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ON EXPOSED POINTS OF LIPSCHITZ FREE SPACES

COLIN PETITJEAN AND ANTONÍN PROCHÁZKA

ABSTRACT. In this note we prove that a molecule $d(x, y)^{-1}(\delta(x) - \delta(y))$ is an exposed point of the unit ball of a Lipschitz free space $\mathcal{F}(M)$ if and only if the metric segment $[x, y] = \{z \in M : d(x, y) = d(z, x) + d(z, y)\}$ is reduced to $\{x, y\}$. This is based on a recent result due to Aliaga and Pernecká which states that the class of Lipschitz free spaces over closed subsets of M is closed under arbitrary intersections when M has finite diameter.

1. INTRODUCTION

For a metric space (M, d) with a distinguished point $0 \in M$, we let $\text{Lip}_0(M)$ be the real Banach space of Lipschitz maps from M to \mathbb{R} which vanish at 0. We recall that the norm of $f \in \text{Lip}_0(M)$, denoted $\|f\|_L$, is the best Lipschitz constant of f , i.e.

$$\|f\|_L = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Next, for $x \in M$, we let $\delta(x) \in \text{Lip}_0(M)^*$ be Dirac measure, i.e. $\langle \delta(x), f \rangle = f(x)$. We then define the Lipschitz free space over M to be the following closed subspace of $\text{Lip}_0(M)^*$:

$$\mathcal{F}(M) := \overline{\text{span}}\{\delta(x) : x \in M\}.$$

It follows from the fundamental linearisation property of Lipschitz free spaces that $\mathcal{F}(M)$ is a canonical predual of $\text{Lip}_0(M)$ (see [7] for more details).

In this note we are interested in extreme points and exposed points of the unit ball of Lipschitz free spaces. If B_X denotes the unit ball of a Banach space X , we recall that $x \in B_X$ is an extreme point of B_X whenever $x \notin \text{conv}(B_X \setminus \{x\})$. Next, x is an exposed point of B_X if there exists a linear functional $f \in X^*$ such that $f(x) > f(z)$ for every $z \in B_X \setminus \{x\}$. In what follows, $\text{ext}(B_X)$ denotes the set of extreme points of B_X while $\text{exp}(B_X)$ denotes the set of exposed points of B_X . It is readily seen that $\text{exp}(B_X) \subset \text{ext}(B_X)$.

The extremal structure of Lipschitz free spaces has already been investigated in a number of articles [1, 2, 5, 6, 9]. In any such study a special attention is dedicated to the elements of $\mathcal{F}(M)$ of the form $m_{xy} = \frac{\delta(x) - \delta(y)}{d(x, y)}$ which we call *molecules* (and which are called elementary molecules in [2]). It is simply a matter of writing down the corresponding convex combination to see that $m_{xy} \in \text{ext}(B_{\mathcal{F}(M)})$ implies that $[x, y] = \{x, y\}$. However, it is only recently that Aliaga and Pernecká [2] managed to prove that, for a complete M , the reverse implication is

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also valid. Here, using one of the ingredients of their proof, we show the following stronger result.

Theorem 1. *Let M be a complete metric space and $p \neq q \in M$ satisfy $[p, q] = \{p, q\}$. Then m_{pq} is an exposed point of $B_{\mathcal{F}(M)}$. It is exposed by the magic function*

$$f_{pq}(t) := \frac{d(x, y)}{2} \left(\frac{d(t, q) - d(t, p)}{d(t, q) + d(t, p)} - \frac{d(0, q) - d(0, p)}{d(0, q) + d(0, p)} \right).$$

2. PROOF OF THE MAIN RESULT

The authors of [2] had the following important insight which is likely to have many more applications in analysis of Lipschitz free spaces.

Proposition 2 (Aliaga and Pernecká [2]). *Let M be a bounded complete metric space. Let $\{M_\alpha \subset M : \alpha \in A\}$ be a collection of closed subsets of M containing 0. Then*

$$\bigcap_{\alpha \in A} \mathcal{F}(M_\alpha) = \mathcal{F}\left(\bigcap_{\alpha \in A} M_\alpha\right).$$

For the proof of Theorem 1 we will need further some notation and few lemmas. Given a metric space M we will set $\widetilde{M} := M \times M \setminus \{(x, x) : x \in M\}$ and $V = \{m_{xy} : (x, y) \in \widetilde{M}\}$ the set of molecules in $\mathcal{F}(M)$. The following folklore fact is also stated in disguise as Lemma 2.1 in [2]. The proof here is different from the one in [2].

Lemma 3. *Let M be a metric space. Let us define $Q : \ell_1(\widetilde{M}) \rightarrow \mathcal{F}(M)$ by $e_{(x,y)} \mapsto m_{xy}$ and linearly on span $\{e_{(x,y)}\}$. Then Q extends to an onto norm-one mapping.*

Proof. The fact that $\|Q\| = 1$ is clear so we can extend Q to the whole space with the same norm. Let us call the extension Q again. We will prove that $B_{\mathcal{F}(M)}^O \subset Q(B_{\ell_1}^O)$, where B_X^O denotes the open unit ball of a Banach space X . For this it is enough to use Lemma 2.23 in [4], i.e. we need to check that $B_{\mathcal{F}(M)}^O \subset \overline{Q(B_{\ell_1}^O)}$. But we have $B_{\mathcal{F}(M)}^O \subset \overline{\text{conv}(V)} \subset \overline{Q(B_{\ell_1})} = \overline{Q(B_{\ell_1}^O)}$. \square

The next lemma is standard.

Lemma 4. *Let $a \in S_{\ell_1}$ and $b \in B_{\ell_\infty}$. Assume that $1 - \alpha\varepsilon \leq \langle a, b \rangle$ for some $0 < \alpha, \varepsilon < 1$. Denote $B = \{n \in \mathbb{N} : |b_n| \leq (1 - \alpha)\}$. Then $\sum_{n \in B} |a_n| \leq \varepsilon$.*

Proof. We denote $G := \mathbb{N} \setminus B$. We have

$$\begin{aligned} 1 - \varepsilon\alpha &\leq \sum_{n=1}^{\infty} a_n b_n \leq \sum_{n \in G} |a_n b_n| + \sum_{n \in B} |a_n b_n| \\ &\leq \sum_{n \in G} |a_n| + (1 - \alpha) \sum_{n \in B} |a_n| \\ &\leq \sum_{n \in \mathbb{N}} |a_n| - \alpha \sum_{n \in B} |a_n| = 1 - \alpha \sum_{n \in B} |a_n|. \end{aligned}$$

It follows that $\sum_{n \in B} |a_n| \leq \varepsilon$. \square

For a metric space M , points $p, q \in M$ and $\varepsilon > 0$ we will denote

$$[p, q]_\varepsilon := \left\{ x \in M : d(p, x) + d(x, q) \leq \frac{1}{1-\varepsilon} d(p, q) \right\}.$$

The properties of the magic function collected in the following lemma have been proved already in [8].

Lemma 5. *Let $(p, q) \in \widetilde{M}$. We have*

- (1) f_{pq} is Lipschitz and $\|f_{pq}\|_L \leq 1$.
- (2) Let $u \neq v \in M$ and $\varepsilon > 0$ be such that $\frac{f_{pq}(u) - f_{pq}(v)}{d(u, v)} > 1 - \varepsilon$. Then both $u, v \in [p, q]_\varepsilon$.
- (3) If $(u, v) \in \widetilde{M}$ and $\frac{f_{pq}(u) - f_{pq}(v)}{d(u, v)} = 1$, then both $u, v \in [x, y]$.

Let us remark at this point that if $[p, q] = \{p, q\}$, then f_{pq} exposes m_{pq} among molecules (immediate from Lemma 5 (3)) and also among those $\mu \in B_{\mathcal{F}(M)}$ which have finite support (or more generally such that $\|\mu\| = \|a\|_1$ in the representation coming from Lemma 3). The next lemma prepares the ground for the remaining cases.

Lemma 6. *Let M be a metric space with the base point $0 = q$ and let $p \neq q \in M$ be such that $[p, q] = \{p, q\}$. Assume that $\mu \in B_{\mathcal{F}(M)}$ satisfies $\langle \mu, f_{pq} \rangle = 1$. Then for every $\varepsilon, \alpha \in (0, \frac{1}{2})$ we have $\mu \in \mathcal{F}([p, q]_\alpha) + 2\varepsilon B_{\mathcal{F}(M)}$.*

Proof. Let us observe right away that by the hypothesis $\|\mu\| = 1$. Let $\varepsilon, \alpha \in (0, \frac{1}{2})$ be fixed. By Lemma 3 there exist $a = (a_i) \in \ell_1$ and $(p_i), (q_i) \subset M$ such that $\mu = \sum_{i=1}^{\infty} a_i m_{p_i q_i}$ and $\|a\|_1 \leq \|\mu\| + \frac{\varepsilon \alpha}{1 - \varepsilon \alpha}$. We have

$$1 - \varepsilon \alpha \leq \frac{1}{\|a\|_1} = \left\langle \frac{\mu}{\|a\|_1}, f_{pq} \right\rangle = \sum_{i=1}^{\infty} \frac{a_i}{\|a\|_1} \langle m_{p_i q_i}, f_{pq} \rangle.$$

Now if we denote $B = \{i \in \mathbb{N} : |\langle m_{p_i q_i}, f_{pq} \rangle| \leq (1 - \alpha)\}$, then Lemma 4 yields that $\sum_{i \in B} \left| \frac{a_i}{\|a\|_1} \right| \leq \varepsilon$ and so $\sum_{i \in B} |a_i| \leq 2\varepsilon$. It follows from Lemma 5 (2) that for every $i \in \mathbb{N} \setminus B$ we have $p_i, q_i \in [p, q]_\alpha$. The conclusion is now immediate. \square

Proof of Theorem 1. We can assume without loss of generality that $0 = q$. Indeed, a change of the base point in M induces a linear isometry between the corresponding Lipschitz free spaces which preserves the molecules. Lemma 6 shows that if $\mu \in B_{\mathcal{F}(M)}$ satisfies $\langle \mu, f_{pq} \rangle = 1$ then $\mu \in \bigcap_{\alpha > 0} \mathcal{F}([p, q]_\alpha)$. Since $[p, q]_1$ is bounded, Proposition 2 yields that $\mu \in \mathcal{F}([p, q]) = \mathcal{F}(\{p, q\})$. This is a 1-dimensional vector space so $\mu = \pm m_{pq}$ but only the choice of the plus sign is reasonable. \square

Remark 7. Apart from the obvious fact that Theorem 1 strengthens and generalizes some of the results in [5] let us also point out that one of the proofs of the main result in [3] (i.e. the characterization of M such that $\mathcal{F}(M) = \ell_1(\Gamma)$) becomes now much simpler.

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