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# ON EXPOSED POINTS OF LIPSCHITZ FREE SPACES

COLIN PETITJEAN AND ANTONÍN PROCHÁZKA

ABSTRACT. In this note we prove that a molecule  $d(x, y)^{-1}(\delta(x) - \delta(y))$  is an exposed point of the unit ball of a Lipschitz free space  $\mathcal{F}(M)$  if and only if the metric segment  $[x, y] = \{z \in M : d(x, y) = d(z, x) + d(z, y)\}$  is reduced to  $\{x, y\}$ . This is based on a recent result due to Aliaga and Pernecká which states that the class of Lipschitz free spaces over closed subsets of  $M$  is closed under arbitrary intersections when  $M$  has finite diameter.

## 1. INTRODUCTION

For a metric space  $(M, d)$  with a distinguished point  $0 \in M$ , we let  $\text{Lip}_0(M)$  be the real Banach space of Lipschitz maps from  $M$  to  $\mathbb{R}$  which vanish at 0. We recall that the norm of  $f \in \text{Lip}_0(M)$ , denoted  $\|f\|_L$ , is the best Lipschitz constant of  $f$ , i.e.

$$\|f\|_L = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Next, for  $x \in M$ , we let  $\delta(x) \in \text{Lip}_0(M)^*$  be Dirac measure, i.e.  $\langle \delta(x), f \rangle = f(x)$ . We then define the Lipschitz free space over  $M$  to be the following closed subspace of  $\text{Lip}_0(M)^*$ :

$$\mathcal{F}(M) := \overline{\text{span}}\{\delta(x) : x \in M\}.$$

It follows from the fundamental linearisation property of Lipschitz free spaces that  $\mathcal{F}(M)$  is a canonical predual of  $\text{Lip}_0(M)$  (see [7] for more details).

In this note we are interested in extreme points and exposed points of the unit ball of Lipschitz free spaces. If  $B_X$  denotes the unit ball of a Banach space  $X$ , we recall that  $x \in B_X$  is an extreme point of  $B_X$  whenever  $x \notin \text{conv}(B_X \setminus \{x\})$ . Next,  $x$  is an exposed point of  $B_X$  if there exists a linear functional  $f \in X^*$  such that  $f(x) > f(z)$  for every  $z \in B_X \setminus \{x\}$ . In what follows,  $\text{ext}(B_X)$  denotes the set of extreme points of  $B_X$  while  $\text{exp}(B_X)$  denotes the set of exposed points of  $B_X$ . It is readily seen that  $\text{exp}(B_X) \subset \text{ext}(B_X)$ .

The extremal structure of Lipschitz free spaces has already been investigated in a number of articles [1, 2, 5, 6, 9]. In any such study a special attention is dedicated to the elements of  $\mathcal{F}(M)$  of the form  $m_{xy} = \frac{\delta(x) - \delta(y)}{d(x, y)}$  which we call *molecules* (and which are called elementary molecules in [2]). It is simply a matter of writing down the corresponding convex combination to see that  $m_{xy} \in \text{ext}(B_{\mathcal{F}(M)})$  implies that  $[x, y] = \{x, y\}$ . However, it is only recently that Aliaga and Pernecká [2] managed to prove that, for a complete  $M$ , the reverse implication is

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also valid. Here, using one of the ingredients of their proof, we show the following stronger result.

**Theorem 1.** *Let  $M$  be a complete metric space and  $p \neq q \in M$  satisfy  $[p, q] = \{p, q\}$ . Then  $m_{pq}$  is an exposed point of  $B_{\mathcal{F}(M)}$ . It is exposed by the magic function*

$$f_{pq}(t) := \frac{d(x, y)}{2} \left( \frac{d(t, q) - d(t, p)}{d(t, q) + d(t, p)} - \frac{d(0, q) - d(0, p)}{d(0, q) + d(0, p)} \right).$$

## 2. PROOF OF THE MAIN RESULT

The authors of [2] had the following important insight which is likely to have many more applications in analysis of Lipschitz free spaces.

**Proposition 2** (Aliaga and Pernecká [2]). *Let  $M$  be a bounded complete metric space. Let  $\{M_\alpha \subset M : \alpha \in A\}$  be a collection of closed subsets of  $M$  containing 0. Then*

$$\bigcap_{\alpha \in A} \mathcal{F}(M_\alpha) = \mathcal{F}\left(\bigcap_{\alpha \in A} M_\alpha\right).$$

For the proof of Theorem 1 we will need further some notation and few lemmas. Given a metric space  $M$  we will set  $\widetilde{M} := M \times M \setminus \{(x, x) : x \in M\}$  and  $V = \{m_{xy} : (x, y) \in \widetilde{M}\}$  the set of molecules in  $\mathcal{F}(M)$ . The following folklore fact is also stated in disguise as Lemma 2.1 in [2]. The proof here is different from the one in [2].

**Lemma 3.** *Let  $M$  be a metric space. Let us define  $Q : \ell_1(\widetilde{M}) \rightarrow \mathcal{F}(M)$  by  $e_{(x,y)} \mapsto m_{xy}$  and linearly on span  $\{e_{(x,y)}\}$ . Then  $Q$  extends to an onto norm-one mapping.*

*Proof.* The fact that  $\|Q\| = 1$  is clear so we can extend  $Q$  to the whole space with the same norm. Let us call the extension  $Q$  again. We will prove that  $B_{\mathcal{F}(M)}^O \subset Q(B_{\ell_1}^O)$ , where  $B_X^O$  denotes the open unit ball of a Banach space  $X$ . For this it is enough to use Lemma 2.23 in [4], i.e. we need to check that  $B_{\mathcal{F}(M)}^O \subset \overline{Q(B_{\ell_1}^O)}$ . But we have  $B_{\mathcal{F}(M)}^O \subset \overline{\text{conv}(V)} \subset \overline{Q(B_{\ell_1})} = \overline{Q(B_{\ell_1}^O)}$ .  $\square$

The next lemma is standard.

**Lemma 4.** *Let  $a \in S_{\ell_1}$  and  $b \in B_{\ell_\infty}$ . Assume that  $1 - \alpha\varepsilon \leq \langle a, b \rangle$  for some  $0 < \alpha, \varepsilon < 1$ . Denote  $B = \{n \in \mathbb{N} : |b_n| \leq (1 - \alpha)\}$ . Then  $\sum_{n \in B} |a_n| \leq \varepsilon$ .*

*Proof.* We denote  $G := \mathbb{N} \setminus B$ . We have

$$\begin{aligned} 1 - \varepsilon\alpha &\leq \sum_{n=1}^{\infty} a_n b_n \leq \sum_{n \in G} |a_n b_n| + \sum_{n \in B} |a_n b_n| \\ &\leq \sum_{n \in G} |a_n| + (1 - \alpha) \sum_{n \in B} |a_n| \\ &\leq \sum_{n \in \mathbb{N}} |a_n| - \alpha \sum_{n \in B} |a_n| = 1 - \alpha \sum_{n \in B} |a_n|. \end{aligned}$$

It follows that  $\sum_{n \in B} |a_n| \leq \varepsilon$ .  $\square$

For a metric space  $M$ , points  $p, q \in M$  and  $\varepsilon > 0$  we will denote

$$[p, q]_\varepsilon := \left\{ x \in M : d(p, x) + d(x, q) \leq \frac{1}{1-\varepsilon} d(p, q) \right\}.$$

The properties of the magic function collected in the following lemma have been proved already in [8].

**Lemma 5.** *Let  $(p, q) \in \widetilde{M}$ . We have*

- (1)  $f_{pq}$  is Lipschitz and  $\|f_{pq}\|_L \leq 1$ .
- (2) Let  $u \neq v \in M$  and  $\varepsilon > 0$  be such that  $\frac{f_{pq}(u) - f_{pq}(v)}{d(u, v)} > 1 - \varepsilon$ . Then both  $u, v \in [p, q]_\varepsilon$ .
- (3) If  $(u, v) \in \widetilde{M}$  and  $\frac{f_{pq}(u) - f_{pq}(v)}{d(u, v)} = 1$ , then both  $u, v \in [x, y]$ .

Let us remark at this point that if  $[p, q] = \{p, q\}$ , then  $f_{pq}$  exposes  $m_{pq}$  among molecules (immediate from Lemma 5 (3)) and also among those  $\mu \in B_{\mathcal{F}(M)}$  which have finite support (or more generally such that  $\|\mu\| = \|a\|_1$  in the representation coming from Lemma 3). The next lemma prepares the ground for the remaining cases.

**Lemma 6.** *Let  $M$  be a metric space with the base point  $0 = q$  and let  $p \neq q \in M$  be such that  $[p, q] = \{p, q\}$ . Assume that  $\mu \in B_{\mathcal{F}(M)}$  satisfies  $\langle \mu, f_{pq} \rangle = 1$ . Then for every  $\varepsilon, \alpha \in (0, \frac{1}{2})$  we have  $\mu \in \mathcal{F}([p, q]_\alpha) + 2\varepsilon B_{\mathcal{F}(M)}$ .*

*Proof.* Let us observe right away that by the hypothesis  $\|\mu\| = 1$ . Let  $\varepsilon, \alpha \in (0, \frac{1}{2})$  be fixed. By Lemma 3 there exist  $a = (a_i) \in \ell_1$  and  $(p_i), (q_i) \subset M$  such that  $\mu = \sum_{i=1}^{\infty} a_i m_{p_i q_i}$  and  $\|a\|_1 \leq \|\mu\| + \frac{\varepsilon \alpha}{1 - \varepsilon \alpha}$ . We have

$$1 - \varepsilon \alpha \leq \frac{1}{\|a\|_1} = \left\langle \frac{\mu}{\|a\|_1}, f_{pq} \right\rangle = \sum_{i=1}^{\infty} \frac{a_i}{\|a\|_1} \langle m_{p_i q_i}, f_{pq} \rangle.$$

Now if we denote  $B = \{i \in \mathbb{N} : |\langle m_{p_i q_i}, f_{pq} \rangle| \leq (1 - \alpha)\}$ , then Lemma 4 yields that  $\sum_{i \in B} \left| \frac{a_i}{\|a\|_1} \right| \leq \varepsilon$  and so  $\sum_{i \in B} |a_i| \leq 2\varepsilon$ . It follows from Lemma 5 (2) that for every  $i \in \mathbb{N} \setminus B$  we have  $p_i, q_i \in [p, q]_\alpha$ . The conclusion is now immediate.  $\square$

*Proof of Theorem 1.* We can assume without loss of generality that  $0 = q$ . Indeed, a change of the base point in  $M$  induces a linear isometry between the corresponding Lipschitz free spaces which preserves the molecules. Lemma 6 shows that if  $\mu \in B_{\mathcal{F}(M)}$  satisfies  $\langle \mu, f_{pq} \rangle = 1$  then  $\mu \in \bigcap_{\alpha > 0} \mathcal{F}([p, q]_\alpha)$ . Since  $[p, q]_1$  is bounded, Proposition 2 yields that  $\mu \in \mathcal{F}([p, q]) = \mathcal{F}(\{p, q\})$ . This is a 1-dimensional vector space so  $\mu = \pm m_{pq}$  but only the choice of the plus sign is reasonable.  $\square$

**Remark 7.** Apart from the obvious fact that Theorem 1 strengthens and generalizes some of the results in [5] let us also point out that one of the proofs of the main result in [3] (i.e. the characterization of  $M$  such that  $\mathcal{F}(M) = \ell_1(\Gamma)$ ) becomes now much simpler.

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