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Elliptic differential-operators with an abstract Robin boundary condition and two spectral parameters

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Abstract

We study the solvability of some boundary-value problems for differential-operator equations of the second order in $L^p(0, 1; X)$, with $1 < p < +\infty$, X being a UMD complex Banach space. The originality of this work lies in the fact that we consider the case where two spectral complex parameters appear in the equation and in abstract Robin boundary conditions. Here, the unbounded linear operator in the equation is not commuting with the one appearing in the boundary conditions. This represents the strong novelty with respect to the existing literature. Existence, uniqueness, representation formula, maximal regularity of the solution, sharp estimates and generation of strongly continuous analytic semigroup are proved. Many concrete applications are given for which our theory applies. This paper improves, in some sense, results by the authors in [7] and it can be viewed as a continuation of the results in [1] studied only in Hilbert spaces.

Key Words and Phrases: Second order boundary value problems with two spectral parameters, Robin boundary conditions, spectral estimates, functional calculus, generation of analytic semigroups.

2020 Mathematics Subject Classification: 35B65, 35C15, 35J25, 47A60, 47D06.

1 Introduction

In this article, we consider a new spectral problem that is given by the equation

$$u''(x) + Au(x) - \lambda u(x) = f(x), \quad x \in (0, 1), \quad (1)$$

together with the abstract Robin boundary conditions

$$u'(0) - Hu(0) - \mu u(0) = d_0, \quad u(1) = u_1. \quad (2)$$

Here, λ , μ are complex parameters, A , H are closed linear operators in a complex Banach space X , f belongs to $L^p(0, 1; X)$ with $1 < p < +\infty$, d_0 , u_1 are given elements of X . We develop a completely different approach from the ones used until now. It allows a more easy verification of the assumptions and their application to concrete problems.

Many boundary value problems with a spectral parameter in the equation and in the boundary conditions arise in different concrete problems. We shall cite some interesting papers related to this research. In one of these last works, see [5], the article considers a class of boundary problems with a spectral parameter in the boundary conditions. In [4], the author considers some second order elliptic boundary value problems on bounded domains

with boundary conditions depending nonlinearly on the spectral parameter. In [2], we find a study, in a separable Hilbert space, of the following boundary-value second-order elliptic differential-operator equation:

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), & x \in (0, 1) \\ \lambda u'(0) - \alpha u(1) = f_1, \quad u(1) = f_2, \end{cases}$$

where α is a complex number with $\operatorname{Re}(\alpha) \geq 0$ and $-A$ is a linear self-adjoint operator guaranteeing the ellipticity of the equation. Note that here, the parameter λ appears in the nonlocal boundary condition. Recently, in [1], the authors consider the following boundary-value problem for an elliptic differential-operator equation of second order

$$\begin{cases} \lambda^2 u(x) - u''(x) + Au(x) = f(x), & x \in (0, 1) \\ u'(0) + \lambda u(1) = f_1, \quad \beta u'(1) + \lambda u(0) = f_2, \end{cases}$$

where the same spectral parameter appears in the equation quadratically; here $-A$ is a closed positive linear operator in a separable complex Hilbert space. In [7], the authors consider Problem (1)-(2) in a complex Banach space X , where $\lambda = \omega$ is a positive spectral parameter and $\mu = 0$. For ω large enough, under some geometrical assumptions on the space X and hypotheses on operators $A - \omega I$ and H , including the fact that they commute in the resolvent sense, the authors furnish necessary and sufficient conditions on the data d_0, u_1 to obtain the existence and uniqueness of a solution u of (1)-(2) with maximal regularity. Recently, in [9], the authors develop an interesting new approach in a non commutative framework, concerning some general Sturm-Liouville problems with the same Robin boundary condition in 0.

In our study of Problem (1)-(2), the ellipticity of the equation is guaranteed by hypothesis (5) below; this assumption allows us to consider, for suitable λ, μ , the operators

$$\begin{cases} \Lambda_{\lambda, \mu} := (Q_\lambda - H_\mu) + e^{2Q_\lambda} (Q_\lambda + H_\mu) \\ Q_\lambda = -\sqrt{-A + \lambda I}, \quad H_\mu = H + \mu I. \end{cases}$$

In all the sequel, for any closed linear operator T on X , $D(T)$ denotes the domain of T and $\rho(T)$ the resolvent set of T . The key point will be to obtain the invertibility of the determinant $\Lambda_{\lambda, \mu}$ of system (1)-(2) with estimates of $\|\Lambda_{\lambda, \mu}^{-1}\|_{\mathcal{L}(X)}$, for appropriate λ, μ . To this end, we consider two different situations:

1. $D(H) \subset D(A)$
2. $D(\sqrt{-A}) \subset D(H)$,

where in the first case, we say that operator H is principal, while in the second case, it is operator $\sqrt{-A}$ which is principal. Concrete applications will illustrate these two cases at the end of this work; the first one is adapted to related problems concerning some heat equations with dynamical boundary conditions of reaction-diffusion type or with Wentzell boundary conditions, whereas the second one will concern, for instance, problems involving the Caputo derivative in the boundary conditions.

Four new and essential results sum up this work.

1. We solve the above equation by giving an explicit and simplified representation of the solution adapted to each case and we show that it verifies the optimal regularity, that is

$$u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A)),$$

see Theorem 2.1 and Theorem 2.4.

2. We give sharp estimates of this solution in each case according to the complex spectral parameters λ, μ belonging to some appropriate precised set, see Theorem 2.2 and Theorem 2.5 .

This part essentially uses the results of [13], where some inequalities on resolvent operators are precised.

3. Thanks to these estimates, we obtain the generation of analytic semigroups corresponding to each case, see Theorem 2.3 and Theorem 2.6.
4. Using the same tools, we study the Dirichlet case and obtain similar results to those obtained with Robin boundary conditions, see Theorem 2.7, Theorem 2.8 and Theorem 2.9.

This article is organized as follows. Section 2 describes the assumptions, including two spectral parameters λ, μ , and enunciates the main results of this paper. In Section 3, we deal with our model without spectral parameter so that we retrieve in a simple manner results of previous works, see [9] and [19]. Section 4 is devoted to some precise estimates of Dore-Yakubov type, which will be useful to analyze our model. Sections 5 and 6 concern the study of our model with spectral parameters λ, μ under two different types of behaviour concerning operators with respect to their domains and to the parameters. Moreover sharp estimates in λ, μ are furnished for the solution. In Section 7, we furnish results for (1) together with Dirichlet boundary conditions. Then, in Section 8, we apply the results of Sections 5, 6, 7 to generation of semigroups. Finally, Section 9 deals with examples of applications.

2 Assumptions and statement of main results

In all this work, we will use the following notation: for $\varphi \in (0, \pi)$, we set

$$S_\varphi := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \varphi\} \cup \{0\}. \quad (3)$$

Our goal is to seek for a classical solution to Problem (1)-(2), that is a function u such that

$$\begin{aligned} i) \quad & u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A)), \\ ii) \quad & u(0) \in D(H), \\ iii) \quad & u \text{ satisfies (1) and (2)}. \end{aligned}$$

We suppose that

$$X \text{ is a } UMD \text{ space.} \quad (4)$$

Recall that X is a UMD space means that for all $q > 1$ the Hilbert transform is continuous from $L^q(\mathbb{R}; X)$ into itself, see [6]; we also assume that

$$\begin{cases} \exists \varphi_0 \in (0, \pi) : S_{\varphi_0} \subset \rho(A) \text{ and } \exists C_A > 0 : \\ \forall \lambda \in S_{\varphi_0}, \quad \|(A - \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_A}{1 + |\lambda|}, \end{cases} \quad (5)$$

and

$$\begin{cases} \forall s \in \mathbb{R}, \quad (-A)^{is} \in \mathcal{L}(X), \quad \exists \theta_A \in (0, \pi) : \\ \sup_{s \in \mathbb{R}} \|e^{-\theta_A |s|} (-A)^{is}\|_{\mathcal{L}(X)} < +\infty. \end{cases} \quad (6)$$

We now set for $\lambda \in S_{\varphi_0}, \mu \in \mathbb{C}$

$$H_\mu = H + \mu I, \quad Q_\lambda = -\sqrt{-A + \lambda I} \quad \text{and} \quad Q = -\sqrt{-A}.$$

The existence of the previous square roots is ensured by subsection 5.1 below and for operator H we consider the two following types of hypotheses:

2.1 First case

$$D(H) \subset D(A), \quad (7)$$

and

$$\begin{cases} \exists \varphi_1 \in (0, \pi), \exists C_H > 0 : \\ S_{\varphi_1} \subset \rho(-H) \text{ and } \sup_{\mu \in S_{\varphi_1}} (1 + |\mu|) \|H_\mu^{-1}\|_{\mathcal{L}(X)} \leq C_H. \end{cases} \quad (8)$$

For $r > 0$, we set

$$\Omega_{\varphi_0, \varphi_1, r} = \left\{ (\lambda, \mu) \in S_{\varphi_0} \times S_{\varphi_1} : |\lambda| \geq r \text{ and } \frac{|\mu|^2}{|\lambda|} \geq r \right\}.$$

Then, we have the following main results:

Theorem 2.1. Assume (4)~(8). Let $d_0, u_1 \in X$ and $f \in L^p(0, 1; X)$ with $p \in (1, +\infty)$. Then, there exists $r_0 > 0$ such that for all $(\lambda, \mu) \in \Omega_{\varphi_0, \varphi_1, r_0}$, the two following statements are equivalent:

1. Problem (1)-(2) has a classical solution u , that is,

$$u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A)), \quad u(0) \in D(H),$$

and u satisfies (1)-(2).

2. $u_1 \in (D(A), X)_{\frac{1}{2p}, p}$.

Moreover in this case u is unique and given by (24) where Q, Λ are replaced by $Q_\lambda, \Lambda_{\lambda, \mu}$.

Here, $(D(A), X)_{\frac{1}{2p}, p}$ denotes the classical real interpolation space equipped with the norm

$$\|w\|_{(D(A), X)_{\frac{1}{2p}, p}} = \|w\| + \left(\int_0^{+\infty} \left\| t^{1-1/2p} A(A-tI)^{-1} w \right\|^p \frac{dt}{t} \right)^{1/p}.$$

Theorem 2.2. Assume (4)~(8), $d_0 \in X$ and $u_1 \in (D(A), X)_{\frac{1}{2p}, p}$. Then, there exists a constant $M > 0$ such that, for $(\lambda, \mu) \in \Omega_{\varphi_0, \varphi_1, r_0}$ and $f \in L^p(0, 1; X)$ with $1 < p < +\infty$, the unique classical solution u of (1)-(2) satisfies

$$\max \left\{ (1 + |\lambda|) \|u\|_{L^p(0,1;X)}, \|u''\|_{L^p(0,1;X)}, \|Q_\lambda^2 u\|_{L^p(0,1;X)} \right\} \leq M \alpha(d_0, u_1, \lambda, \mu, f),$$

where

$$\alpha(d_0, u_1, \lambda, \mu, f) = \frac{1 + |\lambda| + |\mu|}{1 + |\mu|} \left(\|d_0\| + \|f\|_{L^p(0,1;X)} \right) + \|u_1\|_{(D(A), X)_{\frac{1}{2p}, p}} + |\lambda|^{1-\frac{1}{2p}} \|u_1\|.$$

Now, define in $Y = L^p(0, 1; X)$ with $p \in (1, +\infty)$, the following operator

$$\begin{aligned} \mathcal{A} : \quad D(\mathcal{A}) \subset Y &\longrightarrow Y \\ u &\longmapsto A(u(\cdot)), \end{aligned}$$

with domain

$$D(\mathcal{A}) = \{u \in Y : u(x) \in D(A) \text{ a.e. } x \in X \text{ and } A(u(\cdot)) \in Y\}.$$

Here, we consider the Banach space $Z := Y \times X$. For $\mu \in \mathbb{C}$, we build a linear operator $\mathcal{P}_{A,H,\mu}$ on Z , by setting

$$\begin{aligned} \mathcal{P}_{A,H,\mu} : D(\mathcal{P}_{A,H,\mu}) \subset Z &\longrightarrow Z \\ (u, v) &\longmapsto (u'' + \mathcal{A}u, u'(0) - Hv - \mu v), \end{aligned}$$

where $D(\mathcal{P}_{A,H,\mu}) = \{(u, v) \in W \times D(H) : u(1) = 0, u(0) = v\}$ with

$$W = W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A)) \subset Y.$$

We then obtain:

Theorem 2.3. Assume (4)~(8). Set $\varphi_2 := \min\{\varphi_0, \varphi_1\}$. Then, for each $\mu \in \mathbb{C}$ with $|\arg(\mu)| < \pi - \varphi_2$, we have

1. $\mathcal{P}_{A,H,\mu}$ is the infinitesimal generator of a C_0 -semigroup.
2. If $\varphi_2 \in [\pi/2, \pi)$, then $\mathcal{P}_{A,H,\mu}$ is the infinitesimal generator of an analytic semigroup.

2.2 Second case

$$D(Q) \subset D(H), \quad (9)$$

$$\exists \varepsilon \in (0, 1/2], \exists C_{H,Q} > 0, \sup_{t \in [0, +\infty)} (1+t)^\varepsilon \|HQ_t^{-1}\|_{\mathcal{L}(X)} \leq C_{H,Q}, \quad (10)$$

$$(Q - H)^{-1} \left((D(Q), X)_{1/p,p} \right) \subset Q^{-1} \left((D(Q), X)_{1/p,p} \right); \quad (11)$$

here, we have not supposed that $(Q - H)^{-1}$ is an operator but we have used the set-theory notation:

$$(Q - H)^{-1} \left((D(Q), X)_{1/p,p} \right) = \left\{ \xi \in D(Q) : (Q - H)\xi \in (D(Q), X)_{1/p,p} \right\}.$$

In order to obtain spectral estimates for the solution of (1)-(2), we will replace (11) by the new assumption:

$$(Q - H)^{-1} (D(Q)) \subset D(Q^2), \quad (12)$$

where, as above

$$(Q - H)^{-1} (D(Q)) = \{ \xi \in D(Q) : (Q - H)\xi \in D(Q) \}.$$

Now, for $\rho > 0$, we set

$$\Pi_{\varphi_0, \rho} = \left\{ (\lambda, \mu) \in S_{\varphi_0} \times \mathbb{C} : |\lambda| \geq \rho \text{ and } \frac{|\lambda|}{|\mu|^{1/\varepsilon}} \geq \rho \right\}.$$

Then, we have the following main results:

Theorem 2.4. Assume (4)~(6) and (9)~(11). Let $d_0, u_1 \in X$ and $f \in L^p(0, 1; X)$ with $1 < p < +\infty$. Then, there exists $\rho_0 > 0$, such that for all $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$, the two following statements are equivalent:

1. Problem (1)-(2) has a classical solution u , that is,

$$u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A)), \quad u(0) \in D(H),$$

and u satisfies (1), (2).

2. $u_1 \in (D(A), X)_{\frac{1}{2p}, p}$ and $(Q_\lambda - H_\mu)^{-1} d_0 \in (D(A), X)_{\frac{1}{2p}, p}$.

Moreover in this case u is unique and given by (24), where Q, Λ are replaced by $Q_\lambda, \Lambda_{\lambda, \mu}$.

Theorem 2.5. Assume (4)~(6) and (9), (10), (12). Let $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$, $d_0 \in X$ with

$$(Q_\lambda - H_\mu)^{-1} d_0 \in (D(A), X)_{\frac{1}{2p}, p}, \quad u_1 \in (D(A), X)_{\frac{1}{2p}, p} \quad \text{and} \quad f \in L^p(0, 1; X),$$

with $1 < p < +\infty$. Then, there exists a constant $M > 0$, which does not depend on $d_0, u_1, (\lambda, \mu)$ and f , such that the unique classical solution u of (1)-(2) satisfies

$$\max \left\{ (1 + |\lambda|) \|u\|_{L^p(0,1;X)}, \|u''\|_{L^p(0,1;X)}, \|Q_\lambda^2 u\|_{L^p(0,1;X)} \right\} \leq M\beta(d_0, u_1, \lambda, \mu, f),$$

where

$$\begin{aligned} \beta(d_0, u_1, \lambda, \mu, f) &= \|d_0\| + \|f\|_{L^p(0,1;X)} + \left\| (Q_\lambda - H_\mu)^{-1} d_0 \right\|_{(D(A), X)_{\frac{1}{2p}, p}} \\ &\quad + |\lambda|^{1-\frac{1}{2p}} \left\| (Q_\lambda - H_\mu)^{-1} d_0 \right\| + \|u_1\|_{(D(A), X)_{\frac{1}{2p}, p}} + |\lambda|^{1-\frac{1}{2p}} \|u_1\|. \end{aligned}$$

Now, we define for $\mu \in \mathbb{C}$, operators

$$\begin{aligned} \mathcal{L}_{A,H,\mu} : \quad D(\mathcal{L}_{A,H,\mu}) \subset Y &\longrightarrow Y \\ u &\longmapsto u'' + \mathcal{A}u, \end{aligned} \tag{13}$$

where \mathcal{A}, Y are defined above and $D(\mathcal{L}_{A,H,\mu})$ is the space of functions u satisfying

$$\begin{cases} u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A)) \\ u(0) \in D(H) \\ u'(0) - Hu(0) - \mu u(0) = u(1) = 0. \end{cases}$$

We then obtain:

Theorem 2.6. Assume (4)~(6), (9), (10) and (12). Then, for any $\mu \in \mathbb{C}$, we have

1. $\mathcal{L}_{A,H,\mu}$ is the infinitesimal generator of a C_0 -semigroup.
2. If $\varphi_0 \in [\pi/2, \pi)$, then $\mathcal{L}_{A,H,\mu}$ is the infinitesimal generator of an analytic semigroup.

2.3 Dirichlet case

Now we consider the spectral problem

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), & x \in (0, 1) \\ u(0) = u_0, \quad u(1) = u_1. \end{cases} \tag{14}$$

We first state the following result on existence and uniqueness of the solution.

Theorem 2.7. Assume (4)~(6). Let $u_0, u_1 \in X$, $\lambda \in S_{\varphi_0}$ and $f \in L^p(0, 1; X)$ with $1 < p < +\infty$. Then the two following statements are equivalent:

1. Problem (14) has a classical solution u .
2. $u_0, u_1 \in (D(A), X)_{\frac{1}{2p}, p}$.

Moreover in this case u is unique and given by (68) where Q is replaced by Q_λ .

Now, we state our new results on sharp estimates and generation of semigroup.

Theorem 2.8. Assume (4)~(6) and $u_0, u_1 \in (D(A), X)_{\frac{1}{2p}, p}$. Then, there exists a constant $M > 0$ such that, for $\lambda \in S_{\varphi_0}$ and $f \in L^p(0, 1; X)$ with $1 < p < +\infty$, the unique classical solution u of problem (14) satisfies

$$\begin{aligned} & \max \left\{ (1 + |\lambda|) \|u\|_{L^p(0,1;X)}, \|u''\|_{L^p(0,1;X)}, \|Q_\lambda^2 u\|_{L^p(0,1;X)} \right\} \\ & \leq M \left(\|f\|_{L^p(0,1;X)} + \|u_0\|_{(D(A),X)_{\frac{1}{2p},p}} + \|u_1\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} (\|u_0\| + \|u_1\|) \right). \end{aligned}$$

Now, we define operator

$$\begin{aligned} \mathcal{L}_A : D(\mathcal{L}_A) \subset Y & \longrightarrow Y \\ u & \longmapsto u'' + \mathcal{A}u. \end{aligned}$$

where $D(\mathcal{L}_A) = \{u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A)) : u(0) = u(1) = 0\}$. We then obtain

Theorem 2.9. Assume (4)~(6). Then, we have

1. \mathcal{L}_A is the infinitesimal generator of a C_0 -semigroup.
2. If $\varphi_0 \in [\pi/2, \pi)$, then \mathcal{L}_A is the infinitesimal generator of an analytic semigroup.

2.4 Remarks

Remark 2.10. Assume (5) and (9). Let $(\lambda, \mu) \in S_{\varphi_0} \times \mathbb{C}$. Then assumption (12) is equivalent to

$$(Q_\lambda - H_\mu)^{-1} (D(Q)) \subset D(Q^2). \quad (15)$$

In fact, if $\xi \in D(Q)$, there exists $\zeta \in X$ such that $\xi = Q^{-1}\zeta$, so

$$\begin{aligned} (Q - H)\xi &= [Q_\lambda - H_\mu + \mu I + (Q - Q_\lambda)]\xi \\ &= (Q_\lambda - H_\mu)\xi + \mu\xi + (Q - Q_\lambda)Q^{-1}\zeta, \end{aligned}$$

and it will be seen in Lemma 4.4 below that there exists $T_\lambda \in \mathcal{L}(X)$ such that

$$Q = Q_\lambda + T_\lambda \quad \text{and} \quad Q^{-1}T_\lambda = T_\lambda Q^{-1}.$$

Therefore

$$(Q - H)\xi - (Q_\lambda - H_\mu)\xi = \mu\xi + Q^{-1}T_\lambda\zeta \in D(Q).$$

This proves that

$$(Q - H)^{-1} (D(Q)) = (Q_\lambda - H_\mu)^{-1} (D(Q)).$$

Remark 2.11. Assume (5). If there exists $\omega \in [0, 1/2)$ such that $D((-A)^\omega) \subset D(H)$ then, in virtue of Lemma 2.6 statement a) in [13], there exists $C_\omega > 0$ such that, for $t \geq 0$

$$\begin{aligned} \|HQ_t^{-1}\|_{\mathcal{L}(X)} &\leq \|H(-A)^{-\omega}\|_{\mathcal{L}(X)} \|(-A)^\omega (-A + tI)^{-1/2}\|_{\mathcal{L}(X)} \\ &\leq \frac{C_\omega}{(1+t)^{1/2-\omega}}; \end{aligned}$$

so we have (9) and (10) with $\varepsilon = 1/2 - \omega \in (0, 1/2]$.

Remark 2.12. In this paper we have supposed that $0 \in \rho(A)$, but in the theorems above written for $|\lambda|$ large enough, and those concerning generation of semigroups, we can drop this invertibility assumption; more precisely all the previous Theorems remain true if we replace (5) by

$$\begin{cases} \exists \varphi_0 \in (0, \pi), \exists \omega_0 > 0 : S_{\varphi_0} \subset \rho(A - \omega_0 I) \text{ and } \exists C_A > 0 : \\ \forall \lambda \in S_{\varphi_0}, \|(A - \omega_0 I - \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_A}{1 + |\lambda|}. \end{cases}$$

3 Problem without parameters

In this section we study a problem similar to (1)-(2), but without the parameters λ and μ :

$$\begin{cases} u''(x) + Au(x) = f(x), & x \in (0, 1) \\ u'(0) - Hu(0) = d_0, \quad u(1) = u_1. \end{cases} \quad (16)$$

3.1 Hypotheses

Here our hypotheses are

(H₁) X is a *UMD* space,

(H₂) $[0, +\infty) \subset \rho(A)$ and $\sup_{t \in [0, +\infty)} \|(A - tI)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{1+t}$,

(H₃) $\forall s \in \mathbb{R}, (-A)^{is} \in \mathcal{L}(X)$ and

$$\exists \theta_A \in (0, \pi) : \sup_{s \in \mathbb{R}} \|e^{-\theta_A |s|} (-A)^{is}\|_{\mathcal{L}(X)} < +\infty,$$

(H₄) $\Lambda := (Q - H) + e^{2Q}(Q + H)$ is closed and boundedly invertible, where $Q := -\sqrt{-A}$.

(H₅) $Q\Lambda^{-1} \left((D(Q), X)_{1/p,p} \right) \subset (D(Q), X)_{1/p,p}$.

Note that here we are neither in case 1, nor in case 2. In the following remark we discuss about assumption (H₅).

Remark 3.1.

1. Assume (H₂) and (H₄). If we suppose moreover

$$A^{-1}\Lambda^{-1} = \Lambda^{-1}A^{-1}, \quad (17)$$

then (H₅) is satisfied. In fact, the following assertions are equivalent.

- (a) $A^{-1}\Lambda^{-1} = \Lambda^{-1}A^{-1}$,
- (b) $\forall \lambda \in \rho(A), (A - \lambda I)^{-1}\Lambda^{-1} = \Lambda^{-1}(A - \lambda I)^{-1}$,
- (c) $Q^{-1}\Lambda^{-1} = \Lambda^{-1}Q^{-1}$.

Then, under (17), we have $Q(Q - tI)^{-1}Q\Lambda^{-1} = Q\Lambda^{-1}Q(Q - tI)^{-1}$ and since

$$(D(Q), X)_{1/p,p} = \left\{ x \in X : t^{1-1/p}Q(Q - tI)^{-1}x \in L_*^p(\mathbb{R}_+; X) \right\},$$

we get (H₅). Finally we remark that if

$$\forall \zeta \in D(H), \quad A^{-1}\zeta \in D(H) \quad \text{and} \quad A^{-1}H\zeta = HA^{-1}\zeta, \quad (18)$$

then (17) and (H₅) are satisfied.

2. Assume (H₂) and (H₄); if we use the classical notation

$$(D(Q), X)_{1+1/p,p} := \left\{ \psi \in D(Q) : Q\psi \in (D(Q), X)_{1/p,p} \right\},$$

then (H₅) writes

$$\Lambda^{-1} \left((D(Q), X)_{1/p,p} \right) \subset (D(Q), X)_{1+1/p,p}.$$

3. In [8], problem (16) has been studied under more restrictive assumptions, that are $(H_1) \sim (H_4)$ and the commutativity hypothesis

$$\exists \lambda_0 \in \rho(H) : \quad A^{-1} (H - \lambda_0 I)^{-1} = (H - \lambda_0 I)^{-1} A^{-1}, \quad (19)$$

which, from statement 1., implies (H_5) .

4. Assume (H_2) and (H_4) . If $Q - H$ is boundedly invertible then, due to $\Lambda\Lambda^{-1} = I$ and $\Lambda^{-1}\Lambda = I$, we get that

$$\begin{cases} \Lambda^{-1} = (Q - H)^{-1} - (Q - H)^{-1} e^{2Q} (Q + H) \Lambda^{-1} \\ (Q - H)^{-1} = \Lambda^{-1} + \Lambda^{-1} e^{2Q} (Q + H) (Q - H)^{-1}, \end{cases} \quad (20)$$

from which we deduce that for any $\xi \in (D(Q), X)_{1/p,p}$

$$\left[Q\Lambda^{-1}\xi \in (D(Q), X)_{1/p,p} \right] \iff \left[Q(Q - H)^{-1}\xi \in (D(Q), X)_{1/p,p} \right];$$

so we can replace in the previous proposition assumption (H_5) by the equivalent one

$$(H'_5) \quad Q(Q - H)^{-1} \left((D(Q), X)_{1/p,p} \right) \subset (D(Q), X)_{1/p,p}, \quad (21)$$

5. Assume (H_2) and (H_4) . If we suppose that $Q\Lambda^{-1}\xi \in D(Q)$, then we have (H_5) , see Lemma 5 p. 76 in [9]. Similarly, when $Q - H$ is boundedly invertible, using (20) we have that

$$\forall \xi \in D(Q), \quad Q(Q - H)^{-1}\xi \in D(Q),$$

implies (H'_5) and then (H_5) .

3.2 Interpolation spaces

Let us give now some necessary conditions to obtain a classical solution for our problem (16) using known properties of interpolation spaces.

Lemma 3.2. Suppose that Problem (16) has a classical solution u . Then:

1. $u(0), u(1) \in (D(Q^2), X)_{\frac{1}{2p}, p} = (X, D(Q^2))_{1-\frac{1}{2p}, p}$, which implies that

$$u(0), u(1) \in D(Q) \quad \text{and} \quad Qu(0), Qu(1) \in (D(Q), X)_{1/p, p}.$$
2. $u'(0), u'(1) \in (D(Q^2), X)_{\frac{1}{2}+\frac{1}{2p}, p} = (D(Q), X)_{\frac{1}{p}, p}.$

Proof. Suppose that Problem (16) has a classical solution u . Then, from

$$u \in W^{2,p}(0, 1; X) \cap L^p\left(0, 1; D(Q^2)\right), \quad 1 < p < +\infty,$$

we have $u(0), u(1) \in (D(Q^2), X)_{\frac{1}{2p}, p} = (X, D(Q^2))_{1-\frac{1}{2p}, p}$, see [16], Teorema 2', p. 678. But, due to [16], Teorema 6, p. 676, we obtain

$$\left(D(Q^2), X \right)_{1/2p, p} = (D(Q), X)_{1+1/p, p} = \left\{ \varphi \in D(Q) : Q\varphi \in (D(Q), X)_{1/p, p} \right\} \subset D(Q),$$

from which it follows that

$$u(0), u(1) \in D(Q) \quad \text{and} \quad Qu(0), Qu(1) \in (D(Q), X)_{1/p, p}.$$

Similarly, by using Teorema 2', in [16], p. 678, we have

$$u'(0), u'(1) \in \left(D(Q^2), X \right)_{\frac{1}{2}+\frac{1}{2p}, p} = (D(Q), X)_{\frac{1}{p}, p}.$$

□

3.3 Representation formula

Under (H_2) , if u is a classical solution of (16) then there exist $\xi_0, \xi_1 \in X$ such that

$$u(x) = e^{xQ}\xi_0 + e^{(1-x)Q}\xi_1 + I(x) + J(x), \quad x \in [0, 1], \quad (22)$$

where

$$I(x) = \frac{1}{2}Q^{-1} \int_0^x e^{(x-s)Q} f(s) ds \quad \text{and} \quad J(x) = \frac{1}{2}Q^{-1} \int_x^1 e^{(s-x)Q} f(s) ds, \quad (23)$$

see [7], p. 989. Note that here, unlike [7], we do not suppose that A and H commute. Now, taking into account the fact that $I - e^{2Q}$ is invertible, we set $T = (I - e^{2Q})^{-1} \in \mathcal{L}(X)$ and

$$S(x) = T(e^{xQ} - e^{(1-x)Q}e^Q) \in \mathcal{L}(X), \quad x \in [0, 1];$$

then formula (22) writes in the following form

$$u(x) = S(x)\mu_0 + S(1-x)\mu_1 + I(x) + J(x), \quad x \in [0, 1],$$

with $\mu_0 = \xi_0 + e^Q\xi_1, \mu_1 = e^Q\xi_0 + \xi_1$ and we deal with this new writing. We note that

$$\begin{cases} u_0 = u(0) = \mu_0 + J(0) \\ u_1 = u(1) = \mu_1 + I(1) \\ u'(0) = TQ(I + e^{2Q})\mu_0 - 2TQe^Q\mu_1 - QJ(0), \end{cases}$$

and we determine μ_0, μ_1 by using the boundary conditions $u(1) = u_1$ and $u'(0) - Hu(0) = d_0$. So $\mu_1 = u_1 - I(1)$ and

$$TQ(I + e^{2Q})\mu_0 - 2TQe^Q\mu_1 - QJ(0) - H(\mu_0 + J(0)) = d_0;$$

hence

$$\begin{aligned} TQ(I + e^{2Q})(\mu_0 + J(0)) - H(\mu_0 + J(0)) &= d_0 + 2TQe^Q\mu_1 \\ &\quad + QJ(0) + TQ(I + e^{2Q})J(0), \end{aligned}$$

thus

$$\begin{aligned} [Q(I + e^{2Q}) - (I - e^{2Q})H](\mu_0 + J(0)) &= (I - e^{2Q})d_0 + 2Qe^Q\mu_1 \\ &\quad + (I - e^{2Q})QJ(0) + Q(I + e^{2Q})J(0); \end{aligned}$$

but $\Lambda = Q(I + e^{2Q}) - (I - e^{2Q})H$, so

$$\begin{cases} \mu_1 = u_1 - I(1) \\ \mu_0 = \Lambda^{-1} [(I - e^{2Q})d_0 + 2Qe^Q\mu_1 + 2QJ(0)] - J(0). \end{cases}$$

Finally, if u is a classical solution to (16), then

$$u(x) = S(x)\mu_0 + S(1-x)\mu_1 + I(x) + J(x), \quad x \in [0, 1], \quad (24)$$

where

$$\begin{cases} \mu_1 = u_1 - I(1) \\ \mu_0 = \Lambda^{-1} [(I - e^{2Q})d_0 + 2Qe^Q\mu_1 + 2QJ(0)] - J(0) \\ S(x) = (I - e^{2Q})^{-1} (e^{xQ} - e^{(1-x)Q}e^Q). \end{cases} \quad (25)$$

When (19) is satisfied, we can check that this representation formula coincides with the one given in [8] p. 528. We can also, after computations, verify that (24) is the same formula as the one in [9], p. 92 (with $L = M = Q$) and also compare it with (34)~(38) pp. 54-55, in [19].

3.4 Regularity results

The following results will be useful to study the regularity of the solution of (16).

Lemma 3.3. Let $p \in (1, +\infty)$, $\psi \in X$ and $n \in \mathbb{N} \setminus \{0\}$. Then, under (H_2) , we have

1. $x \mapsto e^{xQ}\psi \in L^p(0, 1, X)$.
2. $x \mapsto Q^n e^{xQ}\psi \in L^p(0, 1, X)$ if and only if $\psi \in (D(Q^n), X)_{\frac{1}{np}, p}$.

See for instance [23], Theorem at p. 96.

Lemma 3.4. For $f \in L^p(0, 1, X)$ with $1 < p < +\infty$, under $(H_1) \sim (H_3)$, we have

1. $x \mapsto Q \int_0^x e^{(x-s)Q} f(s) ds \in L^p(0, 1, X)$ and $x \mapsto Q \int_x^1 e^{(s-x)Q} f(s) ds \in L^p(0, 1, X)$.
2. $x \mapsto Q \int_0^1 e^{(x+s)Q} f(s) ds \in L^p(0, 1, X)$.

For statements 1 and 2 which are consequences of the Dore-Venni Theorem, see [15], p. 167-168 and also (24), (25) and (26) in [14].

Lemma 3.5. Let $\psi, \chi \in X$ and $1 < p < +\infty$. Then, under (H_2) , we have

1. $x \mapsto Q^2 S(x)\psi \in L^p(0, 1; X) \iff \psi \in (D(Q^2), X)_{\frac{1}{2p}, p}$,
 $x \mapsto Q^2 S(1-x)\chi \in L^p(0, 1; X) \iff \chi \in (D(Q^2), X)_{\frac{1}{2p}, p}$.
2. $x \mapsto Q^2 S(x)\psi + Q^2 S(1-x)\chi \in L^p(0, 1; X) \iff \psi, \chi \in (D(Q^2), X)_{\frac{1}{2p}, p}$.

Proof.

1. Since $T = (I - e^{2Q})^{-1} = I + e^{2Q}(I - e^{2Q})^{-1}$, we have

$$S(x) = e^{xQ} + (I - e^{2Q})^{-1} e^{xQ} e^{2Q} - \left(I + e^{2Q}(I - e^{2Q})^{-1} \right) e^{(1-x)Q} e^Q;$$

then, by Lemma 3.3

$$\begin{aligned} Q^2 S(\cdot)\psi \in L^p(0, 1; X) &\iff Q^2 e^{\cdot Q} \psi \in L^p(0, 1; X) \\ &\iff \psi \in (D(Q^2), X)_{\frac{1}{2p}, p}. \end{aligned}$$

2. For any $\psi, \chi \in X$, we have

$$Q^2 S(\cdot)\psi + Q^2 S(1-\cdot)\chi \in L^p(0, 1; X), \tag{26}$$

if and only if $Q^2 S(\cdot)\psi \in L^p(0, 1; X)$ and $Q^2 S(1-\cdot)\chi \in L^p(0, 1; X)$.

In fact, assume (26), then

$$Q^2 S(\cdot)\psi + Q^2 S(1-\cdot)\chi \in L^p(1/2, 1; X),$$

but $Q^2 S(\cdot)\psi \in L^p(1/2, 1; X)$, therefore $Q^2 S(1-\cdot)\chi \in L^p(1/2, 1; X)$, which gives

$$Q^2 S(1-\cdot)\chi \in L^p(0, 1; X).$$

□

Lemma 3.6. Consider μ_0, μ_1 defined in (25). Then

$$\begin{cases} \mu_0 \in (D(A), X)_{\frac{1}{2p}, p} \iff \Lambda^{-1}d_0 \in (D(A), X)_{\frac{1}{2p}, p} \\ \mu_1 \in (D(A), X)_{\frac{1}{2p}, p} \iff u_1 \in (D(A), X)_{\frac{1}{2p}, p}. \end{cases}$$

Proof. From [17], Proposition 3.5, p. 1676, we have $J(0), I(1) \in (D(A), X)_{\frac{1}{2p}, p}$, thus:

$$\mu_1 \in (D(A), X)_{\frac{1}{2p}, p} \iff u_1 \in (D(A), X)_{\frac{1}{2p}, p}.$$

Moreover $\mu_0 = \Lambda^{-1}d_0 - \Lambda^{-1} \left[e^{2Q}d_0 + 2Qe^Q\mu_1 + 2QJ(0) \right] - J(0)$, with

$$e^{2Q}d_0 + 2Qe^Q\mu_1 + 2QJ(0) \in (D(Q), X)_{1/p, p},$$

and from (H_5)

$$Q\Lambda^{-1} \left[e^{2Q}d_0 + 2Qe^Q\mu_1 + 2QJ(0) \right] \in (D(Q), X)_{1/p, p},$$

which means that

$$\Lambda^{-1} \left[e^{2Q}d_0 + 2Qe^Q\mu_1 + 2QJ(0) \right] \in (D(Q), X)_{1+1/p, p} = (D(A), X)_{\frac{1}{2p}, p}.$$

Finally: $\mu_0 \in (D(A), X)_{\frac{1}{2p}, p} \iff \Lambda^{-1}d_0 \in (D(A), X)_{\frac{1}{2p}, p}$. □

3.5 Resolution of Problem (16)

Proposition 3.7. Let $f \in L^p(0, 1; X)$ with $1 < p < +\infty$ and assume that $(H_1) \sim (H_5)$ are satisfied. Then the following assertions are equivalent:

1. Problem (16) admits a classical solution u .
2. $u_1, \Lambda^{-1}d_0 \in (D(A), X)_{\frac{1}{2p}, p}$

Moreover in this case u is unique and it is given by (24).

Proof. From the previous study we know that if Problem (16) admits a classical solution u then u is unique and given by (24). Moreover u defined by (24) satisfies

$$u(0) = \mu_0 + J(0) = \Lambda^{-1} \left[(I - e^{2Q})d_0 + 2Qe^Q\mu_1 + 2QJ(0) \right] \in D(H),$$

and then u is a classical solution of (16) if and only if $Q^2u(\cdot) \in L^p(0, 1; X)$.

But from Lemma 3.4, $Q^2I(\cdot), Q^2J(\cdot) \in L^p(0, 1; X)$, so Lemma 3.5 and Lemma 3.6 imply that

$$\begin{aligned} Q^2u(\cdot) \in L^p(0, 1; X) &\iff Q^2S(x)\mu_0 + Q^2S(1-x)\mu_1 \in L^p(0, 1; X) \\ &\iff \mu_0, \mu_1 \in (D(A), X)_{\frac{1}{2p}, p} \\ &\iff u_1, \Lambda^{-1}d_0 \in (D(A), X)_{\frac{1}{2p}, p}; \end{aligned}$$

this proves that statement 1. is equivalent to statement 2. □

Remark 3.8. In the previous Proposition, if moreover, $Q - H$ is boundedly invertible, then using (20) we can replace the condition $\Lambda^{-1}d_0 \in (D(A), X)_{\frac{1}{2p}, p}$ by the simpler one $(Q - H)^{-1}d_0 \in (D(A), X)_{\frac{1}{2p}, p}$.

4 Dore-Yakubov estimates

This section is devoted to Dore-Yakubov Estimates and applications. The results are based on those given in [13] and we have used the definitions and notations from this paper. We consider here a complex Banach space E , which is not necessarily a UMD space.

Definition 4.1. W is an operator of type $\phi \in (0, \pi)$ with bound C , if $W : D(W) \subset E \rightarrow E$ is a closed linear operator such that $S_\phi \subset \rho(-W)$ and

$$\forall \lambda \in S_\phi, \quad \left\| (W + \lambda I)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{C}{1 + |\lambda|},$$

where S_ϕ is defined by (3).

In all this section, we fix φ in $(0, \pi)$, λ in S_φ and $L : D(L) \subset E \rightarrow E$ an operator of type φ with bound C_L , satisfying $D(L) = E$.

We set

$$D_\lambda := L + \lambda I \text{ and } \varepsilon(\varphi) := \min \{ \varphi, \pi - \varphi \} \in (0, \pi/2).$$

Note that $\varepsilon(\varphi) = \varphi$ if $\varphi \in (0, \pi/2]$, and $\varepsilon(\varphi) = \pi - \varphi$ if $\varphi \in [\pi/2, \pi)$.

Lemma 4.2.

1. Let $\theta \in (0, \varepsilon(\varphi))$; then D_λ is an operator of type θ with bound $C_\theta := C_L / \cos\left(\frac{\varphi + \theta}{2}\right)$.

Moreover for $\nu \in S_\theta$ we have : $\left\| (D_\lambda + \nu I)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{C_\theta}{|\lambda| + |\nu| + 1}$.

2. Let $\bar{\theta} \in \mathbb{R}$ and $\nu \in \mathbb{C}$ such that $\theta = |\bar{\theta}| \in (0, \varepsilon(\varphi))$ and $\operatorname{Re}(\nu e^{-i\bar{\theta}/2}) > 0$. Then $D_\lambda^{1/2} + \nu I$ is boundedly invertible and $\left\| (D_\lambda^{1/2} + \nu I)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{C_{\nu, \bar{\theta}}}{|\nu| + \sqrt{|\lambda| + 1}}$, with $C_{\nu, \bar{\theta}} := C_L / \left[\cos\left(\arg(\nu) - \frac{\bar{\theta}}{2}\right) \cos\left(\frac{\varphi + \theta}{2}\right) \right]$.

3. Let $\psi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \frac{\varepsilon(\varphi)}{2}\right)$; then $D_\lambda^{1/2}$ is a linear operator of type ψ with bound

$$K_\psi := C_L / \cos^2(\beta_\psi) \text{ where } \beta_\psi = \frac{\pi}{4} + \frac{\psi - \varepsilon(\varphi)}{2} \in (0, \pi/2).$$

Moreover for $\nu \in S_\psi$ we have

$$\left\| (D_\lambda^{1/2} + \nu I)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{K_\psi}{|\nu| + \sqrt{|\lambda| + 1}}. \quad (27)$$

Proof. This Lemma is essentially based on Lemmas 2.3 and 2.4 in [13]. The novelty is in some precisions given on the estimates in statement 2. and 3., which integrate the behaviour with respect to the complex parameters λ and ν .

1. See [13], (2.1) in Lemma 2.4, p. 99.
2. The idea is to use the calculus given in [13], in Lemma 2.4, at the end of p. 99:

$$\left\| (D_\lambda^{1/2} + \nu I)^{-1} \right\|_{\mathcal{L}(E)} = \left\| \frac{1}{\pi} \int_0^\infty \frac{r^{1/2} e^{i\bar{\theta}/2}}{r e^{i\bar{\theta}} + \nu^2} (D_\lambda + r e^{i\bar{\theta}} I)^{-1} e^{i\bar{\theta}} dr \right\|_{\mathcal{L}(E)},$$

but using $\left\| \left(D_\lambda + r e^{i\bar{\theta}} I \right)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{C_\theta}{|\lambda| + r + 1}$, we get that

$$\begin{aligned} \left\| \left(D_\lambda^{1/2} + \nu I \right)^{-1} \right\|_{\mathcal{L}(E)} &\leq \frac{C_\theta}{\pi} \int_0^{+\infty} \frac{r^{1/2}}{\left| r e^{i\bar{\theta}} + \nu^2 \right|} \frac{1}{|\lambda| + r + 1} dr \\ &\leq \frac{C_\theta}{\pi} \int_0^{+\infty} \frac{r^{1/2}}{\left| \cos \left(\frac{\arg(\nu^2) - \bar{\theta}}{2} \right) \right| (r + |\nu|^2)} \frac{1}{|\lambda| + r + 1} dr \\ &\leq \frac{C_\theta}{\pi \cos \left(\arg(\nu) - \frac{\bar{\theta}}{2} \right)} \int_0^{+\infty} \frac{r^{1/2}}{(r + |\nu|^2)(r + |\lambda| + 1)} dr \\ &= \frac{C_L}{\cos \left(\arg(\nu) - \frac{\bar{\theta}}{2} \right) \cos \left((\varphi + \theta)/2 \right)} \frac{1}{|\nu| + \sqrt{|\lambda| + 1}}. \end{aligned}$$

The last equality, follows from

$$\int_0^{+\infty} \frac{r^{1/2}}{(r + a)(r + b)} dr = \frac{\pi}{\sqrt{b} + \sqrt{a}}, \quad a, b > 0.$$

3. Estimate (27) is deduced from Statement 3. as in [13], Lemma 2.4, p. 100. □

From the previous Lemma, we deduce that

$$\forall \nu \geq 0, \quad \left\| (D_\lambda + \nu I)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{C_0}{|\lambda| + \nu + 1}, \quad (28)$$

and

$$\forall \nu \in S_{\pi/2}, \quad \left\| \left(D_\lambda^{1/2} + \nu I \right)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{K_L}{|\nu| + \sqrt{|\lambda| + 1}}, \quad (29)$$

where $C_0 = C_L / \cos(\varphi/2)$ and $K_L := C_L / \cos^2(\varphi/2)$.

Here, since D_λ is boundedly invertible, we have also that $D_\lambda^{1/2}$ is boundedly invertible and then $\rho(D_\lambda^{1/2})$ contains a ball centered in 0. The following Lemma precises the size of this ball with respect to $\lambda \in S_\varphi$.

Lemma 4.3. We have

1. $\left\| D_\lambda^{-1/2} \right\|_{\mathcal{L}(E)} \leq \frac{K_L}{\sqrt{|\lambda| + 1}}$.
2. For $z \in B \left(0, \frac{\sqrt{|\lambda| + 1}}{2K_L} \right) : z \in \rho(D_\lambda^{1/2})$ and $\left\| (D_\lambda^{1/2} - zI)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{2K_L}{\sqrt{|\lambda| + 1}}$.

Proof. For statement 1., it is enough to consider (29) with $\nu = 0$.

For statement 2, we consider $z \in B \left(0, \frac{\sqrt{|\lambda| + 1}}{2K_L} \right)$, then

$$0 \leq \|z D_\lambda^{-1/2}\|_{\mathcal{L}(E)} = |z| \|D_\lambda^{-1/2}\|_{\mathcal{L}(E)} \leq \frac{\sqrt{|\lambda| + 1}}{2K_L} \frac{K_L}{\sqrt{|\lambda| + 1}} = 1/2 < 1,$$

so $D_\lambda^{1/2} - zI = D_\lambda^{1/2} (I - z D_\lambda^{-1/2})$ is boundedly invertible with

$$\left\| (D_\lambda^{1/2} - zI)^{-1} \right\|_{\mathcal{L}(E)} \leq \left\| D_\lambda^{-1/2} \right\|_{\mathcal{L}(E)} \times \left\| \left(I - z D_\lambda^{-1/2} \right)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{2K_L}{\sqrt{|\lambda| + 1}}.$$

□

Now we will compare $D_\lambda^{1/2}$ and $D_0^{1/2}$. This has been already done for $\lambda > 0$ in [18], Proposition 3.1.7 p. 65. Here λ is a complex parameter: we furnish a precise estimate for the bounded operator T_λ which extends $D_\lambda^{1/2} - D_0^{1/2}$.

Lemma 4.4.

1. There exists a unique $T_\lambda \in \mathcal{L}(E)$ such that

$$D_\lambda^{1/2} = D_0^{1/2} + T_\lambda, \quad (30)$$

2. $T_\lambda D_{\lambda'}^{-1/2} = D_{\lambda'}^{-1/2} T_\lambda$ for any $\lambda' \in S_\varphi$.

3. $\|T_\lambda\|_{\mathcal{L}(E)} \leq C_0 C_L \sqrt{|\lambda|}$.

Proof. First, notice that $L = D_0$ and $D(L) \subset D(D_\lambda^{1/2}) \cap D(D_0^{1/2})$. Thus, if $T_\lambda \in \mathcal{L}(E)$ satisfies (30) then T_λ is unique since $\overline{D(L)} = E$.

We have (see for example [13], p. 100)

$$\begin{aligned} D_\lambda^{-1/2} - D_0^{-1/2} &= \frac{1}{\pi} \int_0^{+\infty} \frac{1}{\sqrt{t}} (D_\lambda + tI)^{-1} dt - \frac{1}{\pi} \int_0^{+\infty} \frac{1}{\sqrt{t}} (D_0 + tI)^{-1} dt \\ &= \frac{1}{\pi} \int_0^{+\infty} \frac{1}{\sqrt{t}} [(D_\lambda + tI)^{-1} - (D_0 + tI)^{-1}] dt \\ &= \frac{-\lambda}{\pi} L^{-1} \int_0^{+\infty} \frac{1}{\sqrt{t}} (L + tI - tI) (D_\lambda + tI)^{-1} (D_0 + tI)^{-1} dt \\ &= \frac{-\lambda}{\pi} L^{-1} \int_0^{+\infty} \frac{1}{\sqrt{t}} (D_\lambda + tI)^{-1} dt \\ &\quad + \frac{\lambda}{\pi} L^{-1} \int_0^{+\infty} \sqrt{t} (D_\lambda + tI)^{-1} (D_0 + tI)^{-1} dt \\ &= -\lambda L^{-1} D_\lambda^{-1/2} + L^{-1} T_\lambda \end{aligned}$$

where $T_\lambda := \frac{\lambda}{\pi} \int_0^{+\infty} \sqrt{t} (D_\lambda + tI)^{-1} (D_0 + tI)^{-1} dt \in \mathcal{L}(E)$.

This proves that $D(D_\lambda^{1/2}) = D(D_0^{1/2}) = D(L^{1/2})$. We then deduce (30) by writing, for $\zeta \in D(L^{1/2})$

$$D_\lambda^{1/2} \zeta - D_0^{1/2} \zeta = (L + \lambda I) D_\lambda^{-1/2} \zeta - L D_0^{-1/2} \zeta = T_\lambda \zeta.$$

Statement 2., is an easy consequence of the definition of T_λ . Statement 3. follows from (28), since

$$\|T_\lambda\|_{\mathcal{L}(E)} \leq \frac{|\lambda|}{\pi} \int_0^{+\infty} \sqrt{t} \frac{C_0}{|\lambda| + t + 1} \frac{C_L}{t + 1} dt = \frac{|\lambda| C_0 C_L}{\sqrt{|\lambda| + 1 + 1}}.$$

□

From Lemma 2.6, p. 103, in [13], for $\lambda \in S_\varphi$, we have that $G_\lambda := -D_\lambda^{1/2}$ generates a semigroup $(e^{tG_\lambda})_{t \geq 0}$ bounded, analytic for $t > 0$ and strongly continuous for $t \geq 0$. Moreover, it satisfies

$$\begin{cases} \exists K_0 > 0, \exists c_0 > 0, \forall t \geq 1/2, \forall \lambda \in S_\varphi : \\ \max \left\{ \|e^{tG_\lambda}\|_{\mathcal{L}(E)}, \|G_\lambda e^{tG_\lambda}\|_{\mathcal{L}(E)} \right\} \leq K_0 e^{-tc_0 |\lambda|^{1/2}}. \end{cases}$$

Lemma 4.5. Let $-\infty < a < b < +\infty$. For $x \in [a, b]$, $\lambda \in S_\varphi$ and $f \in L^p(a, b; E)$, we set

$$U_{\lambda, f}(x) = \int_a^x e^{(x-s)G_\lambda} f(s) ds \quad \text{and} \quad V_{\lambda, f}(x) = \int_x^b e^{(s-x)G_\lambda} f(s) ds. \quad (31)$$

There exists $M_L > 0$ such that for any $f \in L^p(a, b; E)$ and any $\lambda \in S_\varphi$

$$\|U_{\lambda, f}\|_{L^p(a, b; E)} \leq \frac{M_L}{\sqrt{|\lambda| + 1}} \|f\|_{L^p(a, b; E)} \quad \text{and} \quad \|V_{\lambda, f}\|_{L^p(a, b; E)} \leq \frac{M_L}{\sqrt{|\lambda| + 1}} \|f\|_{L^p(a, b; E)}.$$

Proof. We fix $\psi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \frac{\varepsilon(\varphi)}{2}\right)$ and use notations and estimates of Lemma 4.2. We first focus on $U_{\lambda, f}$. Let $x \in [a, b]$. We apply the Dunford-Riesz Calculus to define e^{G_λ} , and obtain

$$\begin{aligned} U_{\lambda, f}(x) &= \frac{1}{2i\pi} \int_a^x \int_\gamma e^{(x-s)z} (zI - G_\lambda)^{-1} f(s) dz ds \\ &= \frac{1}{2i\pi} \int_a^x \int_\gamma e^{(x-s)z} (zI + D_\lambda^{1/2})^{-1} f(s) dz ds, \end{aligned}$$

where the path γ is the boundary positively oriented of $S_\psi \cup B(0, \varepsilon)$ with $\varepsilon = \frac{\sqrt{|\lambda| + 1}}{2K_L}$.

Then

$$\begin{aligned} U_{\lambda, f}(x) &= \frac{1}{2i\pi} \int_a^x \int_\varepsilon^{+\infty} e^{(x-s)re^{i\psi}} (re^{i\psi}I + D_\lambda^{1/2})^{-1} f(s) e^{i\psi} dr ds \\ &\quad + \frac{1}{2i\pi} \int_a^x \int_{2\pi-\psi}^\psi e^{(x-s)\varepsilon e^{i\theta}} (\varepsilon e^{i\theta}I + D_\lambda^{1/2})^{-1} f(s) \varepsilon i e^{i\theta} d\theta ds \\ &\quad - \frac{1}{2i\pi} \int_a^x \int_\varepsilon^{+\infty} e^{(x-s)re^{-i\psi}} (re^{-i\psi}I + D_\lambda^{1/2})^{-1} f(s) e^{-i\psi} dr ds, \end{aligned}$$

hence

$$\begin{aligned} \|U_{\lambda, f}(x)\| &\leq \frac{1}{2\pi} \int_a^x \int_\varepsilon^{+\infty} \|e^{(x-s)re^{i\psi}} f(s)\| \|(re^{i\psi}I + D_\lambda^{1/2})^{-1}\|_{\mathcal{L}(E)} dr ds \\ &\quad + \frac{\varepsilon}{2\pi} \int_a^x \int_\psi^{2\pi-\psi} \|e^{(x-s)\varepsilon e^{i\theta}} f(s)\| \|(\varepsilon e^{i\theta}I + D_\lambda^{1/2})^{-1}\|_{\mathcal{L}(E)} d\theta ds \\ &\quad + \frac{1}{2\pi} \int_a^x \int_\varepsilon^{+\infty} \|e^{(x-s)re^{-i\psi}} f(s)\| \|(re^{-i\psi}I + D_\lambda^{1/2})^{-1}\|_{\mathcal{L}(E)} dr ds. \end{aligned}$$

We deduce, from Lemma 4.2, statement 4. and Lemma 4.3, statement 3., that

$$\begin{aligned} \|U_{\lambda, f}(x)\| &\leq \frac{1}{2\pi} \int_a^x \int_\varepsilon^{+\infty} \|e^{(x-s)re^{i\psi}} f(s)\| \frac{K_\psi}{r + \sqrt{|\lambda| + 1}} dr ds \\ &\quad + \frac{\varepsilon}{2\pi} \int_a^x \int_\psi^{2\pi-\psi} \|e^{(x-s)\varepsilon e^{i\theta}} f(s)\| \frac{2K_L}{\sqrt{|\lambda| + 1}} d\theta ds \\ &\quad + \frac{1}{2\pi} \int_a^x \int_\varepsilon^{+\infty} \|e^{(x-s)re^{-i\psi}} f(s)\| \frac{K_\psi}{r + \sqrt{|\lambda| + 1}} dr ds, \end{aligned}$$

hence

$$\begin{aligned}
\|U_{\lambda,f}(x)\| &\leq \frac{K_\psi}{\pi} \int_a^x \int_\varepsilon^{+\infty} e^{(x-s)r \cos(\psi)} \|f(s)\| \frac{1}{r + \sqrt{|\lambda| + 1}} dr ds \\
&\quad + \frac{\varepsilon K_L}{\pi} \frac{1}{\sqrt{|\lambda| + 1}} \int_a^x \int_\psi^{2\pi-\psi} e^{(x-s)\varepsilon \cos(\theta)} \|f(s)\| d\theta ds \\
&\leq \frac{K_\psi}{\pi} \int_a^x \left(\int_\varepsilon^{+\infty} \frac{e^{(x-s)r \cos(\psi)}}{r + \sqrt{|\lambda| + 1}} dr \right) \|f(s)\| ds \\
&\quad + \frac{\varepsilon K_L}{\pi} \frac{1}{\sqrt{|\lambda| + 1}} \int_\psi^{2\pi-\psi} \int_a^x e^{(x-s)\varepsilon \cos(\psi)} \|f(s)\| ds d\theta \\
&\leq \frac{K_\psi}{\pi} \int_a^x \left(\int_\varepsilon^{+\infty} \frac{e^{(x-s)r \cos(\psi)}}{r + \sqrt{|\lambda| + 1}} dr \right) \|f(s)\| ds \\
&\quad + \frac{2\varepsilon K_L}{\sqrt{|\lambda| + 1}} \int_a^x e^{(x-s)\varepsilon \cos(\psi)} \|f(s)\| ds.
\end{aligned}$$

So, setting

$$\begin{cases} U_{\lambda,f}^1(x) = \frac{K_\psi}{\pi} \int_a^x \left(\int_\varepsilon^{+\infty} \frac{e^{(x-s)r \cos(\psi)}}{r + \sqrt{|\lambda| + 1}} dr \right) \|f(s)\| ds \\ U_{\lambda,f}^2(x) = \frac{2\varepsilon K_L}{\sqrt{|\lambda| + 1}} \int_a^x e^{(x-s)\varepsilon \cos(\psi)} \|f(s)\| ds, \end{cases}$$

we have

$$\|U_{\lambda,f}\|_{L^p(a,b;E)} \leq \|U_{\lambda,f}^1\|_{L^p(a,b)} + \|U_{\lambda,f}^2\|_{L^p(a,b)}. \quad (32)$$

Estimate of $\|U_{\lambda,f}^1\|_{L^p(a,b)}$. Define $g \in L^1(\mathbb{R})$, $F \in L^p(\mathbb{R})$ by

$$g(t) := \begin{cases} \int_\varepsilon^{+\infty} \frac{e^{tr \cos(\psi)}}{r + \sqrt{|\lambda| + 1}} dr & \text{if } t > 0 \\ 0 & \text{elsewhere,} \end{cases} \quad \text{and } F(t) := \begin{cases} \|f(t)\| & \text{if } t \in (a, b) \\ 0 & \text{elsewhere,} \end{cases}$$

and thus

$$\begin{aligned}
U_{\lambda,f}^1(x) &= \frac{K_\psi}{\pi} \int_a^x g(x-s) \|f(s)\| ds + \frac{K_\psi}{\pi} \int_x^b g(x-s) \|f(s)\| ds \\
&= \frac{K_\psi}{\pi} \int_{-\infty}^{+\infty} g(x-s) F(s) ds = \frac{K_\psi}{\pi} (g * F)(x).
\end{aligned}$$

Then, from Young's inequality, we obtain

$$\|U_{\lambda,f}^1\|_{L^p(a,b)} \leq \frac{K_\psi}{\pi} \|g * F\|_{L^p(\mathbb{R})} \leq \frac{K_\psi}{\pi} \|g\|_{L^1(\mathbb{R})} \|F\|_{L^p(\mathbb{R})}.$$

Setting $\ell = \sqrt{|\lambda| + 1}$ and noting that $\varepsilon/\ell = 1/2K$, we have

$$\begin{aligned}
\|g\|_{L^1(\mathbb{R})} &= \int_0^{+\infty} \int_\varepsilon^{+\infty} \frac{e^{tr \cos(\psi)}}{r + \ell} dr dt \\
&= \int_0^{+\infty} \left(\int_{\varepsilon/\ell}^{+\infty} \frac{e^{t\rho \ell \cos(\psi)}}{\rho + 1} d\rho \right) dt \\
&= \frac{1}{\ell \cos(\psi)} \int_{1/2K}^{+\infty} \frac{d\rho}{\rho(\rho + 1)} \\
&= \frac{\ln(2K_L + 1) / \cos(\psi)}{\sqrt{|\lambda| + 1}},
\end{aligned}$$

and finally

$$\|U_{\lambda,f}^1\|_{L^p(a,b)} \leq \frac{K_\psi \ln(2K_L + 1) / \pi \cos(\psi)}{\sqrt{|\lambda| + 1}} \|f\|_{L^p(a,b;E)}.$$

Estimate of $\|U_{\lambda,f}^2\|_{L^p(a,b)}$. Define $F \in L^p(\mathbb{R})$ as above and $h \in L^1(\mathbb{R})$ as follows

$$h(t) := \begin{cases} e^{t\varepsilon \cos(\psi)} & \text{if } t > 0 \\ 0 & \text{elsewhere;} \end{cases}$$

then, as previously

$$U_{\lambda,f}^2(x) = \frac{2\varepsilon K_L}{\sqrt{|\lambda| + 1}} (h * F)(x);$$

therefore from Young's inequality, we get

$$\|U_{\lambda,f}^2\|_{L^p(a,b)} \leq \frac{2K_L / |\cos(\psi)|}{\sqrt{|\lambda| + 1}} \|f\|_{L^p(a,b;E)}.$$

From (32) and the two previous estimates, we obtain the expected result on U_λ . Setting $\tilde{f}(\cdot) := f(\cdot - a - b)$, we note that

$$V_{\lambda,f}(x) = U_{\lambda,\tilde{f}}(b + a - x); \quad (33)$$

then, there exists $M_L > 0$ such that

$$\|V_{\lambda,f}\|_{L^p(a,b;E)} \leq \frac{M_L}{\sqrt{|\lambda| + 1}} \|\tilde{f}\|_{L^p(a,b;E)} = \frac{M_L}{\sqrt{|\lambda| + 1}} \|f\|_{L^p(a,b;E)}.$$

□

Definition 4.6. We say that a closed linear operator \mathcal{A} on E , has the L^p regularity property on $[a, b]$, if the Cauchy problem

$$\begin{cases} u'(t) = \mathcal{A}u(t) + f(t), & t \in (a, b) \\ u(a) = 0, \end{cases}$$

admits, for any $f \in L^p(a, b; E)$, a unique solution $u_f \in W^{1,p}(a, b; E) \cap L^p(a, b; D(\mathcal{A}))$.

In this case, there exists $K > 0$ such that for any $f \in L^p(a, b; E)$

$$\|u_f\|_{L^p(a,b;E)} + \|u_f'\|_{L^p(a,b;E)} + \|\mathcal{A}u_f\|_{L^p(a,b;E)} \leq K \|f\|_{L^p(a,b;E)}.$$

For details on the L^p regularity property we refer to [11] and [12].

Lemma 4.7. Assume that $G = -L^{1/2}$ has the L^p regularity property on $[a, b]$, and consider $U_{\lambda,f}, V_{\lambda,f}$ defined in (31). Let $\lambda \in S_\varphi$, then:

1. The linear operator $G_\lambda = -(-L + \lambda I)^{1/2}$ has the L^p regularity property on $[a, b]$.
2. For any $f \in L^p(a, b; E)$, $U_{\lambda,f}, V_{\lambda,f} \in W^{1,p}(a, b; E) \cap L^p(a, b; D(G))$, $U_{\lambda,f}$ is the unique solution to

$$\begin{cases} v'(t) = G_\lambda v(t) + f(t), & t \in (a, b) \\ v(a) = 0, \end{cases} \quad (34)$$

and $V_{\lambda,f}$ is the unique solution to

$$\begin{cases} v'(t) = -G_\lambda v(t) + f(t), & t \in (a, b) \\ v(b) = 0. \end{cases}$$

3. There exists $\widetilde{M}_L > 0$ (which does not depend on λ) such that for any $f \in L^p(a, b; E)$ we have

$$\begin{cases} \sqrt{|\lambda| + 1} \|U_{\lambda, f}\|_{L^p(a, b; E)} + \|U'_{\lambda, f}\|_{L^p(a, b; E)} + \|G_{\lambda} U_{\lambda, f}\|_{L^p(a, b; E)} \leq \widetilde{M}_L \|f\|_{L^p(a, b; E)} \\ \sqrt{|\lambda| + 1} \|V_{\lambda, f}\|_{L^p(a, b; E)} + \|V'_{\lambda, f}\|_{L^p(a, b; E)} + \|G_{\lambda} V_{\lambda, f}\|_{L^p(a, b; E)} \leq \widetilde{M}_L \|f\|_{L^p(a, b; E)}. \end{cases}$$

Proof. Let $\lambda \in S_{\varphi}$. We consider T_{λ} , defined in Lemma 4.4, statement 1. and due to (30), we have $G_{\lambda} = G - T_{\lambda}$.

1. Let $f \in L^p(a, b; E)$. Here, we want to show that (34) admits a unique solution in $W^{1,p}(a, b; E) \cap L^p(a, b; D(G))$.

- First, we set $g(\cdot) = e^{(\cdot-a)T_{\lambda}} f(\cdot) \in L^p(a, b; E)$.
- Then we consider $U_{0, g}$ defined by (31) which is the solution to

$$\begin{cases} u'(t) = Gu(t) + g(t), & t \in (a, b) \\ u(a) = 0; \end{cases} \quad (35)$$

but G has the L^p regularity property on $[a, b]$, so

$$U_{0, g} \in W^{1,p}(a, b; E) \cap L^p(a, b; D(G)).$$

- Since $T_{\lambda} \in \mathcal{L}(E)$ and $U_{0, g} \in W^{1,p}(a, b; E) \cap L^p(a, b; D(G))$ we get that

$$v := e^{-(\cdot-a)T_{\lambda}} U_{0, g}, \quad (36)$$

is also in $W^{1,p}(a, b; E) \cap L^p(a, b; D(G))$ with

$$v' = -T_{\lambda} e^{-(\cdot-a)T_{\lambda}} U_{0, g} + e^{-(\cdot-a)T_{\lambda}} U'_{0, g}.$$

So using (35) and the fact that $T_{\lambda} G = G T_{\lambda}$ on $D(G)$ (see Lemma 4.4, statement 2.) we deduce that

$$\begin{aligned} v' &= -T_{\lambda} e^{-(\cdot-a)T_{\lambda}} U_{0, g} + e^{-(\cdot-a)T_{\lambda}} (G U_{0, g} + g) \\ &= (G - T_{\lambda}) e^{-(\cdot-a)T_{\lambda}} U_{0, g} + e^{-(\cdot-a)T_{\lambda}} g. \end{aligned}$$

Finally v satisfies

$$\begin{cases} v'(t) = (G - T_{\lambda}) v(t) + f(t), & t \in (a, b) \\ v(a) = 0. \end{cases}$$

- From Lemma 4.2, statement 5, we have $G_{\lambda} = G - T_{\lambda}$ so $v = e^{-(\cdot-a)T_{\lambda}} U_{0, g}$ is a solution of (34) with the expected regularity. Moreover if w is another solution of (34) then $e^{-(\cdot-a)T_{\lambda}} w$ satisfies (35), so $e^{-(\cdot-a)T_{\lambda}} w = U_{0, g}$ and $w = v$; this proves the uniqueness of the solution of (34).
2. From (31) we have that $U_{\lambda, f}$ is a formal solution of (34); then $U_{\lambda, f} = e^{-(\cdot-a)T_{\lambda}} U_{0, g}$ and has the expected regularity. We use (33) to study $V_{\lambda, f}$.

3. Since G has the L^p regularity property on $[a, b]$, there exists $K > 0$ such for any $h \in L^p(a, b; E)$

$$\|U'_{0,h}\|_{L^p(a,b;E)} + \|GU_{0,h}\|_{L^p(a,b;E)} \leq K \|h\|_{L^p(a,b;E)}.$$

Now let $\lambda \in S_\varphi$. $U_{\lambda,f}$ satisfies (34) so

$$\begin{cases} U'_{\lambda,f}(t) = (G - T_\lambda)U_{\lambda,f}(t) + f(t), & t \in (a, b) \\ U_{\lambda,f}(a) = 0; \end{cases}$$

thus setting $h_\lambda = -T_\lambda U_{\lambda,f} + f$

$$\begin{cases} U'_{\lambda,f}(t) = GU_{\lambda,f}(t) + h_\lambda(t), & t \in (a, b) \\ U_{\lambda,f}(a) = 0, \end{cases}$$

then $U_{\lambda,f} = U_{0,h_\lambda}$ and

$$\begin{aligned} \|U'_{\lambda,f}\|_{L^p(a,b;E)} + \|G_\lambda U_{\lambda,f}\|_{L^p(a,b;E)} &= \|U'_{0,h_\lambda}\|_{L^p(a,b;E)} + \|GU_{0,h_\lambda}\|_{L^p(a,b;E)} \\ &\leq \|T_\lambda\|_{\mathcal{L}(E)} \|U_{\lambda,f}\|_{L^p(a,b;E)} + \|f\|_{L^p(a,b;E)} \\ &\leq C_0 C_L \sqrt{|\lambda|} \frac{M_L}{\sqrt{|\lambda|+1}} \|f\|_{L^p(a,b;E)} + \|f\|_{L^p(a,b;E)} \\ &\leq \widetilde{M}_L \|f\|_{L^p(a,b;E)}. \end{aligned}$$

For the estimate of T_λ , we have used Lemma 4.4, statement 3.

Moreover, using again (33) to study $V_{\lambda,f}$, we obtain the expected result. □

5 Spectral problem (1)-(2): first case

5.1 Preliminary estimates

In this subsection we suppose that X, A, H satisfy (4)~(6). Note that the results of Section 4, can be applied to our operator $-A$, since due to (4), (5), $-A$ is densely defined and from (5) we have that $-A$ is an operator of type φ_0 with bound C_A . For $\lambda \in S_{\varphi_0}$, $-A + \lambda I$ is an operator of type θ (for any $\theta \in (0, \varepsilon(\varphi_0))$); in particular if we set $Q_\lambda = -(-A + \lambda I)^{1/2}$, then from Lemma 4.2, statement 2., Q_λ generates a semigroup $(e^{-tQ_\lambda})_{t \geq 0}$ which is bounded, analytic for $t > 0$ and strongly continuous for $t \geq 0$. Moreover, there exists $K > 0$, such that

$$\forall \lambda \in S_{\varphi_0}, \quad \|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \leq \frac{K}{(1 + |\lambda|)^{1/2}}; \quad (37)$$

furthermore, from Lemma 4.3, statement 3., we have $\overline{B(0, 1/2K)} \subset \rho(Q_\lambda)$, so there exists $\delta > 0$, which does not depend on λ such that $Q_\lambda + \delta I$ generates a bounded analytic semigroup; thus, for some $K_1 \geq 1$

$$\forall \lambda \in S_{\varphi_0}, \quad \forall t \geq 0, \quad \|e^{tQ_\lambda}\|_{\mathcal{L}(X)} \leq K_1 e^{-\delta t}. \quad (38)$$

There exist also $K_0, c_0 > 0$ such that

$$\begin{cases} \forall \lambda \in S_{\varphi_0}, \quad \forall t \geq 1/2, \quad \forall j \in \{0, 1, 2\} : \\ \|Q_\lambda^j e^{tQ_\lambda}\|_{\mathcal{L}(X)} \leq K_0 e^{-2c_0|\lambda|^{1/2}}. \end{cases} \quad (39)$$

Lemma 5.1. There exists a constant $M \geq 0$ independent of $\lambda \in S_{\varphi_0}$, such that for any $\lambda \in S_{\varphi_0}$, operators $I \pm e^{2Q_\lambda}$ are invertible in $\mathcal{L}(X)$ and

$$\forall \lambda \in S_{\varphi_0}, \quad \left\| \left(I \pm e^{2Q_\lambda} \right)^{-1} \right\|_{\mathcal{L}(X)} \leq M, \quad (40)$$

Proof. Let $\lambda \in S_{\varphi_0}$. For $t \geq 0$, we have $\|e^{tQ_\lambda}\|_{\mathcal{L}(X)} \leq K_1 e^{-t\delta}$; we choose $k \in \mathbb{N} \setminus \{0\}$ such that $K_1 e^{-2k\delta} \leq 1/2 < 1$. Then $I - e^{2kQ_\lambda}$ is invertible with

$$\left\| \left(I - e^{2kQ_\lambda} \right)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{1}{1 - 1/2} = 2,$$

thus $0 \in \rho(I - e^{2Q_\lambda})$ since

$$\begin{aligned} I &= (I - e^{2Q_\lambda}) \left(I + e^{2Q_\lambda} + \dots + e^{2(k-1)Q_\lambda} \right) (I - e^{2kQ_\lambda})^{-1} \\ &= (I - e^{2kQ_\lambda})^{-1} \left(I + e^{2Q_\lambda} + \dots + e^{2(k-1)Q_\lambda} \right) (I - e^{2Q_\lambda}). \end{aligned}$$

Moreover

$$\begin{aligned} \left\| \left(I - e^{2Q_\lambda} \right)^{-1} \right\|_{\mathcal{L}(X)} &\leq \left\| \left(I + e^{2Q_\lambda} + \dots + e^{2(k-1)Q_\lambda} \right) (I - e^{2kQ_\lambda})^{-1} \right\|_{\mathcal{L}(X)} \\ &\leq \left(1 + \|e^{2Q_\lambda}\|_{\mathcal{L}(X)} + \dots + \|e^{2(k-1)Q_\lambda}\|_{\mathcal{L}(X)}^{k-1} \right) \left\| (I - e^{2kQ_\lambda})^{-1} \right\|_{\mathcal{L}(X)} \\ &\leq 2K_1^k. \end{aligned}$$

We obtain the same result for $I + e^{2Q_\lambda}$. \square

5.2 Spectral estimates

In this subsection we assume that X, A, H satisfy (4)~(8).

Let $\lambda \in S_{\varphi_0}, \mu \in S_{\varphi_1}$. We recall that $H_\mu = H + \mu I$ and furnish estimates concerning operators Q_λ, H_μ which are easy consequences of our assumptions.

In the following, M denotes various constants, independent of λ, μ , which can vary from line to line.

Lemma 5.2. Let $\lambda \in S_{\varphi_0}, \mu \in S_{\varphi_1}$. Then $(-A + \lambda I) H_\mu^{-1} \in \mathcal{L}(X)$; moreover there exists a constant $M > 0$ independent of $\lambda \in S_{\varphi_0}$ and $\mu \in S_{\varphi_1}$ such that

$$\max \left\{ \|HH_\mu^{-1}\|_{\mathcal{L}(X)}, \|AH_\mu^{-1}\|_{\mathcal{L}(X)} \right\} \leq M, \quad (41)$$

$$\|Q_\lambda^2 H_\mu^{-1}\|_{\mathcal{L}(X)} \leq M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|}, \quad (42)$$

and

$$\|Q_\lambda H_\mu^{-1}\|_{\mathcal{L}(X)} \leq M \frac{1 + |\lambda| + |\mu|}{(1 + |\mu|)(1 + |\lambda|)^{1/2}}. \quad (43)$$

Proof. Note that $(-A + \lambda I)$ is closed, so due to (7), $(-A + \lambda I) H_\mu^{-1}$ is bounded. Then

$$\|HH_\mu^{-1}\|_{\mathcal{L}(X)} = \|(H + \mu I) H_\mu^{-1} - \mu H_\mu^{-1}\| \leq \|I\|_{\mathcal{L}(X)} + \|\mu (H + \mu I)^{-1}\|_{\mathcal{L}(X)} \leq M;$$

moreover

$$\|AH_\mu^{-1}\|_{\mathcal{L}(X)} \leq \|AH^{-1}HH_\mu^{-1}\|_{\mathcal{L}(X)} \leq \|AH^{-1}\|_{\mathcal{L}(X)} \|HH_\mu^{-1}\|_{\mathcal{L}(X)} \leq M,$$

and

$$\begin{aligned} \|Q_\lambda^2 H_\mu^{-1}\|_{\mathcal{L}(X)} &= \|(-A + \lambda I) H_\mu^{-1}\|_{\mathcal{L}(X)} \\ &\leq \|A H_\mu^{-1}\|_{\mathcal{L}(X)} + \|\lambda (H + \mu I)^{-1}\|_{\mathcal{L}(X)} \leq M \left(1 + \frac{|\lambda|}{1 + |\mu|}\right). \end{aligned}$$

Finally, since

$$\|Q_\lambda H_\mu^{-1}\|_{\mathcal{L}(X)} = \|Q_\lambda^{-1} Q_\lambda^2 H_\mu^{-1}\|_{\mathcal{L}(X)} \leq \|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \|Q_\lambda^2 H_\mu^{-1}\|_{\mathcal{L}(X)},$$

we deduce (43) from (41) and (37). \square

For $\lambda \in S_{\varphi_0}, \mu \in S_{\varphi_1}$, let us recall that

$$\Lambda_{\lambda,\mu} := (Q_\lambda - H_\mu) + e^{2Q_\lambda} (Q_\lambda + H_\mu).$$

Note that, since $D(H_\mu) \subset D(Q_\lambda^2)$, we have $D(\Lambda_{\lambda,\mu}) = D(H_\mu) = D(H)$. We now introduce, for $r > 0$, the notation

$$\Omega_{\varphi_0, \varphi_1, r} = \left\{ (\lambda, \mu) \in S_{\varphi_0} \times S_{\varphi_1} : |\lambda| \geq r \text{ and } \frac{|\mu|^2}{|\lambda|} \geq r \right\},$$

and furnish results on $\Lambda_{\lambda,\mu}$.

Lemma 5.3. There exist $r_0 > 0$ and $M > 0$ such that for all $(\lambda, \mu) \in \Omega_{\varphi_0, \varphi_1, r_0}$ we have

$$\begin{cases} 0 \in \rho \left((I - e^{2Q_\lambda})^{-1} (I + e^{2Q_\lambda}) Q_\lambda H_\mu^{-1} - I \right) \\ \left\| \left[(I - e^{2Q_\lambda})^{-1} (I + e^{2Q_\lambda}) Q_\lambda H_\mu^{-1} - I \right]^{-1} \right\|_{\mathcal{L}(X)} \leq 2, \end{cases} \quad (44)$$

$$0 \in \rho(\Lambda_{\lambda,\mu}) \quad \text{and} \quad \|\Lambda_{\lambda,\mu}^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\mu|}, \quad (45)$$

and

$$\|Q_\lambda^2 \Lambda_{\lambda,\mu}^{-1}\|_{\mathcal{L}(X)} \leq M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|}. \quad (46)$$

Note that $Q_\lambda^2 \Lambda_{\lambda,\mu}^{-1}$ has the same behaviour as $Q_\lambda H_\mu^{-1}$, see (43) and (52).

Proof. Let $(\lambda, \mu) \in \Omega_{\varphi_0, \varphi_1, r}$ for some $r > 0$. From (7), we have $Q_\lambda H_\mu^{-1} \in \mathcal{L}(X)$, hence $(I - e^{2Q_\lambda})^{-1} (I + e^{2Q_\lambda}) Q_\lambda H_\mu^{-1} - I \in \mathcal{L}(X)$; moreover, from (40) and (43), we obtain

$$\begin{aligned} &\left\| (I - e^{2Q_\lambda})^{-1} (I + e^{2Q_\lambda}) Q_\lambda H_\mu^{-1} \right\|_{\mathcal{L}(X)} \\ &\leq M \|Q_\lambda H_\mu^{-1}\|_{\mathcal{L}(X)} \\ &\leq M \left(\frac{1 + |\mu|}{(1 + |\mu|)(1 + |\lambda|)^{1/2}} + \frac{|\lambda|}{(1 + |\mu|)(1 + |\lambda|)^{1/2}} \right) \\ &\leq M \left(\frac{1}{|\lambda|^{1/2}} + \frac{|\lambda|^{1/2}}{|\mu|} \right) \\ &\leq \frac{2M}{r^{1/2}}. \end{aligned}$$

So there exists $r_0 > 0$ such that for all $(\lambda, \mu) \in \Omega_{\varphi_0, \varphi_1, r_0}$ we have

$$\left\| \left(I - e^{2Q_\lambda} \right)^{-1} \left(I + e^{2Q_\lambda} \right) Q_\lambda H_\mu^{-1} \right\|_{\mathcal{L}(X)} \leq 1/2. \quad (47)$$

Let $(\lambda, \mu) \in \Omega_{\varphi_0, \varphi_1, r_0}$. Then (47) proves (44). We deduce that

$$L_{\lambda, \mu} := \left(I - e^{2Q_\lambda} \right) \left[\left(I - e^{2Q_\lambda} \right)^{-1} \left(I + e^{2Q_\lambda} \right) Q_\lambda H_\mu^{-1} - I \right] \in \mathcal{L}(X),$$

is boundedly invertible. Moreover

$$L_{\lambda, \mu}^{-1} = \left[\left(I - e^{2Q_\lambda} \right)^{-1} \left(I + e^{2Q_\lambda} \right) Q_\lambda H_\mu^{-1} - I \right]^{-1} \left(I - e^{2Q_\lambda} \right)^{-1},$$

satisfies

$$\left\| L_{\lambda, \mu}^{-1} \right\|_{\mathcal{L}(X)} \leq 2M.$$

Now, we write $\Lambda_{\lambda, \mu} = \left(I + e^{2Q_\lambda} \right) Q_\lambda - \left(I - e^{2Q_\lambda} \right) H_\mu = L_{\lambda, \mu} H_\mu$, so $\Lambda_{\lambda, \mu}$ is boundedly invertible with

$$\Lambda_{\lambda, \mu}^{-1} = H_\mu^{-1} L_{\lambda, \mu}^{-1};$$

this furnishes (45). Finally, $\left\| L_{\lambda, \mu}^{-1} \right\|_{\mathcal{L}(X)} \leq 2M$ and (42) gives

$$\left\| Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} \right\|_{\mathcal{L}(X)} = \left\| Q_\lambda^2 H_\mu^{-1} \right\|_{\mathcal{L}(X)} \left\| L_{\lambda, \mu}^{-1} \right\|_{\mathcal{L}(X)} \leq M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|}.$$

□

Lemma 5.4. Assume (4)~(6), let $f \in L^p(0, 1; X)$ with $1 < p < +\infty$ and set for $x \in [0, 1]$

$$I_{\lambda, f}(x) = \frac{1}{2} Q_\lambda^{-1} \int_0^x e^{(x-s)Q_\lambda} f(s) ds \quad \text{and} \quad J_{\lambda, f}(x) = \frac{1}{2} Q_\lambda^{-1} \int_x^1 e^{(s-x)Q_\lambda} f(s) ds; \quad (48)$$

then, there exists $M \geq 0$ (independent of λ and f) such that

$$\left\| Q_\lambda I_{\lambda, f}(1) \right\| \leq M \|f\|_{L^p(0, 1; X)} \quad \text{and} \quad \left\| Q_\lambda J_{\lambda, f}(0) \right\| \leq M \|f\|_{L^p(0, 1; X)}, \quad (49)$$

moreover $I_{\lambda, f}, J_{\lambda, f} \in W^{2, p}(0, 1; X) \cap L^p(0, 1; D(A))$ with

$$\left\| Q_\lambda^2 I_{\lambda, f} \right\|_{L^p(0, 1; X)} \leq M \|f\|_{L^p(0, 1; X)} \quad \text{and} \quad \left\| Q_\lambda^2 J_{\lambda, f} \right\|_{L^p(0, 1; X)} \leq M \|f\|_{L^p(0, 1; X)}.$$

Proof. From (38), we have

$$\begin{cases} \left\| Q_\lambda I_{\lambda, f}(1) \right\| \leq \int_0^1 \left\| e^{(1-s)Q_\lambda} f(s) \right\| ds \leq M \int_0^1 \|f(s)\| ds \leq M \|f\|_{L^p(0, 1; X)} \\ \left\| Q_\lambda J_{\lambda, f}(0) \right\| \leq \int_0^1 \left\| e^{sQ_\lambda} f(s) \right\| ds \leq M \int_0^1 \|f(s)\| ds \leq M \|f\|_{L^p(0, 1; X)}. \end{cases}$$

We apply Lemma 4.7 with $E = X, L = -A, G_\lambda = Q_\lambda, a = 0, b = 1$ so that

$$I_{\lambda, f} = \frac{1}{2} Q_\lambda^{-1} U_{\lambda, f} \quad \text{and} \quad J_{\lambda, f} = \frac{1}{2} Q_\lambda^{-1} V_{\lambda, f};$$

then $Q_\lambda^2 I_{\lambda, f} = \frac{1}{2} Q_\lambda U_{\lambda, f}$ and $Q_\lambda^2 J_{\lambda, f} = \frac{1}{2} Q_\lambda V_{\lambda, f}$ have the desired estimates. □

Lemma 5.5. Assume (4)~(6) and let $f \in L^p(0, 1; X)$ with $1 < p < +\infty$. Set for $x \in [0, 1]$

$$v_0(x) = e^{xQ_\lambda} J_{\lambda, f}(0) \quad \text{and} \quad v_1(x) = e^{xQ_\lambda} I_{\lambda, f}(1).$$

where $I_{\lambda, f}$ and $J_{\lambda, f}$ are given by (48). Moreover, there exists $M > 0$ (independent of λ and f) such that

$$\left\| Q_\lambda^2 v_j \right\|_{L^p(0, 1; X)} \leq M \|f\|_{L^p(0, 1; X)}, \quad j = 0, 1. \quad (50)$$

Proof. From [17], Proposition 3.5, p. 1676, we have

$$J_{\lambda, f}(0) \in (D(Q_\lambda^2), X)_{\frac{1}{2p}, p} = (D(Q_\lambda), X)_{1+\frac{1}{p}, p};$$

so for $x \in (0, 1]$, we can write

$$\begin{aligned} Q_\lambda^2 v_0(x) &= e^{xQ_\lambda} Q_\lambda \int_0^1 e^{sQ_\lambda} f(s) ds \\ &= Q_\lambda \int_0^x e^{(x-s)Q_\lambda} e^{2sQ_\lambda} f(s) ds + e^{2xQ_\lambda} Q_\lambda \int_x^1 e^{(s-x)Q_\lambda} f(s) ds \\ &= 2Q_\lambda^2 I_{\lambda, g}(x) + 2e^{2xQ_\lambda} Q_\lambda^2 J_{\lambda, f}(x), \end{aligned}$$

with $g = e^{2 \cdot Q_\lambda} f(\cdot)$. From Lemma 5.4, statement 1. and (38), we have

$$\begin{cases} \left\| e^{2 \cdot Q_\lambda} Q_\lambda^2 J_{\lambda, f}(\cdot) \right\|_{L^p(0, 1; X)} \leq M \|Q_\lambda^2 J_{\lambda, f}(\cdot)\|_{L^p(0, 1; X)} \leq M \|f\|_{L^p(0, 1; X)} \\ \left\| Q_\lambda^2 I_{\lambda, g} \right\|_{L^p(0, 1; X)} \leq M \|g\|_{L^p(0, 1; X)} \leq M \|f\|_{L^p(0, 1; X)}, \end{cases}$$

from which we deduce $\|Q_\lambda^2 v_0\|_{L^p(0, 1; X)} \leq M \|f\|_{L^p(0, 1; X)}$.

The same estimate runs for v_1 since

$$v_1 = e^{Q_\lambda} J_{\lambda, f(1-\cdot)}(0) \quad \text{and} \quad \|f(1-\cdot)\|_{L^p(0, 1; X)} = \|f\|_{L^p(0, 1; X)}.$$

□

5.3 Proofs to Theorem 2.1 and Theorem 2.2

Let r_0 fixed as in Lemma 5.3.

5.3.1 Proof to Theorem 2.1

We apply Proposition 3.7, with A, H, Q and Λ replaced by

$$A - \lambda I, H + \mu I, Q_\lambda \text{ and } \Lambda_{\lambda, \mu},$$

since in this case Problem (16) becomes Problem (1)-(2). So, it is enough to verify that (4)~(8) imply $(H_1) \sim (H_5)$.

It is clear that (4), (5), (6) imply $(H_1), (H_2), (H_3)$ mentioned in section 3. Moreover, due to (45), assumptions (4)~(7) imply (H_4) . Finally, under (4)~(7)

$$\Lambda_{\lambda, \mu}^{-1}(X) \subset D(Q) \cap D(H) \subset D(Q^2),$$

so that $Q\Lambda_{\lambda, \mu}^{-1}(X) \subset D(Q)$ and then (H_5) is satisfied.

Note that here, the condition $\Lambda_{\lambda, \mu}^{-1} d_0 \in (D(A), X)_{\frac{1}{2p}, p}$ is automatically realized since for any $d_0 \in X$, we have $\Lambda_{\lambda, \mu}^{-1} d_0 \in D(Q) \cap D(H) \subset D(Q^2)$.

5.3.2 Proof to Theorem 2.2

Let $(\lambda, \mu) \in \Omega_{\varphi_0, \varphi_1, r_0}$ and $f \in L^p(0, 1; X)$. We recall that, taking into account the notations (48), we have, for $x \in [0, 1]$

$$u(x) = S_\lambda(x) \mu_0 + S_\lambda(1-x) \mu_1 + I_{\lambda, f}(x) + J_{\lambda, f}(x),$$

where

$$\begin{cases} \mu_1 = u_1 - I_{\lambda, f}(1) \\ \mu_0 = \Lambda_{\lambda, \mu}^{-1} \left[(I - e^{2Q_\lambda}) d_0 + 2Q_\lambda e^{Q_\lambda} \mu_1 + 2Q_\lambda J_{\lambda, f}(0) \right] - J_{\lambda, f}(0) \\ S_\lambda(x) = (I - e^{2Q_\lambda})^{-1} (e^{xQ_\lambda} - e^{(1-x)Q_\lambda} e^{Q_\lambda}) \in \mathcal{L}(X). \end{cases}$$

So we can write $u = h_0 + h_1 - h_2 + h_3 + h_4$ with

$$\begin{cases} h_0(x) = S_\lambda(x) \Lambda_{\lambda, \mu}^{-1} \left[(I - e^{2Q_\lambda}) d_0 + 2Q_\lambda (J_{\lambda, f}(0) - e^{Q_\lambda} I_{\lambda, f}(1)) \right] \\ h_1(x) = 2S_\lambda(x) \Lambda_{\lambda, \mu}^{-1} Q_\lambda e^{Q_\lambda} u_1 \\ h_2(x) = S_\lambda(x) J_{\lambda, f}(0) + S_\lambda(1-x) I_{\lambda, f}(1) \\ h_3(x) = S_\lambda(1-x) u_1 \\ h_4(x) = I_{\lambda, f}(x) + J_{\lambda, f}(x). \end{cases} \quad (51)$$

Estimate of $Q_\lambda^2 h_0$. For $\xi \in X$ and $x \in (0, 1)$, we have

$$\begin{aligned} \left\| Q_\lambda^2 S_\lambda(x) \Lambda_{\lambda, \mu}^{-1} \xi \right\| &= \left\| (I - e^{2Q_\lambda})^{-1} (I - e^{2(1-x)Q_\lambda}) e^{xQ_\lambda} Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} \xi \right\| \\ &\leq \left\| (I - e^{2Q_\lambda})^{-1} (I - e^{2(1-x)Q_\lambda}) \right\|_{\mathcal{L}(X)} \left\| e^{xQ_\lambda} \right\|_{\mathcal{L}(X)} \left\| Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} \right\|_{\mathcal{L}(X)} \|\xi\| \\ &\leq M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|} \|\xi\|, \end{aligned}$$

so, from (46) and (49), we deduce

$$\begin{aligned} \left\| Q_\lambda^2 h_0(x) \right\| &\leq M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|} \left(\|d_0\| + 2 \|Q_\lambda J_{\lambda, f}(0)\| + 2 \left\| e^{Q_\lambda} \right\|_{\mathcal{L}(X)} \|Q_\lambda I_{\lambda, f}(1)\| \right) \\ &\leq M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|} \left(\|d_0\| + \|f\|_{L^p(0,1;X)} \right). \end{aligned}$$

Then

$$\left\| Q_\lambda^2 h_0 \right\|_{L^p(0,1;X)} \leq M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|} \left(\|d_0\| + \|f\|_{L^p(0,1;X)} \right).$$

Estimate of $Q_\lambda^2 h_1$. As above, we have for $\xi \in X$ and $x \in (0, 1)$

$$\left\| Q_\lambda^2 h_1(x) \right\| \leq M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|} \left(\left\| Q_\lambda e^{Q_\lambda} \right\|_{\mathcal{L}(X)} \|u_1\| \right),$$

and from (39), we deduce that $\|Q_\lambda^2 h_1(x)\| \leq M \|u_1\|$, hence

$$\left\| Q_\lambda^2 h_1 \right\|_{L^p(0,1;X)} \leq M \|u_1\|.$$

Estimate of $Q_\lambda^2 h_2$. For $\xi \in X$ and $x \in (0, 1]$, we have

$$\begin{aligned} \left\| Q_\lambda^2 S_\lambda(x) \xi \right\| &= \left\| \left(I - e^{2Q_\lambda} \right)^{-1} \left(I - e^{2(1-x)Q_\lambda} \right) Q_\lambda^2 e^{xQ_\lambda} \xi \right\| \\ &\leq \left\| \left(I - e^{2Q_\lambda} \right)^{-1} \left(I - e^{2(1-x)Q_\lambda} \right) \right\|_{\mathcal{L}(X)} \left\| Q_\lambda^2 e^{xQ_\lambda} \xi \right\| \\ &\leq M \left\| Q_\lambda^2 e^{xQ_\lambda} \xi \right\|, \end{aligned} \quad (52)$$

so

$$\left\| Q_\lambda^2 h_2(x) \right\| \leq M \left(\left\| Q_\lambda^2 e^{xQ_\lambda} J_{\lambda,f}(0) \right\| + \left\| Q_\lambda^2 e^{(1-x)Q_\lambda} I_{\lambda,f}(1) \right\| \right),$$

and then, from (50)

$$\left\| Q_\lambda^2 h_2 \right\|_{L^p(0,1;X)} \leq M \left(\left\| Q^2 v_0 \right\|_{L^p(0,1;X)} + \left\| Q^2 v_1(1-\cdot) \right\|_{L^p(0,1;X)} \right) \leq M \|f\|_{L^p(0,1;X)}.$$

Estimate of $Q_\lambda^2 h_3$. Due to (52), we have, for $x \in [0, 1]$

$$\left\| Q_\lambda^2 h_3(x) \right\| \leq M \left\| Q_\lambda^2 e^{(1-x)Q_\lambda} u_1 \right\|.$$

From Theorem 2.1 in [13], since $u_1 \in (D(A), X)_{\frac{1}{2p}, p} = (X, D(A))_{1-\frac{1}{2p}, p}$, we get

$$\left\| Q_\lambda^2 h_3 \right\|_{L^p(0,1;X)} \leq M \left\| Q_\lambda^2 e^{(1-x)Q_\lambda} u_1 \right\|_{L^p(0,1;X)} \leq M \left(\|u_1\|_{(D(A), X)_{\frac{1}{2p}, p}} + |\lambda|^{1-\frac{1}{2p}} \|u_1\| \right).$$

Estimate of $Q_\lambda^2 h_4$. From Lemma 5.4, we get

$$\left\| Q_\lambda^2 h_4 \right\|_{L^p(0,1;X)} \leq M \|f\|_{L^p(0,1;X)}.$$

Summarizing the previous study we obtain that

$$\left\| Q_\lambda^2 u \right\|_{L^p(0,1;X)} \leq M \alpha(d_0, u_1, \lambda, \mu, f). \quad (53)$$

Moreover, since u satisfies (1), that is,

$$u''(x) - Q_\lambda^2 u(x) = f(x), \quad \text{a.e. } x \in (0, 1),$$

we deduce that

$$\|u''\|_{L^p(0,1;X)} \leq M \alpha(d_0, u_1, \lambda, \mu, f).$$

Writing $u = Q_\lambda^{-2} Q_\lambda^2 u$ and $Q_\lambda u = Q_\lambda^{-1} Q_\lambda^2 u$, we obtain the estimates concerning u and $Q_\lambda u$. Setting, for $x \in [0, 1]$

$$\tilde{S}_\lambda(x) = \left(I - e^{2Q_\lambda} \right)^{-1} \left(e^{xQ_\lambda} + e^{(1-x)Q_\lambda} e^{Q_\lambda} \right) \in \mathcal{L}(X),$$

we have

$$u'(x) = Q_\lambda \tilde{S}_\lambda(x) \mu_0 - Q_\lambda \tilde{S}_\lambda(1-x) \mu_1 + Q I_{\lambda,f}(x) - Q J_{\lambda,f}(x) = Q_\lambda^{-1} Q_\lambda^2 \omega(x),$$

the terms in $\omega(x) = \tilde{S}_\lambda(x) \mu_0 - \tilde{S}_\lambda(1-x) \mu_1 + I_{\lambda,f}(x) - J_{\lambda,f}(x)$ are (in absolute value) those of $u(x)$, so (53) runs when we replace u by ω , this furnishes the estimate for u' .

From Lemma 2.6 a) p. 103 in [13] we have

$$\left\| Q Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} = \left\| (-A)^{1/2} (-A + \lambda I)^{-1/2} \right\|_{\mathcal{L}(X)} \leq M; \quad (54)$$

so writing $Qu = Q Q_\lambda^{-1} Q_\lambda u$, $Q^2 u = \left(Q Q_\lambda^{-1} \right)^2 Q_\lambda^2 u$, we deduce the estimates of $\|Qu\|_{L^p(0,1;X)}$ and $\|Q^2 u\|_{L^p(0,1;X)}$ from those of $\|Q_\lambda u\|_{L^p(0,1;X)}$ and $\|Q_\lambda^2 u\|_{L^p(0,1;X)}$.

Remark 5.6. Under the assumptions of the previous theorem, we obtain moreover that

$$\|u(0)\| \leq \frac{M}{1+|\mu|} \left(\|d_0\| + Me^{-2c_0|\lambda|^{1/2}} \|u_1\| + \|f\|_{L^p(0,1;X)} \right). \quad (55)$$

Indeed

$$u(0) = \Lambda_{\lambda,\mu}^{-1} \left[(I - e^{2Q\lambda}) d_0 + 2Q_\lambda e^{Q\lambda} (u_1 - I_{\lambda,f}(1)) + 2Q_\lambda J_{\lambda,f}(0) \right],$$

so

$$\begin{aligned} \|u(0)\| &\leq \left\| \Lambda_{\lambda,\mu}^{-1} \right\|_{\mathcal{L}(X)} \left\| I - e^{2Q\lambda} \right\|_{\mathcal{L}(X)} \|d_0\| + \left\| \Lambda_{\lambda,\mu}^{-1} \right\|_{\mathcal{L}(X)} \left\| 2Q_\lambda e^{Q\lambda} \right\|_{\mathcal{L}(X)} \|u_1\| \\ &\quad + 2 \left\| \Lambda_{\lambda,\mu}^{-1} \right\|_{\mathcal{L}(X)} \left(\left\| e^{Q\lambda} \right\|_{\mathcal{L}(X)} \|Q_\lambda I_{\lambda,f}(1)\| + \|Q_\lambda J_{\lambda,f}(0)\| \right) \\ &\leq \frac{M}{1+|\mu|} \left(\|d_0\| + Me^{-2c_0|\lambda|^{1/2}} \|u_1\| + \|f\|_{L^p(0,1;X)} \right). \end{aligned}$$

5.3.3 Proof to Theorem 2.3

Remark 5.7. Let $(f, \tau) \in Z$. We consider the following problem

$$\begin{cases} u'' + \mathcal{A}u - \lambda u = f \\ u'(0) - Hu(0) - (\lambda + \mu)u(0) = \tau \\ u(1) = 0; \end{cases} \quad (56)$$

then the two following assertions are equivalent:

1. $(u, v) \in D(\mathcal{P}_{A,H,\mu})$ and $(\mathcal{P}_{A,H,\mu} - \lambda I)(u, v) = (f, \tau)$.
2. $u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A))$ is a classical solution of (56) together with $v = u(0)$.

So to study $\mathcal{P}_{A,H,\mu}$, it remains to solve (56).

We set $\varphi_2 := \min\{\varphi_0, \varphi_1\}$ and define for $\varphi_3 \in (0, \pi - \varphi_2)$, $r_{\varphi_3} \in (r_0, +\infty)$ by

$$r_{\varphi_3} := \frac{r_0}{\cos^2\left(\frac{\varphi_2 + \varphi_3}{2}\right)},$$

for $r_0 > 0$, see Theorem 2.1.

Proposition 5.8. Let $\varphi_3 \in (0, \pi - \varphi_2)$.

1. If $\lambda \in S_{\varphi_2}$, $\mu \in S_{\varphi_3}$ with $|\lambda| \geq r_{\varphi_3}$, then $(\lambda, \lambda + \mu) \in \Omega_{\varphi_0, \varphi_1, r_0}$.
2. Let $\mu \in S_{\varphi_3}$. Then $\mathcal{P}_{A,H,\mu}$ is a closed linear operator on Z with

$$S_{\varphi_2} \setminus B(0, r_{\varphi_3}) \subset \rho(\mathcal{P}_{A,H,\mu}).$$

Moreover, let $\lambda \in S_{\varphi_2} \setminus B(0, r_{\varphi_3})$ and $(f, \tau) \in Z$; then

$$(u, v) = (\mathcal{P}_{A,H,\mu} - \lambda I)^{-1}(f, \tau),$$

satisfies, for $x \in [0, 1]$

$$\begin{cases} u(x) = S_\lambda(x) \Lambda_{\lambda, \lambda + \mu}^{-1} (I - e^{2Q\lambda}) \tau \\ \quad + S_\lambda(x) \left[2\Lambda_{\lambda, \lambda + \mu}^{-1} Q_\lambda [J_{\lambda,f}(0) - e^{Q\lambda} I_{\lambda,f}(1)] - J_{\lambda,f}(0) \right] \\ \quad - S_\lambda(1-x) I_{\lambda,f}(1) + I_{\lambda,f}(x) + J_{\lambda,f}(x) \\ v(x) = u(0), \end{cases}$$

where $S_\lambda(x) = (I - e^{2Q\lambda})^{-1} (e^{xQ\lambda} - e^{(1-x)Q\lambda} e^{Q\lambda}) \in \mathcal{L}(X)$.

3. There exists $M_{A,H,\varphi_3} > 0$ such that for $\lambda \in S_{\varphi_2} \setminus B(0, r_{\varphi_3})$ and $\mu \in S_{\varphi_3}$ we have

$$\|(\mathcal{P}_{A,H,\mu} - \lambda I)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{M_{A,H,\varphi_3}}{1 + |\lambda|}.$$

Proof.

1. We have $(\lambda, \lambda + \mu) \in S_{\varphi_2} \times S_{\varphi_2} \subset S_{\varphi_0} \times S_{\varphi_1}$, moreover $|\lambda| \geq r_{\varphi_3} > r_0$ and, due to [13], Lemma 2.3, p. 98, we have

$$\frac{|\lambda + \mu|^2}{|\lambda|} \geq \cos^2\left(\frac{\varphi_2 + \varphi_3}{2}\right) \frac{(|\lambda| + |\mu|)^2}{|\lambda|} \geq \cos^2\left(\frac{\varphi_2 + \varphi_3}{2}\right) \times |\lambda| \geq r_0.$$

2. It is a consequence of statement 1. and Theorem 2.1.

3. As in statement 1., we have, $|\lambda + \mu| \geq \cos\left(\frac{\varphi_2 + \varphi_3}{2}\right) \times |\lambda|$, so setting

$$C_{\varphi_3} := \frac{1}{\cos\left(\frac{\varphi_2 + \varphi_3}{2}\right)} + 1,$$

we have

$$\frac{1 + |\lambda| + |\lambda + \mu|}{1 + |\lambda + \mu|} \leq \frac{|\lambda|}{1 + |\lambda + \mu|} + \frac{1 + |\lambda + \mu|}{1 + |\lambda + \mu|} \leq C_{\varphi_3}.$$

Let $(f, \tau) \in Z$, then Theorem 2.2 and (55) imply that $(u, v) = (\mathcal{P}_{A,H,\mu} - \lambda I)^{-1}(f, \tau)$, satisfies

$$\|u\|_{L^p(0,1;X)} \leq \frac{MC_{\varphi_3}}{1 + |\lambda|} (\|\tau\| + \|f\|_Y) \quad \text{and} \quad v = \|u(0)\| \leq \frac{MC_{\varphi_3}}{1 + |\lambda|} (\|\tau\| + \|f\|_Y),$$

that is

$$\|(\mathcal{P}_{A,H,\mu} - \lambda I)^{-1}(f, \tau)\|_Z \leq \frac{MC_{\varphi_3}}{1 + |\lambda|} \|(f, \tau)\|_Z.$$

□

The proof of Theorem 2.3 is given by Statement 3 of the previous proposition.

6 Spectral problem (1)-(2): second case

In all this section we suppose that X, A, H satisfy (4)~(6) and (9)~(11).

Note that the results of the previous section obtained under assumption (4)~(6) can be used here, in particular results of subsection 5.1, Lemma 5.4, Lemma 5.5 and also estimate (54).

6.1 Spectral estimates

Let $\lambda \in S_{\varphi_0}, \mu \in \mathbb{C}$. Recall that $H_\mu = H + \mu I$ and $Q_\lambda = -(-A + \lambda I)^{1/2}$. We first furnish estimates concerning operators Q_λ, H_μ which are easy consequences of our assumptions.

Again, in the following M denotes various constants, independent of λ, μ , which can vary from one line to another.

Lemma 6.1. Let $\lambda \in S_{\varphi_0}, \mu \in \mathbb{C}$. Then $H_\mu Q_\lambda^{-1} \in \mathcal{L}(X)$, moreover there exists a constant $M > 0$ independent of λ and μ such that

$$\|H_\mu Q_\lambda^{-1}\|_{\mathcal{L}(X)} \leq M \frac{1 + |\mu|}{(1 + |\lambda|)^\varepsilon}. \quad (57)$$

Proof. From (54), we deduce

$$\begin{aligned}
\|Q_{|\lambda}Q_\lambda^{-1}\|_{\mathcal{L}(X)} &= \|(-A + |\lambda|I)Q^{-1}Q_{|\lambda}^{-1}QQ_\lambda^{-1}\|_{\mathcal{L}(X)} \\
&\leq \|AQ^{-1}Q_{|\lambda}^{-1}QQ_\lambda^{-1}\|_{\mathcal{L}(X)} + \||\lambda|Q^{-1}Q_{|\lambda}^{-1}QQ_\lambda^{-1}\|_{\mathcal{L}(X)} \\
&\leq \|QQ_{|\lambda}^{-1}\|_{\mathcal{L}(X)}\|QQ_\lambda^{-1}\|_{\mathcal{L}(X)} + |\lambda|\|Q_{|\lambda}^{-1}\|_{\mathcal{L}(X)}\|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \\
&\leq M,
\end{aligned}$$

and, from (10)

$$\|HQ_\lambda^{-1}\|_{\mathcal{L}(X)} \leq \|HQ_{|\lambda}^{-1}\|_{\mathcal{L}(X)}\|Q_{|\lambda}Q_\lambda^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{(1+|\lambda|)^\varepsilon}.$$

Moreover

$$\begin{aligned}
\|H_\mu Q_\lambda^{-1}\|_{\mathcal{L}(X)} &\leq \|HQ_\lambda^{-1}\|_{\mathcal{L}(X)} + |\mu|\|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \\
&\leq \frac{M}{(1+|\lambda|)^\varepsilon} + \frac{1+|\mu|}{(1+|\lambda|)^{1/2}} \leq M \frac{1+|\mu|}{(1+|\lambda|)^\varepsilon}.
\end{aligned}$$

□

Remark 6.2. From (57), assumption (10) can be written as

$$\exists \varepsilon \in (0, 1/2], \exists C_{H,Q} > 0, \forall \lambda \in S_{\varphi_0} : \|HQ_\lambda^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_{H,Q}}{(1+|\lambda|)^\varepsilon}.$$

We now introduce the notation : for $\rho > 0$

$$\Pi_{\varphi_0, \rho} = \left\{ (\lambda, \mu) \in S_{\varphi_0} \times \mathbb{C} : |\lambda| \geq \rho \text{ and } \frac{|\lambda|}{|\mu|^{1/\varepsilon}} \geq \rho \right\},$$

where we have set $\frac{|\lambda|}{|\mu|^{1/\varepsilon}} = +\infty$ for $\mu = 0$ and furnished results (see below) on

$$\Lambda_{\lambda, \mu} = (Q_\lambda - H_\mu) + e^{2Q_\lambda}(Q_\lambda + H_\mu),$$

where $\lambda \in S_{\varphi_0}, \mu \in \mathbb{C}$.

Lemma 6.3. There exist $\rho_0 > 0$ and $M > 0$ such that for all $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$:

$$\max \left\{ \|H_\mu Q_\lambda^{-1}\|_{\mathcal{L}(X)}, \left\| (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \right\} \leq 1/2, \quad (58)$$

$$\begin{cases} 0 \in \rho \left(I - (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \right) \\ \left\| \left[I - (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \right]^{-1} \right\|_{\mathcal{L}(X)} \leq 2, \end{cases} \quad (59)$$

$$0 \in \rho(\Lambda_{\lambda, \mu}) \quad \text{and} \quad \|\Lambda_{\lambda, \mu}^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{(1+|\lambda|)^{1/2}}, \quad (60)$$

$$\|Q_\lambda \Lambda_{\lambda, \mu}^{-1}\|_{\mathcal{L}(X)} \leq M, \quad (61)$$

and

$$\left\{ \begin{array}{l} 0 \in \rho(Q_\lambda - H_\mu) \\ \|(Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{(1 + |\lambda|)^{1/2}} \\ \|Q_\lambda (Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \leq M \\ \|(Q_\lambda + H_\mu)(Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \leq M \\ \|e^{2Q_\lambda} (Q_\lambda + H_\mu)(Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \leq 1/2. \end{array} \right. \quad (62)$$

Proof. Let $\rho > 0$ and $(\lambda, \mu) \in \Pi_{\varphi_0, \rho}$. Then

$$I - (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \in \mathcal{L}(X),$$

and from (57) together with Lemma 5.1

$$\begin{aligned} \max \left\{ \|H_\mu Q_\lambda^{-1}\|_{\mathcal{L}(X)}, \left\| (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \right\} &\leq M \|H_\mu Q_\lambda^{-1}\|_{\mathcal{L}(X)} \\ &\leq M \frac{1 + |\mu|}{(1 + |\lambda|)^\varepsilon} \\ &\leq M \left(\left(\frac{1}{|\lambda|} \right)^\varepsilon + \left(\frac{|\mu|^{1/\varepsilon}}{|\lambda|} \right)^\varepsilon \right). \end{aligned}$$

So there exists $\rho_0 > 0$ such that for all $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$: (58) and (59) hold. Now, let $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$. We deduce that

$$\begin{aligned} \Lambda_{\lambda, \mu} &= (I + e^{2Q_\lambda}) Q_\lambda - (I - e^{2Q_\lambda}) H_\mu \\ &= (I + e^{2Q_\lambda}) \left[I - (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \right] Q_\lambda, \end{aligned}$$

is boundedly invertible with

$$\left\{ \begin{array}{l} \Lambda_{\lambda, \mu}^{-1} = Q_\lambda^{-1} \left[I - (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \right]^{-1} (I + e^{2Q_\lambda})^{-1} \\ Q_\lambda \Lambda_{\lambda, \mu}^{-1} = \left[I - (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \right]^{-1} (I + e^{2Q_\lambda})^{-1}, \end{array} \right.$$

so

$$\begin{aligned} \|\Lambda_{\lambda, \mu}^{-1}\|_{\mathcal{L}(X)} &\leq M \|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \left\| I - (I + e^{2Q_\lambda})^{-1} (I - e^{2Q_\lambda}) H_\mu Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \|I + e^{2Q_\lambda}\|_{\mathcal{L}(X)} \\ &\leq \frac{M}{(1 + |\lambda|)^{1/2}}, \end{aligned}$$

and $\|Q_\lambda \Lambda_{\lambda, \mu}^{-1}\|_{\mathcal{L}(X)} \leq M$. Moreover, from (58), $Q_\lambda - H_\mu = (I - H_\mu Q_\lambda^{-1}) Q_\lambda$ is boundedly invertible with

$$(Q_\lambda - H_\mu)^{-1} = Q_\lambda^{-1} (I - H_\mu Q_\lambda^{-1})^{-1},$$

so

$$\left\{ \begin{array}{l} \|(Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \leq \|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \left\| (I - H_\mu Q_\lambda^{-1})^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{M}{(1 + |\lambda|)^{1/2}} \\ \|Q_\lambda (Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} = \left\| (I - H_\mu Q_\lambda^{-1})^{-1} \right\|_{\mathcal{L}(X)} \leq M, \end{array} \right.$$

and

$$\begin{aligned} \left\| (Q_\lambda + H_\mu) (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} &= \left\| (-Q_\lambda + H_\mu + 2Q_\lambda) (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \\ &\leq 1 + 2 \left\| Q_\lambda (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \\ &\leq M. \end{aligned}$$

Finally

$$\begin{aligned} \left\| e^{2Q_\lambda} (Q_\lambda + H_\mu) (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} &\leq \left\| e^{2Q_\lambda} \right\|_{\mathcal{L}(X)} \left\| (Q_\lambda + H_\mu) (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \\ &\leq M \left\| e^{2Q_\lambda} \right\|_{\mathcal{L}(X)}, \end{aligned}$$

and due to (39), we can eventually increase ρ_0 , for $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$, which implies that $|\lambda| \geq \rho_0$, in order to have $M \left\| e^{2Q_\lambda} \right\|_{\mathcal{L}(X)} \leq 1/2$. \square

The proof of the following Lemma will use (15), which is equivalent to (12) from Remark 2.10, to study, for a given $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$, operator $Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}$.

Lemma 6.4. Assume (5), (9), (10) and (12). Fix $(\lambda_1, \mu_1) \in \Pi_{\varphi_0, \rho_0}$. Then, there exists $M > 0$ such that for any $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$, we have

1. $(Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} = (Q_{\lambda_1} - H_{\mu_1})^{-1} Q^{-1} P_{\lambda, \mu}$, where $P_{\lambda, \mu} \in \mathcal{L}(X)$ with

$$\|P_{\lambda, \mu}\|_{\mathcal{L}(X)} \leq M.$$

2. $Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \in \mathcal{L}(X)$ with

$$\left\| Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \leq M. \quad (63)$$

3. There exists $W_{\lambda, \mu} \in \mathcal{L}(X)$ such that

$$\Lambda_{\lambda, \mu}^{-1} = (Q_\lambda - H_\mu)^{-1} \left(I + e^{2Q_\lambda} W_{\lambda, \mu} \right), \quad (64)$$

with

$$\|W_{\lambda, \mu}\|_{\mathcal{L}(X)} \leq M \quad \text{and} \quad \left\| Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \leq M. \quad (65)$$

Proof. Let $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$.

1. We have

$$\begin{aligned} (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} &= (Q_{\lambda_1} - H_{\mu_1})^{-1} (Q_{\lambda_1} - H_{\mu_1}) (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \\ &= (Q_{\lambda_1} - H_{\mu_1})^{-1} [Q_\lambda - H_\mu + (\mu - \mu_1) I] (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \\ &\quad + (Q_{\lambda_1} - H_{\mu_1})^{-1} (Q_{\lambda_1} - Q_\lambda) (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \\ &= (Q_{\lambda_1} - H_{\mu_1})^{-1} \left[Q_\lambda^{-1} + (\mu - \mu_1) (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right] \\ &\quad + (Q_{\lambda_1} - H_{\mu_1})^{-1} (Q_{\lambda_1} - Q_\lambda) (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}, \end{aligned}$$

but, $Q_{\lambda_1} - Q_\lambda = (Q - Q_\lambda) - (Q - Q_{\lambda_1})$ and from Lemma 4.4, there exists $T_{\lambda, \lambda_1} \in \mathcal{L}(X)$ such that $Q_{\lambda_1} = Q_\lambda + T_{\lambda, \lambda_1}$

$$\|T_{\lambda, \lambda_1}\|_{\mathcal{L}(X)} \leq M \left(1 + \sqrt{|\lambda|} \right) \quad \text{and} \quad Q^{-1} T_{\lambda, \lambda_1} = T_{\lambda, \lambda_1} Q^{-1}, \quad (66)$$

so

$$(Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} = (Q_{\lambda_1} - H_{\mu_1})^{-1} Q^{-1} P_{\lambda, \mu},$$

where $P_{\lambda, \mu} \in \mathcal{L}(X)$ is defined by

$$P_{\lambda, \mu} = Q Q_\lambda^{-1} \left[I + (\mu_1 - \mu) Q_\lambda (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} + T_{\lambda, \lambda_1} Q_\lambda (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right].$$

Moreover, using (37), (54), (66) and (62)

$$\begin{aligned} \|P_{\lambda, \mu}\|_{\mathcal{L}(X)} &\leq M \left[1 + (|\mu - \mu_1|) \|Q_\lambda (Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \right] \\ &\quad + M \left[\|T_{\lambda, \lambda_1}\|_{\mathcal{L}(X)} \|Q_\lambda (Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \right] \\ &\leq M \left[1 + \frac{|\mu - \mu_1|}{(1 + |\lambda|)^{1/2}} + \frac{1 + \sqrt{|\lambda|}}{(1 + |\lambda|)^{1/2}} \right]; \end{aligned}$$

but since $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$ we have $1 + |\lambda| \geq 1 + \rho_0 |\mu|^{1/\varepsilon} \geq 1 + \rho_0 |\mu|^2$, thus we obtain $\|P_{\lambda, \mu}\|_{\mathcal{L}(X)} \leq M$.

2. Since Q_λ^2 is closed then, from (15) and the closed graph theorem, we obtain that $Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \in \mathcal{L}(X)$. Moreover we have

$$\begin{aligned} \|Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}\|_{\mathcal{L}(X)} &= \|(-A + \lambda I) (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}\|_{\mathcal{L}(X)} \\ &\leq \|A (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}\|_{\mathcal{L}(X)} \\ &\quad + |\lambda| \|(Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \\ &\leq M. \end{aligned}$$

The last inequality is obtained, from statement 1, which gives

$$\begin{aligned} \|-A (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}\|_{\mathcal{L}(X)} &= \|A (Q_{\lambda_1} - H_{\mu_1})^{-1} Q^{-1} P_{\lambda, \mu}\|_{\mathcal{L}(X)} \\ &\leq \|-A (Q_{\lambda_1} - H_{\mu_1})^{-1} Q^{-1}\|_{\mathcal{L}(X)} \|P_{\lambda, \mu}\|_{\mathcal{L}(X)} \leq M, \end{aligned}$$

and, from (62), (37), which furnishes

$$|\lambda| \|(Q_\lambda - H_\mu)^{-1}\|_{\mathcal{L}(X)} \|Q_\lambda^{-1}\|_{\mathcal{L}(X)} \leq M.$$

3. We set $R_{\lambda, \mu} = (Q_\lambda + H_\mu) (Q_\lambda - H_\mu)^{-1} \in \mathcal{L}(X)$ and write

$$\Lambda_{\lambda, \mu} = (Q_\lambda - H_\mu) + e^{2Q_\lambda} (Q_\lambda + H_\mu) = (I + e^{2Q_\lambda} R_{\lambda, \mu}) (Q_\lambda - H_\mu);$$

but $\Lambda_{\lambda, \mu}, (Q_\lambda - H_\mu)$ are boundedly invertible, so $I + e^{2Q_\lambda} R_{\lambda, \mu}$ is boundedly invertible with

$$(I + e^{2Q_\lambda} R_{\lambda, \mu})^{-1} = I - e^{2Q_\lambda} R_{\lambda, \mu} (I + e^{2Q_\lambda} R_{\lambda, \mu})^{-1}.$$

Now setting $W_{\lambda, \mu} = R_{\lambda, \mu} (I + e^{2Q_\lambda} R_{\lambda, \mu})^{-1} \in \mathcal{L}(X)$ we have

$$\Lambda_{\lambda, \mu}^{-1} = (Q_\lambda - H_\mu)^{-1} (I + e^{2Q_\lambda} R_{\lambda, \mu})^{-1} = (Q_\lambda - H_\mu)^{-1} (I - e^{2Q_\lambda} W_{\lambda, \mu}),$$

and, due to (62), we have

$$\begin{aligned}\|W_{\lambda,\mu}\|_{\mathcal{L}(X)} &\leq \|R_{\lambda,\mu}\|_{\mathcal{L}(X)} \left\| \left(I + e^{2Q_\lambda} R_{\lambda,\mu} \right)^{-1} \right\|_{\mathcal{L}(X)} \\ &\leq \frac{\|R_{\lambda,\mu}\|_{\mathcal{L}(X)}}{1 - \|e^{2Q_\lambda} R_{\lambda,\mu}\|_{\mathcal{L}(X)}} \leq M.\end{aligned}$$

Finally

$$\begin{aligned}\|Q_\lambda^2 \Lambda_{\lambda,\mu}^{-1} Q_\lambda^{-1}\|_{\mathcal{L}(X)} &= \|Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} (I - e^{2Q_\lambda} W_{\lambda,\mu}) Q_\lambda^{-1}\|_{\mathcal{L}(X)} \\ &\leq \|Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}\|_{\mathcal{L}(X)} \\ &\quad + \|Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}\|_{\mathcal{L}(X)} \|e^{2Q_\lambda}\|_{\mathcal{L}(X)} \|W_{\lambda,\mu}\|_{\mathcal{L}(X)} \\ &\leq M.\end{aligned}$$

□

6.2 Proofs to Theorem 2.4 and Theorem 2.5

Let ρ_0 fixed as in Lemma 6.3.

6.2.1 Proof to Theorem 2.4

As in the proof to Theorem 2.1, we want to apply Proposition 3.7, with A, H, Q, Λ replaced by $A - \lambda I, H + \mu I, Q_\lambda, \Lambda_{\lambda,\mu}$. Assumptions $(H_1) \sim (H_4)$ are easily deduced from (4)~(6), (9) and Lemma 6.3. To obtain (H_5) , it is enough to prove (H'_5) given by (21). So, for $\xi \in (D(Q_\lambda), X)_{1/p,p} = (D(Q), X)_{1/p,p}$, we just have to show that

$$\eta = Q_\lambda (Q_\lambda - H_\mu)^{-1} \xi \in (D(Q), X)_{1/p,p},$$

but, from Lemma 4.2, statement 5. we have $Q_\lambda = Q + \lambda(Q_\lambda + Q)^{-1}$, thus

$$(Q_\lambda - H_\mu) Q_\lambda^{-1} = (Q - H) Q_\lambda^{-1} + \lambda(Q_\lambda + Q)^{-1} Q_\lambda^{-1} - \mu Q_\lambda^{-1},$$

so

$$\xi = (Q_\lambda - H_\mu) Q_\lambda^{-1} \eta = (Q - H) Q_\lambda^{-1} \eta + \lambda(Q_\lambda + Q)^{-1} Q_\lambda^{-1} \eta - \mu Q_\lambda^{-1} \eta,$$

and

$$(Q - H) Q_\lambda^{-1} \eta = \xi - \lambda(Q_\lambda + Q)^{-1} Q_\lambda^{-1} \eta + \mu Q_\lambda^{-1} \eta \in (D(Q), X)_{1/p,p},$$

which means that $Q_\lambda^{-1} \eta \in (Q - H)^{-1} \left((D(Q), X)_{1/p,p} \right)$ and, from (11), we get

$$Q_\lambda^{-1} \eta \in Q^{-1} \left((D(Q), X)_{1/p,p} \right),$$

and then $\eta \in (D(Q), X)_{1/p,p}$.

Here the condition $(Q_\lambda - H_\mu)^{-1} d_0 \in (D(A), X)_{\frac{1}{2p},p}$ which is, from Remark 3.8, equivalent to $\Lambda_{\lambda,\mu}^{-1} d_0 \in (D(A), X)_{\frac{1}{2p},p}$, appears naturally, since we have not, as in Theorem 2.1, $\Lambda_{\lambda,\mu}^{-1}(X) \subset D(Q^2)$.

6.2.2 Proof to Theorem 2.5

Assume (4)~(6) and (9), (10), (12). Let $(\lambda, \mu) \in \Pi_{\varphi_0, \rho_0}$. From Lemma 6.4, statement 2., we have

$$\forall \xi \in D(Q_\lambda), \quad Q_\lambda (Q_\lambda - H_\mu)^{-1} \xi \in D(Q_\lambda),$$

then (11) is satisfied, see Remark 3.1 statement 5, and we can apply Theorem 2.4.

Again we adapt the proof of Theorem 2.2 and write $u = k_1 + k_2 + k_3 - h_2 + h_3 + h_4$ with

$$\begin{cases} k_1(x) = S_\lambda(x) \Lambda_{\lambda, \mu}^{-1} Q_\lambda^{-1} \left[-Q_\lambda e^{2Q_\lambda} d_0 + 2Q_\lambda^2 e^{Q_\lambda} u_1 - 2Q_\lambda e^{Q_\lambda} I_{\lambda, f}(1) \right] \\ k_2(x) = 2S_\lambda(x) \Lambda_{\lambda, \mu}^{-1} Q_\lambda J_{\lambda, f}(0) \\ k_3(x) = S_\lambda(x) \Lambda_{\lambda, \mu}^{-1} d_0 \\ h_2(x) = S_\lambda(x) J_{\lambda, f}(0) + S_\lambda(1-x) I_{\lambda, f}(1) \\ h_3(x) = S_\lambda(1-x) u_1 \\ h_4(x) = I_{\lambda, f}(x) + J_{\lambda, f}(x). \end{cases}$$

Estimate of $Q_\lambda^2 k_1$. Due to (65), we have for $\xi \in X$ and $x \in [0, 1]$

$$\begin{aligned} & \left\| Q_\lambda^2 S_\lambda(x) \Lambda_{\lambda, \mu}^{-1} Q_\lambda^{-1} \xi \right\| \\ &= \left\| \left(I - e^{2Q_\lambda} \right)^{-1} \left(I - e^{2(1-x)Q_\lambda} \right) e^{xQ_\lambda} Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} Q_\lambda^{-1} \xi \right\| \\ &\leq \left\| \left(I - e^{2Q_\lambda} \right)^{-1} \left(I - e^{2(1-x)Q_\lambda} \right) \right\|_{\mathcal{L}(X)} \left\| e^{xQ_\lambda} \right\|_{\mathcal{L}(X)} \left\| Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \|\xi\| \\ &\leq M \|\xi\|, \end{aligned}$$

then

$$\left\| Q_\lambda^2 k_1(x) \right\| \leq M \left(\|d_0\| + \|u_1\| + \|f\|_{L^p(0,1;X)} \right),$$

and

$$\left\| Q_\lambda^2 k_1 \right\|_{L^p(0,1;X)} \leq M \left(\|d_0\| + \|u_1\| + \|f\|_{L^p(0,1;X)} \right).$$

Estimate of $Q_\lambda^2 k_2$. We write, for $x \in (0, 1]$

$$\begin{aligned} Q_\lambda^2 e^{xQ_\lambda} \Lambda_{\lambda, \mu}^{-1} Q_\lambda J_{\lambda, f}(0) &= Q_\lambda^2 e^{xQ_\lambda} \Lambda_{\lambda, \mu}^{-1} \int_0^x e^{sQ_\lambda} f(s) ds \\ &\quad + Q_\lambda^2 e^{xQ_\lambda} \Lambda_{\lambda, \mu}^{-1} \int_x^1 e^{sQ_\lambda} f(s) ds \\ &= Q_\lambda \int_0^x e^{(x-s)Q_\lambda} e^{sQ_\lambda} Q_\lambda \Lambda_{\lambda, \mu}^{-1} e^{sQ_\lambda} f(s) ds \\ &\quad + e^{xQ_\lambda} Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} Q_\lambda^{-1} e^{xQ_\lambda} Q_\lambda \int_x^1 e^{(s-x)Q_\lambda} f(s) ds \\ &= Q_\lambda \int_0^x e^{(x-s)Q_\lambda} F_\lambda(s) ds \\ &\quad + e^{xQ_\lambda} Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} Q_\lambda^{-1} e^{xQ_\lambda} Q_\lambda \int_x^1 e^{(s-x)Q_\lambda} f(s) ds, \end{aligned}$$

where $F_\lambda(s) = e^{sQ_\lambda} Q_\lambda \Lambda_{\lambda, \mu}^{-1} e^{sQ_\lambda} f(s)$. So

$$\begin{aligned} \left\| Q_\lambda^2 k_2 \right\|_{L^p(0,1;X)} &\leq M \left\| Q_\lambda \int_0^x e^{(-s)Q_\lambda} F_\lambda(s) ds \right\|_{L^p(0,1;X)} \\ &\quad + M \left\| Q_\lambda^2 \Lambda_{\lambda, \mu}^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \left\| Q_\lambda \int_x^1 e^{(s-x)Q_\lambda} f(s) ds \right\|_{L^p(0,1;X)}; \end{aligned}$$

but, from Lemma 5.4, (38), (62), (63) and (65), we deduce

$$\begin{cases} \left\| Q_\lambda \int_0^{\cdot} e^{(\cdot-s)Q_\lambda} F_\lambda(s) ds \right\|_{L^p(0,1;X)} \leq M \|F_\lambda\|_{L^p(0,1;X)} \leq M \|f\|_{L^p(0,1;X)} \\ \left\| Q_\lambda^2 \Lambda_{\lambda,\mu}^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \left\| Q_\lambda \int_0^{\cdot} e^{(s-\cdot)Q_\lambda} f(s) ds \right\| \leq M \|f\|_{L^p(0,1;X)}; \end{cases}$$

therefore $\|Q_\lambda^2 k_2\|_{L^p(0,1;X)} \leq M \|f\|_{L^p(0,1;X)}$.

Estimate of $Q_\lambda^2 k_3$. Due to (64) we write $k_3 = \widetilde{k}_3 + \overline{k}_3$ with

$$\widetilde{k}_3(x) = S_\lambda(x) (Q_\lambda - H_\mu)^{-1} d_0 \quad \text{and} \quad \overline{k}_3(x) = S_\lambda(x) (Q_\lambda - H_\mu)^{-1} e^{2Q_\lambda} W_{\lambda,\mu} d_0.$$

Due to (52), we have for $x \in (0, 1]$

$$\begin{aligned} \left\| Q_\lambda^2 \widetilde{k}_3(x) \right\| &= \left\| Q_\lambda^2 (I - e^{2Q_\lambda})^{-1} (I - e^{2(1-x)Q_\lambda}) e^{xQ_\lambda} (Q_\lambda - H_\mu)^{-1} d_0 \right\| \\ &\leq M \left\| Q_\lambda^2 e^{xQ_\lambda} (Q_\lambda - H_\mu)^{-1} d_0 \right\|. \end{aligned}$$

From Theorem 2.1 in [13], since $(Q_\lambda - H_\mu)^{-1} d_0 \in (D(A), X)_{\frac{1}{2p}, p}$, we get

$$\left\| Q_\lambda^2 \widetilde{k}_3 \right\|_{L^p(0,1;X)} \leq M \left(\left\| (Q_\lambda - H_\mu)^{-1} d_0 \right\|_{(D(A), X)_{\frac{1}{2p}, p}} + |\lambda|^{1-\frac{1}{2p}} \left\| (Q_\lambda - H_\mu)^{-1} d_0 \right\| \right).$$

We have also, taking into account (38), (63), (39) and (65),

$$\begin{aligned} \left\| Q_\lambda^2 \overline{k}_3(x) \right\| &= \left\| Q_\lambda^2 (I - e^{2Q_\lambda})^{-1} (I - e^{2(1-x)Q_\lambda}) e^{xQ_\lambda} (Q_\lambda - H_\mu)^{-1} e^{2Q_\lambda} W_{\lambda,\mu} d_0 \right\| \\ &\leq M \left\| e^{xQ_\lambda} Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} Q_\lambda e^{2Q_\lambda} W_{\lambda,\mu} d_0 \right\| \\ &\leq M \left\| e^{xQ_\lambda} \right\| \left\| Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right\| \left\| Q_\lambda e^{2Q_\lambda} \right\| \|W_{\lambda,\mu}\| \|d_0\| \\ &\leq M \|d_0\|. \end{aligned}$$

Finally

$$\left\| Q_\lambda^2 \widetilde{k}_3(x) \right\| \leq M \left(\left\| (Q_\lambda - H_\mu)^{-1} d_0 \right\|_{(D(A), X)_{\frac{1}{2p}, p}} + |\lambda|^{1-\frac{1}{2p}} \left\| (Q_\lambda - H_\mu)^{-1} d_0 \right\|_X + \|d_0\| \right).$$

Estimates of $Q_\lambda^2 h_2, Q_\lambda^2 h_3, Q_\lambda^2 h_4$. In these terms, $\Lambda_{\lambda,\mu}^{-1}$ does not appear, so the estimates are the same as in Theorem 2.2.

6.2.3 Proof to Theorem 2.6

Assume that (4)~(6) and (9), (10), (12) hold. From Theorems 2.4 and Theorem 2.5, there exists $M \geq 0$ such that for any $\mu \in \mathbb{C}$

1. $\mathcal{L}_{A,H,\mu}$ is a closed linear operator on Y .
2. $S_{\varphi_0} \setminus B(0, \rho_\mu) \subset \rho(\mathcal{L}_{A,H,\mu})$ where $\rho_\mu := \max \{ \rho_0, \rho_0 |\mu|^{1/\varepsilon} \} > 0$.

3. $\forall \lambda \in S_{\varphi_0} \setminus B(0, \rho_\mu), \forall f \in Y, \forall x \in [0, 1]$

$$\begin{aligned} (\mathcal{L}_{A,H,\mu} - \lambda I)^{-1} f(x) &= S_\lambda(x) \left[2\Lambda_{\lambda,\mu}^{-1} Q_\lambda \left[J_{\lambda,f}(0) - e^{Q_\lambda} I_{\lambda,f}(1) \right] - J_{\lambda,f}(0) \right] \\ &\quad - S_\lambda(1-x) I_{\lambda,f}(1) + I_{\lambda,f}(x) + J_{\lambda,f}(x), \end{aligned}$$

$$\text{where } S_\lambda(x) = (I - e^{2Q_\lambda})^{-1} (e^{xQ_\lambda} - e^{(1-x)Q_\lambda} e^{Q_\lambda}).$$

4. $\forall \lambda \in S_{\varphi_0} \setminus B(0, \rho_\mu): \quad \|(\mathcal{L}_{A,H,\mu} - \lambda I)^{-1}\|_{\mathcal{L}(Y)} \leq \frac{M}{1 + |\lambda|}.$

The proof to Theorem 2.6 is given by Statement 4.

7 Results for Dirichlet boundary conditions

We can find, in [14] and [15], the study of the following problem

$$\begin{cases} u''(x) + Au(x) = f(x), & x \in (0, 1) \\ u(0) = u_0, \quad u(1) = u_1. \end{cases} \quad (67)$$

A classical solution of this problem is a function $u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A))$, satisfying (67).

7.1 Proof to Theorem 2.7

The authors obtain the following result (see Theorem 4, p. 200 in [14] and Theorem 5 p. 173 (with $A = L = M$) in [15]).

Proposition 7.1 ([14],[15]). Let $f \in L^p(0, 1; X)$ with $1 < p < +\infty$ and assume that (4)~(6) are satisfied. Then the following assertions are equivalent:

1. Problem (67) admits a classical solution u .

2. $u_1, u_0 \in (D(A), X)_{\frac{1}{2p}, p}$

Moreover in this case u is unique and given by

$$\begin{aligned} u(x) &= S(x) u_0 + S(1-x) u_1 - S(x) J(0) \\ &\quad - S(1-x) I(1) + I(x) + J(x), \quad x \in (0, 1). \end{aligned} \quad (68)$$

Note that $S(\cdot), I(\cdot), J(\cdot)$ are precised in (23) and (25) with $Q = -\sqrt{-A}$.

Now we are in a position to study, as in Sections 5 and 6, the corresponding spectral problem

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), & x \in (0, 1) \\ u(0) = u_0, \quad u(1) = u_1. \end{cases} \quad (69)$$

Applying the previous Proposition with A replaced by $A - \lambda I$ we obtain Theorem 2.7.

7.2 Proof to Theorem 2.8

Let $\lambda \in S_{\varphi_0}$ and $f \in L^p(0, 1; X)$. Taking into account (48) and (68) with Q_λ replacing Q , for $x \in [0, 1]$, we have

$$u(x) = S_\lambda(x) u_0 + S_\lambda(1-x) u_1 - S(x) J_{\lambda,f}(0) - S(1-x) I_{\lambda,f} + I_{\lambda,f}(x) + J_{\lambda,f}(x).$$

So we can write $u = -h_2 + g_3 + h_3 + h_4$ with

$$\begin{cases} h_2(x) = S_\lambda(x) J_{\lambda,f}(0) + S_\lambda(1-x) I_{\lambda,f}(1) \\ g_3(x) = S_\lambda(x) u_0 \\ h_3(x) = S_\lambda(1-x) u_1 \\ h_4(x) = I_{\lambda,f}(x) + J_{\lambda,f}(x). \end{cases}$$

As in the proof to Theorem 2.2 we get

$$\|Q_\lambda^2 h_2\|_{L^p(0,1;X)} \leq M \|f\|_{L^p(0,1;X)}, \quad \|Q_\lambda^2 h_4\|_{L^p(0,1;X)} \leq M \|f\|_{L^p(0,1;X)},$$

and also

$$\|Q_\lambda^2 h_3\|_{L^p(0,1;X)} \leq M \left(\|u_1\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|u_1\|_X \right).$$

Moreover, $Q_\lambda^2 g_3$ is treated like $Q_\lambda^2 h_3$, so

$$\|Q_\lambda^2 g_3\|_{L^p(0,1;X)} \leq M \left(\|u_0\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|u_0\|_X \right).$$

We finish as in the proof of Theorem 2.2.

7.3 Proof to Theorem 2.9

Assume (4) ~ (6). From Theorems 2.7 and Theorem 2.8, there exists $M \geq 0$ such that

1. \mathcal{L}_A is a closed linear operator on Y and $S_{\varphi_0} \subset \rho(\mathcal{L}_A)$.
2. $\forall \lambda \in S_{\varphi_0}$

$$(\mathcal{L}_A - \lambda I)^{-1} f(x) = -S_\lambda(x) J_{\lambda,f}(0) - S_\lambda(1-x) I_{\lambda,f}(1) + I_{\lambda,f}(x) + J_{\lambda,f}(x),$$

$$\text{where } S_\lambda(x) = (I - e^{2Q_\lambda})^{-1} (e^{xQ_\lambda} - e^{(1-x)Q_\lambda} e^{Q_\lambda}) \in \mathcal{L}(X).$$

3. $\forall \lambda \in S_{\varphi_0} : \quad \|(\mathcal{L}_A - \lambda I)^{-1}\|_{\mathcal{L}(Y)} \leq \frac{M}{1 + |\lambda|}$.

The proof to Theorem 2.9 is given by Statement 4.

8 Applications

8.1 A model example for the first case

In view to illustrate the results obtained in this work, we will consider the concrete problem of the heat equation in the square domain $\Omega = (0, 1) \times (0, 1)$ with a dynamical-Wentzell condition in one of its lateral boundaries

$$(P) \begin{cases} \frac{\partial u}{\partial t}(t, x, y) = \Delta_{x,y}u(t, x, y), & (t, x, y) \in (0, +\infty) \times \Omega \\ \frac{\partial u}{\partial t}(t, 0, y) = \frac{\partial u}{\partial x}(t, 0, y) + \frac{\partial^2 u}{\partial y^2}(t, 0, y), & (t, 0, y) \in (0, +\infty) \times \Gamma_0 \\ u(t, 1, y) = 0, & (t, 1, y) \in (0, +\infty) \times \Gamma_1 \\ u(t, x, 0) = u(t, x, 1) = 0, & x \in (0, 1) \\ u(0, x, y) = u_0(x, y) & (x, y) \in (0, 1) \times (0, 1), \end{cases}$$

where

$$\begin{cases} \Gamma_0 = \{0\} \times (0, 1), & \Gamma_1 = \{1\} \times (0, 1), \\ \gamma_0 = (0, 1) \times \{0\}, & \gamma_1 = (0, 1) \times \{1\}. \end{cases}$$

Here $\frac{\partial^2}{\partial y^2}$ is the Laplace-Beltrami operator on Γ_0 . Physically, $-\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial x}$ represent the interaction between the domain Ω and the lateral boundaries while $\frac{\partial^2 u}{\partial y^2}$ is the boundary diffusion.

Set $\mathcal{E} = L^p(\Omega) \times L^p(\Gamma_0)$; this Banach space is well defined and endowed with its natural norm. Define operator \mathcal{P} by

$$\begin{cases} D(\mathcal{P}) = \left\{ w = (u, v_0) : u, \Delta_{x,y}u \in L^p(\Omega), v_0 \in W^{2,p}(\Gamma_0), u|_{\Gamma_0} = v_0, \right. \\ \quad \left. (\Delta_{x,y}u)|_{\Gamma_0} = \left(\frac{\partial u}{\partial x}\right)_{|\Gamma_0} + \frac{\partial^2 v_0}{\partial y^2} \text{ and } u|_{\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0 \right\}, \\ \mathcal{P}w = \left(\Delta_{x,y}u, \left(\frac{\partial u}{\partial x}\right)_{|\Gamma_0} + \frac{\partial^2 v_0}{\partial y^2} \right), \text{ for } w = (u, v_0) \in D(\mathcal{P}). \end{cases}$$

The boundary conditions are defined in $L^p(\Gamma_0)$ and $\mathcal{P}w \in \mathcal{E}$. On the other hand it is not difficult to see that this operator is closed in \mathcal{E} . When we integrate the time variable t , the following Cauchy problem

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial}{\partial t}(u, v_0) = \left(\frac{\partial u}{\partial t}, \frac{\partial v_0}{\partial t}\right) = \mathcal{P}w = \mathcal{P}(u, v_0) \\ w(0) = (u(0, \cdot), v_0(0, \cdot)) \text{ given,} \end{cases}$$

writes

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ \frac{\partial v_0}{\partial t} = \left(\frac{\partial u}{\partial x}\right)_{|\Gamma_0} + \frac{\partial^2 v_0}{\partial y^2} \\ u|_{\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0 \\ (u(0, \cdot), v_0(0, \cdot)) \text{ given;} \end{cases}$$

since $(u, v_0) \in D(\mathcal{P})$ and $\frac{\partial u}{\partial t} = \Delta u$, we obtain

$$(\Delta u)|_{\Gamma_0} = \left(\frac{\partial u}{\partial x}\right)_{|\Gamma_0} + \left(\frac{\partial^2 v_0}{\partial y^2}\right)_{|\Gamma_0} = \left(\frac{\partial u}{\partial t}\right)_{|\Gamma_0},$$

and since $u|_{\Gamma_0} = v_0$, by using the tangential derivative, we obtain

$$\left(\frac{\partial^2 v_0}{\partial y^2} \right)_{|\Gamma_0} = \left(\frac{\partial^2 u}{\partial y^2} \right)_{|\Gamma_0} ;$$

summarizing up, we deduce the same equation as in Problem (P) :

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ \left(\frac{\partial u}{\partial t} \right)_{|\Gamma_0} = \left(\frac{\partial u}{\partial x} \right)_{|\Gamma_0} + \left(\frac{\partial^2 u}{\partial y^2} \right)_{|\Gamma_0} \\ u(0, \cdot) \text{ is given} \\ u|_{\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0. \end{cases}$$

The study of the evolution equation above is based on the study of the following spectral equation

$$\begin{cases} \mathcal{P}(u, v_0) - \lambda(u, v_0) = (h, d_0) \\ (u, v_0) \in D(\mathcal{P}), (h, d_0) \in \mathcal{E}, \end{cases} \quad (70)$$

and since $u|_{\Gamma_0} = v_0$, (70) is equivalent to

$$\begin{cases} \Delta u - \lambda u = h \\ \left(\frac{\partial u}{\partial x} \right)_{|\Gamma_0} + \left(\frac{\partial^2 u}{\partial y^2} \right)_{|\Gamma_0} - \lambda u|_{\Gamma_0} = d_0 \\ u|_{\gamma_0 \cup \gamma_1 \cup \Gamma_0} = 0, \end{cases} \quad (71)$$

which is an elliptic partial differential equation with the same spectral parameter in the equation and in the boundary condition on Γ_0 . We will write (71) in an operational differential form. We consider the Banach space $X = L^p(0, 1)$ and identify \mathcal{E} with $L^p(0, 1; X)$ by writing as usual, for $g \in \mathcal{E}$, $g(x, y) = (g(x))(y)$, $x, y \in (0, 1)$. We define operator A on X by

$$\begin{cases} D(A) = \{\psi \in W^{2,p}(0, 1) : \psi(0) = \psi(1) = 0\} \\ A\psi(y) = \psi''(y), \end{cases} \quad (72)$$

and operator $H := -A$. So, equation $\Delta u(x, y) - \lambda u(x, y) = h(x, y)$, takes the following form in space X

$$u''(x) + Au(x) - \lambda u(x) = h(x), \quad x \in (0, 1),$$

while the boundary condition

$$\left(\frac{\partial u}{\partial x} \right)_{|\Gamma_0} + \left(\frac{\partial^2 u}{\partial y^2} \right)_{|\Gamma_0} - \lambda u|_{\Gamma_0} = d_0,$$

writes as $u'(0) - Hu(0) - \lambda u(0) = d_0$; the condition $u|_{\gamma_0 \cup \gamma_1} = 0$ (which means that $u(0, y)$ and $u(1, y)$ vanish in $y = 0$ and $y = 1$) is implicitly included in the fact that $u(0) := u(0, \cdot)$ and $u(1) := u(1, \cdot)$ are in $D(H)$.

Therefore (71) or equivalently (70), write in the following abstract form

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = h(x), \quad x \in (0, 1) \\ u'(0) - Hu(0) - \lambda u(0) = d_0 \\ u(1) = 0, \end{cases} \quad (73)$$

where $(h, d_0) \in \mathcal{E} \equiv L^p(0, 1; X) \times L^p(X)$, and we are in the situation of Subsection 5.3.3 with $\mu = 0$.

Let u be the classical solution of (73); then $u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A))$ and

$$(u, u(0)) \in D(\mathcal{P});$$

so that $(u, u(0)) = (\mathcal{P} - \lambda I)^{-1}(h, d_0)$.

Taking into account the fact that, here, we can take $\varphi_0 = \pi - \varepsilon$ ($\varepsilon > 0$ as close to 0 as we want), we can use Proposition 5.8 and Theorem 2.3, to obtain :

$$\exists M > 0, \forall \lambda \in S_{\varphi_0} : \forall (h, d_0) \in \mathcal{E}, \quad \left\| (\mathcal{P} - \lambda I)^{-1}(h, d_0) \right\|_{\mathcal{E}} \leq \frac{M}{1 + |\lambda|} \|(h, d_0)\|_{\mathcal{E}},$$

and deduce that our operator \mathcal{P} defined above generates an analytic semigroup in \mathcal{E} .

This example can be extended to the following problem

$$\begin{cases} \Delta u - \lambda u = h \\ a_0 \left(\frac{\partial u}{\partial x} \right)_{|\Gamma_0} + b_0 \frac{\partial^2 v_0}{\partial y^2} - \lambda v_0 = d_0 \\ a_1 \left(\frac{\partial u}{\partial x} \right)_{|\Gamma_1} + b_1 \frac{\partial^2 v_1}{\partial y^2} - \lambda v_1 = d_1 \\ u|_{\gamma_0 \cup \gamma_1} = 0. \end{cases}$$

8.2 Some concrete examples for the second case

8.2.1 Example 1

Here, we set $\Omega = (0, 1) \times (0, 1)$. Our concrete spectral partial differential problem is

$$(P1) \begin{cases} \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) - \lambda u(x, y) = f(x, y), & (x, y) \in \Omega \\ u(1, y) = 0, & y \in (0, 1) \\ \frac{\partial u}{\partial x}(0, y) - \int_0^y \phi(y, \xi) u(0, \xi) d\xi = 0, & y \in (0, 1) \\ u(x, 0) = u(x, 1) = 0, & x \in (0, 1), \end{cases}$$

where we can take $\lambda \in S_{\varphi_0}$ with φ_0 fixed in $(\pi/2, \pi)$.

Define operator A on $X := L^p(0, 1)$, with $1 < p < +\infty$, as in (72); then the square root of the opposite of this operator is well defined and

$$W_0^{1,p}(0, 1) \subset D((-A)^{1/2}) \subset W^{1,p}(0, 1) \quad \text{and} \quad \left\| (-A)^{1/2} \psi \right\| \approx \|\psi'\|_{L^p(0,1)} + \|\psi\|_{L^p(0,1)},$$

see [3]. We know also that $Q = -\sqrt{-A}$ generates an analytic semigroup in X ; on the other hand $Q_\lambda = -\sqrt{-A + \lambda I}$ is well defined and generates an analytic semigroup in X for all $\lambda \in S_{\varphi_0}$.

Now let us define operator H by

$$H\psi(y) = \int_0^y \phi(y, \xi) \psi(\xi) d\xi, \quad \psi \in X, \quad (74)$$

with an appropriate function ϕ having the following properties. Let $q \in (1, +\infty)$ such that $1/q + 1/p = 1$. We then assume that

$$\begin{cases} \phi(y, \cdot), \frac{\partial \phi}{\partial y}(y, \cdot) \in L^q(0, 1), \text{ for a.e. } y \in (0, 1) \\ \phi(1, \cdot) = 0 \\ \Phi_j : y \mapsto \frac{\partial^j \phi}{\partial y^j}(y, \cdot) \in L^p(0, 1; L^q(0, 1)), \text{ for } j = 0, 1 \\ \phi_1 : y \mapsto \phi(y, y) \in L^p(0, 1). \end{cases} \quad (75)$$

We can build a simple example of a function ϕ satisfying (75), setting, for a fixed $n \in \mathbb{N} \setminus \{0\}$

$$\phi(y, \xi) = (1 - y)^n \tilde{\psi}(\xi), \quad \xi, y \in (0, 1),$$

where $\tilde{\psi} \in W^{1,q}(0, 1) \cap W^{1,p}(0, 1)$. We have

$$\begin{aligned} \|H(\psi)\|_X &= \left(\int_0^1 \left| \int_0^y \phi(y, \xi) \psi(\xi) d\xi \right|^p dy \right)^{1/p} \\ &\leq \left(\int_0^1 \left[\left(\int_0^1 |\phi(y, \xi)|^q d\xi \right)^{1/q} \left(\int_0^1 |\psi(\xi)|^p d\xi \right)^{1/p} \right]^p dy \right)^{1/p} \\ &\leq \left(\int_0^1 \|\phi(y, \cdot)\|_{L^q(0,1)}^p dy \right)^{1/p} \|\psi\|_X \\ &\leq \|\Phi\|_{L^p(0,1;L^q(0,1))} \times \|\psi\|_X, \end{aligned}$$

so $H \in \mathcal{L}(X)$.

Our concrete problem (P1) writes in the following abstract form

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), \quad \text{a.e. } x \in (0, 1) \\ u(1) = 0, \quad u'(0) - Hu(0) = 0. \end{cases}$$

The following assumptions are satisfied:

1. X is a UMD space and operator A verifies

$$\begin{cases} \exists \varphi_0 \in (0, \pi) : S_{\varphi_0} \subset \rho(A) \text{ and } \exists C_A > 0 : \\ \forall \lambda \in S_{\varphi_0}, \quad \|(A - \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_A}{1 + |\lambda|}, \\ \\ \left\{ \begin{array}{l} \forall s \in \mathbb{R}, \quad (-A)^{is} \in \mathcal{L}(X), \quad \exists \theta_A \in (0, \pi) : \\ \sup_{s \in \mathbb{R}} \|e^{-\theta_A |s|} (-A)^{is}\|_{\mathcal{L}(X)} < +\infty. \end{array} \right. \end{cases}$$

This last property is proved explicitly in [21].

2. Since H is bounded, from Remark 2.11 statement 1, we get $D(Q) \subset D(H)$ and

$$\exists C_{H,Q} > 0, \quad \sup_{t \in [0, +\infty)} (1+t)^{1/2} \|HQ_t^{-1}\|_{\mathcal{L}(X)} \leq C_{H,Q}.$$

3. We verify that $(Q - H)^{-1}(D(Q)) \subset D(Q^2)$.

Let $\psi \in D(Q)$ such that $(Q - H)(\psi) \in D(Q)$; then $Q\psi - H\psi = g \in D(Q)$, with

$$W_0^{1,p}(0, 1) \subset D(Q) \subset W^{1,p}(0, 1).$$

To obtain $\psi \in D(Q^2)$, it suffices to have $H\psi \in W_0^{1,p}(0,1)$ for $\psi \in D(Q) \subset W^{1,p}(0,1)$. We have

$$H\psi(y) = \int_0^y \phi(y, \xi) \psi(\xi) d\xi;$$

then $H\psi(0) = 0$, and $H\psi(1) = 0$ due to (75) and

$$(H\psi)'(y) = \phi(y, y) \psi(y) + \int_0^y \frac{\partial \phi}{\partial y}(y, \xi) \psi(\xi) d\xi.$$

In virtue of the assumptions verified by ϕ , we then get $H\psi \in W_0^{1,p}(0,1)$. Therefore $\psi \in D(Q^2)$.

Now, we set $Y = L^p(0,1; X) = L^p(\Omega)$ and considering A, H defined by (72) and (74), we build, as in (13)

$$\begin{aligned} \mathcal{L}_{A,H,0} : D(\mathcal{L}_{A,H,0}) \subset Y &\longrightarrow Y \\ u &\longmapsto u'' + A(u(\cdot)). \end{aligned}$$

Note that in this example, in general, operators Q and H do not commute. We can apply Theorem 2.6 (with $\mu = 0$), to obtain that $\mathcal{L}_{A,H,0}$ is the infinitesimal generator of an analytic semigroup. This result allows us to consider and solve the corresponding Cauchy problem with respect to (P1).

8.2.2 Example 2

Here, we are considering a quasi-elliptic problem under an oblique derivative boundary condition. Let $\Omega = (0,1)^2$ and consider the following spectral problem

$$(P2) \begin{cases} \frac{\partial^2 u}{\partial x^2}(x, y) - \frac{\partial^4 u}{\partial y^4}(x, y) - \lambda u(x, y) = f(x, y), & (x, y) \in \Omega \\ u(1, y) = 0, & y \in (0, 1) \\ \frac{\partial u}{\partial x}(0, y) + c(y) \frac{\partial u}{\partial y}(0, y) = 0, & y \in (0, 1) \\ u(x, 0) = u(x, 1) = \frac{\partial^2 u}{\partial y^2}(x, 0) = \frac{\partial^2 u}{\partial y^2}(x, 1) = 0, & x \in (0, 1). \end{cases}$$

We will assume that $c \in \mathcal{C}^2[0,1]$ and $c(0) = c(1) = 0$. Here the boundary condition on $\Gamma = \{0\} \times (0,1)$

$$\frac{\partial u}{\partial x}(0, y) + c(y) \frac{\partial u}{\partial y}(0, y) = 0,$$

can be written as

$$\nabla u(\sigma) \cdot \alpha(\sigma) = 0 \text{ in } \Gamma, \quad (76)$$

with $\alpha(\sigma)$ a vector on Γ equal to $(1, c(y))$ which is pointing inwardly of Ω . It is known that (76) is called oblique derivative boundary condition on Γ . We set, in space $X = L^p(0,1)$, as above

$$\begin{cases} D(A) = \{\psi \in W^{4,p}(0,1) : \psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0\} \\ A\psi(y) = -\psi^{(4)}(y); \end{cases} \quad (77)$$

so, as we have seen

$$\begin{cases} D(\sqrt{-A}) = \{\psi \in W^{2,p}(0,1) : \psi(0) = \psi(1) = 0\} \\ \sqrt{-A}\psi(y) = -\psi''(y), \end{cases}$$

and clearly $Q = -\sqrt{-A}$ and $Q_\lambda = -\sqrt{-A + \lambda I}$, for all $\lambda \in S_\varphi$ generate analytic semigroups in X . We note also that $\sqrt{-Q} = (-A)^{1/4}$ is well defined and

$$W_0^{1,p}(0,1) \subset D((-A)^{1/4}) \subset W^{1,p}(0,1) \quad \text{and} \quad \|(-A)^{1/4}\psi\| \approx \|\psi'\|_{L^p(0,1)} + \|\psi\|_{L^p(0,1)},$$

see [3]. Now, define operator H by setting

$$\begin{cases} D(H) = W^{1,p}(0,1) \\ [H\psi](y) = -c(y)\psi'(y). \end{cases} \quad (78)$$

We then have $D((-A)^{1/4}) \subset D(H)$; therefore, see Remark 2.11, statement 1, with $\omega = 1/4$, there exists $C > 0$ such that, for $t \geq 0$, we have

$$\|HQ_t^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{(1+t)^{1/4}}.$$

Now, we will prove that $(Q - H)^{-1}(D(Q)) \subset D(Q^2)$. To this end, let $\psi \in D(Q)$ such that $(Q - H)(\psi) \in D(Q)$; then

$$\psi'' - c\psi' = g \in D(Q) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1);$$

so $\psi \in W^{4,p}(0,1)$. We have $\psi \in D(Q)$; then $\psi(0) = \psi(1) = 0$. But $g \in D(Q)$ thus $g(0) = g(1) = 0$ and

$$\psi''(j) = (c\psi')(j) + g(j) = 0, \quad j = 0, 1,$$

that is $\psi''(0) = \psi''(1) = 0$, therefore $\psi \in D(Q^2)$. Note that in this example $Q - H$ is boundedly invertible and from equation $Q\psi - H\psi = g$, it follows that

$$\begin{cases} \psi''(y) - c(y)\psi'(y) = g(y) \\ \psi(0) = \psi(1) = 0. \end{cases}$$

Let ψ_1 and ψ_2 two linearly independent solutions to equation $\psi''(y) - c(y)\psi'(y) = 0$, such that $\psi_1(0) = 0$ and $\psi_2(1) = 0$. Then we have

$$\psi(y) = -\psi_2(y) \int_0^y \frac{\psi_1(s)}{W(s)} g(s) ds - \psi_1(y) \int_y^1 \frac{\psi_2(s)}{W(s)} g(s) ds = [(Q - H)^{-1}g](y),$$

where the wronskian W is given by

$$W(s) = \psi_1(s)\psi_2'(s) - \psi_2(s)\psi_1'(s).$$

We have

$$\psi'(y) = -\psi_2'(y) \int_0^y \frac{\psi_1(s)}{W(s)} g(s) ds - \psi_1'(y) \int_y^1 \frac{\psi_2(s)}{W(s)} g(s) ds,$$

and

$$\psi''(y) = -\psi_2''(y) \int_0^y \frac{\psi_1(s)}{W(s)} g(s) ds - \psi_1''(y) \int_y^1 \frac{\psi_2(s)}{W(s)} g(s) ds + g(y).$$

If $g \in D(Q) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1)$, it is clear that $\psi \in W^{4,p}(0,1)$ and

$$\psi''(0) = g(0) - \psi_1''(0) \int_0^1 \frac{\psi_2(s)}{W(s)} g(s) ds = 0 - [c(0)\psi_1'(0)] \int_0^1 \frac{\psi_2(s)}{W(s)} g(s) ds = 0;$$

similarly we obtain $\psi''(1) = 0$.

Again, our concrete problem (P3) writes in the abstract form

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), & \text{for a.e. } x \in (0, 1) \\ u(1) = 0, \quad u'(0) - Hu(0) = 0, \end{cases}$$

with A and H defined by (77), (78) and setting

$$\begin{aligned} \mathcal{L}_{A,H,0} : D(\mathcal{L}_{A,H,0}) \subset Y &\longrightarrow Y \\ u &\longmapsto u'' + A(u(.)). \end{aligned}$$

We can apply Theorem 2.6 (with $\mu = 0$), to obtain that $\mathcal{L}_{A,H,0}$ is the infinitesimal generator of an analytic semigroup.

8.2.3 Example 3

In [20] the authors have considered and studied the following problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}(x, y, t) + \frac{\partial^2 u}{\partial y^2}(x, y, t) = f(x, y, t), & (x, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times (0, T) \\ u(0, y, 0) = f_1(y), & y \in \mathbb{R} \\ \frac{\partial u}{\partial x}(0, y, t) - D_t^\nu u(0, y, t) = f_2(y, t), & (y, t) \in \mathbb{R} \times (0, T), \end{cases}$$

where D_t^ν , for $\nu \in (0, 1)$, denotes the fractional time derivative (or Caputo Derivative) defined, for instance, by

$$D_t^\nu g(\cdot, t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{1}{(t-\tau)^\nu} \frac{\partial g}{\partial \tau}(\cdot, \tau) d\tau,$$

for functions g of classe C^1 with respect to the second variable; for this derivative, see for instance [10]. This derivative has been extended to functions in $L_{loc}^1(\mathbb{R})$ verifying some integrability condition, see [22].

Analysis of the above problem is useful to study the free boundary problem for the Laplace equation in the case of subdiffusion as illustrated by the fractional derivative, see [24]. We recall that this subdiffusion expressed by this Caputo Derivative means that the square displacement of the diffusing species has a behaviour as t^ν for some real number ν . When $\nu \in (0, 1)$, we are in the presence of a subdiffusion.

Our objective is not to study this problem, but it helps us to consider a class of similar problems illustrating our theory of the second case. So, setting $\Omega_T = (0, 1) \times (0, 1) \times (0, T)$, we will take inspiration from this example to consider the following spectral elliptic problem:

$$(P4) \begin{cases} \frac{\partial^2 u}{\partial x^2}(x, y, t) + \frac{\partial^2 u}{\partial y^2}(x, y, t) - \lambda u(x, y, t) = f(x, y, t), & (x, y, t) \in \Omega_T \\ u(1, y, 0) = f_1(y), & y \in (0, 1) \\ \frac{\partial u}{\partial x}(0, y, t) - D_t^\nu u(0, y, t) = f_2(y, t), & (y, t) \in (0, 1) \times (0, T), \end{cases}$$

for $\lambda \in S_{\varphi_0}$ with $\varphi_0 \in (\pi/2, \pi)$.

In view to write this problem in an abstract form, we will hide the variable (y, t) by considering the following anisotropic Sobolev Banach space $X = W_p^{0,1}((0, 1) \times (0, T))$, consisting of all functions $(y, t) \mapsto w(y, t)$ which are in $L^p((0, 1) \times (0, T))$ such that we have $\frac{\partial w}{\partial t} \in L^p((0, 1) \times (0, T))$; it is endowed with the following natural norm

$$\|w\|_X = \|w\|_{L^p((0,1) \times (0,T))} + \left\| \frac{\partial w}{\partial t} \right\|_{L^p((0,1) \times (0,T))}.$$

Now, define operator A in X by

$$\left\{ \begin{array}{l} D(A) = \left\{ w \in X : \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial y^2} \in L^p(\mathbb{R} \times (0, T)) \text{ and } w(0, t) = w(1, t) = 0 \text{ for } t \in (0, T) \right\} \\ [Aw](y, t) = \frac{\partial^2 w}{\partial y^2}(y, t). \end{array} \right.$$

We also define H by

$$\left\{ \begin{array}{l} D(H) = W_p^{0,1}(\mathbb{R} \times (0, T)) = X \\ [Hw](y, t) = D_t^\nu w(y, t). \end{array} \right.$$

This problem can be written in the following abstract form:

$$\left\{ \begin{array}{l} u''(x) + Au(x) - \lambda u(x) = f(x), \quad \text{for a.e. } x \in (0, 1) \\ u(1) = f_1 \\ u'(0) - Hu(0) = f_2, \end{array} \right.$$

where we have used the usual writing $u(x, y, t) = u(x)(y, t)$ and $f(x, y, t) = f(x)(y, t)$. Now we must verify the following statements.

1. X has the UMD property.

In fact, consider the application

$$\begin{array}{ccc} \mathcal{T} : W_p^{0,1}((0, 1) \times (0, T)) & \longrightarrow & Z = [L^p((0, 1) \times (0, T))]^2 \\ w & \longmapsto & \left(w, \frac{\partial w}{\partial t} \right), \end{array}$$

then $\mathcal{T}(W_p^{0,1}((0, 1) \times (0, T)))$ is a closed subspace of Z and thus has a UMD property. Since it is isometric to X , we deduce that X is a UMD space.

2. Operator A verifies

$$\left\{ \begin{array}{l} S_{\varphi_0} \subset \rho(A) \text{ and } \exists C_A > 0 : \\ \forall \lambda \in S_{\varphi_0}, \quad \|(A - \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_A}{1 + |\lambda|}, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \forall s \in \mathbb{R}, \quad (-A)^{is} \in \mathcal{L}(X), \quad \exists \theta_A \in (0, \pi) : \\ \sup_{s \in \mathbb{R}} \|e^{-\theta_A |s|} (-A)^{is}\|_{\mathcal{L}(X)} < +\infty. \end{array} \right.$$

For the first property we note that the spectral properties of operator A are based on the equation

$$\left\{ \begin{array}{l} \frac{\partial^2 w}{\partial y^2}(y, t) - \lambda w(y, t) = h(y, t) \\ w(0, t) = w(1, t) = 0 \text{ for } t \in (0, T), \end{array} \right.$$

where $h \in W_p^{0,1}((0, 1) \times (0, T))$. Then, for all $\lambda \in S_{\varphi_0}$, we have

$$\forall (y, t) \in (0, 1) \times (0, T), \quad w(y, t) = \int_0^1 K_{\sqrt{\lambda}}(y, s) h(s, t) ds,$$

where the kernel $K_{\sqrt{\lambda}}(y, s)$ is well known. Using the Schur Lemma, for all $t \in (0, 1)$, we obtain

$$\int_0^1 |w(y, t)|^p dy \leq \left[\frac{C}{1 + |\lambda|} \right]^p \int_0^1 |h(s, t)|^p ds;$$

then

$$\int_0^T \int_0^1 |w(y, t)|^p dy dt \leq \left[\frac{C}{1 + |\lambda|} \right]^p \int_0^T \int_0^1 |h(s, t)|^p ds dt,$$

that is

$$\|w\|_{L^p((0,1) \times (0,T))} \leq \frac{C}{1 + |\lambda|} \|h\|_{L^p((0,1) \times (0,T))}.$$

Since we have

$$\forall (y, t) \in (0, 1) \times (0, T), \quad \frac{\partial w}{\partial t}(y, t) = \int_0^1 K_{\sqrt{\lambda}}(y, s) \frac{\partial h}{\partial t}(s, t) ds,$$

we deduce

$$\left\| \frac{\partial w}{\partial t} \right\|_{L^p((0,1) \times (0,T))} \leq \frac{C}{1 + |\lambda|} \left\| \frac{\partial h}{\partial t} \right\|_{L^p((0,1) \times (0,T))},$$

and then

$$\|w\|_X \leq \frac{C}{1 + |\lambda|} \|h\|_X.$$

The second property is proved explicitly in [21].

3. Since H is bounded then from Remark 2.11, statement 1, $D(Q) \subset D(H)$ and

$$\exists C_{H,Q} > 0, \quad \sup_{t \in [0, +\infty)} (1+t)^{1/2} \|HQ_t^{-1}\|_{\mathcal{L}(X)} \leq C_{H,Q}.$$

4. Now, we must verify that $(Q - H)^{-1}(D(Q)) \subset D(Q^2)$. It is enough to verify that $D_t^\nu A^{-1} = A^{-1}D_t^\nu$ on X . We have

$$\forall (y, t) \in (0, 1) \times (0, T), \quad [A^{-1}w](y, t) = \int_0^1 G(y, s)w(s, t) ds,$$

where the kernel G is well known. So, for any $(y, t) \in (0, 1) \times (0, T)$

$$[D_t^\nu A^{-1}w](y, t) = \int_0^1 G(y, s)D_t^\nu w(s, t) ds = [A^{-1}D_t^\nu w](y, t).$$

Again, as in the previous examples, we get that $\mathcal{L}_{A,H,0}$ is the infinitesimal generator of an analytic semigroup.

Remark 8.1. We can generalize the above examples by considering operator A defined in an open bounded regular set ω of \mathbb{R}^{n-1} .

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