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# ON KALTON'S INTERLACED GRAPHS AND NONLINEAR EMBEDDINGS INTO DUAL BANACH SPACES

BRUNO DE MENDONA BRAGA, GILLES LANCIEN, COLIN PETITJEAN,  
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ABSTRACT. We study the nonlinear embeddability of Banach spaces and the equi-embeddability of the family of Kalton's interlaced graphs  $([\mathbb{N}]^k, d_{\mathbb{Z}})_k$  into dual spaces. Notably, we define and study a modification of Kalton's property  $\mathcal{Q}$  that we call property  $\mathcal{Q}_p$  (with  $p \in (1, +\infty]$ ). We show that if  $([\mathbb{N}]^k, d_{\mathbb{Z}})_k$  equi-coarsely Lipschitzly embeds into  $X^*$ , then the Szlenk index of  $X$  is greater than  $\omega$ , and that this is optimal, i.e., there exists a separable dual space  $Y^*$  that contains  $([\mathbb{N}]^k, d_{\mathbb{Z}})_k$  equi-Lipschitzly and so that  $Y$  has Szlenk index  $\omega^2$ . We prove that  $c_0$  does not coarsely Lipschitzly embed into a separable dual space by a map with distortion strictly smaller than  $\frac{3}{2}$ . We also show that neither  $c_0$  nor  $L_1$  coarsely embeds into a separable dual by a weak-to-weak\* sequentially continuous map.

## 1. INTRODUCTION

It was a long standing problem in the nonlinear theory of Banach spaces whether every metric space coarsely embeds into a reflexive Banach space (we refer the reader to Section 2 for definitions). Although some partial positive results on this question were obtained for some classes of metric spaces – for instance, N. Brown and E. Guentner showed in [9, Theorem 1] that every metric space with bounded geometry can be coarsely embedded into a reflexive space, and later F. Baudier and the second named author showed that every locally finite metric space Lipschitz embeds into a reflexive space (see [3]) –, a negative answer was only found in 2007. Indeed, N. Kalton exhibited in [17] a property for metric spaces, that he named property  $\mathcal{Q}$ , which serves as an obstruction to coarse embeddability into reflexive spaces (see Section 3 for precise statements). Precisely, its absence is an obstruction to coarse embeddability into reflexive Banach spaces. As it is easily checked,  $c_0$  fails property  $\mathcal{Q}$  and so does not embed into any reflexive Banach space. This property is defined in terms of the behaviour of Lipschitz maps defined on a particular family of metric graphs: *the Kalton's interlaced graphs* (see Section 2.2).

Furthermore, N. Kalton proved the stronger result that any space  $X$  coarsely containing  $c_0$  must have some of its iterated duals nonseparable (see [17, Theorem 3.6]). Let us point out that coarsely containing the James tree space  $\mathcal{JT}$  would have the same impact on the iterated duals of  $X$

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[24, Theorem 6.2]. The result of Kalton raises the following very natural problem:

**Problem 1.1.** Is there a universal  $n \in \mathbb{N}$  so that if  $c_0$  coarsely embeds into a Banach space  $X$ , then its  $n$ -th iterated dual  $X^{(n)}$  is nonseparable?

It is standard in the linear theory of Banach spaces, that  $c_0$  does not isomorphically embed into any separable dual space. So it is also quite natural to wonder the following:

**Problem 1.2.** Does  $c_0$  coarsely embed into a separable dual space?

It is clear that a negative answer to this last problem would represent the strongest possible positive solution for Problem 1.1 (namely  $n = 2$ ). However, this last problem is still open even in the category of *coarse-Lipschitz embeddings* (see Section 2.1 for a precise definition).

**Problem 1.3.** Does  $c_0$  coarsely Lipschitz embed into a separable dual space?

The current paper revolves around these questions. Therefore, inspired by N. Kalton, we not only study different notions of nonlinear embeddability of  $c_0$  into  $X$ , but we also analyse to which extent the equi-embeddability of the interlaced graphs into a Banach space  $X$  forces the dual of  $X$  to be nonseparable.

We now describe the main findings of this paper. Throughout this paper,  $[\mathbb{N}]^k$  denotes the set of all subsets of  $\mathbb{N}$  with  $k$  elements,  $[\mathbb{N}]^{<\omega}$  denotes the set of all finite subsets of  $\mathbb{N}$ , and  $d_{\mathbb{K}} = d_{\mathbb{K},k}$  denotes Kalton's interlaced metric on  $[\mathbb{N}]^k$ . There exists a metric on  $[\mathbb{N}]^{<\omega}$  which extends all metrics  $d_{\mathbb{K},k}$  simultaneously, and we also denote this metric by  $d_{\mathbb{K}}$  (see Subsection 2.2 for precise definitions).

First of all, inspired by [20], we define a modification of Kalton's property  $\mathcal{Q}$  that we call property  $\mathcal{Q}_p$  for  $p \in (1, \infty]$ . In a nutshell, while property  $\mathcal{Q}$  consists in a strong concentration inequality for Lipschitz maps  $f : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow \mathbb{R}$  defined on the interlaced graphs, property  $\mathcal{Q}_p$  is a concentration inequality proportional to  $k^{1/p}$  (see Definition 3.1). In this way, property  $\mathcal{Q}$  may be seen as property  $\mathcal{Q}_\infty$ . It is readily seen that property  $\mathcal{Q}_p$  is a coarse-Lipschitz invariant. The first main result relates this property with asymptotic uniform convexity (see Subsection 2.4 for definitions of asymptotic properties).

**Theorem 3.3.** *Let  $p \in (1, +\infty]$  and let  $q \in [1, \infty)$  be the conjugate exponent of  $p$ . If a dual space  $X^*$  admits an equivalent  $q$ -AUC\* dual norm then  $X^*$  has property  $\mathcal{Q}_p$ .*

We also prove that property  $\mathcal{Q}_q$  is equivalent to reflexivity for a certain class of Banach spaces (namely those having the  $p$ -alternating Banach-Saks property, see Corollary 3.7). These results can be used to rule out the coarse-Lipschitz embeddability between certain Banach spaces (see Corollaries 3.4 and 3.8).

Next, recall that a separable Banach space  $X$  has separable dual if and only if its Szlenk index  $\text{Sz}(X)$  is countable (see Subsection 2.3 for a definition of the Szlenk index). Hence,  $\text{Sz}(X)$  can be seen as a quantitative

measurement of “how close to be nonseparable” is  $X^*$ . We obtain the following relation between containment of Kalton’s interlaced graphs and the Szlenk index.

**Theorem 4.3.** *Let  $X$  be a Banach space. If the family of Kalton’s interlaced graphs  $(([\mathbb{N}]^k, d_{\mathbb{K}}))_{k \in \mathbb{N}}$  equi-coarse Lipschitz embeds into  $X^*$ , then  $\text{Sz}(X) > \omega$ , where  $\omega$  denotes the first infinite ordinal.*

We also prove that if  $X$  has summable Szlenk index then  $X^*$  enjoys property  $\mathcal{Q}$ . Moreover, Theorem 4.3 is actually optimal and the containment of Kalton’s interlaced graphs cannot help us any further in the problem of whether  $c_0$  coarsely embeds into a separable dual. Indeed, we show the following.

**Theorem 5.3.** *The Kalton graph  $([\mathbb{N}]^{<\omega}, d_{\mathbb{K}})$  Lipschitz embeds into a separable dual space  $X^*$  with  $\text{Sz}(X) = \omega^2$ .*

Although we were not able to obtain a negative answer to Problem 1.3, we obtained a restriction for the existence of a coarse Lipschitz embedding from  $c_0$  into  $X^*$  based on the distortion of such embeddings. Before presenting our result, let us recall this definition. Let  $X$  and  $Y$  be Banach spaces and  $f : X \rightarrow Y$  be a coarse Lipschitz embedding. We say that  $f$  has coarse Lipschitz distortion strictly less than  $K$  if there exist  $A, B, C, D > 0$  with  $AC < K$  so that

$$\frac{1}{A} \|x - y\| - B \leq \|f(x) - f(y)\| \leq C \|x - y\| + D$$

for all  $x, y \in X$ . We obtain the following.

**Theorem 6.2.** *If  $c_0$  coarse Lipschitz embeds into a dual space  $X^*$  with coarse Lipschitz distortion strictly less than  $\frac{3}{2}$ , then  $X$  contains an isomorphic copy of  $\ell_1$ .*

On a different direction, we show that Problem 1.2 has a negative answer with the extra assumption that the embedding is weak-to-weak\* sequentially continuous. Moreover, just as in the isomorphic theory, this also holds for the space  $L_1$ .

**Theorem 7.4.** *Neither  $c_0$  nor  $L_1$  can be coarsely (resp. uniformly) embedded into a separable dual Banach space by a map that is weak-to-weak\* sequentially continuous.*

Since the “weak-to-weak\* sequential continuity” hypothesis is nonstandard, a word on Theorem 7.4 is necessary. The first named author has begun the study of coarse and coarse Lipschitz embeddings between Banach spaces which also satisfy some continuity condition with respect to the weak topologies [6, 7]. For instance, in contrast with the famous result of I. Aharoni that  $c_0$  contains a Lipschitz copy of every separable metric space [1, Theorem in page 288], any Banach space not containing  $\ell_1$  which can be coarsely embedded into  $c_0$  by a weakly sequentially continuous map must actually be isomorphic to a subspace of  $c_0$  [7, Theorem 1.6]. Also, although  $\ell_p$  (resp.  $L_p$ ) coarsely embeds into  $\ell_q$  (resp.  $L_q$ ) for all  $p, q \in [1, 2]$ , the same is only true for weak sequentially continuous coarse embeddings  $\ell_p \rightarrow \ell_q$  if  $p \leq q$  [6, Corollary 1.7]. In particular, Theorem 7.4 shows that although the

theory of coarse embeddability for members of the families  $(\ell_p)_{p \in [1,2]}$  and  $(L_p)_{p \in [1,2]}$  are the same, this is not the case for weakly sequentially continuous embeddings. Indeed,  $L_1$  does not coarsely embed into  $L_q$  by a weakly sequentially continuous map for any  $q > 1$ , but  $\ell_p$  does so into  $\ell_q$  for all  $q \geq p$ .

This summarises our main findings. We now give the definitions and terminologies necessary for this paper.

## 2. PRELIMINARIES

**2.1. Embeddings between metric spaces.** Let  $(M, d_M), (N, d_N)$  be two metric spaces and  $f: M \rightarrow N$  be a map. We define the *compression modulus*  $\rho_f$  by letting

$$\rho_f(t) = \inf\{d_N(f(x), f(y)) : d_M(x, y) \geq t\}$$

for each  $t \geq 0$ , and the *expansion modulus*  $\omega_f$  by letting

$$\omega_f(t) = \sup\{d_N(f(x), f(y)) : d_M(x, y) \leq t\}$$

for all  $t \geq 0$ . We adopt the convention  $\sup(\emptyset) = 0$  and  $\inf(\emptyset) = \infty$ . Note that for every  $x, y \in M$ ,

$$\rho_f(d_M(x, y)) \leq d_N(f(x), f(y)) \leq \omega_f(d_M(x, y)).$$

Moreover, the map  $f: M \rightarrow N$  is called

- (i) a *coarse embedding* if  $\lim_{t \rightarrow \infty} \rho_f(t) = \infty$  and  $\omega_f(t) < \infty$  for every  $t \in [0, +\infty)$ ;
- (ii) a *coarse Lipschitz embedding* if there exists  $A, B, C, D > 0$  such that  $\rho_f(t) \geq At - C$  and  $\omega_f(t) \leq Bt + D$  for every  $t \in [0, +\infty)$ ;
- (iii) a *Lipschitz embedding* if there exists  $A, B > 0$  such that  $\rho_f(t) \geq At$  and  $\omega_f(t) \leq Bt$  for every  $t \in [0, +\infty)$ .

Let  $(M_i)_{i \in I}$  be a family of metric spaces. We say that the family  $(M_i)_{i \in I}$  *equi-coarsely embeds* (*equi-coarsely Lipschitz embeds* and *equi-Lipschitz embeds* respectively) into a metric space  $N$  if there exist two maps

$$\rho, \omega: [0, +\infty) \rightarrow [0, +\infty)$$

and a family of maps  $(f_i: M_i \rightarrow N)_{i \in I}$  such that

- (1)  $\rho(t) \leq \rho_{f_i}(t)$  for every  $i \in I$  and  $t \in [0, \infty)$ ,
- (2)  $\omega_{f_i}(t) \leq \omega(t)$  for every  $i \in I$  and  $t \in [0, \infty)$ , and
- (3) the maps  $\rho$  and  $\omega$  satisfy the properties (i) above (respectively (ii) for coarse Lipschitz embedding and (iii) for Lipschitz embedding).

In order to refine the scale of coarse embeddings between Banach spaces, we will also shortly use the following notion. Let  $X$  and  $Y$  be two Banach spaces. We define  $\alpha_Y(X)$  as the supremum of all  $\alpha \in [0, 1)$  for which there exists a coarse embedding  $f: X \rightarrow Y$  and  $A, C$  in  $(0, \infty)$  so that  $\rho_f(t) \geq At^\alpha - C$  for all  $t > 0$ . Note that in this setting, the map  $f$  is automatically coarse Lipschitz (i.e., the expansion modulus  $\omega_f$  is bounded above by an affine map). Then,  $\alpha_Y(X)$  is called the *compression exponent of  $X$  in  $Y$* .

**2.2. Kalton interlaced graphs.** Given  $k \in \mathbb{N}$  and an infinite  $\mathbb{M} \subset \mathbb{N}$ , let  $[\mathbb{M}]^k$  denote the set of all strictly increasing  $k$ -tuples in  $\mathbb{M}$ . Given distinct  $\bar{n} = (n_1, \dots, n_k), \bar{m} = (m_1, \dots, m_k) \in [\mathbb{M}]^k$ , define a graph structure on  $[\mathbb{M}]^k$  by declaring  $\bar{n}$  and  $\bar{m}$  adjacent if and only if either

$$n_1 \leq m_1 \leq n_2 \leq \dots \leq n_k \leq m_k \quad \text{or} \quad m_1 \leq n_1 \leq m_2 \leq \dots \leq m_k \leq n_k.$$

The metric  $d_{\mathbb{K}}^k$  is defined as the shortest path metric in the graph  $[\mathbb{M}]^k$ . The family  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_k$  is the family of *Kalton's interlaced graphs*.

Given  $k \in \mathbb{N}$ ,  $\mathbb{M}_1 \in [\mathbb{N}]^\omega$  and  $\mathbb{M}_2 \in [\mathbb{M}_1]^\omega$ . The distance  $d_{\mathbb{K}}^k$  is independent of the infinite subset of  $\mathbb{N}$  chosen and  $[\mathbb{M}_1]^k$  is naturally a metric subspace of  $[\mathbb{M}_2]^k$ . This is implied by the following proposition obtained in [24], which moreover gives us an explicit formula to compute  $d_{\mathbb{K}}^k$ .

**Proposition 2.1** (Proposition 2.2 of [24]). *Letting*

$$d_{\mathbb{K}}(\bar{n}, \bar{m}) = \sup\{|\bar{n} \cap S| - |\bar{m} \cap S| : S \text{ segment of } \mathbb{N}\}$$

for all  $\bar{n}, \bar{m} \in [\mathbb{N}]^{<\omega}$ , we have that  $d_{\mathbb{K}}^k = d_{\mathbb{K}} \upharpoonright_{[\mathbb{N}]^k}$  for all  $k \in \mathbb{N}$ .

For  $k \in \mathbb{N}$  and  $\mathbb{M}$  an infinite subset of  $\mathbb{N}$ , we put  $[\mathbb{M}]^{\leq k} = \bigcup_{m \leq k} [\mathbb{M}]^m$ ,  $[\mathbb{M}]^{<\omega} = \bigcup_{m \in \mathbb{N}} [\mathbb{M}]^m$  and  $[\mathbb{M}]^\omega = \{S \subset \mathbb{M} : S \text{ is infinite}\}$ . Just as in the finite case, the elements of  $[\mathbb{M}]^\omega$  are always written as strictly increasing infinite tuples, i.e., if  $\bar{n} = (n_1, n_2, \dots) \in [\mathbb{M}]^\omega$ , we always have  $n_j < n_{j+1}$  for all  $j \in \mathbb{N}$ .

The formula from the previous proposition also defines a graph metric on  $[\mathbb{N}]^{<\omega}$  whose restriction to  $[\mathbb{N}]^k$  of course coincides with  $d_{\mathbb{K}}^k$ . From now on we simply denote the interlaced metric by  $d_{\mathbb{K}}$  (thus omitting the reference to  $k$ ).

*Remark 2.2.* It is easy to see that the sequence  $([\mathbb{N}]^k, d_{\mathbb{K}})_k$  equi-coarsely Lipschitz embeds into a Banach space  $X$  if and only if it equi-Lipschitz embeds into  $X$ . Indeed, this follows from the fact that, for any  $k \in \mathbb{N}$ , the map  $f : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow ([\mathbb{N}]^{2k}, \frac{1}{2}d_{\mathbb{K}})$  defined by:

$$\forall \bar{n} = (n_1, \dots, n_k) \in [\mathbb{N}]^k, \quad f(\bar{n}) = (2n_1, 2n_1 + 1, \dots, 2n_k, 2n_k + 1)$$

is an isometry.

For  $\bar{m} = (m_1, m_2, \dots, m_r) \in [\mathbb{N}]^{<\omega}$  and  $\bar{n} = (n_1, n_2, \dots, n_s) \in [\mathbb{N}]^{<\omega}$ , we write  $\bar{m} \prec \bar{n}$ , if  $r < s \leq k$  and  $m_i = n_i$ , for  $i = 1, 2, \dots, r$ , and we write  $\bar{m} \preceq \bar{n}$  if  $\bar{m} \prec \bar{n}$  or  $\bar{m} = \bar{n}$ . Thus  $\bar{m} \preceq \bar{n}$  if  $\bar{m}$  is an initial segment of  $\bar{n}$ . At last, for  $\bar{n} = (n_1, \dots, n_k)$  and  $\bar{m} = (m_1, \dots, m_l)$  in  $[\mathbb{N}]^{<\omega}$ , we write  $\bar{n} < \bar{m}$  if  $n_k < m_1$ .

**2.3. Szlenk index.** Let  $X$  be Banach space and  $K$  be a weak\* compact subset of  $X^*$ . For each  $\varepsilon > 0$ , define

$$s_\varepsilon(K) = K \setminus \{V \subset X^* : V \text{ weak}^* \text{ open and } \text{diam}(V \cap K) < \varepsilon\}.$$

Given an ordinal  $\xi$ ,  $s_\varepsilon^\xi(K)$  is defined inductively by letting  $s_\varepsilon^0(K) = s_\varepsilon(K)$ ,  $s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K))$  and  $s_\varepsilon^\xi(K) = \bigcap_{\zeta < \xi} s_\varepsilon^\zeta(K)$  if  $\xi$  is a limit ordinal. We then define  $\text{Sz}(X, \varepsilon)$  as the least ordinal  $\xi$  so that  $s_\varepsilon^\xi(B_{X^*}) = \emptyset$ , if such ordinal exists, and  $\text{Sz}(X, \varepsilon) = \infty$  otherwise. The *Szlenk index of  $X$*  is defined as

$$\text{Sz}(X) = \sup_{\varepsilon > 0} \text{Sz}(X, \varepsilon).$$

A Banach space  $X$  is said to have *summable Szlenk index* if there exists  $c > 0$  so that for all  $\varepsilon_1, \dots, \varepsilon_n > 0$  the inequality

$$s_{\varepsilon_n}(s_{\varepsilon_{n-1}}(\dots(s_{\varepsilon_2}(s_{\varepsilon_1}(B_{X^*}))\dots)) \neq \emptyset$$

implies  $\varepsilon_1 + \dots + \varepsilon_n \leq c$ . It is known that any subspace of  $c_0$  has summable Szlenk index, but the converse is not true (see details in Section 4).

The Szlenk index of a Banach space is closely related to the behavior of the so-called weak\*-null or weak\*-continuous trees in its dual. So let us give the necessary definitions. For a Banach space  $X$ , we call *tree of height  $k$*  in  $X$  any family  $(x(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$ , with  $x(\bar{n}) \in X$ . Then, if  $\mathbb{M} \in [\mathbb{N}]^\omega$ ,  $(x(\bar{n}))_{\bar{n} \in [\mathbb{M}]^{\leq k}}$  will be called a *full subtree* of  $(x(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$ . For  $\mathbb{M} \in [\mathbb{N}]^\omega$ , a tree  $(x^*(\bar{n}))_{\bar{n} \in [\mathbb{M}]^{\leq k}}$  in  $X^*$  is called *weak\*-null* if for any  $\bar{n} = (n_1, \dots, n_j) \in [\mathbb{M}]^{\leq k-1} \setminus \{\emptyset\}$ , the sequence  $(x^*(n_1, \dots, n_j, t))_{t > n_j, t \in \mathbb{M}}$  is weak\*-null and the sequence  $(x^*(t))_{t \in \mathbb{M}}$  is also weak\*-null. It is called *weak\*-continuous* if for any  $\bar{n} = (n_1, \dots, n_j) \in [\mathbb{M}]^{\leq k-1} \setminus \{\emptyset\}$ , the sequence  $(x^*(n_1, \dots, n_j, t))_{t > n_j, t \in \mathbb{M}}$  is weak\*-converging to  $x^*(n_1, \dots, n_j)$  and the sequence  $(x^*(t))_{t \in \mathbb{M}}$  is also weak\*-converging to  $x_\emptyset^*$ . Then, the following proposition is a direct consequence of the definition of the Szlenk index.

**Proposition 2.3.** *Let  $X$  be a Banach space and assume that  $(x^*(\bar{n}))_{\bar{n} \in [\mathbb{M}]^{\leq k}}$  is a weak\*-continuous tree in  $B_{X^*}$  such that there exist  $i_1 < \dots < i_l$  in  $\{0, \dots, k-1\}$  and  $K_{i_1}, \dots, K_{i_l} > 0$  satisfying*

$$\forall s \in \{1, \dots, l\} \forall \bar{n} \in [\mathbb{M}]^{i_s} \limsup_{t \rightarrow \infty, t \in \mathbb{M}} \|x^*(\bar{n}, t) - x^*(\bar{n})\| \geq K_{i_s}.$$

Then

$$x_\emptyset^* \in s_{K_{i_l}} \dots s_{K_{i_1}}(B_{X^*}).$$

**2.4. Asymptotic uniform smoothness and convexity.** Let  $X$  be a Banach space. We denote the set of all closed subspaces of  $X$  with finite codimension by  $\text{CoFin}(X)$ . We define the *modulus of asymptotic uniform smoothness of  $X$*  by letting

$$\bar{\rho}_X(t) = \sup_{x \in \partial B_X} \inf_{E \in \text{CoFin}(X)} \sup_{y \in \partial B_E} \|x + ty\| - 1$$

for each  $t \geq 0$ . The space  $X$  is called *asymptotic uniformly smooth* (abbreviated by *AUS*) if  $\lim_{t \rightarrow 0^+} \bar{\rho}_X^*(t)/t = 0$  for all  $t > 0$ . Let  $p \in (1, \infty]$ . We say that  $X$  is  *$p$ -asymptotic uniformly smooth* (abbreviated by  *$p$ -AUS*) if there exists  $C > 0$  so that  $\bar{\rho}_X^*(t) \leq Ct^p$  for all  $t \in [0, 1]$ .

Let  $X$  be a dual space. We denote the set of all weak\* closed subspaces of  $X^*$  with finite codimension by  $\text{CoFin}^*(X)$ . We define the *modulus of weak\* asymptotic uniform convexity of  $X$*  by letting

$$\bar{\delta}_{X^*}(t) = \inf_{x^* \in \partial B_{X^*}} \sup_{E \in \text{CoFin}^*(X)} \inf_{y^* \in \partial B_E} \|x^* + ty^*\| - 1$$

for each  $t \geq 0$ . The space  $X^*$  is called *weak\* asymptotic uniformly convex* (abbreviated by *AUC\**) if  $\bar{\delta}_{X^*}(t) > 0$  for all  $t > 0$ . Let  $p \in [1, \infty)$ . We say that  $X$  is *weak\*  $p$ -asymptotic uniformly convex* (abbreviated by  *$p$ -AUC\**) if there exists  $C > 0$  so that  $\bar{\delta}_{X^*}(t) \geq Ct^p$  for all  $t \in [0, 1]$ .

We first recall the following classical duality result concerning these moduli (see [11, Corollary 2.4]).

**Proposition 2.4.** *Let  $X$  be a Banach space.*

- (i) *Then  $\|\cdot\|_X$  is AUS if and only if  $\|\cdot\|_{X^*}$  is  $AUC^*$ .*
- (ii) *If  $p \in (1, \infty]$  and  $q \in [1, \infty)$  are conjugate exponents, then  $\|\cdot\|_X$  is  $p$ -AUS if and only if  $\|\cdot\|_{X^*}$  is  $q$ - $AUC^*$ .*

The next proposition is elementary.

**Proposition 2.5.** *For any weak\*-null sequence  $(x_n^*)_{n=1}^\infty \subset X^*$  and for any  $x^* \in X^* \setminus \{0\}$  we have*

$$\limsup_{n \rightarrow \infty} \|x^* + x_n^*\| \geq \|x^*\| \left( 1 + \bar{\delta}_X^* \left( \frac{\limsup_{n \rightarrow \infty} \|x_n^*\|}{\|x^*\|} \right) \right).$$

By iterating this estimate one can deduce the following property of weak\*-null trees in a  $q$ - $AUC^*$  dual space.

**Proposition 2.6.** *Let  $X$  be a Banach space with a dual  $q$ - $AUC^*$  norm, for some  $q \in [1, \infty)$ . Then, there exists  $c > 0$  such that for any weak\*-null tree  $(x^*(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$  in  $X^*$ , there exists  $\mathbb{M} \in [\mathbb{N}]^\omega$  such that*

$$\forall \bar{n} \in [\mathbb{M}]^k, \left\| \sum_{\bar{m} \preceq \bar{n}} x^*(\bar{m}) \right\|^q \geq c \sum_{\bar{m} \preceq \bar{n}} \|x^*(\bar{m})\|^q.$$

This standard fact can be obtained for instance by [24, Lemma 3.6 and Lemma 3.7] but can also be found in [19, 21].

We conclude this section by recalling the fundamental renorming result for spaces with Szlenk index equal to  $\omega$ . The result is due to H. Knaust, E. Odell and Th. Schlumprecht [21] in the separable case and M. Raja [27] in the non separable setting. The precise quantitative version can be found in [15].

**Theorem 2.7.** *Let  $X$  be a Banach space such that  $\text{Sz}(X) = \omega$ . Then there exists  $p \in (1, \infty)$  such that  $X$  admits an equivalent  $p$ -AUS norm.*

**2.5. General properties of Lipschitz maps into a dual space.** We finish this preliminaries section by gathering a few decomposition properties of Lipschitz maps from  $([\mathbb{N}]^k, d_{\mathbb{K}})$  into a dual Banach space  $X^*$  which will be heavily used throughout these notes. We start with an elementary separable reduction.

**Proposition 2.8.** *Let  $X$  be a Banach space and  $f: [\mathbb{N}]^k \rightarrow X^*$  be a map. Then, there exists a separable subspace  $Y$  of  $X$  such that the closed linear span of  $f([\mathbb{N}]^k)$  isometrically embeds into  $Y^*$ .*

*Proof.* Since  $[\mathbb{N}]^k$  is countable, the closed linear span of  $f([\mathbb{N}]^k)$  is a separable subspace of  $X^*$ ; let us call it  $Z$ . Therefore, there exists a separable subspace  $Y$  of  $X$  such that

$$\forall x^* \in Z, \|x^*\|_{X^*} = \sup_{y \in B_Y} |x^*(y)|.$$

This concludes our proof.  $\square$

The next proposition is [24, Proposition 2.8]. As it is mentioned in [24], its proof follows the ideas of the proof of [4, Lemma 4.1]. As usual  $\text{Lip}(f)$  denotes the best Lipschitz constant of a Lipschitz map  $f$  between metric

spaces; note that if  $f : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow Y$  with  $Y$  being a normed vector space then  $\text{Lip}(f) = \omega_f(1)$ .

**Proposition 2.9.** *Let  $X$  be a separable Banach space,  $k \in \mathbb{N}$ , and  $f : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X^*$  a Lipschitz map. Then there exist  $\mathbb{M} \in [\mathbb{N}]^\omega$  and a weak\*-null tree  $(x^*(\bar{m}))_{\bar{m} \in [\mathbb{M}]^{\leq k}}$  in  $X^*$  with  $\|x^*(\bar{m})\| \leq \text{Lip}(f)$  for all  $\bar{m} \in [\mathbb{M}]^{\leq k} \setminus \{\emptyset\}$  and so that*

$$\forall \bar{n} \in [\mathbb{M}]^k, \quad f(\bar{n}) = x^*_{\emptyset} + \sum_{i=1}^k x^*(n_1, \dots, n_i) = \sum_{\bar{m} \leq \bar{n}} x^*(\bar{m}).$$

Next, we will extract infinite subsets of  $\mathbb{M}$  in order to simplify further the structure of  $f$  restricted to the corresponding graph. So assume, for the sequel of this subsection, that  $X$  is a separable Banach space,  $f : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X^*$  is Lipschitz and  $(x^*(\bar{m}))_{\bar{m} \in [\mathbb{M}]^{\leq k}}$  is as in the conclusion of Proposition 2.9.

**Proposition 2.10.** *Fix  $\varepsilon > 0$ . Then there exists  $\mathbb{M}_1 \in [\mathbb{M}]^\omega$  such that for all  $i \in \{1, \dots, k\}$  there exists  $K_i \in [0, \text{Lip}(f)]$  satisfying*

$$\forall (n_1, \dots, n_i) \in [\mathbb{M}_1]^i, \quad K_i \leq \|x^*(n_1, \dots, n_i)\| \leq K_i + \varepsilon.$$

*Proof.* This is a direct consequence of Ramsey's theorem and the compactness of  $[0, \text{Lip}(f)]$ .  $\square$

Let us now enumerate  $\mathbb{M}_1$ :  $\mathbb{M}_1 = \{l_1 < \dots < l_n < \dots\}$ . Note that, using the weak\*-lower semi-continuity of  $\|\cdot\|_{X^*}$ , we may also assume, by passing to a further subtree that for every  $i \in \{1, \dots, k\}$ :

$$\forall (n_1, \dots, n_i) \in [\mathbb{N}]^i, \quad \|x^*(l_{2n_1}, \dots, l_{2n_i}) - x^*(l_{2n_1+1}, \dots, l_{2n_i+1})\| \geq \frac{K_i}{2}.$$

Then we set  $y^*_{\emptyset} = 0$  and for every  $\bar{n} = (n_1, \dots, n_i) \in [\mathbb{N}]^{\leq k} \setminus \{\emptyset\}$ , we let

$$y^*(\bar{n}) = x^*(l_{2n_1}, \dots, l_{2n_i}) - x^*(l_{2n_1+1}, \dots, l_{2n_i+1}).$$

We have that for every  $\bar{n} \in [\mathbb{N}]^k$ :

$$\left\| \sum_{i=1}^n y^*(n_1, \dots, n_i) \right\| = \|f(l_{2n_1}, \dots, l_{2n_k}) - f(l_{2n_1+1}, \dots, l_{2n_k+1})\| \leq \text{Lip}(f).$$

Thus, we can build a weak\*-continuous tree  $(z^*(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$  in  $\text{Lip}(f)B_{X^*}$  as follows:

$$\forall \bar{n} \in [\mathbb{N}]^{\leq k}, \quad z^*(\bar{n}) = \sum_{\bar{m} \leq \bar{n}} y^*(\bar{m}).$$

To summarise, we have the following.

**Proposition 2.11.** *Let  $X$  be a separable Banach space,  $k \in \mathbb{N}$ ,  $\varepsilon > 0$  and  $f : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X^*$  a Lipschitz map. Then there exist  $\mathbb{M} \in [\mathbb{N}]^\omega$ , a weak\*-null tree  $(x^*(\bar{m}))_{\bar{m} \in [\mathbb{M}]^{\leq k}}$  in  $X^*$  and constants  $K_1, \dots, K_k$  in  $[0, \text{Lip}(f)]$  such that*

- (i) For all  $\bar{m} \in [\mathbb{M}]^{\leq k} \setminus \{\emptyset\}$ ,  $\|x^*(\bar{m})\| \leq \text{Lip}(f)$ .
- (ii) For all  $\bar{n} \in [\mathbb{M}]^k$ ,  $f(\bar{n}) = \sum_{\bar{m} \leq \bar{n}} x^*(\bar{m})$ .

(iii) For all  $i \in \{1, \dots, k\}$  and all  $(n_1, \dots, n_i) \in [\mathbb{M}]^i$ ,

$$K_i \leq \|x^*(n_1, \dots, n_i)\| \leq K_i + \varepsilon.$$

(iv) Denote  $\mathbb{M} = \{l_1 < \dots < l_n < \dots\}$ ,  $y_{\emptyset}^* = z_{\emptyset}^* = 0$  and, for  $\bar{n} = (n_1, \dots, n_i) \in [\mathbb{N}]^{\leq k} \setminus \{\emptyset\}$ ,

$$y^*(\bar{n}) = x^*(l_{2n_1}, \dots, l_{2n_i}) - x^*(l_{2n_1+1}, \dots, l_{2n_i+1})$$

and

$$z^*(\bar{n}) = \sum_{\bar{m} \preceq \bar{n}} y^*(\bar{m}).$$

Then  $(z^*(\bar{n}))_{\bar{n} \in [\mathbb{N}]^{\leq k}}$  is a weak\*-continuous tree in  $\text{Lip}(f)B_{X^*}$  such that for every  $i \in \{1, \dots, k\}$  and every  $(n_1, \dots, n_i) \in [\mathbb{N}]^i$ ,

$$\|y^*(n_1, \dots, n_i)\| \geq \frac{K_i}{2}.$$

### 3. PROPERTY $\mathcal{Q}_p$

N. Kalton proved in [17, Theorem 3.6] that  $c_0$  neither coarsely nor uniformly embeds into any Banach space  $X$  so that all of its iterated duals are separable. In the same paper, N. Kalton introduced the notion of property  $\mathcal{Q}$  for a Banach space and showed that any reflexive Banach space has property  $\mathcal{Q}$ . Recall, a Banach space  $X$  has *property  $\mathcal{Q}$*  if there exists  $C \geq 1$  such that for every  $k \in \mathbb{N}$  and every Lipschitz map  $f: ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  such that

$$\|f(\bar{n}) - f(\bar{m})\| \leq C\omega_f(1)$$

for all  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$ .

In this section, we introduce property  $\mathcal{Q}_p$  for  $p \in (1, \infty]$ , which coincides with property  $\mathcal{Q}$  when  $p = \infty$ . We then give a sufficient condition for a Banach space to have property  $\mathcal{Q}_p$  and use this in order to obtain some applications to the theory of nonlinear embeddings between Banach spaces.

**Definition 3.1.** Let  $p \in (1, +\infty]$ . We say that a Banach space  $X$  has *property  $\mathcal{Q}_p$*  if there exists  $C \geq 1$  such that for every  $k \in \mathbb{N}$  and every Lipschitz map  $f: ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  such that

$$\|f(\bar{n}) - f(\bar{m})\| \leq C\omega_f(1)k^{\frac{1}{p}}$$

for all  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$  (if  $p = \infty$ , we use the convention that  $1/\infty = 0$ ).

Clearly, property  $\mathcal{Q}_p$  implies property  $\mathcal{Q}_q$  for all  $q < p$ . Hence, since every Banach space which either coarsely or uniformly embeds into a reflexive space has property  $\mathcal{Q}$  [17, Corollary 4.3], the same holds for property  $\mathcal{Q}_p$  for any  $p \in (1, \infty]$ .

The next proposition illustrates some simple permanence properties of property  $\mathcal{Q}_p$ . Since its proof is immediate, we choose to omit it.

**Proposition 3.2.** Let  $p \in (1, \infty]$  and let  $X$  be a Banach space with property  $\mathcal{Q}_p$ . The following hold.

- (i) If  $Y$  coarse Lipschitz embeds into  $X$ , then  $Y$  has property  $\mathcal{Q}_p$ .
- (ii) The family  $([\mathbb{N}]^k, d_{\mathbb{K}})_k$  does not equi-coarsely Lipschitz embed into  $X$ .

(iii) If  $p = \infty$ , then  $([\mathbb{N}]^k, d_{\mathbb{K}})_k$  does not equi-coarsely embed into  $X$ .

The next theorem allows us to obtain new examples of spaces with property  $\mathcal{Q}_p$  and relates this property with asymptotic uniform convexity.

**Theorem 3.3.** *Let  $X$  be a Banach space and let  $p \in (1, +\infty]$ . Assume that  $X$  admits an equivalent norm which is  $p$ -AUS (or equivalently whose dual norm is  $q$ -AUC\*, where  $q$  is the conjugate exponent of  $p$ ). Then  $X^*$  has property  $\mathcal{Q}_p$ .*

*Proof.* Assume, as it is allowed by Proposition 2.8, that  $X$  is separable and that its norm is  $p$ -AUS. Therefore, the norm of  $X^*$  is  $q$ -AUC\*, where  $q$  is the conjugate exponent of  $p$ . Let  $f : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X^*$  be a 1-Lipschitz map and fix  $\varepsilon > 0$ . Consider  $\mathbb{M} \in [\mathbb{N}]^\omega$  and  $(K_i)_{i=1}^k$  given by Proposition 2.11. Since  $(x^*(\bar{m}))_{\bar{m} \in [\mathbb{M}]^{\leq k}}$  is a weak\*-null tree in  $X^*$ , it follows from Proposition 2.6 that we can find  $n_1 < m_1 < \dots < n_k < m_k$  in  $\mathbb{M}$  so that we have the following lower  $\ell_q$  estimate:

$$\begin{aligned} \|f(\bar{n}) - f(\bar{m})\|^q &= \left\| \sum_{i=1}^k x^*(n_1, \dots, n_k) - x^*(m_1, \dots, m_k) \right\|^q \\ &\geq c \left( \sum_{i=1}^k \|x^*(n_1, \dots, n_k)\|^q + \|x^*(m_1, \dots, m_k)\|^q \right), \end{aligned}$$

where  $c > 0$ , only depends on the AUC\* modulus of  $X^*$ . Formally, we have applied Proposition 2.6 to the weak\*-null tree  $(u^*(\bar{m}))_{\bar{m} \in [\mathbb{M}]^{\leq 2k}}$  given by

$$u^*(n_1, \dots, n_l) = \begin{cases} x^*(n_1, n_3, \dots, n_l), & \text{if } l \text{ is odd} \\ -x^*(n_2, n_4, \dots, n_l), & \text{if } l \text{ is even.} \end{cases}$$

Since  $f$  is 1-Lipschitz, we deduce that

$$\sum_{i=1}^k K_i^q \leq \frac{1}{2c}.$$

Using Hölder's inequality and item (iii) in Proposition 2.11, this implies that for every  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$ :

$$\|f(\bar{n}) - f(\bar{m})\| \leq 2 \sum_{i=1}^k K_i + 2k\varepsilon \leq \frac{2k^{1/p}}{(2c)^{1/q}} + 2k\varepsilon.$$

If  $\varepsilon$  was initially chosen small enough, this gives us the desired estimate.  $\square$

Let  $p \in (1, \infty)$ . We now recall the definition and some basic properties of the James space  $\mathcal{J}_p$ . We refer the reader to [2, Section 3.4] and references therein for more details on the classical case  $p = 2$ . The James space  $\mathcal{J}_p$  is the real Banach space of all sequences  $x = (x(n))_{n \in \mathbb{N}}$  of real numbers with finite  $p$ -variation and verifying  $\lim_{n \rightarrow \infty} x(n) = 0$ . The space  $\mathcal{J}_p$  is endowed with the following norm

$$\|x\|_{\mathcal{J}_p} = \sup \left\{ \left( \sum_{i=1}^{k-1} |x(p_{i+1}) - x(p_i)|^p \right)^{1/p} : 1 \leq p_1 < p_2 < \dots < p_k \right\}.$$

This is the historical example, constructed for  $p = 2$  by R.C. James, of a quasi-reflexive Banach space which is isomorphic to its bidual. In fact  $\mathcal{J}_p^{**}$

can be seen as the space of all sequences  $x = (x(n))_{n \in \mathbb{N}}$  of real numbers with finite  $p$ -variation, which is  $\mathcal{J}_p \oplus \mathbb{R}e$ , where  $e$  denotes the constant sequence equal to 1.

The standard unit vector basis  $(e_n)_{n=1}^\infty$  is a monotone shrinking basis for  $\mathcal{J}_p$ . Hence, the sequence  $(e_n^*)_{n=1}^\infty$  of the associated coordinate functionals is a basis of its dual  $\mathcal{J}_p^*$ .

N. Kalton also proved that the James space  $\mathcal{J}_2$  and its dual  $\mathcal{J}_2^*$  fail property  $\mathcal{Q}$  (see [17, Proposition 4.7]). On the other hand, it is shown in [24, Corollary 5.3] that the family  $([\mathbb{N}]^k, d_{\mathbb{K}}^k)_k$  does not equi-coarsely embed in  $\mathcal{J}_2$ , nor in  $\mathcal{J}_2^*$ . It is known that, for  $p \in (1, \infty)$ ,  $\mathcal{J}_p$  admits an equivalent  $p$ -AUS norm and  $\mathcal{J}_p^*$  admits an equivalent  $p'$ -AUS norm, where  $p'$  is the conjugate exponent of  $p$  (see [22, 26]). Therefore we can state.

**Corollary 3.4.** *Let  $p \in (1, \infty)$  and  $p'$  be its conjugate exponent. Then  $\mathcal{J}_p$  has property  $\mathcal{Q}_{p'}$  and  $\mathcal{J}_p^*$  has property  $\mathcal{Q}_p$ .*

A Banach space  $X$  is said to have the *alternating Banach-Saks property* if every bounded sequence  $(x_n)_n$  in  $X$  has a subsequence  $(x_{n_j})_j$  so that its sequence of alternating-sign Cesàro means  $(\sum_{j=1}^k (-1)^j x_{n_j})_k$  converges to 0. N. Kalton proved in [17, Theorem 4.5] that a Banach space with the alternating Banach-Saks property which also has property  $\mathcal{Q}$  must be reflexive. We now present the  $p$ -version of this result. For that, we will need the following theorem, which is a version of [17, Theorem 4.4] to property  $\mathcal{Q}_p$ .

**Theorem 3.5.** *Let  $C \geq 1$ ,  $p \in (1, \infty)$  and  $X$  be a Banach space with property  $\mathcal{Q}_p$  with constant  $C$ . Then, for all  $\varepsilon > 0$  and all bounded sequences  $(x_n)_n$  in  $X$  with weak\* cluster point  $x^{**} \in X^{**}$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  so that*

$$\left\| \sum_{j=1}^{2k} (-1)^j x_{n_j} \right\| \geq \frac{(1-\varepsilon)}{C} d(x^{**}, X) k^{1-1/p},$$

for all  $k \in \mathbb{N}$  and all  $n_1 < \dots < n_{2k} \in \mathbb{M}$ .

*Proof.* If  $x^{**} \in X$ , the statement is trivial. Assume that  $\theta = d(x^{**}, X) > 0$ . Let  $B = \sup_{n \in \mathbb{N}} \|x_n\|$  and pick  $\lambda > 1$  and  $\alpha \in (0, 1)$  so that

$$C^{-1} \lambda^{-1} \theta - \alpha - 2B\alpha \geq (1-\varepsilon)C^{-1} \theta.$$

Going to a subsequence of  $(x_n)$ , we can assume that

$$\left\| \sum_{j=1}^k a_j x_{n_j} - \sum_{j=1}^k b_j x_{m_j} \right\| \geq \lambda^{-1} \theta,$$

for all  $k \in \mathbb{N}$ , all  $\bar{n} < \bar{m} \in [\mathbb{N}]^k$  and all  $a_1, \dots, a_k, b_1, \dots, b_k \geq 0$  with  $\sum_{j=1}^k a_j = \sum_{j=1}^k b_j = 1$ .

A simple application of Ramsey theory and a standard diagonalization procedure gives an infinite subset  $\mathbb{M}$  so that for all  $k \in \mathbb{N}$ , there exists  $b_k > 0$  with the following property: for all  $k \in \mathbb{N}$  and all  $\alpha k^{1-1/p} \leq n_1 < \dots < n_{2k}$ ,

we have that

$$\left\| \sum_{j=1}^{2k} (-1)^j x_{n_j} \right\| \in [b_k - \alpha, b_k].$$

Fix  $k \in \mathbb{N}$ , let  $\mathbb{M}_k = \{n \in \mathbb{M} : n \geq \alpha k^{1-1/p}\}$  and define  $f : [\mathbb{M}_k]^k \rightarrow X$  by setting

$$f(\bar{n}) = \sum_{j=1}^k x_{n_j},$$

for all  $\bar{n} \in [\mathbb{M}_k]^k$ . Then  $\omega_f(1) \leq b_k$ . Since  $X$  has property  $\mathcal{Q}_p$ , there exist  $\bar{n} < \bar{m} \in [\mathbb{M}_k]^k$  with

$$\|f(\bar{n}) - f(\bar{m})\| \leq C b_k k^{1/p}.$$

Therefore, this gives us that  $\lambda^{-1}\theta k \leq C b_k k^{1/p}$ . In particular,

$$b_k \geq C^{-1} \lambda^{-1} \theta k^{1-1/p}.$$

Fix  $\bar{n} \in [\mathbb{M}_k]^{2k}$ . Notice that  $\lceil \alpha k^{1-1/p} \rceil \leq 2k$ . Let  $\bar{m} \in [\mathbb{M}_k]^{2k}$  be any element so that  $m_j = n_{\lceil \alpha k^{1-1/p} \rceil + j - 1}$ , for all  $j \in \{1, \dots, 2k - \lceil \alpha k^{1-1/p} \rceil + 1\}$ . Then, we can pick  $\beta \in \{-1, 1\}$  so that

$$\begin{aligned} \left\| \sum_{j=1}^{2k} (-1)^j x_{n_j} + \beta \sum_{j=1}^{2k} (-1)^j x_{m_j} \right\| &\leq \left\| \sum_{j=1}^{\lceil \alpha k^{1-1/p} \rceil - 1} (-1)^j x_{n_j} \right\| \\ &\quad + \left\| \sum_{j=2k - \lceil \alpha k^{1-1/p} \rceil + 2}^{2k} (-1)^j x_{m_j} \right\| \\ &\leq 2B\alpha k^{1-1/p}. \end{aligned}$$

We conclude that

$$\begin{aligned} \left\| \sum_{j=1}^{2k} (-1)^j x_{n_j} \right\| &\geq \left\| \sum_{j=1}^{2k} (-1)^j x_{m_j} \right\| - 2B\alpha k^{1-1/p} \\ &\geq C^{-1} \lambda^{-1} \theta k^{1-1/p} - 2\alpha - 2B\alpha k^{1-1/p} \\ &\geq (1 - \varepsilon) C^{-1} \theta k^{1-1/p}. \end{aligned}$$

□

We now introduce a  $p$ -version of the alternating Banach-Saks property.

**Definition 3.6.** Let  $p \in (1, \infty)$  and  $C > 0$ . We say that  $X$  has the  $p$ -alternating Banach-Saks property with constant  $C > 0$  if for all sequences  $(x_n)_n$  in  $B_X$  and all  $k \in \mathbb{N}$ , there exists an infinite subset  $\mathbb{M} \subset \mathbb{N}$  so that

$$\left\| \sum_{j=1}^k (-1)^j x_{n_j} \right\| \leq C k^{1/p},$$

for all  $n_1 < \dots < n_k \in \mathbb{M}$ .

Notice that the  $p$ -alternating Banach-Saks property implies the alternating Banach-Saks property.

**Corollary 3.7.** *Let  $p, q \in (1, \infty)$  be so that  $q > p/(p-1)$  (i.e.,  $q$  is larger than the conjugate exponent of  $p$ ). Let  $X$  be a Banach space with the  $p$ -alternating Banach-Saks property and with property  $\mathcal{Q}_q$ . Then  $X$  is reflexive.*

*Proof.* Since reflexivity is separably determined, assume that  $X$  is separable. Let  $C \geq 1$  be so that  $X$  has both the  $p$ -alternating Banach-Saks property and property  $\mathcal{Q}_q$  with constant  $C$ . Suppose  $X$  is not reflexive and pick  $x^{**} \in B_{X^{**}} \setminus X$ , so that  $d(x^{**}, X) > 0$ . Let  $(x_n)_n$  be a sequence in  $B_X$  with  $x^{**}$  as a weak\* cluster point.

Fix  $k \in \mathbb{N}$ . Since  $X$  has the  $p$ -alternating Banach-Saks property with constant  $C$ , by going to a subsequence, we can assume that

$$\left\| \sum_{j=1}^{2k} (-1)^j x_{n_j} \right\| \leq 2^{1/p} C k^{1/p},$$

for all  $n_1 < \dots < n_{2k} \in \mathbb{N}$ . Since  $X$  has property  $\mathcal{Q}_q$  with constant  $C$  the previous theorem tells us that, by going to a subsequence, we can assume that

$$\left\| \sum_{j=1}^{2k} (-1)^j x_{n_j} \right\| \geq \frac{1}{2C} k^{1-1/q} d(x^{**}, X),$$

for all  $n_1 < \dots < n_{2k} \in \mathbb{N}$ .

As  $k$  was arbitrary, this shows that

$$\frac{1}{2C} k^{1-1/q} d(x^{**}, X) \leq 2^{1/p} C k^{1/p}$$

for all  $k \in \mathbb{N}$ . As  $1 - 1/q > 1/p$ , this gives us a contradiction.  $\square$

As another application of Theorem 3.5, we can show that Corollary 3.4 is optimal.

**Corollary 3.8.** *Let  $p$  in  $(1, \infty)$  and  $p'$  be its conjugate exponent. Then, for any  $r > p'$ ,  $\mathcal{J}_p$  fails property  $\mathcal{Q}_r$  and for any  $s > p$ ,  $\mathcal{J}_p^*$  fails property  $\mathcal{Q}_s$ .*

*Proof.* We follow the proof of Proposition 4.7 in [17].

First, consider in  $\mathcal{J}_p$  the sequence  $(x_n)_n$  given by  $x_n = \sum_{i=1}^n e_i$  for all  $n \in \mathbb{N}$ . We have that  $(x_n)_n$  converges weak\* to  $e \in \mathcal{J}_p^{**} \setminus \mathcal{J}_p$ . However, it is easy to see that there exists  $C > 0$  such that for any  $n_1 < \dots < n_{2k}$ , we have:

$$\left\| \sum_{i=1}^{2k} (-1)^j x_{n_j} \right\|_{\mathcal{J}_p} \leq C k^{1/p}.$$

For  $r > p$ , according to Theorem 3.5, this prevents  $\mathcal{J}_p$  to have property  $\mathcal{Q}_r$ .

We now consider in  $\mathcal{J}_p^*$  the sequence  $(e_n^*)_n$  which is weak\*-converging to an element  $\lambda \in \mathcal{J}_p^{***} \setminus \mathcal{J}_p^*$ , which is just the functional assigning its limit to any sequence of bounded  $p$ -variation. For  $x \in \mathcal{J}_p$ , we have

$$\left| \left\langle \sum_{i=1}^{2k} (-1)^j e_{n_j}^*, x \right\rangle \right| \leq \sum_{j=1}^k |x(n_{2j}) - x(n_{2j-1})| \leq k^{1/p'} \|x\|_{\mathcal{J}_p}.$$

It follows that  $\left\| \sum_{i=1}^{2k} (-1)^j e_{n_j}^* \right\|_{\mathcal{J}_p^*} \leq k^{1/p'}$ . We then deduce from Theorem 3.5 that  $\mathcal{J}_p^*$  fails property  $\mathcal{Q}_s$  for all  $s > p$ .  $\square$

From Corollaries 3.4 and 3.8 it is natural to get information on some compression exponents of  $\mathcal{J}_q$  in  $\mathcal{J}_p$  or  $\mathcal{J}_q^*$  in  $\mathcal{J}_p^*$ . More precisely we have.

**Proposition 3.9.** *Let  $p, q$  be in  $(1, \infty)$  and  $p', q'$  be their respective conjugate exponents.*

- (1) *If  $p < q$ , then  $\alpha_{\mathcal{J}_p}(\mathcal{J}_q) \leq \frac{q'}{p'}$ .*
- (2) *If  $p > q$ , then  $\alpha_{\mathcal{J}_p^*}(\mathcal{J}_q^*) \leq \frac{q}{p}$ .*

We shall omit the easy details of the proof. However we need to make a few remarks. Estimates on the compression exponents for the “other half” of the values of  $p$  and  $q$  are already known (see [26] or [25]). They are based on concentration properties for Lipschitz maps defined on the Hamming graphs with values in quasi-reflexive  $p$ -AUS spaces. When one wants to use asymptotic convexity as an obstruction for coarse Lipschitz embeddings, it is customary to use the so-called approximate midpoint principle (see for instance [20]). However this method, as far as we know, only allows to show the impossibility of a coarse Lipschitz embedding, but does not provide extra information on the compression modulus. In fact, this method was used by F. Netillard [26] to prove that for  $p < q$ ,  $\mathcal{J}_q$  does not coarse Lipschitz embed in  $\mathcal{J}_p$  and that for  $p > q$ ,  $\mathcal{J}_q^*$  does not coarse Lipschitz embed in  $\mathcal{J}_p^*$ . Our last corollary is an improvement of these results. This shows that Theorem 3.3 can serve as an alternative to the approximate midpoint principle, but only in a non reflexive setting.

#### 4. CONCENTRATION PROPERTIES AND SZLENK INDICES

In this section, we obtain obstructions to the embeddability of Kalton’s graphs into some dual Banach spaces. We emphasize here that, although we will assume in each proof that our Banach space is separable, all the statements of this section are valid for general Banach spaces. This simply follows from Proposition 2.8.

**Theorem 4.1.** *Let  $X$  be a Banach space with summable Szlenk index. Then  $X^*$  has property  $\mathcal{Q}$ .*

*Proof.* Let  $f : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X^*$  a 1-Lipschitz map, fix  $\varepsilon \in (0, \frac{1}{2C})$  and consider  $\mathbb{M} \in [\mathbb{N}]^\omega$  given by Proposition 2.11. Then it clearly follows from item (iv) and Proposition 2.3 that

$$0 \in s_{\frac{K_1}{2}} \dots s_{\frac{K_k}{2}}(B_{X^*}).$$

Since the Szlenk index of  $X$  is summable, we deduce that  $\sum_{i=1}^k K_i \leq 2C$ , where  $C$  is the “summable Szlenk index constant” of  $X$ . Then, we deduce from items (ii) and (iii) that

$$\text{diam}(f([\mathbb{M}]^k)) \leq 2 \sum_{i=1}^k K_i + 2k\varepsilon \leq 4C + 1.$$

□

*Remark 4.2.* Note that Theorem 3.3 insures that if  $X$  admits an equivalent norm whose dual norm is 1-AUC\* then  $X^*$  has property  $\mathcal{Q}$ . It is known [14] that a separable Banach space admits an equivalent norm whose dual

norm is 1-AUC\* if and only if  $X$  is isomorphic to a subspace of  $c_0$ . It is an easy exercise to check that any subspace of  $c_0$  has a summable Szlenk index. However, there are Banach spaces with summable Szlenk index that do not linearly embed into  $c_0$ . Before describing a few of them, let us mention that a Banach space has a summable Szlenk index if and only if it is *asymptotic- $c_0$*  (see [10, Theorem 4.1]). The original Tsirelson space, now denoted  $T^*$ , is an example of a reflexive asymptotic- $c_0$  space. Let us also mention that there exists a non reflexive quasi-reflexive Banach space which is asymptotic- $c_0$  (see Section 7 in [4] and references therein). In conclusion, Theorem 4.1 applies to spaces that are not covered by N. Kalton's work nor by our Theorem 3.3.

**Theorem 4.3.** *Let  $X$  be a Banach space. If the family of Kalton's interlaced graphs  $(([\mathbb{N}]^{<k}, d_{\mathbb{K}}))_{k \in \mathbb{N}}$  equi-coarse Lipschitz embeds into  $X^*$ , then  $\text{Sz}(X) > \omega$ .*

*Proof.* By Remark 2.2, we can assume that  $(([\mathbb{N}]^{<k}, d_{\mathbb{K}}))_{k \in \mathbb{N}}$  equi-Lipschitz embeds into  $X^*$ . Hence, without loss of generality, we may assume that there exists  $A \in (0, 1]$  so that for any  $k \in \mathbb{N}$  there exists  $f_k : ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X^*$  such that

$$\forall \bar{n}, \bar{m} \in [\mathbb{N}]^k, \quad Ad_{\mathbb{K}}(\bar{n}, \bar{m}) \leq \|f(\bar{n}) - f(\bar{m})\| \leq d_{\mathbb{K}}(\bar{n}, \bar{m}).$$

Let  $\bar{n} < \bar{m} \in [\mathbb{N}]^k$  (that is such that  $n_k < m_1$ ). By the triangle inequality we have

$$\left\| \sum_{\bar{s} \preceq \bar{n}} x^*(\bar{s}) \right\| + \left\| \sum_{\bar{s} \preceq \bar{m}} x^*(\bar{s}) \right\| \geq \|f(\bar{n}) - f(\bar{m})\| \geq Ak.$$

For a fixed  $k$  and a given  $\varepsilon > 0 \in (0, \frac{A}{4})$ , we consider  $\mathbb{M} \in [\mathbb{N}]^\omega$  given by Proposition 2.11. It then follows from item (iii) that  $2 \sum_{i=1}^k K_i \geq Ak - 2k\varepsilon$ . Now, if we denote  $I = \{1, \dots, k\}$ ,  $I_1 = \{i \in I, K_i > \frac{A}{8}\}$  and  $N = |I_1|$ , we have that

$$\frac{Ak}{2} - k\varepsilon \leq \sum_{i=1}^k K_i = \sum_{I \setminus I_1} K_i + \sum_{I_1} K_i \leq \frac{A}{8}k + N.$$

From our choice of  $\varepsilon$ , it follows that  $N \geq \frac{Ak}{8}$ . Finally, we deduce from item (iv) in Proposition 2.11 and Proposition 2.3 that

$$0 \in s_{\frac{A}{16}}^N(B_X^*)$$

and therefore that  $\text{Sz}(X, \frac{A}{16}) \geq \frac{Ak}{8}$ . Since  $k$  was arbitrary, this concludes our proof.  $\square$

*Remark 4.4.* As it is recalled in the introduction, a Banach space  $X$  admits an equivalent AUS norm if and only if  $\text{Sz}(X) \leq \omega$  and in that case there exists  $p \in (1, \infty)$  such that  $X$  admits an equivalent  $p$ -AUS norm. Therefore Theorem 3.3 is a quantitative version of Theorem 4.3. In fact Theorem 4.3 is a consequence of Theorem 3.3 and these deep renorming results. We have chosen to present here an independent, self contained elementary proof.

Let us now say that a Banach space  $X$  has *proportional Szlenk index* if there exists  $C > 0$  such that for all  $\varepsilon > 0$ ,  $\text{Sz}(X, \varepsilon) \leq \frac{\varepsilon}{C}$ . It is clear that a Banach space with summable Szlenk index has proportional Szlenk index.

To the best of our knowledge, whether the converse implication is true is an open problem. We do not know either if the dual of a Banach space with a proportional Szlenk index has property  $\mathcal{Q}$ , but we can prove the following weaker concentration estimate.

**Proposition 4.5.** *Let  $X$  be a Banach space with proportional Szlenk index. Then, there exists  $M > 0$  such that for any  $k \in \mathbb{N}$  and every Lipschitz map  $f: ([\mathbb{N}]^k, d_{\mathbb{K}}) \rightarrow X^*$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  such that:*

$$\forall \bar{n}, \bar{m} \in [\mathbb{M}]^k, \|f(\bar{n}) - f(\bar{m})\| \leq M(1 + \log k)\text{Lip}(f).$$

*Proof.* Assume that  $X$  is a separable Banach space such that for all  $\varepsilon > 0$ ,  $\text{Sz}(X, \varepsilon) \leq \frac{\varepsilon}{C}$ , for some  $C > 0$ . Let  $f: ([\mathbb{N}]^k, d_{\mathbb{K}}^k) \rightarrow X^*$  be a 1-Lipschitz map, fix  $\varepsilon > 0$  and consider  $\mathbb{M} \in [\mathbb{N}]^\omega$  given by Proposition 2.11. Denote  $\eta = \frac{C}{k}$  and, discarding as we may the first values of  $k$ , assume that  $\eta \leq \frac{1}{2}$ . Let now  $N \in \mathbb{N}$  such that  $\frac{1}{2} < N < 2^N \eta$ . For  $r \in \mathbb{N}$ , we denote  $I_r$  the set of all  $i$ 's in  $\{1, \dots, k\}$  such that  $2^{r-1} \eta \leq K_i \leq 2^r \eta$  and  $N_r$  the cardinality of  $I_r$ . Now it follows from item (iv) in Proposition 2.11 that  $N_r \leq \text{Sz}(X, 2^{r-2} \eta) \leq \frac{4C}{2^r \eta}$ . Since  $f$  is 1-Lipschitz, we have that for all  $i$ 's,  $K_i \leq 1$ , which implies that  $I_r$  is empty for  $r > N + 1$ . We deduce that

$$\sum_{i=1}^k K_i \leq k\eta + \sum_{r=1}^{N+1} \sum_{i \in I_r} K_i \leq k\eta + 4C(N+1) \leq C + 4C(N+1).$$

Finally, using item (iii) of Proposition 2.11, we get that

$$\text{diam}(f([\mathbb{N}]^k)) \leq C + 4C(N+1) + 2k\varepsilon \leq 2C + 4C(N+1),$$

if  $\varepsilon$  was initially chosen small enough. In view of the definition of  $\eta$  and  $N$ , this clearly yields the conclusion of our proposition.  $\square$

## 5. OPTIMALITY

In the previous section, we proved that if the family of Kalton's interlaced graphs equi-Lipschitz embeds into a dual Banach space  $X^*$ , then the Szlenk index of  $X$  is at least  $\omega^2$ . Indeed, it is known that, when it is well defined, the Szlenk index of a Banach space is always of the form  $\omega^\alpha$  for some ordinal  $\alpha$  (see for instance [23]). Here we show that this result is optimal. That is, we exhibit a separable dual Banach space with Szlenk index  $\omega^2$  and which contains the interlaced graphs. To this aim we will use Lipschitz free spaces. Recall, if  $(M, x_0)$  is a pointed metric space – a Banach space  $X$  is always considered as a pointed metric space with  $x_0 = 0$  –, then  $\text{Lip}_0(M)$  denotes the space of all Lipschitz maps  $f: M \rightarrow \mathbb{R}$  so that  $f(x_0) = 0$ . Endowed with the norm  $\|f\| = \text{Lip}(f)$ ,  $\text{Lip}_0(M)$  is a Banach space. Given  $x \in M$ , the map  $\delta_x: \text{Lip}_0(M) \rightarrow \mathbb{R}$  given by  $\delta_x(f) = f(x)$  for all  $f \in \text{Lip}_0(M)$  belongs to  $\text{Lip}_0(M)^*$ , and we define the *Lipschitz free space of  $M$*  as

$$\mathcal{F}(M) = \overline{\text{span}}\{\delta_x \in \text{Lip}_0(M)^* : x \in M\}.$$

We refer to the monograph [29] for the basic properties of  $\mathcal{F}(M)$ . Just note that the map  $\delta: x \mapsto \delta_x$  is an isometry from  $M$  into  $\mathcal{F}(M)$ .

In order to exhibit a separable dual Banach space with Szlenk index  $\omega^2$ , the strategy will be to consider the Lipschitz free space  $\mathcal{F}(M)$  over a metric

space  $M$  which contains the interlaced graphs, and then prove that  $\mathcal{F}(M)$  has the required properties. In particular, the next corollary from [12] will be useful for proving that  $\mathcal{F}(M)$  is isometrically a dual Banach space. In the following statement,  $\mathcal{C}_\tau(M)$  stands for the set of maps from  $M$  to  $\mathbb{R}$  which are continuous with respect to some other topology  $\tau$  on  $M$ .

**Proposition 5.1** (Corollary 3.7 of [12]). *Let  $(M, d)$  be a uniformly discrete, bounded, separable metric space with a distinguished point  $0 \in M$ . Assume that there is a Hausdorff topology  $\tau$  on  $M$  such that:*

- (i)  $(M, \tau)$  is compact
- (ii)  $d$  is  $\tau$ -lower semicontinuous.

If  $X = \text{Lip}_0(M, d) \cap \mathcal{C}_\tau(M)$  is equipped with the Lipschitz norm  $\|\cdot\|_L$ , then  $X$  is an isometric predual of  $\mathcal{F}(M)$ . Moreover the weak\*-topology induced by  $X$  on  $\mathcal{F}(M)$  coincides with  $\tau$  on  $\delta(M)$ , that we identify with  $M$ .

For any given  $k \in \mathbb{N}$ , a concrete bi-Lipschitz copy of the metric space  $([\mathbb{N}]^{\leq k}, d_{\mathbb{K}})$  into  $c_0$  is given by the map  $f_k : [\mathbb{N}]^{\leq k} \rightarrow c_0$  defined by

$$\forall \bar{n} = (n_1, \dots, n_j) \in [\mathbb{N}]^{\leq k} : f_k(\bar{n}) = \sum_{i=1}^j s_{n_i},$$

where  $(s_n)_{n=1}^\infty$  stands for the summing basis of  $c_0$ . Indeed, one can easily check that

$$(E) \quad \forall \bar{n}, \bar{m} \in [\mathbb{N}]^{\leq k} : \frac{1}{2} d_{\mathbb{K}}(\bar{n}, \bar{m}) \leq \|f_k(\bar{n}) - f_k(\bar{m})\| \leq d_{\mathbb{K}}(\bar{n}, \bar{m})$$

(see for instance [24, Proposition 2.5]).

For each  $k \in \mathbb{N}$ , let

$$\widetilde{M}_k = \overline{f_k([\mathbb{N}]^{\leq k})}^{w^*} \subset \ell_\infty,$$

where the weak\*-topology is of course given by  $\ell_1$ . Letting  $\mathbf{1} \in \ell_\infty$  be the sequence constant and equal to 1, it is readily seen that

$$\widetilde{M}_k = \left\{ \sum_{i=1}^j s_{n_i} + \ell \mathbf{1} : j, \ell \in \mathbb{N} \cup \{0\}, j + \ell \leq k, n_1 < \dots < n_j \in \mathbb{N} \right\}.$$

Hence, considering  $\widetilde{M}_k$  endowed with the usual norm  $\|\cdot\|_\infty$  of  $\ell_\infty$ ,  $\widetilde{M}_k$  is a countable and uniformly discrete metric space. Moreover,  $\widetilde{M}_k$  is weak\*-compact since  $f_k([\mathbb{N}]^{\leq k})$  is bounded, and the norm  $\|\cdot\|_\infty$  is trivially weak\*-lower semicontinuous.

The next corollary is therefore a direct consequence of Proposition 5.1.

**Corollary 5.2.** *For any  $k \in \mathbb{N}$ , the free space  $\mathcal{F}(\widetilde{M}_k, \|\cdot\|_\infty)$  is isometric to a separable dual Banach space  $X_k^*$ , where  $X_k = \text{Lip}_0(\widetilde{M}_k, \|\cdot\|_\infty) \cap \mathcal{C}_{w^*}(\widetilde{M}_k)$ .*

**Theorem 5.3.** *The Kalton graph  $([\mathbb{N}]^{<\omega}, d_{\mathbb{K}})$  Lipschitz embeds into a separable dual space  $X^*$  with  $\text{Sz}(X) = \omega^2$ .*

*Proof.* Let  $M = ([\mathbb{N}]^{<\omega}, d_{\mathbb{K}})$ , and consider the distinguished point  $0 = \emptyset \in [\mathbb{N}]^{<\omega}$ . For each  $k \in \mathbb{N}$ , let

$$M_{2^k} = B(0, 2^k) = [\mathbb{N}]^{\leq 2^k}.$$

So  $M = \bigcup_{n \in \mathbb{N}} M_{2^k}$ . Then we use Kalton's decomposition [16, Proposition 4.3] to deduce that for every  $\varepsilon > 0$ ,  $\mathcal{F}(M)$   $(1 + \varepsilon)$ -linearly embeds into

$$\left( \sum_{k \in \mathbb{N}} \mathcal{F}(M_{2^k}, d_{\mathbb{K}}) \right)_{\ell_1}.$$

For each  $k \in \mathbb{N}$ , we let  $\widetilde{M}_{2^k}$  be the metric subspace of  $\ell_\infty$  as it is defined above. For each  $k \in \mathbb{N}$ , let  $X_{2^k}$  be the predual of  $\mathcal{F}(\widetilde{M}_{2^k}, \|\cdot\|_\infty)$  given by Corollary 5.2. It follows from (E) that each  $\mathcal{F}(M_{2^k}, d_{\mathbb{K}})$  2-linearly embeds into  $X_{2^k}^*$  for all  $k \in \mathbb{N}$ . Since  $M$  isometrically embeds into  $\mathcal{F}(M)$ , we deduce that, for any  $\varepsilon > 0$ ,  $M$  Lipschitz embeds with distortion  $2(1 + \varepsilon)$  into

$$\left( \sum_{k \in \mathbb{N}} \mathcal{F}(\widetilde{M}_{2^k}, \|\cdot\|_\infty) \right)_{\ell_1} \equiv \left( \sum_{k \in \mathbb{N}} X_{2^k} \right)_{c_0}^*.$$

Let  $X := (\sum_{k \in \mathbb{N}} X_{2^k})_{c_0}$ . It remains to prove that  $\text{Sz}(X) = \omega^2$ . By Theorem 4.3 we know that  $\text{Sz}(X) > \omega$  and therefore  $\text{Sz}(X) \geq \omega^2$ , so we only have to prove the reverse inequality. Notice that  $X_{2^k} = \text{Lip}_0(\widetilde{M}_{2^k}, \|\cdot\|_\infty) \cap \mathcal{C}_{w^*}(\widetilde{M}_{2^k})$  equipped with its Lipschitz norm is isomorphic to a subspace of the Banach space  $(\mathcal{C}(\widetilde{M}_{2^k}, w^*), \|\cdot\|_\infty)$  of continuous functions on the compact metrisable space  $(\widetilde{M}_{2^k}, w^*)$ . Indeed, as  $(\widetilde{M}_{2^k}, \|\cdot\|_\infty)$  is bounded and uniformly discrete, we have that the sup-norm and the Lipschitz norm are equivalent on  $X_{2^k}$ . In fact

$$X_{2^k} = \{f \in \mathcal{C}_{w^*}(\widetilde{M}_{2^k}) : f(0) = 0\},$$

which clearly is a hyperplane of  $\mathcal{C}_{w^*}(\widetilde{M}_{2^k})$  and it follows, for instance from [2, Proposition 4.4.1], that  $X_{2^k}$  is actually isomorphic to  $(\mathcal{C}(\widetilde{M}_{2^k}, w^*), \|\cdot\|_\infty)$ .

Next, we claim that the Cantor–Bendixson index of  $\widetilde{M}_{2^k}$  is equal to  $2^k + 1$ . Indeed it is readily seen by induction that the first  $2^k$  derived sets are

$$\widetilde{M}_{2^k}^{(d)} = \left\{ \sum_{i=1}^j s_{n_i} + \ell \mathbf{1} : j, \ell \in \mathbb{N} \cup \{0\}, \ell \geq d, j + \ell \leq 2^k, n_1 < \dots < n_j \in \mathbb{N} \right\},$$

whenever  $d \in \{1, \dots, 2^k\}$  so the claim easily follows. This shows that  $X_{2^k}$  is isomorphic to  $c_0$  (e.g., [2, Theorem 4.5.2]) and therefore that  $\text{Sz}(X_{2^k}) = \omega$ . Finally it follows from [8] that

$$\text{Sz}(X) = \text{Sz}\left(\left(\sum_{k \in \mathbb{Z}} X_k\right)_{c_0}\right) \leq \omega^2,$$

and we are done.  $\square$

*Remark 5.4.* The proof of the last proposition shows that  $\mathcal{F}([\mathbb{N}]^{<\omega}, d_{\mathbb{K}})$  is isomorphic to a subspace of  $X^* = (\sum_{k \in \mathbb{N}} \mathcal{F}(\widetilde{M}_{2^k}, \|\cdot\|_\infty))_{\ell_1}$ . In fact, the image is even complemented in  $X^*$ . Indeed, this follows from the following two facts (we adopt the same notation as in the proof of Theorem 5.3 above). First, in Kalton's decomposition, the image of  $\mathcal{F}(M)$  is complemented in  $(\sum_k \mathcal{F}(M_{2^k}))_{\ell_1}$  (this is proved in detail in [13, Proposition 3.5]).

Second, we claim that  $(\sum \mathcal{F}(M_{2^k}))_{\ell_1}$  is isomorphic to a 1-complemented subspace of  $(\sum \mathcal{F}(\widetilde{M}_{2^k}))_{\ell_1}$ . It is enough to show that  $\mathcal{F}(f_{2^k}(M_{2^k}))$  is 1-complemented in  $\mathcal{F}(\widetilde{M}_{2^k})$  for every  $k \in \mathbb{N}$ , but this simply follows from the

fact that

$$r_{2^k} : \widetilde{M}_{2^k} \rightarrow f_{2^k}(M_{2^k})$$

$$\sum_{i=1}^j s_{n_i} + \ell \mathbf{1} \mapsto \sum_{i=1}^j s_{n_i}$$

is a 1-Lipschitz retraction.  $\square$

*Remark 5.5.* We proved that, for every  $\varepsilon > 0$ ,  $([\mathbb{N}]^{<\omega}, d_{\mathbb{K}})$  Lipschitz embeds with distortion  $2(1 + \varepsilon)$  into a separable dual Banach space. It is actually possible to do it with distortion  $(1 + \varepsilon)$ . To this end, instead of using the natural embeddings of the  $([\mathbb{N}]^k, d_{\mathbb{K}})$ 's into  $c_0$  (which are of distortion 2), one can build concrete metric spaces containing isometrically the interlaced graphs and which satisfy the assumptions of Proposition 5.1. The counterpart is that one has to define by hand the required topology  $\tau$  and then check that the distance is  $\tau$ -lower semicontinuous (which was automatic with the  $w^*$ -topology in  $\ell_\infty$ ). The same optimal estimate on the Szlenk index is obtained.

## 6. LOW DISTORTION EMBEDDING OF THE GRID OF $c_0$ INTO DUALS

In this section, we produce two uniformly discrete countable metric spaces so that if they Lipschitz embeds into  $X^*$  with Lipschitz distortion at most  $3/2$  or  $2$ , respectively, then  $X$  must contain an isomorphic copy of  $\ell_1$ . We use this in order to prove Theorem 6.2.

We define the *integer grid of  $c_0$*  as

$$\text{Grid}(c_0) = \{(x_n)_n \in c_0 : \forall n \in \mathbb{N}, x_n \in \mathbb{Z}\}.$$

So  $\text{Grid}(c_0)$  is a  $(1, 1)$ -net of  $c_0$  (meaning that it is 1-separated and for every  $x \in c_0$ ,  $d(x, \text{Grid}(c_0)) \leq 1$ ). We consider it as a metric space with the metric inherited from  $c_0$ .

**Proposition 6.1.** *Let  $X$  be a Banach space and  $f : \text{Grid}(c_0) \cap 2B_{c_0} \rightarrow X^*$  be a Lipschitz embedding with distortion strictly smaller than  $\frac{3}{2}$ . Then  $X$  contains an isomorphic copy of  $\ell_1$ .*

*Proof.* Replacing  $f$  by  $\lambda f$  for some appropriate  $\lambda > 0$ , we may assume that there exists  $D \in [1, \frac{3}{2})$  so that

$$\|x - y\| \leq \|f(x) - f(y)\| \leq D\|x - y\|$$

for all  $x, y \in \text{Grid}(c_0) \cap B_{c_0}$ . Fix  $\varepsilon > 0$  such that  $3 - \varepsilon - 2D > 0$ .

Let  $(e_n)_n$  be the canonical basis of  $c_0$ . For every  $k \in \mathbb{N}$ , pick  $x_k \in S_X$  such that

$$\langle x_k, f(2e_k) - f(-e_k) \rangle \geq 3 - \varepsilon.$$

We claim that the sequence  $(x_k)_k$  has no weakly Cauchy subsequence. Indeed, let  $\mathbb{M} = \{m_1 < m_2 < \dots\} \in [\mathbb{N}]^\omega$  and set  $A_1 = \{m_{2k+1} : k \in \mathbb{N}\}$  and  $A_2 = \mathbb{N} \setminus A_1$ . Then, for all  $k \in \mathbb{N}$  and all  $m > m_{2k+1}$  we have that

$$\begin{aligned} \langle x_{m_{2k+1}}, f(\mathbf{1}_{A_1 \cap [1, m]}) - f(\mathbf{1}_{A_2 \cap [1, m]}) \rangle &= \left\langle x_{m_{2k+1}}, f(2e_{m_{2k+1}}) - f(-e_{m_{2k+1}}) \right. \\ &\quad \left. + f(\mathbf{1}_{A_1 \cap [1, m]}) - f(2e_{m_{2k+1}}) \right. \\ &\quad \left. + f(-e_{m_{2k+1}}) - f(\mathbf{1}_{A_2 \cap [1, m]}) \right\rangle \\ &\geq 3 - \varepsilon - 2D \end{aligned}$$

and

$$\begin{aligned} \langle x_{m_{2k}}, f(\mathbf{1}_{A_1 \cap [1, m]}) - f(\mathbf{1}_{A_2 \cap [1, m]}) \rangle &= \langle x_{m_{2k}}, f(-e_{m_{2k}}) - f(2e_{m_{2k}}) \\ &\quad + f(\mathbf{1}_{A_1 \cap [1, m]}) - f(-e_{m_{2k}}) \\ &\quad + f(2e_{m_{2k}}) - f(\mathbf{1}_{A_2 \cap [1, m]}) \rangle \\ &\leq -3 + \varepsilon + 2D. \end{aligned}$$

Let  $\mathcal{U}$  be a nonprincipal ultrafilter on  $\mathbb{N}$  and set

$$x^* = w^* \text{-} \lim_{m, \mathcal{U}} (f(\mathbf{1}_{A_1 \cap [1, m]}) - f(\mathbf{1}_{A_2 \cap [1, m]})).$$

The above inequalities imply that for all  $k \in \mathbb{N}$ :

$$x^*(x_{m_{2k+1}}) \geq 3 - \varepsilon - 2D \quad \text{and} \quad x^*(x_{m_{2k}}) \leq -3 + \varepsilon + 2D.$$

This shows that  $(x_{m_k})_k$  is not weakly Cauchy.

By Rosenthal's  $\ell_1$  theorem [28], this implies that  $(x_k)_k$  has a subsequence equivalent to the standard unit basis of  $\ell_1$ . In particular,  $X$  contains an isomorphic copy of  $\ell_1$  and we are done.  $\square$

**Theorem 6.2.** *If  $c_0$  coarse Lipschitz embeds into a dual space  $X^*$  with coarse Lipschitz distortion strictly less than  $\frac{3}{2}$ , then  $X$  contains an isomorphic copy of  $\ell_1$ .*

*Proof.* Assume  $f$  is such a coarse Lipschitz embedding from  $c_0$  into  $X^*$ . Replacing  $f$  with  $x \mapsto f(nx)/n$  for a large enough  $n \in \mathbb{N}$ , the map  $f$  restricted to  $\text{Grid}(c_0) \cap 2B_{c_0}$  becomes a Lipschitz embedding with distortion strictly smaller than  $3/2$ . Then, it follows from Proposition 6.1 that  $X$  contains an isomorphic copy of  $\ell_1$ .  $\square$

We will now show that replacing  $\text{Grid}(c_0) \cap B_{c_0}$  by an appropriate graph  $M$ , the distortion in Proposition 6.1 can be pushed up to 2. It is not clear if the metric space  $M$  needed for this is isometric to a subset of  $c_0$ , but maybe such graphs are better suited for trying to go to larger distortions.

We define the graph  $M$  as follows. Let  $S = [\mathbb{N}]^{<\omega}$ ,  $G = \mathbb{N}$  and  $H = \mathbb{N}$ . Moreover, we write  $G = \{g_i : i \in \mathbb{N}\}$  and  $H = \{h_i : i \in \mathbb{N}\}$ , where  $g_i = h_i = i$  for all  $i \in \mathbb{N}$ . We define  $M$  as the disjoint union

$$M = \{0\} \sqcup S \sqcup G \sqcup H$$

and we define a graph structure on  $M$  by putting an edge

- between 0 and any element of  $S$ ,
- between  $A \in S$  and  $g_k \in G$  iff  $k \in A$ , and
- between  $A \in S$  and  $h_k \in H$  iff  $k \notin A$ .

Then endow  $M$  with the shortest path distance, which we denote by  $d_M$ . It should be clear that  $d_M(A, B) = 2$  if  $A \neq B \in S$  and that  $d_M(g_k, h_k) = 4$  for all  $k \in \mathbb{N}$ .

**Proposition 6.3.** *Let  $f : (M, d_M) \rightarrow X^*$  be a Lipschitz embedding with distortion strictly less than 2. Then  $X$  contains  $\ell_1$ .*

*Proof.* Replacing  $f$  by  $\lambda f$  for some appropriate  $\lambda > 0$ , we may assume that there exists  $D \in [1, 2)$  so that

$$\|x - y\| \leq \|f(x) - f(y)\| \leq D\|x - y\|$$

for all  $x, y \in M$ . Fix  $\varepsilon > 0$  such that  $4 - \varepsilon - 2D > 0$ .

For every  $k \in \mathbb{N}$ , let  $x_k \in S_X$  be such that  $\langle x_k, f(g_k) - f(h_k) \rangle \geq 4 - \varepsilon$ . We claim that  $(x_k)_k$  does not contain any weakly Cauchy subsequence. Rosenthal's  $\ell_1$  theorem thus implies that  $X$  contains  $\ell_1$ . In order to prove our claim, let  $\mathbb{M} = \{m_1 < m_2 < \dots\}$  be an infinite subset of  $\mathbb{N}$ . We set  $A = \{m_{2k+1} : k \in \mathbb{N}\}$  and  $B = \mathbb{N} \setminus A$ . Let  $\mathcal{U}$  be a nonprincipal ultrafilter on  $\mathbb{N}$  and let  $\xi_A = w^*\text{-}\lim_{m, \mathcal{U}} f(A \cap [1, m])$  and  $\xi_B = w^*\text{-}\lim_{m, \mathcal{U}} f(B \cap [1, m])$ . We have

$$\begin{aligned} \langle x_{m_{2k+1}}, \xi_A - \xi_B \rangle &= \left\langle x_{m_{2k+1}}, f(g_{2k+1}) - f(h_{2k+1}) + \xi_A - f(g_{2k+1}) \right. \\ &\quad \left. + f(h_{2k+1}) - \xi_B \right\rangle \\ &\geq 4 - \varepsilon - 2D \end{aligned}$$

and

$$\begin{aligned} \langle x_{m_{2k}}, \xi_A - \xi_B \rangle &= \left\langle x_{m_{2k}}, f(h_{2k}) - f(g_{2k}) + \xi_A - f(h_{2k}) \right. \\ &\quad \left. + f(g_{2k}) - \xi_B \right\rangle \\ &\leq -4 + \varepsilon + 2D \end{aligned}$$

for all  $k \in \mathbb{N}$ . So  $(x_{m_k})_k$  is not weakly Cauchy, and we are done.  $\square$

*Remark 6.4.* The results of this section have a certain importance also for the theory of Lipschitz free spaces. Indeed, it is a well known open problem of N. Kalton (see [18] the remarks after Problem 1) to determine whether the  $\mathcal{F}(M)$  enjoys the metric approximation property (MAP) for every bounded and uniformly discrete metric space  $M$ . One silly approach would be to show that for such  $M$ , the free space  $\mathcal{F}(M)$  is isometrically a dual. Then a stroke of Grothendieck theorem would imply that  $\mathcal{F}(M)$  has the (MAP) since, being isomorphic to  $\ell_1$ , it has the (AP). This approach does not work since not all such  $\mathcal{F}(M)$  are isometrically duals (other than the examples above see Example 5.8 in [12]) and so A. Rueda Zoca proposed a refined strategy which consists in proving that for every bounded uniformly discrete metric space  $(M, d)$  and for every  $0 < \alpha < 1$  the free space  $\mathcal{F}(M, d^\alpha)$  of the  $\alpha$ -snowlaked  $M$  is isometrically a dual. Then again by Grothendieck the space  $\mathcal{F}(M, d^\alpha)$  would enjoy the (MAP) and by approximation when  $\alpha \rightarrow 1$ ,  $\mathcal{F}(M)$  would enjoy the (MAP) too. Strike number two: this approach does not work either since for the metric space from Propostion 6.3 there is  $0 < \alpha_0 < 1$  such that for all  $\alpha_0 < \alpha < 1$  the space  $\mathcal{F}(M, d_M^\alpha)$  is not isometrically a dual. Indeed, it is enough to take  $\alpha_0$  such that the Banach-Mazur distance of  $\mathcal{F}(M)$  and  $\mathcal{F}(M, d_M^\alpha)$  is strictly less than 2. Now, since  $(M, d)$  embeds isometrically into  $\mathcal{F}(M)$ , it will Lipschitz embed into  $\mathcal{F}(M, d_M^\alpha)$  with distortion  $< 2$  whenever  $\alpha_0 < \alpha < 1$ . Proposition 6.3 then implies that  $\mathcal{F}(M, d_M^\alpha)$  cannot be a dual as it is separable.

## 7. WEAK SEQUENTIALLY CONTINUOUS EMBEDDINGS

In this section, we show that Problem 1.2 has a negative answer with the further assumption that the embedding is weak-to-weak\* sequentially continuous.

Let  $X$  be a Banach space,  $K \subset X^*$  and  $x \in K$ . We say that  $x^*$  is a point of weak\*-to-norm continuity of  $K$  if every sequence  $(x_n)_n$  in  $K$  which converges to  $X$  in the weak\* topology converges to  $x$  in the norm topology.

The following is well known (see [2, Lemma 6.3.4]).

**Lemma 7.1.** *Let  $X$  be a Banach space with separable dual and  $K$  be a weak\* compact subset of  $X^*$ . Then  $K$  has a point of weak\*-to-norm continuity.*

Let  $X$  and  $Y$  be Banach spaces,  $E \subset X$ , and  $f : E \rightarrow Y^*$  be a map. We define the  $\text{weak}(f)$  topology on  $E$  as the topology generated by the collection

$$\{f^{-1}(U) \subset E : U \subset Y^* \text{ weak* open}\}.$$

Clearly,  $f$  is  $\text{weak}(f)$ -to-weak\* continuous.

The next result should be compared with [2, Lemma 6.3.5], which is a classical result in the isomorphic theory of Banach spaces.

**Lemma 7.2.** *Suppose  $X$  and  $Y$  are Banach spaces and assume that  $Y^*$  is separable. Let  $f : B_X \rightarrow Y^*$  be a norm-to-weak\* continuous bounded map so that its inverse exists and it is uniformly continuous. Then every closed bounded subset  $F \subset B_X$  contains a point of  $\text{weak}(f)$ -to-norm continuity for  $f|_F$ .*

*Proof.* Let  $F \subset B_X$  be closed and bounded and let  $W$  be the weak\* closure of  $f(F)$ . Since  $f(F)$  is bounded,  $W$  is weak\* compact. By Lemma 7.1, there exists  $y \in W$  which is a point of weak\*-to-norm continuity of  $W$ . Let  $(y_n)_n$  be a sequence in  $f(F)$  converging to  $y$  in the weak\* topology. By the choice of  $y$ ,  $(y_n)_n$  converges to  $y$  in norm. For each  $n \in \mathbb{N}$ , pick  $x_n \in F$  so that  $f(x_n) = y_n$ . Since  $(y_n)_n$  is a Cauchy sequence and  $f^{-1}$  is uniformly continuous, it follows that  $(x_n)_n$  is Cauchy. Let  $x = \lim_n x_n$ . As  $F$  is closed,  $x \in F$ . Since  $f$  is norm-to-weak\* continuous, we have

$$f(x) = w^* \text{-} \lim_n f(x_n) = w^* \text{-} \lim_n y_n = y.$$

Fix  $\varepsilon > 0$ . Since  $y$  is a point of weak\*-to-norm continuity of  $K$ , there exists a weak\* open set  $U$  containing  $y$  so that  $\|y - w\| < \varepsilon$  for all  $w \in U \cap W$ . Hence,  $\|y - f(v)\| < \varepsilon$  for all  $v \in f^{-1}(U) \cap F$ . Since  $f(x) = y$ , this shows that  $x$  is a point of  $\text{weak}(F)$ -to-norm continuity of  $f|_F$ .  $\square$

**Lemma 7.3.** *Let  $X$  be either  $L_1$  or  $c_0$ , and  $Y$  be a Banach space with separable dual. There is no weak-to-weak\* sequentially continuous bounded map  $B_X \rightarrow Y$  with a uniformly continuous inverse.*

*Proof.* Suppose  $f : B_X \rightarrow Y^*$  is a weak-to-weak\* sequentially continuous bounded map with uniformly continuous inverse. In particular,  $f$  is norm-to-weak\* continuous and Lemma 7.2 gives a point  $x \in F$  of  $\text{weak}(f)$ -to-norm continuity for  $f|_F$ , where  $F = \{x \in X : \|x\| \in [1/2, 1]\}$ .

**Claim 1.** *There exists a sequence  $(x_n)_n$  in  $B_X$  converging to  $x$  in the weak topology.*

*Proof.* If  $X = c_0$ , let  $z_n = e_n$  for all  $n \in \mathbb{N}$ , where  $(e_n)_n$  is the standard basis of  $c_0$ . If  $X = L_1$ , let  $(r_n)_n$  be the sequence of Rademacher functions and

let  $z_n = r_n x$  for all  $n \in \mathbb{N}$ . In both cases, it follows that  $(z_n)_n$  is weakly null and

$$(*) \quad \lim_n \|x + z_n\| \in [\|x\|, 1].$$

Indeed, if  $X = c_0$  this is trivial that  $\lim_n \|x + z_n\| = 1$  and if  $X = L_1$  it follows from the classic properties of the Rademacher functions that  $\lim_n \|x + z_n\| = \|x\|$  (e.g., see [2, Lemma 6.3.2]).

By  $(*)$ , there exists a sequence  $(\alpha_n)_n$  of positive reals converging to 1 so that  $\alpha_n(x + z_n) \in F$  for all  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$ , set  $x_n = \alpha_n(x + z_n)$ . Since  $\lim_n \alpha_n = 1$  and  $w\text{-}\lim_n z_n = 0$ ,  $(x_n)_n$  converges to  $x$  in the weak topology.  $\square$

Let  $(x_n)_n$  be the sequence given by the claim above. Since  $f$  is weak-to-weak\* sequentially continuous and  $x = w\text{-}\lim_n x_n$ , it follows that  $(x_n)_n$  converges to  $x$  in the weak( $f$ ) topology. As  $x$  is a point of weak( $f$ )-to-norm continuity of  $f|_F$  and  $(x_n)_n$  is in  $F$ , we have that  $\lim_n f(x_n) = f(x)$ . As  $f^{-1}$  is continuous,  $\lim_n x_n = x$ . Notice that for each  $n \in \mathbb{N}$ , if  $X = c_0$ ,  $\|z_n\| = 1$  and, if  $X = L_1$ ,  $\|z_n\| = \|x\|$ . So,

$$0 = \lim_n \|x - x_n\| \geq \lim_n |\alpha_n| \cdot \|z_n\| - \lim_n \|\alpha_n x - x\| \geq \|x\|,$$

which is a contradiction.  $\square$

**Theorem 7.4.** *Neither  $c_0$  nor  $L_1$  can be coarsely (resp. uniformly) embedded into a separable dual Banach space by a map that is weak-to-weak\* sequentially continuous.*

*Proof.* Let  $X$  be either  $L_1$  and  $c_0$ ,  $Y$  be a Banach space with separable dual and assume that there exists a weak-to-weak\* sequentially continuous map  $f : X \rightarrow Y^*$  which is either a coarse or a uniform embedding.

**Claim 2.** *There exists a coarse map  $g : X \rightarrow \ell_2(Y^*)$  which is weak-to-weak\* sequentially continuous and so that  $g^{-1}$  exists and is uniformly continuous.*

*Proof.* If  $f : X \rightarrow Y^*$  is uniformly continuous there is nothing to be done. Indeed, we may simply take  $g = i \circ f$ , where  $i : Y^* \rightarrow \ell_2(Y^*)$  is a linearly isometric inclusion.

Suppose  $f$  is a coarse embedding. Without loss of generality, assume that  $f(0) = 0$ . By [5, Lemma 5.1], there exist sequences of positive reals  $(a_n)_n$  and  $(b_n)_n$  so that the map  $g : X \rightarrow \ell_2(Y^*)$  given by  $g(x) = (f(a_n x)/b_n)_n$ , for all  $x \in X$ , is a well defined coarse embedding with uniformly continuous inverse.<sup>1</sup> Since  $f$  is weak-to-weak\* sequentially continuous, so is  $g$ .  $\square$

Since  $\ell_2(Y^*)$  is the dual of  $\ell_2(Y)$  and separable, the result follows from Lemma 7.3.  $\square$

Notice that Theorem 7.4 implies that  $L_1$  does not coarse (resp. uniform) embeds into either  $\ell_p$  or  $L_p$  by a weakly sequentially continuous map, for all  $p > 1$ . In contrast with that,  $\ell_q$  strongly embeds into  $\ell_p$  by a weakly sequentially continuous map for all  $q \leq p$  (see [6, Theorem 1.8]).

<sup>1</sup>Notice that the hypothesis of [5, Lemma 5.1] also demand the map to be norm continuous. However, this is only used in order to guarantee that  $g$  is norm continuous.

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## REFERENCES

- [1] I. Aharoni, *Every separable metric space is Lipschitz equivalent to a subset of  $c_0^+$* , Israel J. Math. **19** (1974), 284–291. MR0511661
- [2] F. Albiac and N. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006. MR2192298
- [3] F. Baudier and G. Lancien, *Embeddings of locally finite metric spaces into Banach spaces*, Proc. Amer. Math. Soc. **136** (2008), 1029–1033. MR2361878
- [4] F. Baudier, G. Lancien, P. Motakis, and Th. Schlumprecht, *A new coarsely rigid class of Banach spaces*, arXiv e-prints (2018Jun), arXiv:1806.00702, available at [1806.00702](#).
- [5] B. M. Braga, *Coarse and uniform embeddings*, J. Funct. Anal. **272** (2017), no. 5, 1852–1875. MR3596709
- [6] ———, *Nonlinear Weakly Sequentially Continuous Embeddings Between Banach Spaces*, International Mathematics Research Notices (2018).
- [7] ———, *On asymptotic uniform smoothness and nonlinear geometry of banach spaces*, Journal of the Institute of Mathematics of Jussieu (2019), 138.
- [8] P.A.H. Brooker, *Direct sums and the Szlenk index*, J. Funct. Anal. **260** (2011), no. 8, 2222–2246.
- [9] N. Brown and E. Guentner, *Uniform embeddings of bounded geometry spaces into reflexive Banach space*, Proc. Amer. Math. Soc. **133** (2005), no. 7, 2045–2050. MR2137870
- [10] R.M. Causey, *Concerning summable Szlenk index*, Glasgow Math. Journal (2018), 1–15.
- [11] S. J. Dilworth, D. Kutzarova, G. Lancien, and N. L. Randrianarivony, *Equivalent norms with the property ( $\beta$ ) of Rolewicz*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111** (2017), no. 1, 101–113. MR3596040
- [12] L. García-Lirola, C. Petitjean, A. Procházka, and A. Rueda Zoca, *Extremal structure and duality of Lipschitz free spaces*, Mediterr. J. Math. **15** (2018), no. 2, Art. 69, 23. MR3778926
- [13] L. García-Lirola, C. Petitjean, and A. Rueda Zoca, *On the structure of spaces of vector-valued Lipschitz functions*, Studia Math. **239** (2017), no. 3, 249–271. MR3693571
- [14] G. Godefroy, N. Kalton, and G. Lancien, *Subspaces of  $c_0(\mathbb{N})$  and Lipschitz isomorphisms*, Geom. Funct. Anal. **10** (2000), no. 4, 798–820.
- [15] ———, *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. **353** (2001), no. 10, 38953918.
- [16] N. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*, Collect. Math. **55** (2004), no. 2, 171–217. MR2068975
- [17] ———, *Coarse and uniform embeddings into reflexive spaces*, Q. J. Math. **58** (2007), no. 3, 393–414. MR2354924
- [18] ———, *The uniform structure of Banach spaces*, Math. Ann. **354** (2012), no. 4, 1247–1288. MR2992997
- [19] ———, *Uniform homeomorphisms of Banach spaces and asymptotic structure*, Trans. Amer. Math. Soc. **365** (2013), no. 2, 1051–1079. MR2995383
- [20] N. Kalton and L. Randrianarivony, *The coarse Lipschitz geometry of  $l_p \oplus l_q$* , Math. Ann. **341** (2008), no. 1, 223–237. MR2377476

- [21] H. Knaust, E. Odell, and Th. Schlumprecht, *On asymptotic structure, the Szlenk index and UKK properties in Banach spaces*, Positivity **3** (1999), no. 2, 173–199. MR1702641
- [22] G. Lancien, *Réflexivité et normes duales possédant la propriété de Kadec-Klee*, Publications Mathématiques de Besançon **14** (1993/94).
- [23] ———, *A survey on the Szlenk index and some of its applications*, Rev. R. Acad. Cien. Serie A Mat. **100** (2006), no. 1-2, 209–235.
- [24] G. Lancien, C. Petitjean, and A. Procházka, *On the coarse geometry of James spaces*, arXiv e-prints (2018May), arXiv:1805.05171, available at [1805.05171](https://arxiv.org/abs/1805.05171).
- [25] G. Lancien and M. Raja, *Asymptotic and coarse Lipschitz structures of quasi-reflexive Banach spaces*, ArXiv e-prints (May 2017), available at [1705.00577](https://arxiv.org/abs/1705.00577).
- [26] F. Netillard, *Coarse lipschitz embeddings of james spaces*, Bull. Belg. Math. Soc. Simon Stevin. **25** (2018), no. 1, 71–84. MR3784507
- [27] M. Raja, *On asymptotically uniformly smooth Banach spaces*, J. Funct. Anal. **264** (2013), no. 2, 479–492. MR2997388
- [28] H. Rosenthal, *A characterization of Banach spaces containing  $l^1$* , Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413. MR0358307
- [29] N. Weaver, *Lipschitz algebras*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018. Second edition of [MR1832645]. MR3792558

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