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Elliptic differential-operator with an abstract Robin boundary condition containing two spectral parameters, study in a non commutative framework

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Abstract

We study the solvability of boundary-value problems for differential-operator equations of the second order in $L^p(0,1;X)$, with $1 < p < +\infty$, X being a UMD complex Banach space. The originality of this work lies in the fact that we have considered the case when spectral complex parameters appear in the equation and in the abstract Robin boundary condition illustrated by some unbounded operator non commuting with the one used in the equation. Existence, uniqueness, representation formula, maximal regularity of the solution, sharp estimates and generation of strongly continuous analytic semigroup are proved. Many concrete applications are given for which our theory applies. This work gives news considerations with respect to all those studied by the authors in [7] and is a continuation, in some sense, of the results in [1] studied in Hilbertian spaces.
Key Words and Phrases: Second order boundary value problem with two spectral parameters, Robin boundary condition, spectral estimates, functional calculus, generation of analytic semigroups.
2020 Mathematics Subject Classification: 35B65, 35C15, 35J25, 47A60, 47D06.

1 Introduction

In this article, we consider a new spectral problem that is equation

$$u''(x) + Au(x) - \lambda u(x) = f(x), \quad x \in (0,1), \tag{1}$$

together with the abstract Robin boundary conditions

$$u'(0) - Hu(0) - \mu u(0) = d_0, \quad u(1) = u_1. \tag{2}$$

Here, λ, μ are complex parameters; A, H are closed linear operators in a complex Banach space X; f belongs to $L^p(0,1;X)$ with $1 < p < +\infty$; d_0, u_1 are given elements of X. We develop a completely different approach from those used until now where it is more easy to verify the assumptions and to apply this approach to concrete problems.

Many boundary value problems with a spectral parameter in the equation and in the boundary conditions arise in different concrete problems. Let us cite some interesting studied problems. One of the first works, was treated in [5], where the author have considered a class of boundary problems with a spectral parameter in the boundary conditions. In [4],

the author has considered some second order elliptic boundary value problems on bounded domains with boundary conditions depending nonlinearly on the spectral parameter. In [2], we find a study, in a separable Hilbert space, of the following boundary-value second-order elliptic differential-operator equation:

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), & x \in (0,1) \\ \lambda u'(0) - \alpha u(1) = f_1, \ u(1) = f_2, \end{cases}$$

where λ is the spectral parameter, α is a complex number with $\mathrm{Re}(\alpha) \geqslant 0$ and $-A$ is a linear self-adjoint operator garanteeing the ellipticity of the equation. Note that here, the parameter λ appears in the nonlocal boundary condition. Recently, in [1], the authors have considered the following boundary-value problem for an elliptic differential-operator equation of second order

$$\begin{cases} \lambda^2 u(x) - u''(x) + Au(x) = f(x), & x \in (0,1) \\ u'(0) + \lambda u(1) = f_1, \ \beta u'(1) + \lambda u(0) = f_2, \end{cases}$$

where the same spectral parameter appears in the equation quadratically; here $-A$ is a closed positive linear operator in some separable complex Hilbert space. In [7], the authors have considered in a complex Banach space X, Problem (1)-(2) where $\lambda = \omega$ is some positive spectral parameter and $\mu = 0$. For ω large enough, under some geometrical assumption on the space X and hypotheses on operators $A - \omega I$ and H including the fact that they commute in the resolvent sense, the authors have furnished necessary and sufficient conditions on the data d_0, u_1 to obtain the existence and uniqueness of a solution u of (1)-(2) with maximal regularity. Recently, in [9], the authors have developed an interesting new approach in a non commutative framework, concerning some general Sturm-Liouville problems with the same Robin boundary condition in 0.

In our study of Problem (1)-(2), the ellipticity of the equation is guaranteed by hypothesis (5) below, this assumption allows us to consider, for suitable λ, μ, the operators

$$\begin{cases} \Lambda_{\lambda,\mu} := (Q_\lambda - H_\mu) + e^{2Q_\lambda}(Q_\lambda + H_\mu) \\ Q_\lambda = -\sqrt{-A + \lambda I}, \quad H_\mu = H + \mu I. \end{cases}$$

In all the sequel, for any closed linear operator T on X, $D(T)$ denotes the domain of T and $\rho(T)$ the resolvent set of T. The key point will be to obtain the invertibility of the determinant $\Lambda_{\lambda,\mu}$ of system (1)-(2) with estimates of $\left\|\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}$, for appropriate λ, μ. To this end, we consider two different situations:

1. $D(H) \subset D(A)$
2. $D(\sqrt{-A}) \subset D(H)$,

where in the first case, we say that operator H is principal, while in the second case, it is operator $\sqrt{-A}$ which is principal. Concrete applications will illustrate these two cases in the end of this work: the first one is adapted to related problems concerning some heat equations with dynamical boundary conditions of reactive-diffusion type or with Wentzell boundary conditions, whereas the second one will concern, for instance, problems involving the Caputo derivative in the boundary conditions. Moreover, using the same tools, we study the Dirichlet case and obtain similar results to those obtained with Robin boundary conditions.

Three new and essential results sum up this work. First, we solve the above equation by giving an explicit and simplified representation of the solution adapted to each case and we show that it is verifying the optimal regularity that is

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)).$$

Then we give sharp estimates of this solution in each case according to the complex spectral parameters λ, μ belonging to some appropriate precised set. This part essencially uses the results of [14] where some inequalities on resolvent operators are precised. Finally, using these estimates, we obtain the generation of two kinds of analytic semigroups each corresponding to a situation when we specify the complex parameters λ, μ.

This article is organized as follows. Section 2 describes the assumptions, including two spectral parameters λ, μ, studied in this work. In Section 3, we deal with our model without spectral parameter, we retrieve in a simple manner results of previous works. Section 4 is devoted to some precise estimates of Dore-Yakubov type, which will be useful to analyze our model. Sections 5 and 6 concern the study of our model with spectral parameters λ, μ under two different types of behaviour concerning operators with respect to their domains and to the parameters. Moreover sharp estimates in λ, μ are furnished for the solution. In Section 7, we furnish results for (1) together with Dirichlet boundary conditions. Then, in Section 8, we apply the results of Sections 5, 6, 7 to generation of semigroups. Finally, Section 9 deals with examples of applications.

2 Assumptions

In all this work, we will use the following notation: for $\varphi \in (0, \pi)$, we set

$$S_\varphi := \{z \in \mathbb{C} \backslash \{0\} : |\arg(z)| \leqslant \varphi\} \cup \{0\}. \tag{3}$$

Our goal is to seek for a classical solution of Problem (1)-(2), that is a function u such that

$$\begin{array}{l} i)\ u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)), \\ ii)\ u(0) \in D(H), \\ iii)\ u \text{ satisfies (1)-(2)}. \end{array}$$

We suppose that

$$X \text{ is a } UMD \text{ space}, \tag{4}$$

Recall that X is a UMD space means that for all $q > 1$ the Hilbert transform is continuous from $L^q(\mathbb{R}; X)$ into itself, see [6].

$$\begin{cases} \exists\ \varphi_0 \in (0,\pi) :\ S_{\varphi_0} \subset \rho(A) \ \text{ and } \exists C_A > 0 : \\ \forall \lambda \in S_{\varphi_0}, \quad \left\|(A - \lambda I)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \dfrac{C_A}{1 + |\lambda|}, \end{cases} \tag{5}$$

$$\begin{cases} \forall s \in \mathbb{R},\ \ (-A)^{is} \in \mathcal{L}(X),\ \exists \theta_A \in (0,\pi) : \\ \sup\limits_{s \in \mathbb{R}} \left\| e^{-\theta_A |s|} (-A)^{is} \right\|_{\mathcal{L}(X)} < +\infty. \end{cases} \tag{6}$$

We now set for $\lambda \in S_{\varphi_0}, \mu \in \mathbb{C}$

$$H_\mu = H + \mu I, \quad Q_\lambda = -\sqrt{-A + \lambda I} \quad \text{and} \quad Q = -\sqrt{-A},$$

The existence of the previous square roots is ensured by subsection 5.1 below and for operator H we consider the two following types of hypotheses:

First case

$$D(H) \subset D(A), \tag{7}$$

and

$$\begin{cases} \exists \varphi_1 \in (0,\pi), \exists C_H > 0 : \\ S_{\varphi_1} \subset \rho(-H) \ \text{ and } \sup\limits_{\mu \in S_{\varphi_1}} (1 + |\mu|) \left\| H_\mu^{-1} \right\|_{\mathcal{L}(X)} \leqslant C_H. \end{cases} \tag{8}$$

Second case

$$D(Q) \subset D(H), \tag{9}$$

$$\exists \varepsilon \in (0,1/2],\ \exists C_{H,Q} > 0,\ \sup_{t\in[0,+\infty)} (1+t)^{\varepsilon} \parallel HQ_t^{-1} \parallel_{\mathcal{L}(X)} \leqslant C_{H,Q}, \tag{10}$$

$$(Q-H)^{-1}\left((D(Q),X)_{1/p,p}\right) \subset Q^{-1}\left((D(Q),X)_{1/p,p}\right), \tag{11}$$

here $Q-H$ is not supposed to be boundedly invertible and

$$(Q-H)^{-1}\left((D(Q),X)_{1/p,p}\right) = \left\{\xi \in D(Q) : (Q-H)\,\xi \in (D(Q),X)_{1/p,p}\right\}.$$

In order to obtain spectral estimates for the solutions of (1)-(2) we will replace assumption (11) by the new assumption:

$$(Q-H)^{-1}(D(Q)) \subset D\left(Q^2\right), \tag{12}$$

where $(Q-H)^{-1}(D(Q)) = \{\xi \in D(Q) : (Q-H)\,\xi \in D(Q)\}$.

Remark 2.1. In these two cases we do not suppose the commutativity between the resolvent operators of H and A. On the other hand, there is no reason a priori so that the solution u verifies $u(0) \in D(H)$.

Remark 2.2. Assume (5) and (9).

1. If we have the commutativity assumption

$$\forall \zeta \in D(H), \quad Q^{-1}H\zeta = HQ^{-1}\zeta,$$

then (12) is satisfied.

2. Let $(\lambda,\mu) \in S_{\varphi_0} \times \mathbb{C}$. If $\xi \in D(Q)$, there exists $\zeta \in X$ such that $\xi = Q^{-1}\zeta$, so

$$\begin{aligned}(Q-H)\,\xi &= [Q_\lambda - H_\mu + \mu I + (Q-Q_\lambda)]\,\xi \\ &= (Q_\lambda - H_\mu)\,\xi + \mu\xi + (Q-Q_\lambda)\,Q^{-1}\zeta,\end{aligned}$$

and it will be seen in Lemma 4.4 below that there exists $T_\lambda \in \mathcal{L}(X)$ such that

$$Q = Q_\lambda + T_\lambda \quad \text{and} \quad Q^{-1}T_\lambda = T_\lambda Q^{-1}.$$

then

$$(Q-H)\,\xi - (Q_\lambda - H_\mu)\,\xi = \mu\xi + Q^{-1}T_\lambda\zeta \in D(Q).$$

This proves that $(Q-H)^{-1}(D(Q)) = (Q_\lambda - H_\mu)^{-1}(D(Q))$ and then (12) is equivalent to

$$(Q_\lambda - H_\mu)^{-1}(D(Q)) \subset D\left(Q^2\right). \tag{13}$$

Remark 2.3. Assume (5).

1. If $H \in \mathcal{L}(X)$ then, for $t \geqslant 0$

$$\parallel HQ_t^{-1} \parallel_{\mathcal{L}(X)} \leqslant \parallel H \parallel_{\mathcal{L}(X)} \parallel Q_t^{-1} \parallel_{\mathcal{L}(X)},$$

so due to (37) below, we have (9) and (10) with $\varepsilon = 1/2$.

2. If there exists $\omega \in [0,1/2)$ such that $D((-A)^{\omega}) \subset D(H)$ then, in virtue of Lemma 2.6 statement a) in [14], there exists $C_\omega > 0$ such that, for $t \geqslant 0$

$$\begin{aligned} \| HQ_t^{-1} \|_{\mathcal{L}(X)} &\leqslant \| H(-A)^{-\omega} \|_{\mathcal{L}(X)} \| (-A)^{\omega} (-A+tI)^{-1/2} \|_{\mathcal{L}(X)} \\ &\leqslant \frac{C_\omega}{(1+t)^{1/2-\omega}}, \end{aligned}$$

so we have (9) and (10) with $\varepsilon = 1/2 - \omega \in (0,1/2]$.

3. It will be seen, see (57) below, that (10) can be written as

$$\exists \varepsilon \in (0,1/2],\ \exists C_{H,Q} > 0,\ \forall \lambda \in S_{\varphi_0} :\ \| HQ_\lambda^{-1} \|_{\mathcal{L}(X)} \leqslant \frac{C_{H,Q}}{(1+|\lambda|)^{\varepsilon}}.$$

4. In [20], the authors have considered (1)-(2) (with the complex number λ replaced by $\omega > 0$ and $\mu = 0$) under assumptions (4)$\sim$(6) together with

$$\begin{cases} D(A) \subset D(H) \text{ and } \exists \omega_0 \geqslant 0, \exists \nu \in (0,1/2), \exists C > 0, \forall \sigma \geqslant 0, \forall \omega \geqslant \omega_0 : \\ \left\| H(A-(\omega+\sigma)I)^{-1} \right\|_{\mathcal{L}(X)} \leqslant \dfrac{C}{|\omega+\sigma|^{1/2+\nu}}. \end{cases} \tag{14}$$

Then they prove the existence of $\omega_1 > \omega_0$ such that for any $\omega \geqslant \omega_1$

$$0 \in \rho(Q_\omega \pm H), \quad D(Q_\omega) \subset D(H) \quad \text{and} \quad \| HQ_\omega^{-1} \|_{\mathcal{L}(X)} \leqslant \frac{C}{\omega^{\nu}}.$$

They have supposed moreover that

$$\forall \omega \geqslant \omega_1, \quad Q_\omega (Q_\omega \pm H)^{-1} \left((D(Q_\omega), X)_{1/p,p} \right) \subset (D(Q_\omega), X)_{1/p,p}. \tag{15}$$

These assumptions are, in some sense, stronger than (9)$\sim$(11).

5. For suitable λ, μ we can observe that in the first case

$$Q_\lambda - H_\mu = \left(Q_\lambda H_\mu^{-1} - I \right) H_\mu,$$

is boundedly invertible and in the second case

$$Q_\lambda - H_\mu = \left(I - H_\mu Q_\lambda^{-1} \right) Q_\lambda,$$

is also boundedly invertible. In fact, in the two cases we can prove that

$$\Lambda_{\lambda,\mu} := (Q_\lambda - H_\mu) + e^{2Q_\lambda} (Q_\lambda + H_\mu),$$

is boundedly invertible which will allow us to build a representation formula of the classical solution u. So we could generalize these two cases by considering λ, μ for which $Q_\lambda - H_\mu$ is boundedly invertible and $\left\| (Q_\lambda + H_\mu)^{-1} \right\|_{\mathcal{L}(X)}$ is small enough in order to get that

$$\Lambda_{\lambda,\mu} = \left(I - e^{2Q_\lambda} \right) \left(I - 2 \left(I - e^{2Q_\lambda} \right)^{-1} Q_\lambda e^{2Q_\lambda} (Q_\lambda + H_\mu)^{-1} \right),$$

is boundedly invertible, see for example [7]. In this generalization we suppose neither $D(H) \subset D(A)$ nor $D(Q) \subset D(H)$, but it could be difficult to obtain estimates of $\left\| (Q_\lambda + H_\mu)^{-1} \right\|_{\mathcal{L}(X)}$ with respect to λ, μ.

3 Problem without parameters

In this section we study a problem similar to (1)-(2), but without the parameters λ and μ, that is

$$\begin{cases} u''(x) + Au(x) = f(x), \quad x \in (0,1) \\ u'(0) - Hu(0) = d_0, \ u(1) = u_1. \end{cases} \tag{16}$$

3.1 Hypotheses

Here our hypotheses are

(H_1) X is a UMD space,

(H_2) $[0,+\infty) \subset \rho(A)$ and $\sup_{t\in[0,+\infty)} \left\|(A-tI)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \dfrac{C}{1+t}$,

(H_3) $\forall s \in \mathbb{R}$, $(-A)^{is} \in \mathcal{L}(X)$ and

$$\exists \theta_A \in (0,\pi): \quad \sup_{s\in\mathbb{R}} \left\|e^{-\theta_A|s|}(-A)^{is}\right\|_{\mathcal{L}(X)} < +\infty,$$

(H_4) $\Lambda := (Q-H) + e^{2Q}(Q+H)$ is closed and boundedly invertible, where $Q := -\sqrt{-A}$.

(H_5) $Q\Lambda^{-1}\left((D(Q),X)_{1/p,p}\right) \subset (D(Q),X)_{1/p,p}$.

Note that here we are neither in case 1, nor in case 2.
In the following remark we discuss about assumption (H_5).

Remark 3.1.

1. Assume (H_2) and (H_4). If we suppose moreover

$$A^{-1}\Lambda^{-1} = \Lambda^{-1}A^{-1}, \tag{17}$$

then (H_5) is satisfied. In fact, the following assertions are equivalent.

(a) $A^{-1}\Lambda^{-1} = \Lambda^{-1}A^{-1}$,
(b) $\forall \lambda \in \rho(A)$, $(A-\lambda I)^{-1}\Lambda^{-1} = \Lambda^{-1}(A-\lambda I)^{-1}$,
(c) $Q^{-1}\Lambda^{-1} = \Lambda^{-1}Q^{-1}$.

Then, under (17), we have

$$Q(Q-tI)^{-1}Q\Lambda^{-1} = Q\Lambda^{-1}Q(Q-tI)^{-1},$$

and since

$$(D(Q),X)_{1/p,p} = \left\{x \in X : t^{1-1/p}Q(Q-tI)^{-1}x \in L^p_*(\mathbb{R}_+;X)\right\},$$

we get (H_5).

Finally we remark that if

$$\forall \zeta \in D(H), \quad A^{-1}\zeta \in D(H) \quad \text{and} \quad A^{-1}H\zeta = HA^{-1}\zeta, \tag{18}$$

then (17) and (H_5) are satisfied.

2. Assume (H_2) and (H_4); then (H_5) is equivalent to

$$\Lambda^{-1}\left(\left(D\left(Q\right),X\right)_{1/p,p}\right)\subset\left(D\left(Q\right),X\right)_{1+1/p,p}.$$

3. In [8], problem (16) has been studied under more restrictive assumptions, that are $(H_1)\sim(H_4)$ and the commutativity hypothesis

$$\exists\lambda_0\in\rho(H):\quad A^{-1}\left(H-\lambda_0 I\right)^{-1}=\left(H-\lambda_0 I\right)^{-1}A^{-1}, \tag{19}$$

which, from statement 1., implies (H_5).

4. Assume (H_2) and (H_4). If $Q-H$ is boundedly invertible then, due to $\Lambda\Lambda^{-1}=I$ and $\Lambda^{-1}\Lambda=I$, we get that

$$\begin{cases}\Lambda^{-1}=\left(Q-H\right)^{-1}-\left(Q-H\right)^{-1}e^{2Q}\left(Q+H\right)\Lambda^{-1}\\ \left(Q-H\right)^{-1}=\Lambda^{-1}+\Lambda^{-1}e^{2Q}\left(Q+H\right)\left(Q-H\right)^{-1},\end{cases} \tag{20}$$

from which we deduce that for any $\xi\in\left(D\left(Q\right),X\right)_{1/p,p}$

$$\left[Q\Lambda^{-1}\xi\in\left(D\left(Q\right),X\right)_{1/p,p}\right]\Longleftrightarrow\left[Q\left(Q-H\right)^{-1}\xi\in\left(D\left(Q\right),X\right)_{1/p,p}\right],$$

so we can replace in the previous proposition assumption (H_5) by the equivalent one

$$(H_5')\quad Q\left(Q-H\right)^{-1}\left(\left(D\left(Q\right),X\right)_{1/p,p}\right)\subset\left(D\left(Q\right),X\right)_{1/p,p}, \tag{21}$$

5. Assume (H_2) and (H_4). If we suppose that

$$\forall\xi\in D(Q),\quad Q\Lambda^{-1}\xi\in D(Q),$$

then we have (H_5), see Lemma 5 p. 76 in ([9]).

Similarly, when $Q-H$ is boundedly invertible, using (20) we have that

$$\forall\xi\in D(Q),\quad Q\left(Q-H\right)^{-1}\xi\in D(Q),$$

implies (H_5') and then (H_5).

Remark 3.2. Problem (16) has been already studied (with or without spectral parameters):

- In [9], with hypotheses similar to those given here, as a particular case ($B=0$) of a more general problem

$$\begin{cases}u''(x)+2Bu'\left(x\right)+Au(x)=f(x),\quad x\in(0,1)\\ u'(0)-Hu(0)=d_0,\ u(1)=u_1.\end{cases}$$

where A and B are not supposed to commute. The representation formula of the solution given in [9] was obtained after long calculations.

- In [20], the authors also furnishes a representation formula of the solution of

$$\begin{cases}u''(x)+Au(x)-\omega u\left(x\right)=f(x),\quad x\in(0,1)\\ u'(0)-Hu(0)=d_0,\ u(1)=u_1,\end{cases}$$

under (4)$\sim$(6), (14) and (15), but the method is quite long and laborious.

We propose in the following, see subsection 3.3, a simple way to obtain in our case a representation formula written in a condensed form, which will be useful for the study of the problem with a spectral parameter.

3.2 Interpolation spaces

Let us give now some necessary conditions to obtain a classical solution for our problem (16) using known properties of interpolation spaces.

Lemma 3.3. Suppose that Problem (16) has a classical solution u. Then:

1. $u(0), u(1) \in \left(D(Q^2), X\right)_{\frac{1}{2p},p} = \left(X, D(Q^2)\right)_{1-\frac{1}{2p},p}$, which implies that

$$u(0),\ u(1) \in D(Q) \quad \text{and} \quad Qu(0),\ Qu(1) \in \left(D(Q), X\right)_{1/p,p}.$$

2. $u'(0),\ u'(1) \in \left(D(Q^2), X\right)_{\frac{1}{2}+\frac{1}{2p},p} = \left(D(Q), X\right)_{\frac{1}{p},p}$.

Proof. Suppose that Problem (16) has a classical solution u. Then, from

$$u \in W^{2,p}(0,1;X) \cap L^p\left(0,1;D(Q^2)\right), \quad 1 < p < +\infty,$$

we have $u(0), u(1) \in \left(D(Q^2), X\right)_{\frac{1}{2p},p} = \left(X, D(Q^2)\right)_{1-\frac{1}{2p},p}$, (see [17], Teorema 2', p. 678). But

$$\begin{aligned}\left(D\left(Q^2\right), X\right)_{1/2p,p} &= D_{Q^2}(1-1/2p, p) = \left(D(Q), X\right)_{1+1/p,p} \\ &= \left\{\varphi \in D(Q) : Q\varphi \in \left(D(Q), X\right)_{1/p,p}\right\} \subset D(Q),\end{aligned}$$

from which it follows that

$$u(0), u(1) \in D(Q) \quad \text{and} \quad Qu(0), Qu(1) \in \left(D(Q), X\right)_{1/p,p}.$$

Similarly, by using Teorema 2', in [17], p. 678, we have

$$u'(0), u'(1) \in \left(D(Q^2), X\right)_{\frac{1}{2}+\frac{1}{2p},p} = \left(D(Q), X\right)_{\frac{1}{p},p}.$$

□

3.3 Representation formula

Under (H_2), if u is a classical solution of (16) then there exist $\xi_0, \xi_1 \in X$ such that

$$u(x) = e^{xQ}\xi_0 + e^{(1-x)Q}\xi_1 + I(x) + J(x), \quad x \in [0,1], \tag{22}$$

where

$$I(x) = \frac{1}{2}Q^{-1}\int_0^x e^{(x-s)Q} f(s)ds \quad \text{and} \quad J(x) = \frac{1}{2}Q^{-1}\int_x^1 e^{(s-x)Q} f(s)ds, \tag{23}$$

see [7], p. 989. Note that here, unlike [7], we do not supposed that A and H commute. Now, taking into account the fact that $I - e^{2Q}$ is invertible, we set $T = \left(I - e^{2Q}\right)^{-1} \in \mathcal{L}(X)$ and

$$S(x) = T\left(e^{xQ} - e^{(1-x)Q}e^{Q}\right) \in \mathcal{L}(X), \quad x \in [0,1],$$

then formula (22) writes in the following form

$$u(x) = S(x)\mu_0 + S(1-x)\mu_1 + I(x) + J(x), \quad x \in [0,1],$$

with $\mu_0 = \xi_0 + e^Q\xi_1, \mu_1 = e^Q\xi_0 + \xi_1$ and we deal with this new writing. We note that

$$\begin{cases} u_0 = u(0) = \mu_0 + J(0) \\ u_1 = u(1) = \mu_1 + I(1) \\ u'(0) = TQ\left(I + e^{2Q}\right)\mu_0 - 2TQe^Q\mu_1 - QJ(0), \end{cases}$$

and now we determine μ_0, μ_1 by using the boundary conditions

$$u(1) = u_1 \quad \text{and} \quad u'(0) - Hu(0) = d_0.$$

So $\mu_1 = u_1 - I(1)$ and

$$TQ\left(I + e^{2Q}\right)\mu_0 - 2TQe^Q\mu_1 - QJ(0) - H(\mu_0 + J(0)) = d_0,$$

hence

$$\begin{aligned} TQ\left(I + e^{2Q}\right)(\mu_0 + J(0)) - H(\mu_0 + J(0)) &= d_0 + 2TQe^Q\mu_1 \\ &\quad + QJ(0) + TQ\left(I + e^{2Q}\right)J(0), \end{aligned}$$

thus

$$\begin{aligned} \left[Q\left(I + e^{2Q}\right) - \left(I - e^{2Q}\right)H\right](\mu_0 + J(0)) &= \left(I - e^{2Q}\right)d_0 + 2Qe^Q\mu_1 \\ &\quad + \left(I - e^{2Q}\right)QJ(0) + Q\left(I + e^{2Q}\right)J(0), \end{aligned}$$

but $\Lambda = Q\left(I + e^{2Q}\right) - \left(I - e^{2Q}\right)H$ so

$$\begin{cases} \mu_1 = u_1 - I(1) \\ \mu_0 = \Lambda^{-1}\left[\left(I - e^{2Q}\right)d_0 + 2Qe^Q\mu_1 + 2QJ(0)\right] - J(0). \end{cases}$$

Finally, if u is a classical solution of (16) then

$$u(x) = S(x)\mu_0 + S(1-x)\mu_1 + I(x) + J(x), \quad x \in [0,1], \tag{24}$$

where

$$\begin{cases} \mu_1 = u_1 - I(1) \\ \mu_0 = \Lambda^{-1}\left[\left(I - e^{2Q}\right)d_0 + 2Qe^Q\mu_1 + 2QJ(0)\right] - J(0) \\ S(x) = \left(I - e^{2Q}\right)^{-1}\left(e^{xQ} - e^{(1-x)Q}e^Q\right). \end{cases} \tag{25}$$

When (19) is satisfied, we can check that this representation formula coincides with the one given in [8] p. 528. We can also, after computations, verify that (24) is the same formula as the one p. 92 (with $L = M = Q$) in [9] and also compare it with (34)$\sim$(38) pp. 54-55, in [20].

3.4 Regularity results

The following results will be useful to study the regularity of the solution of (16).

Lemma 3.4. Let $p \in (1, +\infty), \psi \in X$ and $n \in \mathbb{N}\setminus\{0\}$. Then, under (H_2), we have

1. $x \mapsto e^{xQ}\psi \in L^p(0,1,X)$.
2. $x \mapsto Q^n e^{xQ}\psi \in L^p(0,1,X)$ if and only if $\psi \in (D(Q^n), X)_{\frac{1}{np},p}$.

See for instance [25], Theorem, p. 96.

Lemma 3.5. For $f \in L^p(0,1,X)$ with $1 < p < +\infty$, under $(H_1) \sim (H_3)$, we have

1. $x \mapsto Q\int_0^x e^{(x-s)Q} f(s)\,ds \in L^p(0,1,X)$,

   $x \mapsto Q\int_x^1 e^{(s-x)Q} f(s)\,ds \in L^p(0,1,X)$.

2. $x \mapsto Q\int_0^1 e^{(x+s)Q} f(s)\,ds \in L^p(0,1,X)$.

For statements 1 and 2 which are consequences of the Dore-Venni Theorem, see [16], p. 167-168 and also (24), (25) and (26) in [15].

Lemma 3.6. Let $\psi, \chi \in X$ and $1 < p < +\infty$. Then, under (H_2), we have

1. $x \mapsto Q^2 S(x)\psi \in L^p(0,1;X) \iff \psi \in \left(D\left(Q^2\right),X\right)_{\frac{1}{2p},p}$,

   $x \mapsto Q^2 S(1-x)\chi \in L^p(0,1;X) \iff \chi \in \left(D\left(Q^2\right),X\right)_{\frac{1}{2p},p}$.

2. $x \mapsto Q^2 S(x)\psi + Q^2 S(1-x)\chi \in L^p(0,1;X) \iff \psi, \chi \in \left(D\left(Q^2\right),X\right)_{\frac{1}{2p},p}$.

Proof.

1. $T = \left(I - e^{2Q}\right)^{-1} = I + e^{2Q}\left(I - e^{2Q}\right)^{-1}$ so

$$S(x) = e^{xQ} + \left(I - e^{2Q}\right)^{-1} e^{xQ} e^{2Q} - \left(I + e^{2Q}\left(I - e^{2Q}\right)^{-1}\right) e^{(1-x)Q} e^{Q};$$

   then, by lemma 3.4

$$\begin{aligned} Q^2 S(\cdot)\psi \in L^p(0,1;X) \quad &\iff \quad Q^2 e^{\cdot Q}\psi \in L^p(0,1;X) \\ &\iff \quad \psi \in \left(D\left(Q^2\right),X\right)_{\frac{1}{2p},p}. \end{aligned}$$

2. For any $\psi, \chi \in X$, we have

$$Q^2 S(\cdot)\psi \in L^p(1/2,1;X) \quad \text{and} \quad Q^2 S(1-\cdot)\chi \in L^p(0,1/2;X).$$

   Then

$$Q^2 S(\cdot)\psi + Q^2 S(1-\cdot)\chi \in L^p(0,1;X),$$

   if and only if $Q^2 S(\cdot)\psi \in L^p(0,1;X)$ and $Q^2 S(1-\cdot)\chi \in L^p(0,1;X)$.

□

Lemma 3.7. Consider μ_0, μ_1 defined in (25). Then

$$\begin{cases} \mu_0 \in \left(D(A),X\right)_{\frac{1}{2p},p} \iff \Lambda^{-1} d_0 \in \left(D(A),X\right)_{\frac{1}{2p},p} \\ \mu_1 \in \left(D(A),X\right)_{\frac{1}{2p},p} \iff u_1 \in \left(D(A),X\right)_{\frac{1}{2p},p}. \end{cases}$$

Proof. From [18], Proposition 3.5, p. 1676, we have $J(0), I(1) \in (D(A), X)_{\frac{1}{2p},p}$, thus:

$$\mu_1 \in (D(A), X)_{\frac{1}{2p},p} \iff u_1 \in (D(A), X)_{\frac{1}{2p},p}.$$

Moreover $\mu_0 = \Lambda^{-1} d_0 - \Lambda^{-1}\left[e^{2Q} d_0 + 2Qe^{Q}\mu_1 + 2QJ(0)\right] - J(0)$, with

$$e^{2Q} d_0 + 2Qe^{Q}\mu_1 + 2QJ(0) \in (D(Q), X)_{1/p,p},$$

and from (H_5)

$$Q\Lambda^{-1}\left[e^{2Q} d_0 + 2Qe^{Q}\mu_1 + 2QJ(0)\right] \in (D(Q), X)_{1/p,p},$$

which means that

$$\Lambda^{-1}\left[e^{2Q} d_0 + 2Qe^{Q}\mu_1 + 2QJ(0)\right] \in (D(Q), X)_{1+1/p,p} = (D(A), X)_{\frac{1}{2p},p}.$$

Finally: $\mu_0 \in (D(A), X)_{\frac{1}{2p},p} \iff \Lambda^{-1} d_0 \in (D(A), X)_{\frac{1}{2p},p}$. □

3.5 Resolution of Problem (16)

We are now in a position to retrieve (in the case $L = M$) in a simple manner, the result of Corollary 1, p. 92 in [9] and also Theorem 2, p. 56, in [20].

Proposition 3.8. Let $f \in L^p(0,1;X)$ with $1 < p < +\infty$ and assume that $(H_1) \sim (H_5)$ are satisfied. Then the following assertions are equivalent:

1. Problem (16) admits a classical solution u.
2. $u_1, \Lambda^{-1} d_0 \in (D(A), X)_{\frac{1}{2p},p}$

   Moreover in this case u is unique and it is given by (24).

Proof. From the previous study we know that if Problem (16) admits a classical solution u then u is unique and given by (24). Moreover u defined by (24) satisfies

$$u(0) = \mu_0 + J(0) = \Lambda^{-1}\left[\left(I - e^{2Q}\right) d_0 + 2Qe^{Q}\mu_1 + 2QJ(0)\right] \in D(H),$$

and then u is a classical solution of (16) if and only if $Q^2 u(\cdot) \in L^p(0,1;X)$.

But from Lemma 3.5, $Q^2 I(\cdot), Q^2 J(\cdot) \in L^p(0,1;X)$, so Lemma 3.6 and Lemma 3.7 imply that

$$\begin{aligned} Q^2 u(\cdot) \in L^p(0,1;X) &\iff Q^2 S(x)\mu_0 + Q^2 S(1-x)\mu_1 \in L^p(0,1;X) \\ &\iff \mu_0, \mu_1 \in (D(A), X)_{\frac{1}{2p},p} \\ &\iff u_1, \Lambda^{-1} d_0 \in (D(A), X)_{\frac{1}{2p},p}, \end{aligned}$$

this proves that statement 1. is equivalent to statement 2. □

Remark 3.9. In the previous Proposition, if moreover, $Q - H$ is boundedly invertible, then using (20) we can replace the condition $\Lambda^{-1} d_0 \in (D(A), X)_{\frac{1}{2p},p}$ by the simplest one $(Q - H)^{-1} d_0 \in (D(A), X)_{\frac{1}{2p},p}$.

4 Dore-Yakubov estimates

This section is devoted to Dore-Yakubov Estimates and applications. The results are essentially based on those given in [14] and we have used the definitions and notations of this paper. We consider here a complex Banach space E.

In the following we fix φ in $(0,\pi)$ and $L : D(L) \subset E \longrightarrow E$ is a closed densely defined linear operator. Note that here E is not supposed to be a UMD space and there is no BIP assumption on operator L.

Definition 4.1. L is said to be an operator of type φ with bound C_L if $S_\varphi \subset \rho(-L)$ and

$$\forall \lambda \in S_\varphi, \quad \left\|(L+\lambda I)^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{C_L}{1+|\lambda|},$$

where S_φ is defined by (3).

In all this section L is an operator of type φ with bound C_L. We fix $\lambda \in S_\varphi$ and we set

$$D_\lambda := L + \lambda I,$$

and

$$\varepsilon(\varphi) := \min\{\varphi, \pi-\varphi\} = \begin{cases} \varepsilon(\varphi) = \varphi & \text{if} \quad \varphi \in (0,\pi/2] \\ \varepsilon(\varphi) = \pi-\varphi & \text{if} \quad \varphi \in [\pi/2,\pi), \end{cases}$$

note that $\varepsilon(\varphi) \in (0,\pi/2)$. The first Lemma below is essentially based on Lemmas 2.3 and 2.4 in [14]. The novelty is in some precisions given on the estimate of

$$\left\|\left((L+\lambda I)^{1/2}+\nu I\right)^{-1}\right\|_{\mathcal{L}(E)},$$

which integrates the behaviour with respect to the complex parameters λ and ν.

Lemma 4.2.

1. Let $\theta \in (0,\varepsilon(\varphi))$; then D_λ is an operator of type θ with bound

$$C_\theta := C_L/\cos\left(\frac{\varphi+\theta}{2}\right) \in (0,+\infty).$$

Moreover for $\nu \in S_\theta$ we have

$$\left\|(D_\lambda+\nu I)^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{C_\theta}{|\lambda|+|\nu|+1}. \tag{26}$$

In particular, setting $C_0 = C/\cos(\varphi/2)$, we have for any $\nu \geqslant 0$

$$\left\|(D_\lambda+\nu I)^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{C_0}{|\lambda|+\nu+1}. \tag{27}$$

2. $D_\lambda^{1/2}$ is well defined and $-D_\lambda^{1/2}$ generates a semigroup $\left(e^{-tD_\lambda^{1/2}}\right)_{t\geqslant 0}$ which is bounded, analytic for $t>0$ and strongly continuous for $t \geqslant 0$.

3. Let $\overline{\theta} \in \mathbb{R}$ such that $\theta = \left|\overline{\theta}\right| \in (0,\varepsilon(\varphi))$. Then, for a complex number ν with $\mathrm{Re}\left(\nu e^{-i\overline{\theta}/2}\right) > 0$ we get : $D_\lambda^{1/2}+\nu I$ is boundedly invertible and

$$\left\|\left(D_\lambda^{1/2}+\nu I\right)^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{C_{\nu,\overline{\theta}}}{|\nu|+\sqrt{|\lambda|+1}},$$

where $C_{\nu,\overline{\theta}} := C_L/\left[\cos\left(\arg(\nu)-\overline{\theta}/2\right)\cos\left(\frac{\varphi+\theta}{2}\right)\right]$.

4. Let $\psi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \frac{\varepsilon(\varphi)}{2}\right)$; then $D_\lambda^{1/2}$ is of type ψ with bound $K_\psi := C_L/\cos^2(\beta_\psi)$ where
$$\beta_\psi = \frac{\pi}{4} + \frac{\psi - \varepsilon(\varphi)}{2} \in (0, \pi/2).$$
Moreover for $\nu \in S_\psi$ we have
$$\left\| \left(D_\lambda^{1/2} + \nu I\right)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{K_\psi}{|\nu| + \sqrt{|\lambda| + 1}}. \tag{28}$$
In particular, setting $K_L := C_L/\cos^2(\varphi/2)$, we have for any $\nu \in S_{\pi/2}$
$$\left\| \left(D_\lambda^{1/2} + \nu I\right)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{K_L}{|\nu| + \sqrt{|\lambda| + 1}}. \tag{29}$$

Proof.

1. See [14], (2.1) in Lemma 2.4, p. 99 and also Theorem 10.3, p. 320 in [21]. Remark that we have $\cos\left(\frac{\varphi + \theta}{2}\right) > 0$. Moreover, noting that for any $\nu \geqslant 0$ we have
$$\nu \in \bigcap_{\theta \in (0, \varepsilon(\varphi))} S_\theta,$$
we get (27) with $C_0 = \lim_{\theta \longrightarrow 0^+} C_\theta$.

2. See [14], Lemma 2.4.

3. The idea is to use the calculus given in [14], in Lemma 2.4, at the end of p. 99
$$\begin{aligned}
\left\| \left(D_\lambda^{1/2} + \nu I\right)^{-1} \right\|_{\mathcal{L}(E)} &= \left\| \frac{1}{\pi} \int_0^{+\infty e^{i\bar{\theta}}} \frac{z^{1/2}}{z + \nu^2} (D_\lambda + zI)^{-1} dz \right\|_{\mathcal{L}(E)} \\
&= \left\| \frac{1}{\pi} \int_0^{+\infty} \frac{r^{1/2} e^{i\bar{\theta}/2}}{re^{i\bar{\theta}} + \nu^2} \left(D_\lambda + re^{i\bar{\theta}} I\right)^{-1} e^{i\bar{\theta}} dr \right\|_{\mathcal{L}(E)},
\end{aligned}$$
but now we apply the estimate $\left\| \left(D_\lambda + re^{i\bar{\theta}} I\right)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{C_\theta}{|\lambda| + r + 1}$, instead of
$$\left\| \left(D_\lambda + re^{i\bar{\theta}} I\right)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{C_\theta}{r + 1},$$
and so
$$\begin{aligned}
\left\| \left(D_\lambda^{1/2} + \nu I\right)^{-1} \right\|_{\mathcal{L}(E)} &\leqslant \frac{C_\theta}{\pi} \int_0^{+\infty} \frac{r^{1/2}}{\left|re^{i\bar{\theta}} + \nu^2\right|} \frac{1}{|\lambda| + r + 1} dr \\
&\leqslant \frac{C_\theta}{\pi} \int_0^{+\infty} \frac{r^{1/2}}{\left|\cos\left(\frac{\arg(\nu^2) - \bar{\theta}}{2}\right)\right| \left(r + |\nu|^2\right)} \frac{1}{|\lambda| + r + 1} dr \\
&\leqslant \frac{C_\theta}{\pi \cos\left(\arg(\nu) - \frac{\bar{\theta}}{2}\right)} \int_0^{+\infty} \frac{r^{1/2}}{\left(r + |\nu|^2\right)(r + |\lambda| + 1)} dr.
\end{aligned}$$

For $a, b > 0, a \neq b$, we have

$$\begin{aligned}\int_0^{+\infty} \frac{r^{1/2}}{(r+a)(r+b)} dr &= \frac{2b}{b-a}\int_0^{+\infty} \frac{1}{x^2+b} dx - \frac{2a}{b-a}\int_0^{+\infty} \frac{1}{x^2+a} dx \\ &= \frac{\pi}{\sqrt{b}+\sqrt{a}},\end{aligned}$$

from which we deduce

$$\left\| \left(D_\lambda^{1/2} + \nu I \right)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{C_L}{\pi \cos\left(\arg(\nu) - \overline{\theta}/2 \right) \cos\left((\varphi+\theta)/2 \right)} \frac{\pi}{|\nu| + \sqrt{|\lambda|+1}}.$$

4. Estimate (28) is deduced from statement 2. as in [14], Lemma 2.4, p. 100. Since $S_{\pi/2} \subset S_\psi$ for any $\psi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \frac{\varepsilon(\varphi)}{2} \right)$, we get (29) with

$$K_L := \lim_{\psi \longrightarrow \pi/2^+} K_\psi = \frac{C_L}{\cos^2\left((\pi - \varepsilon(\varphi))/2 \right)} = \frac{2C_L}{1+\cos(\varphi)} = \frac{C_L}{\cos^2(\varphi/2)}.$$

□

Here, since D_λ is boundedly invertible, we have also that $D_\lambda^{1/2}$ is boundedly invertible and then $\rho\left(D_\lambda^{1/2} \right)$ contains a ball centered in 0. The following Lemma precises the size of this ball with respect to $\lambda \in S_\varphi$.

Lemma 4.3. Setting $r_\lambda = \left\| D_\lambda^{-1/2} \right\|_{\mathcal{L}(E)}^{-1} > 0$ and $K_L := C_L / \cos^2(\varphi/2)$, we have

1. $\left\| D_\lambda^{-1/2} \right\|_{\mathcal{L}(E)} \leqslant \frac{K_L}{\sqrt{|\lambda|+1}}$.
2. $B(0, r_\lambda) \subset \rho(D_\lambda^{1/2})$ and for $z \in B(0, r_\lambda)$
$$\|(D_\lambda^{1/2} - zI)^{-1}\|_{\mathcal{L}(E)} \leqslant \frac{K_L}{\sqrt{|\lambda|+1}} \times \frac{1}{1 - \|z D_\lambda^{-1/2}\|_{\mathcal{L}(E)}}.$$
3. $\overline{B\left(0, \frac{\sqrt{|\lambda|+1}}{2K_L}\right)} \subset B(0, r_\lambda)$ and for $z \in \overline{B\left(0, \frac{\sqrt{|\lambda|+1}}{2K_L}\right)}$
$$\|(D_\lambda^{1/2} - zI)^{-1}\|_{\mathcal{L}(E)} \leqslant \frac{2K_L}{\sqrt{|\lambda|+1}}.$$

Proof.

1. It enough to consider (29) with $\nu = 0$. Note that we can obtain a best estimate: $\left\| \left(D_\lambda^{1/2} \right)^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{C_L/\cos(\varphi/2)}{\sqrt{|\lambda|+1}}$, by using similar arguments as in [14], (2.4) p. 100.
2. We consider $z \in B(0, r_\lambda)$. Then $\|z D_\lambda^{-1/2}\|_{\mathcal{L}(E)} < 1$, so
$$D_\lambda^{1/2} - zI = D_\lambda^{1/2}\left(I - z D_\lambda^{-1/2} \right),$$
is boundedly invertible with
$$\begin{aligned}\|(D_\lambda^{1/2} - zI)^{-1}\|_{\mathcal{L}(E)} &\leqslant \|D_\lambda^{-1/2}\|_{\mathcal{L}(E)} \times \left\| \left(I - z D_\lambda^{-1/2} \right)^{-1} \right\|_{\mathcal{L}(E)} \\ &\leqslant \frac{K_L}{\sqrt{|\lambda|+1}} \times \frac{1}{1 - \|z D_\lambda^{-1/2}\|_{\mathcal{L}(E)}}.\end{aligned}$$

3. We consider $z \in \overline{B\left(0, \frac{\sqrt{|\lambda|+1}}{2K}\right)}$. Then

$$0 \leqslant \|zD_\lambda^{-1/2}\|_{\mathcal{L}(E)} = |z| \, \|D_\lambda^{-1/2}\|_{\mathcal{L}(E)} \leqslant \frac{\sqrt{|\lambda|+1}}{2K_L} \frac{K_L}{\sqrt{|\lambda|+1}} = 1/2 < 1,$$

so $z \in B(0, r_\lambda)$ and

$$\|(D_\lambda^{1/2} - zI)^{-1}\|_{\mathcal{L}(E)} \leqslant \frac{K_L}{\sqrt{|\lambda|+1}} \times \frac{1}{1 - \|zD_\lambda^{-1/2}\|_{\mathcal{L}(E)}} \leqslant \frac{2K_L}{\sqrt{|\lambda|+1}}.$$

□

Now we will compare $D_\lambda^{1/2}$ and $D_0^{1/2}$. This has been already done for $\lambda > 0$ in [19], Proposition 3.1.7 p. 65. Here λ is a complex parameter: we furnish a precise estimate for the bounded operator T_λ which extends $D_\lambda^{1/2} - D_0^{1/2}$; we give also a new writing of T_λ with respect to $\left(D_\lambda^{1/2} + D_0^{1/2}\right)^{-1}$.

Lemma 4.4.

1. There exists a unique $T_\lambda \in \mathcal{L}(E)$ such that

$$D_\lambda^{1/2} = D_0^{1/2} + T_\lambda, \tag{30}$$

moreover $T_\lambda = \frac{\lambda}{2i\pi} \int_\gamma \sqrt{z} \, (zI - D_\lambda)^{-1} \, (zI - L)^{-1} \, dz \in \mathcal{L}(E)$, where γ is the boundary of S_η, positively oriented, with η fixed in $(\pi - \varepsilon(\varphi), \pi)$, and T_λ does not depend on η.

2. $T_\lambda D_{\lambda'}^{-1/2} = D_{\lambda'}^{-1/2} T_\lambda$ for any $\lambda' \in S_\varphi$.

3. $\|T_\lambda\|_{\mathcal{L}(E)} \leqslant \frac{C_L^2}{\cos((\varphi+\pi)/2)} \sqrt{|\lambda|}$.

4. $0 \in \rho\left(D_\lambda^{1/2} + D_0^{1/2}\right)$ and

$$T_\lambda = \lambda \left(D_\lambda^{1/2} + D_0^{1/2}\right)^{-1} = \frac{\lambda}{2i\pi} \int_\gamma \frac{1}{\sqrt{z+\lambda} + \sqrt{z}} \, (zI - L)^{-1} \, dz,$$

so that

$$\forall \xi \in D\left(L^{1/2}\right), \quad \left(D_\lambda^{1/2} - D_0^{1/2}\right) \xi = \lambda \left(D_\lambda^{1/2} + D_0^{1/2}\right)^{-1} \xi \in D(L).$$

Proof.

1. First, notice that $L = D_0$ and $D(L) \subset D\left(D_\lambda^{1/2}\right) \cap D\left(D_0^{1/2}\right)$. Thus, if $T_\lambda \in \mathcal{L}(E)$ satisfies (30) then T_λ is unique since $\overline{D(L)} = E$ and

$$T_\lambda \xi = D_\lambda^{1/2} \xi - D_0^{1/2} \xi, \quad \xi \in D(L),$$

We fix $\eta > 0$ such that $\pi - \varepsilon(\varphi) < \eta < \pi$. Then $\pi - \eta \in (0, \varepsilon(\varphi))$ and, from Lemma 4.2, statement 1., $L = D_0$ and D_λ are operators of type $\pi - \eta$; in particular for $\omega > 0$ small enough we have

$$\sigma(D_0) \cup \sigma(D_\lambda) \subset \omega + S_\eta \subset S_\eta.$$

So, we can define $D_\lambda^{-1/2}, D_0^{-1/2}$ by using functional calculus and considering γ, the positively oriented boundary of S_η:

$$D_\lambda^{-1/2} = \frac{1}{2i\pi}\int_\gamma \frac{1}{\sqrt{z}}\left(zI - D_\lambda\right)^{-1} dz \quad \text{and} \quad D_0^{-1/2} = \frac{1}{2i\pi}\int_\gamma \frac{1}{\sqrt{z}}\left(zI - D_0\right)^{-1} dz;$$

it is well known that these integrals do not depend on η. We write

$$\begin{aligned}
D_\lambda^{-1/2} - D_0^{-1/2} &= \frac{1}{2i\pi}\int_\gamma \frac{1}{\sqrt{z}}\left[\left(zI - D_\lambda\right)^{-1} - \left(zI - D_0\right)^{-1}\right] dz \\
&= \frac{\lambda}{2i\pi}\int_\gamma \frac{1}{\sqrt{z}}\left(zI - D_\lambda\right)^{-1}\left(zI - D_0\right)^{-1} dz \\
&= \frac{\lambda}{2i\pi}L^{-1}\int_\gamma \frac{1}{\sqrt{z}}\left(zI - D_\lambda\right)^{-1}\left(L - zI + zI\right)\left(zI - L\right)^{-1} dz \\
&= -\frac{\lambda}{2i\pi}L^{-1}\int_\gamma \frac{1}{\sqrt{z}}\left(zI - D_\lambda\right)^{-1} dz \\
&\quad + \frac{\lambda}{2i\pi}L^{-1}\int_\gamma \frac{1}{\sqrt{z}}\left(zI - D_\lambda\right)^{-1} z\left(zI - L\right)^{-1} dz \\
&= L^{-1}\left(-\lambda D_\lambda^{-1/2} + \frac{\lambda}{2i\pi}\int_\gamma \sqrt{z}\left(zI - D_\lambda\right)^{-1}\left(zI - L\right)^{-1} dz\right) \\
&= -\lambda L^{-1} D_\lambda^{-1/2} + L^{-1} T_\lambda,
\end{aligned}$$

where $T_\lambda := \dfrac{\lambda}{2i\pi}\displaystyle\int_\gamma \sqrt{z}\left(zI - D_\lambda\right)^{-1}\left(zI - L\right)^{-1} dz \in \mathcal{L}(E)$.

This proves that for $\xi \in E$

$$\begin{cases} D_\lambda^{-1/2}\xi = D_0^{-1/2}\xi - \lambda L^{-1} D_\lambda^{-1/2}\xi + L^{-1}T_\lambda\xi \in D\left(D_0^{1/2}\right) \\ D_0^{-1/2}\xi = D_\lambda^{-1/2}\xi + \lambda L^{-1} D_\lambda^{-1/2}\xi - L^{-1}T_\lambda\xi \in D\left(D_\lambda^{1/2}\right); \end{cases}$$

thus we obtain $D\left(D_\lambda^{1/2}\right) = D\left(D_0^{1/2}\right) = D\left(L^{1/2}\right)$. We then deduce (30) by writing, for $\zeta \in D\left(L^{1/2}\right)$

$$\begin{aligned}
D_\lambda^{1/2}\zeta - D_0^{1/2}\zeta &= \left(L + \lambda I\right) D_\lambda^{-1/2}\zeta - L D_0^{-1/2}\zeta \\
&= L\left(D_\lambda^{-1/2} - D_0^{-1/2}\right)\zeta + \lambda D_\lambda^{-1/2}\zeta \\
&= \frac{\lambda}{2i\pi}\int_\gamma \sqrt{z}\left(zI - D_\lambda\right)^{-1}\left(zI - L\right)^{-1}\zeta dz \\
&= T_\lambda\zeta.
\end{aligned}$$

2. This is an easy consequence of the above definition of T_λ.

3. We have

$$\begin{aligned}
T_\lambda &= -\frac{\lambda}{2i\pi}\int_0^{+\infty} \sqrt{re^{i\eta}}\left(re^{i\eta}I - D_\lambda\right)^{-1}\left(re^{i\eta}I - L\right)^{-1} e^{i\eta} dr \\
&\quad + \frac{\lambda}{2i\pi}\int_0^{+\infty} \sqrt{re^{-i\eta}}\left(re^{-i\eta}I - D_\lambda\right)^{-1}\left(re^{-i\eta}I - L\right)^{-1} e^{-i\eta} dr \\
&= \frac{\lambda}{2i\pi}\int_0^{+\infty} \sqrt{re^{i\eta}}\left(re^{i(\eta-\pi)}I + D_\lambda\right)^{-1}\left(re^{i(\eta-\pi)}I + L\right)^{-1} e^{i\eta} dr \\
&\quad - \frac{\lambda}{2i\pi}\int_0^{+\infty} \sqrt{re^{-i\eta}}\left(re^{i(\pi-\eta)}I + D_\lambda\right)^{-1}\left(re^{i(\pi-\eta)}I + L\right)^{-1} e^{-i\eta} dr,
\end{aligned}$$

but $re^{\pm i(\eta-\pi)} \in S_{\pi-\eta}$ and using (26) with $C_{\pi-\eta} = \dfrac{C_L}{\cos\left(\left(\varphi+\pi-\eta\right)/2\right)}$ we get

$$\begin{aligned}
\|T_\lambda\|_{\mathcal{L}(E)} &\leqslant \frac{|\lambda|}{2\pi}\int_0^{+\infty}\sqrt{r}\frac{C_{\pi-\eta}}{|\lambda|+r+1}\frac{C_L}{1+r}dr \\
&\quad + \frac{|\lambda|}{2\pi}\int_0^{+\infty}\sqrt{r}\frac{C_{\pi-\eta}}{|\lambda|+r+1}\frac{C_L}{1+r}dr \\
&= \frac{|\lambda|\,C_{\pi-\eta}C_L}{\pi}\int_0^{+\infty}\frac{\sqrt{r}}{\left(r+|\lambda|+1\right)\left(r+1\right)}dr \\
&= \frac{|\lambda|\,C_{\pi-\eta}C_L}{\sqrt{|\lambda|+1}+1} \\
&\leqslant \frac{C_L^2}{\cos\left(\left(\varphi+\pi-\eta\right)/2\right)}\sqrt{|\lambda|}.
\end{aligned}$$

Moreover, since T_λ does not depend on η, then

$$\|T_\lambda\|_{\mathcal{L}(E)} \leqslant \lim_{\eta\longrightarrow\pi^-}\frac{C_L^2}{\cos\left(\left(\varphi+\pi-\eta\right)/2\right)}\sqrt{|\lambda|} = \frac{C_L^2}{\cos\left(\varphi/2\right)}\sqrt{|\lambda|}.$$

4. The integral below is absolutely convergent, then $0 \in \rho\left(D_\lambda^{1/2}+D_0^{1/2}\right)$ and

$$\left(D_\lambda^{1/2}+D_0^{1/2}\right)^{-1} = \frac{1}{2i\pi}\int_\gamma \frac{1}{\sqrt{z+\lambda}+\sqrt{z}}\left(zI-L\right)^{-1}dz.$$

Moreover for $\xi \in D\left(L\right)$

$$\begin{aligned}
D_\lambda^{1/2}\xi - D_0^{1/2}\xi &= \left(D_\lambda^{1/2}-D_0^{1/2}\right)\left(D_\lambda^{1/2}+D_0^{1/2}\right)\left(D_\lambda^{1/2}+D_0^{1/2}\right)^{-1}\xi \\
&= \lambda\left(D_\lambda^{1/2}+D_0^{1/2}\right)^{-1}\xi,
\end{aligned}$$

and from the uniqueness of $T_\lambda \in \mathcal{L}(E)$ satisfying (30) we get :

$$T_\lambda = \lambda\left(D_\lambda^{1/2}+D_0^{1/2}\right)^{-1}.$$

□

Remark 4.5.

1. From the previous Lemma, statement 4, we see that $D_\lambda^{1/2}+D_0^{1/2}$ is boundedly invertible but, $D_\lambda^{1/2}-D_0^{1/2}$ is not boundedly invertible unless $L \in \mathcal{L}(E)$.
2. In the three previous Lemmas the most important fact is that the constants C_θ, C_0, $C_{\nu,\bar{\theta}}$, K_ψ, K_L depend only on L and φ, but do not depend on λ.

Lemma 4.6. Let $-\infty < a < b < +\infty$. Then:

1. For $\lambda \in S_\varphi$, $G_\lambda = -D_\lambda^{1/2}$ which generates a semigroup $\left(e^{tG_\lambda}\right)_{t\geqslant 0}$ bounded, analytic for $t > 0$ and strongly continuous for $t \geqslant 0$ satisfies moreover

$$\begin{cases}
\exists K_0 > 0,\ \exists c_0 > 0,\ \forall x \geqslant 1/2,\ \forall \lambda \in S_\varphi : \\
\max\left\{\| e^{xG_\lambda} \|_{\mathcal{L}(E)}, \| G_\lambda e^{xG_\lambda} \|_{\mathcal{L}(E)}\right\} \leqslant K_0 e^{-xc_0|\lambda|^{1/2}}.
\end{cases}$$

2. For $x \in [a,b]$, $\lambda \in S_\varphi$ and $f \in L^p(a,b;E)$ we set,

$$\begin{cases} U_{\lambda,f}(x) = \displaystyle\int_a^x e^{(x-s)G_\lambda} f(s)ds, & a \leqslant x \\ V_{\lambda,f}(x) = \displaystyle\int_x^b e^{(s-x)G_\lambda} f(s)ds, & x \leqslant b. \end{cases} \tag{31}$$

There exists $M_L > 0$ such that for any $f \in L^p(a,b;E)$ and any $\lambda \in S_\varphi$

$$\begin{cases} \|U_{\lambda,f}\|_{L^p(a,b;E)} \leqslant \dfrac{M_L}{\sqrt{|\lambda|+1}} \|f\|_{L^p(a,b;E)} \\ \|V_{\lambda,f}\|_{L^p(a,b;E)} \leqslant \dfrac{M_L}{\sqrt{|\lambda|+1}} \|f\|_{L^p(a,b;E)}. \end{cases}$$

Proof. We fix $\psi \in \left(\dfrac{\pi}{2}, \dfrac{\pi}{2} + \dfrac{\varepsilon(\varphi)}{2}\right)$ and use notations and estimates of Lemma 4.2.

1. See Lemma 2.6, p. 103, in [14].

2. We first focus on $U_{\lambda,f}$. Let $x \in [a,b]$. We apply the Dunford-Riesz Calculus to define $e^{\cdot Q_\lambda}$, and obtain

$$\begin{aligned} U_{\lambda,f}(x) &= \frac{1}{2i\pi} \int_a^x \int_\gamma e^{(x-s)z} (zI - G_\lambda)^{-1} f(s)\, dz\, ds \\ &= \frac{1}{2i\pi} \int_a^x \int_\gamma e^{(x-s)z} (zI + D_\lambda^{1/2})^{-1} f(s)\, dz\, ds, \quad x \geqslant a, \end{aligned}$$

where the path γ is the boundary positively oriented of $S_\psi \cup B(0,\varepsilon)$ with $\varepsilon = \dfrac{\sqrt{|\lambda|+1}}{2K_L}$.

Then

$$\begin{aligned} U_{\lambda,f}(x) &= \frac{1}{2i\pi} \int_a^x \int_\varepsilon^{+\infty} e^{(x-s)re^{i\psi}} (re^{i\psi} I + D_\lambda^{1/2})^{-1} f(s) e^{i\psi} dr ds \\ &+ \frac{1}{2i\pi} \int_a^x \int_{2\pi-\psi}^{\psi} e^{(x-s)\varepsilon e^{i\theta}} (\varepsilon e^{i\theta} I + D_\lambda^{1/2})^{-1} f(s) \varepsilon i e^{i\theta} d\theta ds \\ &- \frac{1}{2i\pi} \int_a^x \int_\varepsilon^{+\infty} e^{(x-s)re^{-i\psi}} (re^{-i\psi} I + D_\lambda^{1/2})^{-1} f(s) e^{-i\psi} dr ds, \end{aligned}$$

hence

$$\begin{aligned} \|U_{\lambda,f}(x)\| &\leqslant \frac{1}{2\pi} \int_a^x \int_\varepsilon^{+\infty} \left\| e^{(x-s)re^{i\psi}} f(s) \right\| \left\| (re^{i\psi} I + D_\lambda^{1/2})^{-1} \right\|_{\mathcal{L}(E)} dr ds \\ &+ \frac{\varepsilon}{2\pi} \int_a^x \int_\psi^{2\pi-\psi} \left\| e^{(x-s)\varepsilon e^{i\theta}} f(s) \right\| \left\| (\varepsilon e^{i\theta} I + D_\lambda^{1/2})^{-1} \right\|_{\mathcal{L}(E)} d\theta ds \\ &+ \frac{1}{2\pi} \int_a^x \int_\varepsilon^{+\infty} \left\| e^{(x-s)re^{-i\psi}} f(s) \right\| \left\| (re^{-i\psi} I + D_\lambda^{1/2})^{-1} \right\|_{\mathcal{L}(E)} dr ds. \end{aligned}$$

We deduce, from Lemma 4.2, statement 4. and Lemma 4.3, statement 3., that

$$\begin{aligned}
\|U_{\lambda,f}(x)\| &\leqslant \frac{1}{2\pi}\int_a^x\int_\varepsilon^{+\infty}\left\|e^{(x-s)re^{i\psi}}f(s)\right\|\frac{K_\psi}{r+\sqrt{|\lambda|+1}}drds\\
&\quad+\frac{\varepsilon}{2\pi}\int_a^x\int_\psi^{2\pi-\psi}\left\|e^{(x-s)\varepsilon e^{i\theta}}f(s)\right\|\frac{2K_L}{\sqrt{|\lambda|+1}}d\theta ds\\
&\quad+\frac{1}{2\pi}\int_a^x\int_\varepsilon^{+\infty}\left\|e^{(x-s)re^{-i\psi}}f(s)\right\|\frac{K_\psi}{r+\sqrt{|\lambda|+1}}drds\\
&\leqslant \frac{K_\psi}{\pi}\int_a^x\int_\varepsilon^{+\infty}e^{(x-s)r\cos(\psi)}\|f(s)\|\frac{1}{r+\sqrt{|\lambda|+1}}drds\\
&\quad+\frac{\varepsilon K_L}{\pi}\frac{1}{\sqrt{|\lambda|+1}}\int_a^x\int_\psi^{2\pi-\psi}e^{(x-s)\varepsilon\cos(\theta)}\|f(s)\|\,d\theta ds\\
&\leqslant \frac{K_\psi}{\pi}\int_a^x\left(\int_\varepsilon^{+\infty}\frac{e^{(x-s)r\cos(\psi)}}{r+\sqrt{|\lambda|+1}}dr\right)\|f(s)\|\,ds\\
&\quad+\frac{\varepsilon K_L}{\pi}\frac{1}{\sqrt{|\lambda|+1}}\int_\psi^{2\pi-\psi}\int_a^x e^{(x-s)\varepsilon\cos(\psi)}\|f(s)\|\,dsd\theta\\
&\leqslant \frac{K_\psi}{\pi}\int_a^x\left(\int_\varepsilon^{+\infty}\frac{e^{(x-s)r\cos(\psi)}}{r+\sqrt{|\lambda|+1}}dr\right)\|f(s)\|\,ds\\
&\quad+\frac{2\varepsilon K_L}{\sqrt{|\lambda|+1}}\int_a^x e^{(x-s)\varepsilon\cos(\psi)}\|f(s)\|\,ds.
\end{aligned}$$

So, setting

$$\begin{cases}
U^1_{\lambda,f}(x)=\dfrac{K_\psi}{\pi}\displaystyle\int_a^x\left(\int_\varepsilon^{+\infty}\frac{e^{(x-s)r\cos(\psi)}}{r+\sqrt{|\lambda|+1}}dr\right)\|f(s)\|\,ds\\
U^2_{\lambda,f}(x)=\dfrac{2\varepsilon K_L}{\sqrt{|\lambda|+1}}\displaystyle\int_a^x e^{(x-s)\varepsilon\cos(\psi)}\|f(s)\|\,ds,
\end{cases}$$

we have

$$\|U_{\lambda,f}\|_{L^p(a,b;E)}\leqslant\left\|U^1_{\lambda,f}\right\|_{L^p(a,b)}+\left\|U^2_{\lambda,f}\right\|_{L^p(a,b)}. \tag{32}$$

Estimate of $\left\|U^1_{\lambda,f}\right\|_{L^p(a,b)}$. Define $g\in L^1(\mathbb{R}), F\in L^p(\mathbb{R})$ by

$$g(t):=\begin{cases}\displaystyle\int_\varepsilon^{+\infty}\frac{e^{tr\cos(\psi)}}{r+\sqrt{|\lambda|+1}}dr & \text{if } t>0\\ 0 & \text{else,}\end{cases}\quad\text{and}\quad F(t):=\begin{cases}\|f(t)\| & \text{if } t\in(a,b)\\ 0 & \text{else,}\end{cases}$$

thus

$$\begin{aligned}
U^1_{\lambda,f}(x) &= \frac{K_\psi}{\pi}\int_a^x g(x-s)\|f(s)\|\,ds+\frac{K_\psi}{\pi}\int_x^b g(x-s)\|f(s)\|\,ds\\
&= \frac{K_\psi}{\pi}\int_a^b g(x-s)\|f(s)\|\,ds\\
&= \frac{K_\psi}{\pi}\int_{-\infty}^{+\infty}g(x-s)F(s)\,ds\\
&= \frac{K_\psi}{\pi}(g*F)(x),
\end{aligned}$$

then from the Young inequality, we obtain

$$\begin{aligned}\left\|U^1_{\lambda,f}\right\|_{L^p(a,b)} &\leqslant \frac{K_\psi}{\pi}\left\|g*F\right\|_{L^p(\mathbb{R})}\\ &\leqslant \frac{K_\psi}{\pi}\left\|g\right\|_{L^1(\mathbb{R})}\left\|F\right\|_{L^p(\mathbb{R})}.\end{aligned}$$

Setting $\ell=\sqrt{|\lambda|+1}$ and noting that $\varepsilon/\ell=1/2K$, we have

$$\begin{aligned}\left\|g\right\|_{L^1(\mathbb{R})} &= \int_0^{+\infty}\int_\varepsilon^{+\infty}\frac{e^{tr\cos(\psi)}}{r+\ell}drdt\\ &= \int_0^{+\infty}\left(\int_\varepsilon^{+\infty}\frac{e^{t(r/\ell)\ell\cos(\psi)}}{r/\ell+1}\frac{dr}{\ell}\right)dt\\ &= \int_0^{+\infty}\left(\int_{\varepsilon/\ell}^{+\infty}\frac{e^{t\rho\ell\cos(\psi)}}{\rho+1}d\rho\right)dt\\ &= \int_{\varepsilon/\ell}^{+\infty}\left(\int_0^{+\infty}e^{t\rho\ell\cos(\psi)}dt\right)\frac{d\rho}{\rho+1}\\ &= \int_{1/2K}^{+\infty}\left[\frac{e^{t\rho\ell\cos(\psi)}}{\rho\ell\cos(\psi)}\right]_0^{+\infty}\frac{d\rho}{\rho+1}\\ &= \frac{1}{\ell\cos(\psi)}\int_{1/2K_L}^{+\infty}\frac{d\rho}{\rho(\rho+1)}\\ &= \frac{\ln(2K_L+1)/\cos(\psi)}{\sqrt{|\lambda|+1}},\end{aligned}$$

finally

$$\left\|U^1_{\lambda,f}\right\|_{L^p(a,b)}\leqslant\frac{K_\psi\ln(2K_L+1)/\pi\cos(\psi)}{\sqrt{|\lambda|+1}}\left\|f\right\|_{L^p(a,b;E)}.$$

Estimate of $\left\|U^2_{\lambda,f}\right\|_{L^p(a,b)}$. Define $h\in L^1(\mathbb{R}),F\in L^p(\mathbb{R})$ by

$$h(t):=\begin{cases}e^{t\varepsilon\cos(\psi)} & \text{if } t>0\\ 0 & \text{else}\end{cases}\quad\text{and}\quad F(t):=\begin{cases}\|f(t)\| & \text{if } t\in(a,b)\\ 0 & \text{else;}\end{cases}$$

then

$$\begin{aligned}U^2_{\lambda,f}(x) &= \frac{2\varepsilon K_L}{\sqrt{|\lambda|+1}}\left(\int_a^x h(x-s)\|f(s)\|\,ds+\int_x^b g(x-s)\|f(s)\|\,ds\right)\\ &= \frac{2\varepsilon K_L}{\sqrt{|\lambda|+1}}\int_a^b h(x-s)\|f(s)\|\,ds\\ &= \frac{2\varepsilon K_L}{\sqrt{|\lambda|+1}}\int_{-\infty}^{+\infty}h(x-s)F(s)\,ds\\ &= \frac{2\varepsilon K_L}{\sqrt{|\lambda|+1}}(h*F)(x);\end{aligned}$$

therefore from the Young inequality, we get

$$\begin{aligned}\left\|U^2_{\lambda,f}\right\|_{L^p(a,b)} &\leqslant \frac{2\varepsilon K_L}{\sqrt{|\lambda|+1}}\left\|h*F\right\|_{L^p(\mathbb{R})}\\ &\leqslant \frac{2\varepsilon K_L}{\sqrt{|\lambda|+1}}\left\|h\right\|_{L^1(\mathbb{R})}\left\|F\right\|_{L^p(\mathbb{R})}\\ &= \frac{2K_L/|\cos(\psi)|}{\sqrt{|\lambda|+1}}\left\|f\right\|_{L^p(a,b;E)}.\end{aligned}$$

From (32) and the two previous estimates, there exists $M_L \geqslant 0$ such that for any $f \in L^p(a,b;E)$

$$\|U_{\lambda,f}\|_{L^p(a,b;E)} \leqslant \frac{M_L}{\sqrt{|\lambda|+1}} \|f\|_{L^p(a,b;E)} .$$

We note that

$$V_{\lambda,f}(x) = U_{\lambda,\widetilde{f}}(b+a-x), \tag{33}$$

with $\widetilde{f}(\cdot) := f(\cdot - a - b)$; then

$$\|V_{\lambda,f}\|_{L^p(a,b;E)} \leqslant \frac{M_L}{\sqrt{|\lambda|+1}} \left\|\widetilde{f}\right\|_{L^p(a,b;E)} = \frac{M_L}{\sqrt{|\lambda|+1}} \|f\|_{L^p(a,b;E)} .$$

□

Definition 4.7. We say that a closed linear operator $\mathcal{A}$ on E, has the L^p regularity property on $[a,b]$, if the Cauchy problem

$$\begin{cases} u'(t) = \mathcal{A}u(t) + f(t), & t \in (a,b) \\ u(a) = 0, \end{cases}$$

admits, for any $f \in L^p(a,b;E)$, a unique solution

$$u_f \in W^{1,p}(a,b;E) \cap L^p(a,b;D(\mathcal{A})) .$$

In this case, there exists $K > 0$ such that for any $f \in L^p(a,b;E)$

$$\|u_f\|_{L^p(a,b;E)} + \left\|u_f'\right\|_{L^p(a,b;E)} + \|\mathcal{A}u_f\|_{L^p(a,b;E)} \leqslant K \|f\|_{L^p(a,b;E)} .$$

For details on the L^p regularity property we refer to [12] and [13].

Lemma 4.8. Assume that $G = -L^{1/2}$ has the L^p regularity property on $[a,b]$, and consider $U_{\lambda,f}, V_{\lambda,f}$ defined in (31). Let $\lambda \in S_\varphi$, then:

1. The linear operator $G_\lambda = -(-L + \lambda I)^{1/2}$ has the L^p regularity property on $[a,b]$.
2. For any $f \in L^p(a,b;E)$, $U_{\lambda,f}, V_{\lambda,f} \in W^{1,p}(a,b;E) \cap L^p(a,b;D(G))$, $U_{\lambda,f}$ is the unique solution of
$$\begin{cases} v'(t) = G_\lambda v(t) + f(t), & t \in (a,b) \\ v(a) = 0, \end{cases} \tag{34}$$
and $V_{\lambda,f}$ is the unique solution of
$$\begin{cases} v'(t) = -G_\lambda v(t) + f(t), & t \in (a,b) \\ v(b) = 0. \end{cases}$$
3. There exists $\widetilde{M}_L > 0$ (which does not depend on λ) such that for any $f \in L^p(a,b;E)$ we have
$$\begin{cases} \sqrt{|\lambda|+1}\, \|U_{\lambda,f}\|_{L^p(a,b;E)} + \left\|U_{\lambda,f}'\right\|_{L^p(a,b;E)} + \|G_\lambda U_{\lambda,f}\|_{L^p(a,b;E)} \leqslant \widetilde{M}_L \|f\|_{L^p(a,b;E)} \\ \sqrt{|\lambda|+1}\, \|V_{\lambda,f}\|_{L^p(a,b;E)} + \left\|V_{\lambda,f}'\right\|_{L^p(a,b;E)} + \|G_\lambda V_{\lambda,f}\|_{L^p(a,b;E)} \leqslant \widetilde{M}_L \|f\|_{L^p(a,b;E)} . \end{cases}$$

Proof. Let $\lambda \in S_\varphi$. We consider T_λ, defined in Lemma 4.4, statement 1. and due to (30), we have $G_\lambda = G - T_\lambda$.

1. Let $f \in L^p(a,b;E)$. Here, we want to show that (34) admits a unique solution in $W^{1,p}(a,b;E) \cap L^p(a,b;D(G))$.

   - First, we define $g \in L^p(a,b;E)$ by
   $$g(t) = e^{(t-a)T_\lambda} f(t), \quad t \in (a,b).$$
   - Then we consider $U_{0,g}$ defined by (31) which is the solution of
   $$\begin{cases} u'(t) = Gu(t) + g(t), & t \in (a,b) \\ u(a) = 0, \end{cases} \tag{35}$$
   but G has the L^p regularity property on $[a,b]$, so
   $$U_{0,g} \in W^{1,p}(a,b;E) \cap L^p(a,b;D(G)).$$
   - Since $T_\lambda \in \mathcal{L}(E)$ and $U_{0,g} \in W^{1,p}(a,b;E) \cap L^p(a,b;D(G))$ we get that
   $$v := e^{-(\cdot-a)T_\lambda} U_{0,g}, \tag{36}$$
   is also in $W^{1,p}(a,b;E) \cap L^p(a,b;D(G))$ with
   $$v' = -T_\lambda e^{-(\cdot-a)T_\lambda} U_{0,g} + e^{-(\cdot-a)T_\lambda} U'_{0,g}.$$
   So using (35) and the fact that $T_\lambda G = G T_\lambda$ on $D(G)$ (see Lemma 4.4, statement 2.) we deduce that
   $$\begin{aligned} v' &= -T_\lambda e^{-(\cdot-a)T_\lambda} U_{0,g} + e^{-(\cdot-a)T_\lambda} (GU_{0,g} + g) \\ &= (G - T_\lambda) e^{-(\cdot-a)T_\lambda} U_{0,g} + e^{-(\cdot-a)T_\lambda} g. \end{aligned}$$
   Finally v satisfies
   $$\begin{cases} v'(t) = (G - T_\lambda) v(t) + f(t), & t \in (a,b) \\ v(a) = 0. \end{cases}$$
   - From Lemma 4.2, statement 5, we have $G_\lambda = G - T_\lambda$ so $v = e^{-(\cdot-a)T_\lambda} U_{0,g}$ is a solution of (34) with the expected regularity. Moreover if w is another solution of (34) then $e^{-(\cdot-a)T_\lambda} w$ satisfies (35), so $e^{-(\cdot-a)T_\lambda} w = U_{0,g}$ and $w = v$; this proves the uniqueness of the solution of (34).

2. From (31) we have that $U_{\lambda,f}$ is a formal solution of (34) then
   $$U_{\lambda,f} = e^{-(\cdot-a)T_\lambda} U_{0,g},$$
   and has the expected regularity. We use (33) to study $V_{\lambda,f}$.

3. Since G has the L^p regularity property on $[a,b]$, there exists $K \geqslant 0$ such for any $h \in L^p(a,b;E)$
   $$\left\| U'_{0,h} \right\|_{L^p(a,b;E)} + \| GU_{0,h} \|_{L^p(a,b;E)} \leqslant K \| h \|_{L^p(a,b;E)}.$$

Now let $\lambda \in S_\varphi$. $U_{\lambda,f}$ satisfies (34) so

$$\begin{cases} U'_{\lambda,f}(t) = (G - T_\lambda)\, U_{\lambda,f}(t) + f(t), & t \in (a,b) \\ U_{\lambda,f}(a) = 0, \end{cases}$$

thus setting $h_\lambda = -T_\lambda U_{\lambda,f} + f$

$$\begin{cases} U'_{\lambda,f}(t) = G U_{\lambda,f}(t) + h_\lambda(t), & t \in (a,b) \\ U_{\lambda,f}(a) = 0, \end{cases}$$

then $U_{\lambda,f} = U_{0,h_\lambda}$ and

$$\begin{aligned} \left\|U'_{\lambda,f}\right\|_{L^p(a,b;E)} + \|G_\lambda U_{\lambda,f}\|_{L^p(a,b;E)} &= \left\|U'_{0,h_\lambda}\right\|_{L^p(a,b;E)} + \|G U_{0,h_\lambda}\|_{L^p(a,b;E)} \\ &\leqslant K \|h_\lambda\|_{L^p(a,b;E)} \\ &\leqslant \|T_\lambda\|_{\mathcal{L}(E)} \|U_{\lambda,f}\|_{L^p(a,b;E)} + \|f\|_{L^p(a,b;E)} \\ &\leqslant \frac{C_L^2}{\cos((\varphi+\pi)/2)} \sqrt{|\lambda|} \frac{M_L}{\sqrt{|\lambda|+1}} \|f\|_{L^p(a,b;E)} \\ &\quad + \|f\|_{L^p(a,b;E)} \\ &\leqslant \widetilde{M}_L \|f\|_{L^p(a,b;E)}. \end{aligned}$$

We use again (33) to study $V_{\lambda,f}$.

□

5 Spectral problem (1)-(2): first case

5.1 Preliminary estimates

In this subsection we suppose that X, A, H satisfy (4)∼(6). Note that the results of Section 4, can be applied to our operator $-A$, since due to (4), (5), $-A$ is densely defined and from (5) we have that $-A$ is an operator of type φ_0 with bound C_A. For $\lambda \in S_{\varphi_0}$, $-A + \lambda I$ is an operator of type θ (for any $\theta \in (0, \varepsilon(\varphi_0))$; in particular if we set $Q_\lambda = -(-A + \lambda I)^{1/2}$, then from Lemma 4.2, statement 2., Q_λ generates a semigroup $\left(e^{-tQ_\lambda}\right)_{t \geqslant 0}$ which is bounded, analytic for $t > 0$ and strongly continuous for $t \geqslant 0$. Moreover, there exists $K > 0$, such that

$$\forall \lambda \in S_{\varphi_0}, \quad \| Q_\lambda^{-1} \|_{\mathcal{L}(X)} \leqslant \frac{K}{(1+|\lambda|)^{1/2}}; \tag{37}$$

furthermore, from Lemma 4.3, statement 3., we have

$$\overline{B(0, 1/2K)} \subset \rho(Q_\lambda),$$

so there exists $\delta > 0$, which does not depend on λ such that $Q_\lambda + \delta I$ generates a bounded analytic semigroup thus, for some $K_1 \geqslant 1$

$$\forall \lambda \in S_{\varphi_0}, \ \forall x \geqslant 0, \quad \| e^{xQ_\lambda} \|_{\mathcal{L}(X)} \leqslant K_1 e^{-\delta x}. \tag{38}$$

There exist also $K_0 \geqslant 0$ and $c_0 > 0$ such that

$$\begin{cases} \forall \lambda \in S_{\varphi_0}, \ \forall x \geqslant 1/2, \ \forall j \in \{0,1,2\} : \\ \| Q_\lambda^j e^{xQ_\lambda} \|_{\mathcal{L}(X)} \leqslant K_0 e^{-2c_0|\lambda|^{1/2}}. \end{cases} \tag{39}$$

Lemma 5.1. There exists a constant $M \geqslant 0$ independent of $\lambda \in S_{\varphi_0}$, such that for any $\lambda \in S_{\varphi_0}$, operators $I \pm e^{2Q_\lambda}$ are invertible in $\mathcal{L}(X)$ and

$$\forall \lambda \in S_{\varphi_0}, \quad \left\| \left(I \pm e^{2Q_\lambda} \right)^{-1} \right\|_{\mathcal{L}(X)} \leqslant M, \tag{40}$$

Proof. Let $\lambda \in S_{\varphi_0}$. For $x \geqslant 0$, we have

$$\left\| e^{xQ_\lambda} \right\|_{\mathcal{L}(X)} \leqslant K_1 e^{-x\delta},$$

we choose $k \in \mathbb{N}\backslash\{0\}$ such that $K_1 e^{-2k\delta} \leqslant 1/2 < 1$. Then $I - e^{2kQ_\lambda}$ is invertible with

$$\left\| \left(I - e^{2kQ_\lambda} \right)^{-1} \right\|_{\mathcal{L}(X)} \leqslant \frac{1}{1 - 1/2} = 2,$$

thus $0 \in \rho(I - e^{2Q_\lambda})$ since

$$\begin{aligned} I &= (I - e^{2Q_\lambda}) \left(I + e^{2Q_\lambda} + \cdots + e^{2(k-1)Q_\lambda} \right) (I - e^{2kQ_\lambda})^{-1} \\ &= (I - e^{2kQ_\lambda})^{-1} \left(I + e^{2Q_\lambda} + \cdots + e^{2(k-1)Q_\lambda} \right) (I - e^{2Q_\lambda}). \end{aligned}$$

Moreover

$$\begin{aligned} \left\| \left(I - e^{2Q_\lambda} \right)^{-1} \right\|_{\mathcal{L}(X)} &\leqslant \left\| \left(I + e^{2Q_\lambda} + \cdots + e^{2(k-1)Q_\lambda} \right) (I - e^{2kQ_\lambda})^{-1} \right\|_{\mathcal{L}(X)} \\ &\leqslant \left(1 + \left\| e^{2Q_\lambda} \right\|_{\mathcal{L}(X)} + \cdots + \left\| e^{2Q_\lambda} \right\|_{\mathcal{L}(X)}^{k-1} \right) \left\| (I - e^{2kQ_\lambda})^{-1} \right\|_{\mathcal{L}(X)} \\ &\leqslant 2K_1^k. \end{aligned}$$

We obtain the result for $I + e^{2Q_\lambda} = I - \left(-e^{2Q_\lambda} \right)$ if we replace $e^{2kQ_\lambda}, e^{2Q_\lambda}$ by $-e^{2kQ_\lambda}, -e^{2Q_\lambda}$ in the above proof. □

5.2 Spectral estimates

In this subsection we assume that X, A, H satisfy (4)∼(8).

Let $\lambda \in S_{\varphi_0}, \mu \in S_{\varphi_1}$. We recall that $H_\mu = H + \mu I$ and furnish estimates concerning operators Q_λ, H_μ which are easy consequences of our assumptions.

In the following M denotes various constants, independent of λ, μ, which can vary from line to line.

Lemma 5.2. Let $\lambda \in S_{\varphi_0}, \mu \in S_{\varphi_1}$. Then $(-A + \lambda I) H_\mu^{-1} \in \mathcal{L}(X)$, moreover there exists a constant $M > 0$ independent of $\lambda \in S_{\varphi_0}$ and $\mu \in S_{\varphi_1}$ such that

$$\max \left\{ \left\| H H_\mu^{-1} \right\|_{\mathcal{L}(X)}, \left\| A H_\mu^{-1} \right\|_{\mathcal{L}(X)} \right\} \leqslant M, \tag{41}$$

$$\left\| Q_\lambda^2 H_\mu^{-1} \right\|_{\mathcal{L}(X)} \leqslant M \frac{1 + |\lambda| + |\mu|}{1 + |\mu|}, \tag{42}$$

and

$$\left\| Q_\lambda H_\mu^{-1} \right\|_{\mathcal{L}(X)} \leqslant M \frac{1 + |\lambda| + |\mu|}{(1 + |\mu|)(1 + |\lambda|)^{1/2}}. \tag{43}$$

Proof. Note that $(-A+\lambda I)$ is closed, so due to (7), $(-A+\lambda I)\,H_\mu^{-1}$ is bounded. Then

$$\begin{aligned}\left\|HH_\mu^{-1}\right\|_{\mathcal{L}(X)} &= \left\|(H+\mu I)\,H_\mu^{-1}-\mu H_\mu^{-1}\right\| \\ &\leqslant \|I\|_{\mathcal{L}(X)}+\left\|\mu\,(H+\mu I)^{-1}\right\|_{\mathcal{L}(X)} \leqslant 1+C_H,\end{aligned}$$

moreover

$$\begin{aligned}\left\|AH_\mu^{-1}\right\|_{\mathcal{L}(X)} &\leqslant \left\|AH^{-1}HH_\mu^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant \left\|AH^{-1}\right\|_{\mathcal{L}(X)}\left\|HH_\mu^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant \left\|AH^{-1}\right\|_{\mathcal{L}(X)}(1+C_H)=C',\end{aligned}$$

and

$$\begin{aligned}\left\|Q_\lambda^2H_\mu^{-1}\right\|_{\mathcal{L}(X)} &= \left\|(-A+\lambda I)\,H_\mu^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant \left\|AH_\mu^{-1}\right\|_{\mathcal{L}(X)}+\left\|\lambda\,(H+\mu I)^{-1}\right\|_{\mathcal{L}(X)} \leqslant C'+C_H\frac{|\lambda|}{1+|\mu|}.\end{aligned}$$

Finally, since

$$\left\|Q_\lambda H_\mu^{-1}\right\|_{\mathcal{L}(X)}=\left\|Q_\lambda^{-1}Q_\lambda^2H_\mu^{-1}\right\|_{\mathcal{L}(X)} \leqslant \left\|Q_\lambda^{-1}\right\|_{\mathcal{L}(X)}\left\|Q_\lambda^2H_\mu^{-1}\right\|_{\mathcal{L}(X)},$$

we deduce (43) from (41) and (37). □

For $\lambda\in S_{\varphi_0},\mu\in S_{\varphi_1}$, let us recall that

$$\Lambda_{\lambda,\mu}:=(Q_\lambda-H_\mu)+e^{2Q_\lambda}\,(Q_\lambda+H_\mu).$$

Note that, since $D\,(H_\mu)\subset D(Q_\lambda^2)$, we have $D\,(\Lambda_{\lambda,\mu})=D\,(H_\mu)=D(H)$. We now introduce, for $r>0$, the notation

$$\Omega_{\varphi_0,\varphi_1,r}=\left\{(\lambda,\mu)\in S_{\varphi_0}\times S_{\varphi_1}:|\lambda|\geqslant r \text{ and } \frac{|\mu|^2}{|\lambda|}\geqslant r\right\},$$

and furnish results on $\Lambda_{\lambda,\mu}$.

Lemma 5.3. There exist $r_0>0$ and $M>0$ such that for all $(\lambda,\mu)\in\Omega_{\varphi_0,\varphi_1,r_0}$ we have

$$\begin{cases}0\in\rho\left(\left(I-e^{2Q_\lambda}\right)^{-1}\left(I+e^{2Q_\lambda}\right)Q_\lambda H_\mu^{-1}-I\right) \\ \left\|\left[\left(I-e^{2Q_\lambda}\right)^{-1}\left(I+e^{2Q_\lambda}\right)Q_\lambda H_\mu^{-1}-I\right]^{-1}\right\|_{\mathcal{L}(X)}\leqslant 2,\end{cases} \tag{44}$$

$$0\in\rho\,(\Lambda_{\lambda,\mu}) \quad\text{and}\quad \left\|\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}\leqslant\frac{M}{1+|\mu|}, \tag{45}$$

and

$$\left\|Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}\leqslant M\frac{1+|\lambda|+|\mu|}{1+|\mu|}. \tag{46}$$

Note that $Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}$ has the same behaviour as $Q_\lambda H_\mu^{-1}$, see (43) and (53).

Proof. Let $(\lambda,\mu)\in\Omega_{\varphi_0,\varphi_1,r}$ for some $r>0$. From (7), we have $Q_\lambda H_\mu^{-1}\in\mathcal{L}(X)$, hence $\left(I-e^{2Q_\lambda}\right)^{-1}\left(I+e^{2Q_\lambda}\right)Q_\lambda H_\mu^{-1}-I\in\mathcal{L}(X)$; moreover, from (39), (40) and (43), we obtain

$$\begin{aligned}\left\|\left(I-e^{2Q_\lambda}\right)^{-1}\left(I+e^{2Q_\lambda}\right)Q_\lambda H_\mu^{-1}\right\|_{\mathcal{L}(X)} &\leqslant M\left\|Q_\lambda H_\mu^{-1}\right\|_{\mathcal{L}(X)}\\ &\leqslant M\left(\frac{1}{|\lambda|^{1/2}}+\frac{|\lambda|^{1/2}}{|\mu|}\right)\\ &\leqslant \frac{2M}{r^{1/2}}.\end{aligned}$$

So there exists $r_0>0$ such that for all $(\lambda,\mu)\in\Omega_{\varphi_0,\varphi_1,r_0}$ we have

$$\left\|\left(I-e^{2Q_\lambda}\right)^{-1}\left(I+e^{2Q_\lambda}\right)Q_\lambda H_\mu^{-1}\right\|_{\mathcal{L}(X)}\leqslant 1/2. \tag{47}$$

Let $(\lambda,\mu)\in\Omega_{\varphi_0,\varphi_1,r_0}$. Then (47) proves (44). We deduce that

$$L_{\lambda,\mu}:=\left(I-e^{2Q_\lambda}\right)\left[\left(I-e^{2Q_\lambda}\right)^{-1}\left(I+e^{2Q_\lambda}\right)Q_\lambda H_\mu^{-1}-I\right]\in\mathcal{L}(X),$$

is boundedly invertible. Moreover

$$L_{\lambda,\mu}^{-1}=\left[\left(I-e^{2Q_\lambda}\right)^{-1}\left(I+e^{2Q_\lambda}\right)Q_\lambda H_\mu^{-1}-I\right]^{-1}\left(I-e^{2Q_\lambda}\right)^{-1},$$

satisfies

$$\left\|L_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}\leqslant 2M.$$

Now, we write $\Lambda_{\lambda,\mu}=\left(I+e^{2Q_\lambda}\right)Q_\lambda-\left(I-e^{2Q_\lambda}\right)H_\mu=L_{\lambda,\mu}H_\mu$, so $\Lambda_{\lambda,\mu}$ is boundedly invertible with

$$\Lambda_{\lambda,\mu}^{-1}=H_\mu^{-1}L_{\lambda,\mu}^{-1},$$

this furnish (45). Finally, $\left\|L_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}\leqslant 2M$ and (42) gives

$$\left\|Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}=\left\|Q_\lambda^2 H_\mu^{-1}\right\|_{\mathcal{L}(X)}\left\|L_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}\leqslant M\frac{1+|\lambda|+|\mu|}{1+|\mu|}.$$

□

5.3 Main results

Let r_0 fixed as in Lemma 5.3.

Theorem 5.4. Assume (4)∼(8). Let $d_0\in X, u_1\in X, (\lambda,\mu)\in\Omega_{\varphi_0,\varphi_1,r_0}$ and $f\in L^p(0,1;X)$ with $1<p<+\infty$; the two following statements are equivalent:

1. Problem (1)-(2) has a classical solution u, that is,

$$u\in W^{2,p}(0,1;X)\cap L^p(0,1;D(A)),\ u(0)\in D(H),$$

and u satisfies (1)-(2).

2. $u_1\in(D(A),X)_{\frac{1}{2p},p}$.

Moreover in this case u is unique and given by (24) where Q,Λ are replaced by $Q_\lambda,\Lambda_{\lambda,\mu}$.

Proof. We apply Proposition 3.8, with A, H, Q and Λ replaced by

$$A-\lambda I, H+\mu I, Q_\lambda \text{ and } \Lambda_{\lambda,\mu},$$

since in this case Problem (16) becomes Problem (1)-(2). So, it is enough to verify that (4)∼(8) imply $(H_1) \sim (H_5)$.

It is clear that (4), (5), (6) imply $(H_1), (H_2), (H_3)$ mentioned in section 3. Moreover, due to (45), assumptions (4)∼(7) imply (H_4). Finally, under (4)∼(7)

$$\Lambda_{\lambda,\mu}^{-1}(X) \subset D(Q) \cap D(H) \subset D\left(Q^2\right),$$

so that $Q\Lambda_{\lambda,\mu}^{-1}(X) \subset D(Q)$ and then (H_5) is satisfied.

Note that here, the condition $\Lambda_{\lambda,\mu}^{-1} d_0 \in (D(A), X)_{\frac{1}{2p},p}$ is automatically realized since for any $d_0 \in X$, we have $\Lambda_{\lambda,\mu}^{-1} d_0 \in D(Q) \cap D(H) \subset D\left(Q^2\right)$. □

Lemma 5.5. Assume (4)∼(6), let $f \in L^p(0,1;X)$ with $1 < p < +\infty$ and set for $x \in [0,1]$

$$\begin{cases} I_{\lambda,f}(x) = \dfrac{1}{2} Q_\lambda^{-1} \displaystyle\int_0^x e^{(x-s)Q_\lambda} f(s) ds \\ J_{\lambda,f}(x) = \dfrac{1}{2} Q_\lambda^{-1} \displaystyle\int_x^1 e^{(s-x)Q_\lambda} f(s) ds, \end{cases} \tag{48}$$

then, there exists $M \geqslant 0$ (independent of λ and f) such that

$$\|Q_\lambda I_{\lambda,f}(1)\| \leqslant M \|f\|_{L^p(0,1;X)} \quad \text{and} \quad \|Q_\lambda J_{\lambda,f}(0)\| \leqslant M \|f\|_{L^p(0,1;X)}, \tag{49}$$

moreover $I_{\lambda,f}, J_{\lambda,f} \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$ with

$$\left\|Q_\lambda^2 I_{\lambda,f}\right\|_{L^p(0,1;X)} \leqslant M \|f\|_{L^p(0,1;X)} \quad \text{and} \quad \left\|Q_\lambda^2 J_{\lambda,f}\right\|_{L^p(0,1;X)} \leqslant M \|f\|_{L^p(0,1;X)}.$$

Proof. From (38), we have

$$\begin{cases} \|Q_\lambda I_{\lambda,f}(1)\| \leqslant \displaystyle\int_0^1 \left\|e^{(1-s)Q_\lambda} f(s)\right\| ds \leqslant M \int_0^1 \|f(s)\| ds \leqslant M \|f\|_{L^p(0,1;X)} \\ \|Q_\lambda J_{\lambda,f}(0)\| \leqslant \displaystyle\int_0^1 \left\|e^{sQ_\lambda} f(s)\right\| ds \leqslant M \int_0^1 \|f(s)\| ds \leqslant M \|f\|_{L^p(0,1;X)}. \end{cases}$$

We apply Lemma 4.8 with $E = X, L = -A$, $G_\lambda = Q_\lambda$, $a = 0$, $b = 1$ so that

$$I_{\lambda,f} = \frac{1}{2} Q_\lambda^{-1} U_{\lambda,f} \quad \text{and} \quad J_{\lambda,f} = \frac{1}{2} Q_\lambda^{-1} V_{\lambda,f};$$

then $Q_\lambda^2 I_{\lambda,f} = \dfrac{1}{2} Q_\lambda U_{\lambda,f}$ and $Q_\lambda^2 J_{\lambda,f} = \dfrac{1}{2} Q_\lambda V_{\lambda,f}$ have the desired estimates. □

Lemma 5.6. Assume (4)∼(6) and let $f \in L^p(0,1;X)$ with $1 < p < +\infty$. We use notations (48) and set for $x \in [0,1]$

$$v_0(x) = e^{xQ_\lambda} J_{\lambda,f}(0) \quad \text{and} \quad v_1(x) = e^{xQ_\lambda} I_{\lambda,f}(1).$$

Then, for $j \in \{0,1\}$, $v_j \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$ with

$$v_j' = Q_\lambda v_j \quad \text{and} \quad v_j'' = Q_\lambda^2 v_j.$$

Moreover, there exists $M \geqslant 0$ (independent of λ and f) such that

$$\left\|Q_\lambda^2 v_j\right\|_{L^p(0,1;X)} \leqslant M \|f\|_{L^p(0,1;X)}, \ j = 0,1. \tag{50}$$

Proof. We write, for $x \in]0,1]$

$$\begin{aligned} Q_\lambda^2 v_0(x) &= e^{xQ_\lambda} Q_\lambda \int_0^1 e^{sQ_\lambda} f(s)\, ds \\ &= e^{2xQ_\lambda} Q_\lambda \int_0^x e^{(s-x)Q_\lambda} f(s)\, ds + Q_\lambda \int_x^1 e^{(x-s)Q_\lambda} e^{2sQ_\lambda} f(s)\, ds \\ &= 2e^{2xQ_\lambda} Q_\lambda^2 I_{\lambda,f}(x) + 2Q_\lambda^2 J_{\lambda,g}(x), \end{aligned}$$

with $g = e^{2\cdot Q_\lambda} f(\cdot)$. From Lemma 5.5, statement 1. and (38), we have

$$\begin{cases} \left\| e^{2\cdot Q_\lambda} Q_\lambda^2 I_{\lambda,f}(\cdot) \right\|_{L^p(0,1;X)} \leqslant M \left\| Q_\lambda^2 I_{\lambda,f}(\cdot) \right\|_{L^p(0,1;X)} \leqslant M \|f\|_{L^p(0,1;X)} \\ \|Q_\lambda^2 J_{\lambda,f}\|_{L^p(0,1;X)} \leqslant M \|g\|_{L^p(0,1;X)} \leqslant M \|f\|_{L^p(0,1;X)}, \end{cases}$$

from which we deduce $\|Q_\lambda^2 v_0\|_{L^p(0,1;X)} \leqslant M \|f\|_{L^p(0,1;X)}$.

The same estimate runs for v_1 since

$$v_1 = e^{\cdot Q_\lambda} J_{\lambda,f(1-\cdot)}(0) \quad \text{and} \quad \|f(1-\cdot)\|_{L^p(0,1;X)} = \|f\|_{L^p(0,1;X)}.$$

□

Recall that X is a Banach space with norm $\|\cdot\|$ and $(D(A),X)_{\frac{1}{2p},p}$ is the Banach space equipped with the norm defined by

$$\|\varphi\|_{(D(A),X)_{\frac{1}{2p},p}} = \|\varphi\| + \left(\int_0^{+\infty} \left\| t^{1-1/2p} A (A - tI)^{-1} \varphi \right\|^p \frac{dt}{t} \right)^{1/p}. \tag{51}$$

Theorem 5.7. Assume (4)∼(8), $d_0 \in X$ and $u_1 \in (D(A),X)_{\frac{1}{2p},p}$. Then, there exists a constant $M \geqslant 0$ such that, for $(\lambda,\mu) \in \Omega_{\varphi_0,\varphi_1,r_0}$ and $f \in L^p(0,1;X)$ with $1 < p < +\infty$, the unique classical solution u of (1)-(2) satisfies

$$\begin{cases} \|u\|_{L^p(0,1;X)} \leqslant \dfrac{M\alpha(d_0,u_1,\lambda,\mu,f)}{1+|\lambda|} \\ \max\left\{ \|u'\|_{L^p(0,1;X)}, \|Qu\|_{L^p(0,1;X)}, \|Q_\lambda u\|_{L^p(0,1;X)} \right\} \leqslant \dfrac{M\alpha(d_0,u_1,\lambda,\mu,f)}{\sqrt{1+|\lambda|}} \\ \max\left\{ \|u''\|_{L^p(0,1;X)}, \|Q^2 u\|_{L^p(0,1;X)}, \|Q_\lambda^2 u\|_{L^p(0,1;X)} \right\} \leqslant M\alpha(d_0,u_1,\lambda,\mu,f), \end{cases}$$

where

$$\alpha(d_0,u_1,\lambda,\mu,f) = \frac{1+|\lambda|+|\mu|}{1+|\mu|} \left(\|d_0\| + \|f\|_{L^p(0,1;X)} \right) + \|u_1\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|u_1\|.$$

Proof. Let $(\lambda,\mu) \in \Omega_{\varphi_0,\varphi_1,r_0}$ and $f \in L^p(0,1;X)$. We recall that, taking into account the notations (48), we have, for $x \in [0,1]$

$$u(x) = S_\lambda(x)\mu_0 + S_\lambda(1-x)\mu_1 + I_{\lambda,f}(x) + J_{\lambda,f}(x),$$

where

$$\begin{cases} \mu_1 = u_1 - I_{\lambda,f}(1) \\ \mu_0 = \Lambda_{\lambda,\mu}^{-1} \left[\left(I - e^{2Q_\lambda} \right) d_0 + 2Q_\lambda e^{Q_\lambda} \mu_1 + 2Q_\lambda J_{\lambda,f}(0) \right] - J_{\lambda,f}(0) \\ S_\lambda(x) = \left(I - e^{2Q_\lambda} \right)^{-1} \left(e^{xQ_\lambda} - e^{(1-x)Q_\lambda} e^{Q_\lambda} \right) \in \mathcal{L}(X). \end{cases}$$

So we can write $u = h_0 + h_1 - h_2 + h_3 + h_4$ with

$$\begin{cases} h_0(x) = S_\lambda(x)\,\Lambda_{\lambda,\mu}^{-1}\left[\left(I - e^{2Q_\lambda}\right)d_0 + 2Q_\lambda\left(J_{\lambda,f}(0) - e^{Q_\lambda} I_{\lambda,f}(1)\right)\right] \\ h_1(x) = 2S_\lambda(x)\,\Lambda_{\lambda,\mu}^{-1} Q_\lambda e^{Q_\lambda} u_1 \\ h_2(x) = S_\lambda(x)\,J_{\lambda,f}(0) + S_\lambda(1-x)\,I_{\lambda,f}(1) \\ h_3(x) = S_\lambda(1-x)\,u_1 \\ h_4(x) = I_{\lambda,f}(x) + J_{\lambda,f}(x). \end{cases} \tag{52}$$

Estimate of $Q_\lambda^2 h_0$. For $\xi \in X$ and $x \in (0,1)$, we have

$$\begin{aligned} \left\|Q_\lambda^2 S_\lambda(x)\,\Lambda_{\lambda,\mu}^{-1}\xi\right\| &= \left\|\left(I - e^{2Q_\lambda}\right)^{-1}\left(I - e^{2(1-x)Q_\lambda}\right)e^{xQ_\lambda} Q_\lambda^2 \Lambda_{\lambda,\mu}^{-1}\xi\right\| \\ &\leqslant \left\|\left(I - e^{2Q_\lambda}\right)^{-1}\left(I - e^{2(1-x)Q_\lambda}\right)\right\|_{\mathcal{L}(X)} \left\|e^{xQ_\lambda}\right\|_{\mathcal{L}(X)} \left\|Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)} \|\xi\| \\ &\leqslant M\frac{1+|\lambda|+|\mu|}{1+|\mu|}\,\|\xi\|, \end{aligned}$$

so, from (46) and (49), we deduce

$$\begin{aligned} \left\|Q_\lambda^2 h_0(x)\right\| &\leqslant M\frac{1+|\lambda|+|\mu|}{1+|\mu|}\left(\|d_0\| + 2\|Q_\lambda J_{\lambda,f}(0)\| + 2\left\|e^{Q_\lambda}\right\|_{\mathcal{L}(X)}\|Q_\lambda I_{\lambda,f}(1)\|\right) \\ &\leqslant M\frac{1+|\lambda|+|\mu|}{1+|\mu|}\left(\|d_0\| + \|f\|_{L^p(0,1;X)}\right). \end{aligned}$$

Then

$$\left\|Q_\lambda^2 h_0\right\|_{L^p(0,1;X)} \leqslant M\frac{1+|\lambda|+|\mu|}{1+|\mu|}\left(\|d_0\| + \|f\|_{L^p(0,1;X)}\right).$$

Estimate of $Q_\lambda^2 h_1$. As above, we have for $\xi \in X$ and $x \in (0,1)$

$$\left\|Q_\lambda^2 h_1(x)\right\| \leqslant M\frac{1+|\lambda|+|\mu|}{1+|\mu|}\left(\left\|Q_\lambda e^{Q_\lambda}\right\|_{\mathcal{L}(X)}\|u_1\|\right),$$

and from (39), we deduce that

$$\left\|Q_\lambda^2 h_1(x)\right\| \leqslant M\|u_1\|,$$

hence

$$\left\|Q_\lambda^2 h_1\right\|_{L^p(0,1;X)} \leqslant M\|u_1\|.$$

Estimate of $Q_\lambda^2 h_2$. For $\xi \in X$ and $x \in]0,1]$, we have

$$\begin{aligned} \left\|Q_\lambda^2 S_\lambda(x)\,\xi\right\| &= \left\|\left(I - e^{2Q_\lambda}\right)^{-1}\left(I - e^{2(1-x)Q_\lambda}\right)Q_\lambda^2 e^{xQ_\lambda}\xi\right\| \\ &\leqslant \left\|\left(I - e^{2Q_\lambda}\right)^{-1}\left(I - e^{2(1-x)Q_\lambda}\right)\right\|_{\mathcal{L}(X)}\left\|Q_\lambda^2 e^{xQ_\lambda}\xi\right\| \\ &\leqslant M\left\|Q_\lambda^2 e^{xQ_\lambda}\xi\right\|, \end{aligned} \tag{53}$$

so

$$\left\|Q_\lambda^2 h_2(x)\right\| \leqslant M\left(\left\|Q_\lambda^2 e^{xQ_\lambda} J_{\lambda,f}(0)\right\| + \left\|Q_\lambda^2 e^{(1-x)Q_\lambda} I_{\lambda,f}(1)\right\|\right),$$

and then, from (50)

$$\begin{aligned} \left\|Q_\lambda^2 h_2\right\|_{L^p(0,1;X)} &\leqslant M\left(\left\|Q^2 v_0\right\|_{L^p(0,1;X)} + \left\|Q^2 v_1(1-\cdot)\right\|_{L^p(0,1;X)}\right) \\ &\leqslant M\|f\|_{L^p(0,1;X)}. \end{aligned}$$

Estimate of $Q_\lambda^2 h_3$. Due to (53), we have, for $x \in [0,1[$

$$\left\|Q_\lambda^2 h_3(x)\right\| \leqslant M \left\|Q_\lambda^2 e^{(1-x)Q_\lambda} u_1\right\|.$$

From Theorem 2.1 in [14], since

$$u_1 \in (D(A),X)_{\frac{1}{2p},p} = (X,D(A))_{1-\frac{1}{2p},p},$$

we get

$$\begin{aligned}\left\|Q_\lambda^2 h_3\right\|_{L^p(0,1;X)} &\leqslant M \left\|Q_\lambda^2 e^{(1-x)Q_\lambda} u_1\right\|_{L^p(0,1;X)} \\ &\leqslant M\left(\|u_1\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|u_1\|\right).\end{aligned}$$

Estimate of $Q_\lambda^2 h_4$. From Lemma 5.5, we get

$$\left\|Q_\lambda^2 h_4\right\|_{L^p(0,1;X)} \leqslant M\|f\|_{L^p(0,1;X)}.$$

Summarizing the previous study we obtain that

$$\left\|Q_\lambda^2 u\right\|_{L^p(0,1;X)} \leqslant M\alpha(d_0,u_1,\lambda,\mu,f). \tag{54}$$

Moreover since u satisfies (1) that is

$$u''(x) - Q_\lambda^2 u(x) = f(x), \quad \text{a.e. } x \in (0,1),$$

we deduce that

$$\|u''\|_{L^p(0,1;X)} \leqslant M\alpha(d_0,u_1,\lambda,\mu,f).$$

Writing $u = Q_\lambda^{-2} Q_\lambda^2 u$ and $Q_\lambda u = Q_\lambda^{-1} Q_\lambda^2 u$, we obtain the estimates concerning u and $Q_\lambda u$. Setting, for $x \in [0,1]$

$$\widetilde{S}_\lambda(x) = \left(I - e^{2Q_\lambda}\right)^{-1}\left(e^{xQ_\lambda} + e^{(1-x)Q_\lambda}e^{Q_\lambda}\right) \in \mathcal{L}(X),$$

we have

$$u'(x) = Q_\lambda \widetilde{S}_\lambda(x)\mu_0 - Q_\lambda \widetilde{S}_\lambda(1-x)\mu_1 + QI_{\lambda,f}(x) - QJ_{\lambda,f}(x) = Q_\lambda^{-1}Q_\lambda^2\omega(x),$$

the terms in $\omega(x) = \widetilde{S}_\lambda(x)\mu_0 - \widetilde{S}_\lambda(1-x)\mu_1 + I_{\lambda,f}(x) - J_{\lambda,f}(x)$ are (in absolute value) those of $u(x)$, so (54) runs when we replace u by ω , this furnishes the estimate for u'.

From Lemma 2.6 a) p. 103 in [14] we have

$$\left\|QQ_\lambda^{-1}\right\|_{\mathcal{L}(X)} = \left\|(-A)^{1/2}(-A+\lambda I)^{-1/2}\right\|_{\mathcal{L}(X)} \leqslant M; \tag{55}$$

so writing $Qu = QQ_\lambda^{-1}Q_\lambda u, Q^2 u = \left(QQ_\lambda^{-1}\right)^2 Q_\lambda^2 u$, we deduce the estimates of $\|Qu\|_{L^p(0,1;X)}$ and $\|Q^2 u\|_{L^p(0,1;X)}$ from those of $\|Q_\lambda u\|_{L^p(0,1;X)}$ and $\|Q_\lambda^2 u\|_{L^p(0,1;X)}$. □

Remark 5.8. Under the assumptions of the previous theorem, we obtain moreover that

$$\|u(0)\| \leqslant \frac{M}{1+|\mu|}\left(\|d_0\| + Me^{-2c_0|\lambda|^{1/2}}\|u_1\| + \|f\|_{L^p(0,1;X)}\right). \tag{56}$$

Indeed

$$u(0) = \Lambda_{\lambda,\mu}^{-1}\left[\left(I - e^{2Q_\lambda}\right)d_0 + 2Q_\lambda e^{Q_\lambda}\left(u_1 - I_{\lambda,f}(1)\right) + 2Q_\lambda J_{\lambda,f}(0)\right],$$

so

$$\begin{aligned}\|u(0)\| &\leqslant \left\|\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}\left\|I - e^{2Q_\lambda}\right\|_{\mathcal{L}(X)}\|d_0\| + \left\|\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}\left\|2Q_\lambda e^{Q_\lambda}\right\|_{\mathcal{L}(X)}\|u_1\| \\ &\quad +2\left\|\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)}\left(\left\|e^{Q_\lambda}\right\|_{\mathcal{L}(X)}\|Q_\lambda I_{\lambda,f}(1)\| + \|Q_\lambda J_{\lambda,f}(0)\|\right) \\ &\leqslant \frac{M}{1+|\mu|}\left(\|d_0\| + Me^{-2c_0|\lambda|^{1/2}}\|u_1\| + \|f\|_{L^p(0,1;X)}\right).\end{aligned}$$

6 Spectral problem (1)-(2): second case

In all this section we suppose that X, A, H satisfy (4)$\sim$(6) and (9)$\sim$(11).

Note that the results of the previous section obtained under assumption (4)$\sim$(6) can be used here, in particular results of subsection 5.1, Lemma 5.5, Lemma 5.6 and also estimate (55).

6.1 Spectral estimates

Let $\lambda \in S_{\varphi_0}, \mu \in \mathbb{C}$. Recall that $H_\mu = H + \mu I$ and $Q_\lambda = -(-A + \lambda I)^{1/2}$. We first furnish estimates concerning operators Q_λ, H_μ which are easy consequences of our assumptions.

Again, in the following M denotes various constants, independent of λ, μ, which can vary from one line to another.

Lemma 6.1. Let $\lambda \in S_{\varphi_0}, \mu \in \mathbb{C}$. Then $H_\mu Q_\lambda^{-1} \in \mathcal{L}(X)$, moreover there exists a constant $M \geqslant 0$ independent of $\lambda \in S_{\varphi_0}$ and $\mu \in \mathbb{C}$ such that

$$\left\|H_\mu Q_\lambda^{-1}\right\|_{\mathcal{L}(X)} \leqslant M\frac{1+|\mu|}{(1+|\lambda|)^\varepsilon}. \tag{57}$$

Proof. From (55), we deduce

$$\begin{aligned}\left\|Q_{|\lambda|}Q_\lambda^{-1}\right\|_{\mathcal{L}(X)} &= \left\|(-A + |\lambda| I)\, Q^{-1}Q_{|\lambda|}^{-1}QQ_\lambda^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant \left\|AQ^{-1}Q_{|\lambda|}^{-1}QQ_\lambda^{-1}\right\|_{\mathcal{L}(X)} + \left\||\lambda|\, Q^{-1}Q_{|\lambda|}^{-1}QQ_\lambda^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant \left\|QQ_{|\lambda|}^{-1}\right\|_{\mathcal{L}(X)}\left\|QQ_\lambda^{-1}\right\|_{\mathcal{L}(X)} + |\lambda|\left\|Q_{|\lambda|}^{-1}\right\|_{\mathcal{L}(X)}\left\|Q_\lambda^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant M,\end{aligned}$$

and, from (10)

$$\|\, HQ_\lambda^{-1}\,\|_{\mathcal{L}(X)} \leqslant \|\, HQ_{|\lambda|}^{-1}\,\|_{\mathcal{L}(X)}\|\, Q_{|\lambda|}Q_\lambda^{-1}\,\|_{\mathcal{L}(X)} \leqslant \frac{M}{(1+|\lambda|)^\varepsilon}.$$

Moreover

$$\begin{aligned}\left\|H_\mu Q_\lambda^{-1}\right\|_{\mathcal{L}(X)} &\leqslant \left\|HQ_\lambda^{-1}\right\|_{\mathcal{L}(X)} + |\mu|\left\|Q_\lambda^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant \frac{M}{(1+|\lambda|)^\varepsilon} + \frac{1+|\mu|}{(1+|\lambda|)^{1/2}} \leqslant M\frac{1+|\mu|}{(1+|\lambda|)^\varepsilon}.\end{aligned}$$

□

We now introduce the notation : for $\rho > 0$

$$\Pi_{\varphi_0,\rho} = \left\{ (\lambda,\mu) \in S_{\varphi_0} \times \mathbb{C} : |\lambda| \geqslant \rho \ \text{ and } \ \frac{|\lambda|}{|\mu|^{1/\varepsilon}} \geqslant \rho \right\},$$

where we have set $\dfrac{|\lambda|}{|\mu|^{1/\varepsilon}} = +\infty$ for $\mu = 0$ and furnished results on

$$\Lambda_{\lambda,\mu} = (Q_\lambda - H_\mu) + e^{2Q_\lambda}(Q_\lambda + H_\mu),$$

where $\lambda \in S_{\varphi_0}, \mu \in \mathbb{C}$.

Lemma 6.2. There exist $\rho_0 > 0$ and $M \geqslant 0$ such that for all $(\lambda,\mu) \in \Pi_{\varphi_0,\rho_0}$:

$$\max\left\{ \left\| H_\mu Q_\lambda^{-1} \right\|_{\mathcal{L}(X)}, \left\| \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \right\} \leqslant 1/2, \tag{58}$$

$$\begin{cases} 0 \in \rho\left(I - \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1} \right) \\ \left\| \left[I - \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1} \right]^{-1} \right\|_{\mathcal{L}(X)} \leqslant 2, \end{cases} \tag{59}$$

$$0 \in \rho\left(\Lambda_{\lambda,\mu}\right) \quad \text{and} \quad \left\| \Lambda_{\lambda,\mu}^{-1} \right\|_{\mathcal{L}(X)} \leqslant \frac{M}{(1+|\lambda|)^{1/2}}, \tag{60}$$

$$\left\| Q_\lambda \Lambda_{\lambda,\mu}^{-1} \right\|_{\mathcal{L}(X)} \leqslant M, \tag{61}$$

and

$$\begin{cases} 0 \in \rho\left(Q_\lambda - H_\mu\right) \\ \left\| (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \leqslant \dfrac{M}{(1+|\lambda|)^{1/2}} \\ \left\| Q_\lambda (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \leqslant M \\ \left\| (Q_\lambda + H_\mu)(Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \leqslant M \\ \left\| e^{2Q_\lambda}(Q_\lambda + H_\mu)(Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \leqslant 1/2. \end{cases} \tag{62}$$

Proof. Let $\rho > 0$ and $(\lambda,\mu) \in \Pi_{\varphi_0,\rho}$. Then

$$I - \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1} \in \mathcal{L}(X),$$

and from (57) together with Lemma 5.1

$$\begin{aligned} \max\left\{ \left\| H_\mu Q_\lambda^{-1} \right\|_{\mathcal{L}(X)}, \left\| \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \right\} &\leqslant M \left\| H_\mu Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \\ &\leqslant M \frac{1+|\mu|}{(1+|\lambda|)^{\varepsilon}} \\ &\leqslant M \left(\left(\frac{1}{|\lambda|}\right)^{\varepsilon} + \left(\frac{|\mu|^{1/\varepsilon}}{|\lambda|}\right)^{\varepsilon} \right). \end{aligned}$$

So there exists $\rho_0 > 0$ such that for all $(\lambda,\mu) \in \Pi_{\varphi_0,\rho_0}$: (58) and (59) hold. Now, let $(\lambda,\mu) \in \Pi_{\varphi_0,\rho_0}$. We deduce that

$$\begin{aligned}\Lambda_{\lambda,\mu} &= \left(I + e^{2Q_\lambda}\right) Q_\lambda - \left(I - e^{2Q_\lambda}\right) H_\mu \\ &= \left(I + e^{2Q_\lambda}\right) \left[I - \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1}\right] Q_\lambda,\end{aligned}$$

is boundedly invertible with

$$\begin{cases} \Lambda_{\lambda,\mu}^{-1} = Q_\lambda^{-1} \left[I - \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1}\right]^{-1} \left(I + e^{2Q_\lambda}\right)^{-1} \\ Q_\lambda \Lambda_{\lambda,\mu}^{-1} = \left[I - \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1}\right]^{-1} \left(I + e^{2Q_\lambda}\right)^{-1}, \end{cases}$$

so

$$\begin{aligned}\left\|\Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)} &\leqslant M \left\|Q_\lambda^{-1}\right\|_{\mathcal{L}(X)} \left\|I - \left(I + e^{2Q_\lambda}\right)^{-1} \left(I - e^{2Q_\lambda}\right) H_\mu Q_\lambda^{-1}\right\|_{\mathcal{L}(X)} \left\|I + e^{2Q_\lambda}\right\|_{\mathcal{L}(X)} \\ &\leqslant \frac{M}{(1+|\lambda|)^{1/2}},\end{aligned}$$

and $\left\|Q_\lambda \Lambda_{\lambda,\mu}^{-1}\right\|_{\mathcal{L}(X)} \leqslant M$. Moreover, from (58), $Q_\lambda - H_\mu = \left(I - H_\mu Q_\lambda^{-1}\right) Q_\lambda$ is boundedly invertible with

$$(Q_\lambda - H_\mu)^{-1} = Q_\lambda^{-1} \left(I - H_\mu Q_\lambda^{-1}\right)^{-1},$$

so

$$\begin{cases} \left\|(Q_\lambda - H_\mu)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \left\|Q_\lambda^{-1}\right\|_{\mathcal{L}(X)} \left\|\left(I - H_\mu Q_\lambda^{-1}\right)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \dfrac{M}{(1+|\lambda|)^{1/2}} \\ \left\|Q_\lambda (Q_\lambda - H_\mu)^{-1}\right\|_{\mathcal{L}(X)} = \left\|\left(I - H_\mu Q_\lambda^{-1}\right)^{-1}\right\|_{\mathcal{L}(X)} \leqslant M, \end{cases}$$

and

$$\begin{aligned}\left\|(Q_\lambda + H_\mu)(Q_\lambda - H_\mu)^{-1}\right\|_{\mathcal{L}(X)} &= \left\|(-Q_\lambda + H_\mu + 2Q_\lambda)(Q_\lambda - H_\mu)^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant 1 + 2\left\|Q_\lambda (Q_\lambda - H_\mu)^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant M.\end{aligned}$$

Finally

$$\begin{aligned}\left\|e^{2Q_\lambda}(Q_\lambda + H_\mu)(Q_\lambda - H_\mu)^{-1}\right\|_{\mathcal{L}(X)} &\leqslant \left\|e^{2Q_\lambda}\right\|_{\mathcal{L}(X)} \left\|(Q_\lambda + H_\mu)(Q_\lambda - H_\mu)^{-1}\right\|_{\mathcal{L}(X)} \\ &\leqslant M \left\|e^{2Q_\lambda}\right\|_{\mathcal{L}(X)},\end{aligned}$$

and due to (39), we can eventually increase ρ_0, for $(\lambda,\mu) \in \Pi_{\varphi_0,\rho_0}$, which implies that $|\lambda| \geqslant \rho_0$, in order to have

$$M \left\|e^{2Q_\lambda}\right\|_{\mathcal{L}(X)} \leqslant 1/2.$$

□

6.2 Main results

Let ρ_0 fixed as in Lemma 6.2.

Theorem 6.3. Assume (4)$\sim$(6) and (9)$\sim$(11). Let

$$d_0 \in X,\ u_1 \in X,\ \ (\lambda,\mu) \in \Pi_{\varphi_0,\rho_0},$$

and $f \in L^p(0,1;X)$ with $1 < p < +\infty$. Then, the two following statements are equivalent:

1. Problem (1)-(2) has a classical solution u, that is,

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)),\ u(0) \in D(H),$$

and u satisfies (1)-(2).

2. $u_1 \in (D(A),X)_{\frac{1}{2p},p}$ and $(Q_\lambda - H_\mu)^{-1} d_0 \in (D(A),X)_{\frac{1}{2p},p}$.

Moreover in this case u is unique and given by (24) where Q, Λ are replaced by $Q_\lambda, \Lambda_{\lambda,\mu}$.

Proof. As in the proof of Theorem 5.4, we want to apply Proposition 3.8, with A, H, Q, Λ replaced by $A - \lambda I, H + \mu I, Q_\lambda, \Lambda_{\lambda,\mu}$. Assumptions $(H_1) \sim (H_4)$ are easily deduced from (4)$\sim$(6), (9) and Lemma 6.2. To obtain (H_5), it is enough, due to (21), to prove (H_5'). So, for $\xi \in (D(Q_\lambda),X)_{1/p,p} = (D(Q),X)_{1/p,p}$, we just have to show that

$$\eta = Q_\lambda (Q_\lambda - H_\mu)^{-1} \xi \in (D(Q),X)_{1/p,p},$$

but, from Lemma 4.2, statement 5. we have $Q_\lambda = Q + \lambda (Q_\lambda + Q)^{-1}$, thus

$$(Q_\lambda - H_\mu) Q_\lambda^{-1} = (Q - H) Q_\lambda^{-1} + \lambda (Q_\lambda + Q)^{-1} Q_\lambda^{-1} - \mu Q_\lambda^{-1},$$

so

$$\xi = (Q_\lambda - H_\mu) Q_\lambda^{-1} \eta = (Q - H) Q_\lambda^{-1} \eta + \lambda (Q_\lambda + Q)^{-1} Q_\lambda^{-1} \eta - \mu Q_\lambda^{-1} \eta,$$

and

$$(Q - H) Q_\lambda^{-1} \eta = \xi - \lambda (Q_\lambda + Q)^{-1} Q_\lambda^{-1} \eta + \mu Q_\lambda^{-1} \eta \in (D(Q),X)_{1/p,p},$$

which means that $Q_\lambda^{-1}\eta \in (Q-H)^{-1}\left((D(Q),X)_{1/p,p}\right)$ and, from (11), we get

$$Q_\lambda^{-1}\eta \in Q^{-1}\left((D(Q),X)_{1/p,p}\right),$$

then $\eta \in (D(Q),X)_{1/p,p}$.

Here the condition $(Q_\lambda - H_\mu)^{-1} d_0 \in (D(A),X)_{\frac{1}{2p},p}$ which is, from Remark 3.9, equivalent to $\Lambda_{\lambda,\mu}^{-1} d_0 \in (D(A),X)_{\frac{1}{2p},p}$, appears naturally, since we have not, as in Theorem 5.4, $\Lambda_{\lambda,\mu}^{-1}(X) \subset D(Q^2)$. □

The proof of the following Lemma will use (13), wich is equivalent to (12) from Remark 2.2, to study, for a given $(\lambda,\mu) \in \Pi_{\varphi_0,\rho_0}$, the operator $Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1}$.

Lemma 6.4. Assume (5), (9),(10) and (12). Fix $(\lambda_1,\mu_1) \in \Pi_{\varphi_0,\rho_0}$. Then, there exists $M \geqslant 0$ such that for any $(\lambda,\mu) \in \Pi_{\varphi_0,\rho_0}$, we have

1. $(Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} = (Q_{\lambda_1} - H_{\mu_1})^{-1} Q^{-1} P_{\lambda,\mu}$, where $P_{\lambda,\mu} \in \mathcal{L}(X)$ with

$$\|P_{\lambda,\mu}\|_{\mathcal{L}(X)} \leqslant M.$$

2. $Q_\lambda^2 \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1} \in \mathcal{L}(X)$ with

$$\left\| Q_\lambda^2 \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \leqslant M. \tag{63}$$

3. There exists $W_{\lambda,\mu} \in \mathcal{L}(X)$ such that

$$\Lambda_{\lambda,\mu}^{-1} = \left(Q_\lambda - H_\mu\right)^{-1} \left(I + e^{2Q_\lambda} W_{\lambda,\mu}\right), \tag{64}$$

with

$$\|W_{\lambda,\mu}\|_{\mathcal{L}(X)} \leqslant M \quad \text{and} \quad \left\| Q_\lambda^2 \Lambda_{\lambda,\mu}^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \leqslant M. \tag{65}$$

Proof. Let $(\lambda, \mu) \in \Pi_{\varphi_0,\rho_0}$.

1. We have

$$\begin{aligned}
\left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1} &= \left(Q_{\lambda_1} - H_{\mu_1}\right)^{-1} \left(Q_{\lambda_1} - H_{\mu_1}\right) \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1} \\
&= \left(Q_{\lambda_1} - H_{\mu_1}\right)^{-1} \left[Q_\lambda - H_\mu + (\mu - \mu_1) I\right] \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1} \\
&\quad + \left(Q_{\lambda_1} - H_{\mu_1}\right)^{-1} \left(Q_{\lambda_1} - Q_\lambda\right) \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1} \\
&= \left(Q_{\lambda_1} - H_{\mu_1}\right)^{-1} \left[Q_\lambda^{-1} + (\mu - \mu_1) \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1}\right] \\
&\quad + \left(Q_{\lambda_1} - H_{\mu_1}\right)^{-1} \left(Q_{\lambda_1} - Q_\lambda\right) \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1},
\end{aligned}$$

but, $Q_{\lambda_1} - Q_\lambda = (Q - Q_\lambda) - (Q - Q_{\lambda_1})$ and from Lemma 4.4, there exists $T_{\lambda,\lambda_1} \in \mathcal{L}(X)$ such that $Q_{\lambda_1} = Q_\lambda + T_\lambda$

$$\|T_{\lambda,\lambda_1}\|_{\mathcal{L}(X)} \leqslant M \left(1 + \sqrt{|\lambda|}\right) \quad \text{and} \quad Q^{-1} T_{\lambda,\lambda_1} = T_{\lambda,\lambda_1} Q^{-1}, \tag{66}$$

so

$$\left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1} = \left(Q_{\lambda_1} - H_{\mu_1}\right)^{-1} Q^{-1} P_{\lambda,\mu},$$

where $P_{\lambda,\mu} \in \mathcal{L}(X)$ is defined by

$$P_{\lambda,\mu} = Q Q_\lambda^{-1} \left[I + (\mu_1 - \mu) Q_\lambda \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1} + T_{\lambda,\lambda_1} Q_\lambda \left(Q_\lambda - H_\mu\right)^{-1} Q_\lambda^{-1}\right].$$

Moreover, using (37), (55), (66) and (62)

$$\begin{aligned}
\|P_{\lambda,\mu}\|_{\mathcal{L}(X)} &\leqslant M \left[1 + (|\mu - \mu_1|) \left\| Q_\lambda \left(Q_\lambda - H_\mu\right)^{-1} \right\|_{\mathcal{L}(X)} \left\| Q_\lambda^{-1} \right\|_{\mathcal{L}(X)}\right] \\
&\quad + M \left[\|T_{\lambda,\lambda_1}\|_{\mathcal{L}(X)} \left\| Q_\lambda \left(Q_\lambda - H_\mu\right)^{-1} \right\|_{\mathcal{L}(X)} \left\| Q_\lambda^{-1} \right\|_{\mathcal{L}(X)}\right] \\
&\leqslant M \left[1 + \frac{|\mu - \mu_1|}{(1 + |\lambda|)^{1/2}} + \frac{1 + \sqrt{|\lambda|}}{(1 + |\lambda|)^{1/2}}\right],
\end{aligned}$$

but since $(\lambda, \mu) \in \Pi_{\varphi_0,\rho_0}$ we have

$$1 + |\lambda| \geqslant 1 + \rho_0 |\mu|^{1/\varepsilon} \geqslant 1 + \rho_0 |\mu|^2,$$

thus $\|P_{\lambda,\mu}\|_{\mathcal{L}(X)} \leqslant M$.

2. Since Q_λ^2 is closed then, from (13) and the closed graph theorem, we obtain that $Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \in \mathcal{L}(X)$. Moreover we have

$$\begin{aligned}\left\| Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} &= \left\| (-A + \lambda I)(Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \\ &\leqslant \left\| A (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \\ &\quad + |\lambda| \left\| (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \left\| Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \\ &\leqslant M.\end{aligned}$$

The last inequality is obtained, from statement 1, which gives

$$\begin{aligned}\left\| -A (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} &= \left\| A (Q_{\lambda_1} - H_{\mu_1})^{-1} Q^{-1} P_{\lambda,\mu} \right\|_{\mathcal{L}(X)} \\ &\leqslant \left\| -A (Q_{\lambda_1} - H_{\mu_1})^{-1} Q^{-1} \right\|_{\mathcal{L}(X)} \|P_{\lambda,\mu}\|_{\mathcal{L}(X)} \\ &\leqslant M,\end{aligned}$$

and, from (62), (37), which furnishes

$$|\lambda| \left\| (Q_\lambda - H_\mu)^{-1} \right\|_{\mathcal{L}(X)} \left\| Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \leqslant M.$$

3. We set $R_{\lambda,\mu} = (Q_\lambda + H_\mu)(Q_\lambda - H_\mu)^{-1} \in \mathcal{L}(X)$ and write

$$\Lambda_{\lambda,\mu} = (Q_\lambda - H_\mu) + e^{2Q_\lambda}(Q_\lambda + H_\mu) = \left(I + e^{2Q_\lambda} R_{\lambda,\mu}\right)(Q_\lambda - H_\mu),$$

but $\Lambda_{\lambda,\mu}, (Q_\lambda - H_\mu)$ are boundedly invertible, so $I + e^{2Q_\lambda} R_{\lambda,\mu}$ is boundedly invertible with

$$\left(I + e^{2Q_\lambda} R_{\lambda,\mu}\right)^{-1} = I - e^{2Q_\lambda} R_{\lambda,\mu} \left(I + e^{2Q_\lambda} R_{\lambda,\mu}\right)^{-1}.$$

Now setting $W_{\lambda,\mu} = R_{\lambda,\mu} \left(I + e^{2Q_\lambda} R_{\lambda,\mu}\right)^{-1} \in \mathcal{L}(X)$ we have

$$\Lambda_{\lambda,\mu}^{-1} = (Q_\lambda - H_\mu)^{-1} \left(I + e^{2Q_\lambda} R_{\lambda,\mu}\right)^{-1} = (Q_\lambda - H_\mu)^{-1} \left(I - e^{2Q_\lambda} W_{\lambda,\mu}\right),$$

and, due to (62), we have

$$\begin{aligned}\|W_{\lambda,\mu}\|_{\mathcal{L}(X)} &\leqslant \|R_{\lambda,\mu}\|_{\mathcal{L}(X)} \left\| \left(I + e^{2Q_\lambda} R_{\lambda,\mu}\right)^{-1} \right\|_{\mathcal{L}(X)} \\ &\leqslant \frac{\|R_{\lambda,\mu}\|_{\mathcal{L}(X)}}{1 - \|e^{2Q_\lambda} R_{\lambda,\mu}\|_{\mathcal{L}(X)}} \\ &\leqslant M.\end{aligned}$$

Finally

$$\begin{aligned}\left\| Q_\lambda^2 \Lambda_{\lambda,\mu}^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} &= \left\| Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} \left(I - e^{2Q_\lambda} W_{\lambda,\mu}\right) Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \\ &\leqslant \left\| Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \\ &\quad + \left\| Q_\lambda^2 (Q_\lambda - H_\mu)^{-1} Q_\lambda^{-1} \right\|_{\mathcal{L}(X)} \left\| e^{2Q_\lambda} \right\|_{\mathcal{L}(X)} \|W_{\lambda,\mu}\|_{\mathcal{L}(X)} \\ &\leqslant M.\end{aligned}$$

□

Remark 6.5. Assume (4)∼(6) and (9), (10), (12). Let $(\lambda,\mu)\in\Pi_{\varphi_0,\rho_0}$. From Lemma 6.4, statement 2., we have

$$\forall\xi\in D(Q_\lambda),\quad Q_\lambda\left(Q_\lambda-H_\mu\right)^{-1}\xi\in D(Q_\lambda),$$

then (11) is satisfied, see Remark 3.1 statement 5, and we can apply Theorem 6.3.

Theorem 6.6. Assume (4)∼(6) and (9), (10), (12). Let $(\lambda,\mu)\in\Pi_{\varphi_0,\rho_0}, d_0\in X$ with

$$\left(Q_\lambda-H_\mu\right)^{-1}d_0\in\left(D\left(A\right),X\right)_{\frac{1}{2p},p},\quad u_1\in\left(D(A),X\right)_{\frac{1}{2p},p}\quad\text{and}\quad f\in L^p\left(0,1;X\right),$$

with $1<p<+\infty$. Then, there exists a constant $M\geqslant 0$, which does not depend on $d_0,u_1,(\lambda,\mu)$ and f, such that the unique classical solution u of (1)-(2) satisfies

$$\begin{cases}\|u\|_{L^p(0,1;X)}\leqslant\dfrac{M\beta\left(d_0,u_1,\lambda,\mu,f\right)}{1+|\lambda|}\\ \max\left\{\|u'\|_{L^p(0,1;X)},\|Qu\|_{L^p(0,1;X)},\|Q_\lambda u\|_{L^p(0,1;X)}\right\}\leqslant\dfrac{M\beta\left(d_0,u_1,\lambda,\mu,f\right)}{\sqrt{1+|\lambda|}}\\ \max\left\{\|u''\|_{L^p(0,1;X)},\left\|Q^2u\right\|_{L^p(0,1;X)},\left\|Q_\lambda^2u\right\|_{L^p(0,1;X)}\right\}\leqslant M\beta\left(d_0,u_1,\lambda,\mu,f\right),\end{cases}$$

where

$$\begin{aligned}\beta\left(d_0,u_1,\lambda,\mu,f\right)&=\|d_0\|+\|f\|_{L^p(0,1;X)}+\left\|\left(Q_\lambda-H_\mu\right)^{-1}d_0\right\|_{(D(A),X)_{\frac{1}{2p},p}}\\&\quad+|\lambda|^{1-\frac{1}{2p}}\left\|\left(Q_\lambda-H_\mu\right)^{-1}d_0\right\|_X+\|u_1\|_{(D(A),X)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\|u_1\|.\end{aligned}$$

Note that $\beta\left(d_0,u_1,\lambda,\mu,f\right)$ contains $\|u_1\|$ since it is in $\|u_1\|_{(D(A),X)_{\frac{1}{2p},p}}$ see (51).

Proof. Again we adapt the proof of Theorem 5.7 and write $u=k_1+k_2+k_3-h_2+h_3+h_4$ with

$$\begin{cases}k_1\left(x\right)=S_\lambda\left(x\right)\Lambda_{\lambda,\mu}^{-1}Q_\lambda^{-1}\left[-Q_\lambda e^{2Q_\lambda}d_0+2Q_\lambda^2e^{Q_\lambda}u_1-2Q_\lambda e^{Q_\lambda}I_{\lambda,f}\left(1\right)\right]\\ k_2\left(x\right)=2S_\lambda\left(x\right)\Lambda_{\lambda,\mu}^{-1}Q_\lambda J_{\lambda,f}\left(0\right)\\ k_3\left(x\right)=S_\lambda\left(x\right)\Lambda_{\lambda,\mu}^{-1}d_0\\ h_2\left(x\right)=S_\lambda\left(x\right)J_{\lambda,f}\left(0\right)+S_\lambda\left(1-x\right)I_{\lambda,f}\left(1\right)\\ h_3\left(x\right)=S_\lambda\left(1-x\right)u_1\\ h_4\left(x\right)=I_{\lambda,f}\left(x\right)+J_{\lambda,f}\left(x\right).\end{cases}$$

Estimate of $Q_\lambda^2k_1$. Due to (65), we have for $\xi\in X$ and $x\in[0,1]$

$$\begin{aligned}&\left\|Q_\lambda^2S_\lambda\left(x\right)\Lambda_{\lambda,\mu}^{-1}Q_\lambda^{-1}\xi\right\|\\&\quad=\left\|\left(I-e^{2Q_\lambda}\right)^{-1}\left(I-e^{2(1-x)Q_\lambda}\right)e^{xQ_\lambda}Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}Q_\lambda^{-1}\xi\right\|\\&\quad\leqslant\left\|\left(I-e^{2Q_\lambda}\right)^{-1}\left(I-e^{2(1-x)Q_\lambda}\right)\right\|_{\mathcal{L}(X)}\left\|e^{xQ_\lambda}\right\|_{\mathcal{L}(X)}\left\|Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}Q_\lambda^{-1}\right\|_{\mathcal{L}(X)}\|\xi\|\\&\quad\leqslant M\|\xi\|,\end{aligned}$$

then

$$\left\|Q_\lambda^2k_1\left(x\right)\right\|\leqslant M\left(\|d_0\|+\|u_1\|+\|f\|_{L^p(0,1;X)}\right),$$

and

$$\left\|Q_\lambda^2k_1\right\|_{L^p(0,1;X)}\leqslant M\left(\|d_0\|+\|u_1\|+\|f\|_{L^p(0,1;X)}\right).$$

Estimate of $Q_\lambda^2 k_2$. We write, for $x \in]0,1]$

$$\begin{aligned} Q_\lambda^2 e^{xQ_\lambda}\Lambda_{\lambda,\mu}^{-1}Q_\lambda J_{\lambda,f}(0) &= Q_\lambda^2 e^{xQ_\lambda}\Lambda_{\lambda,\mu}^{-1}\int_0^x e^{sQ_\lambda}f(s)ds \\ &\quad + Q_\lambda^2 e^{xQ_\lambda}\Lambda_{\lambda,\mu}^{-1}\int_x^1 e^{sQ_\lambda}f(s)ds \\ &= Q_\lambda \int_0^x e^{(x-s)Q_\lambda}e^{sQ_\lambda}Q_\lambda\Lambda_{\lambda,\mu}^{-1}e^{sQ_\lambda}f(s)ds \\ &\quad + e^{xQ_\lambda}Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}Q_\lambda^{-1}e^{xQ_\lambda}Q_\lambda\int_x^1 e^{(s-x)Q_\lambda}f(s)ds \\ &= Q_\lambda \int_0^x e^{(x-s)Q_\lambda}F_\lambda(s)\,ds \\ &\quad + e^{xQ_\lambda}Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}Q_\lambda^{-1}e^{xQ_\lambda}Q_\lambda\int_x^1 e^{(s-x)Q_\lambda}f(s)ds, \end{aligned}$$

where $F_\lambda(s) = e^{sQ_\lambda}Q_\lambda\Lambda_{\lambda,\mu}^{-1}e^{sQ_\lambda}f(s)$. So

$$\begin{aligned} \left\|Q_\lambda^2 k_2\right\|_{L^p(0,1;X)} &\leqslant M\left\|Q_\lambda\int_0^\cdot e^{(\cdot-s)Q_\lambda}F_\lambda(s)\,ds\right\|_{L^p(0,1;X)} \\ &\quad + M\left\|Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}Q_\lambda^{-1}\right\|_{\mathcal{L}(X)}\left\|Q_\lambda\int_\cdot^1 e^{(s-\cdot)Q_\lambda}f(s)\,ds\right\|_{L^p(0,1;X)}, \end{aligned}$$

but, from Lemma 5.5, (38), (62), (63) and (65), we deduce

$$\begin{cases} \left\|Q_\lambda\int_0^\cdot e^{(\cdot-s)Q_\lambda}F_\lambda(s)\,ds\right\|_{L^p(0,1;X)} \leqslant M\|F_\lambda\|_{L^p(0,1;X)} \leqslant M\|f\|_{L^p(0,1;X)} \\ \left\|Q_\lambda^2\Lambda_{\lambda,\mu}^{-1}Q_\lambda^{-1}\right\|_{\mathcal{L}(X)}\left\|Q_\lambda\int_\cdot^1 e^{(s-\cdot)Q_\lambda}f(s)\,ds\right\| \leqslant M\|f\|_{L^p(0,1;X)}, \end{cases}$$

therefore

$$\left\|Q_\lambda^2 k_2\right\|_{L^p(0,1;X)} \leqslant M\|f\|_{L^p(0,1;X)}.$$

Estimate of $Q_\lambda^2 k_3$. Due to (64) we write $k_3 = \widetilde{k_3} + \overline{k_3}$ with

$$\begin{cases} \widetilde{k_3}(x) = S_\lambda(x)(Q_\lambda - H_\mu)^{-1}d_0 \\ \overline{k_3}(x) = S_\lambda(x)(Q_\lambda - H_\mu)^{-1}e^{2Q_\lambda}W_{\lambda,\mu}d_0. \end{cases}$$

Due to (53), we have for $x \in]0,1]$

$$\begin{aligned} \left\|Q_\lambda^2\widetilde{k_3}(x)\right\| &= \left\|Q_\lambda^2\left(I - e^{2Q_\lambda}\right)^{-1}\left(I - e^{2(1-x)Q_\lambda}\right)e^{xQ_\lambda}(Q_\lambda - H_\mu)^{-1}d_0\right\| \\ &\leqslant M\left\|Q_\lambda^2 e^{xQ_\lambda}(Q_\lambda - H_\mu)^{-1}d_0\right\|. \end{aligned}$$

From Theorem 2.1 in [14], since

$$(Q_\lambda - H_\mu)^{-1}d_0 \in (D(A),X)_{\frac{1}{2p},p} = (X,D(A))_{1-\frac{1}{2p},p},$$

we get

$$\left\|Q_\lambda^2\widetilde{k_3}\right\|_{L^p(0,1;X)} \leqslant M\left(\left\|(Q_\lambda - H_\mu)^{-1}d_0\right\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\left\|(Q_\lambda - H_\mu)^{-1}d_0\right\|\right).$$

We have also, taking into account (38), (63), (39) and (65)

$$\begin{aligned}\left\|Q_\lambda^2\overline{k_3}(x)\right\| &= \left\|Q_\lambda^2\left(I-e^{2Q_\lambda}\right)^{-1}\left(I-e^{2(1-x)Q_\lambda}\right)e^{xQ_\lambda}\left(Q_\lambda-H_\mu\right)^{-1}e^{2Q_\lambda}W_{\lambda,\mu}d_0\right\| \\ &\leqslant M\left\|e^{xQ_\lambda}Q_\lambda^2\left(Q_\lambda-H_\mu\right)^{-1}Q_\lambda^{-1}Q_\lambda e^{2Q_\lambda}W_{\lambda,\mu}d_0\right\| \\ &\leqslant M\left\|e^{xQ_\lambda}\right\|\left\|Q_\lambda^2\left(Q_\lambda-H_\mu\right)^{-1}Q_\lambda^{-1}\right\|\left\|Q_\lambda e^{2Q_\lambda}\right\|\|W_{\lambda,\mu}\|\|d_0\| \\ &\leqslant \|d_0\|.\end{aligned}$$

Finally

$$\left\|Q_\lambda^2\widetilde{k_3}(x)\right\| \leqslant M\left(\left\|\left(Q_\lambda-H_\mu\right)^{-1}d_0\right\|_{(D(A),X)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\left\|\left(Q_\lambda-H_\mu\right)^{-1}d_0\right\|_X+\|d_0\|\right).$$

Estimate of $Q_\lambda^2h_2, Q_\lambda^2h_3, Q_\lambda^2h_4$. In theses terms, $\Lambda_{\lambda,\mu}^{-1}$ does not appear so the estimates are the same as in Theorem 5.7.

□

7 Results for Dirichlet boundary conditions

We can find, in [15] and [16], the study of the following problem

$$\begin{cases} u''(x)+Au(x)=f(x), & x\in(0,1) \\ u(0)=u_0,\ u(1)=u_1. \end{cases} \tag{67}$$

A classical solution of this problem is a function $u\in W^{2,p}(0,1;X)\cap L^p(0,1;D(A))$, satisfying (67). The authors obtain the following result (see Theorem 4, p. 200 in [15] and Theorem 5 p. 173 (with $A=L=M$) in [16]).

Proposition 7.1 ([15],[16])**.** Let $f\in L^p(0,1;X)$ with $1<p<+\infty$ and assume that $(H_1)\sim(H_3)$ are satisfied. Then the following assertions are equivalent:

1. Problem (67) admits a classical solution u.
2. $u_1,u_0\in\left(D(A),X\right)_{\frac{1}{2p},p}$

Moreover in this case u is unique and given by

$$\begin{aligned}u(x) &= S(x)u_0+S(1-x)u_1-S(x)J(0) \\ &\quad -S(1-x)I(1)+I(x)+J(x),\quad x\in(0,1).\end{aligned} \tag{68}$$

Note that assumptions $(H_1)\sim(H_3)$ are described in Section 3 and $S(\cdot),I(\cdot),J(\cdot)$ are precised in (23) and (25) with $Q=-\sqrt{-A}$.

Now we are in position to study, as in Sections 5 and 6, the spectral corresponding problem

$$\begin{cases} u''(x)+Au(x)-\lambda u(x)=f(x), & x\in(0,1) \\ u(0)=u_0,\ u(1)=u_1. \end{cases} \tag{69}$$

Applying the previous Theorem with A replaced by $A-\lambda I$ we obtain

Theorem 7.2. Assume that (4)∼(6) hold. Let $u_0,\ u_1\in X$, $\lambda\in S_{\varphi_0}$ and $f\in L^p(0,1;X)$ with $1<p<+\infty$; the two following statements are equivalent:

1. Problem (69) has a classical solution u.

2. $u_0, u_1 \in (D(A), X)_{\frac{1}{2p},p}$.

   Moreover in this case u is unique and given by (68) where Q is replaced by Q_λ.

We have also

Theorem 7.3. Assume (4)$\sim$(6) and $u_0, u_1 \in (D(A), X)_{\frac{1}{2p},p}$. Then, there exists a constant $M \geqslant 0$ such that, for $\lambda \in S_{\varphi_0}$ and $f \in L^p(0,1;X)$ with $1 < p < +\infty$, the unique classical solution u of (1)-(2) satisfies

$$\begin{cases} \|u\|_{L^p(0,1;X)} \leqslant \dfrac{M\delta(u_0,u_1,\lambda,f)}{1+|\lambda|} \\ \max\left\{\|u'\|_{L^p(0,1;X)}, \|Qu\|_{L^p(0,1;X)}, \|Q_\lambda u\|_{L^p(0,1;X)}\right\} \leqslant \dfrac{M\delta(u_0,u_1,\lambda,f)}{\sqrt{1+|\lambda|}} \\ \max\left\{\|u''\|_{L^p(0,1;X)}, \|Q^2 u\|_{L^p(0,1;X)}, \|Q_\lambda^2 u\|_{L^p(0,1;X)}\right\} \leqslant M\delta(u_0,u_1,\lambda,f), \end{cases}$$

where

$$\begin{aligned} \delta(u_0,u_1,\lambda,f) &= \|f\|_{L^p(0,1;X)} + \|u_0\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|u_0\|_X \\ &\quad + \|u_1\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|u_1\|. \end{aligned}$$

Proof. Let $\lambda \in S_{\varphi_0}$ and $f \in L^p(0,1;X)$. Taking into account (48) and (68) with Q_λ replacing Q, for $x \in [0,1]$, we have

$$u(x) = S_\lambda(x)u_0 + S_\lambda(1-x)u_1 - S(x)J_{\lambda,f}(0) - S(1-x)I_{\lambda,f} + I_{\lambda,f}(x) + J_{\lambda,f}(x).$$

So we can write $u = -h_2 + g_3 + h_3 + h_4$ with

$$\begin{cases} h_2(x) = S_\lambda(x)J_{\lambda,f}(0) + S_\lambda(1-x)I_{\lambda,f}(1) \\ g_3(x) = S_\lambda(x)u_0 \\ h_3(x) = S_\lambda(1-x)u_1 \\ h_4(x) = I_{\lambda,f}(x) + J_{\lambda,f}(x). \end{cases}$$

As in the proof of Theorem 5.7 we get

$$\left\|Q_\lambda^2 h_2\right\|_{L^p(0,1;X)} \leqslant M\|f\|_{L^p(0,1;X)}, \quad \left\|Q_\lambda^2 h_4\right\|_{L^p(0,1;X)} \leqslant M\|f\|_{L^p(0,1;X)},$$

and also

$$\left\|Q_\lambda^2 h_3\right\|_{L^p(0,1;X)} \leqslant M\left(\|u_1\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|u_1\|_X\right).$$

Moreover, $Q_\lambda^2 g_3$ is treated like $Q_\lambda^2 h_3$ so

$$\left\|Q_\lambda^2 g_3\right\|_{L^p(0,1;X)} \leqslant M\left(\|u_0\|_{(D(A),X)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|u_0\|_X\right).$$

We finish as in the proof of Theorem 5.7. □

8 Generation of Semigroups

In this section, we set $Y = L^p(0,1;X)$ with $p \in (1,+\infty)$ and, under (5), we consider operator

$$\begin{array}{rccc} \mathcal{A}: & D(\mathcal{A}) \subset Y & \longrightarrow & Y \\ & u & \longmapsto & A(u(\cdot)), \end{array}$$

with the following domain

$$D(\mathcal{A}) = \{u \in Y : u(x) \in D(A) \text{ a.e. } x \in X \text{ and } A(u(\cdot)) \in Y\}.$$

Note that this operator is well defined from [11], Proposition 7.1, p. 359.

8.1 First case

Here, under (4)$\sim$(8), we consider the Banach space $Z := Y \times X$ normed by

$$\|(u,v)\|_Z := \|u\|_Y + \|v\|, \quad (u,v) \in Z.$$

For $\mu \in \mathbb{C}$, we build a linear operator $\mathcal{P}_{A,H,\mu}$ on Z, by setting

$$\begin{array}{rccc} \mathcal{P}_{A,H,\mu}: & D(\mathcal{P}_{A,H,\mu}) \subset Z & \longrightarrow & Z \\ & (u,v) & \longmapsto & (u'' + \mathcal{A}u, u'(0) - Hv - \mu v), \end{array}$$

where $D(\mathcal{P}_{A,H,\mu}) = \{(u,v) \in W \times D(H) : u(1) = 0, u(0) = v\}$ with

$$W = W^{2,p}(0,1;X) \cap L^p(0,1;D(A)) \subset Y.$$

Remark 8.1. Let $(f,\tau) \in Z$. We consider the following problem

$$\begin{cases} u'' + \mathcal{A}u - \lambda u = f \\ u'(0) - Hu(0) - (\lambda+\mu)u(0) = \tau \\ u(1) = 0, \end{cases} \tag{70}$$

then the two following assertions are equivalent:

1. $(u,v) \in D(\mathcal{P}_{A,H,\mu})$ and $(\mathcal{P}_{A,H,\mu} - \lambda I)(u,v) = (f,\tau)$.
2. $u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$ is a classical solution of (70) together with $v = u(0)$.

So to study $\mathcal{P}_{A,H,\mu}$, it remains to solve (70).

We set $\varphi_2 := \min\{\varphi_0, \varphi_1\}$, we define for $\varphi_3 \in (0, \pi - \varphi_2)$, $r_{\varphi_3} \in (r_0, +\infty)$ by

$$r_{\varphi_3} := \frac{r_0}{\cos^2\left(\frac{\varphi_2+\varphi_3}{2}\right)}.$$

Proposition 8.2. Let $\varphi_3 \in (0, \pi - \varphi_2)$.

1. If $\lambda \in S_{\varphi_2}, \mu \in S_{\varphi_3}$ with $|\lambda| \geqslant r_{\varphi_3}$, then $(\lambda, \lambda+\mu) \in \Omega_{\varphi_0,\varphi_1,r_0}$.

2. Let $\mu \in S_{\varphi_3}$. Then $\mathcal{P}_{A,H,\mu}$ is a closed linear operator on Z with

$$S_{\varphi_2} \backslash B\left(0, r_{\varphi_3}\right) \subset \rho\left(\mathcal{P}_{A,H,\mu}\right).$$

Moreover, let $\lambda \in S_{\varphi_2} \backslash B\left(0, r_{\varphi_3}\right)$ and $(f, \tau) \in Z$, then

$$(u, v) = \left(\mathcal{P}_{A,H,\mu} - \lambda I\right)^{-1} (f, \tau),$$

satisfies, for $x \in [0,1]$

$$\begin{cases} u(x) &= S_\lambda(x) \Lambda_{\lambda,\lambda+\mu}^{-1} \left(I - e^{2Q_\lambda}\right) \tau \\ & \quad + S_\lambda(x) \left[2\Lambda_{\lambda,\lambda+\mu}^{-1} Q_\lambda \left[J_{\lambda,f}(0) - e^{Q_\lambda} I_{\lambda,f}(1)\right] - J_{\lambda,f}(0)\right] \\ & \quad - S_\lambda(1-x) I_{\lambda,f}(1) + I_{\lambda,f}(x) + J_{\lambda,f}(x) \\ v(x) &= u(0), \end{cases}$$

where $S_\lambda(x) = \left(I - e^{2Q_\lambda}\right)^{-1} \left(e^{xQ_\lambda} - e^{(1-x)Q_\lambda} e^{Q_\lambda}\right) \in \mathcal{L}(X)$.

3. There exists $M_{A,H,\varphi_3} > 0$ such that for $\lambda \in S_{\varphi_2} \backslash B\left(0, r_{\varphi_3}\right)$ and $\mu \in S_{\varphi_3}$ we have

$$\left\| \left(\mathcal{P}_{A,H,\mu} - \lambda I\right)^{-1} \right\|_{\mathcal{L}(Z)} \leqslant \frac{M_{A,H,\varphi_3}}{1 + |\lambda|}.$$

Proof.

1. We have $(\lambda, \lambda + \mu) \in S_{\varphi_2} \times S_{\varphi_2} \subset S_{\varphi_0} \times S_{\varphi_1}$, moreover $|\lambda| \geqslant r_{\varphi_3} > r_0$ and, due to [14], Lemma 2.3, p. 98, we have

$$\frac{|\lambda + \mu|^2}{|\lambda|} \geqslant \cos^2\left(\tfrac{\varphi_2 + \varphi_3}{2}\right) \frac{\left(|\lambda| + |\mu|\right)^2}{|\lambda|} \geqslant \cos^2\left(\tfrac{\varphi_2 + \varphi_3}{2}\right) \times |\lambda| \geqslant r_0.$$

2. It is a consequence of statement 1. and Theorem 5.4.

3. As in statement 1., we have, $|\lambda + \mu| \geqslant \cos\left(\frac{\varphi_2 + \varphi_3}{2}\right) \times |\lambda|$ so setting

$$C_{\varphi_3} := \frac{1}{\cos\left(\frac{\varphi_2 + \varphi_3}{2}\right)} + 1,$$

we have

$$\frac{1 + |\lambda| + |\lambda + \mu|}{1 + |\lambda + \mu|} \leqslant \frac{|\lambda|}{1 + |\lambda + \mu|} + \frac{1 + |\lambda + \mu|}{1 + |\lambda + \mu|} \leqslant C_{\varphi_3}.$$

Let $(f, \tau) \in Z$, then Theorem 5.7 and (56) imply that

$$(u, v) = \left(\mathcal{P}_{A,H,\mu} - \lambda I\right)^{-1} (f, \tau),$$

satisfies

$$\begin{cases} \|u\|_{L^p(0,1;X)} \leqslant \dfrac{MC_{\varphi_3}}{1 + |\lambda|} \left(\|\tau\| + \|f\|_Y\right) \\ \\ v = \|u(0)\| \leqslant \dfrac{MC_{\varphi_3}}{1 + |\lambda|} \left(\|\tau\| + \|f\|_Y\right), \end{cases}$$

that is

$$\left\| \left(\mathcal{P}_{A,H,\mu} - \lambda I\right)^{-1} (f, \tau) \right\|_Z \leqslant \frac{MC_{\varphi_3}}{1 + |\lambda|} \|(f, \tau)\|_Z.$$

□

We then obtain:

Theorem 8.3. Assume (4)∼(8). For each $\mu \in \mathbb{C}$ with $|\arg(\mu)| < \pi - \varphi_2$.

1. $\mathcal{P}_{A,H,\mu}$ is the infinitesimal generator of a C_0-semigroup.
2. Moreover, if $\varphi_2 \in [\pi/2, \pi)$, then $\mathcal{P}_{A,H,\mu}$ is the infinitesimal generator of an analytic semigroup.

8.2 Second case

Assume that (4)∼(6) and (9), (10), (12) hold.

We define for $\mu \in \mathbb{C}$, operators

$$\begin{array}{rccc} \mathcal{L}_{A,H,\mu}: & D(\mathcal{L}_{A,H,\mu}) \subset Y & \longrightarrow & Y \\ & u & \longmapsto & u'' + \mathcal{A}u, \end{array} \tag{71}$$

where $D(\mathcal{L}_{A,H,\mu})$ is the subspace of Y of the functions u satisfying

$$\begin{cases} u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)) \\ u(0) \in D(H) \\ u'(0) - Hu(0) - \mu u(0) = u(1) = 0. \end{cases}$$

Remark 8.4. From Theorems 6.3 and Theorem 6.6, there exists $M \geqslant 0$ such that for any $\mu \in \mathbb{C}$

1. $\mathcal{L}_{A,H,\mu}$ is a closed linear operator on Y.
2. $S_{\varphi_0} \backslash B(0, \rho_\mu) \subset \rho(\mathcal{L}_{A,H,\mu})$ where $\rho_\mu := \max\left\{\rho_0, \rho_0 |\mu|^{1/\varepsilon}\right\} > 0$.
3. $\forall \lambda \in S_{\varphi_0} \backslash B(0, \rho_\mu)$, $\forall f \in Y$, $\forall x \in [0,1]$

$$\begin{array}{rcl} \left(\mathcal{L}_{A,H,\mu} - \lambda I\right)^{-1} f)(x) & = & S_\lambda(x)\left[2\Lambda_{\lambda,\mu}^{-1} Q_\lambda \left[J_{\lambda,f}(0) - e^{Q_\lambda} I_{\lambda,f}(1)\right] - J_{\lambda,f}(0)\right] \\ & & -S_\lambda(1-x) I_{\lambda,f}(1) + I_{\lambda,f}(x) + J_{\lambda,f}(x), \end{array}$$

where $S_\lambda(x) = \left(I - e^{2Q_\lambda}\right)^{-1}\left(e^{xQ_\lambda} - e^{(1-x)Q_\lambda} e^{Q_\lambda}\right)$.

4. $\forall \lambda \in S_{\varphi_0} \backslash B(0, \rho_\mu)$: $\left\|(\mathcal{L}_{A,H,\mu} - \lambda I)^{-1}\right\|_{\mathcal{L}(Y)} \leqslant \dfrac{M}{1+|\lambda|}$.

We then obtain:

Theorem 8.5. Assume (4)∼(6), (9), (10) and (12). Then, for any $\mu \in \mathbb{C}$:

1. $\mathcal{L}_{A,H,\mu}$ is the infinitesimal generator of a C_0-semigroup.
2. Moreover, if $\varphi_0 \in [\pi/2, \pi)$, then $\mathcal{L}_{A,H,\mu}$ is the infinitesimal generator of an analytic semigroup.

8.3 Dirichlet case

Assume $(H_1) \sim (H_3)$. We define operator

$$\begin{array}{rccc} \mathcal{L}_A: & D(\mathcal{L}_A) \subset Y & \longrightarrow & Y \\ & u & \longmapsto & u'' + \mathcal{A}u. \end{array}$$

where $D(\mathcal{L}_A) = \{u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)) : u(0) = u(1) = 0\}$.

Remark 8.6. From Theorems 5.7 and Theorem 6.6, there exists $M \geqslant 0$ such that

1. $\mathcal{L}_A$ is a closed linear operator on Y.
2. $S_{\varphi_0} \subset \rho(\mathcal{L}_A)$.
3. $\forall \lambda \in S_{\varphi_0}$

$$\begin{array}{rcl} \left(\mathcal{L}_{A,H,\mu} - \lambda I)^{-1} f\right)(x) & = & -S_\lambda(x) J_{\lambda,f}(0) - S_\lambda(1-x) I_{\lambda,f}(1) \\ & & + I_{\lambda,f}(x) + J_{\lambda,f}(x), \end{array}$$

   where $S_\lambda(x) = \left(I - e^{2Q_\lambda}\right)^{-1}\left(e^{xQ_\lambda} - e^{(1-x)Q_\lambda} e^{Q_\lambda}\right) \in \mathcal{L}(X)$.

4. $\forall \lambda \in S_{\varphi_0}: \quad \|(\mathcal{L}_A - \lambda I)^{-1}\|_{\mathcal{L}(Y)} \leqslant \dfrac{M}{1+|\lambda|}$.

We then obtain:

Theorem 8.7. Assume $(H_1) \sim (H_3)$. We have

1. $\mathcal{L}_A$ is the infinitesimal generator of a C_0-semigroup.
2. Moreover, if $\varphi_0 \in [\pi/2, \pi)$, then $\mathcal{L}_A$ is the infinitesimal generator of an analytic semigroup.

Remark 8.8. For simplicity, in this paper we have supposed that $0 \in \rho(A)$, see (5), but in the theorems above written for $|\lambda|$ large enough, and those concerning generation of semigroups, we can drop this invertibility assumption; more precisely Theorems 5.4, 5.7, 6.3, 6.6, 8.3, 8.5, 8.7) remain true if we replace (5) by

$$\begin{cases} \exists \varphi_0 \in (0,\pi), \ \exists \omega_0 > 0: \ S_{\varphi_0} \subset \rho(A - \omega_0 I) \text{ and } \exists C_A > 0: \\ \forall \lambda \in S_{\varphi_0}, \quad \left\|(A - \omega_0 I - \lambda I)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \dfrac{C_A}{1+|\lambda|}, \end{cases}$$

In fact it is enough to write (1) in the following form

$$u''(x) + (A - \omega_0 I)u(x) - (\lambda - \omega_0)u(x) = f(x), \quad x \in (0,1),$$

to apply our results replacing A by $A - \omega_0 I$, λ by $\lambda - \omega_0$ and to notice that for $|\lambda| \geqslant 2\omega_0$ we have:

$$\frac{1}{2}|\lambda| \leqslant |\lambda - \omega_0| \leqslant \frac{3}{2}|\lambda|,$$

of course the constants r_0, ρ_0 may change.

9 Applications

9.1 A model example for the first case

In view to illustrate the results obtained in this work, we will consider the concrete problem of the heat equation in the square domain $\Omega = (0,1) \times (0,1)$ with a dynamical-Wentzell condition in one of its lateral boundaries

$$(P) \left\{ \begin{array}{lcll} \dfrac{\partial u}{\partial t}(t,x,y) & = & \Delta_{x,y} u(t,x,y), & (t,x,y) \in (0,+\infty) \times \Omega \\ \dfrac{\partial u}{\partial t}(t,0,y) & = & \dfrac{\partial u}{\partial x}(t,0,y) + \dfrac{\partial^2 u}{\partial y^2}(t,0,y), & (t,0,y) \in (0,+\infty) \times \Gamma_0 \\ u(t,1,y) & = & 0, & (t,1,y) \in (0,+\infty) \times \Gamma_1 \\ u(t,x,0) & = & u(t,x,1) = 0, & x \in (0,1) \\ u(0,x,y) & = & u_0(x,y) & (x,y) \in (0,1) \times (0,1), \end{array} \right.$$

where

$$\left\{ \begin{array}{ll} \Gamma_0 = \{0\} \times (0,1), & \Gamma_1 = \{1\} \times (0,1), \\ \gamma_0 = (0,1) \times \{0\}, & \gamma_1 = (0,1) \times \{1\}. \end{array} \right.$$

Here $\dfrac{\partial^2}{\partial y^2}$ is the Laplace-Beltrami operator on Γ_0. Physically, $-\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial x}$ represent the interaction between the domain Ω and the lateral boundaries while $\dfrac{\partial^2 u}{\partial y^2}$ is the boundary diffusion.

Set $\mathcal{E} = L^p(\Omega) \times L^p(\Gamma_0)$; this Banach space is well defined and endowed with its natural norm. Define operator $\mathcal{P}$ by

$$\left\{ \begin{array}{rcl} D(\mathcal{P}) & = & \Big\{ w = (u, v_0) : u, \Delta_{x,y} u \in L^p(\Omega), v_0 \in W^{2,p}(\Gamma_0), u_{|\Gamma_0} = v_0, \\ & & \qquad (\Delta_{x,y} u)_{|\Gamma_0} = \left(\frac{\partial u}{\partial x}\right)_{|\Gamma_0} + \dfrac{\partial^2 v_0}{\partial y^2} \ \text{ and } \ u_{|\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0 \Big\}, \\ \mathcal{P} w & = & \left(\Delta_{x,y} u, \left(\frac{\partial u}{\partial x}\right)_{|\Gamma_0} + \dfrac{\partial^2 v_0}{\partial y^2} \right), \quad \text{for } w = (u, v_0) \in D(\mathcal{P}). \end{array} \right.$$

The boundary conditions are defined in $L^p(\Gamma_0)$ and $\mathcal{P} w \in \mathcal{E}$.

On the other hand it is not difficult to see that this operator is closed since all the actions which describe the boundary conditions are "closed". We conclude that $\mathcal{P}$ is closed and well defined on $\mathcal{E}$.

When we integrate the time variable t, the following Cauchy problem

$$\left\{ \begin{array}{l} \dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial t}(u, v_0) = \left(\dfrac{\partial u}{\partial t}, \dfrac{\partial v_0}{\partial t} \right) = \mathcal{P} w = \mathcal{P}(u, v_0) \\ w(0) = (u(0,.), v_0(0,.)) \ \text{ given}, \end{array} \right.$$

writes

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} = \Delta u \\ \dfrac{\partial v_0}{\partial t} = (\partial u / \partial x)_{|\Gamma_0} + \dfrac{\partial^2 v_0}{\partial y^2} \\ u_{|\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0 \\ (u(0,.), v_0(0,.)) \ \text{ given}; \end{array} \right.$$

since $(u, v_0) \in D(\mathcal{P})$ and $\dfrac{\partial u}{\partial t} = \Delta u$, we obtain

$$(\Delta u)_{|\Gamma_0} = \left(\frac{\partial u}{\partial x}\right)_{|\Gamma_0} + \left(\frac{\partial^2 v_0}{\partial y^2}\right)_{|\Gamma_0} = \left(\frac{\partial u}{\partial t}\right)_{|\Gamma_0},$$

and since $u_{|\Gamma_0} = v_0$, by using the tangential derivative, we obtain

$$\left(\frac{\partial^2 v_0}{\partial y^2}\right)_{|\Gamma_0} = \left(\frac{\partial^2 u}{\partial y^2}\right)_{|\Gamma_0},$$

summarizing we deduce the same equation as in Problem (P) :

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u \\ \left(\dfrac{\partial u}{\partial t}\right)_{|\Gamma_0} = \left(\dfrac{\partial u}{\partial x}\right)_{|\Gamma_0} + \left(\dfrac{\partial^2 u}{\partial y^2}\right)_{|\Gamma_0} \\ u(0,.) \text{ is given} \\ u_{|\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0. \end{cases}$$

Now it is well known that the complete study of the abstract evolution equation above is based on the study of the following spectral equation

$$\begin{cases} \mathcal{P}(u, v_0) - \lambda(u, v_0) = (h, d_0) \\ (u, v_0) \in D(\mathcal{P}), (h, d_0) \in \mathcal{E}, \end{cases} \tag{72}$$

and since $u_{|\Gamma_0} = v_0$, (72) is equivalent to

$$\begin{cases} \Delta u - \lambda u = h \\ \left(\dfrac{\partial u}{\partial x}\right)_{|\Gamma_0} + \left(\dfrac{\partial^2 u}{\partial y^2}\right)_{|\Gamma_0} - \lambda u_{|\Gamma_0} = d_0 \\ u_{|\gamma_0 \cup \gamma_1 \cup \Gamma_0} = 0, \end{cases} \tag{73}$$

which is an elliptic partial differential equation with the same spectral parameter in the equation and in the boundary condition on Γ_0.

We will write (73) in an operational differential form. We consider the Banach space $X = L^p(0,1)$ and identify $\mathcal{E}$ with $L^p(0,1;X)$ by writing as usual, for $g \in \mathcal{E}$

$$g(x,y) = (g(x))(y), \quad x, y \in (0,1).$$

We define operator A on X by

$$\begin{cases} D(A) = \{\psi \in W^{2,p}(0,1) : \psi(0) = \psi(1) = 0\} \\ A\psi(y) = \psi''(y), \end{cases} \tag{74}$$

and operator $H := -A$. So, equation

$$\Delta u(x,y) - \lambda u(x,y) = h(x,y),$$

takes the following form in space X

$$u''(x) + Au(x) - \lambda u(x) = h(x), \ x \in (0,1),$$

while the boundary condition

$$\left(\frac{\partial u}{\partial x}\right)_{|\Gamma_0} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{|\Gamma_0} - \lambda u_{|\Gamma_0} = d_0,$$

writes as

$$u'(0) - Hu(0) - \lambda u(0) = d_0;$$

the condition $u_{|\gamma_0 \cup \gamma_1} = 0$ (which means that $u(0,y)$ and $u(1,y)$ vanish in $y=0$ and $y=1$) is implicitely included in the fact that $u(0) := u(0,.)$ and $u(1) := u(1,.)$ are in $D(H)$.

Therefore (73) or equivalently (72), write in the following abstract form

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = h(x), \ x \in (0,1) \\ u'(0) - Hu(0) - \lambda u(0) = d_0 \\ u(1) = 0, \end{cases} \tag{75}$$

where $(h, d_0) \in \mathcal{E} \equiv L^p(0,1;X) \times L^p(X)$ and we are in the situation of Subsection 8.1 with $\mu = 0$.

Let u be the classical solution of (75), then $u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$ and

$$(u, u(0)) \in D(\mathcal{P}),$$

so that $(u, u(0)) = (\mathcal{P} - \lambda I)^{-1}(h, d_0)$.

Taking into account the fact that, here, we can take $\varphi_0 = \pi - \varepsilon$ ($\varepsilon > 0$ as close to 0 as we want), we can use Proposition 8.2 and Theorem 8.3, to obtain :

$$\exists M > 0, \ \forall \lambda \in S_{\varphi_0} : \forall (h, d_0) \in \mathcal{E}, \quad \left\| (\mathcal{P} - \lambda I)^{-1}(h, d_0) \right\|_{\mathcal{E}} \leqslant \frac{M}{1 + |\lambda|} \|(h, d_0)\|_{\mathcal{E}},$$

and deduce that our operator $\mathcal{P}$ defined above generates an analytic semigroup in $\mathcal{E}$.

This example can be extended to the following problem

$$\begin{cases} \Delta u - \lambda u = h \\ a_0 \left(\partial u / \partial x\right)_{|\Gamma_0} + b_0 \dfrac{\partial^2 v_0}{\partial y^2} - \lambda v_0 = d_0 \\ a_1 \left(\partial u / \partial x\right)_{|\Gamma_1} + b_1 \dfrac{\partial^2 v_1}{\partial y^2} - \lambda v_1 = d_1 \\ u_{|\gamma_0 \cup \gamma_1} = 0. \end{cases}$$

9.2 Some concrete examples for the second case

9.2.1 Example 1

Here, we set $\Omega = (0,1) \times (0,1)$. Our concrete spectral partial differential problem is

$$(P1) \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) - \lambda u(x,y) = f(x,y), & (x,y) \in \Omega \\ u(1,y) = 0, & y \in (0,1) \\ \dfrac{\partial u}{\partial x}(0,y) - \displaystyle\int_0^y \phi(y,\xi)\, u(0,\xi)\, d\xi = 0, & y \in (0,1) \\ u(x,0) = u(x,1) = 0, & x \in (0,1), \end{cases}$$

where we can take $\lambda \in S_{\varphi_0}$ with φ_0 fixed in $(\pi/2, \pi)$.

Define operator A on $X := L^p(0,1)$, with $1 < p < +\infty$, as in (74), then the square root of the opposite of this operator is well defined and

$$W_0^{1,p}(0,1) \subset D((-A)^{1/2}) \subset W^{1,p}(0,1) \quad \text{and} \quad \left\|(-A)^{1/2}\psi\right\| \approx \|\psi'\|_{L^p(0,1)} + \|\psi\|_{L^p(0,1)},$$

see [3]. We know also that $Q = -\sqrt{-A}$ generates an analytic semigroup in X, on the other hand $Q_\lambda = -\sqrt{-A+\lambda I}$ is well defined and generates an analytic semigroup in X for all $\lambda \in S_{\varphi_0}$.

Now let us define operator H by

$$H\psi(y) = \int_0^y \phi(y,\xi)\psi(\xi)d\xi, \psi \in X, \tag{76}$$

with an appropiate function ϕ having the following properties. Let $q \in (1,+\infty)$ such that $1/q + 1/p = 1$. We then assume that

$$\begin{cases} \phi(y,\cdot), \dfrac{\partial\phi}{\partial y}(y,\cdot) \in L^q(0,1), \text{ for a.e. } y \in (0,1) \\ \phi(1,\cdot) = 0 \\ \Phi_j : y \longmapsto \dfrac{\partial^j \phi}{\partial y^j}(y,\cdot) \in L^p(0,1;L^q(0,1)), \text{ for } j = 0,1 \\ \phi_1 : y \longmapsto \phi(y,y) \in L^p(0,1). \end{cases} \tag{77}$$

We can build a simple example of a function ϕ satisfying (77), setting, for a fixed $n \in \mathbb{N}\setminus\{0\}$

$$\phi(y,\xi) = (1-y)^n \widetilde{\psi}(\xi), \ \xi, y \in (0,1),$$

where $\widetilde{\psi} \in W^{1,q}(0,1) \cap W^{1,p}(0,1)$. We have

$$\begin{aligned} \|H(\psi)\|_X &= \left(\int_0^1 \left|\int_0^y \phi(y,\xi)\psi(\xi)d\xi\right|^p dy\right)^{1/p} \\ &\leqslant \left(\int_0^1 \left[\left(\int_0^1 |\phi(y,\xi)|^q d\xi\right)^{1/q}\left(\int_0^1 |\psi(\xi)|^p d\xi\right)^{1/p}\right]^p dy\right)^{1/p} \\ &\leqslant \left(\int_0^1 \left[\left(\int_0^1 |\phi(y,\xi)|^q d\xi\right)^{1/q}\right]^p dy\right)^{1/p}\left(\int_0^1 |\psi(\xi)|^p d\xi\right)^{1/p} \\ &\leqslant \left(\int_0^1 \|\phi(y,\cdot)\|^p_{L^q(0,1)} dy\right)^{1/p} \|\psi\|_X \\ &\leqslant \|\Phi\|_{L^p(0,1;L^q(0,1))} \times \|\psi\|_X, \end{aligned}$$

so $H \in \mathcal{L}(X)$.

Our concrete problem $(P1)$ writes in the following abstract form

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), \quad \text{a.e. } x \in (0,1) \\ u(1) = 0, \ \ u'(0) - Hu(0) = 0. \end{cases}$$

The following assumptions are satisfied:

1. X is a UMD space and operator A verifies

$$\begin{cases} \exists\ \varphi_0 \in (0,\pi) : \ S_{\varphi_0} \subset \rho(A) \ \text{ and } \exists C_A > 0 : \\ \forall \lambda \in S_{\varphi_0}, \quad \left\|(A-\lambda I)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \dfrac{C_A}{1+|\lambda|}, \end{cases}$$

$$\begin{cases} \forall s \in \mathbb{R},\ \ (-A)^{is} \in \mathcal{L}(X),\ \exists \theta_A \in (0,\pi): \\ \sup\limits_{s\in\mathbb{R}} \left\| e^{-\theta_A |s|} (-A)^{is} \right\|_{\mathcal{L}(X)} < +\infty. \end{cases}$$

this last property is proved explicitely in [23].

2. Since H is bounded, from Remark 2.3, statement 1, we get $D(Q) \subset D(H)$ and,

$$\exists C_{H,Q} > 0, \quad \sup_{t\in[0,+\infty)} (1+t)^{1/2} \parallel HQ_t^{-1} \parallel_{\mathcal{L}(X)} \leqslant C_{H,Q}.$$

3. We verifiy that $(Q-H)^{-1}(D(Q)) \subset D(Q^2)$.
   Let $\psi \in D(Q)$ such that $(Q-H)(\psi) \in D(Q)$, then

$$Q\psi - H\psi = g \in D(Q),$$

   with

$$W_0^{1,p}(0,1) \subset D(Q) \subset W^{1,p}(0,1).$$

   To obtain $\psi \in D(Q^2)$, it suffices to have $H\psi \in W_0^{1,p}(0,1)$ for $\psi \in D(Q) \subset W^{1,p}(0,1)$. We have

$$H\psi(y) = \int_0^y \phi(y,\xi)\,\psi(\xi) d\xi,$$

   then $H\psi(0) = 0$, and $H\psi(1) = 0$ due to (77) and

$$(H\psi)'(y) = \phi(y,y)\,\psi(y) + \int_0^y \frac{\partial \phi}{\partial y}(y,\xi)\,\psi(\xi) d\xi.$$

   In virtue of the assumptions verified by ϕ, we then get $H\psi \in W_0^{1,p}(0,1)$. Therefore $\psi \in D(Q^2)$.

Now, we set $Y = L^p(0,1;X) = L^p(\Omega)$ and considering A, H defined by (74) and (76), we build, as in (71)

$$\begin{array}{rccc} \mathcal{L}_{A,H,0}: & D(\mathcal{L}_{A,H,0}) \subset Y & \longrightarrow & Y \\ & u & \longmapsto & u'' + A(u(.)). \end{array}$$

Note that in this example, in general, operators Q and H do not commute.

We can apply Theorems 8.5 (with $\mu = 0$), to obtain that $\mathcal{L}_{A,H,0}$ is the infinitesimal generator of an analytic semigroup.

This result allows us to consider and solve the corresponding Cauchy problem with respect to $(P1)$.

9.2.2 Example 2

Here, we are considering a quasi-elliptic problem under an oblique derivative boundary condition. Let $\Omega = (0,1)^2$ and consider the following spectral problem

$$(P2) \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) - \dfrac{\partial^4 u}{\partial y^4}(x,y) - \lambda u(x,y) = f(x,y), & (x,y) \in \Omega \\ u(1,y) = 0, & y \in (0,1) \\ \dfrac{\partial u}{\partial x}(0,y) + c(y)\dfrac{\partial u}{\partial y}(0,y) = 0, & y \in (0,1) \\ u(x,0) = u(x,1) = \dfrac{\partial^2 u}{\partial y^2}(x,0) = \dfrac{\partial^2 u}{\partial y^2}(x,0) = 0, & x \in (0,1). \end{cases}$$

We will assume that

$$c \in \mathcal{C}^2[0,1] : c(0) = c(1) = 0.$$

Here the boundary condition on $\Gamma = \{0\} \times (0,1)$:

$$\frac{\partial u}{\partial x}(0,y) + c(y)\frac{\partial u}{\partial y}(0,y) = 0,$$

can be written as

$$\nabla u(\sigma) \cdot \alpha(\sigma) = 0 \text{ in } \Gamma, \tag{78}$$

with $\alpha(\sigma)$ a vector on Γ equal to $(1, c(y))$ which is pointing inwardly of Ω. It is known that (78) is called oblique derivative boundary condition on Γ. We set, in space $X = L^p(0,1)$, as above

$$\begin{cases} D(A) = \{\psi \in W^{4,p}(0,1) : \psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0\} \\ A\psi(y) = -\psi^{(4)}(y), \end{cases} \tag{79}$$

so, as we have seen

$$\begin{cases} D(\sqrt{-A}) = \{\psi \in W^{2,p}(0,1) : \psi(0) = \psi(1) = 0\} \\ \sqrt{-A}\psi(y) = -\psi''(y), \end{cases}$$

and clearly $Q = -\sqrt{-A}$ and $Q_\lambda = -\sqrt{-A + \lambda I}$, for all $\lambda \in S_\varphi$ generate analytic semigroups in X. We note also that $\sqrt{-Q} = (-A)^{1/4}$ is well defined and

$$W_0^{1,p}(0,1) \subset D((-A)^{1/4}) \subset W^{1,p}(0,1) \quad \text{and} \quad \left\|(-A)^{1/4}\psi\right\| \approx \|\psi'\|_{L^p(0,1)} + \|\psi\|_{L^p(0,1)},$$

see [3]. Now, define operator H by setting and

$$\begin{cases} D(H) = W^{1,p}(0,1) \\ [H\psi](y) = -c(y)\psi'(y). \end{cases} \tag{80}$$

We then have $D((-A)^{1/4}) \subset D(H)$, therefore, see Remark 2.3, statement 2, there exists $C > 0$ such that, for $t \geqslant 0$, we have

$$\begin{aligned} \| HQ_t^{-1} \|_{\mathcal{L}(X)} &\leqslant \| H(-A)^{-1/4} \|_{\mathcal{L}(X)} \| (-A)^{1/4}(-A + tI)^{-1/2} \|_{\mathcal{L}(X)} \\ &\leqslant \frac{C}{(1+t)^{1/4}}. \end{aligned}$$

Now, we will prove that $(Q - H)^{-1}(D(Q)) \subset D(Q^2)$. To this end, let $\psi \in D(Q)$ such that $(Q - H)(\psi) \in D(Q)$, then

$$\psi'' - c\psi' = g \in D(Q) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1),$$

so $\psi \in W^{4,p}(0,1)$. We have $\psi \in D(Q)$, then $\psi(0) = \psi(1) = 0$. But $g \in D(Q)$ thus $g(0) = g(1) = 0$ and

$$\psi''(j) = (c\psi')(j) + g(j) = 0, \quad j = 0, 1,$$

that is $\psi''(0) = \psi''(1) = 0$, therefore $\psi \in D(Q^2)$.

Note that in this example $Q - H$ is boundedly invertible and from equation $Q\psi - H\psi = g$, it follows that

$$\begin{cases} \psi''(y) - c(y)\psi'(y) = g(y) \\ \psi(0) = \psi(1) = 0. \end{cases}$$

Let $\psi_1$and ψ_2 two linearly independent solutions of

$$\psi''(y) - c(y)\psi'(y) = 0,$$

such that $\psi_1(0) = 0$ and $\psi_2(1) = 0$. Then we have

$$\begin{aligned}\psi(y) &= -\psi_2(y)\int_0^y \frac{\psi_1(s)}{W(s)}g(s)ds - \psi_1(y)\int_y^1 \frac{\psi_2(s)}{W(s)}g(s)ds \\ &= \left[(Q-H)^{-1}g\right](y),\end{aligned}$$

where the wronskian W is given by

$$W(s) = \psi_1(s)\psi_2'(s) - \psi_2(s)\psi_1'(s).$$

We have

$$\psi'(y) = -\psi_2'(y)\int_0^y \frac{\psi_1(s)}{W(s)}g(s)ds - \psi_1'(y)\int_y^1 \frac{\psi_2(s)}{W(s)}g(s)ds,$$

and

$$\psi''(y) = -\psi_2''(y)\int_0^y \frac{\psi_1(s)}{W(s)}g(s)ds - \psi_1''(y)\int_y^1 \frac{\psi_2(s)}{W(s)}g(s)ds + g(y).$$

If $g \in D(Q) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1)$, it is clear that $\psi \in W^{4,p}(0,1)$ and

$$\begin{aligned}\psi''(0) &= g(0) - \psi_1''(0)\int_0^1 \frac{\psi_2(s)}{W(s)}g(s)ds \\ &= 0 - [c(0)\psi_1'(0)]\int_0^1 \frac{\psi_2(s)}{W(s)}g(s)ds \\ &= 0;\end{aligned}$$

similarly we obtain $\psi''(1) = 0$.

Again, our concrete problem $(P3)$ writes in the abstract form

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), & \text{for a.e. } x \in (0,1) \\ u(1) = 0, \;\; u'(0) - Hu(0) = 0, \end{cases}$$

with A and H defined by (79) (80) and setting

$$\begin{array}{rccc} \mathcal{L}_{A,H,0}: & D(\mathcal{L}_{A,H,0}) \subset Y & \longrightarrow & Y \\ & u & \longmapsto & u'' + A(u(.)). \end{array}$$

We can apply Theorem 8.5 (with $\mu = 0$), to obtain that $\mathcal{L}_{A,H,0}$ is the infinitesimal generator of an analytic semigroup.

9.2.3 Example 3

Here, we are considering a quasi-elliptic problem under an oblique derivative boundary condition. Let $\Omega = (0,1) \times \mathbb{R}$ and consider the following spectral problem

$$(P3)\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) - \dfrac{\partial^4 u}{\partial y^4}(x,y) - \lambda u(x,y) = f(x,y), & (x,y) \in \Omega \\ u(1,y) = 0, & y \in \mathbb{R} \\ \dfrac{\partial u}{\partial x}(0,y) + c(y)\dfrac{\partial u}{\partial y}(0,y) = 0, & y \in \mathbb{R}, \end{cases}$$

where the function c is assumed to be a positive regular function. Here the boundary condition on $\Gamma = \{0\} \times \mathbb{R}$:

$$\frac{\partial u}{\partial x}(0,y) + c(y)\frac{\partial u}{\partial y}(0,y) = 0,$$

can be written as

$$\nabla u(\sigma) \cdot \alpha(\sigma) = 0 \text{ in } \Gamma,$$

with $\alpha(\sigma)$ a vector equal to $(1, c(y))$ which is pointing inwardly of Ω. It is known that condition is called oblique derivative boundary condition on Γ. We set, in space $X = L^2(\mathbb{R})$:

$$\begin{cases} D(A) = H^4(\mathbb{R}) \\ A\psi(y) = -\psi^{(4)}(y), \end{cases}$$

so

$$\begin{cases} D(\sqrt{-A}) = H^2(\mathbb{R}) \\ \sqrt{-A}\psi(y) = -\psi''(y). \end{cases}$$

So, we know that the spectrum of $\sqrt{-A}$ is exactly $[0, +\infty)$. This follows from the fact that the Fourier transform:

$$\sqrt{-A}\psi - \lambda\psi = g \Longleftrightarrow \left(4\pi^2\xi^2 - \lambda\right)\widehat{\psi} = \widehat{g},$$

gives

$$\left(\sqrt{-A} - \lambda I\right)^{-1} g = \mathcal{F}^{-1}\left(\left(4\pi^2\xi^2 - \lambda\right)^{-1}\widehat{g}\right),$$

for all $\lambda \in \mathbb{C}\backslash[0, +\infty)$. Same arguments are used to see that A verify our assumption (5) but only on $S_\varphi \backslash \{0\}$. We do not have the invertibility of A.

Clearly $Q = -\sqrt{-A}$ and $Q_\lambda = -\sqrt{-A + \lambda I}$, for all $\lambda \in S_\varphi$ (with $\lambda \neq 0$) generate analytic semigroups in X. We note also that $\sqrt{-Q} = (-A)^{1/4}$ is well defined. Note that $(-A + tI)^{-1/2}$ and $(-A + tI)^{-1/4}$ are well defined and bounded for all $t > 0$.

$$D\left((-A)^{1/4}\right) = H^1(\mathbb{R}) \quad \text{and} \quad \left\|(-A)^{1/4}\psi\right\| \approx \left\|\psi'\right\|_X,$$

see Theorem 3.1, p.5 in [3].

Now, define

$$\begin{cases} D(H) = H^1(\mathbb{R}), \\ [H\psi](y) = -c(y)\psi'(y), \end{cases}$$

we then have

$$D\left((-A)^{1/4}\right) = D(H).$$

Therefore, as we have seen, there exists $C > 0$ such that, for $t > 0$

$$\begin{aligned} \| HQ_t^{-1} \|_{\mathcal{L}(X)} &\leqslant \| H(-A + I)^{-1/4} \|_{\mathcal{L}(X)} \| (-A + I)^{1/4}(-A + tI)^{-1/2} \|_{\mathcal{L}(X)} \\ &\leqslant \frac{C}{(1+t)^{1/4}}. \end{aligned}$$

Now, we will prove that

$$(Q - H)^{-1}(D(Q)) \subset D\left(Q^2\right).$$

Let $\psi \in D(Q) = H^2(\mathbb{R})$ such that $(Q - H)(\psi) \in D(Q)$. Then $\psi'' - c\psi' = g \in H^2(\mathbb{R})$, therefore $\psi'' = c\psi' + g \in H^1(\mathbb{R})$ and

$$\psi''' = c\psi'' + c'\psi' + g' = c^2\psi' + cg + c'\psi' + g' \in H^1(\mathbb{R}),$$

from which it follows that

$$\psi'''' = c^2\psi'' + 2cc'\psi' + cg' + c'g + c'\psi'' + c''\psi' + g'',$$

so $\psi \in H^4(\mathbb{R}) = D(Q^2)$.

Here $0 \notin \rho(A)$, but using Remark 8.8, as in the two previous examples, we get that $\mathcal{L}_{A,H,0}$ is the infinitesimal generator of an analytic semigroup.

Note that in this example, operators Q and H do not commute necessarily. Remark that in this example we can also take $X = L^p(\mathbb{R})$, with $1 < p < +\infty$, the same arguments apply.

9.2.4 Example 4

In [22] the authors have considered and studied the following problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y,t) + \dfrac{\partial^2 u}{\partial y^2}(x,y,t) = f(x,y,t), & (x,y,t) \in \mathbb{R}_+ \times \mathbb{R} \times (0,T) \\ u(0,y,0) = f_1(y), & y \in \mathbb{R} \\ \dfrac{\partial u}{\partial x}(0,y,t) - D_t^\nu u(0,y,t) = f_2(y,t), & (y,t) \in \mathbb{R} \times (0,T), \end{cases}$$

where D_t^ν, for $\nu \in (0,1)$, denotes the fractional time derivative (or Caputo Derivative) defined, for instance, by

$$D_t^\nu g(.,t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{1}{(t-\tau)^\nu} \frac{\partial g}{\partial \tau}(.,\tau) d\tau,$$

for functions g of classe C^1 with respect to the second variable; for this derivative, see for instance [10]. This derivative has been extended to functions in $L^1_{loc}(\mathbb{R})$ verifying some integrability condition, see [24].

Analysis of the above problem is useful to study the free boundary problem for the Laplace equation in the case of subdiffusion as illustrated by the fractional derivative, see [26]. We recall that this subdiffusion expressed by this Caputo Derivative means that the square displacement of the diffusing species has a behaviour as t^ν for some real number ν. When $\nu \in (0,1)$, we are in the presence of a subdiffusion.

Our objective is not to study this problem, but it helps us to consider a class of similar problems illustrating our theory of the second case. So, setting $\Omega_T = (0,1) \times (0,1) \times (0,T)$, we will take inspiration from this example to consider the following spectral elliptic problem:

$$(P4) \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y,t) + \dfrac{\partial^2 u}{\partial y^2}(x,y,t) - \lambda u(x,y,t) = f(x,y,t), & (x,y,t) \in \Omega_T \\ u(1,y,0) = f_1(y), & y \in (0,1) \\ \dfrac{\partial u}{\partial x}(0,y,t) - D_t^\nu u(0,y,t) = f_2(y,t), & (y,t) \in (0,1) \times (0,T), \end{cases}$$

for $\lambda \in S_{\varphi_0}$with $\varphi_0 \in (\pi/2, \pi)$.

In view to write this problem in an abstract form, we will hide the variable (y,t) by considering the following anisotropic Sobolev Banach space

$$X = W_p^{0,1}((0,1) \times (0,T)),$$

consisting of all functions $(y,t) \longmapsto w(y,t)$ which are in $L^p((0,1) \times (0,T))$ such that we have $\dfrac{\partial w}{\partial t} \in L^p((0,1) \times (0,T))$; it is endowed with the following natural norm

$$\|w\|_X = \|w\|_{L^p((0,1)\times(0,T))} + \left\| \frac{\partial w}{\partial t} \right\|_{L^p((0,1)\times(0,T))}.$$

Now, define operator A in X by

$$\begin{cases} D(A) = \left\{ w \in X : \dfrac{\partial w}{\partial y}, \dfrac{\partial^2 w}{\partial y^2} \in L^p(\mathbb{R} \times (0,T)) \text{ and } w(0,t) = w(1,t) = 0 \text{ for } t \in (0,T) \right\} \\ [Aw](y,t) = \dfrac{\partial^2 w}{\partial y^2}(y,t). \end{cases}$$

We also define H by

$$\begin{cases} D(H) = W_p^{0,1}(\mathbb{R} \times (0,T)) = X \\ [Hw](y,t) = D_t^{\nu} w(y,t). \end{cases}$$

This problem can be written in the following abstract form:

$$\begin{cases} u''(x) + Au(x) - \lambda u(x) = f(x), \quad \text{for a.e. } x \in (0,1) \\ u(1) = f_1 \\ u'(0) - Hu(0) = f_2, \end{cases}$$

where we have used the usual abstract writting

$$u(x,y,t) = u(x)(y,t) \quad \text{and} \quad f(x,y,t) = f(x)(y,t).$$

Now we must verify the following statements.

1. X is a UMD space.

   In fact, consider the application

   $$\begin{array}{rccc} \mathcal{T} & : & W_p^{0,1}((0,1) \times (0,T)) & \longrightarrow & Z = [L^p((0,1) \times (0,T))]^2 \\ & & w & \longmapsto & \left(w, \dfrac{\partial w}{\partial t} \right), \end{array}$$

   then $\mathcal{T}\left(W_p^{0,1}((0,1) \times (0,T))\right)$ is a closed subspace of Z and thus has a UMD property. Since it is isometric to X , we deduce that X is a UMD space.

2. Operator A verifies

   $$\begin{cases} S_{\varphi_0} \subset \rho(A) \text{ and } \exists C_A > 0 : \\ \forall \lambda \in S_{\varphi_0}, \ \left\| (A - \lambda I)^{-1} \right\|_{\mathcal{L}(X)} \leqslant \dfrac{C_A}{1 + |\lambda|}, \end{cases}$$

   and

   $$\begin{cases} \forall s \in \mathbb{R}, \ (-A)^{is} \in \mathcal{L}(X), \ \exists \theta_A \in (0,\pi) : \\ \sup\limits_{s \in \mathbb{R}} \left\| e^{-\theta_A |s|} (-A)^{is} \right\|_{\mathcal{L}(X)} < +\infty. \end{cases}$$

   For the first property we note that the spectral properties on operator A are based on the equation

   $$\begin{cases} \dfrac{\partial^2 w}{\partial y^2}(y,t) - \lambda w(y,t) = h(y,t) \\ w(0,t) = w(1,t) = 0 \text{ for } t \in (0,T), \end{cases}$$

   where $h \in W_p^{0,1}((0,1) \times (0,T))$. Then, for all $\lambda \in S_{\varphi_0}$, we have

   $$\forall (y,t) \in (0,1) \times (0,T), \quad w(y,t) = \int_0^1 K_{\sqrt{\lambda}}(y,s) h(s,t) ds,$$

where the kernel $K_{\sqrt{\lambda}}(y,s)$ is well known. Using the Schur Lemma, we obtain, for all $t \in (0,1)$, we obtain

$$\int_0^1 |w(y,t)|^p \, dy \leqslant \left[\frac{C}{1+|\lambda|}\right]^p \int_0^1 |h(s,t)|^p \, ds;$$

then

$$\int_0^T \int_0^1 |w(y,t)|^p \, dydt \leqslant \left[\frac{C}{1+|\lambda|}\right]^p \int_0^T \int_0^1 |h(s,t)|^p \, dsdt,$$

that is

$$\|w\|_{L^p((0,1)\times(0,T))} \leqslant \frac{C}{1+|\lambda|} \|h\|_{L^p((0,1)\times(0,T))}.$$

Since we have

$$\forall (y,t) \in (0,1)\times(0,T), \quad \frac{\partial w}{\partial t}(y,t) = \int_0^1 K_{\sqrt{\lambda}}(y,s) \frac{\partial h}{\partial t}(s,t) ds,$$

we deduce

$$\left\| \frac{\partial w}{\partial t} \right\|_{L^p((0,1)\times(0,T))} \leqslant \frac{C}{1+|\lambda|} \left\| \frac{\partial h}{\partial t} \right\|_{L^p((0,1)\times(0,T))},$$

and then

$$\|w\|_X \leqslant \frac{C}{1+|\lambda|} \|h\|_X.$$

The second property is proved explicitely in [23].

3. Since H is bounded then from Remark 2.3, statement 1, $D(Q) \subset D(H)$ and

$$\exists C_{H,Q} > 0, \quad \sup_{t\in[0,+\infty)} (1+t)^{1/2} \parallel HQ_t^{-1} \parallel_{\mathcal{L}(X)} \leqslant C_{H,Q}.$$

4. Now, we must verifiy that $(Q-H)^{-1}(D(Q)) \subset D(Q^2)$. It is enough to verify that $D_t^\nu A^{-1} = A^{-1} D_t^\nu$ on X. We have

$$\forall (y,t) \in (0,1)\times(0,T), \quad \left[A^{-1}w\right](y,t) = \int_0^1 G(y,s) w(s,t) ds,$$

where the kernel G is well known. So, for any $(y,t) \in (0,1)\times(0,T)$

$$\left[D_t^\nu A^{-1} w\right](y,t) = \int_0^1 G(y,s) D_t^\nu w(s,t) ds = \left[A^{-1} D_t^\nu\right] w(y,t).$$

Again, as in the previous examples , we get that $\mathcal{L}_{A,H,0}$ is the infinitesimal generator of an analytic semigroup.

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