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**Semiclassical Green functions and Lagrangian intersection.  
Applications to the propagation of Bessel beams in non-homogeneous media**

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*Abstract:* We study semi-classical asymptotics for problems with localized right-hand sides by considering a Hamiltonian  $H(x, p)$  positively homogeneous of degree  $m \geq 1$  on  $T^*\mathbf{R}^2 \setminus 0$ . The energy shell is  $E = 1$ , and the right-hand side  $f_h$  is microlocalized: (1) on the vertical plane  $\Lambda = \{x = x_0\}$ ; (2) on the “cylinder”  $\Lambda = \{(X, P) = (\varphi\omega(\psi), \omega(\psi)); \varphi \in \mathbf{R}, \omega(\psi) = (\cos \psi, \sin \psi)\}$ . We restrict essentially to the isotropic case  $H(x, p) = \frac{|p|^m}{\rho(x)}$ , with  $\rho$  a smooth positive function. In case (2),  $\Lambda$  is the frequency set of Bessel function  $J_0(\frac{|x|}{h})$ , and the solution  $u_h$  of  $(H(x, hD_x) - E)u_h = f_h$  when  $m = 1$ , is called a “Bessel beam”, and arises in the theory of optical fibers.

## 1. Introduction

Let  $M = \mathbf{R}^n$ ,  $H(x, hD_x; h)$  a  $h$ -PDO with symbol  $H(x, p) \sim H_0(x, p) + hH_1(x, p) + \dots$ , and  $E \neq 0$  be a non critical energy level for Hamiltonian  $H_0$ . The general problem of “semi-classical Green functions” consists in solving  $(H - E)u_h = f_h$ , where  $f_h$  is a Lagrangian distribution supported microlocally on some Lagrangian manifold  $\Lambda$ , i.e.  $\text{WF}_h f_h \subset \Lambda$  (here  $\text{WF}_h$  denotes the semi-classical wave front set, we recall in Sect.3). When  $\Lambda = \{x = x_0\} = T_{x_0}^*M$ ,  $x_0 \in M$ , we call the “vertical plane”, is the conormal bundle to  $\{x_0\}$ , we say simply that  $f_h$  is a “localized function” at  $x_0$ , and to fix the ideas, assume  $x_0 = 0$ . Moreover, we require  $u_h = E_+ f_h$  to be *outgoing* at infinity. Here are some examples of  $f$  (expressed in a single chart):

*Examples 1.1:*

(1) WKB functions in Fourier representation

$$(1.1) \quad f(x; h) = \frac{e^{-i\pi n/4}}{(2\pi h)^{n/2}} \int e^{i(xp+S(p))/h} A(p; h) dp$$

here  $\Lambda = \{(-\partial_p S(p), p) : p \in \mathbf{R}^n\}$ . Although a canonical transformation can reduce (locally)  $S(p)$  to  $S_0(p) = -\langle x_0, p \rangle$  and  $\Lambda$  to  $\Lambda_0 = \{x = x_0\}$ , it has no reason to preserve homogeneity of the Hamiltonian. But we may still attempt to adapt our constructions to this case.

Whenever no confusion may occur such an oscillatory integral, conveniently normalized including the phase factor, will be written as  $\int^*(\dots)$  (see [CdV, Def.19 pdf p.75]). The advantage is two-fold: on the one hand we recover the usual notation for Maslov canonical operator  $\int^*(\dots) = K_\Lambda^h(A)$ ; on the other hand, if the phase  $xp + S(p)$  has a non-degenerate critical point at  $p_0$ , then asymptotic stationary phase readily gives at first order

$$(1.2) \quad \int^*(\dots) = e^{i(xp_0+S(p_0))/h} e^{-im\pi/2} |\det(S''(p_0))|^{-1/2} A_0(p) + \mathcal{O}(h)$$

where  $m$  is Morse index (number of negative squares) of  $S''(p_0)$ .

(2) A conormal distribution  $f(x; h) = \int^* e^{ix_n p_n/h} A(x', p_n) dp_n$  with respect to the hypersurface  $N = \{x_n = 0\}$ , i.e.  $\Lambda = T_N^* \mathbf{R}^n$ , or of more general type.

(3)  $f_h$  identifies with a “Bessel beam”, and

$$(1.3) \quad \Lambda = \{x = X(\varphi, \psi) = \varphi\omega(\psi), p = P(\varphi, \psi) = \omega(\psi), \varphi \in \mathbf{R}\}$$

is called the “cylinder” ; here  $\omega \in \mathbf{S}^{n-1}$  is the unit vector parametrized by  $\psi$ , see Sect. 5.

(4) Coherent states  $f(x; h) = \frac{1}{h^n} \exp(-\omega^2 \cdot (x - x_0)^2/2h) \exp(ix\xi_0/h)$ , and  $\Lambda = \{x = x_0\} \times \mathbf{R}^n \cup \mathbf{R}^n \times \{\xi = \xi_0\}$ , or more generally a Lagrangian distributions with a complex phase in the sense of Melin-Sjöstrand) [MeSj], equivalently a complex germ in the sense of Maslov. This case however requires some special treatment and will not be considered here.

The “initial manifold”  $\Lambda$  we shall be mostly interested in are the vertical plane and the cylinder, but our technics hopefully extend to other examples above. It is essential however that  $H_0$  be positively homogeneous with respect to  $p$  on  $T^*M \setminus 0$ .

By well known results on propagation of singularities,  $\text{WF}_h E_+ f \subset \Lambda \cup \Lambda_+$ , where  $\Lambda_+$  is the flow-out of the Hamilton vector field  $v_H$  in the energy surface  $H = E$ . More generally, we can construct  $E_+ f_h$  microlocally, since  $H$  can be reduced microlocally to  $hD_{x_n}$  near  $H = E$  while preserving Lagrangian intersection. This follows from a straightforward extension of local constructions of [MelUhl] to the semi-classical case.

However, global existence of an outgoing solution at infinity provided suitable hypotheses on the Hamilton vector flow, such as Lagrangian intersection, the non-trapping (in  $x$ -space) and the non-return conditions, is more involved. The main strategy has been set up in [MelUhl] and has received a more systematic treatment in relatively recent works [Ca], [JF-Bo], [KlCa], when  $H(x, hD_x)$  is semi-classical Helmholtz (Schrödinger) operator  $-h^2\Delta + V(x)$ ,  $V \leq 0$  and  $f_h$  a localized function at  $x_0$ . In this case, the non-return condition is a condition on the set

$$(1.4) \quad \mathcal{R} = \{(p, \eta) : p^2 + V(0) = \eta^2 + V(0) = 0; \exists t > 0 : X(t, p) = 0, P(t, p) = \eta\}$$

where  $(X(t, p), P(t, p)) = \exp tv_H(0, p)$  is the trajectory issued from  $(0, p) \in T^*M$ . Outgoing solutions  $u_h$ , are characterized by Sommerfeld radiation condition of the form

$$\frac{x}{|x|} \nabla_x w(x) + i\sqrt{-V(x_0)} w(x) \rightarrow 0, \quad |x| \rightarrow \infty, \quad n \geq 2$$

where  $w = \lim_{h \rightarrow 0} w_h$ ,  $w_h(x) = h^{d/2} u_h(hx)$ , is the unique solution of  $(-\Delta + V(0))w = f$ . It relates in a non trivial way the behavior of  $u_h$  at infinity with the value of the potential  $V$  at  $x_0$ . Proving Sommerfeld radiation condition requires careful estimates on  $U_h(t) = e^{itH/h}$ , or  $U_h(t)f_h$ , along with a discussion according to the relative magnitude of  $t$  and  $h$ . The proof consists in testing  $U_h(t)f_h$  against some fixed  $\phi \in \mathcal{S}(\mathbf{R}^n)$ , and show that  $\langle u_h, \phi \rangle \rightarrow \langle w, \phi \rangle$  as  $h \rightarrow 0$ . In particular one needs to know asymptotics of  $u_h$  in a  $h$ -dependent neighbhd of  $\Lambda$ .

In this paper, given  $H$  and  $\Lambda$ , we content ourselves to present, in the sense of *formal asymptotics*, a “close form” for the outgoing solution of

$$(1.5) \quad (H(x, hD_x; h) - E)u_h(x) = f_h(x), \quad u_h(x) = E_+ f_h(x; h) = \int_0^\infty e^{-it(H-E)/h} f_h dt$$

in term of Maslov canonical operator for bi-Lagrangian distributions. By formal asymptotics [Ler] we mean that, in principle, our approximate solution has no reason to be equal to  $E_+ f(x; h) \bmod \mathcal{O}(h)$ . In practice however numerical simulations show that Maslov canonical operator provides an excellent agreement with the “exact solution”.

We assume that Hamiltonian  $H_0$  (the principal symbol of  $H$ ) is positively homogeneous of degree  $m \geq 1$  with respect to momentum variable  $p$ , see Sect.2. Note that these are not suitable symbols for Pseudo-differential Calculus because of the singularity at  $p = 0$ , but this is relatively harmless : since  $E \neq 0$  we may assume  $0 \notin \text{supp } A_0(0, \cdot)$  in Example 1.1 (1), or more generally  $(x_0, 0) \notin \text{WF}_h(f_h)$ .

Our constructions will be essentially geometric (solving Hamilton-Jacobi and transport equations) using adapted coordinates on the initial manifold  $\Lambda$ , so to minimize the number of required canonical charts. Introducing also eikonal coordinates, we shall perform, by asymptotic stationary phase, the integration over  $t$  and coordinates on  $\partial\Lambda_+$ , and try to split  $u_h$  according to its components supported on  $\Lambda$  and  $\Lambda_+$  respectively.

By Maupertuis-Jacobi correspondence, our arguments can be translated to other types of operators, including Helmholtz operator, or the water-waves operator, see e.g. [DoRo], [DoMiRo], [ReMiKaDo]. In particular we recover the situation of [Ca] with energy surface  $p^2 + V(x) = 0$ , if we choose  $V(x) = -(E\rho(x))^{2/m} < 0$ .

### 1.1 Lagrangian intersection and microlocal Green functions.

As in [MelUhl] our constructions make use of symbolic calculus adapted to Lagrangian intersection. So we need first to translate some notions relative to asymptotics with respect to smoothness (or “standard pseudo-differential calculus”), to the framework of asymptotics with respect to small parameter  $h$  (or “ $h$ -pseudo-differential calculus”), in particular to allow for general phase functions (without homogeneity in the momentum variable).

Let  $\iota_0 : \Lambda_0 \rightarrow T^*M$  be a smooth embedded Lagrangian manifold, and  $\iota_1 : \Lambda_1 \rightarrow T^*M$  be a smooth embedded Lagrangian manifold with smooth boundary  $\partial\Lambda_1$  (isotropic manifold). Following [MelUhl] we say that  $(\Lambda_0, \Lambda_1)$  is an *intersecting pair* of Lagrangian manifolds iff  $\Lambda_0 \cap \Lambda_1 = \partial\Lambda_1$  and the intersection is *clean*, i.e.

$$\forall z \in \partial\Lambda_1 \quad T_z\Lambda_0 \cap T_z\Lambda_1 = T_z\partial\Lambda_1$$

(in particular  $\Lambda_0$  and  $\Lambda_1$  cannot be transverse). On the set of intersecting pairs of Lagrangian manifolds we define an equivalence relation by saying that  $(\Lambda_0, \Lambda_1) \sim (\Lambda'_0, \Lambda'_1)$  iff near any  $z \in \partial\Lambda_1$ ,  $z' \in \partial\Lambda'_1$ , there is a symplectic map  $\kappa$  such that  $\kappa(z) = z'$ , and a neighbhd  $V \subset T^*M$  of  $z$  such that  $\kappa(\Lambda_0 \cap V) \subset \Lambda'_0$ ,  $\kappa(\Lambda_1 \cap V) \subset \Lambda'_1$ . We have the following result which readily extends this of [MelUhl] to the semi-classical case:

**Lemma 1.1:** *All intersecting pairs of manifolds in  $T^*M$  are locally equivalent. More precisely near each  $z \in T^*M$ , there exists a canonical map  $\kappa : T^*M \rightarrow T^*\mathbf{R}^n$  such that  $\kappa(z) = (0, \xi_0)$ ,  $\xi_0 = (\xi', 0) \in \mathbf{R}^n$ ,  $\kappa(\Lambda_0 \cap V) \subset T_0^*\mathbf{R}^n$  (the conormal bundle to  $0 \in \mathbf{R}^n$ ), and  $\kappa(\Lambda_1 \cap V) \subset \Lambda_+$ ,  $\Lambda_+^0$  being the flow-out of  $T_0^*\mathbf{R}^n$  by the Hamilton vector field  $v_{K_0} = ((0, \dots, 0, 1), 0)$  of  $K_0(x, \xi) = \xi_n$ , passing through  $x = 0, \xi_0 = (\xi', 0)$  i.e.*

$$(1.6) \quad \Lambda_+^0 = \{(x, \xi) \in T^*\mathbf{R}^n : x = (0, x_n), \xi = (\xi', 0), x_n \geq 0\}$$

Given some Hamiltonian  $H_0 \in C^\infty(T^*M)$ , and  $\Sigma_E = H_0^{-1}(E)$  a non critical energy surface, we shall consider intersecting pairs  $(\Lambda, \Lambda_+)$  where

$$(1.7) \quad \Lambda_+ = \{(x, \xi) \in T^*M, \exists t \geq 0, \exists z = (y, \eta) \in \Lambda \cap \Sigma_E, (x, \xi) = \exp tv_{H_0}(z)\}$$

is the flow-out of  $\Lambda$  by  $v_{H_0}$  in  $\Sigma_E$ , provided  $v_{H_0}$  is transverse to  $\Lambda$  along  $\Lambda \cap \Sigma_E$ . For small  $\tau$ , we can replace  $\Lambda_+$  by  $\Lambda_+(\tau)$  in the energy level  $E - \tau$ . We denote by

$$\Lambda_t = \{z(t) = (X(t, z), P(t, z)) = \exp tv_{H_0}(z), t \geq 0, z \in \Lambda\}$$

By Lemma 1.1, near each  $z \in \partial\Lambda_+(E)$  we are reduced microlocally to the case where  $\Lambda = T_0^*\mathbf{R}^n$ ,  $\Lambda_+ = \Lambda_+^0$  as in (1.6) and  $H_0 = K_0$  is the “model Hamiltonian”.

Consider now  $H^w(x, hD_x; h)$  a  $h$ -PDO with principal symbol  $H_0$  and non singular energy surface  $\Sigma_E$ . Let  $(\Lambda, \Lambda_+)$  be a Lagrangian pair,  $\Lambda_+ \cap \Lambda = \partial\Lambda_+$ . Using Lemma 1.1, it is well known (constructing the higher order terms) that  $H^w(x; hD_x; h)$  can be reduced microlocally near  $\partial\Lambda_+$  to  $K(x, hD_x; h) = hD_{x_n}$ . As a warm-up, let us construct  $u = E_+ f \bmod \mathcal{O}(h^\infty)$  for  $K(x, hD_x; h) = hD_{x_n}$ , and  $f(x; h) = \int^* e^{ix\xi/h} A(\xi) d\xi$ . By a gauge transformation and a shift of  $\text{supp } A_0(\cdot)$  in (1.1) we may assume  $E = 0$ . So we just need to compute a primitive of  $f(x; h)$ . Let  $T > 0$  and  $\theta_T \in C^\infty(\mathbf{R})$  vanishing near  $+\infty$  and  $\theta_T(t) = 1$  for  $t \leq T$ . We consider

$$(1.8) \quad u(x, h) = \frac{i}{h} \int_0^\infty \theta_T(t) dt \int^* e^{i(x'\xi' + (x_n - t)\xi_n)/h} A(\xi) d\xi$$

Provided  $x_n \leq T/2$  (say) it is easy to check (by using integration by parts and a non-stationary phase argument) that

$$(1.9) \quad hD_{x_n} u(x, h) = f(x, h) + \mathcal{O}(h^\infty)$$

For simplicity we assume  $n = 1$  and denote  $x_n$  by  $x$ . The phase function  $(t, \xi) \mapsto \Phi(t, x, \xi) = (x - t)\xi$  is critical at  $t = x$  and  $\xi = 0$ , and  $(x, 0)$  is a non-degenerate critical point. So when  $x > 0$ , (1.2) shows that  $u(x; h) = A(0) + \mathcal{O}(h^\infty)$ , while  $u(x; h) = \mathcal{O}(h^\infty)$  for  $x < 0$ . For the constant term we have  $\text{WF}_h(A(0)) \subset \mathbf{R} \times \{0\}$  with equality iff  $A(0) \neq 0$ . This gives the component microlocally supported on  $\Lambda_+^0$ . The component supported on  $\Lambda$  can be simply defined as  $u(x; h) - A(0)$ .

This procedure gives already a microlocal description of the forward parametrix  $E_+$ , which we want to make global by constructing Maslov canonical operator associated with the pair  $(\Lambda, \Lambda_+)$ .

## 1.2 Hypothesis and main results.

Here we mostly pay attention to the case of the vertical plane. Note that Lagrangian intersection always holds on  $\Lambda$ , for if  $\partial_p H(x_0, p) = 0$ , we would have  $\langle \partial_p H(x_0, p), p \rangle = mH = 0$  which contradicts  $E \neq 0$ . We also give some hints on the case of the “cylinder” (1.3) where Lagrangian intersection plays a role ; this will be presented in more detail in Sect.5.

- *Case of a general Hamiltonian homogeneous with respect to  $p$ .*

By constructing (1.5) we mean in particular to determine the Lagrangian singularities of  $u_h$ . The first step is to find solution  $\Phi(t, x, \theta)$  of Hamilton-Jacobi equation parametrizing  $\Lambda_+$ , and a positive density on  $\Lambda_+$ . Here  $\theta = (\psi, \lambda)$  are coordinates on  $\Lambda$ , near  $\partial\Lambda_+$ ,  $\psi$  being a  $n-1$ -vector of coordinates along  $\partial\Lambda_+$ , and  $\lambda$  a “radial coordinate” related with the energy parameter  $\tau$  in  $H = E - \tau$ . Actually we can identify  $\theta$  with the initial point  $z \in \Lambda$  as in (1.7).

A *focal point* for a Lagrangian embedding  $\iota : \tilde{\Lambda} \rightarrow T^*M$  (where  $\tilde{\Lambda}$  is either  $\Lambda$ ,  $\Lambda_t = \exp tv_H(\Lambda)$ , or  $\Lambda_+$ ) is a point  $z \in \tilde{\Lambda}$  such that  $d\pi_x : \tilde{\Lambda} \rightarrow M$  has rank  $< n$ . The set of focal points is denoted by  $\mathcal{F}(\tilde{\Lambda})$ . There is a covering of  $\tilde{\Lambda}$  by canonical charts  $U$  where rank  $d\pi_x(z) \geq k$  for all  $z \in U$ . These  $U$  for which  $k = n$  are called *regular* charts, and those for which  $k < n$  *singular* charts.

We get virtually any kind of Lagrangian singularities, but due to homogeneity of  $H$  with respect to  $p$ , it is convenient to introduce another type of points.

**Definition 1.2:** Let  $H$  be positively homogeneous of degree  $m$ . We call a point  $z = (x, p)$  such that  $-\partial_x H(z) \neq 0$  an *ordinary point* if  $\langle -\partial_x H(z), p \rangle \neq 0$ , and a *special point* otherwise. If  $-\partial_x H(z) = 0$  we call  $z$  *residual point*.

*Example 1.2:* For Tricomi Hamiltonian,  $H(x, p) = x_2 p_1^2 + p_2^2$ , the residual points are those for  $p_1 = 0$ , the special points those for  $p_1 \neq 0$  but  $p_2 = 0$ , and the ordinary points those for  $p_1 p_2 \neq 0$ .

For  $t \geq 0$ , we denote by  $\mathcal{S}(\Lambda_t)$  the set of special points on  $\Lambda_t$ . Note that for Hamiltonian (1.15)  $z(t) = (x, p) \in \mathcal{S}(\Lambda_t)$  means that  $v_H$  is tangent to the level curves of  $\rho$  at  $z(t)$ .

We shall partition point  $z \in \Lambda_+$  according to the following values: (1)  $z$  is a focal (or non-focal) point; (2)  $z$  is a special (or ordinary, or residual) point. Thus each canonical chart splits again into ordinary, special or residual points.

At an ordinary point we have

$$\partial_t \Phi(t, x, \theta) = 0 \implies \partial_t^2 \Phi(t, x, \theta) \neq 0$$

so that we can perform asymptotic stationary phase in  $t$  to simplify (1.4) at  $x = X(t, \theta)$  when  $\langle \dot{P}, \dot{X} \rangle \neq 0$ . Of course, several values of parameters  $(t, \theta)$  can contribute. Then  $\Phi(x, t, \theta)$  reduces to a phase function  $\Psi(x, \theta)$  and we can further reduce the number of variables  $\theta$  in a standard way, according to the fact that  $z(t)$  is a focal point or not.

At a special point instead, the situation looks like (1.8) and we have

$$\partial_t \Phi(t, x, \theta) = \partial_\lambda \Phi(t, x, \theta) = 0 \implies \det \Phi''_{(t, \lambda), (t, \lambda)}(t, x, \theta) \neq 0$$

where  $\lambda$  is one of the “ $\theta$ ”-variables (constrained to be equal to 1 on the critical set  $C_\Phi$ ), so that we can perform asymptotic stationary phase with respect to  $(t, \lambda)$ . For a special point  $\langle \dot{P}, \dot{X} \rangle = 0$ , and we discuss according to  $\langle \dot{P}, X_\psi \rangle$  and  $\langle P_\psi, X_\psi \rangle$  are 0 or not.

Introducing Maslov indices and densities, we can thus describe the singularities of the leading (or wave) front  $z(t)$  near  $t = 0$ .

In this generality, we only succeed to describe the contribution to (1.5) (by asymptotic stationary phase) of short times  $t$  (“near field”), see Proposition 4.3. Thus we obtain the “germ” of Maslov canonical operator for bi-Lagrangian distributions. This is actually sufficient to compute  $u_h$  microlocally near  $\Lambda$ . We could obtain further information relative to the asymptotics of  $u_h$  microlocally in a  $h^\delta$ -neighbhd of  $\Lambda$ , for any  $0 < \delta < 1$ .

- *Special case of the vertical plane and a “conformal metric”.* We get more complete results in this case : under suitable hypothesis we can allow for larger times, and thus, invoking also some general geometric “stability” arguments (see e.g. Assumption (2.18)), construct (1.5) given any  $x \in M$ .

Let  $\rho$  be a smooth positive function on  $M$ ,  $m \geq 1$ , and

$$(1.15) \quad H(x, p) = |p|^m \frac{1}{\rho(x)}$$

(we could allow for more general Hamiltonians of the form  $F(x, |p|)$ , see Example 2.1.) We shall always assume that there is no finite motion

$$(1.16) \quad |X(t, \theta)| \rightarrow \infty \text{ as } t \rightarrow \infty$$

If  $\rho$  is bounded, a sufficient condition for (1.16) is that energy  $E$  is non trapping, see [GeSj]. In (4.32) we provide a stronger “convexity” condition on  $\rho$  which is natural from the point of view of classification of Lagrangian singularities. It clearly implies (1.16) and we expect it to provide an information on the “return set” in (1.4), e.g.  $\mathcal{R} = \emptyset$ . It also implies that special and residual points are “exceptional” compared to ordinary points, in the same way singular points are “exceptional” with respect to regular points. This allows a natural subdivision of canonical charts into ordinary and special (residual) points, so we can speak of a regular-ordinary chart, or regular-special chart, etc. . .

Once we know the phase function, we will adapt to the semi-classical case the constructions of [MelUhl], using symbols satisfying the compatibility condition. By gluing the different components of  $u$  we will get Maslov operator  $K_+^h$  for a bi-Lagrangian distribution associated with the pair of Lagrangian manifolds  $(\Lambda, \Lambda_+)$ . Our main result can be formulated as follows:

**Theorem 1.3:** Let  $f_h$  as in (1.1) or, at leading order in  $h$ ,  $f_h = [K_\Lambda^h A_0]$ . Under hypotheses above, with we notations (3.26), we can express (1.5) in terms of Maslov canonical operator for bi-Lagrangian distributions as

$$(1.18) \quad u_h(x) = [K_{\Lambda, \Lambda_+}^h \tilde{\sigma}(B_0)](x; h), \quad x \neq 0$$

where  $B_0$  depends linearly on  $A_0$ .

*Remark 1.1:* In case (1.15) with  $m = 1$  and  $f_h = 0$ , the asymptotic solution of  $H(x, hD_x)u_h = Eu_h$  has been constructed in [DoMaNaTu1], Example 6, in term of Bessel functions.

- *Special case of the cylinder and a “conformal metric”.* To the former “ $\theta$ -variables”  $(t, \psi, \lambda)$ , one has now to add  $\varphi$  as a parameter. Note that  $\Lambda$  has a Lagrangian singularity at  $\varphi = 0$ .

The initial condition constraints somehow  $\rho$  to be radial on  $\partial\Lambda_+$ , namely  $\nabla\rho(\varphi\omega(\psi)), \omega^\perp(\psi)\rangle = 0$ . Thus  $z = (x, p) \in \partial\Lambda_+$  is a special point iff  $\nabla\rho(\varphi\omega(\psi)) = 0$  but  $\varphi \neq 0$ . Because of this constraint, implying a relation between  $\varphi$  and energy parameter  $\tau$ , we are led to assume  $m = 1$ .

The result is similar to Theorem 1.2 but less simple to write, so we refer to Sect.5 for a more complete statement.

### *1.3 Some open problems.*

- Semi-classical structure of the Green function in a  $h^\delta$ -ngnbhd of  $\Lambda$ .
- Other types of initial Lagrangian manifolds, e.g. in Example 1.1 general WKB form in case (1); cases (2) & (4).
- Other types of Hamiltonians, in particular those to which we can apply Maupertuis-Jacobi correspondence, see e.g. [DoRo], [DoMiRo], [ReMiKaDo].
- Structure of the Green function near residual points, in particular glancing points, where Lagrangian intersection fails to be transverse.
- Hyperbolic equations ( $\partial\Lambda_+$  non compact)
- Case of multiple characteristics, involving Lagrangian manifolds with boundary  $\partial\Lambda_+$  and corner  $c\Lambda_+$  [MelUhl].
- Complex phases as in Remark 4.2.

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## 2. Hamiltonians and phase functions

In this Sect. we consider integral manifolds for positively homogeneous Hamiltonians on  $T^*M \setminus 0$ , which is the first step in constructing semi-classical Green kernels. The material is quite standard, but more specific points will be discussed in Sect. 4 and 5.

### 2.1 Eikonal coordinates

We shall deal with positively homogeneous Hamiltonians of degree  $m \geq 1$  with respect to  $p$  on the cotangent bundle  $T^*M \setminus 0$  ( $M = \mathbf{R}^n$  for simplicity), and eventually restrict to “conformal metrics” to get most explicit results. We have in mind elliptic Hamiltonians, but our formal constructions could as well allow for “Minkowski metrics”, as was originally devised in [MelUhl]. One of the main differences is that  $\partial\Lambda_+$  is no longer compact. In this Sect. we write  $H$  for  $H_0$ .

*Examples 2.1:*

(1)  $m = 2$  if  $H(x, p)$  is a geodesic flow associated with a Riemannian metric  $ds^2 = g_{ij}(x) dx^i \otimes dx^j$ . In the Riemannian case, when  $E = 1$ , geodesics are parametrized by arc-length.

(2)  $m = 1$  if  $H(x, p)$  is a “Randers symbol”, associated with a Finsler metric [Tay], [DoRo2] and reference therein.

(3)  $H(x, p) = F(x, |p|)$ , where  $F$  is positively homogeneous of degree  $m \geq 1$  with respect to second variable, see [DoMiRo]. In particular,  $H$  is of the form (1.15) with  $m \geq 1$ . Hamilton equations  $(\dot{x}, \dot{p}) = v_H(x, p)$  then read

$$(2.2) \quad \dot{x} = \partial_p H = m|p|^{m-1} \frac{1}{\rho(x)} \frac{p}{|p|}, \quad \dot{p} = -\partial_x H = |p|^m \frac{\nabla \rho(x)}{\rho(x)^2}$$

Note that when  $m$  is not an even integer,  $H$  is not smooth at  $p = 0$ , but we will take the energy level away from  $E = 0$ . Our most complete results hold for such Hamiltonians with  $n = 2$ .

First we recall some general facts on canonical coordinates near Lagrangian manifolds, see [DoMaNaTu1], [DoNaSh]. Let  $\iota : \tilde{\Lambda}_+ \rightarrow T^*M$  be a smooth embedded Lagrangian manifold. The 1-form  $p dx$  is closed on  $\tilde{\Lambda}$ , so locally exact, and  $p dx = dS$  on any simply connected domain  $U$  (so called canonical chart). Such a  $S$  is called an *eikonal* (or action) and is defined up to a constant. If  $p dx = dS \neq 0$  on  $U$ ,  $S$  can thus be chosen as a coordinate on  $U$ .

• *Case of the vertical plane.* Here  $\Lambda_+$  is the flow-out of  $H$  with initial data on  $\Lambda = T_0^*M$ , see Example 1.1. Let also  $\psi \in \mathbf{R}^{n-1}$  be smooth coordinates on  $\Lambda$ , which we complete by  $\tau$ , the dual coordinate of  $t$ , so that  $\partial\Lambda_+$  is given in  $\Lambda$  by  $\tau = 0$ , and in  $\Lambda_+$  by  $t = 0$ .

In the special case  $H(x, p) = \frac{|p|^m}{\rho(x)}$ , we have  $P(\psi, \tau) = |P|_\tau \omega(\psi)$ , with

$$(2.3) \quad |P|_\tau = (H\rho(0))^{1/m} = ((E - \tau)\rho(0))^{1/m}$$

Sections are defined as follows: for small  $\tau$ , let  $\Lambda_+(\tau)$  be the Lagrangian manifold in the energy shell  $\tau + H(x, p) = E$  issued from  $\Lambda$  at  $t = 0$ . We consider the isotropic manifold  $\partial\Lambda_+(\tau) = \Lambda \cap \Lambda_+(\tau)$ , viewing  $\Lambda_+(\tau)$  as a manifold with boundary. When  $\tau = 0$ , we simply write  $\Lambda_+(0) = \Lambda_+$ .

We assume that  $\iota : \Lambda_+ \rightarrow T^*M$  is an embedding. For  $t \geq 0$ , let  $\Lambda_t = \exp t v_H(\Lambda)$ , we have the group property  $\Lambda_{t+t'} = \exp t v_H(\Lambda_{t'})$  for all  $t, t' \geq 0$ . We define in a similar way the family of isotropic manifolds  $\partial\Lambda_t(\tau) = \exp t \partial\Lambda_+(\tau)$ .

We compute the eikonal  $S$  on  $\Lambda_+(\tau)$  by integrating along a path in  $\Lambda$  (where  $dx = 0$ ), connecting the base point (say  $(0,0)$ ) to  $(0, P(\psi)) \in \partial\Lambda_+(\tau)$ ,  $\psi \in \mathbf{R}^{n-1}$ , followed by the integral curve  $x = X(t, \psi), p = P(t, \psi)$  of  $v_H$  starting at  $(0, P(\psi))$ , where  $dS = p dx|_{\Lambda_+(\tau)} = \langle P(t, \psi), dX(t, \psi) \rangle$ . Because  $\Lambda_+(\tau)$  is Lagrangian,  $S$  doesn't depend on that choice. Since  $S(0, \psi) = \text{Const.} = S_0$  on  $\Lambda$ , we get

$$(2.4) \quad \begin{aligned} S(t, \psi) &= S(0, \psi) + \int_{(0, \psi)}^{(t, \psi)} p dx|_{\Lambda_+} = S(0, \psi) + \int_{(0, \psi)}^{(t, \psi)} P(s, \psi, \tau) dX(s, \psi, \tau) \\ &= S_0 + \int_0^t \langle P(s, \psi, \tau), \dot{X}(s, \psi, \tau) \rangle ds \end{aligned}$$

By Hamilton equations and Euler identity  $\langle P(s, \psi, \tau), \dot{X}(s, \psi, \tau) \rangle = \langle P, \partial_p H \rangle = mH = m(E - \tau)$ . Now  $S(t, \psi) = S_0 + m(E - \tau)t = S_0 + mHt$  is the action on  $\Lambda_+$ , and the eikonal coordinate is just  $S(t, \psi) = mHt$  up to a constant  $S_0$ . From the identity  $dS = d(m(E - \tau)t) = m(E - \tau)dt$  (which does not involve  $d\tau$  since we stay on  $\Lambda_+(\tau)$ ) we get

$$(2.5) \quad m(E - \tau)dt = \langle P(t, \psi, \tau), dX(t, \psi, \tau) \rangle = \langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau) \rangle dt + \langle P(t, \psi, \tau), X_\psi(t, \psi, \tau) \rangle d\psi$$

it follows (Huygens' principle) that

$$(2.6) \quad \langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau) \rangle = mH, \quad \langle P(t, \psi, \tau), \partial_\psi X(t, \psi, \tau) \rangle = 0$$

We will denote for short  $(X, P) = (X(t, \psi, \tau), P(t, \psi, \tau))$  or  $X(t, \psi), P(t, \psi)$  when  $\tau = 0$ . This is called the *leading front*.

• *Case of the cylinder.* Here  $\Lambda_+(\tau)$  is the flow-out of  $H$  at energy  $E - \tau$  with initial data on (1.3). Computing the action we find  $p dx|_\Lambda = d\varphi$  so coordinate  $\varphi$  will play the role of  $x$  in Sect.4. As in (2.4) along  $\Lambda_+(\tau)$  we have

$$\begin{aligned} S(t, \varphi, \psi, \tau) &= S(0, \varphi, \psi, \tau) + \int_{(0, \varphi, \psi)}^{(t, \varphi, \psi)} p dx|_{\Lambda_+} = S(0, \varphi, \psi, \tau) + \int_{(0, \varphi, \psi)}^{(t, \varphi, \psi)} P(s, \varphi, \psi, \tau) dX(s, \varphi, \psi, \tau) = \\ &= S_0 + \varphi + \int_0^t \langle P(s, \varphi, \psi, \tau), \dot{X}(s, \varphi, \psi, \tau) \rangle ds \end{aligned}$$

As before,  $\langle P, \dot{X} \rangle = mH$ , so on  $\Lambda_+(\tau)$

$$(2.10) \quad S(t, \varphi, \psi, \tau) = \varphi + m(E - \tau)t + S_0$$

Identifying the differential of  $S$  we get (omitting again variables)

$$(2.11) \quad \langle P, \dot{X} \rangle = mH, \quad \langle P, \partial_\psi X \rangle = 0, \quad \langle P, \partial_\varphi X \rangle = 1$$

At this point it is important to notice that  $\varphi$  is a variable on  $\Lambda$ , but only a parameter on  $\Lambda_+(\tau)$  because  $\partial\Lambda_+(\tau)$  is  $n - 1$ -dimensional. In case (1.15), as well as in the case of other special geometries,

$\psi$  turns out to be a variable both on  $\Lambda$  and  $\Lambda_+(\tau)$ . However,  $\varphi, \psi$  are independent variables on  $\Lambda_t = \exp tv_H(\Lambda)$ . In Sect.5 we develop a slightly different point of view, extending the phase-space to  $T^*(M \times \mathbf{R})$ , which amount to introduce variable  $\tau$  as the dual variable of  $t$ , and change  $\langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau) \rangle$  to  $\langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau) \rangle + \tau dt$  accordingly. This enables to treat initially  $\varphi$  as a variable, which is eventually constrained by a relation between  $\tau, \psi, \varphi$  (in case of (1.15) this relation connects only  $\tau$  with  $\varphi$ ).

## 2.2 Hamilton-Jacobi equation for small $t$ and non degenerate phase functions

Because of focal points in general we cannot readily find a phase-function  $\Psi(x)$  such that  $H(x, \partial_x \Psi(x)) = E$ , so we obtain it as a critical value (with respect to an auxiliary time-variable  $t$ ) of a phase  $\Phi(x, t)$ . To do this we solve Hamilton-Jacobi equation (HJ) in the extended phase space  $T^*(M \times \mathbf{R})$ , which is the suitable framework to vary  $(t, \tau)$  as well ( $\tau$  being set eventually to 0). So we look for a phase function  $\Phi(x, t)$  satisfying

$$(2.15) \quad \partial_t \Phi + H(x, \partial_x \Phi) = E, \quad \Phi|_{t=0} = \phi$$

with given  $\phi$  (to be chosen later on), and prescribed  $\partial_t \Phi(x_0, 0) = \tau_0, \partial_x \Phi(x_0, 0) = \eta_0$  satisfying  $\tau_0 + H(x_0, \eta_0) = E$ . By Hamilton Eq.,  $\tau = \tau_0$  is a constant of the motion. It is well-known [Hö, Thm6.4.5] that (2.15) as a unique solution for small  $t$ . This is the generating function of the Lagrangian manifold the extended phase-space

$$\tilde{\Lambda}_+ = \{p = \partial_x \Phi(x, t), \tau = \partial_t \Phi(x, t), x, t \in M \times \mathbf{R}_+\} \subset T^*(M \times \mathbf{R})$$

constructed along the integral curves of  $v_H$  starting at  $t = 0$  from the Lagrangian manifold  $\Lambda_\phi$  in  $T^*M$  given by  $p = \partial_x \phi$ . Its section at fixed  $t, \tau$  is the Lagrangian manifold

$$\Lambda_{\Phi, t, \tau} = \{p = \partial_x \Phi(x, t), x \in M \times \mathbf{R}\} \subset T^*M$$

which is simply the flow out  $\Lambda_t(\tau)$  of  $\{p = \phi'(x)\}$  in  $H(x, p) = E - \tau$  at time  $t$ . In turn,  $\phi$  usually depends on other variables  $\theta$ , so does  $\Phi$  and we need again to find the critical value of  $\Phi$  with respect to  $\theta$ , which is function on

$$(2.16) \quad C_\Phi = \{(x, t, \theta) : \partial_\theta \Phi = \partial_t \Phi = 0\}$$

We shall consider *reduced phase functions*, with least number of variables  $\theta$ . This will be recalled in Sect.3. We choose the initial condition to be the standard pseudo-differential phase function of the form  $\phi(x) = x\eta$ . Here  $\eta$  is a parameter, we choose so that  $\phi'|_\Lambda = P(\tilde{\psi})$  where  $\tilde{\psi}$  are coordinates on  $\Lambda$ , that could be taken of the form  $\tilde{\psi} = (\psi, \tau)$ , where  $\psi$  are coordinates on  $\partial\Lambda_+$ .

The phase  $\Phi(x, t)$  has the property (for a general Hamiltonian) that along each of these curves

$$(2.17) \quad \Phi(x(t), t) = \phi(x) + \int_0^t [\langle \frac{\partial H}{\partial p}(x(s), p(s)), p(s) \rangle + \tau(s)] ds$$

which is  $\phi(x) + (mH + \tau)t$  for  $H$  positively homogeneous of degree  $m$ . This is also the action  $\int_{(0,x_0)}^{(t,x)} L(q(s), \dot{q}(s)) ds = \int_{(0,x_0)}^{(t,x)} \langle p, dq \rangle - H dt$ , where the integral is computed along an integral curve of  $v_H$  from  $x_0$  to  $x$ , see e.g. [Ar, Sect.46].

*Remark 2.1:* Adding an imaginary quadratic term would allow to treat Maslov operators with complex phase. For instance, if  $\phi(x) = x^2/2$ , and  $H(x, p) = K_0(x, p) = p_n$ , then  $\Phi(x, t) = \frac{1}{2}(x'^2 + (x_n - t)^2)$ . As we shall see, the solution  $\Phi$  of HJ equation at given energy  $E > 0$  is easily constructed for positively homogeneous Hamiltonians, and related with the distance associated with  $H$  for Finsler metric ( $m = 1$ ), or Riemann metric ( $m = 2$ ).

*Remark 2.2:* Hypotheses such as Lagrangian intersection are not required to solve (2.15), and phase functions written in eikonal coordinates such as (4.8) and (5.16), when written with auxiliary “ $\theta$ -parameters”, always parametrize  $\Lambda_+$ . Additional assumptions are necessary only when eliminating  $t$  or some of the “ $\theta$ -parameters”. A similar situation appears in diffraction theory, see e.g. [Sj].

### 2.3 The phase functions “in the large”

So far we have described the phase function when “moving along”  $\Lambda_+$  for small  $t$ . Thus the critical point of  $(t, \theta) \mapsto \Phi(x, t, \theta)$ , is such that  $x = X(t, \theta)$ . A “dual” point of view is to fix  $x$  and find the set of  $(t, \theta)$  with  $x = X(t, \theta)$  with  $(x, t, \theta) \in C_\Phi$ .

We discuss the case of the vertical plane  $\Lambda = T_{x_0}^* M$ , which reduces to standard variational problems in the space variable.

Assume that for  $\tau = 0$  the initial surface,  $\partial\Lambda_+ \subset \Sigma_E$  is compact and parametrized by  $(x, p) = (X(\psi) = 0, P(\psi) = \eta)$ ,  $\psi \in \mathbf{S}^{n-1}$ . So we want to consider the map  $(t, \psi) \mapsto x = \exp tv_H(0, P(\psi))$ , or which is the same,  $(t, \eta) \mapsto x = \exp tv_H(0, \eta)$ . In the Riemannian case ( $m = 2$ ) this is related to the problem of geodesic completeness, which holds locally. Namely if  $|x|$  is small enough, there is a unique  $(t, \psi)$  such that  $x = \exp tv_H(0, P(\psi))$ . This holds globally if the Riemannian manifold ( $m = 2$ ) is geodesically convex. Otherwise, the “inverse map”  $x \mapsto (t, \eta)$  may be multivalued.

It is well known [CdV, pdf p.132] that the global geodesic convexity can be relaxed (locally) to a non-degeneracy condition. Namely, let  $H$  be associated with a Lagrangian  $\mathcal{L}(x, \dot{x})$  strictly convex with respect to  $\dot{x}$ , in particular if  $H$  is positively homogeneous of degree  $m > 1$  with respect to  $p$ .

Let  $k \subset \mathbf{R}_p^n$  be a compact set, which will be identified with the support of  $A(p; h)$  in (1.1). Denote by  $\text{Exp}_{x_0}^t \eta = \pi_x(\exp tv_H(x_0, \eta))$  the projection of the bicharacteristic of  $H$  starting from  $(x_0, \eta)$  at time 0 with  $\eta \in k$ . For fixed  $(x, t)$  we make the generic assumption :

(2.18) *For all  $\eta \in k$  such that  $x = \text{Exp}_{x_0}^t \eta$ , the map  $\mathbf{R}^n \rightarrow \mathbf{R}^n$ ,  $\xi \mapsto \text{Exp}_{x_0}^t \xi$  is a local diffeomorphism near  $\eta$  : in other terms,  $x_0$  and  $x$  are not conjugated along any trajectory that links them together within time  $t$ , with initial momentum  $\xi$ .*

The set of such  $(x, t)$  is an open set  $\Omega_{x,t} \subset \mathbf{R}^{n+1}$ , and its complement has Lebesgue measure 0.

Fixing  $(x, t)$ , (2.18) implies by Morse theory that  $\eta \mapsto x = \exp tv_H(0, \eta)$  has a discrete set of pre-images  $\eta$ .

Fixing  $x$ , consider now the pre-images of  $(t, \eta) \mapsto x = \exp tv_H(0, \eta)$ . It can happen that the integral manifold of  $v_H$  has several sheets over  $x$ , so several values of  $t$  contribute to the same

$x = X(t, \psi)$ . However, under the non-trapping condition  $|X(t, \psi)| \rightarrow \infty$  as  $t \rightarrow \infty$ , there is again, generically, a finite number of such  $t_j$ . Namely, it suffices that (2.18) holds with a time  $T$  such that for  $t \geq T$ ,  $X(t, \psi)$  will never coincide again with  $x$ . Moreover these  $t_j$  are non-degenerate critical points of  $t \mapsto \Phi(x, t, \eta)$ .

In particular this is the case when  $x_0$  and  $x$  are connected by (possibly several) minimal geodesics for the Riemannian metric associated to  $H$ , each indexed by some  $\eta_\alpha$ .

#### 2.4 Semi-classical Cauchy problem.

To get (1.5) one first needs to construct  $v_h = e^{-it(H-E)/h} f_h$ , which solves Cauchy problem  $(hD_t + H - E)v_h(t, x) = 0$ ,  $v_h(0, x) = f_h$ . When  $f_h(x) = \int^* e^{i(xp+S(p))/h} A(p; h) dp$ , at least for small  $t$

$$(2.20) \quad v_h(t, x) = \int^* e^{i\Phi(t, x, p)/h} a(t, x, p; h) dp$$

where the phase function  $\Phi$  is as above with initial condition  $\Phi|_{t=0} = xp$ , and the symbol  $a(t, x, p; h)$  verifies  $a(0, x, p; h) = A(p)$ , and solves some transport equations along the integral curves of  $v_H$ . One may address the problem of a semi-classical “close form” of (2.20), i.e. of performing the integration with respect to  $p$ , so that the final expression is given by oscillating functions (as in WKB expansions). Under Assumption (2.18) the answer to this problem is given by Van Vleck formula [CdV, pdf p.132] which gives  $u_h$  as a finite sum

$$(2.21) \quad \sum_{\alpha} \frac{A(\eta_{\alpha})}{\sqrt{\text{Jac}_{x_0}(\eta_{\alpha})}} e^{i\Phi(t, x, \eta_{\alpha})/h} e^{-i\pi \text{ind}(\gamma_{\alpha})/2}$$

indexed by all  $\eta_{\alpha} \in \text{supp } A$  such that  $\text{Exp}_{x_0}^t \eta = x$ . Here  $\text{Jac}_{x_0}(\eta_{\alpha})$  is the Jacobian of  $\text{Exp}_{x_0}^t$  at  $\eta_{\alpha}$  and  $\text{ind}(\gamma_{\alpha})$  Morse index of the integral curve  $s \mapsto \exp s v_H(x_0, \eta_{\alpha})$ ,  $s \in [0, t]$ .

In other words, under Assumption (2.18) it suffices to use only non singular charts on  $\Lambda_+$  over  $x$ , and the solution is expressed in term of finitely many oscillating functions. When Assumption (2.18) is not met, i.e.  $x = x_*$  is conjugated to  $x_0$ , then there is at least one focal point  $(x_*, p_*)$  over  $x_*$  in  $\Lambda_+$ . The construction of the canonical operator (see Sect.3) necessarily uses a singular chart in a neighborhood of  $(x_*, p_*)$ , and the solution in a neighborhood of  $x_*$  involves not only simple oscillating functions corresponding to nonsingular charts as in (2.21) (if any) but also an integral of an oscillating function over some of the momenta (or “ $\theta$ ”-variables). The total number of singular and nonsingular charts over  $x_*$  however remains finite, and so (generically) only a finite sum of integrals and simple oscillating functions contribute (one summand per each chart). See also [HeSj], [LiYau].

#### 2.5 Distances and generating functions

When  $\Lambda$  is the vertical plane, the phase function  $\Phi$  is related to the “distance” to  $x_0$  for the “metric” implied by  $H_0$ , which is of special interest. We make here some general remarks, mainly following [CdV], [GuSt].

In Sect.4-5 we shall discuss how to parametrize, by a non degenerate phase function, the flow of  $v_H$  out of some Lagrangian plane, when  $H$  is positively homegeneous of degree  $m$ . It includes the

case  $m = 1$  which plays an important role because of Finsler metrics. So we begin with a general discussion on corresponding symplectic maps.

Let  $H$  be a positively homogeneous Hamiltonian of degree  $m$  with respect to  $p$ , defined on  $T^*M \setminus 0$ , and  $\Gamma \subset T^*M \setminus 0 \times T^*M \setminus 0$  be the graph of  $\exp v_H$  (time-1 flow). Recall from [GuSt, formula (5.6) pdf p.138 and Thm 5.4.1] that

$$(2.29) \quad (\exp v_H)^*(p dx) - p dx = (m - 1)dH$$

Integrating over a path  $\gamma_E \subset \{H = E\}$ , we recover the fact that  $\int_{\gamma_E} (\exp v_H)^*(p dx) = \int_{\gamma_E} p dx$ .

So when  $m = 1$ , not only the 2-form, but also the 1-form  $p dx$  are preserved by  $v_H$ . In this case,  $v_H$  is actually the lift of a vector field on  $M$ . When  $m > 1$ , formula (2.29) gives a generating function for  $\Gamma$  under the following assumption (geodesic convexity):

(2.30) *Let  $\pi_{M \times M} : T^*(M \times M) \rightarrow M \times M$  be the natural projection, and assume  $\pi_{M \times M} : \Gamma \rightarrow M \times M$  is a diffeomorphism, i.e. for all  $(x, y) \in M \times M$ , there is a unique  $\xi \in T_x^*M$  such that  $y = \exp v_H(x, \xi)$ .*

In case of a geodesic flow ( $m = 2$ ) (2.30) holds true when  $M$  is geodesically convex. Provided (2.30),  $\Gamma$  has a generating function  $\chi$ , i.e.  $d\chi = \text{pr}_2(p dx) - \text{pr}_1(p dx)$ , where  $\text{pr}_i : T^*(M \times M) = T^*M \times T^*M$  onto the  $i$ :th factor, and  $\text{pr}_1 \circ (\pi_{M \times M}|_\Gamma)^{-1}$  is a diffeomorphism  $M \times M \rightarrow T^*M$ . Moreover we can then represent  $\chi$  as

$$(2.31) \quad \chi = (\text{pr}_1 \circ (\pi_{M \times M}|_\Gamma)^{-1})^* H$$

In case of the geodesic flow ( $m = 2$ )  $\chi(x, y) = \frac{1}{2} \text{dist}(x, y)^2$ . Formula (2.29) is related to *exact symplectic twist maps* as follows. An exact symplectic twist map [Ar], [Ka], [GuSt], [CdV7]  $F : T^*M \rightarrow T^*M$  is a symplectic map with a generating function  $S_1 : M \times M \rightarrow \mathbf{R}$ ,  $(x, X) \mapsto S_1(x, X)$  which satisfies

$$(2.32) \quad F^*(p dx) - p dx = P dX - p dx = dS_1(x, X)$$

$(p, x)$  and  $(P, X)$  are related by  $p = -\partial_x S_1, P = \partial_X S_1$ . In notation  $S_1$  the subscript 1 refers to time-1 flow. In case  $H(x, p) = p^2$  (flat metric on  $\mathbf{R}^n$ ), comparing (2.32) with (2.29), i.e.  $dS_1 = (m - 1)dH$ , we get  $S_1(x, X) = \frac{1}{4}(x - X)^2$ , and more generally, if  $H(x, p) = |p|^m$ , with  $m > 1$ ,  $S_1(x, X) = \left(\frac{m-1}{m}\right)^{m/(m-1)} |x - X|^{m/(m-1)}$ .

Again,  $S_1$  is not well defined when  $m = 1$ . More generally  $F(x, y)$  coincides with  $\chi(x, y)$  above for the geodesic flow.

For Hamilton-Jacobi (HJ) equation we have the following Proposition, extending (2.15) for large  $t$ . Assume  $H$  is associated with a Lagrangian convex with respect to  $\dot{x}$ . Let  $x_0, y_0 \in M$  be non conjugate points along an extremal curve  $\gamma_0(t)$  such that  $x_0 = \gamma_0(0)$  and  $y_0 = \gamma_0(t_0)$ , and  $(x_0, \xi_0), (y_0, \eta_0)$  the corresponding points in  $T^*M$ .

**Proposition 2.2** [CdV, Thm 14, pdf p.45]: *Let  $(t_0, x_0, y_0)$  be as above. Then for any  $(x, y)$  close to  $(x_0, y_0)$ , and  $t$  close to  $t_0$ , there is a unique extremal curve  $\gamma$  such that  $x = \gamma(0)$  and  $y = \gamma(t)$ . Let*

$\tilde{S}(t, x, y)$  be the action along these curves (minimizing the Lagrangian action) This is a generating function for the Hamiltonian flow near  $(x_0, \xi_0)$ , verifying HJ equation

$$(2.33) \quad \partial_t \tilde{S} + H(y, \partial_y \tilde{S}) = 0$$

This is verified in the Riemannian case  $\tilde{S}(t, x, y) = F(x, y)/2t = \text{dist}^2(x, y)/2t$  where  $F$  is the exact symplectic twist map considered above, and can be identified with the phase in the Heat kernel. We can check (2.33) trivially when  $H = \frac{1}{2}p^2$ . This holds also under assumption (2.30). Clearly under Hypothesis (2.18), (2.33) extends (2.15) for large times.

So far we have assumed some convexity of  $H$  with respect to  $p$ . The case  $m = 1$  (Finsler metric and Randers symbols) is investigated in [Ta] : it turns out that similar results hold when the *square* of Finsler metric or Randers symbol enjoys some convexity property, so for a “conformal metric” the case  $m = 1$  makes no difference.

## 2.6 Maupertuis-Jacobi correspondence

Geodesic flows are often used in Maupertuis-Jacobi correspondence. Let  $M$  be a smooth manifold, and  $\mathcal{H}_0, H_0 \in C^\infty(T^*M)$  two Hamiltonians, possessing a common regular energy surface  $\Sigma = \{\mathcal{H}_0 = \mathcal{E}\} = \{H_0 = E\}$ . Then  $\mathcal{H}_0, H_0$  have the same integral curves  $(\mathcal{X}(\tau), \mathcal{P}(\tau)) = (X(t), P(t))$  on  $\Sigma$ , up to a reparametrization of time. Hamiltonian vector fields are related by  $v_{\mathcal{H}_0} = \mathcal{G}(x, p)v_{H_0}$ , parametrizations by  $dt = \mathcal{G}(\tau)d\tau$  for some smooth (positive) function  $\mathcal{G}$ . We say that  $(\mathcal{H}_0, \mathcal{E})$  and  $(H, E)$  satisfy Maupertuis-Jacobi correspondence and write  $(\mathcal{H}, \mathcal{E}) \sim (H, E)$ .

*Example 2.3:*  $H(x, p) = p^2 + V$  at energy  $E$  and  $\mathcal{H}_E(x, p) = \frac{p^2}{E - V(x)}$  at energy 1 satisfy Maupertuis-Jacobi correspondence.

*Example 2.4:* Let  $H(x, p) = \frac{p^2}{1+x^2}$  and  $\mathcal{H}(x, p) = p^2 - x^2$ . Since  $(H, 1)$  and  $(\mathcal{H}, 1)$  are mapped to each other by Maupertuis-Jacobi correspondence, we know that  $P(t, \psi) = \omega(\psi) \cosh f(t)$ ,  $X(t, \psi) = \omega(\psi) \sinh f(t)$ , where  $f$  satisfies the ODE  $f'(t) = \frac{2}{\cosh^2 f(t)}$ ,  $f(0) = 0$ . Integrating, we find  $f + \frac{1}{2} \sinh(2f) = 4t$ , so  $f(t) > 0$  for all  $t > 0$ . Another example with a frequency vector is  $H(x, p) = \frac{p^2}{1+\nu^2 \cdot x^2}$ .

*Example 2.5:*  $H(x, p) = |p|^m \frac{1}{\rho(x)}$  above;  $\mathcal{H}$  is for instance the water-wave Hamiltonian  $\mathcal{H}(x, p) = |p|(1 + \mu(x)p^2) \tanh(D(x)|p|)$ , see [DoRo], [DoMiRo], [ReDoKaMi].

Semi-classical Green functions for  $(\mathcal{H}, \mathcal{E})$  and  $(H, E)$ , i.e. solutions of (1.5) with different Hamiltonians are mapped diffeomorphically to each other. It could be interesting to compare the construction of corresponding Maslov canonical operators at each step (phase function, density and so on...)

### 3. Maslov canonical operators and bi-Lagrangian distributions

Our purpose is to describe the solution globally, including unfolding of Lagrangian singularities; this is of primary importance in the context of wave propagation. Among many references to the subject we make use in particular of [M], [Hö], [Du], [Iv], [BaWe], [CdV], [GuSt], [DoZh], [DoNaSh], [DoRo]

#### 3.1. Lagrange immersions and non-degenerate phase functions:

Consider a vector bundle  $\pi_{\mathcal{B}} : \mathcal{B} \rightarrow M$ , with local section  $\mathbf{R}^N$ , for some  $N$  that may depend on the base point  $x \in M$ . The function  $\Phi : (x, \theta) \mapsto \Phi(x, \theta)$  on  $\mathcal{B}$ ,  $\theta \in \mathbf{R}^N$ , defined near  $(x_0, \theta_0)$  with  $\xi_0 = \partial_x \Phi(x_0, \theta_0)$  is called a non-degenerate phase function (definition of [Hö]) iff  $d_{x, \theta} \Phi(x_0, \theta_0) \neq 0$ , and  $d\partial_{\theta_1} \Phi, \dots, d\partial_{\theta_N} \Phi$  are linearly independent on the critical set (as in (2.16))

$$(3.5) \quad C_{\Phi} = \{(x, \theta) \in \mathcal{B} : \frac{\partial \Phi}{\partial \theta}(x, \theta) = 0\}$$

This last condition amounts (definition of [Iv]) to say that the rank of the  $(N+n) \times N$ -matrix  $(\phi''_{\theta x}, \phi''_{\theta \theta})$  equals to  $N$  on  $C_{\Phi}$ . Or again, (definition of [BaWe], [CdV]), that the Lagrangian submanifold  $d\phi(\mathcal{B})$  of  $T^*\mathcal{B}$  be transverse to  $E^{\perp}$ , where  $E = \text{Ker}(\pi_{\mathcal{B}})$ , i.e. the fiber-derivative of  $\phi$  is transverse to the zero-section of  $\mathcal{B}$  (transversality in the sense of Bott). See also [GuSt2, Sect.4.9&5] for a systematic approach.

Let  $\iota : \Lambda \rightarrow T^*M$  be a Lagrangian immersion. We know that there exists a covering of  $\Lambda$  by canonical charts  $U$ , such that  $\Lambda$  is parametrized in each  $U$  by a non-degenerate phase function. Thus the bundle  $\mathcal{B} = M \times \mathbf{R}^N \rightarrow M$ , has constant fiber  $\mathbf{R}^N$  over  $U$ .

Let also  $\pi_{\Lambda} : \Lambda \rightarrow M$  be the natural projection. The Lagrangian immersions  $\iota$  and

$$(3.7) \quad \iota_{\Phi} : C_{\Phi} \rightarrow T^*M, \quad (x, \theta) \mapsto (x, \frac{\partial \Phi}{\partial x})$$

have the same image on  $U$  and  $C_{\phi}$  is a submanifold of  $\mathcal{B}$  of dimension  $n$ . In particular,  $\iota_{\Phi} : C_{\Phi} \rightarrow \Lambda$  is a diffeomorphism onto its image. It is easy to prove [Iv, (1.2.7)] that

$$(3.9) \quad N - \text{rank } \Phi''_{\theta \theta} = n - \text{rank } d\pi_{\Lambda}(\iota_{\Phi}(x, \theta))$$

We set  $k = \text{rank } d\pi_{\Lambda}(\iota_{\Phi}(x, \theta))$ , and say that  $U$  has rank  $k$ . If  $k = n$ ,  $U$  is called a “regular” chart, and  $\Lambda$  is called “projectable” or “horizontal” on  $U$ . On the other extreme, if  $k = 0$ ,  $U$  is called a “maximally singular” chart, and  $\Lambda$  is called “vertical” on  $U$ .

#### 3.2 Maslov canonical operator for Lagrangian distributions.

Let  $f_h = f(\cdot; h)$  be a semi-classical Lagrangian distribution (or oscillatory integral) i.e. locally

$$(3.11) \quad f(x; h) = \int^* e^{i\varphi(x, \theta)/h} a(x, \theta; h) d\theta$$

where  $\varphi(x, \theta)$  is a non-degenerate phase function in the sense of [Hö], and  $a(x, \theta; h) = a_0(x, \theta) + ha_1(x, \theta) + \dots$  an amplitude. With  $f$  we associate the critical set  $C_{\varphi} = \{(x, \theta) \in M \times \mathbf{R}^N : \partial_{\theta} \varphi = 0\}$



and  $\iota_\varphi : C_\varphi \rightarrow T^*M$  with image the Lagrangian submanifold  $\Lambda_\varphi = \{(x, \partial_x \varphi) : (x, \theta) \in C_\varphi\}$ . Choose local coordinates  $\xi \in \mathbf{R}^n$  on  $\Lambda_\varphi$  and define the half-density in the local chart  $(C_\varphi, \iota_\varphi)$  by  $\sqrt{d\mu_\varphi} = |\det \varphi''|^{-1/2} |d\xi|^{1/2}$ . The (oscillating) principal symbol of  $f$  in  $\Lambda_\varphi$  is then defined as

$$(3.12) \quad e^{i\phi(\xi)/h} A_0(\xi) = e^{i\phi(\xi)/h} e^{i\pi \operatorname{sgn} \varphi''/4} a_0(x(\xi), \theta(\xi)) \sqrt{d\mu_\varphi}$$

where  $\phi$  is a “reduced phase function”. Conversely, assume  $\iota : \Lambda \rightarrow T^*M$  is a smooth Lagrangian immersion, with a smooth positive half-density  $\sqrt{d\mu}$ , it can be parametrized locally by phase functions  $\varphi$  in canonical charts  $\iota_\varphi : U_\varphi \rightarrow T^*M$ . These phases can be chosen coherently, and define a class of “reduced phase functions”  $\phi$ , parametrizing  $\iota$  locally. This gives the fibre bundle of phases  $\mathcal{L}_h$ , including Maslov indices, equipped with transition functions. We are also given local smooth half-densities  $|d\mu_\varphi|^{1/2}$  on  $\Lambda$ , defining the fibre bundle of half-densities  $\Omega^{1/2}$ , equipped with transition functions. The collection of these objects make a fibre bundle  $\Omega^{1/2} \otimes \mathcal{L}_h$  over  $\Lambda$ . A section of  $\Omega^{1/2} \otimes \mathcal{L}_h$  will be written as

$$(3.13) \quad f(x; h) = [K_{(\Lambda, \mu)}^h A](x; h)$$

where  $K_{(\Lambda, \mu)}^h$  is called Maslov canonical operator. The set of such Lagrangian distributions microlocally supported on  $\Lambda$  will be sometimes denoted by  $I(M; \Lambda)$ . The “reduced phase function” and the “principal symbol” of  $f$  are defined invariantly. See e.g. [DoNaSh] for details.

Recall in particular the following standard result see e.g. [Iv, Thm.1.2.8]. Let  $u \in I(M, \Lambda)$  be a Lagrangian distribution locally of the form  $u = \int e^{i\Phi(x, \theta)/h} a(x, \theta; h) d\theta$  and  $B$  a  $h$ -PDO with principal symbol  $B(x, p)$ . Then the same holds for  $Bu$ , its amplitude  $b(x, \theta; h)$  has leading term

$$(3.15) \quad b_0(x, \theta) = B(x, \partial_x \Phi(x, \theta)) a_0(x, \theta)$$

and its principal symbol on  $\Lambda$  is related to this of  $u$  by  $b^0(x, p) = B(x, p) a^0(x, p)$ . Moreover if  $B$  has sub-principal symbol  $B_1$ , and  $B(x, p) = 0$  on  $\Lambda$ , then  $hBu$  is again a Lagrangian distribution with principal symbol on  $\Lambda$

$$(3.16) \quad c^0(x, p) = \frac{1}{i} \{B, a^0\} + B_1 a^0$$

In order that (3.11) solves a PDE such that  $(H - E)u_h = f_h \bmod \mathcal{O}(h^2)$ ,  $\Phi$  needs to satisfy HJ equation, and  $a_0$  a transport equation. If  $H(x, p; h) = H_0(x, p) + hH_1(x, p) + \dots$ , the principal symbol of (3.11) is given by  $\frac{1}{i} \mathcal{L}_{v_{H_0}} a + H_1 a \bmod \mathcal{O}(h)$ , which we require to vanish  $\bmod \mathcal{O}(h)$  on  $C_\Phi$ ; this gives

$$(3.17) \quad \left( \frac{1}{i} \mathcal{L}_{v_{H_0}}|_\Lambda + H_1 \right) a_0(x(\xi), \theta(\xi)) \sqrt{d\mu_\varphi} = 0$$

Note that on  $\Lambda_+$ , we can replace Lie derivative  $\mathcal{L}_{v_{H_0}}$  by  $d/dt$ . Provided  $\Phi$  is a non-degenerate phase-function, (3.17) admits a global solution, computed on each canonical chart. For instance on a regular chart, this is just WKB construction. In a totally singular chart instead, we solve (3.17) in Fourier representation, the example of Helmholtz (Schrödinger) operator is given e.g. in [DoRo].

The function  $a_0(x(\xi), \theta(\xi))$  is smooth in  $\xi$ , but of course when expressed in  $x$ -variable, singularities may occur due to singular Jacobians at focal points. For the sake of simplicity, we have discarded Maslov indices, see e.g. [Iv], [BaWe, Sect.4], [DoNaSh] for a complete description.

### 3.3 Maslov canonical operator for bi-Lagrangian distributions and symbolic calculus.

We want to describe oscillatory integrals microlocally supported near a Lagrangian intersection, i.e. on  $\Lambda \cup \Lambda^+$ . As in [MelUhl], we extend Maslov canonical operator by introducing amplitudes (or symbols) verifying the “compatibility” condition. Semi-classical distributions used in [MelUhl] (adapted to the semi-classical case), are locally of the form (see (1.8))

$$(3.18) \quad u(x; h) = \int_0^\infty dt \int^* e^{i(x'\xi' + (x_n - t)\xi_n)/h} b(t, x, \xi) d\xi$$

where  $b$  is a classical symbol. We say that  $u_h(x) = u(x; h)$  is a *bi-Lagrangian distribution* on the intersecting pair  $(\Lambda, \Lambda_+^0)$ .

Recall  $\text{WF}_h(u_h) \subset \Lambda \cup \Lambda_+^0$ , which is the conormal bundle of the manifold with boundary  $x' = 0, x_n \geq 0$ .

Let  $I(M, \Lambda, \Lambda_+^0)$  be the set of bi-Lagrangian distributions. Using in particular Lemma 1.1 it can be shown that it is stable under  $h$ -FIO's preserving the Lagrangian intersection, so there is no loss of generality here in assuming  $\Lambda^+ = \Lambda_+^0$ .

Ignoring non-vanishing densities and Maslov contributions, we find that, if  $u_h$  is a bi-Lagrangian distribution on  $(\Lambda, \Lambda_+^0)$ , then the principal symbol of  $u_h$  as a Lagrangian distribution near  $\Lambda_+^0 \setminus L_0$ ,  $L_0 = \partial\Lambda_+^0$ , is given by  $\sigma^+(u_h) = a(x_n, (0, x_n), (\xi', 0))$ , while the principal symbol of  $u_h$  as a Lagrangian distribution near  $\Lambda \setminus L_0$  equals  $\sigma(u_h) = -ia(0, 0, \xi)/\xi_n$  on  $\Lambda \setminus L_0$ . Observe that  $\sigma^+$  is smooth up to  $L_0$  and  $\xi_n \sigma(u_h) = -i\sigma^+(u_h)$  on  $L_0$ , which we call the *compatibility condition*.

The space of principal symbols that satisfy the compatibility condition is stable under the action of  $h$ -PDO, and denoted by  $S(\Lambda, \Lambda_+^0, \Omega^{1/2} \otimes \mathcal{L}_h)$ . As in [MelUhl] one can show that

$$(3.20) \quad I(M, \Lambda, \Lambda_+^0) \rightarrow S(\Lambda, \Lambda_+^0, \Omega^{1/2} \otimes \mathcal{L}_h), \quad u_h \mapsto \tilde{\sigma}(u_h) = \tilde{\sigma}(u_h)$$

is surjective. This allows to define the “restriction operator”  $R : \tilde{\sigma}(u_h) \mapsto \sigma^+(u)|_{L_0}$ . Thus there is a well-defined Maslov bundle over  $\Lambda, \Lambda_+^0$ .

At this stage, we prefer to write  $\tilde{\sigma}(b)$  instead of  $\tilde{\sigma}(u_h)$ , since  $\tilde{\sigma}(u_h)$  depends only on the amplitude.

Adapting [MelUhl, Prop.2.3] to asymptotics with respect to  $h$ , we can show that if  $u$  is a bi-Lagrangian distribution on  $(\Lambda, \Lambda^+)$ , and  $B$  is a  $h$ -PDO with  $\text{WF}_h(B) \cap \Lambda = \text{WF}'_h(B) \cap \Lambda = \emptyset$ , then  $Bu$  is a Lagrangian distribution supported on  $\Lambda_+^0$ , while if  $B$  is a  $h$ -PDO with  $\text{WF}_h(B) \cap \Lambda_+^0 = \emptyset$ , then  $Bu$  is a Lagrangian distribution supported on  $\Lambda$ . The main result on symbolic calculus for bi-Lagrangian distributions [MelUhl, Prop.5.4] is an extension of (3.15)-(3.16), which we rephrase as follows:

**Proposition 3.3:** *Let  $u_h \in I(M, \Lambda, \Lambda_+^0)$  be a bi-Lagrangian distribution with symbol*

$$\tilde{\sigma}(u_h) \in S(\Lambda, \Lambda_+^0, \Omega^{1/2} \otimes \mathcal{L}_h)$$

(i) Let  $B(x, hD_x; h)$  be a  $h$ -PDO with principal symbol  $B(x, p)$ . Then  $Bu_h \in I(M, \Lambda, \Lambda_+^0)$  is again a bi-Lagrangian distribution with symbol  $B(x, p)\tilde{\sigma}(u_h) \in S(\Lambda, \Lambda_+^0, \Omega^{1/2} \otimes \mathcal{L}_h)$ , identified with  $(B|_{\Lambda}\sigma(u_h), B|_{\Lambda_+^0}\sigma^+(u_h))$ .

(ii) Assume moreover  $B(x, p)$  vanishes on  $\Lambda_+^0$  and  $B(x, hD_x; h)$  has sub-principal symbol  $B_1$ . Then  $Bu_h = v_h + hw_h$ , where  $v_h \in I(M, \Lambda)$  and  $w_h \in I(M, \Lambda, \Lambda_+^0)$ . Their principal symbols (with the identification above) are given by

$$(3.23) \quad \begin{aligned} \tilde{\sigma}(v_h) &= B(x, p)\tilde{\sigma}(u_h)|_{\Lambda} \approx (B(x, p)\sigma(u_h), 0) \\ \tilde{\sigma}(w_h)|_{\Lambda_+^0} &\approx \left(\frac{1}{i}\{B, \tilde{\sigma}(u_h)\} + B_1\tilde{\sigma}(u_h)\right)|_{\Lambda_+^0} \end{aligned}$$

From this Proposition, it is easy to recover (at least locally) as in [UhlMel, Prop.6.6], the solution  $u_h$  of  $H(x, hD_x)u_h = f_h$  as a bi-Lagrangian distribution  $u_h \in I(M, \Lambda, \Lambda_+^0)$ .

Let us try to formulate these ideas more globally, replacing  $\Lambda_+^0$  by  $\Lambda_+$ . Consider the fibration  $\Lambda_+ = \bigcup_{t \geq 0} \Lambda_t$ . Define as in (3.18) (forgetting about densities and Maslov indices)

$$u_h(x) = \int_0^\infty dt [K_{\Lambda_t}^h b](t, x; h)$$

where  $b$  is an amplitude, whose principal symbol in a canonical chart takes the form  $B_0(\xi)$  as in (3.12). Let

$$[K_{\Lambda, \Lambda_+}^h b](x; h) = \int_0^\infty [K_{\Lambda_t}^h b](t, x; h)$$

Then the map  $b \mapsto [K_{\Lambda, \Lambda_+}^h b]$  is linear, and defines Maslov canonical operator for bi-Lagrangian distributions at first order in  $h$  as

$$(3.25) \quad [K_{\Lambda, \Lambda_+}^h b](x; h) = [K_{\Lambda, \Lambda_+}^h \tilde{\sigma}(B_0)](x; h)$$

where in (3.21), we have changed  $\tilde{\sigma}(u_h)$  to  $\tilde{\sigma}(b)$  according to our convention.

Let us write (1.5) in terms of Maslov canonical operator for bi-Lagrangian distributions. At first order in  $h$  we can write the initial data as  $f_h = [K_{\Lambda}^h A_0]$ . We construct the solution according to the standard scheme (HJ and transport equations) which determines an amplitude  $b$ , depending linearly on  $a$ . We write symbolically, according to (3.25)

$$(3.26) \quad u_h(x) = [K_{\Lambda, \Lambda_+}^h \tilde{\sigma}(B_0)](x; h)$$

where  $B_0$  depends linearly on  $A_0$ . Actually we shall determine  $B_0$  by constructing, at first order, the pair  $(\sigma(u_h), \sigma^+(u_h))$ .

### 3.4 The constant coefficient case.

In general it is difficult to obtain a decomposition of  $u_h$  adapted to the splitting (3.21), i.e.  $u_h = u_h^0 + u_h^1$  where  $\text{WF}_h(u_h^0) \subset \Lambda$  and  $\text{WF}_h(u_h^1) \subset \Lambda_+^0$ . A “weak form” of this property is given in [MelUh, Prop.2.3]. In [AnDoNaRo3] we consider Helmholtz operator with constant coefficient in 2-D, when  $f$  is radially symmetric and has compact support (instead of  $\mathcal{F}f$ ); its Fourier transform

$g = \mathcal{F}_1 f$  is again of the form  $g(p) = g(|p|) = g(r)$  and extends holomorphically to  $\mathbf{C}^2$ , so that we can compute explicitly  $u_h$  by contour integrals when  $E = k^2$ ,  $k > 0$ . As a result (after taking  $\varepsilon \rightarrow 0$ ),

$$u_h^0(x) = \frac{i\pi g(k)}{(2\pi h)^2} \int_{-\pi/2}^{\pi/2} \exp[i|x|k \cos \theta/h] d\theta$$

with  $\text{WF}_h u_h^0 \subset \{x = 0\} \cup \{(x, k\frac{x}{|x|}), x \neq 0\} = \Lambda \cup \Lambda_+$ . Consider now  $u_h^1$ , after taking  $\varepsilon \rightarrow 0$ ,

$$u_h^1(x) = \frac{1}{(2\pi h)^2} \int \exp[i|x|r \cos \theta/h] \frac{g(r)}{r + k + i\varepsilon} dr d\theta$$

We set  $\tilde{g}(r) = \frac{g(r)}{r(r+k)}$ , and assume for simplicity  $\tilde{g}(r)\sqrt{r} \in L^1(\mathbf{R}_+)$ , we have  $u_1(x) = H_0(\tilde{g})(\frac{|x|}{h})$ , where  $H_0$  denotes Hankel transform of order 0. Let  $\chi \in C_0^\infty(\mathbf{R}^2)$  be radially symmetric, and equal to 1 near 0, since  $\text{WF}_h f_h = \{x = 0\}$ , we have

$$g = \mathcal{F}_h(\chi f_h) + \mathcal{O}(h^\infty) = (2\pi h)^{-2} \mathcal{F}_h(\chi) * g + \mathcal{O}(h^\infty)$$

so in the expression for  $u_1$  we may replace mod  $\mathcal{O}(h^\infty)$ ,  $\tilde{g}(r)$  by a constant times  $\hat{g}(r) = \frac{(\mathcal{F}_h(\chi) * g)(r)}{r(r+m)}$  (see [Bad] for 2-D convolution and Fourier transform in polar coordinates). To estimate  $\text{WF}_h u_1$ , we compute again the Fourier transform of  $(1 - \tilde{\chi})\hat{g}$  where  $\tilde{\chi}$  is a cut-off equal to 1 near 0, and we find it is again  $\mathcal{O}(h^\infty)$  if  $\chi \equiv 1$  on  $\text{supp } \tilde{\chi}$ . This shows that  $\text{WF}_h u_1 \subset \{x = 0\}$ .

Note that this Example makes use of Bessel function  $J_0(\frac{|x|}{h})$ , we shall return to such “localized functions” in Sect.5.

#### 4. $f_h$ is supported microlocally on the vertical plane

Consider the case where  $H_0$  is positively homogeneous of degree  $m \geq 1$  with respect to  $p$  and  $f_h$  is microlocally concentrated on the vertical plane  $\Lambda = \{x = 0\}$ , e.g.  $f_h(x) = h^{-n}f(\frac{x}{h})$ , with  $\mathcal{F}_1 f \in \mathcal{S}(\mathbf{R}^n)$ .

##### 4.1 Some non-degeneracy condition.

Recall first from [DoNaSh, Lemma 6] the following result. Let  $\tilde{\iota} : \tilde{\Lambda} \rightarrow T^*\tilde{M}$  be a Lagrangian embedding of dimension  $\tilde{n}$ ,  $U \subset \tilde{\Lambda}$  a connected simply connected open set,

$$(\tilde{\phi}, \tilde{\psi}) = (\phi_1, \dots, \phi_k, \psi_1, \dots, \psi_{\tilde{n}-k})$$

local coordinates on  $U$ . Here we don't assume that  $k$  is the rank of  $d\pi_x : \tilde{\Lambda} \rightarrow \tilde{M}$ . Thus  $\tilde{\Lambda}$  is defined by  $x = X(\tilde{\phi}, \tilde{\psi}), p = P(\tilde{\phi}, \tilde{\psi})$  in the chart  $U$ . Let  $\Pi(\tilde{\phi}, \tilde{\psi})$  be a smooth  $\tilde{n} \times k$  matrix function defined in  $U$  such that:

$$(4.1) \quad \Pi^*(\tilde{\phi}, \tilde{\psi}) X_{\tilde{\psi}}(\tilde{\phi}, \tilde{\psi}) = \text{Id}_{k \times k}$$

$$(4.2) \quad \kappa : (\tilde{\phi}, \tilde{\psi}) \mapsto (X(\tilde{\phi}, \tilde{\psi}), \tilde{\psi}) \text{ is an embedding}$$

Then there is a neighbhd  $V$  of  $\kappa(U)$  such that the system

$$(4.3) \quad \Pi^*(\tilde{\phi}, \tilde{\psi})(x - X(\tilde{\phi}, \tilde{\psi})) = 0, \quad (x, \tilde{\psi}) \in V$$

has a unique smooth solution  $\tilde{\phi} = \tilde{\phi}(x, \tilde{\psi})$  satisfying the condition  $X(\tilde{\phi}(x, \tilde{\psi}), \tilde{\psi}) = x$ , when  $(x, \tilde{\psi}) \in \kappa(U)$ .

For  $(\tilde{\phi}, \tilde{\psi}) \in U$ , define the  $\tilde{n} \times \tilde{n}$  matrix

$$(4.4) \quad \mathcal{M}(\tilde{\phi}, \tilde{\psi}) = (\Pi(\tilde{\phi}, \tilde{\psi}); P_{\tilde{\psi}}(\tilde{\phi}, \tilde{\psi}) - P_{\tilde{\phi}}(\tilde{\phi}, \tilde{\psi})\Pi^*(\tilde{\phi}, \tilde{\psi})X_{\tilde{\psi}}(\tilde{\phi}, \tilde{\psi}))$$

As we shall see, invertibility of  $\mathcal{M}(\tilde{\phi}, \tilde{\psi})$  plays an important role [DoMaNa2], [DoNaSh].

Consider now our special setting where  $H$  is positively homogeneous of degree  $m$ ,  $\Lambda$  is the vertical plane, and recall  $\langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau) \rangle = mH$  from (2.6). Here  $\tau$  is taken as a parameter, everything depends smoothly on  $\tau$  and  $\Lambda_+(0) = \Lambda_+$ . So  $\Pi(t, \psi, \tau) = \frac{1}{mH}P(t, \psi, \tau)$  is a left inverse of  $\dot{X}$ :  $\Pi^*\dot{X} = \frac{1}{mH}\langle P, \dot{X} \rangle = 1$ . Further, the map  $\Lambda_+ \rightarrow \mathbf{R}^n$ ,  $(t, \psi) \mapsto (X(t, \psi, \tau), \psi)$  is clearly an embedding. This fulfills conditions (4.1), (4.2) above for  $\tilde{\Lambda} = \Lambda_+(\tau)$ , with  $\tilde{n} = n$ ,  $k = 1$  and  $\tilde{\phi} = t$  and  $\tilde{\psi} = \psi$ . So the system  $\Pi^*(t, \psi, \tau)(x - X(t, \psi, \tau)) = \langle P, x - X(t, \psi, \tau) \rangle = 0$  has a unique solution  $t = t_0(x, \psi, \tau)$  satisfying the condition  $X(t_0(x, \psi, \tau), \psi, \tau) = x$ , and this solution is a smooth function.

Moreover by (2.6) again, the matrix

$$(4.5) \quad \mathcal{M}(t, \psi) = (\Pi(t, \psi), P_{\psi}(t, \psi) - \dot{P} \frac{1}{mH} {}^t P X_{\psi}) = (\frac{1}{mH} P, P_{\psi})$$

has determinant  $\frac{1}{mH} \det(P, P_{\psi})$ . As we shall see, it turns out that  $\det \mathcal{M}$  gives the invariant (inverse) density on  $\Lambda_+$ .

*Example 4.1:* Let us compute  $\det(P, P_\psi)$  at  $t = 0$  for a geodesic flow  $H(x, p)$ , on the energy shell  $E = 1$  when  $n = 2$  or  $n = 3$ . When  $n = 2$ , up to a change of  $x$  coordinates such that at  $x = 0$ , the metric  $H(0, p)$  takes the diagonal form  $H(x, p) = \frac{p_1^2}{a_1^2} + \frac{p_2^2}{a_2^2}$  (elliptic polarization), and  $P = (a_1 \cos \psi, a_2 \sin \psi)$ . Hence  $\det(P, P_\psi) = a_1 a_2$  at  $x = 0$ , and for small  $|t|$  we have  $\det(P, P_\psi) > 0$ . For  $n = 3$ ,  $H(0, p) = \frac{p_1^2}{a_1^2} + \frac{p_2^2}{a_2^2} + \frac{p_3^2}{a_3^2}$ , and in spherical coordinates  $(\psi_1, \psi_2)$  where  $0 < \psi_1 < \pi$ , we find  $\det(P, P_\psi) = a_1 a_2 a_3 \sin \psi_1 > 0$  for small  $t$  and away from the poles  $(0, 0, \pm 1)$ .

*Example 4.2:* When  $H(x, p) = \frac{|p|^m}{\rho(x)}$ , recall from (2.3) that  $P(\psi, \tau) = |P|_\tau \omega(\psi)$  at  $t = 0$ . Since  $\det(\omega(\psi), \omega^\perp(\psi)) = 1$ , again we have  $\det(P, P_\psi) \neq 0$  for small  $t$ .

#### 4.2 Construction of the phase function and half-density.

First we construct a generating function  $\Phi$  of  $\Lambda_+$  using HJ theory in eikonal coordinates, and verifying  $\Phi|_{t=0} = \langle x, \omega(\psi) \rangle$ . There are several ways to do that. The first one gives

$$(4.7) \quad \Phi_0(x, t, \psi, \tau) = mEt + \langle P(t, \psi, \tau), x - X(t, \psi, \tau) \rangle$$

with initial condition  $\Phi(x, 0) = \langle p, x \rangle$ , where again  $\tau + H = E$ . The “ $\theta$  variables” in Hörmander’s definition are then  $(\psi, \tau)$ . In the simple example where  $n = 1$ ,  $\tau + H(x, p) = \tau + p = E$ , (there are no variable  $\psi$ ),  $\Phi_0 = Et + p(x - t)$ .

The second one consists [DoMaNaTu2] in choosing a new coordinate  $\lambda = \lambda(\tau)$ ,  $\lambda(0) = 1$ , on  $\Lambda$  completing the  $\psi$  variables, such that  $\partial\Lambda_+$  is given by  $\lambda = 1$ ; it will play the role of a “Lagrange multiplier”. We define

$$(4.8) \quad \Phi(x, t, \psi, \lambda) = mEt + \lambda \langle P(t, \psi), x - X(t, \psi) \rangle$$

where now  $(X, P)$  are evaluated on  $\Lambda_+$  (and not on  $\Lambda_+(\tau)$ ). The “ $\theta$  variables” in Hörmander’s definition are then  $(\psi, \lambda)$ . In the Example above,  $\Phi = Et + \lambda p(x - t)$ .

It is more convenient to work with  $\Phi$  than with  $\Phi_0$ . Variables  $\tau$  and  $\lambda$  are diffeomorphically mapped onto each other. In case (1.15) this goes as follows as follows : comparing (4.7) with (4.8) at  $t = 0$ , we get  $P(\psi, \tau) = \lambda P(\psi)$ , with  $P(\psi, \tau)$  as in (2.3). But  $|P|_\tau^m = \rho(0)(E - \tau)$ , while  $|P|^m = \rho(0)E$ , hence

$$(4.9) \quad \lambda = \left(1 - \frac{\tau}{E}\right)^{1/m}$$

A similar scaling holds in Example 4.1.

**Proposition 4.1:** *Let  $H(x, p)$  be positively homogeneous of degree  $m \geq 1$  with respect to  $p$  on  $T^*M \setminus 0$ , and  $\det(P, P_\psi) > 0$ . Then  $\Phi(x, t, \psi, \lambda)$  given in (4.8) is a non-degenerate phase function defining  $\Lambda_+$ , and solves HJ Eq. (2.5), with initial condition  $\Phi|_{t=0} = \langle x, p \rangle$ . The positive invariant (inverse) density on  $\Lambda_+$  is given by*

$$(4.10) \quad F[\Phi, d\mu_+]|_{C_\Phi} = mE \det(P, P_\psi) > 0$$

The critical set  $C_\Phi$  is then determined by  $x = X(t, \psi)$  (which can be inverted as  $t = t(x, \psi)$ ) and  $\lambda = 1$ . It coincides with the set  $\kappa(U)$  defined in (4.2)-(4.3).

*Proof:* We have using (2.6)

$$(4.11) \quad \partial_t \Phi = \dot{\Phi} = mE + \lambda \langle \dot{P}, x - X(t, \psi) \rangle - \lambda \langle P, \dot{X} \rangle = mE(1 - \lambda) + \lambda \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle$$

so  $\partial_t \Phi = 0$  and  $\partial_x \Phi = P(t, \psi)$  along  $x = X(t, \psi)$  when  $\lambda = 1$ . We are left to show that  $\Phi$  is non degenerate phase function, with  $(\psi, \lambda)$  as “ $\theta$ -parameter”s. From (4.8)

$$(4.12) \quad \begin{aligned} \partial_\psi \Phi &= \lambda \langle \partial_\psi P(t, \psi), x - X(t, \psi) \rangle \\ \partial_\lambda \Phi &= \langle P(t, \psi), x - X(t, \psi) \rangle \end{aligned}$$

Let us add  $t$  to the “ $\theta$ -variables”, and consider the variational system  $\partial_t \Phi = \partial_\psi \Phi = \partial_\lambda \Phi = 0$ , which determines the critical set  $C_\Phi$ . Last 2 equations  $\partial_\psi \Phi = 0, \partial_\lambda \Phi = 0$  give an homogeneous linear system in  $x - X(t, \psi)$  with determinant  $\det(P, P_\psi)$ .

So at least for small  $t$  the phase is critical with respect to  $(\psi, \lambda)$  precisely for  $\lambda = 1$  and  $x = X(t, \psi)$ , in particular it is critical along  $\partial\Lambda_+$  when  $\lambda = 1$ . By the discussion above and [DoNaSh, Lemma 6], we find that  $\langle P(t, \psi), x - X(t, \psi, \tau) \rangle = 0$  when  $\lambda = 1$  has a unique solution  $t = t_1(x, \psi) = t_0(x, \psi, \tau = 0)$  satisfying the condition: if  $(x; \psi, \lambda = 1) \in C_\Phi$ , then  $X(t_1(x, \psi), \psi) = x$ . We recall  $t_0$  is defined in the discussion after (4.4). Moreover  $t_1$  is the critical point of  $t \mapsto \Phi$  when  $\lambda = 1$ .

Condition  $\det(P, P_\psi) > 0$  (which holds at  $t = 0$ ) actually ensures that  $\Phi$  is a non degenerate phase function, i.e. the vectors  $(d\partial_t \Phi, d\partial_\psi \Phi, d\partial_\lambda \Phi,)$  are linearly independent on the set  $x = X(t, \psi)$ . Namely, look at the variational system and compute on  $C_\Phi$  the differentials

$$(4.13) \quad \begin{aligned} d\dot{\Phi} &= -mE d\lambda + \lambda \langle \dot{P}(t, \psi), dx - dX(t, \psi) \rangle \\ d(\partial_\psi \Phi) &= \lambda \langle P_\psi(t, \psi), dx - dX(t, \psi) \rangle \\ d(\partial_\lambda \Phi) &= \langle P, dx - dX(t, \psi) \rangle \end{aligned}$$

Introduce the Jacobian (see e.g. [NaStSh], [DoNaSh] and references therein)

$$(4.14) \quad F[\Phi, d\mu_+]|_{C_\Phi} = \frac{dt \wedge d\psi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi)}{dx \wedge dt \wedge d\psi \wedge d\lambda}$$

Here  $dx$  is the volume form. Substituting (4.13) into  $\omega = dt \wedge d\psi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi)$  we get

$$\begin{aligned} \omega &= -mE d\psi \wedge dt \wedge d\lambda \wedge \langle P, dx - dX(t, \psi) \rangle \wedge \langle P_\psi, dx - dX(t, \psi) \rangle + \\ & d\psi \wedge dt \wedge \langle \dot{P}, dx - dX(t, \psi) \rangle \wedge \langle P, dx - dX(t, \psi) \rangle \wedge \langle P_\psi, dx - dX(t, \psi) \rangle \end{aligned}$$

Writing  $dX = \dot{X} dt + X_\psi d\psi$ , we check that the second term vanishes, so we are left with

$$\omega = -mE dt \wedge d\psi \wedge d\lambda \wedge dx \det(P, P_\psi)$$

which gives (4.10). So if  $\det(P, P_\psi) > 0$ ,  $\Phi$  is a non-degenerate phase function, and (4.10) the invariant (inverse) density on  $\Lambda_+$ . ♣

*Remark 4.1:* Let  $y$  be coordinates on  $C_\Phi = \{(x, t, \psi, \lambda) : \frac{\partial \Phi}{\partial(t, \psi, \lambda)} = 0\}$  we can also write (4.14) in the form [GrSj, Sect.11]

$$F[\Phi, d\mu_+]|_{C_\Phi} = \det \left( \begin{array}{cc} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial(t, \psi, \lambda)} \\ \frac{\partial^2 \Phi}{\partial x \partial(t, \psi, \lambda)} & \frac{\partial^2 \Phi}{\partial(t, \psi, \lambda)^2} \end{array} \right) |_{C_\Phi}$$

This formula makes use of the implicit relation between  $x$  and  $(t, \psi)$  such that  $x = X(t, \psi)$  when  $\lambda = 1$ . It is actually independent of the choice of coordinates on  $C_\Phi$  but it does depend on the choice of local coordinates  $x$ .

*Remark 4.2:* As mentionned in the last Example 1.1, we can also consider complex phase functions. For instance replace (4.8) by

$$\Phi(x, t, \psi, \lambda) = mEt + \lambda \langle P(t, \psi), x - X(t, \psi) \rangle + i \frac{\lambda \mu}{2} \langle P(t, \psi), x - X(t, \psi) \rangle^2$$

As in (4.11)-(4.12) we get

$$\begin{aligned} \partial_t \Phi &= mE(1 - \lambda - i\mu\lambda) + \langle \dot{P}, x - X \rangle + i\lambda\mu \langle P, x - X \rangle \langle \dot{P}, x - X \rangle \\ \partial_\psi \Phi &= \lambda \langle \partial_\psi P, x - X \rangle + i\lambda\mu \langle P, x - X \rangle \langle \partial_\psi P, x - X \rangle \\ \partial_\lambda \Phi &= \langle P, x - X \rangle + i \frac{\mu}{2} \langle P, x - X \rangle^2 \end{aligned}$$

so it has the same critical point as the real phase when  $\lambda = 1$ , with the initial condition  $\Phi_{t=0} = \langle x, \omega(\psi) \rangle + i \frac{\mu}{2} \langle x, \omega(\psi) \rangle^2$ .

Now we look for the critical points of the phase, and describe the Lagrangian singularities. Let (see Proposition A.1)

$$(4.16) \quad a = \langle \dot{P}, X_\psi \rangle = b = \langle P_\psi, \dot{X} \rangle, \quad c = \langle P_\psi, X_\psi \rangle$$

In the following we investigate some configurations of  $\Lambda_+$  near  $t = 0$ .

**Proposition 4.3:** *Let  $n = 2$  for simplicity. Let  $z \in \Lambda_+$  (possibly on  $\partial\Lambda_+$ ) and assume  $\Phi$  is a non-degenerate phase function near  $z$  (which holds true when  $z \in \partial\Lambda_+$ ). We have:*

(i) *Let  $z \in \Lambda_+$  such that  $\langle -\partial_x H(z), \partial_p H(z) \rangle = \langle \dot{X}, \dot{P} \rangle \neq 0$ . Then near  $z$  the rank of  $d\pi|_{\Lambda_+}$  is 1 when  $c = 0$ , 2 when  $c \neq 0$ .*

(ii) *Let  $z_0 = (X(t_0, \psi_0), P(t_0, \psi_0)) \in \Lambda_+$  be a special point for some  $(t_0, \psi_0)$ . Then the rank of  $d\pi|_{\Lambda_+}(z)$  is 1 or 2. When the rank is 1, the tangent space of the caustics at  $X(t_0, \psi_0)$  takes the form*

$$(4.17) \quad \frac{\partial x_1}{\partial \psi} \frac{\partial \lambda}{\partial x_1} + \frac{\partial x_2}{\partial \psi} \frac{\partial \lambda}{\partial x_2} = 0$$

where  $\frac{\partial \lambda}{\partial x}(x, \psi) \neq 0$ .

(iii) *Let  $z_0 = (X(t_0, \psi_0), P(t_0, \psi_0)) \in \Lambda_+$  be a residual point for some  $(t_0, \psi_0)$ , i.e.  $\dot{P}(t_0, \psi_0) = 0$ . Then the eikonal is  $mE dt = \langle P, dx \rangle \neq 0$  at  $z_0$ .*

*Proof:* (i) On  $C_\Phi$  we have  $\partial_t^2 \Phi = -\langle \dot{P}, \dot{X} \rangle \neq 0$ , so implicit function theorem shows that (for small  $t$ )  $\dot{\Phi} = 0$  is equivalent to  $t = t(\psi, \lambda)$ . Since we have eliminated  $t$ , the “ $\theta$ -parameters” are now  $(\psi, \lambda)$ , and we set

$$\Psi(x, \psi, \lambda) = \Phi(x, t(x, \psi, \lambda), \psi, \lambda)$$

Differentiating the relation  $\partial_t \Phi = 0$ , we get that on  $C_\Phi$  and  $\lambda = 1$

$$\frac{\partial t}{\partial \psi} = -\frac{\langle P_\psi, \dot{X} \rangle}{\langle \dot{P}, \dot{X} \rangle}, \quad \frac{\partial t}{\partial \lambda} = -\frac{mE}{\langle \dot{P}, \dot{X} \rangle}$$



and a straightforward computation using (4.16) and (2.6) yields

$$(4.18) \quad \Psi''_{\theta\theta} = \Psi''_{(\psi,\lambda)} = \begin{pmatrix} \frac{\langle \dot{P}, X_\psi \rangle^2}{\langle \dot{P}, \dot{X} \rangle} - \langle P_\psi, X_\psi \rangle & mE \frac{\langle \dot{P}, X_\psi \rangle}{\langle \dot{P}, \dot{X} \rangle} \\ * & \frac{(mE)^2}{\langle \dot{P}, \dot{X} \rangle} \end{pmatrix}$$

Applying (3.9) to the non-degenerate phase function  $\Phi$  with  $N = n = 2$ , we find that the rank of  $d\pi|_{\Lambda_+}(z)$  is 1 when  $c = 0$  or 2 when  $c \neq 0$ .

(ii) We could attempt to solve  $\partial_t \Phi(x, t, \psi, \lambda) = \partial_\psi \Phi(x, t, \psi, \lambda) = 0$  but already for  $t = 0$ , the determinant of the Hessian of  $\Phi$  with respect to  $(t, \psi)$  vanishes on  $\Lambda_+$ . But it turns out that we can solve instead (locally)  $\Phi'_{t,\lambda}(x, t, \psi, \lambda) = (\partial_t \Phi, \partial_\lambda \Phi) = 0$  which is equivalent to  $t = t(x, \psi)$ ,  $\lambda = \lambda(x, \psi)$ . Namely can eliminate variables  $(t, \lambda)$ , since  $\frac{\partial^2 \Phi}{\partial t \partial \lambda} = -mE \neq 0$ . The implicit function theorem gives  $(t, \lambda) = (t(x, \psi), \lambda(x, \psi))$ .

We want to keep  $\lambda(x, \psi) = 1$ . Differentiating  $\Phi'_{t,\lambda} = 0$  along  $\Lambda_+$  with respect to  $x$  and  $\psi$  we find, using (2.6) and Hamilton equations

$$(4.20) \quad \begin{aligned} \langle \dot{P}, \dot{X} \rangle \frac{\partial t}{\partial x} + mE \frac{\partial \lambda}{\partial x} &= {}^t \dot{P}, \quad mE \frac{\partial t}{\partial x} = {}^t P \\ \langle \dot{P}, \dot{X} \rangle \frac{\partial t}{\partial \psi} + mE \frac{\partial \lambda}{\partial \psi} &= -\langle \dot{P}, X_\psi \rangle = -a, \quad mE \frac{\partial t}{\partial \psi} = 0 \end{aligned}$$

Assume  $\frac{\partial \lambda}{\partial x} = 0$  at  $z_0$ . This implies  $\dot{P} = \frac{\langle \dot{P}, \dot{X} \rangle}{mE} P$ , i.e.  $\partial_x H + \frac{1}{mE} \langle -\partial_x H, \dot{X} \rangle P = 0$ . Taking scalar product with  $P \neq 0$  we find  $\langle \partial_x H, P \rangle + \frac{|P|^2}{mE} \langle -\partial_x H, \dot{X} \rangle P = 0$ , and since  $z$  is a special point,  $\langle -\partial_x H, \dot{X} \rangle P = 0$ . It follows that  $\partial_x H = 0$  which is a contradiction ( $z$  is not a residual point).

Now we need  $\lambda = \lambda(x, \psi) = 1$ ; since  $\partial_x \lambda \neq 0$ , the implicit functions theorem shows that (possibly after renumbering the coordinates) that  $x_2 = x_2(x_1, \psi)$ . By second line (4.20) we have  $\frac{\partial t}{\partial \psi} = 0$ , and  $-mE \frac{\partial \lambda}{\partial \psi} = a = \langle \dot{P}, X_\psi \rangle$ . Since we have eliminated  $t, \lambda$ , the “ $\theta$ -parameter” is simply  $\psi$ , and we set

$$(4.21) \quad \Psi(x_1, \psi) = \Phi(x_1, x_2(x_1, \psi), t_2(x_1, x_2(x_1, \psi)), \psi, \lambda(x_1, x_2(x_1, \psi), \psi))$$

A straightforward calculation on  $C_\Phi$  gives

$$\begin{aligned} \partial_\psi \Psi(x_1, \psi) &= \frac{\partial x_2}{\partial \psi} P_2 + \left( \frac{\partial \lambda}{\partial x_2} \frac{\partial x_2}{\partial \psi} + \frac{\partial \lambda}{\partial \psi} \right) \langle P, x - X \rangle + \\ &\lambda \left[ \left( \frac{1}{mE} \frac{\partial x_2}{\partial \psi} \dot{P}_1 P_2 + \partial_\psi P_1 \right) (x_1 - X_1) + \left( \frac{1}{mE} \frac{\partial x_2}{\partial \psi} \dot{P}_2 P_2 + \partial_\psi P_2 \right) (x_2 - X_2) - \right. \\ &\left. P_1 \left( \frac{1}{mE} \frac{\partial x_2}{\partial \psi} P_2 \dot{X}_1 + \partial_\psi X_1 \right) + P_2 \left( \frac{\partial x_2}{\partial \psi} - \frac{1}{mE} \frac{\partial x_2}{\partial \psi} P_2 \dot{X}_2 + \partial_\psi X_2 \right) \right] \end{aligned}$$

where we recall  $\frac{\partial \lambda}{\partial \psi} = 0$  at  $z(t)$ , and a similar expression for  $\partial_\psi^2 \Psi(x_1, \psi)$ . By (3.9) with  $N = 1, n = 2$ , it follows that  $\text{rank } d\pi_x = 2$  if  $\partial_\psi^2 \Psi(x_1, \psi)|_{x_1=X_1} \neq 0$ , and  $\text{rank } d\pi_x = 1$  if  $\partial_\psi^2 \Psi(x_1, \psi)|_{x_1=X_1} = 0$  ( $X_1$  being evaluated at  $(t, \psi) = (t_0, \psi_0)$ ). In the latter case, differentiating  $\lambda = \lambda(x, \psi) = 1$  gives  $\frac{\partial \lambda}{\partial \psi} + \frac{\partial \lambda}{\partial x} \frac{\partial x}{\partial \psi} = 0$ . Since  $\frac{\partial \lambda}{\partial x} = 0$  at point  $z$ , (4.17) easily follows.

(iii) We consider a residual point as a limit of special points, for which  $\frac{\partial \lambda}{\partial x} = 0$ . Since  $a = 0$ , we have  $\frac{\partial \lambda}{\partial \psi} = \frac{\partial \lambda}{\partial x} = 0$  at  $z_0$ , and  $\lambda(x, \psi) = 1 + \mathcal{O}(|x - X(t_0, \psi_0), \psi - \psi_0|^2)$ . Then (4.20) reduces to  $mE \frac{\partial t}{\partial x} = {}^t P$  at  $z_0$ , which can be cast in the form  $dt = \langle P, dx \rangle \neq 0$ . ♣.

Let us make a few comments. Since the flow out of  $\partial\Lambda_+$  looks like a “cylinder” in the direction  $t$ , generically  $\text{rank } d\pi_x \Lambda_+ = 2$  in statement (i), and after eliminating again  $\lambda$  (since the lower right metrix element in (4.18) is non vanishing  $\lambda = 1$  is a non-degenerate critical point), we are left with a phase depending on  $x$  and  $\psi$  alone. In general  $\psi$  can be eventually eliminated, which gives caustics along the projection of the “turning lines” of the cylinder. So the phase depends on  $x$  alone, and near such a  $z$  takes the form of a WKB solution. In the case (ii) of a special point, the situation is reminiscent of (1.8).

For residual points Proposition 4.3 tells nothing however about  $\text{rank } d\pi_x(z_0)$ . For instance, if  $\dot{P} = 0$ , hence  $\langle \dot{P}, \dot{X} \rangle = 0$  and  $\partial_t \Phi = \partial_t^2 \Phi = 0$ , we could have  $\partial_t^3 \Phi \neq 0$ , and we have a cusp described by Percy functions (see e.g. [DoMaNaTu1, App.2]). Alternatively we could think of Hamiltonian  $p^2$  for which  $\text{rank } d\pi_x(z_0) = n$  is maximal, or of Hamiltonian  $H(x, p) = p_1$  for which  $\text{rank } d\pi_x(z_0) = 1$ .

#### 4.3 Reduced parametrizations of $\Lambda_+$ in case of the conformal metric, $n = 2$ .

In case of the conformal metric we can make the results more precise (at least for  $n = 2$ ), due to fact that  $\dot{X}$  is parallel to  $P$ . First information is related with the density. So far we have shown that (4.8) is a non-degenerate phase function parametrizing  $\Lambda_+$  for small  $t$ . We want to allow for larger values of  $t$  (the far field). Actually Proposition 4.1 and its proof show that  $\Phi(x, t, \psi, \lambda)$  is a non-degenerate phase function parametrizing  $\Lambda_+$  so long as  $\det(P, P_\psi) > 0$ . We have :

**Proposition 4.4:** *Let  $H(x, p)$  be as in (1.15),  $n = 2$ . Then representation (4.8) defines globally a non degenerate phase function parametrizing  $\Lambda_+$ , and the (inverse) density on  $\Lambda_+$  is  $mE \det(P, P_\psi) > 0$ .*

*Proof:* By general Hamilton-Jacobi theory we recalled in Sect.4.2, there exists a well-defined density on the regular part of  $\Lambda_+$ , which must coincide with (4.10). So we are left to show that (4.10) holds near focal points: but this follows from Lemmas A.2-3. ♣

We have no direct proof that (4.10) is valid everywhere on  $\Lambda_+$ . See however [DoMaNaTu1], Example 6, in case case  $m = 1$ , but then for any  $m$  by Maupertuis-Jacobi correspondence which preserves the Hamiltonian flow up to a reparametrization of time.

Next information is related to elimination of extra “ $\theta$ -variables” in the phase function and determination of  $\text{rank } d\psi_x : \Lambda_+ \rightarrow M$ . As we have seen, we can proceed to find the critical value of  $t \mapsto \Phi(x, t, \psi, \lambda)$  when  $z(t) = (X(t, \psi), P(t, \psi))$  is an ordinary point. Although this is not essential to our argument, in a simple scenario there would be at most one special or residual point on each bicharacteristic. We present some condition implying such a scenario. Recall  $\partial_t \Phi(x, t, \psi, \lambda = 1) = \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle = 0$  This holds on  $\Lambda_+$ , i.e. for  $x = X(t, \psi)$ . Taking second derivative at critical point gives

$$(4.30) \quad \partial_t^2 \Phi(X(t, \psi), t, \psi, \lambda = 1) = -\frac{m|P(t, \psi)|^{2m-2}}{\rho(X(t, \psi))^3} \langle \nabla \rho(X(t, \psi)), P(t, \psi) \rangle$$

so we have to take into account the set of  $\psi$  such that  $\langle \nabla \rho(X(t, \psi)), P(t, \psi) \rangle = 0$ , i.e. of the special or residual points. Consider  $f(t, \psi) = \langle \nabla \rho(X(t, \psi)), P(t, \psi) \rangle$ , so that  $f = 0$  if  $(X(t, \psi), P(t, \psi))$  is special or residual. Using Hamilton equations, we find

$$(4.31) \quad \partial_t f(t, \psi) = \frac{m|P(t, \psi)|^{m-2}}{\rho(X(t, \psi))} [\langle \nabla^2 \rho(X(t, \psi)) \cdot P(t, \psi), P(t, \psi) \rangle + \frac{|\nabla \rho(X(t, \psi))|^2}{m\rho(X(t, \psi))} |P(t, \psi)|^2]$$

Let  $z(s)$  be a special (or residual) point for some  $s \geq 0$ , then whenever  $\partial_t \Phi(t, x, \theta) = 0$  at some  $t > s$  (this occurs when the bicharacteristic  $t \mapsto z(t)$  projects again on  $x$ ),  $z(t)$  is no longer special (or residual). This holds under the strong “defocussing” condition

$$(4.32) \quad G(\rho)(x, p) = \langle \nabla^2 \rho(x) \cdot p, p \rangle + \frac{|\nabla \rho(x)|^2}{m\rho(x)} |p|^2 > 0$$

In particular the only critical point of  $\rho$  is a non degenerate minimum. This excludes however natural potentials having a limit as  $|x| \rightarrow \infty$ , since the second term in (4.32) cannot rescue the lack of convexity at infinity, as shows the example  $\rho(x) = \rho_0 + \langle x \rangle^{-\varepsilon}$ . By assumption (4.32),  $\partial_t f(t, \psi) > 0$ , and  $t \mapsto \partial_t^2 \Phi(X(t, \psi), t, \psi, \lambda = 1)$  is strictly decaying on  $\Lambda_+$ .

• *Ordinary critical points.* They correspond to non degenerate critical points of  $t \mapsto \Phi(x, t, \psi, \lambda = 1)$ .

**Lemma 4.6:** *Assume (4.32),  $n = 2$  (no condition on  $\nabla \rho$  is required here). Then  $I_\psi$  is an interval, and*

$$(4.33) \quad \forall t \in I_\psi, x = X(t, \psi) \iff t = t_1(x, \psi) \quad \text{on } C_\Phi$$

where  $t_1$  is a smooth function. Moreover  $\pi_x : \Lambda_+ \rightarrow M$  at every ordinary critical point has same rank as the symmetric matrix (4.18) i.e.  $d\pi_x$  has rank 1 ( $\langle P_\psi, X_\psi \rangle = 0$ ) or 2 ( $\langle P_\psi, X_\psi \rangle \neq 0$ ).

*Proof:* Note that when  $t = 0$ ,  $f(0, \psi) = \langle \nabla \rho(0), P(\psi) \rangle$ . So when  $f(0, \psi) > 0$ , 0 is non-degenerate critical point of  $t \mapsto \Phi(x, t, \psi, \lambda)$ , and the implicit function theorem shows that (4.33) holds. Since  $t \mapsto f(t, \psi)$  is increasing, this holds for all  $t$  in the maximal interval of definition of the integral curve starting at  $(0, P(\psi))$ . When  $f(0, \psi) < 0$  instead, (4.33) holds on an interval ending at some  $s$  such that  $f(s, \psi) = 0$ . Let us compute the rank of  $\pi : \Lambda_+ \rightarrow M$  at an ordinary point. Let  $U = \{(t, \psi) : t \in I_\psi, z(t) = (X(t, \psi), P(t, \psi)) \in \mathcal{S}^c(\Lambda_t)\}$ , then the same computation as in (4.18) shows that  $U$  is a canonical chart rank 1 or 2, which gives the Lemma. ♣

• *Special and residual critical points.* Near the end point  $s$  of  $I_\psi$  we can solve (locally) as in Proposition 4.3,  $\partial_t \Phi(x, t, \psi, \lambda) = \partial_\lambda \Phi(x, t, \psi, \lambda) = 0$  which gives  $t = t(x, \psi)$  and  $\lambda = \lambda(x, \psi)$ . Namely, the Hessian of  $\Phi$  with respect to  $(t, \lambda)$  at  $(s, 1)$  has determinant  $-\left(\frac{\partial^2 \Phi}{\partial t \partial \lambda}\right)^2 = -(mE)^2 < 0$  on  $\Lambda_+$ . So if  $(X(s, \psi), P(s, \psi))$  is a special point then  $(s, 1)$  is a non degenerate point of  $(t, \lambda) \mapsto \Phi(x, t, \psi, \lambda)$ . Integrating Hamilton equations also for  $t < 0$  gives the Lagrangian manifold  $\Lambda_- \cup \Lambda_+$ . So there is no loss of generality in assuming the special point is at  $s = 0$ . The following Lemma strengthens Proposition 4.3 in case  $X_\psi \neq 0$ .

**Lemma 4.7:** *Let  $n = 2$  and  $H$  be as in (1.15).*

(i) *Assume  $z(s) = (X(s, \psi), P(s, \psi)) \in \Lambda_+$  be a special point (hence  $\nabla \rho(x(s)) \neq 0$ ). If  $X_\psi = 0$ , then  $\text{rank } d\pi_x(z(s)) = 1$  as in Proposition 4.3 (i). If  $X_\psi \neq 0$ , then  $ac \neq 0$  so that  $\text{rank } d\pi_x(z) = 2$ . Near  $z(s)$ ,  $\Lambda_+$  is given by  $t = t(x)$ ,  $\psi = \psi(x)$ , and  $\frac{\partial t}{\partial x} \neq 0$ ,  $\frac{\partial \psi}{\partial x} \neq 0$ . The constraint  $\lambda = 1$  takes the form*

$$(4.41) \quad \frac{\partial \lambda}{\partial \psi} = - \left| \frac{\partial \psi}{\partial x} \right|^{-2} \left\langle \frac{\partial \lambda}{\partial x}, \frac{\partial \psi}{\partial x} \right\rangle$$

(ii) Assume  $z(s) = (X(s, \psi), P(s, \psi)) \in \Lambda_+$  be a residual point (i.e.  $\nabla \rho(x(s)) = 0$ ). If  $X_\psi \neq 0$ , then  $c \neq 0$  and  $\text{rank } d\pi_x(z) = 2$ .

*Proof:* As in Proposition 4.3, the relations  $\partial_t \Phi = \partial_\lambda \Phi = 0$  being given by  $(t, \lambda) = (t(x, \psi), \lambda(x, \psi))$  we use (4.20). Since  $X_\psi \neq 0$ ,  $\langle P, X_\psi \rangle = 0$  and  $\det(P, P_\psi) > 0$  everywhere by Proposition 4.4, we have  $c \neq 0$ .

(i) By the same geometric argument we have,  $\dot{P} \neq 0$  by (2.2), and since  $\langle \nabla \rho, P \rangle = 0$ , the relation  $a = \langle \dot{P}, X_\psi \rangle = 0$  would contradict  $X_\psi \neq 0$ . So by second line (4.20),  $-mE\partial_\psi \lambda = \langle \dot{P}, X_\psi \rangle \neq 0$ , or  $\partial_\psi \lambda \neq 0$ . Now we need  $\lambda = \lambda(x, \psi) = 1$ ; since  $\partial_\psi \lambda \neq 0$ , the implicit functions theorem shows that  $\psi = \psi(x)$ . Then we have

$$(4.42) \quad x = X(t, \psi) \iff t = t(x), \psi = \psi(x) \text{ on } \partial_t \Phi(x, t, \psi, \lambda = 1) = \partial_\lambda \Phi(x, t, \psi, \lambda = 1) = 0$$

Differentiating  $\lambda = \lambda(x, \psi)$  gives

$$\frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial \psi} \frac{\partial \psi}{\partial x} = 0$$

and together with the first equation (4.20)

$$\langle P_\psi, \dot{X} \rangle^t \left( \frac{\partial \psi}{\partial x} \right) = mE\dot{P} - \langle \dot{P}, \dot{X} \rangle P$$

Let us show that  $\frac{\partial \psi}{\partial x} \neq 0$ . Otherwise, we would have  $\langle P, \dot{X} \rangle \dot{P} = \langle \dot{P}, \dot{X} \rangle P$ , and since we know that  $\langle \nabla \rho(X(s, \psi)), P(s, \psi) \rangle = 0$ , this would contradict the fact that  $\dot{P}$  is parallel to  $\nabla \rho(X(s, \psi))$ . Moreover  $\frac{\partial t}{\partial \psi} = 0$ ,  $\frac{\partial t}{\partial x} = \frac{1}{mH} {}^t P \neq 0$ , which readily gives (4.41). To compute the rank of  $\pi$  at a special point, we are left to compute second derivative of (4.21), namely  $-\partial_\psi^2 \Psi = ac \neq 0$ , so we conclude as in Proposition 4.3 that  $\text{rank } \pi|_{\Lambda_+}$  is 2.

(ii) Thinking of a residual point as the limit of special points, (4.20) shows that  $\frac{\partial \lambda}{\partial x} = \frac{\partial \lambda}{\partial \psi} = 0$ , and (4.20) reduces to  $mE\frac{\partial t}{\partial x} = {}^t P$ ,  $\frac{\partial t}{\partial \psi} = 0$ . Note that on  $C_\Phi$  (this holds for general  $H$  positively homogeneous of degree  $m$  on  $T^*M \setminus 0$ )

$$(4.43) \quad -\text{Hess}_{(t, \psi, \lambda)} \Phi = \begin{pmatrix} \langle \dot{P}, \dot{X} \rangle & \langle \dot{P}, X_\psi \rangle & \langle P, \dot{X} \rangle \\ \langle \dot{P}, X_\psi \rangle & \langle P_\psi, X_\psi \rangle & 0 \\ \langle P, \dot{X} \rangle & 0 & 0 \end{pmatrix}$$

thus  $\det \text{Hess}_{(t, \psi, \lambda)} \Phi = (mE)^2 \langle P_\psi, X_\psi \rangle \neq 0$ , so by the implicit functions theorem

$$(t, \psi, \lambda) = (t_2(x), \psi_2(x), \lambda_2(x))$$

Let us check again that  $\lambda_2(x) = 1$ : Differentiating  $\Phi'_{t, \psi, \lambda}(x, t, \psi, \lambda) = 0$  with respect to  $x, \lambda$  gives the triangular system

$$\begin{aligned} \langle P_\psi, \dot{X} \rangle^t \left( \frac{\partial t_2}{\partial x} \right) + \langle P_\psi, X_\psi \rangle^t \left( \frac{\partial \psi_2}{\partial x} \right) &= P_\psi \\ \langle \dot{P}, \dot{X} \rangle^t \left( \frac{\partial t_2}{\partial x} \right) + \langle \dot{P}, X_\psi \rangle^t \left( \frac{\partial \psi_2}{\partial x} \right) + mH^t \left( \frac{\partial \lambda_2}{\partial x} \right) &= \dot{P} \\ mH^t \left( \frac{\partial t_2}{\partial x} \right) &= P \end{aligned}$$

Since  $\dot{P} = 0$ , using  $a = b = 0$  this reduces to

$$c^t\left(\frac{\partial\psi_2}{\partial x}\right) = P_\psi, \quad \frac{\partial\lambda_2}{\partial x} = 0, \quad mH^t\left(\frac{\partial t_2}{\partial x}\right) = P$$

and in particular  $\lambda = \lambda_2(x) = 1$ . There are no “ $\theta$ ”-parameters left and so  $\text{rank } d\pi_x(z(s)) = 2$ . By Proposition 4.4,  $\det\left(\frac{\partial\psi_2}{\partial x}, \frac{\partial t_2}{\partial x}\right) \neq 0$ . ♣

Note that if  $X_\psi = 0$  at a focal point of  $\Lambda_+$ , then  $P_\psi \neq 0$  (otherwise this would violate property (3) of Proposition A.1).

From Lemma 4.7, the set of focal points which are also special points is  $\mathcal{S}(\Lambda_t) \cap \mathcal{F}(\Lambda_t) = \{\psi : \langle P_\psi, X_\psi \rangle = 0\}$ . In Example 2.4, we find  $\langle P_\psi, X_\psi \rangle = \frac{1}{2} \sinh(2f(t)) > 0$  for all  $t > 0$  and vanishes at  $t = 0$ .

Now to find the canonical charts for the phase functions, we use a connectedness argument. Assume (4.32), and let  $s$  be the supremum of  $I_\psi$ , we have  $f(s, \psi) = 0$ . Since  $G(\rho)(s, \psi) > 0$ , we have  $f(t, \psi) > 0$  for all  $t > s$ , so all points  $z(t) = (X(t, \psi), P(t, \psi))$  for  $t > s$  are ordinary points. So far we proved:

**Proposition 4.8:** *Let  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ . Then there exists globally a smooth solution  $\Psi$  of  $HJ$  equation  $H(x, \partial_x \Phi) = E$ . Let  $C_\Phi = \{(t, x, \psi), \psi \in \mathbf{S}^1 : \partial_t \Phi = \partial_\lambda \Phi = \partial_\psi \Phi = 0\}$ . Then the embedding*

$$\iota_\Phi : C_\Phi \rightarrow T^*\mathbf{R}^2, \quad (t, x, \psi, \lambda) \mapsto (x, \partial_x \Phi(x, t, \psi, \lambda = 1))$$

*such that  $\iota_\Phi(C_\Phi) \subset \Lambda_+$  consists in charts of rank 1 or 2 [the rank is never 0 since  $p \neq 0$  in the energy shell  $H = E$ ]. Under hypothesis (4.32) these charts can intersect the set of special points  $\mathcal{S}$  only along a line.*

*Remark 4.9:* The canonical charts in  $\Lambda_+$  where  $\Phi = \Phi(x)$ , i.e. of WKB type are of course of maximal rank 2, in particular there is a WKB solution near a special point  $z$  such that  $\langle X_\psi, P_\psi \rangle \neq 0$ .

#### 4.4 Construction of Maslov canonical operator for bi-Lagrangian distributions.

Assume already we have constructed the amplitude  $a(x, t, \psi, \lambda)$ : thus we got a solution  $v(t, x; h)$  to the Cauchy problem that can be expressed as an oscillatory integral  $\int e^{i\Phi(x, t, \psi, \lambda)/h} \tilde{a}(x, t, \psi, \lambda) d\psi d\lambda$  with a suitable  $\tilde{a}$ . If we content to the first order approximation of the outgoing solution to (1.5), we need to compute  $u(x; h) = \int_0^\infty v(t, x; h) dt$  i.e.

$$(4.45) \quad u(x; h) = \int_0^\infty \int e^{i\Phi(x, t, \psi, \lambda)/h} a(x, t, \psi, \lambda) d\psi d\lambda dt$$

(lateron we will choose  $a$  independent of  $x$ ). We shall rely on the symbolic calculus set up in Sect.3, but proceed with a specific choice of symbols.

Given  $f_h$ , we construct  $(\sigma, \sigma^+)$ , and a bi-Lagrangian distribution  $u$  solution of  $H(x, hD_x; h)u = f_h \bmod \mathcal{O}(h)$ . It is obtained by choosing in (4.45) a suitable symbol  $a$  “interpolating” between  $\sigma$  and  $\sigma^+$ . Since our previous constructions are global, we shall obtain a generalization of Maslov canonical operator associated with the pair  $(\Lambda, \Lambda_+)$ .

Let  $\sigma : p \mapsto \sigma(p)$  be a function on  $\Lambda \setminus \partial\Lambda_+(\tau)$  and  $\sigma^+ : (t, \psi) \mapsto \sigma^+(t, \psi)$  a function on  $\Lambda_+$ , that we consider as the restriction of  $\tilde{\sigma}^+ : (t, \psi, \lambda) \mapsto \tilde{\sigma}^+(t, \psi, \lambda)$  as a function on  $\tilde{\Lambda}_+$  in the extended phase-space. We introduce densities as in (4.10) but ignore Maslov contributions. Let  $\delta(\psi, \lambda) d\psi d\lambda$  and

$$\delta^+(t, \psi) dt d\psi = (F[\Phi, d\mu_+]|_{C_\Phi})^{-1} dt d\psi = (mH \det(P, P_\psi))^{-1} dt d\psi$$

be densities on  $\Lambda$  and  $\Lambda_+$  respectively. We shall actually deal with the square root of these densities (half-densities). We can extend  $\delta^+(t, \psi) dt d\psi$  as a density  $\delta^+(t, \psi, \tau) dt d\psi$  or  $\delta^+(t, \psi, \lambda) dt d\psi$  on  $\Lambda_+(\tau)$ , where we recall  $\lambda$  and  $\tau$  are related by (4.9) in case of Hamiltonian (1.15). Now we take  $\tilde{\sigma}^+$  the solution of transport equation (3.17) with initial value  $\tilde{\sigma}^+(0, \psi, \lambda) = 1$ , i.e.

$$\left(\frac{1}{i} \frac{d}{dt} + H_1\right) a_0(t, \psi, \lambda) \sqrt{\delta^+(t, \psi, \lambda)} = 0$$

Let  $a(t, \psi, \lambda)$  be a continuous function, smooth outside  $\partial\Lambda_+$ , such that

$$(4.50) \quad a(0, \psi, \lambda) = (\delta(\psi, \lambda))^{1/2} (\lambda - 1) \sigma(P(\psi, \lambda)), \quad a(t, \psi, \lambda = 1) = (\delta^+(t, \psi))^{1/2} \sigma^+(t, \psi)$$

The existence of  $a$  is ensured by the *compatibility condition*

$$(4.51) \quad \lim_{\lambda \rightarrow 1} (\delta(\psi, \lambda))^{1/2} (\lambda - 1) \sigma(P(\psi, \lambda)) = (\delta^+(0, \psi))^{1/2} \sigma^+(0, \psi)$$

(so  $\sigma(P(\psi, \lambda))$  may blow up as  $\lambda \rightarrow 1$ ), and we can choose

$$(4.52) \quad a(t, \psi, \lambda) = (\delta(\psi, \lambda))^{1/2} (\lambda - 1) \sigma(P(\psi, \lambda)) + (\delta^+(t, \psi))^{1/2} \sigma^+(t, \psi) - (\delta^+(0, \psi))^{1/2} \sigma^+(0, \psi)$$

which verifies

$$\lim_{(t, \lambda) \rightarrow (0, 1)} a(t, \psi, \lambda) = (\delta^+(0, \psi))^{1/2} \sigma^+(0, \psi)$$

In the special case (1.15) and  $n = 2$ , (2.3) shows that in polar coordinates we can choose the density on  $\Lambda$  of the form  $dp = \lambda d\lambda d\psi$  (Lebesgue measure). With these choices, (4.51) writes

$$(4.53) \quad \lim_{\lambda \rightarrow 1} (\lambda - 1) \sigma(P(\psi, \lambda)) = \sigma^+(0, \psi)$$

**Definition 4.11:** *With  $a$  as in (4.52), and ignoring for the moment Maslov contributions, we call Maslov canonical operator for the pair  $(\sigma, \sigma^+)$*

$$(4.54) \quad \mathcal{K}_+(\sigma, \sigma^+)(x; h) = \int_0^\infty dt \int e^{i\Phi(x, t, \psi, \lambda)/h} a(t, \psi, \lambda) d\psi d\lambda$$

We shall simplify expressions (4.54) using stationary phase, but this will be possible only when  $t \neq 0$ , i.e.  $x \neq 0$ . Consider again  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ , where results are most complete.

First we prove consistency of Definition 4.11 relative to standard Maslov canonical operators.

**Proposition 4.12:** *Let  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ .*

(i) If  $\sigma = 0$  and  $\text{supp } \sigma^+ \cap \partial\Lambda_+ = \emptyset$ , then (up to a constant factor)  $\mathcal{K}_+(0, \sigma^+) = \mathcal{K}_{\Lambda_+} \sigma^+ + \mathcal{O}(h)$ , where  $\mathcal{K}_{\Lambda_+}$  is the standard canonical operator on  $\Lambda_+ \setminus \partial\Lambda_+$ .

(ii) If  $\sigma^+ = 0$  and  $\text{singsupp } \sigma \cap \partial\Lambda_+ = \emptyset$  (singular support), then (up to a constant factor)  $\mathcal{K}_+(\sigma, 0) = \mathcal{K}_\Lambda \sigma + \mathcal{O}(h)$ , where  $\mathcal{K}_\Lambda$  is the standard canonical operator on  $\Lambda$ .

Moreover the expressions for  $\mathcal{K}_+(0, \sigma^+)$  and  $\mathcal{K}_+(\sigma, 0)$  can be simplified by standard stationary phase methods.

*Proof:*

(i) *Wave part  $\mathcal{K}_{\Lambda_+}(A)$ .* By (4.52) we have  $a(t, \psi, \lambda) = \sigma^+(t, \psi)$ . To find the pre-images of  $(t, \psi) \mapsto x = X(t, \psi)$ , we need to discuss according to the case  $\pi_x : \Lambda \rightarrow M$  is of maximum rank or not. For small  $x$ , local geodesic completeness (at least when  $m = 2$ ) shows that  $(t, \psi) \mapsto x$  is 1-1. Otherwise, under Assumption (2.18) (i.e. if  $x$  and  $x_0 = 0$  are not conjugated along any trajectory with initial momentum  $\eta = P(\psi) \in \text{supp } A \subset \Lambda$ , in particular if  $x$  is not a focal point), then  $x = X(t, \psi)$  for a finite number (locally constant near such an  $x$ ) of  $(t, \psi)$ . We discuss according to  $(t, \eta)$  is ordinary or not.

•  $X(t, \psi)$  is an ordinary point, we apply Lemma 4.6. First we perform stationary phase in  $t$ , at  $t = t(x, \psi)$ , which is a non-degenerate point ( $\langle \dot{P}, \dot{X} \rangle \neq 0$ ), this introduces in the density the additional factor

$$\delta_1^+(t(x, \psi), \psi) = \left( \frac{\partial_t^2 \Phi}{2i\pi h} \right)^{-1} = \left( \frac{\langle \dot{P}, \dot{X} \rangle}{2i\pi h} \right)^{-1}$$

Next we perform stationary phase in  $\lambda$  at  $\lambda = 1$  which is a non-degenerate point ( $mE \neq 0$ ), multiplying the density by

$$\delta_2^+(t(x, \psi), \psi) = \left( \frac{\partial_\lambda^2 \Phi}{2i\pi h} \right)^{-1} = \left( \frac{1}{mE} \frac{\langle \dot{P}, \dot{X} \rangle}{2i\pi h} \right)^{-1}$$

As a result

$$(4.58) \quad \mathcal{K}_+(\sigma, \sigma^+)(x; h) = \frac{2\pi h}{mE} \int e^{i\Phi(x, t(x, \psi), \psi, 1)/h} (\delta^+(t(x, \psi), \psi) \delta_1^+(t(x, \psi), \psi) \delta_2^+(t(x, \psi), \psi))^{1/2} \sigma^+(t(x, \psi), \psi) d\psi + \mathcal{O}(h)$$

This can be further simplified by Lemma 4.6, i.e. if  $\langle P_\psi, X_\psi \rangle \neq 0$  the integral is of WKB type, otherwise  $\psi$  plays the role of momentum in case of a fold, and the integral is given by Airy functions. In both case, we introduce a new density depending on  $x$  alone (these density factors eventually merge to a single one, due to chain rule). Of course we could have applied stationary phase with respect to  $t, \psi, \lambda$  simultaneously.

•  $X(t, \psi)$  is a special point (i.e.  $\langle \nabla \rho(X(t, \psi), P(t, \psi)) \rangle = 0$ ). There we apply Lemma 4.7. If  $X_\psi \neq 0$  (non focal point), the critical point is  $(t, \psi, \lambda) = (t(x), \psi(x), 1)$ , and by (4.43)

$$(\delta^+(t, \psi))^{1/2} = \det \text{Hess}_{(t, \psi, \lambda)} \Phi(x, t(x), \psi(x), 1) = (mH)^2 \langle X_\psi, P_\psi \rangle \neq 0$$

By stationary phase formula

$$(4.55) \quad \mathcal{K}_+(\sigma, \sigma^+)(x; h) = \frac{1}{mE} e^{i\Phi(x, t(x), \psi(x), 1)/h} \left( \det \left( \frac{\langle X_\psi, P_\psi \rangle}{2i\pi h} \right) \right)^{-1/2} (\delta^+ \sigma^+)^{1/2}(t(x), \psi(x)) + \mathcal{O}(h)$$

i.e.  $\mathcal{K}_+(\sigma, \sigma^+)(x; h)$  is of WKB type. If  $X_\psi = 0$  (focal point),  $\mathcal{K}_+(\sigma, \sigma^+)(x; h)$  is again of Airy type, after eliminating  $\psi$ .

- If  $X(t, \psi)$  is a residual point, we apply instead Lemma 4.7(ii).

To sum up all contributions of the oscillating integrals we use canonical charts and the discussion in Sect.2.4. Actually we have to take into account all the pre-images of  $X(t, \psi)$ . In case of a non-focal point, Assumption (2.18) ensures that  $\mathcal{K}_+(\sigma, \sigma^+)(x; h)$  can be written as a finite sum (over all pre-images  $\eta_\alpha = P(\psi_\alpha), t = t_\alpha$ ) of WKB expansions of the form  $\mathcal{K}_+(\sigma_\alpha, \sigma_\alpha^+)(x; h)$  above. This carries (generically) to the more general case of a focal point.

(ii) *Boundary part*  $\mathcal{K}_\Lambda(A)$ . By (4.52)  $a(t, \psi, \lambda) = (\delta(\psi, \lambda))^{1/2}(\lambda - 1)\sigma(P(\psi, \lambda))$ . We have

$$\frac{h}{i} \frac{d}{dt} e^{i\Phi(x, t, \psi, \lambda)/h} = e^{i\Phi(x, t, \psi, \lambda)/h} [mE(1 - \lambda) + \lambda \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle]$$

so integrating over  $t$  (cutting-off large times  $t$ ), we get

$$\begin{aligned} \mathcal{K}_+(\sigma_0, 0) &= \frac{h}{imE} \int e^{i\Phi(x, 0, \psi, \lambda)/h} (\delta(\psi, \lambda))^{1/2} \sigma(P(\psi, \lambda)) d\psi d\lambda + \\ &\frac{1}{mE} \int_0^t dt \int e^{i\Phi(x, t, \psi, \lambda)/h} \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle \delta(\psi, \lambda) \sigma(P(\psi, \lambda)) d\psi d\lambda \end{aligned}$$

To the second integral we apply stationary phase, so using  $x = X(t, \psi)$  on  $C_\psi$ , we find this term is  $\mathcal{O}(h^{3/2})$ . We are left with the integral over  $\Lambda$ , where we change variables  $(\psi, \lambda)$  to  $p$ . Using the initial condition  $\Phi(x, 0, \psi, \lambda) = \langle x, P(\psi, \lambda) \rangle$ , we find this term is nothing but  $\int e^{ipx/h} \sigma(p) dp = \mathcal{K}_\Lambda \sigma(x; h)$ .

So we have found reduced representations of  $\mathcal{K}(\sigma, \sigma^+)(x; h)$ . ♣

*Remark 4.7:* By linearity, if  $(\text{singsupp } \sigma \cup \text{supp } \sigma^+) \cap \partial\Lambda_+ = \emptyset$  we obtain also an asymptotics for  $\mathcal{K}_+(\sigma, \sigma^+)$ . We needn't assume  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$  if we content with asymptotics for small  $x$ .

Next we apply symbolic calculus on  $I(M, \Lambda, \Lambda^+)$  adapted from [MelUhl, Prop.5.4& Prop.6.6] in Sect.3, and take advantage of using symbols of the form (4.52). This yields the main result of this section:

**Proposition 4.13** (commutation formula): We have

$$H(x, hD_x; h) \mathcal{K}_+(\sigma, \sigma^+) = \mathcal{K}_+(H(x, hD_x; h) \sigma; \frac{d\sigma^+}{dt} + iH_1 \sigma^+) + \mathcal{O}(h^2)$$

and the solution of  $H(x, hD_x; h)u_h = f_h = \int e^{ixp/h} A(p) dp$  is given at first order in  $h$  by

$$\mathcal{K}_+(\frac{A(p)}{\lambda - 1}; \sigma^+)(x; h)$$

where  $\sigma^+$  solves  $\frac{d\sigma^+}{dt} + iH_1 \sigma^+ = 0$ .

*Proof:* We apply the  $h$ -PDO  $H(x, hD_x)$  under the integral sign in (4.54). By eikonal equation (2.15)

$$\begin{aligned} (H(x, hD_x) - E) e^{i\Phi(x, t, \psi, \lambda)/h} &= e^{i\Phi(x, t, \psi, \lambda)/h} (H(x, \partial_x \Phi) - E) = \\ &- e^{i\Phi(x, t, \psi, \lambda)/h} \frac{\partial \Phi}{\partial t} = -h D_t e^{i\Phi(x, t, \psi, \lambda)/h} \end{aligned}$$



all equalities being understood mod  $\mathcal{O}(h)$ , whatever the quantization rule is. Let  $A(p)$  be the symbol of  $f_h$  on  $\Lambda$ , we put (we can make  $\sigma$  depend on  $\lambda$  also)

$$\sigma(P(\psi, \lambda), \lambda) = \frac{A(P(\psi, \lambda))}{\lambda - 1}$$

Integrating by parts in  $t$ , the first term in  $H(x, hD_x)\mathcal{K}_+(\sigma, \sigma^+)(x; h)$  is (change of variables)

$$-\frac{h}{i} \int e^{i\Phi(x, 0, \psi, \lambda)/h} A(P(\psi, \lambda)) d\psi d\lambda = -\frac{h}{i} \int e^{ixp/h} A(p) dp$$

*4.5 Maslov canonical operator in a  $h^\delta$ -nghbhd of  $x_0$ .*

If we expect a formal expression  $u_h$  like (1.5) with data  $f_h$  microlocalized on  $T_{x_0}^*M$  to satisfy Sommerfeld radiation condition, we need in particular a precise knowledge of  $u_h$  in a  $h$ -dependent nghbhd  $U_h$  of  $x_0$ , which reveals for instance the important role played by the “return set” (1.4), see [Ca]. So it is useful to derive an asymptotic formula for (1.18) in  $U_h$ , which amounts to “freeze” the coefficients at  $x_0$ . See [AnDoNaRo,Thm2] for a first result.

**5.  $f_h$  is supported microlocally on the “cylinder”  $\Lambda = \{X = \varphi\omega(\psi), P = \omega(\psi)\}$**

Here we assume

$$(5.1) \quad \Lambda = \{x = X(\varphi, \psi) = \varphi\omega(\psi), p = P(\varphi, \psi) = \omega(\psi), \varphi \in \mathbf{R}, \omega \in \mathbf{S}^{n-1}\}$$

When  $n = 2$  this is the wave-front set of Bessel function  $f_h(x; h) = J_0(\frac{|x|}{h})$ ; such functions arise in the wave beam theory (see [Ki], [DoMaNa] and references therein), so we call  $f_h$  a “Bessel beam”. The new difficulty lies in  $\partial\Lambda_+$ , since  $\varphi$  and  $\psi$  are not independent variables on the energy surfaces  $H = E - \tau$ .

*5.1 Non degeneracy conditions*

Let us check first the Lagrangian intersection. By analogy with the terminology used in Diffraction Theory [Hö], we consider the case when the pair  $(\Lambda, \Lambda_+)$  fails to be a Lagrangian intersection everywhere.

**Definition 5.1:** *The point  $z \in \Lambda$  is called glancing if  $v_H(z) \in T_z\Lambda$ . We denote by  $\mathcal{G}(\Lambda)$  the set of glancing points on  $\Lambda$ .*

So  $(\Lambda, \Lambda_+)$  is an intersecting pair whenever  $\mathcal{G}(\Lambda) = \emptyset$ . Hamiltonian flow preserves the set of glancing points, i.e.  $\exp tv_H(\mathcal{G}(\Lambda)) = \mathcal{G}(\Lambda_t)$ .

**Proposition 5.2:** *Let  $H$  be homogeneous of degree  $m$ , and  $\Lambda$  be the cylinder. With the notation above,  $z \in \partial\Lambda_+$  is a glancing point iff*

$$(5.2) \quad \langle \partial_p H + \varphi \partial_x H, \omega^\perp(\psi) \rangle = 0, \quad \langle -\partial_x H, \omega(\psi) \rangle = 0, \quad H(z) = E$$

More generally ( $n = 2$  for simplicity) assume  $z(t) \in \Lambda_t$  is a glancing point,  $(\varphi, \psi)$  are local coordinates on  $\Lambda_t$  near  $z(t)$ , and  $(P, P_\psi)$  is a basis of  $\mathbf{R}^2$  at  $z(t) = (X(t, \varphi, \psi), P(t, \varphi, \psi))$ . Let

$$R = R(t, \varphi, \psi) = |X_\psi|^2 (\langle -\partial_x H, P_\psi \rangle - mH \langle P_\varphi, P_\psi \rangle) - |P_\psi|^2 (\langle \partial_p H, X_\psi \rangle - mH \langle X_\varphi, X_\psi \rangle)$$

Then  $H(z(t)) = E$ , and

$$(5.2)_t \quad \begin{aligned} \langle P, P_\psi \rangle \neq 0 &\implies \langle -\partial_x H - mHP_\varphi, P_\psi^\perp \rangle = 0 \text{ and } R = 0 \\ \langle P, P_\psi \rangle = 0 &\implies \langle -\partial_x H, P \rangle = mH \langle P, P_\varphi \rangle \text{ and } R = 0 \end{aligned}$$

*Proof:* We complete  $\omega(\psi)$  in  $\mathbf{S}^{n-1}$  into a (direct) orthonormal basis  $\omega^\perp(\psi) = (\omega_1(\psi), \dots, \omega_{n-1}(\psi))$  of  $\mathbf{R}^n$ , and denote by  $\omega^\perp(\psi)\delta\psi = \omega_1(\psi)\delta\psi_1 + \dots + \omega_{n-1}(\psi)\delta\psi_{n-1}$  a section of  $T\mathbf{S}^{n-1}$ ,  $\delta\psi_j \in \mathbf{R}$ . The tangent space  $T_z\Lambda$  has the parametric equations

$$\delta X = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi, \quad \delta P = \omega^\perp(\psi)\delta\psi, \quad (\delta\varphi, \delta\psi) \in \mathbf{R}^n$$

so  $v_H \in T_z\Lambda$  iff there exist  $(\delta\varphi, \delta\psi)$  such that

$$\partial_p H = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi, \quad -\partial_x H = \omega^\perp(\psi)\delta\psi$$

Taking scalar products with  $\omega(\psi), \omega^\perp(\psi)$ , and using Euler identity, we get  $\delta\varphi = \langle \partial_p H, P(\psi) \rangle = mH$ ,  $\delta\psi = \langle -\partial_x H, \omega^\perp(\psi) \rangle$ . Since  $(\omega(\psi), \omega^\perp(\psi))$  form a basis of  $\mathbf{R}^n$ , relations (5.2) are necessary and sufficient for  $v_H \in T_z \Lambda_+ \cap T_z \Lambda = T_z \partial \Lambda_+$ .

Let now  $t > 0$ ,  $(\delta X, \delta P) \in T_{z(t)} \Lambda(t)$  is given by  $\delta X = X_\varphi \delta\varphi + X_\psi \delta\psi$ ,  $\delta P = P_\varphi \delta\varphi + P_\psi \delta\psi$ , so if  $v_H \in T_{z(t)} \Lambda(t)$ , due to Euler identity and (2.11) we still have  $\delta\varphi = mH$  and

$$\partial_p H = mH X_\varphi + X_\psi \delta\psi, \quad -\partial_x H = mH P_\varphi + P_\psi \delta\psi$$

Taking scalar product of the first equation by  $X_\psi$  gives

$$|X_\psi|^2 \delta\psi = (\langle \partial_p H, X_\psi \rangle - mH \langle X_\varphi, X_\psi \rangle)$$

Taking scalar product of the second equation with  $P_\psi$  and  $P$  gives

$$\delta\psi = |P_\psi|^{-2} (\langle -\partial_x H, P_\psi \rangle - mH \langle P, P_\varphi \rangle), \quad \langle -\partial_x H, P \rangle = mH \langle P, P_\varphi \rangle + \langle P, P_\psi \rangle \delta\psi$$

Discussing according to the fact  $P$  is orthogonal to  $P_\psi$  or not readily gives (5.2)<sub>t</sub>. ♣

Since the glancing property is invariant under the Hamiltonian flow, (5.2) and (5.2)<sub>t</sub> are actually equivalent.

*Example 5.1:* When  $H = p^2$ , all points are glancing. When  $H = \frac{|p|^m}{\rho(x)}$ ,  $z(0)$  is a glancing point iff

$$(5.3) \quad \text{either : } \varphi \neq 0 \text{ and } \nabla \rho = 0, \text{ or : } \varphi = 0 \text{ and } \langle \nabla \rho(0), \omega(\psi) \rangle = 0$$

Second condition means that if  $z(0) = (0, \omega(\psi))$  is a special point. Assuming (4.32) it follows that if  $z(0)$  is a glancing point,  $z(t)$  will be glancing but never special at later  $t > 0$ .

*Example 5.2:* Consider  $H(x, p) = \langle \mu, p \rangle$ , where  $\mu \in \mathbf{R}^2$  (constant coefficients case), then (5.2) will always occur at some point  $\psi$ . Namely,  $X(t, \psi) = \mu t + \varphi \omega(\psi)$ ,  $P(t, \psi) = \omega(\psi)$ . The system  $x = X(t, \varphi, \psi)$  has a unique solution  $(t_1(x, \psi), \varphi_1(x, \psi), \psi)$  provided  $\langle \mu, \omega^\perp \rangle \neq 0$ , i.e.  $v_H$  is transverse to  $\Lambda$ , and  $\psi = \psi_0$  belongs to  $\{\tau = 0\}$ , i.e.  $\langle \mu, \omega(\psi_0) \rangle = E$ . It is given by

$$t = \frac{\langle x, \omega^\perp(\psi_0) \rangle}{\langle \mu, \omega^\perp(\psi_0) \rangle}, \quad \varphi = \frac{1}{\langle \mu, \omega^\perp(\psi_0) \rangle} \det \begin{pmatrix} \langle x, \omega(\psi_0) \rangle & \langle x, \omega^\perp(\psi_0) \rangle \\ \langle \mu, \omega(\psi_0) \rangle & \langle \mu, \omega^\perp(\psi_0) \rangle \end{pmatrix}$$

In particular  $\varphi_1(x, \psi = \psi_0) = 0$  if the determinant of Gram matrix vanishes, i.e. when  $x$  is parallel to  $\mu$ .

*Example 5.3:* Assume again  $M = \mathbf{R}^2$ . For  $a, b \in \mathbf{R}$ , let  $\rho(x) = 1 + ax_1^2 + bx_2^2$ . Consider Hamiltonian on  $T^*\mathbf{R}^2$  of the form

$$H(x, p) = \frac{\langle \mu, p \rangle}{\rho(x)}$$

with  $\mu = (1, 0)$ . A computation shows that, with  $x_1 = \varphi \cos \psi$ ,  $x_2 = \varphi \sin \psi$

$$\begin{aligned} \langle -\partial_x H, \omega(\psi) \rangle &= \frac{2\varphi \cos \psi}{\rho^2(x)} (a \cos^2 \psi + b \sin^2 \psi) \\ -\rho^2(x) \langle -\partial_x H, \omega(\psi) \rangle &= (\rho(x) + 2\varphi^2 \cos^2 \psi (a - b)) \sin \psi \end{aligned}$$

so choosing  $a > b > 0$  and  $E > 1$ , we see that  $\mathcal{G}(\Lambda) = \emptyset$ . Otherwise, there may be a unique glancing point near  $x = 0$ . Such a non-transversality is called a *kiss* in [ElGr].

Let us first describe  $\partial\Lambda_+(\tau)$  in case (1.15). The intersection of  $\Lambda$  with the energy surface  $H = E - \tau$ , is given by the implicit equation  $H(\varphi\omega(\psi), \omega(\psi)) = E - \tau$ , which usually defines a smooth 1-D isotropic submanifold. To avoid boundaries, we assume  $\partial\Lambda_+(\tau)$  is a closed isotropic manifold of dimension  $n - 1$ , i.e.  $\partial\Lambda_+(\tau)$  is defined by a map

$$(5.4) \quad \mathbf{R} \times \mathbf{T}^{n-1} \rightarrow \partial\Lambda_+(\tau) : (\tau, \varpi) \mapsto (\varphi, \psi) = (\varphi(\tau, \varpi), \psi(\tau, \varpi))$$

verifying

$$(5.5) \quad \rho(\varphi(\tau, \varpi)\omega \circ \psi(\tau, \varpi)) = \frac{1}{E - \tau}$$

It is clear in this case that  $\Lambda_+$  can be parametrized by  $(t, \psi)$ . Recall from Proposition A.1 that  $\langle \dot{X}, P_\psi \rangle = \langle \dot{P}, X_\psi \rangle$ . So taking the limit  $t \rightarrow 0_+$  readily implies

$$(5.6) \quad \varphi \langle \nabla \rho(\varphi\omega(\psi), \omega^\perp(\psi)) \rangle = 0$$

on  $\partial\Lambda_+$ . By Proposition 5.2, if  $z \in \partial\Lambda_+$  is ordinary (in the sense of Definition 1.2), and  $\varphi \neq 0$ , then  $z$  is not glancing, and by (5.6)  $\langle \nabla \rho(\varphi\omega(\psi), \omega(\psi)) \rangle = \pm |\nabla \rho| \neq 0$ . Taking derivative of (5.5) with respect to  $\tau, \varpi$  we get

$$\partial_\tau \varphi(\tau, \varpi) = \pm |\nabla \rho|^{-1} (E - \tau)^{-2}, \quad \partial_\varpi \varphi(\tau, \varpi) = 0$$

So  $\varphi = \varphi(\tau)$ , which can be inverted by implicit functions theorem, so  $\tau = \tau(\varphi)$  and it follows that we can take  $\psi(\tau, \varpi) = \varpi$ , i.e.  $\varpi = \psi$ . So in case (1.15)  $\Lambda_+$  can be parametrized by  $(t, \psi)$ , and  $\varphi = \varphi(\tau)$ .

In the general case, we replace (5.5) by

$$(5.8) \quad H(\varphi(\tau, \varpi)\omega \circ \psi(\tau, \varpi), \omega \circ \psi(\tau, \varpi)) = E - \tau$$

Assume  $\Lambda_+$  can be again parametrized by  $(t, \psi)$ , then we still have  $\langle -\partial_x H, X_\psi \rangle = \langle \partial_p H, P_\psi \rangle$ . If  $z \in \partial\Lambda_+$  is an ordinary point, then differentiating (5.8) with respect to  $\tau, \varpi$  gives

$$(5.9) \quad \partial_\tau \varphi = \frac{1 + \varphi(\partial_\tau \psi - 1) \langle \partial_x H, \omega^\perp(\psi) \rangle}{\langle -\partial_x H, \omega(\psi) \rangle}, \quad \partial_\varpi \varphi = \frac{\varphi(\partial_\varpi \psi - 1) \langle \partial_x H, \omega^\perp(\psi) \rangle}{\langle -\partial_x H, \omega(\psi) \rangle}$$

which simplifies again if  $\varpi = \psi$ , so that  $\partial_\tau \psi = 0$ .

Next we discuss the properties of Hamiltonian flow issued from  $\Lambda$  using eikonal coordinates as in Sect.4.1. Consider Hamiltonian  $\tau + H(x, p)$  on  $T^*(\mathbf{R}^n \times \mathbf{R}_+)$  and the Lagrangian manifold  $\tilde{\Lambda}_+ = \bigcup_\tau \Lambda_+(\tau)$  in the extended phase-space, i.e.

$$(5.10) \quad \begin{aligned} \tilde{\Lambda}_+ &= \{(x, p; t, \tau) : \tau + H(x, p) = E, z(t) = (X(t, \varphi, \psi, \tau), P(t, \varphi, \psi, \tau)), \\ &\quad z(0) \in \partial\Lambda_+(\tau), t \geq 0\} \subset T^*(M \times \mathbf{R}_+) \end{aligned}$$

Assume that

$$\tilde{\iota}: \tilde{\Lambda}_+ \rightarrow T^*(M \times \mathbf{R}_+)$$

is a Lagrangian embedding (this holds true if we take  $t, \tau$  small enough). Let  $\tilde{X} = (X(t, \varphi, \psi, \tau))^t$ , and  $\tilde{P} = (P(t, \varphi, \psi, \tau))^\tau$  in the extended phase-space. The action on  $\tilde{\Lambda}_+$  is of the form

$$(5.11) \quad \langle \tilde{P}, d\tilde{X} \rangle = \langle P(t, \varphi, \psi, \tau), dX(t, \varphi, \psi, \tau) \rangle + \tau dt$$

Here  $\varphi$  is still considered as a variable.

With the notations of (4.1)-(4.2), let  $\tilde{n} = n + 1$ ,  $k = 2$ ,  $\tilde{\psi} = \psi$ ,  $\tilde{\phi} = (t, \varphi)$ . By the preceding discussion, we may assume  $\tau = \tau(\varphi, \psi)$ , which simplifies further to  $\tau = \tau(\varphi)$  in case of Hamiltonian (1.15). Recall  $S(t, \varphi, \psi, \tau) = \varphi + m(E - \tau)t + S_0$  from (2.10). In the extended phase-space, we set  $\tilde{S}(t, \varphi, \psi, \tau) = S(t, \varphi, \psi, \tau) + t\tau$ , one has also to differentiate with respect to  $\tau$ , so that  $dS + \tau dt + t d\tau = \langle P, dX \rangle + \tau dt$ , or

$$d\varphi + m(E - \tau)dt + (1 - m)t d\tau = \langle P, \dot{X} \rangle dt + X_\varphi d\varphi + X_\psi d\psi + X_\tau d\tau$$

using  $d\tau = \tau_\varphi d\varphi + \tau_\psi d\psi$  we get by identification (still with the notation  $y_x = \frac{\partial y}{\partial x}$ )

$$(5.12) \quad \begin{aligned} \langle P, X_\varphi \rangle + \tau_\varphi \langle P, X_\tau \rangle &= 1 + (1 - m)t\tau_\varphi \\ \langle P, X_\psi \rangle + \tau_\psi \langle P, X_\tau \rangle &= (1 - m)t\tau_\psi \\ \langle P, \dot{X} \rangle &= mH \end{aligned}$$

We look for a “left inverse” of

$$(\dot{X}, \tilde{X}_\varphi) = \begin{pmatrix} 1 & 0 \\ \dot{X} & X_\varphi + X_\tau \tau_\varphi \end{pmatrix}$$

We try  $\tilde{\Pi}_1 = \begin{pmatrix} 1 & -mH \\ 0 & P \end{pmatrix}$ , which gives, using (5.12)

$$\tilde{\Pi}_1^* \begin{pmatrix} 1 & 0 \\ \dot{X} & X_\varphi + X_\tau \tau_\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + (1 - m)t\tau_\varphi \end{pmatrix}$$

so we choose

$$(5.13) \quad \tilde{\Pi} = \tilde{\Pi}_1 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1 - (m-1)t\tau_\varphi} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha mH \\ 0 & \alpha P \end{pmatrix}, \quad \alpha = \alpha(t, \varphi, \psi) = (1 + (1 - m)\tau_\varphi)^{-1}$$

which is well defined when  $m = 1$  or whenever  $1 - (m - 1)t\tau_\varphi > 0$ . By [DoNaSh, Lemma 6] (which doesn't assume  $\tilde{k}$  to be the rank of  $\tilde{\Lambda}_+$ ), we know in particular that the equation

$$(5.14) \quad \tilde{\Pi}^* \begin{pmatrix} 0 \\ x - X(t, \varphi, \psi, \tau) \end{pmatrix} = 0 \iff \langle P(t, \varphi, \psi, \tau), x - X(t, \varphi, \psi, \tau) \rangle = 0$$

has a unique solution  $t = t_1(x, \psi, \tau)$ ,  $\varphi = \varphi_1(x, \psi, \tau)$ . To meet the initial condition, we have seen that  $\tau = \tau(\varphi, \psi)$ , which can be inverted as  $\varphi = \varphi(\tau, \psi)$ . So we interpret  $\varphi$  as a parameter we have to adjust so to stay on the energy surface, say  $\tau = 0$ , in the same way (Proposition 4.3(ii)) that  $\lambda$  was a

function of  $(x, \psi)$ , so that  $\lambda(x, \psi) = 1$ . So we can omit  $\tau$  from the notations, its role being played by  $\varphi$ . Thus as in [DoNaSh, Lemma 6] we can prove the following result, at least for small  $t > 0$ .

**Proposition 5.3:** *There is a open set  $U \subset \mathbf{R}^{n+1}$ ,  $(t, \varphi, \psi) \in U$ , such that*

$$(5.15) \quad \kappa : U \mapsto \mathbf{R}_{t,x}^{n+1} \times \mathbf{R}^{n-1}, \quad (t, \varphi, \psi) \mapsto (\tilde{X}(t, \varphi, \psi), \psi)$$

*is an embedding. Then there is a neighbhd  $V$  of  $C = \kappa(U)$ , such that*

$$(5.16) \quad \langle P(t, \varphi, \psi), x - X(t, \varphi, \psi) \rangle = 0, \quad (t, x, \psi) \in V$$

*has a unique solution  $t = t_1(x, \psi)$ ,  $\varphi = \varphi_1(x, \psi)$  satisfying the condition*

$$(5.17) \quad (t, x, \psi) \in C \implies x = X(t_1(x, \psi), \varphi_1(x, \psi), \psi)$$

Consider now  $\mathcal{M}(t, \varphi, \psi, \tau) = (\tilde{\Pi}; \tilde{P}_{\tilde{\psi}} - \tilde{P}_{\tilde{\phi}} \tilde{\Pi}^* \tilde{X}_{\tilde{\psi}})$  as in (4.4). In [DoNaSh, Lemma 9],  $\mathcal{M}$  is constructed in a canonical chart of rank  $\tilde{k}$  for the Lagrangian embedding (without boundary)  $\iota : \Lambda \rightarrow T^*M$ , so that  $\mathcal{M}$  becomes an invertible matrix. Here we consider instead (4.4) as an Ansatz, and check that  $\det \mathcal{M}$  defines the (inverse) density on  $\Lambda_+$ . With  $\alpha$  as in (5.13) we have

$$\mathcal{M} = \begin{pmatrix} 1 & -mH\alpha & \tau_{\psi} - \alpha(m-1)\tau_{\varphi}\tau_{\psi}t \\ 0 & \alpha P & P_{\psi} + \tau_{\psi}P_{\tau} + \alpha(m-1)\tau_{\psi}t(P_{\varphi} + \tau_{\varphi}P_{\tau}) \end{pmatrix}$$

so that

$$(5.19) \quad \det \mathcal{M} = \alpha(t, \varphi, \psi) \det(P, P_{\psi} + \alpha\tau_{\psi}(P_{\tau} + (m-1)tP_{\varphi}))$$

Note that when  $m = 1$  (which implies that  $\tau + H(x, p)$  is homogeneous of degree 1 as an Hamiltonian on  $T^*(M \times \mathbf{R})$ ),  $\det \mathcal{M} = \det(P, d_{\psi}P)$ , where  $d_{\psi}P = P_{\psi} + \tau_{\psi}P_{\tau}$ . On the other hand when  $\tau_{\psi} = 0$ ,  $\det \mathcal{M} = \alpha(t, \varphi, \psi) \det(P, P_{\psi})$  is non zero for small  $t$ . By the discussion above, this holds for Hamiltonian (1.15), so for  $m = 1$  we recover the same (inverse) density on  $\Lambda_+$  as in Sect.4. Moreover  $\det \mathcal{M} = \det(P, P_{\psi}) > 0$  for all  $t > 0$  since the proof of Proposition 4.4 carries to this case.

## 5.2 Construction of the phase function in the extended phase-space

From now on we consider a general Hamiltonian  $H(x, p)$  positively homogeneous of degree  $m = 1$ , so that  $\det \mathcal{M} = \det(P, d_{\psi}P)$  can be interpreted as the (inverse) density on  $\Lambda_+$  when  $\det(P, d_{\psi}P) \neq 0$ .

Consider HJ equation for the phase function parametrizing  $\Lambda_+$  as in (2.5)

$$(5.25) \quad \partial_t \Phi + H(x, \partial_x \Phi) = E, \quad \Phi|_{t=0} = \langle x, \omega(\psi) \rangle$$

We find as before an integral manifold  $\Lambda_{\Phi,t} = \{p = \partial_x \Phi(t, x)\} \subset T^*\mathbf{R}^n$ . As before we let  $\Phi$  depend on additional " $\theta$ -variables. As in (4.7), we could make the choice

$$\Phi_0(x, t, \psi, \varphi, \tau) = m(E - \tau)t + \varphi + \langle P(t, \varphi, \psi, \tau), x - X(t, \varphi, \psi, \tau) \rangle$$

as in [DoNaSh, formula (2.40)], which allows for some uniformity in  $\tau$ . But as in Sect.4, we add  $\lambda$  to the “ $\theta$ ”-variables, and put instead

$$(5.26) \quad \Phi(x, t, \psi, \varphi, \lambda, \tau) = mEt + \varphi + \lambda \langle P(t, \varphi, \psi, \tau), x - X(t, \varphi, \psi, \tau) \rangle$$

where the integral curve  $(X, P)$  will be eventually computed on  $\tau = 0$ . Remember here  $\tau = \tau(\varphi, \psi)$ . We check that for  $\lambda = 1$

$$\Phi|_{t=0} = \int P dX + \langle \omega(\psi), x \rangle - \varphi = \varphi + \langle \omega(\psi), x \rangle - \varphi = \langle \omega(\psi), x \rangle$$

so  $\Phi|_{t=0} = \langle \omega(\psi), x \rangle$  satisfies the initial condition. The dependence on  $\lambda$  in the phase thus becomes linear, but then we need to restrict to  $\tau = 0$ . Variables  $(x, t, \varphi, \psi, \lambda)$  will be used in various combinations, reflecting the role of the boundary  $\partial\Lambda_+$ , or more generally  $\partial\Lambda_t$ . In the sequel we keep  $m > 1$  as a free parameter, but this really makes sense when  $m = 1$ , in order to interpret  $\det(P, d_\psi P)$  as an (inverse) density on  $\Lambda_+$ .

Taking partial derivatives in (5.26) with respect to variables  $t, \psi, \lambda$ , with  $\partial_t \int \langle P, dX \rangle = mE$ ,  $\partial_\varphi \int \langle P, dX \rangle = 1$ ,  $\partial_t \int \langle P, dX \rangle = 0$ , we find

$$(5.27) \quad \begin{aligned} \partial_t \Phi &= (1 - \lambda)mE + \lambda \langle \dot{P}, x - X \rangle \\ \partial_\psi \Phi &= \lambda \langle P_\psi, x - X \rangle \\ \partial_\lambda \Phi &= \langle P, x - X \rangle \end{aligned}$$

To this set of equations we could add

$$\partial_\varphi \Phi = 1 - \lambda + \lambda \langle P_\varphi, x - X \rangle$$

but actually we will not use this equation. The critical point  $x = X(t, \varphi, \psi)$  is uniquely determined (for small  $t$ ) : namely the determinant of the  $2 \times 2$  system for the last 2 Eq. is given by  $\det(P, P_\psi)$ , and this is non zero at  $t = 0$ . So when  $\lambda = 1$  and  $x = X(t, \varphi, \psi)$  belongs to the critical set  $C_\Phi$ . Taking differential on  $C_\Phi$  gives

$$(5.28) \quad \begin{aligned} d\partial_t \Phi &= -Ed\lambda + \langle \dot{P}, dx - dX \rangle \\ d\partial_\psi \Phi &= \langle P_\psi, dx - dX \rangle \\ d\partial_\lambda \Phi &= \langle P, dx - dX \rangle \end{aligned}$$

which, again, we could implement by

$$d\partial_\varphi \Phi = -d\lambda + \langle P_\varphi, dx - dX \rangle$$

It is natural to expect the (inverse) density  $F[\Phi, d\mu_+]|_{C_\Phi}$  as in (4.10) (up to a constant factor) to be  $\det(P, d_\psi P)$  computed above. Namely we have:

**Proposition 5.4:** *Let  $H(x, p)$  be positively homogeneous of degree 1 with respect to  $p$  on  $T^*M \setminus 0$ . Then  $\Phi(x, t, \varphi, \psi, \lambda, \tau)$  given in (5.26) solves HJ Eq. (2.15). We have*

$$(5.29) \quad F[\Phi, d\mu_+]|_{C_\Phi} = \frac{dt \wedge d\psi \wedge d\varphi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi)}{dx \wedge dt \wedge d\psi \wedge d\varphi \wedge d\lambda} = mE \det(P, d_\psi P)$$

Then the critical set  $C_\Phi$  is determined as in Proposition 5.2 by  $x = X(t, \varphi, \psi)$  (which can be inverted as  $t = t_1(x, \psi)$ ,  $\varphi = \varphi_1(x, \psi)$ ) and  $\lambda = 1$ . It coincides with the set  $\kappa(U)$  defined in Proposition 5.3. When  $\det(P, d_\psi P) \neq 0$ , this is (inverse) density on  $\Lambda_+$ , and  $\Phi$  is a non-degenerate phase function defining  $\Lambda_+$ .

*Proof:* We assume  $n = 2$  for simplicity and start to compute  $F[\Phi, d\mu_+]|_{C_\Phi}$ . We expand the  $2n+2$ -form

$$\omega = dt \wedge d\psi \wedge d\varphi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi) = \omega_1 + \omega_2$$

where

$$\omega_1 = -mEdt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge \langle d_\psi P, dx - dX \rangle \wedge \langle P, dx - dX \rangle$$

$$\omega_2 = dt \wedge d\psi \wedge d\varphi \wedge \langle \dot{P}, dx - dX \rangle \wedge \langle d_\psi P, dx - dX \rangle \wedge \langle P, dx - dX \rangle$$

Using (5.28) a short calculation gives  $\omega_2 = 0$ , while only the term  $-mEdt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge \langle d_\psi P, dx \rangle \wedge \langle P, dx \rangle$  contributes in  $\omega_1$ , so  $\omega_1 = -mE \det(d_\psi P, P) dt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge dx$ . So we get (5.29), and the Proposition follows as in Proposition 4.1. See also [DoNaSh, Lemma 10]. ♣

Assuming  $\varphi = \varphi(\tau)$  as for Hamiltonian (1.15) and fixing  $\tau = 0$ , there is no longer dependence on  $\varphi$ . Thus we can proceed as in Sect.4, define ordinary and special points precisely as in Definition 2.1, and extend Proposition 4.3 to this case. The only requirement is that the density computed in Proposition 5.3 is non vanishing. In the same way, Lemmas 4.6 and 4.7 for Hamiltonian (1.15) extend to this case.

It may be more convenient to determine simultaneously the critical point  $(t, \psi)$  instead of  $(t, \lambda)$ . Namely consider the system  $(\frac{\partial_t \Phi}{\partial_\psi \Phi}) = 0$ , i.e.  $\Phi'_{(t, \psi)} = 0$ . We have

$$(5.31) \quad \Phi''_{t, \psi} = \begin{pmatrix} \frac{\partial^2 \Phi}{\partial t^2} & \frac{\partial^2 \Phi}{\partial t \partial \psi} \\ \frac{\partial^2 \Phi}{\partial \psi \partial t} & \frac{\partial^2 \Phi}{\partial \psi^2} \end{pmatrix} = \begin{pmatrix} \langle \dot{P}, \dot{X} \rangle & \langle P_\psi, \dot{X} \rangle \\ \langle P_\psi, \dot{X} \rangle & \langle P_\psi, X_\psi \rangle \end{pmatrix}$$

where  $\langle P_\psi, \dot{X} \rangle = \langle X_\psi, \dot{P} \rangle$ . Let

$$(5.32) \quad D(z(t)) = \det \Phi''_{t, \psi}$$

(for short we omit other variables in the notation  $z(t)$ ) and

$$\Omega(t) = \{\psi : z(t) = (X(t, \varphi, \psi), P(t, \varphi, \psi)), D(z(t)) \neq 0\}$$

For  $t = 0$ , we have  $\det D(z(0)) = \varphi \langle -\partial_x H, \partial_p H \rangle - \langle \partial_p H, \omega(\psi)^\perp \rangle^2$ , so for Hamiltonian (1.15) this is non vanishing when  $z(0)$  is an ordinary point where  $\varphi \neq 0$ , and  $D(z(0)) = 0$  if  $z(0)$  is glancing. So when  $D(z(t)) \neq 0$ , implicit function theorem shows that  $\Phi'_{(t, \psi)} = 0$  is equivalent to  $t = t_0(x, \varphi)$ ,  $\psi = \psi_0(x, \varphi)$ . We conjecture that  $D(z(t)) = 0$  iff  $z(0)$  is a glancing point.

In other words, for the vertical plane (Proposition 4.3(ii)) we express the two variables  $(t, \lambda)$  as a function of the  $2n - 1$  variables  $(x, \psi)$ , while for the cylinder we express the  $n$  variables  $(t, \psi)$  as a function of the  $n + 1$  variables  $(x, \varphi)$  (when  $n = 2$  there is the same number of variables in the source and target spaces).



*Remark 5.1:* Assume  $\det(\dot{X}, X_\psi) \neq 0$  (i.e.  $z(t)$  is not a focal point), then  $\Phi''_{t,\psi}$  is just Gram matrix of  $(\dot{P}, P_\psi)$  in the basis  $\dot{X}, X_\psi$ , and  $\det(\dot{P}, P_\psi) = 0$  iff  $D(z(t)) = 0$ . We can interchange of course the role of the pairs of vectors.

Again it is instructive to look at the model Hamiltonian  $H = \langle p, \mu \rangle$ , where  $X(t, \psi) = \mu t + \varphi \omega(\psi)$ ,  $P(t, \psi) = \omega(\psi)$ , and  $\Phi(t, \varphi, \psi, \lambda) = Et + (1 - \lambda)\varphi + \lambda \langle \omega(\psi), x - \mu t \rangle$  is independent of  $\varphi$  when  $\lambda = 1$ . Since  $\langle \mu, \omega(\psi) \rangle = E$ , we have  $\dot{\Phi} = E - \lambda \langle \mu, \omega(\psi) \rangle = (1 - \lambda)E$ , and  $\partial_\psi \Phi = \lambda \langle \omega^\perp(\psi), x - \mu t \rangle$ . So on the critical set  $x = X(t, \varphi, \psi)$ ,  $\lambda = 1$ , we have  $\dot{\Phi} = \partial_\psi \Phi = 0$ ; assuming Lagrangian intersection, the Hessian of  $\Phi$  with respect to  $(t, \psi)$  has determinant  $D(z(t)) = -(\frac{\partial^2 \Phi}{\partial t \partial \psi})^2 = -\langle \mu, \omega^\perp(\psi) \rangle^2$ . So if  $\langle \mu, \omega^\perp(\psi) \rangle \neq 0$ ,  $\psi = \psi_0$  such that  $\langle \mu, \psi_0 \rangle = E$ , i.e.  $z(t)$  is non glancing, then  $D(z(t)) \neq 0$  is a constant. We notice also that the time component of the critical point is precisely  $t_1(x, \psi)$  computed in Proposition 5.2.

### 5.3 Reduced parametrizations of $\Lambda_+$ in case of the “conformal metric”.

As in Sect.4 we could retrieve global informations in case of Hamiltonian (1.15) when  $m = 1$ , due to the fact that the (inverse) density  $\det(P, P_\psi)$  on  $\Lambda_+$  is globally positive. But we content ourselves to find the critical point and the critical value of  $(t, \psi) \mapsto \Phi(x, t, \varphi, \psi, \lambda = 1)$ , whenever  $D(z(t)) \neq 0$  (i.e. away from special points and  $\varphi = 0$ ). This will provide the microlocal structure of (1.5) near those points of  $\partial\Lambda_+$ .

Take polar coordinates on  $M$  of the form  $(r, \theta)$  such that  $(r, \theta) = (\varphi, \psi)$  parametrize a point on  $\Lambda$  near  $\partial\Lambda_+$ . We make the identification  $x = (r, \theta)$ .

• *Ordinary points.* We show that near an ordinary point  $z(0) \in \partial\Lambda_+$ ,  $\text{rank } d\pi_x(z(0)) = 2$ . Namely we have:

**Proposition 5.5:** *Let  $\Phi'_{t,\psi}(x, t, \varphi, \psi, \lambda = 1) = (\partial_t \Phi, \partial_\psi \Phi)$ , and assume  $D(z(0)) \neq 0$  as in (5.32) i.e.  $z(0)$  is an ordinary point with  $\varphi \neq 0$ . Then for  $x$  sufficiently close to  $\varphi\omega(\psi)$ , the system  $\Phi'_{t,\psi}(x, t, \psi, \lambda = 1) = 0$  is equivalent to  $t = t(x), \psi = \psi(x)$ , and  $\varphi = \varphi(x)$  when  $x$  is sufficiently close to  $\varphi\omega(\psi)$ ,  $(\varphi, \psi) \in \Omega(z(0))$ . The critical value of  $\Phi$  takes the form*

$$(5.34) \quad \Psi(r, \theta; \varphi, \psi) = r \cos(\theta - \psi(x)) + \mathcal{O}(|r - \varphi(x), \theta - \psi(x)|) = r + \mathcal{O}(|r - \varphi(x), \theta - \psi(x)|)$$

*In particular  $\text{rank } d\pi_x(z(0)) = 2$ .*

*Proof:* Since  $D(z(0)) \neq 0$ , implicit function theorem shows that  $\Phi'_{t,\psi} = 0$  (we omitted  $\lambda = 1$ ) has a unique solution  $t = t_0(x, \varphi), \psi = \psi_0(x, \varphi)$ , i.e.  $t = t_0(r, \theta, \varphi), \psi = \psi_0(r, \theta, \varphi)$ . Differentiating  $\Phi'_{t,\psi} = 0$

along  $\Lambda_+$  with respect to  $r, \theta$  and  $\varphi$  we find

$$\begin{aligned}
(5.35) \quad & \langle \dot{P}, \dot{X} \rangle \frac{\partial t_0}{\partial r} + \langle \dot{P}, X_\psi \rangle \frac{\partial \psi_0}{\partial r} = \langle \dot{P}, \omega(\theta) \rangle \\
& \langle P_\psi, \dot{X} \rangle \frac{\partial t_0}{\partial r} + \langle P_\psi, X_\psi \rangle \frac{\partial \psi_0}{\partial r} = \langle P_\psi, \omega(\theta) \rangle \\
& \langle \dot{P}, \dot{X} \rangle \frac{1}{r} \frac{\partial t_0}{\partial \theta} + \langle \dot{P}, X_\psi \rangle \frac{1}{r} \frac{\partial \psi_0}{\partial r} = \langle \dot{P}, \omega^\perp(\theta) \rangle \\
& \langle P_\psi, \dot{X} \rangle \frac{1}{r} \frac{\partial t_0}{\partial r} + \langle P_\psi, X_\psi \rangle \frac{1}{r} \frac{\partial \psi_0}{\partial r} = \langle P_\psi, \omega^\perp(\theta) \rangle \\
& \langle \dot{P}, \dot{X} \rangle \frac{\partial t_0}{\partial \varphi} + \langle \dot{P}, X_\psi \rangle \frac{\partial \psi_0}{\partial \varphi} = -\langle \dot{P}, X_\varphi \rangle \\
& \langle P_\psi, \dot{X} \rangle \frac{\partial t_0}{\partial \varphi} + \langle P_\psi, X_\psi \rangle \frac{\partial \psi_0}{\partial \varphi} = -\langle P_\psi, X_\varphi \rangle
\end{aligned}$$

Using the relation  $\langle \dot{P}, X_\psi \rangle = \langle P_\psi, \dot{X} \rangle$ , the 3 sub-systems have determinant

$$D(z(t)) = \det \begin{pmatrix} \langle \dot{P}, \dot{X} \rangle & \langle P_\psi, \dot{X} \rangle \\ \langle P_\psi, \dot{X} \rangle & \langle P_\psi, X_\psi \rangle \end{pmatrix} \neq 0$$

So for  $(\varphi, \psi) \in \Omega(z(s))$ , (5.35) has the unique solution with the condition

$$(5.36) \quad t_0(\varphi, \psi, \varphi) = 0, \quad \psi_0(\varphi, \psi, \varphi) = \psi, \quad (\varphi, \psi) \in \Omega(0)$$

On the other hand we know from Proposition 5.2 that on  $C_\Phi$ , i.e. when  $x = X(t, \varphi, \psi)$ , we have  $\varphi = \varphi_1(x, \psi)$  and  $t = t_1(x, \psi)$ . This gives :

$$g(x, \psi) = \psi - \psi_0(x, \varphi_1(x, \psi)) = 0$$

Compute  $\partial_\psi g(x, \psi) = 1 - \frac{\partial \psi_0}{\partial \varphi} \frac{\partial \varphi_1}{\partial \psi}$  at  $x = \varphi \omega(\psi)$ . Combining Proposition 5.3 and [DoNaSh, Lemma 7] (which still doesn't assume  $\tilde{k} = 2$  to be the rank of  $\tilde{\Lambda}_+$ ), we get

$$\partial_\psi \begin{pmatrix} t_1 \\ \varphi_1 \end{pmatrix} (x, \psi) = - \begin{pmatrix} 1 & 0 \\ -H & {}^t P \end{pmatrix} \begin{pmatrix} 0 \\ X_\psi \end{pmatrix} + \mathcal{O}(|x - X(t_1(x, \psi), \varphi_1(x, \psi), \psi)|)$$

Since  $\langle P, X_\psi \rangle = 0$ , we get in particular

$$\frac{\partial \varphi_1}{\partial \psi}(x, \psi) = \mathcal{O}(|x - X(t_1(x, \psi), \varphi_1(x, \psi), \psi)|) = o(1)$$

Thus  $\partial_\psi g(x, \psi) \neq 0$  along  $C_\Phi$  and implicit function theorem gives  $\psi = \psi_1(x)$ . Substituting into  $t = t_1(x, \psi) = t_0(x, \varphi)$  we get also  $t = t_1(x, \psi_1(x)) = t_0(x, \varphi_1(x, \psi_1(x)))$ . Substituting into (5.26) (where we have assumed  $m = 1$ ) gives the critical value (for  $\lambda = 1$ ) where all “ $\theta$ -variables” have been eliminated

$$(5.37) \quad \Psi(r, \theta) = Et_1(t, \theta) + \varphi_1(r, \theta) + \langle P(t_1, \varphi_1, \psi_1), r\omega(\theta) - X(t_1, \varphi_1, \psi_1) \rangle$$

which we expand around  $(r, \theta) = (\varphi, \psi)$  using (5.36). With

$$t_1(t, \theta) = o(1), \quad \varphi_1(t, \theta) = \varphi + o(1), \quad P(t_1, \varphi_1, \psi_1) = \omega(\psi) + o(1), \quad X(t_1, \varphi_1, \psi_1) = \varphi\omega(\psi) + o(1)$$

substituting into (5.37) we find  $\Psi(r, \theta) = r \cos(\theta - \psi) + o(1)$ , where

$$o(1) = \mathcal{O}(|x - X(t_1(x, \psi), \varphi_1(x, \psi), \psi)|) = \mathcal{O}(|r - \varphi, \theta - \psi|)$$

This proves (5.34) (no “ $\theta$ -parameters”). ♣

• *Special points.* We know that a special point at  $t = 0$  is also a critical point of  $\rho$ . We assume  $\varphi \neq 0$ , so it is not glancing (or residual).

**Proposition 5.6:** *Let  $\Phi'_{t,\psi,\lambda}(x, t, \varphi, \psi, \lambda) = (\partial_t \Phi, \partial_\psi \Phi, \partial_\lambda \Phi)$ , and assume  $z(0)$  is an special point i.e.  $\nabla \rho(\varphi\omega(\psi)) = 0$ , but  $\varphi \neq 0$ . Then for  $x$  sufficiently close to  $\varphi\omega(\psi)$ , the system  $\Phi'_{t,\psi,\lambda}(x, t, \psi, \lambda) = 0$  is equivalent to  $t = t_2(x, \varphi), \psi = \psi_2(x, \varphi)$ . The critical value of  $\Phi$  takes the form*

$$(5.39) \quad \Psi(x, t, \varphi, \psi, \lambda = 1) = mHt_2(x, \varphi) + \varphi + \langle P(t, \varphi, \psi), x - X(t, \varphi, \psi) \rangle|_{t=t_2(x, \varphi), \psi=\psi_2(x, \varphi)}$$

In particular  $\text{rank } d\pi_x(z(0)) = 1$ .

*Proof:* We have as in (4.44)

$$\det \frac{\partial^2 \Phi}{\partial(t, \psi, \lambda)}|_{t=0} = -(mH)^2 \langle P_\psi, X_\psi \rangle|_{t=0} = -(mH)^2 \varphi$$

this is non zero for non zero  $\varphi$ , so by implicit functions theorem  $(t, \psi, \lambda) = (t_2(x, \varphi), \psi_2(x, \varphi), \lambda_2(x, \varphi))$ . Differentiating  $\Phi'_{t,\psi,\lambda}(x, t, \varphi, \psi, \lambda) = 0$  with respect to  $x, \varphi$  gives the triangular systems

$$(5.40) \quad \begin{aligned} \langle P_\psi, \dot{X} \rangle^t \left( \frac{\partial t_2}{\partial x} \right) + \langle P_\psi, X_\psi \rangle^t \left( \frac{\partial \psi_2}{\partial x} \right) &= P_\psi \\ \langle \dot{P}, \dot{X} \rangle^t \left( \frac{\partial t_2}{\partial x} \right) + \langle \dot{P}, X_\psi \rangle^t \left( \frac{\partial \psi_2}{\partial x} \right) + mH^t \left( \frac{\partial \lambda_2}{\partial x} \right) &= \dot{P} \\ mH^t \left( \frac{\partial t_2}{\partial x} \right) &= P \end{aligned}$$

and

$$(5.41) \quad \begin{aligned} \langle \dot{P}, \dot{X} \rangle \frac{\partial t_2}{\partial \varphi} + \langle \dot{P}, X_\psi \rangle \frac{\partial \psi_2}{\partial \varphi} + mH \frac{\partial \lambda_2}{\partial \varphi} &= -\langle \dot{P}, X_\varphi \rangle \\ \langle P_\psi, \dot{X} \rangle \frac{\partial t_2}{\partial \varphi} + \langle P_\psi, X_\psi \rangle \frac{\partial \psi_2}{\partial \varphi} &= -\langle P_\psi, X_\varphi \rangle \\ mH \frac{\partial t_2}{\partial \varphi} &= -1 \end{aligned}$$

We first solve (5.41), using  $a = b$ , this gives  $\frac{\partial \psi_2}{\partial \varphi} = -\frac{\langle P_\psi, X_\varphi \rangle}{c}$ , and  $\frac{\partial \lambda_2}{\partial \varphi} = 0$ , as expected since  $\lambda = 1$  and  $\partial \Lambda_+(\tau)$  is now parametrized by  $\varphi$ . Then (5.40), still evaluated at  $t = 0$ , gives  $\frac{\partial t_2}{\partial x} = \frac{P}{mH}$ ,  $\frac{\partial \psi_2}{\partial x} = \frac{P_\psi}{c}$

and again  $\frac{\partial \lambda_2}{\partial x} = 0$ . Now we put as before  $\Psi(x, \varphi) = \Phi(x, t, \varphi, \psi, \lambda = 1)|_{t=t_2(x, \varphi), \psi=\psi_2(x, \varphi)}$ , with  $\Phi$  given in (5.39). Taking derivative with respect to  $\varphi$  gives (we evaluate at  $t = 0$ )

$$(5.43) \quad \partial_\varphi \Psi(x, \varphi) = \frac{\partial \psi_2}{\partial \varphi} \langle \omega^\perp(\psi), x - X(t_2(x, \varphi), \varphi, \psi_2(x, \varphi)) \rangle$$

and differentiating again (5.43) with respect to  $\varphi$

$$-\partial_\varphi^2 \Psi(x, \varphi) = \frac{\langle P_\psi, X_\varphi \rangle^2}{c} = 0$$

So (3.9) ensures that  $\text{rank } d\psi_x(z(0)) = 1$ . ♣

Note that the 3:rd Eqn's in (5.40) and (5.41) yield the condition that  $P(t_2(x, \varphi), \varphi, \psi_2(x, \varphi))$  be independent of  $\varphi$ , and  $t_2, \psi_2$  satisfy the integral equation

$$mHt_2(x, \varphi) + \varphi = \int^x \langle P, dx \rangle$$

$P$  being evaluated along  $t = t_2(x, \varphi), \psi = \psi_2(x, \varphi)$ , which gives the eikonal ; cf. also Proposition 4.3 (iii).

Of course, these methods elaborated for computing the critical values of  $\Phi$  do not extend to a glancing point  $z(t)$  ( $\varphi = 0$ ), where all second derivatives of  $\Phi$  vanish. We already notice that, since  $P_\psi \neq 0$  at  $t = 0$ , the first Eq. (5.40) is singular in the limit  $\varphi \rightarrow 0$ , so there is no hope to recover the variational system for (5.40) and (5.41), differentiating  $\Phi'_{t, \psi, \lambda} = 0$  once again along  $\Lambda_+$  with respect to  $x$  and  $\varphi$ .

#### 5.4 Construction of Maslov canonical operator.

It goes the same way as in Sect. 4., using Propositions 5.5 and 5.6 instead of Proposition 4.3, and Lemmas 4.6 & 4.7 in the discussion according to ordinary or special points. For simplicity we work with the “conformal metric” (1.15) where we recall  $m = 1$ . To avoid glancing points, we will assume that  $\varphi \neq 0$  on  $\text{WF}_h f_h$ , i.e.  $\text{WF}_h f_h \cap T_0^* M = \emptyset$ .

We look for  $u_h$  of the form

$$u_h(x) = \int_0^\infty \int e^{i\Phi(x, t, \varphi, \psi, \lambda, \tau)/h} a(x, t, \varphi, \psi, \lambda, \tau) d\mu(\psi, \varphi, \lambda) dt$$

where  $\Phi$  as in (5.26), and  $d\mu(\psi, \varphi, \lambda)$  are defined in some charts: in a neighbhd of an ordinary point  $z(t)$ , we take  $d\mu(\psi, \lambda) = \mu_1(\psi, \lambda) d\psi d\lambda$ , and consider  $\Phi(x, t, \varphi, \psi, \lambda, \tau)$  as  $\Phi(x, t, \psi, \lambda)$  with fixed  $\varphi$ , as in Proposition 5.5. In a neighbhd of a special or residual point  $z(t)$  instead, we take  $d\mu(\psi, \varphi) = \mu_2(\psi, \varphi) d\psi d\varphi$ , and think of  $\Phi(x, t, \varphi, \psi, \lambda, \tau)$  as  $\Phi(x, t, \varphi, \psi)$  with  $\lambda = 1$ , as in Proposition 5.6. In both cases, we omit  $\tau$  from the notations, since it is related to  $\lambda$  and  $\varphi$  as before. These phases and densities can be chosen coherently, and define a “new” Maslov bundle, subordinated to the additional partition between ordinary/ special or residual points.

In an ordinary chart, we proceed as in Sect.4.

#### 5.5 Radially symmetric “conformal metric”

We choose  $m = 1$  for simplicity, and  $\rho(x)$  radially symmetric, i.e.  $\rho = \rho(|x|)$  with  $\rho$  a smooth function on  $\mathbf{R}_+$ .

We have  $\rho(|x|) = \rho(\varphi)$  (choosing the branch  $\varphi > 0$ ),  $\varphi = \varphi_0 > 0$  on  $L = \Lambda \cap \{H = E\}$ , and  $p dx = d\varphi$  on  $\Lambda$ . The hypothesis  $\varphi_0 > 0$  may be relaxed. We assume also  $\rho'(\varphi_0) \neq 0$ .

The phase function is now given by (5.16) with  $\varphi = \varphi_0$ . The variational system is obtained directly in the  $x - X$  variables as

$$\begin{aligned} d\partial_t \Phi &= -mH d\lambda + (dx - dX)^* \partial_t P \\ d\partial_\psi \Phi &= (dx - dX)^* P \partial_\psi P \\ d\partial_\lambda \Phi &= (dx - dX)^* P \end{aligned}$$

We check the linear independance. Let

$$d(\partial_t \Phi)\alpha + \langle d(\partial_\psi \Phi), \beta \rangle + d(\partial_\lambda \Phi)\gamma = 0$$

Identifying the term in  $d\lambda$  we find  $\alpha = 0$ , then  $(dx - dX)^*(\langle \partial_\psi P, \beta \rangle + \gamma P) = 0$ , which readily implies  $\beta = \gamma = 0$ , for small  $t$  since  $P|_{t=0} = \omega(\psi)$  and  $\partial_\psi P|_{t=0} = \omega(\psi)^\perp$ .

In a second step, we seek for a reduced generating function for  $\Lambda_+$  by eliminating  $t$  when  $t > 0$  by stationary phase. This follows easily from implicit functions theorem and the value at  $t = 0$

$$\partial_t^2 \Phi|_{x=X(\psi)} = -\langle \partial_t P, \partial_\psi X \rangle = \text{Const.} \langle \omega(\psi), \omega(\psi) \rangle \neq 0$$

(without any further assumption).

It follows that  $u_h$  has only “wave-part” component, until the Hamilton flow meets  $\varphi = 0$  where the analysis has slightly to be changed to take into account the singularity of  $\rho(|x|)$  at this point.

We complete this Example by the integration of motion (at least in 2-D), since the system is integrable (with  $H$  as in (1.15)). Introduce polar coordinates  $(r, \theta)$  and  $(s, \psi)$  on  $T^*\mathbf{R}^2$  (same parameter  $\psi$  as in the definition of  $\Lambda$ ), so that

$$(5.50) \quad x = r \omega(\theta), \quad p = s \omega(\psi)$$

and

$$(5.51) \quad p dx = s \cos(\psi - \theta) dr + rs \sin(\psi - \theta) d\theta$$

On  $H = \frac{s}{\rho(r)}$ , Hamilton equations are given by

$$\dot{r} = \frac{1}{\rho(r)}, \quad \dot{s} = \frac{\rho'(r)}{\rho(r)^2} s$$

that is

$$\frac{\dot{s}}{s} = \frac{\rho'(r)}{\rho(r)^2} = \frac{\rho'(r)}{\rho(r)} \dot{r}$$

Integrating these equations we find  $s(t) = s_0 \rho(r(t))$  so

$$(4.52) \quad P(t, \psi) = s_0 \frac{\rho(r(t))}{\rho(r_0)} \omega(\psi)$$

If the trajectory leaves  $\Lambda$  at  $r(0) = r_0 = \varphi_0$ , then  $s_0 = 1$ ,  $\frac{1}{\rho(r_0)} = E$  (which fixes  $r_0$ ). The first Eq. (4.51) has separated variables  $\rho(r) dr = dt$ , so if  $N(r) = \frac{1}{E} + \int_{r_0}^r \rho(r') dr'$ , we get  $N(r) = T$ , which is solved implicitly by  $r = R(t)$ . Finally,  $\theta = \text{Const.}$ ,  $\psi = \text{Const.}$ , so the solution of Hamilton equations with data on  $L = \partial\Lambda_+$  are simply

$$(5.53) \quad \begin{aligned} \theta &= \psi \\ x &= X(t, r_0, \psi) = R(t) \omega(\psi) \\ p &= P(t, r_0, \psi) = \frac{\rho(R(t))}{\rho(r_0)} \omega(\psi) \end{aligned}$$

These are the parametric equations of  $L_t = g^t(L)$ . We are left to investigate the “collision” on  $r = 0$ , where the solution becomes singular, and the “boundary part” comes into play.

### 5.6 Example in the constant coefficient case

To close this section we consider as in Remark 3.1,  $H = -h^2 \Delta$ ,  $n = 2$  and

$$f_h(x) = J_0\left(\frac{|x|}{h}\right) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i\langle x, \omega(\psi) \rangle} d\psi = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i|x| \sin \psi} d\psi$$

The outgoing solution is given by the oscillating integral

$$u_h(x) = (4\pi^2 h)^{-1/2} \int e^{i\Phi(x, y, \psi)/h} (\xi^2 - E - i0)^{-1} d\psi dy d\xi$$

which we compute (formally) by stationary phase in  $(y, \xi)$ . The critical point is given by  $y = x$ ,  $\xi = \frac{x}{|x|} \sin \psi$ , and

$$(5.61) \quad u_h(x) = h^{1/2} \int_{-\pi}^{\pi} e^{i|x| \sin \psi/h} \frac{1}{\sin^2 \psi - E - i0} d\psi + \mathcal{O}(h^{3/2})$$

and the leading term can be simply evaluated by contour integrals. Obviously energy  $E = 1$  plays a special role. The phase of course is the same as in Bessel function. We can also consider more general  $f_h$  and stick in an amplitude of the form (see [DoMaNaTu])

$$A(x, \psi) = \frac{1}{2} (a(|x|, \psi) + a(-|x|, \psi)) + \frac{\langle x, \omega(\psi) \rangle}{2|x|} (a(|x|, \psi) - a(-|x|, \psi))$$

so that

$$f_h(x) = \left(\frac{i}{2\pi h}\right)^{1/2} \int_{-\pi}^{\pi} e^{i\langle x, \omega(\psi) \rangle} A(x, \psi) d\psi$$

As in (5.61) we get

$$(5.62) \quad u_h(x) = h^{1/2} \int_{-\pi}^{\pi} e^{i|x| \sin \psi/h} \frac{A(x, \psi)}{\sin^2 \psi - E - i0} d\psi + \mathcal{O}(h)$$

When  $A$  is independent of  $\psi$ , this can lead to significant simplifications.

## Appendix Lagrange immersions and global half-densities

Recall first some well-known properties of Lagrangian immersions (see e.g. [DoZh], [DoNaSh]) :

**Proposition A.1:** *Let  $\iota : \Lambda \rightarrow T^*M$ , be a Lagrangian immersion, parametrized on a canonical chart  $U$  by  $\varphi \mapsto z = \iota(\varphi) = (X(\varphi), P(\varphi))$ . Introduce the Jacobian matrices  $B(z) = \frac{\partial P}{\partial \varphi}$ ,  $C(z) = \frac{\partial X}{\partial \varphi}$ . Then:*

- (1) *the matrix  $(B(z), C(z))$  is of rank  $n$ .*
- (2) *the matrix  ${}^tC(z)B(z)$  is symmetric.*
- (3)  *$C(z) \pm iB(z)$  is non degenerate.*

The symmetry of  ${}^tC(z)B(z)$  expresses for instance in the situation of Sect.4 (resp. Sect.5) as the symmetry of Gram matrices

$$(A.1) \quad \begin{pmatrix} \langle \dot{X}, \dot{P} \rangle & \langle \dot{X}, P_\psi \rangle \\ \langle X_\psi, \dot{P} \rangle & \langle X_\psi, P_\psi \rangle \end{pmatrix}, \quad \begin{pmatrix} \langle \dot{X}, \dot{P} \rangle & \langle \dot{X}, P_\varphi \rangle & \langle \dot{X}, P_\psi \rangle \\ \langle X_\varphi, \dot{P} \rangle & \langle X_\varphi, P_\varphi \rangle & \langle X_\varphi, P_\psi \rangle \\ \langle X_\psi, \dot{P} \rangle & \langle X_\psi, P_\varphi \rangle & \langle X_\psi, P_\psi \rangle \end{pmatrix}$$

We consider the rank of projections  $\pi_x : \Lambda_+ \rightarrow M$ . It is equal to the rank of  $\pi_{x,t} : \tilde{\Lambda}_+ \rightarrow M \times \mathbf{R}_t$ .

In general we call *focal point* a point  $z \in \Lambda$  where  $\pi_* : T\Lambda \rightarrow TM$  is singular, and *caustics* the projection  $\mathcal{C}$  of the set of focal points onto  $M$ . Assume  $n = 2$ , and let  $z$  be a focal point, so  $C(z)$  cannot be of rank 2, and by property (1) above either  $B(z)$  is of rank 2 (and  $C(z)$  has rank at most 1, since the projection  $\pi : \Lambda_+ \mapsto M$  is not a diffeomorphism at  $z$ ) or both  $C(z)$  and  $B(z)$  are of rank 1.

Assume now  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ . Let  $\Lambda = \Lambda_+$  be an integral manifold of  $v_H$  in the energy shell  $H(x, p) = E$ , and  $U \subset \Lambda_+$  be a canonical chart parametrized by  $\varphi = (t, \psi)$ , i.e.  $z = (X(t, \psi), P(t, \psi))$  verifies  $\dot{X} = \partial_x H(X, P)$ ,  $\dot{P} = -\partial_p H(X, P)$ , such that  $U \rightarrow T^*M$  is an immersion (not necessarily an embedding). Recall  $\langle P, \dot{X} \rangle = mH$ ,  $\langle P, X_\psi \rangle = 0$ . Actually  $(X(t, \psi), P(t, \psi))$  may depend on additional parameters, as in Sect.5, but here only  $t, \psi$  matter. Consider the quantity  $\det(P, P_\psi)$ .

**Lemma A.2:** *Let  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ . Assume that at some point  $z = (X(t, \psi), P(t, \psi))$ , we have  $\det(P, P_\psi) = 0$ .*

1) *If  $\nabla \rho(X(t, \psi)) \neq 0$ , then either  $|P(t, \psi)| = 1$ , or  $P_\psi(t, \psi) = 0$ . In the latter case  $C(z) = (\dot{X}, X_\psi)$  has rank 2, i.e.  $\pi_x$  is regular at  $z$ .*

2) *If  $\rho$  has a critical point at some  $x_0 = X(t, \psi)$ , then  $\pi_x$  is regular at  $z = (x_0, P(t, \psi))$*

*Proof:* We apply Proposition A.1 to the Lagrangian immersion  $\iota : U \rightarrow T^*M$ ,  $\varphi = (t, \psi)$ . Assume  $\det(P, P_\psi) = 0$  at some point  $z = (X(t, \psi), P(t, \psi))$ . Then either  $P_\psi = 0$ , or by Hamilton equations (3.2), there is  $\alpha \in \mathbf{R}$  such that  $\dot{X} = \alpha P_\psi$ . Let first  $\nabla \rho \neq 0$ .

(i) Let  $P_\psi = 0$ . In this case  $X_\psi \neq 0$ , for otherwise this would contradict property (3) of Proposition 3.2. The symmetry of  ${}^tC(z)B(z)$  shows that  $\langle \dot{P}, X_\psi \rangle = 0$ , or by (3.2)  $\langle \nabla \rho(X), X_\psi \rangle = 0$ . Since  $\langle P, X_\psi \rangle = 0$ , we find that  $\dot{X}, P, \dot{P}, \nabla \rho(X)$  are parallel, and all orthogonal to  $X_\psi$ . In particular,  $C(z) = (\dot{X}, X_\psi)$  is of rank 2.

(ii) Let  $P_\psi = 0$ . By property (2) of Proposition A.1, the matrix

$${}^tC(z)B(z) = \begin{pmatrix} * & \dot{X}_1 \partial_\psi P_1 + \dot{X}_2 \partial_\psi P_2 \\ \dot{P}_1 \partial_\psi X_1 + \dot{P}_2 \partial_\psi X_2 & * \end{pmatrix}$$

has to be symmetric. If  $\dot{X} = \alpha P_\psi$ , this implies  $\alpha = \langle \dot{P}, \partial_\psi X \rangle |P_\psi|^{-2}$ , and hence differentiating the dispersion relation (2.2)

$$(A.5) \quad \dot{X} = \langle \dot{P}, \partial_\psi X \rangle |P_\psi|^{-2} P_\psi = \frac{|P|^m}{\rho(X)^2} \langle \nabla \rho(X), X_\psi \rangle |P_\psi|^{-2} P_\psi$$

• Assume  $B(z)$  is of rank 2, and  $C(z)$  of rank at most 1. We know that  $\langle P(t, \psi), \dot{X}(t, \psi) \rangle = m(E - \tau) \neq 0$ ,  $\dot{X} \neq 0$  parallel to  $P \neq 0$ , and  $\langle P(t, \psi), \partial_\psi X(t, \psi) \rangle = 0$ .

Assume  $X_\psi \neq 0$ . Since  $C(z)$  is of rank at most 1,  $X_\psi$  parallel to  $\dot{X}$ , which is itself parallel to  $P$ . So  $P$  is parallel to  $X_\psi$ , which contradicts  $\langle P, X_\psi \rangle = 0$ . Hence  $X_\psi = 0$ . By first equality (A.6) we have  $\langle P, P_\psi \rangle = 0$ . If  $\det(P, P_\psi) = 0$ , then we would have  $P$  both orthogonal to  $X_\psi$ , and parallel to  $P_\psi \neq 0$ . But  $B(z)$  is of rank 2, which is a contradiction.

• So by property (3) of Proposition A.1 we must have  $B(z)$  and  $C(z)$  of rank 1. So either  $\dot{X} = \lambda X_\psi$  and  $\dot{P} = \mu P_\psi$  for some  $\lambda, \mu \in \mathbf{R}$ , or  $X_\psi = 0, \dot{P} = \mu P_\psi$ , or  $P_\psi = 0, \dot{X} = \lambda X_\psi$ , or  $X_\psi = P_\psi = 0$ .

Examine the first case: Identifying the off-diagonal terms of  ${}^tB(z)C(z)$ , which is symmetric by property (2) of Proposition A.1, we find that either  $\lambda = \mu$  or  $\frac{\partial X}{\partial \psi} \perp \frac{\partial P}{\partial \psi}$ . But  $\lambda \neq \mu$  since otherwise the complex matrices  $C(z) \pm iB(z)$  would be degenerate, which violates property (3) of Proposition 3.2. So  $\frac{\partial X}{\partial \psi}$  and  $\frac{\partial P}{\partial \psi}$  are colinear, and orthogonal to both  $\frac{\partial P}{\partial \psi}$  and  $\frac{\partial P}{\partial t}$ . Assume first  $\lambda\mu \neq 0$ . Using again (A.6), we find that  $P \perp P_\psi$ , since  $P_\psi \neq 0$ , we find that  $\det(P, P_\psi) \neq 0$ . Assume then  $\lambda\mu = 0$ . If  $\lambda = 0, \mu \neq 0$ , we would have  $\dot{X} = 0$ , which is impossible. Let now  $\lambda \neq 0$ , then  $B + iC = (P_\psi + iX_\psi, i\lambda X_\psi)$  has rank 2, so has  $(P_\psi, X_\psi)$ . So by (A.6)  $P, \frac{\partial X}{\partial \psi}$  and  $\frac{\partial X}{\partial t}$  are colinear, and orthogonal to both  $\frac{\partial P}{\partial \psi}$  and  $\nabla \rho$ . As before, this implies  $\det(P, P_\psi) \neq 0$ .

Examine the second case  $X_\psi = 0, \dot{P} = \mu P_\psi$ . Writing that  ${}^tB(z)C(z)$  is symmetric, we find  $\dot{X} \perp P_\psi$ . Since  $B(z) + iC(z)$  has rank 2,  $P_\psi \neq 0$ . Then  $P$  and  $\dot{X}$  are parallel, and both orthogonal to  $\nabla \rho, P_\psi$  and  $\dot{P}$ . As before, we find  $\det(P, P_\psi) \neq 0$ .

The last two cases are similar. So  $\det(P, P_\psi) \neq 0$  at any focal point where  $\nabla \rho \neq 0$ .

(iii) Let at last,  $\nabla \rho(x_0) = 0$ . The first situation above cannot hold, since this would imply  $\dot{X} = 0$  by (A.6), hence  $P = 0$ . Hence  $\dot{P} = P_\psi = 0$  which implies again  $C(z)$  of rank 2 by Proposition A.1. ♣



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