mathbb{B}_{\star, (K)}^{\text{sn}}$ to obtain (40g). In the same way, we also use (38e) with $\Gamma_K = \mathbb{S}_{\star, (K)}^{\text{sn}}$ to obtain (40h). We use (38f) with $\|\cdot\|_{(K)}^{\mathcal{R}} = \|\cdot\|_{\star, (K)}^{\text{sn}}$ to obtain (40i).

This ends the proof. \square

4.2 Variational formulation for nondecreasing FSM

As a straightforward application of Proposition 15, we obtain our second main result, namely a variational formulation for suitable nondecreasing FSM.

Theorem 16 *Let $\|\cdot\|$ be the source norm with associated family $\left\{\|\cdot\|_{\star, (K)}^{\text{sn}}\right\}_{K \subset V}$ of generalized local- K -support dual norms, as in Definition 9.*

Suppose that both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_{\star}$ are orthant-strictly monotonic. Then, for any nondecreasing finite-valued set function $F : 2^V \rightarrow \mathbb{R}$ such that $F(\emptyset) = 0$, we have the equality

$$(F \circ \text{supp})(x) = \frac{1}{\|x\|} \min_{\substack{z \in \mathbb{R}_V^{\setminus \emptyset} \\ \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{\star, (K)}^{\text{sn}} \leq \|x\|}} \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{\star, (K)}^{\text{sn}} F(K), \quad \forall x \in \mathbb{R}^d \setminus \{0\}. \quad (41)$$

When $\text{supp}(x) = L \neq \emptyset$, the minimum in (41) is achieved at $z \in \mathbb{R}_V^{\setminus \emptyset}$ such that $z_{(K)} = 0$ for $K \neq L$ and $z_{(L)} = x$.

Proof. Equation (41) derives from (40c) and (40i).

When $\text{supp}(x) = L \neq \emptyset$, we have $\|x\| = \|x\|_{(L)}^{\mathcal{R}}$ by (61e). Moreover, $\|\cdot\|_{(K)}^{\mathcal{R}} = \|\cdot\|_{\star, (K)}^{\text{sn}}$ by Equation (69) in Proposition 25, since the norm $\|\cdot\|$ is orthant-strictly monotonic, hence is orthant-monotonic. As a consequence, we have that $\|x\| = \|x\|_{\star, (L)}^{\text{sn}}$. Therefore, $z \in \mathbb{R}_V^{\setminus \emptyset}$ such that $z_{(L)} = x$ and $z_{(K)} = 0$ for $K \neq L$ is admissible for the minimization problem (41). We deduce that $F(L) = (F \circ \text{supp})(x) \leq \frac{1}{\|x\|} F(L) \|x\|_{\star, (L)}^{\text{sn}} = F(L)$.

This ends the proof. \square

As an illustration, Theorem 16 applies when the norm $\|\cdot\|$ is any of the ℓ_p -norms $\|\cdot\|_p$ on the space \mathbb{R}^d , for $p \in]1, \infty[$, giving:

$$(F \circ \text{supp})(x) = \frac{1}{\|x\|_p} \min_{\substack{z \in \mathbb{R}_V^{\setminus \emptyset} \\ \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{p, \star, (K)}^{\text{sn}} \leq \|x\|_p \\ \sum_{\emptyset \subsetneq K \subset V} z_{(K)} = x}} \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{p, \star, (K)}^{\text{sn}} F(K), \quad \forall x \in \mathbb{R}^d \setminus \{0\}. \quad (42)$$

Indeed, when $p \in]1, \infty[$, the ℓ_p -norm $\|\cdot\|_p$ is orthant-strictly monotonic, and so is its dual norm $\|\cdot\|_{\star} = \|\cdot\|_q$ where $1/p + 1/q = 1$ as easily seen. When $p = \infty$, the ℓ_{∞} -norm $\|\cdot\|_{\infty}$ is not orthant-strictly monotonic. When $p = 1$, the ℓ_1 -norm $\|\cdot\|_1$ is orthant-strictly monotonic, but the dual norm $\|\cdot\|_{\star} = \|\cdot\|_{\infty}$ is not.

Applications to sparse optimization. Supposing that the assumptions of Theorem 16 are satisfied, we get the two following reformulations for exact sparse optimization problems. Let $F : 2^V \rightarrow \mathbb{R}_+$ be a nondecreasing finite-valued set function such that $F(\emptyset) = 0$.

Let $C \subset \mathbb{R}^d$ be such that $0 \notin C$ (to avoid a division by zero, and also because the minimum would be achieved at zero). Then, we have that

$$\min_{x \in C} F(\text{supp}(x)) = \min_{x \in C} \frac{1}{\|x\|} \underbrace{\min_{\substack{z \in \mathbb{R}_V^{\setminus \emptyset} \\ \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{\star, (K)}^{\text{sn}} \leq \|x\| \\ \sum_{\emptyset \subsetneq K \subset V} z_{(K)} = x}} \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{\star, (K)}^{\text{sn}} F(K)}_{\text{convex optimization problem}}. \quad (43)$$

For any $\alpha \in \mathbb{R}$, we have that

$$\min_{(F \circ \text{supp})(x) \leq \alpha} f(x) = \min_{\substack{x \in \mathbb{R}^d, z \in \mathbb{R}_V^{\setminus \emptyset} \\ \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{\star, (K)}^{\text{sn}} \leq \|x\| \\ \sum_{\emptyset \subsetneq K \subset V} z_{(K)} = x \\ \sum_{\emptyset \subsetneq K \subset V} F(K) \|z_{(K)}\|_{\star, (K)}^{\text{sn}} \leq \alpha \|x\|}} f(x), \quad (44a)$$

$$= \min_{\substack{z \in \mathbb{R}_V^{\setminus \emptyset} \\ \sum_{\emptyset \subsetneq K \subset V} \|z_{(K)}\|_{\star, (K)}^{\text{sn}} \leq \sum_{\emptyset \subsetneq K \subset V} z_{(K)} \\ \sum_{\emptyset \subsetneq K \subset V} F(K) \|z_{(K)}\|_{\star, (K)}^{\text{sn}} \leq \alpha \sum_{\emptyset \subsetneq K \subset V} z_{(K)} \|z_{(K)}\|_{\star, (K)}^{\text{sn}}}} f\left(\sum_{\emptyset \subsetneq K \subset V} z_{(K)}\right). \quad (44b)$$

4.3 Upper and lower bounds for nondecreasing FSM as norm ratios

We now show that the variational formulation obtained in §4.2 yields a family of lower and upper bounds for suitable nondecreasing FSM as a ratio between two norms — the denominator norm being any orthant-strictly monotonic norm with orthant-strictly monotonic dual norm.

Proposition 17 *Let $\|\cdot\|$ be the source norm with associated families $\left\{\|\cdot\|_{\star,(K)}^{\text{tn}}\right\}_{K \subset V}$ of generalized local-top- K dual norms and $\left\{\|\cdot\|_{\star,(K)}^{\text{sn}}\right\}_{K \subset V}$ of generalized local- K -support dual norms, as in Definition 9.*

Suppose that both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_{\star}$ are orthant-strictly monotonic. Let $F : 2^V \rightarrow \mathbb{R}_+$ be a nondecreasing nonnegative set function, such that $F(K) > F(\emptyset) = 0$ for all $\emptyset \subsetneq K \subset V$. Then, we have the inequalities

$$\frac{\|x\|_{\star,[F]}^{\text{sn}}}{\|x\|} \leq F(\text{supp}(x)) \leq \min_{K \subset V} \frac{F(K) \|x\|_{\star,(K)}^{\text{sn}}}{\|x\|}, \quad \forall x \in \mathbb{R}^d \setminus \{0\}, \quad (45)$$

where the norm $\|\cdot\|_{\star,[F]}^{\text{sn}}$ is characterized

- either by its dual norm $\|\cdot\|_{\star,[F]}^{\text{tn}} = (\|\cdot\|_{\star,[F]}^{\text{sn}})^{\star}$ which has unit ball $\bigcap_{K \subset V} F(K) \mathbb{B}_{\star,(K)}^{\text{tn}}$, that is,

$$\|\cdot\|_{\star,[F]}^{\text{sn}} = (\|\cdot\|_{\star,[F]}^{\text{tn}})^{\star} = \sigma_{\mathbb{B}_{\star,[F]}^{\text{tn}}} \quad \text{with} \quad \mathbb{B}_{\star,[F]}^{\text{tn}} = \bigcap_{K \subset V} F(K) \mathbb{B}_{\star,(K)}^{\text{tn}}, \quad (46a)$$

or, equivalently, by the formula

$$(\|\cdot\|_{\star,[F]}^{\text{sn}})^{\star}(y) = \|y\|_{\star,[F]}^{\text{tn}} = \sup_{K \subset V} \frac{\|y\|_{\star,(K)}^{\text{tn}}}{F(K)}, \quad \forall y \in \mathbb{R}^d, \quad (46b)$$

- or by the inf-convolution

$$\|\cdot\|_{\star,[F]}^{\text{sn}} = \bigsqcap_{K \subset V} \left(F(K) \|\cdot\|_{\star,(K)}^{\text{sn}} \right), \quad (46c)$$

$$\text{that is, } \|x\|_{\star,[F]}^{\text{sn}} = \inf_{\substack{z \in \mathbb{R}_V^{\setminus \emptyset} \\ \sum_{K=1}^d z^{(K)} = x}} \sum_{\emptyset \subsetneq K \subset V} F(K) \|z^{(K)}\|_{\star,(K)}^{\text{sn}}, \quad \forall x \in \mathbb{R}^d. \quad (46d)$$

Proof. The equivalence between the four equivalent formulations in Equation (46) is straightforward (See [3, Proposition 15] for a similar proof) as it is immediate to verify that $\|\cdot\|_{\star,[F]}^{\text{sn}}$ is a norm since $F(K) > F(\emptyset) = 0$ for all $\emptyset \subsetneq K \subset V$.

Now, the two inequalities in Equation (45) follow from Equation (41), as the assumptions of Theorem 16 are satisfied.

To get the right hand side (upper bound) inequality in Equation (45), it suffices to consider the Equality (41) and to deduce an upper bound by reducing the set over which the minimization is done to all the vectors $z \in \mathbb{R}_V^{\setminus \emptyset}$ such that $z_{(K)} = 0$ for $K \neq L$ and $z_{(L)} = x$, for all $L \subset V$.

To get the left hand side (lower bound) inequality in Equation (45), it suffices to consider the Equality (41) and to deduce a lower bound by extending the set over which the minimization is done to all the vectors $z \in \mathbb{R}_V^{\setminus \emptyset}$ such that $\sum_{\emptyset \subsetneq K \subset V} z_{(K)} = x$. Then, we recognize the inf-convolution formulation in Equation (46d).

This ends the proof. \square

Equation (45) can also be written as

$$\|x\|_{\star, [F]}^{\text{sn}} \leq F(\text{supp}(x)) \|x\| \leq \min_{K \subset V} \left(F(K) \|x\|_{\star, (K)}^{\text{sn}} \right), \quad \forall x \in \mathbb{R}^d. \quad (47)$$

When the norm $\|\cdot\|$ is the ℓ_p -norm ($1 < p < +\infty$), the left hand side inequality in (47) coincides with the inequality obtained in [13, Proposition 2]; moreover, Equation (46b) corresponds to [13, Equation (2)], and Equation (46d) to [13, Equation (3)]. So, we extend the results in [13, Proposition 2] beyond the ℓ_p -norms, and we also provide an upper bound (right hand side inequality in (47)).

5 Conclusion

The combinatorial expression of the support mapping makes it difficult to handle it as such in continuous optimization problems, and we have seen that the Fenchel conjugacy is not adapted. In this paper, we have introduced a class of conjugacies that are suitable for functions of the support (FSM). Each conjugacy is induced by a Capra-coupling that depends on a given source norm on \mathbb{R}^d . Our main result is that any nondecreasing finite-valued FSM is a Capra-convex function (that is, is equal to its Capra-biconjugate) when both the source norm and its dual norm are orthant-strictly monotonic. We have also shown the surprising consequence that any nondecreasing finite-valued FSM coincides, on the unit sphere of the source norm, with a proper convex lsc function. From there, we have obtained exact variational formulations for normalized nondecreasing finite-valued FSM. For this purpose, we have introduced sequences of generalized local-top- K and local- K -support dual norms, generated from the source norm on \mathbb{R}^d .

The reformulations that we propose for exact sparse optimization problems make use of 2^d new (latent) vectors, making a direct numerical implementation out of reach. However, the variational formulation may suggest approximations of the FSM or algorithms making use of the partial convexity that our analysis has put to light. Moreover, we have provided expressions for the Capra-subdifferential of nondecreasing finite-valued FSM, which can inspire “gradient-like” algorithms. In all cases, the variational formulation obtained yields a new family of lower and upper bounds for null at zero nondecreasing finite-valued FSM as a ratio between two norms; this may lead to new smooth sparsity inducing terms, proxies for the FSM.

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A Properties of the constant along primal rays coupling (Capra)

We recall properties of the Capra-conjugacy, induced by the coupling Capra in Definition 4 (see §2.2.1 and [3]).

Here are expressions for Capra-conjugates and biconjugates. We recall that, in convex analysis, for any subset $S \subset \mathbb{R}^d$, $\sigma_S : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ denotes the *support function*⁶ of the subset S :

$$\sigma_S(y) = \sup_{x \in S} \langle x, y \rangle, \quad \forall y \in \mathbb{R}^d. \quad (48)$$

Proposition 18 ([3, Proposition 9]) *For any function $g : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$, the \mathfrak{C}' -Fenchel-Moreau conjugate is the function $g^{\mathfrak{C}'} : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ given by*

$$g^{\mathfrak{C}'} = g^* \circ n. \quad (49)$$

For any function $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$, the \mathfrak{C} -Fenchel-Moreau conjugate is the function $f^{\mathfrak{C}} : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ given by

$$f^{\mathfrak{C}} = (\inf [f \mid n])^*, \quad (50)$$

*where the conditional infimum $\inf [f \mid n]$ has the expression*⁷

$$\inf [f \mid n](x) = \inf \{f(x') \mid n(x') = x\} = \begin{cases} \inf_{\lambda > 0} f(\lambda x) & \text{if } x \in \mathbb{S} \cup \{0\}, \\ +\infty & \text{if } x \notin \mathbb{S} \cup \{0\}, \end{cases} \quad (51)$$

and the \mathfrak{C} -Fenchel-Moreau biconjugate is the function $f^{\mathfrak{C}\mathfrak{C}'} : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ given by

$$f^{\mathfrak{C}\mathfrak{C}'} = (f^{\mathfrak{C}})^{\star'} \circ n = (\inf [f \mid n])^{\star\star'} \circ n. \quad (52)$$

For any subset $W \subset \mathbb{W}$, the \mathfrak{C} -Fenchel-Moreau conjugate of the characteristic function (9) of the set W is given by

$$\delta_W^{\mathfrak{C}} = \sigma_{n(W)}. \quad (53)$$

Here are characterizations of the Capra-convex functions (see Definition 3).

⁶The support *function* (of a subset) in (48) should be distinguished from the support *mapping* in (2).

⁷The name “conditional infimum” comes from [17]. We chose the notation to stress the analogy with a conditional expectation. We adopt the convention that $\inf \emptyset = +\infty$.

Proposition 19 ([3, Proposition 10]) *A function is $\dot{\mathcal{C}}$ -convex if and only if it is the composition of a closed convex function on \mathbb{R}^d with the normalization mapping (17). More precisely, for any function $h : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$, we have the equivalences*

$$\begin{aligned} h \text{ is } \dot{\mathcal{C}}\text{-convex} &\Leftrightarrow h = h^{\dot{\mathcal{C}}\dot{\mathcal{C}}'} \\ &\Leftrightarrow h = (h^{\dot{\mathcal{C}}})^{\star'} \circ n \text{ (where } (h^{\dot{\mathcal{C}}})^{\star'} \text{ is a closed convex function)} \\ &\Leftrightarrow \text{there exists a closed convex function } f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}} \text{ such that } h = f \circ n . \end{aligned}$$

Following the definition of the subdifferential of a function with respect to a duality in [1], the *Capra-subdifferential* of the function $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ at $x \in \mathbb{R}^d$ has the following expressions

$$\partial_{\dot{\mathcal{C}}} f(x) = \{y \in \mathbb{R}^d \mid \dot{\mathcal{C}}(x', y) \dot{+} (-f(x')) \leq \dot{\mathcal{C}}(x, y) \dot{+} (-f(x)) , \forall x' \in \mathbb{R}^d\} \quad (55a)$$

$$= \{y \in \mathbb{R}^d \mid f^{\dot{\mathcal{C}}}(y) = \dot{\mathcal{C}}(x, y) \dot{+} (-f(x))\} \quad (55b)$$

$$= \{y \in \mathbb{R}^d \mid (\inf [f \mid n])^{\star}(y) = \langle n(x), y \rangle \dot{+} (-f(x))\} , \quad (55c)$$

so that, thanks to the definition (17) of the normalization mapping n , we deduce that

$$\partial_{\dot{\mathcal{C}}} f(0) = \{y \in \mathbb{R}^d \mid (\inf [f \mid n])^{\star}(y) = -f(0)\} \quad (55d)$$

$$\partial_{\dot{\mathcal{C}}} f(x) = \{y \in \mathbb{R}^d \mid (\inf [f \mid n])^{\star}(y) = \frac{\langle x, y \rangle}{\|x\|} \dot{+} (-f(x))\} , \forall x \in \mathbb{R}^d \setminus \{0\} . \quad (55e)$$

B Material on local-coordinate- K and generalized local-top- K norms

We start, in §B.1, by providing background on orthant-monotonic and orthant-strictly monotonic norms. Then, in §B.2, we introduce local-coordinate- K norms and dual local-coordinate- K norms, and in §B.3, we introduce generalized local-top- K and local- K -support dual norms; they are norms on the subspaces \mathcal{R}_K of \mathbb{R}^d in (6) constructed from a source norm.

B.1 Orthant-monotonic and orthant-strictly monotonic norms

Dual norms. We recall that the expression $\|y\|_{\star} = \sup_{\|x\| \leq 1} \langle x, y \rangle$, $\forall y \in \mathbb{R}^d$, defines a norm on \mathbb{R}^d , called the *dual norm* $\|\cdot\|_{\star}$. By definition of the dual norm, we have the inequality

$$\langle x, y \rangle \leq \|x\| \times \|y\|_{\star} , \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d . \quad (56)$$

We denote the unit sphere \mathbb{S}_{\star} and the unit ball \mathbb{B}_{\star} of the dual norm $\|\cdot\|_{\star}$ by

$$\mathbb{S}_{\star} = \{y \in \mathbb{R}^d \mid \|y\|_{\star} = 1\} , \quad \mathbb{B}_{\star} = \{y \in \mathbb{R}^d \mid \|y\|_{\star} \leq 1\} . \quad (57)$$

Denoting by σ_S the *support function* of the set $S \subset \mathbb{R}^d$ ($\sigma_S(y) = \sup_{x \in S} \langle x, y \rangle$), we have

$$\|\cdot\| = \sigma_{\mathbb{B}_\star} = \sigma_{\mathbb{S}_\star} \text{ and } \|\cdot\|_\star = \sigma_{\mathbb{B}} = \sigma_{\mathbb{S}}, \quad (58)$$

where $\mathbb{B}_\star = \mathbb{B}^\circ = \{y \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1, \forall x \in \mathbb{B}\}$ is the polar set \mathbb{B}° of the unit ball \mathbb{B} .

We recall properties of orthant-monotonic and orthant-strictly monotonic norms.

Proposition 20 ([5, Proposition 6], [6, Theorem 2.26], [9, Theorem 3.2]) *Let $\|\cdot\|$ be a norm on \mathbb{R}^d . The following assertions are equivalent.*

1. *The norm $\|\cdot\|$ is orthant-monotonic.*
2. *The dual norm $\|\cdot\|_\star$ is orthant-monotonic.*
3. *The norm $\|\cdot\|$ is increasing with the coordinate subspaces, in the sense that, for any $x \in \mathbb{R}^d$ and any $J \subset K \subset V$, we have $\|x_J\| \leq \|x_K\|$.*

Proposition 21 ([5, Proposition 8]) *Let $\|\cdot\|$ be a norm on \mathbb{R}^d . The following assertions are equivalent.*

1. *The norm $\|\cdot\|$ is orthant-strictly monotonic.*
2. *The norm $\|\cdot\|$ is strictly increasing with the coordinate subspaces in the sense that, for any $x \in \mathbb{R}^d$ and any $J \subsetneq K \subset V$, we have $x_J \neq x_K \implies \|x_J\| < \|x_K\|$.*
3. *For any vector $u \in \mathbb{R}^d \setminus \{0\}$, there exists a vector $v \in \mathbb{R}^d \setminus \{0\}$ such that $\text{supp}(v) = \text{supp}(u)$, that $u \circ v \geq 0$, and that v is $\|\cdot\|$ -dual to u , that is, $\langle u, v \rangle = \|u\| \times \|v\|_\star$.*

B.2 Local-coordinate- K and dual local-coordinate- K norms

After their definitions in §B.2.1, we study properties of dual local-coordinate- K norms and of local-coordinate- K norms in §B.2.2.

B.2.1 Definition of local-coordinate- K and dual local-coordinate- K norms

For the sake of completeness, we duplicate the Definition 8 here. For any subset $K \subset V$, we call *local-coordinate- K norm* (associated with the source norm $\|\cdot\|$) the norm $\|\cdot\|_{(K)}^{\mathcal{R}}$ (on the subspace \mathcal{R}_K in (6)) given by

$$\|\cdot\|_{(K)}^{\mathcal{R}} = (\|\cdot\|_{(K),\star}^{\mathcal{R}})_\star,$$

that is, whose dual norm (on the subspace \mathcal{R}_K) is the *dual local-coordinate- K norm*, denoted by $\|\cdot\|_{(K),\star}^{\mathcal{R}}$, with expression

$$\|y\|_{(K),\star}^{\mathcal{R}} = \sup_{J \subset K} \|y_J\|_{J,\star}, \quad \forall y \in \mathcal{R}_K,$$

where the (K, \star) -norm $\|\cdot\|_{K, \star}$ is given in Definition 7. We will also use the following extension (defined for all $y \in \mathbb{R}^d$ and not only for $y \in \mathcal{R}_K$)

$$\|y\|_{(K), \star}^{\mathcal{R}} = \|y_K\|_{(K), \star}^{\mathcal{R}}, \quad \forall y \in \mathbb{R}^d, \quad (59a)$$

which defines a *seminorm*⁸ on \mathbb{R}^d , and not a norm. The local-coordinate- K norm $\|\cdot\|_{(K)}^{\mathcal{R}}$ on \mathcal{R}_K can also be extended to a seminorm on \mathbb{R}^d by

$$\|x\|_{(K)}^{\mathcal{R}} = \|x_K\|_{(K)}^{\mathcal{R}}, \quad \forall x \in \mathbb{R}^d. \quad (59b)$$

Now, to establish results in Sect. 3, we provide properties of local-coordinate- K and dual local-coordinate- K norms.

B.2.2 Properties of dual local-coordinate- K norms and of local-coordinate- K norms

Properties of dual local-coordinate- K norms.

Proposition 22

- For any $K \subset V$, the dual local-coordinate- K norm satisfies

$$\|y\|_{(K), \star}^{\mathcal{R}} = \sup_{J \subset K} \sigma_{(\mathcal{R}_J \cap \mathbb{S})}(y) = \sigma_{(\text{supp}^{\subset K} \cap \mathbb{S})}(y) = \sigma_{(\text{supp}^{\subset K} \cap \mathbb{S})}(y), \quad \forall y \in \mathcal{R}_K. \quad (60a)$$

- We have the equality

$$\|y\|_{(V), \star}^{\mathcal{R}} = \|y\|_{\star}, \quad \forall y \in \mathbb{R}^d. \quad (60b)$$

- The family $\left\{ \|\cdot\|_{(K), \star}^{\mathcal{R}} \right\}_{K \subset V}$ of dual local-coordinate- K norms in Definition 8 is nondecreasing, that is,

$$J \subset K \subset V \implies \|y\|_{(J), \star}^{\mathcal{R}} \leq \|y\|_{(K), \star}^{\mathcal{R}} \leq \|y\|_{(V), \star}^{\mathcal{R}} = \|y\|_{\star}, \quad \forall y \in \mathcal{R}_J. \quad (60c)$$

- The family $\left\{ \mathbb{B}_{(K), \star}^{\mathcal{R}} \right\}_{K \subset V}$ of units balls of the dual local-coordinate- K norms in Definition 8 is nonincreasing, that is,

$$J \subset K \subset V \implies \mathbb{B}_{(J), \star}^{\mathcal{R}} \supset \mathbb{B}_{(K), \star}^{\mathcal{R}} \supset \mathbb{B}_{(V), \star}^{\mathcal{R}} = \mathbb{B}_{\star}. \quad (60d)$$

Proof.

- For any $y \in \mathbb{R}^d$, we have

$$\begin{aligned} \|y\|_{(K), \star}^{\mathcal{R}} &= \sup_{J \subset K} \|y_J\|_{J, \star} && \text{(by definition (23))} \\ &= \sup_{J \subset K} \sigma_{(\mathcal{R}_J \cap \mathbb{S})}(y) && \text{(by (21))} \\ &= \sigma_{\bigcup_{J \subset K} (\mathcal{R}_J \cap \mathbb{S})}(y) \\ &= \sigma_{(\text{supp}^{\subset K} \cap \mathbb{S})}(y). && \text{(as we have } \text{supp}^{\subset K} \cap \mathbb{S} = \bigcup_{J \subset K} (\mathcal{R}_J \cap \mathbb{S}) \text{ by (8))} \end{aligned}$$

⁸A seminorm satisfies all the axioms of a norm, except that other vectors than the null vector give zero.

To finish, we will now prove that $\sigma_{\text{supp}^{<K} \cap \mathbb{S}} = \sigma_{\text{supp}^{=K} \cap \mathbb{S}}$. For this purpose, we show in two steps that $\text{supp}^{<K} \cap \mathbb{S} = \overline{\text{supp}^{=K} \cap \mathbb{S}}$.

First, we establish that $\overline{\text{supp}^{=K}} = \text{supp}^{<K}$. The inclusion $\overline{\text{supp}^{=K}} \subset \text{supp}^{<K}$ is easy. Indeed, that the level set $\text{supp}^{<K}$ is closed is straightforward: if a sequence $(x_n)_{n \in \mathbb{N}} \in \text{supp}^{<K}$ is converging towards x , we get that $(x)_{V \setminus K} = 0$ since $(x_n)_{V \setminus K} = 0$ for all $n \in \mathbb{N}$; thus, we obtain that $x \in \text{supp}^{<K}$. There remains to prove the reverse inclusion $\text{supp}^{<K} \subset \overline{\text{supp}^{=K}}$. For this purpose, we consider $x \in \text{supp}^{<K}$. If $x \in \text{supp}^{=K}$, obviously $x \in \overline{\text{supp}^{=K}}$. Therefore, we suppose that $\text{supp}(x) = L \subsetneq K$ and we consider the sequence $(x_n)_{n \in \mathbb{N}^*}$ defined by $x_n = x + (1/n)\mathbf{1}_{K \setminus L}$, where $\mathbf{1}$ is the vector of \mathbb{R}^d made of ones. We have that $\text{supp}(x_n) = K$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$ when n goes to infinity, thus $x \in \overline{\text{supp}^{=K}}$. This proves that $\text{supp}^{<K} \subset \overline{\text{supp}^{=K}}$.

Second, we prove that $\overline{\text{supp}^{<K} \cap \mathbb{S}} = \overline{\text{supp}^{=K} \cap \mathbb{S}}$. The inclusion $\overline{\text{supp}^{=K} \cap \mathbb{S}} \subset \overline{\text{supp}^{<K} \cap \mathbb{S}}$ is easy. Indeed, we have $\overline{\text{supp}^{=K} \cap \mathbb{S}} \subset \overline{\text{supp}^{=K} \cap \mathbb{S}} = \text{supp}^{<K} \cap \mathbb{S}$ since we have just proved that $\overline{\text{supp}^{=K}} = \text{supp}^{<K}$. To prove the reverse inclusion $\overline{\text{supp}^{<K} \cap \mathbb{S}} \subset \overline{\text{supp}^{=K} \cap \mathbb{S}}$, we consider $x \in \overline{\text{supp}^{<K} \cap \mathbb{S}}$. As we have just seen that $\text{supp}^{<K} = \overline{\text{supp}^{=K}}$, we deduce that $x \in \overline{\text{supp}^{=K}}$. Therefore, there exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $\text{supp}^{=K}$ such that $z_n \rightarrow x$ when $n \rightarrow +\infty$. Since $x \in \mathbb{S}$, we can always suppose that $z_n \neq 0$, for all $n \in \mathbb{N}$. Therefore $z_n/\|z_n\|$ is well defined and, when $n \rightarrow +\infty$, we have $z_n/\|z_n\| \rightarrow x/\|x\| = x$ since $x \in \mathbb{S} = \{x \in \mathbb{X} \mid \|x\| = 1\}$. Now, on the one hand, $z_n/\|z_n\| \in \text{supp}^{=K}$, for all $n \in \mathbb{N}$, and, on the other hand, $z_n/\|z_n\| \in \mathbb{S}$. As a consequence $z_n/\|z_n\| \in \overline{\text{supp}^{=K} \cap \mathbb{S}}$, and we conclude that $x \in \overline{\text{supp}^{=K} \cap \mathbb{S}}$. Thus, we have proved that $\overline{\text{supp}^{<K} \cap \mathbb{S}} \subset \overline{\text{supp}^{=K} \cap \mathbb{S}}$.

From $\text{supp}^{<K} \cap \mathbb{S} = \overline{\text{supp}^{=K} \cap \mathbb{S}}$, we get that $\sigma_{\text{supp}^{<K} \cap \mathbb{S}} = \sigma_{\overline{\text{supp}^{=K} \cap \mathbb{S}}} = \sigma_{\text{supp}^{=K} \cap \mathbb{S}}$.

Thus, we have proved all equalities in (60a).

- By the equality $\|y\|_{(K),*}^{\mathcal{R}} = \sigma_{(\text{supp}^{<K} \cap \mathbb{S})}(y)$ in (60a), we get that, for all $y \in \mathbb{R}^d$, $\|y\|_{(V),*}^{\mathcal{R}} = \sigma_{(\text{supp}^{<V} \cap \mathbb{S})}(y) = \sigma_{\mathbb{S}}(y) = \|y\|_*$, since $\text{supp}^{<V} = \mathbb{R}^d$ and by (58). Thus, we have proved the equality (60b).
- The inequalities in (60c) easily derive from the very definition (23) of the dual local-coordinate- K norms $\|\cdot\|_{(K),*}^{\mathcal{R}}$.
- The inclusions and equality in (60d) directly follow from the equality and the inequalities between norms in (60b) and (60c).

This ends the proof. □

Properties of local-coordinate- K norms.

Proposition 23

- For any subset $K \subset V$, the local-coordinate- K norm $\|\cdot\|_{(K)}^{\mathcal{R}}$ has unit ball (in the subspace \mathcal{R}_K in (6))

$$\mathbb{B}_{(K)}^{\mathcal{R}} = \overline{\text{co}}(\mathcal{R}_K \cap \mathbb{S}) \subset \mathcal{R}_K, \quad (61a)$$

where $\overline{\text{co}}(S)$ denotes the closed convex hull of a subset $S \subset \mathbb{R}^d$.

- We have the equality

$$\|x\|_{(V)}^{\mathcal{R}} = \|x\|, \quad \forall x \in \mathbb{R}^d. \quad (61b)$$

- The family $\left\{ \|\cdot\|_{(K)}^{\mathcal{R}} \right\}_{K \subset V}$ of local-coordinate- K norms in Definition 8 is nonincreasing, that is,

$$J \subset K \subset V \implies \|x\|_{(J)}^{\mathcal{R}} \geq \|x\|_{(K)}^{\mathcal{R}} \geq \|x\|_{(V)}^{\mathcal{R}} = \|x\|, \quad \forall x \in \mathcal{R}_J. \quad (61c)$$

- The family $\left\{ \mathbb{B}_{(K)}^{\mathcal{R}} \right\}_{K \subset V}$ of unit balls of the local-coordinate- K norms in (24b) is non-decreasing, that is,

$$J \subset K \subset V \implies \mathbb{B}_{(J)}^{\mathcal{R}} \subset \mathbb{B}_{(K)}^{\mathcal{R}} \subset \mathbb{B}_{(V)}^{\mathcal{R}} = \mathbb{B}. \quad (61d)$$

- We have the implication

$$x \in \mathcal{R}_K \implies \|x\| = \|x\|_{(K)}^{\mathcal{R}}. \quad (61e)$$

Proof.

- For any $y \in \mathcal{R}_K$, we have

$$\begin{aligned} \|y\|_{(K),\star}^{\mathcal{R}} &= \sup_{J \subset K} \sigma_{(\mathcal{R}_J \cap \mathbb{S})}(y) \\ &= \sigma_{(\mathcal{R}_K \cap \mathbb{S})}(y) \end{aligned} \quad (\text{by (21)})$$

since $\mathcal{R}_J \subset \mathcal{R}_K$ for $J \subset K$ and by definition (48) of the support functions $\sigma_{(\mathcal{R}_J \cap \mathbb{S})}$

$$= \sigma_{\overline{\text{co}}(\mathcal{R}_K \cap \mathbb{S})}(y) \quad (\text{by [2, Prop. 7.13]})$$

As $\overline{\text{co}}(\bigcup_{J \subset K} (\mathcal{R}_J \cap \mathbb{S}))$ is a closed convex subset of \mathcal{R}_K , we conclude that $\mathbb{B}_{(K)}^{\mathcal{R}} = \overline{\text{co}}(\bigcup_{J \subset K} (\mathcal{R}_J \cap \mathbb{S}))$ by (58). Thus, we have proven (61a).

- From the equality (60b), we deduce the equality (61b) between the dual norms by definition of the dual norm.

- The equality and inequalities between norms in (61c) easily derive from the inclusions and equality between unit balls in (61d).

- The inclusions and equality between unit balls in (61d) directly follow from the inclusions and equality between unit balls in (60d) and as the unit ball of the dual norm is $\mathbb{B}_{(K)}^{\mathcal{R}} = (\mathbb{B}_{(K),\star}^{\mathcal{R}})^{\odot}$, the polar set of $\mathbb{B}_{(K),\star}^{\mathcal{R}}$.

- We prove (61e) using the fact that $x \in \mathcal{R}_K \Leftrightarrow x \in \text{supp}^{\subset K}$. For any $x \in \mathbb{R}^d$ and for any $K \subset V$, we have⁹

$$x \in \text{supp}^{\subset K} \Leftrightarrow x = 0 \text{ or } \frac{x}{\|x\|} \in \text{supp}^{\subset K}$$

⁹In what follows, by “or”, we mean the so-called *exclusive or* (exclusive disjunction). Thus, every “or” should be understood as “or $y \neq 0$ and”.

by 0-homogeneity (4) of the supp mapping, and by definition (5a) of $\text{supp}^{\subset K}$

$$\begin{aligned}
&\Leftrightarrow x = 0 \text{ or } \frac{x}{\|x\|} \in \text{supp}^{\subset K} \cap \mathbb{S} && (\text{as } \frac{x}{\|x\|} \in \mathbb{S}) \\
&\Leftrightarrow x = 0 \text{ or } \frac{x}{\|x\|} \in \bigcup_{J \subset K} (\mathcal{R}_J \cap \mathbb{S}) && (\text{as } \text{supp}^{\subset K} = \bigcup_{J \subset K} \mathcal{R}_J \text{ by (8)}) \\
&\Rightarrow x = 0 \text{ or } \frac{x}{\|x\|} \in \mathbb{B}_{(K)}^{\mathcal{R}} && (\text{as } \mathbb{B}_{(K)}^{\mathcal{R}} = \overline{\text{co}}(\bigcup_{J \subset K} (\mathcal{R}_J \cap \mathbb{S})) \text{ by (61a)}) \\
&\Rightarrow x = 0 \text{ or } \left\| \frac{x}{\|x\|} \right\|_{(K)}^{\mathcal{R}} \leq 1
\end{aligned}$$

since $\mathbb{B}_{(K)}^{\mathcal{R}}$ is the unit ball of the norm $\|\cdot\|_{(K)}^{\mathcal{R}}$ by (24b)

$$\begin{aligned}
&\Rightarrow \|x\|_{(K)}^{\mathcal{R}} \leq \|x\| \\
&\Rightarrow \|x\|_{(K)}^{\mathcal{R}} \leq \|x\| = \|x\|_{(V)}^{\mathcal{R}} && (\text{where the last equality comes from (61c)}) \\
&\Rightarrow \|x\|_{(K)}^{\mathcal{R}} = \|x\|_{(V)}^{\mathcal{R}} . && (\text{as } \|x\|_{(V)}^{\mathcal{R}} \leq \|x\|_{(K)}^{\mathcal{R}} \text{ by (61c)})
\end{aligned}$$

Therefore, we have obtained (61e).

This ends the proof. \square

B.3 Generalized local-top- K and local- K -support dual norms

After the definitions of generalized local-top- K and local- K -support dual norms in §B.3.1, we study their properties in §B.3.2. The key Proposition 26 is used in the proof of Proposition 13

B.3.1 Definition of generalized local-top- K and local- K -support dual norms

Definition 24 For $K \subset V$, we call generalized local-top- K norm (associated with the source norm $\|\cdot\|$) the norm (on the subspace \mathcal{R}_K in (6)) defined by

$$\|x\|_{(K)}^{\text{tn}} = \sup_{J \subset K} \|x_J\| = \sup_{J \subset K} \|x_J\|_J, \quad \forall x \in \mathcal{R}_K. \quad (62)$$

We call generalized local- K -support norm (associated with the source norm $\|\cdot\|$) the dual norm (on the subspace \mathcal{R}_K in (6)) of the generalized local-top- K norm, denoted by¹⁰ $\|\cdot\|_{(K)}^{\star \text{sn}}$:

$$\|\cdot\|_{(K)}^{\star \text{sn}} = (\|\cdot\|_{(K)}^{\text{tn}})_{\star}. \quad (63)$$

Now, we do the same but with the dual norm $\|\cdot\|_{\star}$ in lieu of the source norm $\|\cdot\|$. For the sake of completeness, we duplicate the Definition 9 here.

¹⁰We use the symbol \star in the superscript to indicate that the generalized local- K -support norm $\|\cdot\|_{(K)}^{\star \text{sn}}$ is a dual norm.

For any subset $K \subset V$, we call *generalized local-top- K dual norm* (on the subspace \mathcal{R}_K in (6)) the local norm defined by

$$\|y\|_{\star, (K)}^{\text{tn}} = \sup_{J \subset K} \|y_J\|_{\star} = \sup_{J \subset K} \|y_J\|_{\star, J}, \quad \forall y \in \mathcal{R}_K. \quad (64)$$

We call *generalized local- K -support dual norm* the dual norm (on the subspace \mathcal{R}_K) of the generalized local-top- K dual norm, denoted by¹¹ $\|\cdot\|_{\star, (K)}^{\star\text{sn}}$:

$$\|\cdot\|_{\star, (K)}^{\star\text{sn}} = \left(\|\cdot\|_{\star, (K)}^{\text{tn}}\right)_{\star}. \quad (65)$$

We adopt the convention $\|\cdot\|_{\star, (\emptyset)}^{\text{tn}} = 0$ (although this is not a norm, but a seminorm). We denote the unit sphere and the unit ball of the generalized local- K -support dual norm $\|\cdot\|_{\star, (K)}^{\star\text{sn}}$ by

$$\mathbb{S}_{\star, (K)}^{\star\text{sn}} = \{x \in \mathcal{R}_K \mid \|x\|_{\star, (K)}^{\star\text{sn}} = 1\}, \quad \forall K \subset V, \quad (66a)$$

$$\mathbb{B}_{\star, (K)}^{\star\text{sn}} = \{x \in \mathcal{R}_K \mid \|x\|_{\star, (K)}^{\star\text{sn}} \leq 1\}, \quad \forall K \subset V. \quad (66b)$$

We will also use the following extension of the generalized local-top- K dual norm $\|\cdot\|_{\star, (K)}^{\text{tn}}$ on \mathcal{R}_K into a seminorm on \mathbb{R}^d by

$$\|y\|_{\star, (K)}^{\text{tn}} = \|y_K\|_{\star, (K)}^{\text{tn}}, \quad \forall y \in \mathbb{R}^d, \quad (67a)$$

and the extension of the generalized local- K -support dual norm $\|\cdot\|_{\star, (K)}^{\star\text{sn}}$ on \mathcal{R}_K into a seminorm on \mathbb{R}^d by

$$\|x\|_{\star, (K)}^{\star\text{sn}} = \|x_K\|_{\star, (K)}^{\star\text{sn}}, \quad \forall x \in \mathbb{R}^d. \quad (67b)$$

B.3.2 Properties of local-coordinate- K and dual local-coordinate- K norms

Proposition 25 *Let $\|\cdot\|$ be the source norm with associated families $\left\{\|\cdot\|_{(K)}^{\mathcal{R}}\right\}_{K \subset V}$ of local-coordinate- K norms and $\left\{\|\cdot\|_{(K), \star}^{\mathcal{R}}\right\}_{K \subset V}$ of dual local-coordinate- K norms, as in Definition 8, and with associated families $\left\{\|\cdot\|_{\star, (K)}^{\text{tn}}\right\}_{K \subset V}$ of generalized local-top- K dual norms and $\left\{\|\cdot\|_{\star, (K)}^{\star\text{sn}}\right\}_{K \subset V}$ of generalized local- K -support dual norms, as in Definition 9.*

We have that local-coordinate- K norms are always lower than local- K -support dual norms, that is,

$$\|x\|_{(K)}^{\mathcal{R}} \leq \|x\|_{\star, (K)}^{\star\text{sn}}, \quad \forall x \in \mathcal{R}_K, \quad \forall K \subset V, \quad (68a)$$

¹¹We use the symbol \star in the superscript to indicate that the generalized local- K -support dual norm $\|\cdot\|_{\star, (K)}^{\star\text{sn}}$ is a dual norm.

whereas dual local-coordinate- K norms are always greater than generalized local-top- K dual norms, that is,

$$\|y\|_{(K),\star}^{\mathcal{R}} \geq \|y\|_{\star,(K)}^{\text{tn}}, \quad \forall y \in \mathcal{R}_K, \quad \forall K \subset V. \quad (68b)$$

If the source norm $\|\cdot\|$ is orthant-monotonic, then equalities in (68) hold true, that is,

$$\|\cdot\| \text{ is orthant-monotonic} \implies \forall K \subset V \quad \begin{cases} \|\cdot\|_{(K)}^{\mathcal{R}} &= \|\cdot\|_{\star,(K)}^{\text{sn}} , \\ \|\cdot\|_{(K),\star}^{\mathcal{R}} &= \|\cdot\|_{\star,(K)}^{\text{tn}} . \end{cases} \quad (69)$$

The two equalities in (69) are equalities between local norms (on the subspace \mathcal{R}_K in (6)), but they remain valid as equalities between seminorms (on \mathbb{R}^d).

Proof. It is easily established that, for any subset $K \subset V$, we have the inequality $\|\cdot\|_{K,\star} \leq \|\cdot\|_{\star,K}$ [5, Lemma 2]. From the definition (26) of the generalized local-top- K dual norm, and the definition (23) of the dual local-coordinate- K norm, we obtain (68b). By taking the dual norms, we get (68a).

The norms for which the equality $\|\cdot\|_{K,\star} = \|\cdot\|_{\star,K}$ holds true for all subsets $K \subset V$ are the orthant-monotonic norms ([6, Theorem 2.26], [9, Theorem 3.2]) Therefore, if the norm $\|\cdot\|$ is orthant-monotonic, we get that, for any $y \in \mathcal{R}_K$,

$$\begin{aligned} \|y\|_{(K),\star}^{\mathcal{R}} &= \sup_{J \subset K} \|y_J\|_{J,\star} && \text{(by definition (23) of } \|y\|_{(K),\star}^{\mathcal{R}} \text{)} \\ &= \sup_{J \subset K} \|y_J\|_{\star,J} && \text{(because the norm } \|\cdot\| \text{ is orthant-monotonic)} \\ &= \|y\|_{\star,(K)}^{\text{tn}} && \text{(by definition (26) of } \|y\|_{\star,(K)}^{\text{tn}} \text{)} \end{aligned}$$

Thus, we have obtained the lower equality in the right hand side of (69).

The upper equality in the right hand side of (69) follows by taking the dual norms.

The two equalities in (69) remain valid as equalities between seminorms (on \mathbb{R}^d) because of the definitions (59a), (59b), (67a) and (67b) of the four corresponding seminorms.

This ends the proof. \square

We end this part with Proposition 26, which is key to prove Proposition 13.

Proposition 26 *We consider a norm $\|\cdot\|$ whose dual norm $\|\cdot\|_{\star}$ is strictly orthant-monotonic. Then, we have the implications (that involve seminorms)*

$$(\forall y \in \mathbb{R}^d) \quad \text{supp}(y) = L \subset K \subset V \implies \|y\|_{\star,(K)}^{\text{tn}} = \|y\|_{(K),\star}^{\mathcal{R}} = \|y\|_{(L),\star}^{\mathcal{R}} = \|y\|_{\star,(L)}^{\text{tn}}, \quad (70a)$$

$$(\forall y \in \mathbb{R}^d) \quad \text{supp}(y) = L \not\subset K \subset V \implies \|y\|_{\star,(K)}^{\text{tn}} = \|y\|_{(K),\star}^{\mathcal{R}} < \|y\|_{(L),\star}^{\mathcal{R}} = \|y\|_{\star,(L)}^{\text{tn}}. \quad (70b)$$

Proof. The proof is in four points.

- Since we have supposed that the dual norm $\|\cdot\|_{\star}$ is strictly orthant-monotonic, it is orthant-monotonic and, by Proposition 20, we get that the source norm $\|\cdot\|$ is also orthant-monotonic. Thus, using Equation (69) in Proposition 25, we obtain the equality $\|\cdot\|_{(K),\star}^{\mathcal{R}} = \|\cdot\|_{\star,(K)}^{\text{tn}}$ between

the two local norms, for any $K \subset V$. As this equality between local norms is valid as an equality between seminorms, we get that

$$\|y\|_{(K),\star}^{\mathcal{R}} = \|y\|_{\star,(K)}^{\text{tn}}, \quad \forall y \in \mathbb{R}^d. \quad (71)$$

• We prove Implication (70a). Let $K \subset V$ and $y \in \mathbb{R}^d$ be such that $\text{supp}(y) = L \subset K$. Since $L = \text{supp}(y)$, we have that $y \in \mathcal{R}_L$. Now, by Equation (60c) of Proposition 22, we have that

$$\|y\|_{(L),\star}^{\mathcal{R}} \leq \|y\|_{(K),\star}^{\mathcal{R}} \leq \|y\|_{(V),\star}^{\mathcal{R}}. \quad (72)$$

Thus, to prove (70a) it is enough to prove that $\|y\|_{(L),\star}^{\mathcal{R}} = \|y\|_{(V),\star}^{\mathcal{R}}$ which, by Equation (71), is equivalent to prove that $\|y\|_{\star,(L)}^{\text{tn}} = \|y\|_{\star,(V)}^{\text{tn}}$. On the one hand, we show that $\|y\|_{\star} \leq \|y\|_{\star,(L)}^{\text{tn}}$. Indeed, since $y = y_L$, we have $\|y\|_{\star} = \|y_L\|_{\star} = \|y_L\|_{\star,L} \leq \|y\|_{\star,(L)}^{\text{tn}}$, by the very definition (26) of the generalized local-top- L dual norm $\|\cdot\|_{\star,(L)}^{\text{tn}}$. On the other hand, we show that $\|y\|_{\star,(L)}^{\text{tn}} \leq \|y\|_{\star}$. Indeed, replacing $\|y\|_{(\cdot),\star}^{\mathcal{R}}$ by $\|y\|_{\star,(\cdot)}^{\text{tn}}$ in Equation (72) we have that $\|y\|_{\star,(L)}^{\text{tn}} \leq \|y\|_{\star,(V)}^{\text{tn}} = \|y\|_{\star}$, where the last equality comes from replacing $\|y\|_{(V),\star}^{\mathcal{R}}$ by $\|y\|_{\star,(V)}^{\text{tn}}$ in Equation (60b). Hence, we deduce that $\|y\|_{\star,(L)}^{\text{tn}} = \|y\|_{\star,(V)}^{\text{tn}}$.

• We prove Implication (70b). Let $K \subset V$ and $y \in \mathbb{R}^d$ be such that $\text{supp}(y) = L \not\subset K$.
– First, we prove an intermediate result: for any K' such that $K' \subsetneq L = \text{supp}(y)$, we have that $\|y\|_{\star,(K')}^{\text{tn}} < \|y\|_{\star,(L)}^{\text{tn}}$. Indeed, we have successively

$$\begin{aligned} \|y\|_{\star,(K')}^{\text{tn}} &= \sup_{J \subset K'} \|y_J\|_{\star} && \text{(by definition (26) of the generalized local-top-} K' \text{ dual norm)} \\ &< \|y_L\|_{\star} \end{aligned}$$

since $K' \subsetneq L$, and $L = \text{supp}(y)$ for any $J \subset K'$ we have $y_J \neq y_L$, and thus $\|y_J\|_{\star} < \|y_L\|_{\star}$ by strict orthant-monotonicity of the dual norm and Item 2 in Proposition 21, and thus $\sup_{J \subset K'} \|y_J\|_{\star} < \|y_L\|_{\star}$,

$$\begin{aligned} &\leq \sup_{J \subset L} \|y_J\|_{\star} && \text{(obviously since } L \in \{J \mid J \subset L\}) \\ &= \|y\|_{\star,(L)}^{\text{tn}}. && \text{(by definition (26) of the generalized local-top-} L \text{ dual norm)} \end{aligned}$$

– Second, we apply the previous inequality with the subset $K' = K \cap L$ and we obtain that

$$\|y\|_{\star,(K \cap L)}^{\text{tn}} < \|y\|_{\star,(L)}^{\text{tn}}, \quad (73)$$

because $K' = K \cap L \subsetneq L$ since $L \not\subset K$ by assumption.

– Third, we prove that $\|y\|_{\star,(K)}^{\text{tn}} = \|y\|_{\star,(K \cap L)}^{\text{tn}}$. We have

$$\begin{aligned} \|y\|_{\star,(K)}^{\text{tn}} &= \sup_{J \subset K} \|y_J\|_{\star} && \text{(by definition (26) of the generalized local-top-} K \text{ dual norm)} \\ &= \|y_{J^\#}\|_{\star} \end{aligned}$$

where $J^\sharp \subset K$ exists by definition (26) of the generalized local-top- K dual norm

$$\begin{aligned}
&= \|y_{J^\sharp \cap L}\|_\star && \text{(we have that } y = y_L \text{ since } L = \text{supp}(y)\text{)} \\
&\leq \sup_{J \subset K \cap L} \|y_J\|_\star && (J' = J^\sharp \cap L \text{ is such that } J' \subset K \cap L \text{ since } J^\sharp \subset K) \\
&= \|y\|_{\star, (K \cap L)}^{\text{tn}} && \text{(by definition (26) of the generalized local-top-}(K \cap L)\text{ dual norm)} \\
&\leq \|y\|_{\star, (K)}^{\text{tn}} && \text{(by (74), as proved below)}
\end{aligned}$$

so that we obtain the equality $\|y\|_{\star, (K)}^{\text{tn}} = \|y\|_{\star, (K \cap L)}^{\text{tn}}$.

– Fourth, combining the above equality $\|y\|_{\star, (K)}^{\text{tn}} = \|y\|_{\star, (K \cap L)}^{\text{tn}}$ with Equation (73), we obtain the inequality $\|y\|_{\star, (K)}^{\text{tn}} < \|y\|_{\star, (L)}^{\text{tn}}$. Then, using again Equation (69) — that is, $\|\cdot\|_{\star, (K)}^{\text{tn}} = \|\cdot\|_{(K), \star}^{\mathcal{R}}$ for all $K \subset V$ — we finally obtain the inequality $\|y\|_{(K), \star}^{\mathcal{R}} < \|y\|_{(L), \star}^{\mathcal{R}}$. Thus, we have proved Implication (70b).

- We are going to show the implication (that involves seminorms)

$$J \subset K \subset V \implies \|y\|_{\star, (J)}^{\text{tn}} \leq \|y\|_{\star, (K)}^{\text{tn}}, \quad \forall y \in \mathbb{R}^d. \quad (74)$$

Let J and K be two subsets such that $J \subset K \subset V$. For any $y \in \mathbb{R}^d$, we have

$$\begin{aligned}
\|y\|_{\star, (J)}^{\text{tn}} &= \sup_{J' \subset J} \|y_{J'}\|_{\star, J'} && \text{(by definition (26) of } \|y\|_{\star, (J)}^{\text{tn}}\text{)} \\
&= \sup_{J' \subset J} \|(y_J)_{J'}\|_{\star, J'} && \text{(since } J' \subset J\text{)} \\
&= \|y_J\|_{\star, (J)}^{\text{tn}} && \text{(again by definition (26) of } \|y_J\|_{\star, (J)}^{\text{tn}}\text{)} \\
&\leq \|y_J\|_{\star, (K)}^{\text{tn}} && \text{(since } y_J \in \mathcal{R}_J \text{ and using the implication (60c))} \\
&= \sup_{K' \subset K} \|(y_J)_{K'}\|_{\star, K'} && \text{(again by definition (26) of } \|y_J\|_{\star, (K)}^{\text{tn}}\text{)} \\
&\leq \|y_K\|_{\star, K}
\end{aligned}$$

since $\|(y_J)_{K'}\|_{\star, K'} = \|y_{J \cap K'}\|_{\star, K'} \leq \|y_K\|_{\star, K}$ by orthant-monotonicity of $\|\cdot\|_\star$ (Item 3 in Proposition 20), using the fact that $J \cap K' \subset K$ for all the considered K' , and the expression of the restriction norm $\|\cdot\|_{\star, K}$ in Definition 7

$$\begin{aligned}
&\leq \sup_{K' \subset K} \|y_{K'}\|_{\star, K'} && \text{(since } K \in \{K' \mid K' \subset K\}\text{)} \\
&= \|y\|_{\star, (K)}^{\text{tn}} && \text{(again by definition (26) of } \|y\|_{\star, (K)}^{\text{tn}}\text{)}
\end{aligned}$$

Thus, we have proven the Implication (74).

This ends the proof. \square

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