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**Semiclassical Green functions and Lagrangian intersection.**  
**Applications to the propagation of Bessel beams in non-homogeneous media**

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*Abstract:* We study semi-classical asymptotics for problems with localized right-hand sides by considering a Hamiltonian  $H(x, p)$  positively homogeneous of degree  $m \geq 1$  on  $T^*\mathbf{R}^2 \setminus 0$ . The energy shell is  $E = 1$ , and the right-hand side  $f_h$  is microlocalized: (1) on the vertical plane  $\Lambda = \{x = x_0\}$ ; (2) on the “cylinder”  $\Lambda = \{(X, P) = (\varphi\omega(\psi), \omega(\psi)); \varphi \in \mathbf{R}, \omega(\psi) = (\cos \psi, \sin \psi)\}$ . We restrict essentially to the isotropic case  $H(x, p) = \frac{|p|^m}{\rho(x)}$ , with  $\rho$  a smooth positive function. In case (2),  $\Lambda$  is the frequency set of Bessel function  $J_0(\frac{|x|}{h})$ , and the solution  $u_h$  of  $(H(x, hD_x) - 1)u_h = f_h$ , which is called a “Bessel beam”, arises in the theory of optical fibers.

**1. Introduction**

Let  $M = \mathbf{R}^n$ ,  $H(x, hD_x; h)$  a  $h$ -PDO with symbol  $H(x, p) \sim H_0(x, p) + hH_1(x, p) + \dots$ , and  $E \neq 0$  be a non critical energy level for Hamiltonian  $H_0$ . The general problem of “semi-classical wave functions” consists in solving  $(H - E)u_h = f_h$ , where  $f_h$  is a Lagrangian distribution supported microlocally on some Lagrangian manifold  $\Lambda$ , i.e.  $\text{WF}_h f_h \subset \Lambda$  (here  $\text{WF}_h$  denotes the semi-classical wave front set, we recall in Sect.3). When  $\Lambda = \{x = x_0\} = T_{x_0}^*M$ ,  $x_0 \in M$ , we call the “vertical plane”, is the conormal bundle to  $\{x_0\}$ , we say simply that  $f_h$  is a “localized function” at  $x_0$ , and to fix the ideas, assume  $x_0 = 0$ . Moreover, we require  $u_h = E_+ f_h$  to be *outgoing* at infinity. Here are some examples of  $f$  (expressed in a single chart):

*Examples 1.1:*

(1) WKB functions in Fourier representation

$$(1.1) \quad f(x; h) = \frac{e^{-i\pi n/4}}{(2\pi h)^{n/2}} \int e^{i(xp+S(p))/h} A(p; h) dp$$

here  $\Lambda = \{(-\partial_p S(p), p) : p \in \mathbf{R}^n\}$ . Although a canonical transformation can reduce (locally)  $S(p)$  to  $S_0(p) = -\langle x_0, p \rangle$  and  $\Lambda$  to  $\Lambda_0 = \{x = x_0\}$ , it has no reason to preserve homogeneity of the Hamiltonian. But we may still attempt to adapt our constructions to this case.

Whenever no confusion may occur such an oscillatory integral, conveniently normalized including the phase factor, will be written as  $\int^*(\dots)$  (see [CdV, Def.19 pdf p.75]). The advantage is two-fold: on the one hand we recover the usual notation for Maslov canonical operator  $\int^*(\dots) = K_\Lambda^h(A)$ ; on the other hand, if the phase  $xp + S(p)$  has a non-degenerate critical point at  $p_0$ , then asymptotic stationary phase readily gives at first order

$$(1.2) \quad \int^*(\dots) = e^{i(xp_0+S(p_0))/h} e^{-im\pi/2} |\det(S''(p_0))|^{-1/2} A_0(p) + \mathcal{O}(h)$$

where  $m$  is Morse index (number of negative squares) of  $S''(p_0)$ .

(2) A conormal distribution  $f(x; h) = \int^* e^{ix_n p_n/h} A(x', p_n) dp_n$  with respect to the hypersurface  $N = \{x_n = 0\}$ , i.e.  $\Lambda = T_N^* \mathbf{R}^n$ , or of more general type.

(3)  $f_h$  identifies with a ‘‘Bessel beam’’, and

$$(1.3) \quad \Lambda = \{x = X(\varphi, \psi) = \varphi\omega(\psi), p = P(\varphi, \psi) = \omega(\psi), \varphi \in \mathbf{R}\}$$

is called the ‘‘cylinder’’ ; here  $\omega \in \mathbf{S}^{n-1}$  is the unit vector parametrized by  $\psi$ , see Sect. 5.

(4) Coherent states  $f(x; h) = \frac{1}{h^n} \exp(-\omega^2 \cdot (x - x_0)^2/2h) \exp(ix\xi_0/h)$ , and  $\Lambda = \{x = x_0\} \times \mathbf{R}^n \cup \mathbf{R}^n \times \{\xi = \xi_0\}$ , or more generally a Lagrangian distributions with a complex phase in the sense of Melin-Sjöstrand) [MeSj], equivalently a complex germ in the sense of Maslov. This case however requires some special treatment and will not be considered here.

The ‘‘initial manifold’’  $\Lambda$  we shall be mostly interested in are the vertical plane and the cylinder, but our technics hopefully extend to other examples above. It is essential however that  $H_0$  be positively homogeneous with respect to  $p$  on  $T^*M \setminus 0$ .

By well known results on propagation of singularities,  $\text{WF}_h E_+ f \subset \Lambda \cup \Lambda_+$ , where  $\Lambda_+$  is the flow-out of the Hamilton vector field  $v_H$  in the energy surface  $H = E$ . More generally, we can construct  $E_+ f_h$  microlocally, since  $H$  can be reduced microlocally to  $hD_{x_n}$  near  $H = E$  while preserving Lagrangian intersection. This follows from a straightforward extension of local constructions of [MelUhl] to the semi-classical case.

However, global existence of an outgoing solution at infinity provided suitable hypotheses on the Hamilton vector flow, such as Lagrangian intersection, the non-trapping (in  $x$ -space) and the non-return conditions, is more involved. The main strategy has been set up in [MelUhl] and has received a more systematic treatment in relatively recent works [Ca], [JF-Bo], [KlCa], when  $H(x, hD_x)$  is semi-classical Helmholtz (Schrödinger) operator  $-h^2\Delta + V(x)$ ,  $V \leq 0$  and  $f_h$  a localized function at  $x_0$ . In this case, the non-return condition is a condition on the set

$$\{(p, \eta) : p^2 + V(0) = \eta^2 + V(0) = 0; \exists t > 0 : X(t, p) = 0, P(t, p) = \eta\}$$

where  $(X(t, p), P(t, p)) = \exp t v_H(0, p)$  is the trajectory issued from  $(0, p) \in T^*M$ . Outgoing solutions  $u_h$ , are characterized by Sommerfeld radiation condition of the form

$$\frac{x}{|x|} \nabla_x w(x) + i\sqrt{-V(x_0)} w(x) \rightarrow 0, \quad |x| \rightarrow \infty, \quad n \geq 2$$

where  $w = \lim_{h \rightarrow 0} w_h$ ,  $w_h(x) = h^{d/2} u_h(hx)$ , is the unique solution of  $(-\Delta + V(0))w = f$ . It relates in a non trivial way the behavior of  $u_h$  at infinity with the value of the potential  $V$  at  $x_0$ . Proving Sommerfeld radiation condition requires careful estimates on  $U_h(t) = e^{itH/h}$ , or  $U_h(t)f_h$ , along with a discussion according to the relative magnitude of  $t$  and  $h$ . The proof consists in testing  $U_h(t)f_h$  against some fixed  $\phi \in \mathcal{S}(\mathbf{R}^n)$ , and show that  $\langle u_h, \phi \rangle \rightarrow \langle w, \phi \rangle$  as  $h \rightarrow 0$ . In particular one needs to know asymptotics of  $u_h$  in a  $h$ -dependent neighbhd of  $\Lambda$ .

In this paper, given  $H$  and  $\Lambda$ , we content ourselves to present, in the sense of *formal asymptotics*, a “close form” for the outgoing solution

$$(1.5) \quad u_h(x) = E_+ f_h(x; h) = \int_0^\infty e^{-it(H-E)/h} f_h dt$$

in term of Maslov canonical operator for bi-Lagrangian distributions. By formal asymptotics [Ler] we mean that, in principle, our approximate solution has no reason to be equal to  $E_+ f(x; h) \bmod \mathcal{O}(h)$ . In practice however numerical simulations show that Maslov canonical operator provides an excellent agreement with the “exact solution”.

We assume that Hamiltonian  $H_0$  (the principal symbol of  $H$ ) is positively homogeneous of degree  $m \geq 1$  with respect to momentum variable  $p$ . Note that these are not suitable symbols for Pseudo-differential Calculus because of the singularity at  $p = 0$ , but since  $E \neq 0$  this is relatively harmless.

Our constructions will be essentially geometric (solving Hamilton-Jacobi and transport equations) using adapted coordinates on the initial manifold  $\Lambda$ , so to minimize the number of required canonical charts. Introducing also eikonal coordinates, we shall perform, by asymptotic stationary phase, the integration over  $t$  and also over extra variables, such as coordinates on  $\Lambda \cap \Lambda_+$ , and try to split  $u_h$  according to its components supported on  $\Lambda$  and  $\Lambda_+$  respectively.

By Maupertuis-Jacobi correspondence, our arguments can be translated to other types of operators, including Helmholtz operator, or the water-waves operator [DoRo].

### 1.1 Lagrangian intersection and microlocal Green functions.

As in [MelUhl] our constructions make use of symbolic calculus adapted to Lagrangian intersection. So we need first to translate some notions relative to asymptotics with respect to smoothness (or “standard pseudo-differential calculus”), to the framework of asymptotics with respect to small parameter  $h$  (or “ $h$ -pseudo-differential calculus”), in particular to allow for general phase functions (without homogeneity in the momentum variable).

Let  $\iota_0 : \Lambda_0 \rightarrow T^*M$  be a smooth embedded Lagrangian manifold, and  $\iota_1 : \Lambda_1 \rightarrow T^*M$  be a smooth embedded Lagrangian manifold with smooth boundary  $\partial\Lambda_1$  (isotropic manifold). Following [MelUhl] we say that  $(\Lambda_0, \Lambda_1)$  is an *intersecting pair* of Lagrangian manifolds iff  $\Lambda_0 \cap \Lambda_1 = \partial\Lambda_1$  and the intersection is *clean*, i.e.

$$\forall z \in \partial\Lambda_1 \quad T_z\Lambda_0 \cap T_z\Lambda_1 = T_z\partial\Lambda_1$$

(in particular  $\Lambda_0$  and  $\Lambda_1$  cannot be transverse). On the set of intersecting pairs of Lagrangian manifolds we define an equivalence relation by saying that  $(\Lambda_0, \Lambda_1) \sim (\Lambda'_0, \Lambda'_1)$  iff near any  $z \in \partial\Lambda_1$ ,  $z' \in \partial\Lambda'_1$ , there is a symplectic map  $\kappa$  such that  $\kappa(z) = z'$ , and a neighbhd  $V \subset T^*M$  of  $z$  such that  $\kappa(\Lambda_0 \cap V) \subset \Lambda'_0$ ,  $\kappa(\Lambda_1 \cap V) \subset \Lambda'_1$ . We have the following result which readily extends this of [MelUhl] to the semi-classical case:

**Lemma 1.1:** *All intersecting pairs of manifolds in  $T^*M$  are locally equivalent. More precisely near each  $z \in T^*M$ , there exists a canonical map  $\kappa : T^*M \rightarrow T^*\mathbf{R}^n$  such that  $\kappa(z) = (0, \xi_0)$ ,  $\xi_0 = (\xi', 0) \in \mathbf{R}^n$ ,  $\kappa(\Lambda_0 \cap V) \subset T_0^*\mathbf{R}^n$  (the conormal bundle to  $0 \in \mathbf{R}^n$ ), and  $\kappa(\Lambda_1 \cap V) \subset \Lambda_+$ ,  $\Lambda_+^0$  being the flow-out of  $T_0^*\mathbf{R}^n$  by the Hamilton vector field  $v_{K_0} = ((0, \dots, 0, 1), 0)$  of  $K_0(x, \xi) = \xi_n$ , passing through*

$x = 0, \xi_0 = (\xi', 0)$  i.e.

$$(1.6) \quad \Lambda_+^0 = \{(x, \xi) \in T^*\mathbf{R}^n : x = (0, x_n), \xi = (\xi', 0), x_n \geq 0\}$$

Given some Hamiltonian  $H_0 \in C^\infty(T^*M)$ , and  $\Sigma_E = H_0^{-1}(E)$  a non critical energy surface, we shall consider intersecting pairs  $(\Lambda, \Lambda_+)$  where

$$(1.7) \quad \Lambda_+ = \{(x, \xi) \in T^*M, \exists t \geq 0, \exists z = (y, \eta) \in \Lambda \cap \Sigma_E, (x, \xi) = \exp tv_{H_0}(z)\}$$

is the flow-out of  $\Lambda$  by  $v_{H_0}$ , provided  $v_{H_0}$  is transverse to  $\Lambda$  along  $\Lambda \cap \Sigma_E$ .

By Lemma 1.1, near each  $z \in \partial\Lambda_+(E)$  we are reduced microlocally to the case where  $\Lambda = T_0^*\mathbf{R}^n$ ,  $\Lambda_+ = \Lambda_+^0$  as in (1.6) and  $H_0 = K_0$  is the “model Hamiltonian”.

Consider now  $H^w(x, hD_x; h)$  a  $h$ -PDO with principal symbol  $H_0$  and non singular energy surface  $\Sigma_E$ . Let  $(\Lambda, \Lambda_+)$  be a Lagrangian pair,  $\Lambda_+ \cap \Lambda = \partial\Lambda_+$ . Using Lemma 1.1, it is well known (constructing the higher order terms) that  $H^w(x; hD_x; h)$  can be reduced microlocally near  $\partial\Lambda_+$  to  $K(x, hD_x; h) = hD_{x_n}$ . As a warm-up, let us construct  $u = E_+ f \bmod \mathcal{O}(h^\infty)$  for  $K(x, hD_x; h) = hD_{x_n}$ , and  $f(x; h) = \int^* e^{ix\xi/h} A(\xi) d\xi$ . By the gauge transformation  $e^{it(K-E)/h} = e^{iEx_n/h} e^{itK/h} e^{-iEx_n/h}$ , we can assume  $E = 0$ . So we just need to compute a primitive of  $f(x; h)$ . Let  $T > 0$  and  $\theta_T \in C^\infty(\mathbf{R})$  vanishing near  $+\infty$  and  $\theta_T(t) = 1$  for  $t \leq T$ . We consider

$$(1.8) \quad u(x, h) = \frac{i}{h} \int_0^\infty \theta_T(t) dt \int^* e^{i(x'\xi' + (x_n - t)\xi_n)/h} A(\xi) d\xi$$

Provided  $x_n \leq T/2$  (say) it is easy to check (by using integration by parts and a non-stationary phase argument) that

$$(1.9) \quad hD_{x_n} u(x, h) = f(x, h) + \mathcal{O}(h^\infty)$$

For simplicity we assume  $n = 1$  and denote  $x_n$  by  $x$ . The phase function  $(t, \xi) \mapsto \Phi(t, x, \xi) = (x - t)\xi$  is critical at  $t = x$  and  $\xi = 0$ , and  $(x, 0)$  is a non-degenerate critical point. So when  $x > 0$ , (1.2) shows that  $u(x; h) = A(0) + \mathcal{O}(h^\infty)$ , while  $u(x; h) = \mathcal{O}(h^\infty)$  for  $x < 0$ . For the constant term we have  $\text{WF}_h(A(0)) \subset \mathbf{R} \times \{0\}$  with equality iff  $A(0) \neq 0$ . This gives the component microlocally supported on  $\Lambda_+^0$ . The component supported on  $\Lambda$  can be simply defined as  $u(x; h) - A(0)$ .

This procedure gives already a microlocal description of the forward parametrix  $E_+$ , which we want to make global by constructing Maslov canonical operator associated with the pair  $(\Lambda, \Lambda_+)$ . It gives actually a general flavour of our main arguments.

## 1.2 Hypothesis and main results.

We mostly restrict our attention to the case of the vertical plane. Note that Lagrangian intersection always holds on  $\Lambda$ , for if  $\partial_p H(x_0, p) = 0$ , we would have  $\langle \partial_p H(x_0, p), p \rangle = mH = 0$  which contradicts  $E \neq 0$ . We also give some hints on the case of the “cylinder” (1.3) where Lagrangian intersection plays a role, is less intuitive and will be presented in Sect.5. Still, the results and technics are reminiscent of those of the vertical plane.

- *Case of a general Hamiltonian homogeneous with respect to  $p$ .* We shall assume Lagrangian intersection, which is used in Proposition 5.2 (case of the cylinder) to characterize the critical set  $C_\Phi$ .

The first step is to construct a solution  $\Phi(t, x, \theta)$  of Hamilton-Jacobi equation parametrizing  $\Lambda_+$ , and a (globally) positive density on  $\Lambda_+$ . We shall partition points  $z \in \Lambda_+$  according to the following values: (1)  $z$  is a focal (or non-focal) point; (2)  $z$  is a special (or ordinary) point; (3) (only in the cylinder case)  $z$  is a glancing (or non-glancing) point. We say that  $z = z(t) = (X(t, \theta), P(x, \theta))$  on some integral curve of Hamilton vector field  $v_H$  is an *ordinary point* iff  $v_H$  is not tangent at  $z(t)$  to the level sets of  $\rho$ , and a *special point* otherwise. Here  $\theta$  parametrizes the initial value of  $X$  on  $\Lambda$ .

At an ordinary point we have

$$\partial_t \Phi(t, x, \theta) = 0 \implies \partial_t^2 \Phi(t, x, \theta) \neq 0$$

so that we can perform asymptotic stationary phase in  $t$  to simplify (1.4) at  $x = X(t, \theta)$ . Of course, several values of parameters  $(t, \theta)$  can contribute. Then  $\Phi(x, t, \theta)$  reduces to a phase function  $\Psi(x, \theta)$  and we can further reduce the number of variables  $\theta$  in a standard way, according to the fact that  $z(t)$  is a focal point or not.

At a special point instead, the situation looks like (1.8) and we have

$$\partial_t \Phi(t, x, \theta) = \partial_\lambda \Phi(t, x, \theta) = 0 \implies \frac{\partial^2 \Phi}{\partial t \partial \lambda}(t, x, \theta) \neq 0$$

where  $\lambda$  is one of the “ $\theta$ ”-variables (constrained to be equal to 1 on the critical set  $C_\Phi$ ), so that we can perform asymptotic stationary phase with respect to  $(t, \lambda)$ .

Introducing Maslov indices and densities, we can thus describe the singularities of the leading (or wave) front near  $t = 0$ .

In this generality, we only succeed to describe the contribution to (1.5) (by asymptotic stationary phase) of short times  $t$ , see Proposition 4.3. Thus we obtain the “germ” of Maslov canonical operator for bi-Lagrangian distributions. This is actually sufficient to compute  $u_h$  microlocally near  $\Lambda$ . We could obtain further information relative to the asymptotics of  $u_h$  microlocally in a  $h^\delta$ -neighbhd of  $\Lambda$ , for any  $0 < \delta < 1$ .

- *Special case of the vertical plane and a “conformal metric”.* We get much more complete results in this case : under suitable hypothesis we can allow for larger times, and thus, invoking also some general geometric “stability” arguments (see e.g. Assumption (2.18)), construct (1.5) given any  $x \in M$ .

Let  $\rho$  be a smooth positive function on  $M$ ,  $m \geq 1$ , and

$$(1.15) \quad H(x, p) = |p|^m \frac{1}{\rho(x)}$$

We shall always assume that there is no finite motion

$$(1.16) \quad |X(t, \theta)| \rightarrow \infty \text{ as } t \rightarrow \infty$$

If  $\rho$  is bounded, a sufficient condition for (1.16) is that energy  $E$  is non trapping, see [GeSj].

Once we know the phase function, we will adapt to the semi-classical case the constructions of [MelUhl], using symbols satisfying the compatibility condition. By gluing the different components of  $u$  we will get Maslov operator  $K_+^h$  for a bi-Lagrangian distribution associated with the pair of Lagrangian manifolds  $(\Lambda, \Lambda_+)$ . Thus we get

**Theorem 1.2:** Let  $f_h$  as in (1.1) or, at leading order in  $h$ ,  $f_h = [K_\Lambda^h A_0]$ . Under hypotheses above, with we notations (3.26), we can express (1.5) in terms of Maslov canonical operator for bi-Lagrangian distributions as

$$u_h(x) = [K_{\Lambda, \Lambda_+}^h \tilde{\sigma}(B_0)](x; h), \quad x \neq 0$$

where  $B_0$  depends linearly on  $A_0$ .

- *Special case of the cylinder and a “conformal metric”.* To the former “ $\theta$ -variables”  $(t, \psi, \lambda)$ , one has now to add  $\varphi$  ( $\Lambda$  has a Lagrangian singularity at  $\varphi = 0$ ).

The initial condition constraints somehow  $\rho$  to be radial on  $\partial\Lambda_+$ , namely  $\nabla\rho(\varphi\omega(\psi), \omega^\perp(\psi)) = 0$ . Thus  $z = (x, p) \in \partial\Lambda_+$  is a special point iff  $\nabla\rho(\varphi\omega(\psi)) = 0$ .

The result is similar to Theorem 1.2 but less simple to write. We refer to Sect.5 for a more complete statement.

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## 2. Hamiltonians and phase functions

In this Sect. we consider integral manifolds for positively homogeneous Hamiltonians on  $T^*M \setminus 0$ , which is the first step in constructing semi-classical Green kernels. The material is quite standard, but more specific points will be discussed in Sect. 4 and 5.

### 2.1 Eikonal coordinates

We shall deal with positively homogeneous Hamiltonians of degree  $m \geq 1$  with respect to  $p$  on the cotangent bundle  $T^*M \setminus 0$  ( $M = \mathbf{R}^n$  for simplicity), and eventually restrict to “conformal metrics” to get most explicit results. We have in mind elliptic Hamiltonians, but our formal constructions could as well allow for “Minkowski metrics”, as was originally devised in [MelUhl]. One of the main differences is that  $\partial\Lambda_+$  is no longer compact. In this Sect. we write  $H$  for  $H_0$ .

*Examples 2.1:*

- (1)  $m = 2$  if  $H(x, p)$  is a geodesic flow associated with a Riemannian metric  $ds^2 = g_{ij}(x) dx^i \otimes dx^j$ . In the Riemannian case, when  $E = 1$ , geodesics are parametrized by arc-length.
- (2)  $m = 1$  if  $H(x, p)$  is a “Randers symbol”, associated with a Finsler metric [Tay], [DoRo2].
- (3) Let  $m \geq 1$ , and  $H$  as in (1.15). Hamilton equations  $(\dot{x}, \dot{p}) = v_H(x, p)$  then read

$$(2.2) \quad \dot{x} = \partial_p H = m|p|^{m-1} \frac{1}{\rho(x)} \frac{p}{|p|}, \quad \dot{p} = -\partial_x H = |p|^m \frac{\nabla \rho(x)}{\rho(x)^2}$$

Note that when  $m$  is not an even integer,  $H$  is not smooth at  $p = 0$ , but we will take the energy level away from  $E = 0$ . Our most complete results hold for Hamiltonian (1.15) with  $n = 2$ .

First we recall some general facts on canonical coordinates near Lagrangian manifolds, see [DoNaSh]. Let  $\iota : \Lambda_+ \rightarrow T^*M$  be a smooth embedded Lagrangian manifold. The 1-form  $p dx$  is closed on  $\Lambda_+$ , so locally exact, and  $p dx = dS$  on any simply connected domain  $U$  (so called canonical chart). Such a  $S$  is called an *eikonal* (or action) and is defined up to a constant. It can be chosen as a coordinate on  $U$ .

• *Case of the vertical plane.* Here  $\Lambda_+$  is the flow-out of  $H$  with initial data on  $\Lambda = T_0^*M$ , see Example 1.1. Let also  $\psi \in \mathbf{R}^{n-1}$  be smooth coordinates on  $\Lambda$ , which we complete by  $\tau$ , the dual coordinate of  $t$ , so that  $\partial\Lambda_+$  is given in  $\Lambda$  by  $\tau = 0$ , and in  $\Lambda_+$  by  $t = 0$ .

In the special case  $H(x, p) = \frac{|p|^m}{\rho(x)}$ , we have  $P(\psi, \tau) = |P|_\tau \omega(\psi)$ , with

$$(2.3) \quad |P|_\tau = (H\rho(0))^{1/m} = ((E - \tau)\rho(0))^{1/m}$$

Sections are defined as follows: for small  $\tau$ , let  $\Lambda_+(\tau)$  be the Lagrangian manifold in the energy shell  $\tau + H(x, p) = E$  issued from  $\Lambda$  at  $t = 0$ . We consider the isotropic manifold  $\partial\Lambda_+(\tau) = \Lambda \cap \Lambda_+(\tau)$ , viewing  $\Lambda_+(\tau)$  as a manifold with boundary. When  $\tau = 0$ , we simply write  $\Lambda_+(0) = \Lambda_+$ .

We assume that  $\iota : \Lambda_+ \rightarrow T^*M$  is an embedding. For  $t \geq 0$ , let  $\Lambda_t = \exp t v_H(\Lambda)$ , we have  $\Lambda_{t+s} = \exp t v_H(\Lambda_s)$  for all  $t, s \geq 0$ .

We compute the eikonal  $S$  on  $\Lambda_+(\tau)$  by integrating along a path in  $\Lambda$  (where  $dx = 0$ ), connecting the base point (say  $(0,0)$ ) to  $(0, P(\psi)) \in \partial\Lambda_+(\tau)$ ,  $\psi \in \mathbf{R}^{n-1}$ , followed by the integral curve  $x =$

$X(t, \psi), p = P(t, \psi)$  of  $v_H$  starting at  $(0, P(\psi))$ , where  $dS = p dx|_{\Lambda_+(\tau)} = \langle P(t, \psi), dX(t, \psi) \rangle$ . Because  $\Lambda_+(\tau)$  is Lagrangian,  $S$  doesn't depend on that choice. Since  $S(0, \psi) = \text{Const.} = S_0$  on  $\Lambda$ , we get

$$(2.4) \quad \begin{aligned} S(t, \psi) &= S(0, \psi) + \int_{(0, \psi)}^{(t, \psi)} p dx|_{\Lambda_+} = S(0, \psi) + \int_{(0, \psi)}^{(t, \psi)} P(s, \psi, \tau) dX(s, \psi, \tau) \\ &= S_0 + \int_0^t \langle P(s, \psi, \tau), \dot{X}(s, \psi, \tau) \rangle ds \end{aligned}$$

By Hamilton equations and Euler identity  $\langle P(s, \psi), \dot{X}(s, \psi) \rangle = \langle P, \partial_p H \rangle = mH = m(E - \tau)$ . Now  $S(t, \psi) = S_0 + m(E - \tau)t = S_0 + mHt$  is the action on  $\Lambda_+$ , and the eikonal coordinate is just  $S(t, \psi) = mHt$  up to a constant  $S_0$ . From the identity  $dS = d(m(E - \tau)t) = m(E - \tau)dt$  (which does not involve  $d\tau$  since we stay on  $\Lambda_+(\tau)$ ) we get

$$(2.5) \quad m(E - \tau)dt = \langle P(t, \psi, \tau), dX(t, \psi, \tau) \rangle = \langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau) \rangle dt + \langle P(t, \psi, \tau), X_\psi(t, \psi, \tau) \rangle d\psi$$

it follows that

$$(2.6) \quad \langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau) \rangle = mH, \quad \langle P(t, \psi, \tau), \partial_\psi X(t, \psi, \tau) \rangle = 0$$

We will denote for short  $(X, P) = (X(t, \psi, \tau), P(t, \psi, \tau))$  or  $X(t, \psi), P(t, \psi)$  when  $\tau = 0$ . This is called the *leading front*.

• *Case of the cylinder.* Here  $\Lambda_+$  is the flow-out of  $H$  with initial data on (1.3). Computing the action we find  $p dx|_\Lambda = d\varphi$  so coordinate  $\varphi$  will play the role of  $x$  in Sect.4. As in (2.4) along  $\Lambda_+(\tau)$  we have

$$\begin{aligned} S(t, \varphi, \psi, \tau) &= S(0, \varphi, \psi, \tau) + \int_{(0, \varphi, \psi)}^{(t, \varphi, \psi)} p dx|_{\Lambda_+} = S(0, \varphi, \psi, \tau) + \int_{(0, \varphi, \psi)}^{(t, \varphi, \psi)} P(s, \varphi, \psi, \tau) dX(s, \varphi, \psi, \tau) = \\ &= S_0 + \varphi + \int_0^t \langle P(s, \varphi, \psi, \tau), \dot{X}(s, \varphi, \psi, \tau) \rangle ds \end{aligned}$$

As before,  $\langle P, \dot{X} \rangle = mH$ , so on  $\Lambda_+(\tau)$

$$(2.10) \quad S(t, \varphi, \psi, \tau) = \varphi + m(E - \tau)t + S_0$$

Identifying the differential of  $S$  we get (omitting again variables)

$$(2.11) \quad \langle P, \dot{X} \rangle = mH, \quad \langle P, \partial_\psi X \rangle = 0, \quad \langle P, \partial_\varphi X \rangle = 1$$

## 2.2 Hamilton-Jacobi equation for small $t$ and non degenerate phase functions

Because of focal points in general we cannot readily find a phase-function  $\Psi(x)$  such that  $H(x, \partial_x \Psi(x)) = E$ , so we obtain it as a critical value (with respect to an auxiliary time-variable  $t$ ) of a phase  $\Phi(x, t)$ . To do this we solve Hamilton-Jacobi equation (HJ) in the extended phase space  $T^*(M \times \mathbf{R})$ , which is the suitable framework to vary  $(t, \tau)$  as well ( $\tau$  being set eventually to 0). So we look for a phase function  $\Phi(x, t)$  satisfying

$$(2.15) \quad \partial_t \Phi + H(x, \partial_x \Phi) = E, \quad \Phi|_{t=0} = \phi$$

with given  $\phi$  (to be chosen later on), and prescribed  $\partial_t \Phi(x_0, 0) = \tau_0, \partial_x \Phi(x_0, 0) = \eta_0$  satisfying  $\tau_0 + H(x_0, \eta_0) = E$ . By Hamilton Eq.,  $\tau = \tau_0$  is a constant of the motion. It is well-known [Hö, Thm6.4.5] that (2.15) as a unique solution for small  $t$ . This is the generating function of the Lagrangian manifold the extended phase-space

$$\tilde{\Lambda}_+ = \{p = \partial_x \Phi(x, t), \tau = \partial_t \Phi(x, t), x, t \in M \times \mathbf{R}_+\} \subset T^*(M \times \mathbf{R})$$

constructed along the integral curves of  $v_H$  starting at  $t = 0$  from the Lagrangian manifold  $\Lambda_\phi$  in  $T^*M$  given by  $p = \partial_x \phi$ . Its section at fixed  $t, \tau$  is the Lagrangian manifold

$$\Lambda_{\Phi, t, \tau} = \{p = \partial_x \Phi(x, t), x \in M \times \mathbf{R}\} \subset T^*M$$

which is simply the flow out  $\Lambda_t(\tau)$  of  $\{p = \phi'(x)\}$  in  $H(x, p) = E - \tau$  at time  $t$ . In turn,  $\phi$  usually depends on other variables  $\theta$ , so does  $\Phi$  and we need again to find the critical value of  $\Phi$  with respect to  $\theta$ , which is function on

$$(2.16) \quad C_\Phi = \{(x, t, \theta) : \partial_\theta \Phi = \partial_t \Phi = 0\}$$

We shall consider *reduced phase functions*, with least number of variables  $\theta$ . This will be recalled in Sect.3. We choose the initial condition to be the standard pseudo-differential phase function of the form  $\phi(x) = x\eta$ . Here  $\eta$  is a parameter, we choose so that  $\phi'|_\Lambda = P(\tilde{\psi})$  where  $\tilde{\psi}$  are coordinates on  $\Lambda$ , that could be taken of the form  $\tilde{\psi} = (\psi, \tau)$ , where  $\psi$  are coordinates on  $\partial\Lambda_+$ .

The phase  $\Phi(x, t)$  has the property (for a general Hamiltonian) that along each of these curves

$$(2.17) \quad \Phi(x(t), t) = \phi(x) + \int_0^t \left[ \left\langle \frac{\partial H}{\partial p}(x(s), p(s)), p(s) \right\rangle + \tau(s) \right] ds$$

which is  $\phi(x) + (mH + \tau)t$  for  $H$  positively homogeneous of degree  $m$ .

*Remark 2.1:* Adding an imaginary quadratic term would allow to treat Maslov operators with complex phase. For instance, if  $\phi(x) = x^2/2$ , and  $H(x, p) = K_0(x, p) = p_n$ , then  $\Phi(x, t) = \frac{1}{2}(x'^2 + (x_n - t)^2)$ . As we shall see, the solution  $\Phi$  of HJ equation at given energy  $E > 0$  is easily constructed for positively homogeneous Hamiltonians, and related with the distance associated with  $H$  for Finsler metric ( $m = 1$ ), or Riemann metric ( $m = 2$ ).

*Remark 2.2:* Hypotheses such as Lagrangian intersection are not required to solve (2.15), and phase functions written in eikonal coordinates such as (4.8) and (5.16), when written with auxiliary “ $\theta$ -parameters”, always parametrize  $\Lambda_+$ . Additional assumptions are necessary only when eliminating  $t$  or some of the “ $\theta$ -parameters”. A similar situation appears in diffraction theory, see e.g. [Sj].

### 2.3 The phase functions “in the large”

So far we have described the phase function when “moving along”  $\Lambda_+$  for small  $t$ . Thus the critical point of  $(t, \theta) \mapsto \Phi(x, t, \theta)$ , is such that  $x = X(t, \theta)$ . A “dual” point of vue is to fix  $x$  and find the set of  $(t, \theta)$  with  $x = X(t, \theta)$  with  $(x, t, \theta) \in C_\Phi$ .

We discuss the case of the vertical plane  $\Lambda = T_{x_0}^* M$ , which reduces to standard variational problems in the space variable.

Assume that for  $\tau = 0$  the initial surface,  $\partial\Lambda_+ \subset \Sigma_E$  is compact and parametrized by  $(x, p) = (X(\psi) = 0, P(\psi) = \eta)$ ,  $\psi \in \mathbf{S}^{n-1}$ . So we want to consider the map  $(t, \psi) \mapsto x = \exp tv_H(0, P(\psi))$ , or which is the same,  $(t, \eta) \mapsto x = \exp tv_H(0, \eta)$ . In the Riemannian case ( $m = 2$ ) this is related to the problem of geodesic completeness, which holds locally. Namely if  $|x|$  is small enough, there is a unique  $(t, \psi)$  such that  $x = \exp tv_H(0, P(\psi))$ . This holds globally if the Riemannian manifold ( $m = 2$ ) is geodesically convex. Otherwise, the “inverse map”  $x \mapsto (t, \eta)$  may be multivalued.

It is well known [CdV, pdf p.132] that the global geodesic convexity can be relaxed (locally) to a non-degeneracy condition. Namely, let  $H$  be associated with a Lagrangian  $\mathcal{L}(x, \dot{x})$  strictly convex with respect to  $\dot{x}$ , in particular if  $H$  is positively homogeneous of degree  $m > 1$  with respect to  $p$ .

Let  $k \subset \mathbf{R}_p^n$  be a compact set, which will be identified with the support of  $A(p; h)$  in (1.1). Denote by  $\text{Exp}_{x_0}^t \eta = \pi_x(\exp tv_H(x_0, \eta))$  the projection of the bicharacteristic of  $H$  starting from  $(x_0, \eta)$  at time 0 with  $\eta \in k$ . For fixed  $(x, t)$  we make the generic assumption :

(2.18) *For all  $\eta \in k$  such that  $x = \text{Exp}_{x_0}^t \eta$ , the map  $\mathbf{R}^n \rightarrow \mathbf{R}^n$ ,  $\xi \mapsto \text{Exp}_{x_0}^t \xi$  is a local diffeomorphism near  $\eta$  : in other terms,  $x_0$  and  $x$  are not conjugated along any trajectory that links them together within time  $t$ , with initial momentum  $\xi$ .*

The set of such  $(x, t)$  is an open set  $\Omega_{x,t} \subset \mathbf{R}^{n+1}$ , and its complement has Lebesgue measure 0.

Fixing  $(x, t)$ , (2.18) implies by Morse theory that  $\eta \mapsto x = \exp tv_H(0, \eta)$  has a discrete set of pre-images  $\eta$ .

Fixing  $x$ , consider now the pre-images of  $(t, \eta) \mapsto x = \exp tv_H(0, \eta)$ . It can happen that the integral manifold of  $v_H$  has several sheets over  $x$ , so several values of  $t$  contribute to the same  $x = X(t, \psi)$ . However, under the non-trapping condition  $|X(t, \psi)| \rightarrow \infty$  as  $t \rightarrow \infty$ , there is again, generically, a finite number of such  $t_j$ . Namely, it suffices that (2.18) holds with a time  $T$  such that for  $t \geq T$ ,  $X(t, \psi)$  will never coincide again with  $x$ . Moreover these  $t_j$  are non-degenerate critical points of  $t \mapsto \Phi(x, t, \eta)$ .

In particular this is the case when  $x_0$  and  $x$  are connected by (possibly several) minimal geodesics for the Riemannian metric associated to  $H$ , each indexed by some  $\eta_\alpha$ .

#### 2.4 Semi-classical Cauchy problem.

To get (1.5) one first needs to construct  $v_h = e^{-it(H-E)/h} f_h$ , which solves Cauchy problem  $(hD_t + H - E)v_h(t, x) = 0$ ,  $v_h(0, x) = f_h$ . When  $f_h(x) = \int^* e^{i(xp+S(p))/h} A(p; h) dp$ , at least for small  $t$

$$(2.20) \quad v_h(t, x) = \int^* e^{i\Phi(t, x, p)/h} a(t, x, p; h) dp$$

where the phase function  $\Phi$  is as above with initial condition  $\Phi|_{t=0} = xp$ , and the symbol  $a(t, x, p; h)$  verifies  $a(0, x, p; h) = A(p)$ , and solves some transport equations along the integral curves of  $v_H$ . One may address the problem of a semi-classical “close form” of (2.20), i.e. of performing the integration with respect to  $p$ , so that the final expression is given by oscillating functions (as in WKB expansions).

Under Assumption (2.18) the answer to this problem is given by Van Vleck formula [CdV, pdf p.132] which gives  $u_h$  as a finite sum

$$(2.21) \quad \sum_{\alpha} \frac{A(\eta_{\alpha})}{\sqrt{\text{Jac}_{x_0}(\eta_{\alpha})}} e^{i\Phi(t,x,\eta_{\alpha})/h} e^{-i\pi \text{ind}(\gamma_{\alpha})/2}$$

indexed by all  $\eta_{\alpha} \in \text{supp } A$  such that  $\text{Exp}_{x_0}^t \eta = x$ . Here  $\text{Jac}_{x_0}(\eta_{\alpha})$  is the Jacobian of  $\text{Exp}_{x_0}^t$  at  $\eta_{\alpha}$  and  $\text{ind}(\gamma_{\alpha})$  Morse index of the integral curve  $s \mapsto \exp sv_H(x_0, \eta_{\alpha})$ ,  $s \in [0, t]$ .

In other words, under Assumption (2.18) it suffices to use only non singular charts on  $\Lambda_+$  over  $x$ , and the solution is expressed in term of finitely many oscillating functions. When Assumption (2.18) is not met, i.e.  $x = x_*$  is conjugated to  $x_0$ , then there is at least one focal point  $(x_*, p_*)$  over  $x_*$  in  $\Lambda_+$ . The construction of the canonical operator (see Sect.3) necessarily uses a singular chart in a neighborhood of  $(x_*, p_*)$ , and the solution in a neighborhood of  $x_*$  involves not only simple oscillating functions corresponding to nonsingular charts as in (2.21) (if any) but also an integral of an oscillating function over some of the momenta (or “ $\theta$ ”-variables). The total number of singular and nonsingular charts over  $x_*$  however remains finite, and so (generically) only a finite sum of integrals and simple oscillating functions contribute (one summand per each chart). See also [HeSj], [LiYau].

### 2.5 Distances and generating functions

When  $\Lambda$  is the vertical plane, the phase function  $\Phi$  is related to the “distance” to  $x_0$  for the “metric” implied by  $H_0$ , which is of special interest. We make here some general remarks, mainly following [CdV], [GuSt].

In Sect.4-5 we shall discuss how to parametrize, by a non degenerate phase function, the flow of  $v_H$  out of some Lagrangian plane, when  $H$  is positively homogeneous of degree  $m$ . It includes the case  $m = 1$  which plays an important role because of Finsler metrics. So we begin with a general discussion on corresponding symplectic maps.

Let  $H$  be a positively homogeneous Hamiltonian of degree  $m$  with respect to  $p$ , defined on  $T^*M \setminus 0$ , and  $\Gamma \subset T^*M \setminus 0 \times T^*M \setminus 0$  be the graph of  $\exp v_H$  (time-1 flow). Recall from [GuSt, formula (5.6) pdf p.138 and Thm 5.4.1] that

$$(2.29) \quad (\exp v_H)^*(p dx) - p dx = (m - 1)dH$$

Integrating over a path  $\gamma_E \subset \{H = E\}$ , we recover the fact that  $\int_{\gamma_E} (\exp v_H)^*(p dx) = \int_{\gamma_E} p dx$ .

So when  $m = 1$ , not only the 2-form, but also the 1-form  $p dx$  are preserved by  $v_H$ . In this case,  $v_H$  is actually the lift of a vector field on  $M$ . When  $m > 1$ , formula (2.29) gives a generating function for  $\Gamma$  under the following assumption (geodesic convexity):

(2.30) *Let  $\pi_{M \times M} : T^*(M \times M) \rightarrow M \times M$  be the natural projection, and assume  $\pi_{M \times M} : \Gamma \rightarrow M \times M$  is a diffeomorphism, i.e. for all  $(x, y) \in M \times M$ , there is a unique  $\xi \in T_x^*M$  such that  $y = \exp v_H(x, \xi)$ .*

In case of a geodesic flow ( $m = 2$ ) (2.30) holds true when  $M$  is geodesically convex. Provided (2.30),  $\Gamma$  has a generating function  $\chi$ , i.e.  $d\chi = \text{pr}_2(p dx) - \text{pr}_1(p dx)$ , where  $\text{pr}_i : T^*(M \times M) = T^*M \times T^*M$  onto the  $i$ :th factor, and  $\text{pr}_1 \circ (\pi_{M \times M}|_{\Gamma})^{-1}$  is a diffeomorphism  $M \times M \rightarrow T^*M$ . Moreover we can then represent  $\chi$  as

$$(2.31) \quad \chi = (\text{pr}_1 \circ (\pi_{M \times M}|_{\Gamma})^{-1})^* H$$

In case of the geodesic flow ( $m = 2$ )  $\chi(x, y) = \frac{1}{2} \text{dist}(x, y)^2$ . Formula (2.29) is related to *exact symplectic twist maps* as follows. An exact symplectic twist map [Ar], [Ka], [GuSt], [CdV7]  $F : T^*M \rightarrow T^*M$  is a symplectic map with a generating function  $S_1 : M \times M \rightarrow \mathbf{R}$ ,  $(x, X) \mapsto S_1(x, X)$  which satisfies

$$(2.32) \quad F^*(p dx) - p dx = P dX - p dx = dS_1(x, X)$$

$(p, x)$  and  $(P, X)$  are related by  $p = -\partial_x S_1, P = \partial_X S_1$ . In notation  $S_1$  the subscript 1 refers to time-1 flow. In case  $H(x, p) = p^2$  (flat metric on  $\mathbf{R}^n$ ), comparing (2.32) with (2.29), i.e.  $dS_1 = (m-1)dH$ , we get  $S_1(x, X) = \frac{1}{4}(x - X)^2$ , and more generally, if  $H(x, p) = |p|^m$ , with  $m > 1$ ,  $S_1(x, X) = \left(\frac{m-1}{m}\right)^{m/(m-1)} |x - X|^{m/(m-1)}$ .

Again,  $S_1$  is not well defined when  $m = 1$ . More generally  $F(x, y)$  coincides with  $\chi(x, y)$  above for the geodesic flow.

For Hamilton-Jacobi (HJ) equation we have the following Proposition, extending (2.15) for large  $t$ . Assume  $H$  is associated with a Lagrangian convex with respect to  $\dot{x}$ . Let  $x_0, y_0 \in M$  be non conjugate points along an extremal curve  $\gamma_0(t)$  such that  $x_0 = \gamma_0(0)$  and  $y_0 = \gamma_0(t_0)$ , and  $(x_0, \xi_0), (y_0, \eta_0)$  the corresponding points in  $T^*M$ .

**Proposition 2.2** [CdV, Thm 14, pdf p.45]: *Let  $(t_0, x_0, y_0)$  be as above. Then for any  $(x, y)$  close to  $(x_0, y_0)$ , and  $t$  close to  $t_0$ , there is a unique extremal curve  $\gamma$  such that  $x = \gamma(0)$  and  $y = \gamma(t)$ . Let  $\tilde{S}(t, x, y)$  be the action along these curves (minimizing the Lagrangian action) This is a generating function for the Hamiltonian flow near  $(x_0, \xi_0)$ , verifying HJ equation*

$$(2.33) \quad \partial_t \tilde{S} + H(y, \partial_y \tilde{S}) = 0$$

This is verified in the Riemannian case  $\tilde{S}(t, x, y) = F(x, y)/2t = \text{dist}^2(x, y)/2t$  where  $F$  is the exact symplectic twist map considered above, and can be identified with the phase in the Heat kernel. We can check (2.33) trivially when  $H = \frac{1}{2}p^2$ . This holds also under assumption (2.30). Clearly under Hypothesis (2.18), (2.33) extends (2.15) for large times.

So far we have assumed some convexity of  $H$  with respect to  $p$ . The case  $m = 1$  (Finsler metric and Randers symbols) is investigated in [Ta] : it turns out that similar results hold when the *square* of Finsler metric or Randers symbol enjoys some convexity property, so for a “conformal metric” the case  $m = 1$  makes no difference.

## 2.6 Maupertuis-Jacobi correspondence

Geodesic flows are often used in Maupertuis-Jacobi correspondence. Let  $M$  be a smooth manifold, and  $\mathcal{H}, H \in C^\infty(T^*M)$  two Hamiltonians, possessing a common regular energy surface  $\Sigma = \{\mathcal{H} = \mathcal{E}\} = \{H = E\}$ . Then  $\mathcal{H}, H$  have the same integral curves  $(\mathcal{X}(\tau), \mathcal{P}(\tau)) = (X(t), P(t))$  on  $\Sigma$ , up to a reparametrization of time. Hamiltonian vector fields are related by  $X_{\mathcal{H}} = \mathcal{G}(x, p)X_H$ , parametrizations by  $dt = \mathcal{G}(\tau)d\tau$  for some smooth (positive) function  $\mathcal{G}$ . We say that  $(\mathcal{H}, \mathcal{E})$  satisfy and write  $(\mathcal{H}, \mathcal{E}) \sim (H, E)$ .

*Example 2.3:*  $H(x, p) = p^2 + V$  at energy  $E$  and  $\mathcal{H}_E(x, p) = \frac{p^2}{E - V(x)}$  at energy 1 satisfy Maupertuis-Jacobi correspondence.

*Example 2.4:* Let  $H(x, p) = \frac{p^2}{1+x^2}$  and  $\mathcal{H}(x, p) = p^2 - x^2$ . Since  $(H, 1)$  and  $(\mathcal{H}, 1)$  are mapped to each other by Maupertuis-Jacobi correspondence, we know that  $P(t, \psi) = \omega(\psi) \cosh f(t)$ ,  $X(t, \psi) = \omega(\psi) \sinh f(t)$ , where  $f$  satisfies the ODE  $f'(t) = \frac{2}{\cosh^2 f(t)}$ ,  $f(0) = 0$ . Integrating, we find  $f + \frac{1}{2} \sinh(2f) = 4t$ , so  $f(t) > 0$  for all  $t > 0$ . Another example with a frequency vector is  $H(x, p) = \frac{p^2}{1+\nu^2 \cdot x^2}$ .

*Example 2.5:*  $H(x, p) = |p|^m \frac{1}{\rho(x)}$  above;  $\mathcal{H}$  is for instance the water-wave Hamiltonian  $\mathcal{H}(x, p) = |p|(1 + \mu(x)p^2) \tanh(D(x)|p|)$ , see [DoRo], [DoMiRo], [ReDoKaMi].

Solutions of  $(\mathcal{H} - \mathcal{E})u_h = f_h$  and  $(H - E)v_h = f_h$  are related by a smooth diffeomorphism. Nevertheless it is interesting to construct  $u_h$  and  $v_h$  simultaneously. As we have seen, the first step is the resolution of HJ equations. We have:

**Proposition 2.3:** *For  $E$  in some interval, let  $(H, E) \sim (\mathcal{H}_E, 0)$  be a smooth family of smooth Hamiltonians, that satisfy Maupertuis-Jacobi correspondence, and consider HJ equations*

$$(2.35) \quad \begin{aligned} \partial_t \Phi_E(t, x) + H(x, \partial_x \Phi_E(t, x)) &= E, \quad \Phi_E|_{t=0} = \phi(x) \\ \partial_t \Psi_E(s, x) + \mathcal{H}_E(x, \partial_x \Psi_E(s, x)) &= 0, \quad \Psi_E|_{s=0} = \phi(x) \end{aligned}$$

For  $t$  sufficiently small, the maps  $t \mapsto \Phi_E(x, t)$  and  $s \mapsto \Psi_E(x, s)$  have respectively a critical point at  $t = t_E(x)$  and  $s = s_E(x)$ ; moreover  $\Phi(x, t_E(x)) = \Psi(x, s_E(x))$ .

*Proof:* To fix the ideas, we assume  $\Lambda$  is the vertical plane  $x = 0$ . Let  $\Lambda_{\Phi_E, t} = \{p = \partial_x \Phi_E(t, x)\}$ , and  $\Lambda_{\Psi_E, t} = \{p = \partial_x \Psi_E(t, x)\}$  be the integral manifolds of (2.35) in  $T^*\mathbf{R}^n$ , and consider  $L_t = \Lambda_{\Phi_E, t} \cap \Lambda_{\Psi_E, t}$ . The initial manifold is the isotropic manifold  $L_0 = \partial\Lambda_+$ , so for each  $t > 0$ ,  $L_t = \exp tv_H(L_0) = \exp tv_{\mathcal{H}_E}(L_0)$  is isotropic. Assume that at  $(x, t) = 0$ ,  $\frac{\partial \Phi_E}{\partial t} = \frac{\partial \Psi_E}{\partial t} = 0$ , so at this point,  $\Lambda_{\Phi_E, t}$  and  $\Lambda_{\Psi_E, t}$  are tangent to the common integral manifold  $\Lambda_+$  for  $H$  and  $\mathcal{H}_E$ . Since  $v_{\mathcal{H}_E}$  and  $v_H$  are parallel there, both  $\frac{\partial \Phi_E}{\partial t}$  and  $\frac{\partial \Psi_E}{\partial t}$  will “remain small” in this direction, so by implicit functions theorem, there exists surfaces  $t = t_E(x)$  and  $s = s_E(x)$  in  $T^*M$  such that  $\frac{\partial \Phi_E}{\partial t}(x, t_E(x)) = \frac{\partial \Psi_E}{\partial s}(x, s_E(x)) = 0$ . Since  $\Phi_E(x, 0) = \Psi_E(x, 0) = \phi(x)$ , we get also by integration  $\Phi_E(x, t_E(x)) = \Psi_E(x, s_E(x))$ . ♣

So the critical values of  $\Psi_E$  and  $\Phi_E$  are simply related by a diffeomorphism. Changing parametrization of time also induces a change in the half-densities.

### 3. Maslov canonical operators and bi-Lagrangian distributions

Our purpose is to describe the solution globally, including unfolding of Lagrangian singularities; this is of primary importance in the context of wave propagation. Among many references to the subject we make use in particular of [M], [Hö], [Du], [Iv], [BaWe], [CdV], [GuSt], [DoZh], [DoNaSh], [DoRo]

#### 3.1. Lagrange immersions and non-degenerate phase functions:

Consider a vector bundle  $\pi_{\mathcal{B}} : \mathcal{B} \rightarrow M$ , with local section  $\mathbf{R}^N$ , for some  $N$  that may depend on the base point  $x \in M$ . The function  $\Phi : (x, \theta) \mapsto \Phi(x, \theta)$  on  $\mathcal{B}$ ,  $\theta \in \mathbf{R}^N$ , defined near  $(x_0, \theta_0)$  with  $\xi_0 = \partial_x \Phi(x_0, \theta_0)$  is called a non-degenerate phase function (definition of [Hö]) iff  $d_{x, \theta} \Phi(x_0, \theta_0) \neq 0$ , and  $d\partial_{\theta_1} \Phi, \dots, d\partial_{\theta_N} \Phi$  are linearly independent on the critical set (as in (2.16))

$$(3.5) \quad C_{\Phi} = \{(x, \theta) \in \mathcal{B} : \frac{\partial \Phi}{\partial \theta}(x, \theta) = 0\}$$

This last condition amounts (definition of [Iv]) to say that the rank of the  $(N+n) \times N$ -matrix  $(\phi''_{\theta x}, \phi''_{\theta \theta})$  equals to  $N$  on  $C_{\Phi}$ . Or again, (definition of [BaWe], [CdV]), that the Lagrangian submanifold  $d\phi(\mathcal{B})$  of  $T^*\mathcal{B}$  be transverse to  $E^{\perp}$ , where  $E = \text{Ker}(\pi_{\mathcal{B}})$ , i.e. the fiber-derivative of  $\phi$  is transverse to the zero-section of  $\mathcal{B}$  (transversality in the sense of Bott). See also [GuSt2, Sect.4.9&5] for a systematic approach.

Let  $\iota : \Lambda \rightarrow T^*M$  be a Lagrangian immersion. We know that there exists a covering of  $\Lambda$  by canonical charts  $U$ , such that  $\Lambda$  is parametrized in each  $U$  by a non-degenerate phase function. Thus the bundle  $\mathcal{B} = M \times \mathbf{R}^N \rightarrow M$ , has constant fiber  $\mathbf{R}^N$  over  $U$ .

Let also  $\pi_{\Lambda} : \Lambda \rightarrow M$  be the natural projection. The Lagrangian immersions  $\iota$  and

$$(3.7) \quad \iota_{\Phi} : C_{\Phi} \rightarrow T^*M, \quad (x, \theta) \mapsto (x, \frac{\partial \Phi}{\partial x})$$

have the same image on  $U$  and  $C_{\Phi}$  is a submanifold of  $\mathcal{B}$  of dimension  $n$ . In particular,  $\iota_{\Phi} : C_{\Phi} \rightarrow \Lambda$  is a diffeomorphism onto its image. It is easy to prove [Iv,(1.2.7)] that

$$(3.9) \quad N - \text{rank } \Phi''_{\theta \theta} = n - \text{rank } d\pi_{\Lambda}(\iota_{\Phi}(x, \theta))$$

We set  $k = \text{rank } d\pi_{\Lambda}(\iota_{\Phi}(x, \theta))$ , and say that  $U$  has rank  $k$ . If  $k = n$ ,  $U$  is called a “regular” chart, and  $\Lambda$  is called “projectable” or “horizontal” on  $U$ . On the other extreme, if  $k = 0$ ,  $U$  is called a “maximally singular” chart, and  $\Lambda$  is called “vertical” on  $U$ .

#### 3.2 Maslov canonical operator for Lagrangian distributions.

Let  $f_h = f(\cdot; h)$  be a semi-classical Lagrangian distribution (or oscillatory integral) i.e. locally

$$(3.11) \quad f(x; h) = \int^* e^{i\varphi(x, \theta)/h} a(x, \theta; h) d\theta$$

where  $\varphi(x, \theta)$  is a non-degenerate phase function in the sense of [Hö], and  $a(x, \theta; h) = a_0(x, \theta) + ha_1(x, \theta) + \dots$  an amplitude. With  $f$  we associate the critical set  $C_{\varphi} = \{(x, \theta) \in M \times \mathbf{R}^N : \partial_{\theta} \varphi = 0\}$

and  $\iota_\varphi : C_\varphi \rightarrow T^*M$  with image the Lagrangian submanifold  $\Lambda_\varphi = \{(x, \partial_x \varphi) : (x, \theta) \in C_\varphi\}$ . Choose local coordinates  $\xi \in \mathbf{R}^n$  on  $\Lambda_\varphi$  and define the half-density in the local chart  $(C_\varphi, \iota_\varphi)$  by  $\sqrt{d\mu_\varphi} = |\det \varphi''|^{-1/2} |d\xi|^{1/2}$ . The (oscillating) principal symbol of  $f$  in  $\Lambda_\varphi$  is then defined as

$$(3.12) \quad e^{i\phi(\xi)/h} A_0(\xi) = e^{i\phi(\xi)/h} e^{i\pi \operatorname{sgn} \varphi''/4} a_0(x(\xi), \theta(\xi)) \sqrt{d\mu_\varphi}$$

where  $\phi$  is a “reduced phase function”. Conversely, assume  $\iota : \Lambda \rightarrow T^*M$  is a smooth Lagrangian immersion, with a smooth positive half-density  $\sqrt{d\mu}$ , it can be parametrized locally by phase functions  $\varphi$  in canonical charts  $\iota_\varphi : U_\varphi \rightarrow T^*M$ . These phases can be chosen coherently, and define a class of “reduced phase functions”  $\phi$ , parametrizing  $\iota$  locally. This gives the fibre bundle of phases  $\mathcal{L}_h$ , including Maslov indices, equipped with transition functions. We are also given local smooth half-densities  $|d\mu_\varphi|^{1/2}$  on  $\Lambda$ , defining the fibre bundle of half-densities  $\Omega^{1/2}$ , equipped with transition functions. The collection of these objects make a fibre bundle  $\Omega^{1/2} \otimes \mathcal{L}_h$  over  $\Lambda$ . A section of  $\Omega^{1/2} \otimes \mathcal{L}_h$  will be written as

$$(3.13) \quad f(x; h) = [K_{(\Lambda, \mu)}^h A](x; h)$$

where  $K_{(\Lambda, \mu)}^h$  is called Maslov canonical operator. The set of such Lagrangian distributions microlocally supported on  $\Lambda$  will be sometimes denoted by  $I(M; \Lambda)$ . The “reduced phase function” and the “principal symbol” of  $f$  are defined invariantly. See e.g. [DoNaSh] for details.

Recall in particular the following standard result see e.g. [Iv, Thm.1.2.8]. Let  $u \in I(M, \Lambda)$  be a Lagrangian distribution locally of the form  $u = \int e^{i\Phi(x, \theta)/h} a(x, \theta; h) d\theta$  and  $B$  a  $h$ -PDO with principal symbol  $B(x, p)$ . Then the same holds for  $Bu$ , its amplitude  $b(x, \theta; h)$  has leading term

$$(3.15) \quad b_0(x, \theta) = B(x, \partial_x \Phi(x, \theta)) a_0(x, \theta)$$

and its principal symbol on  $\Lambda$  is related to this of  $u$  by  $b^0(x, p) = B(x, p) a^0(x, p)$ . Moreover if  $B$  has sub-principal symbol  $B_1$ , and  $B(x, p) = 0$  on  $\Lambda$ , then  $hBu$  is again a Lagrangian distribution with principal symbol on  $\Lambda$

$$(3.16) \quad c^0(x, p) = \frac{1}{i} \{B, a^0\} + B_1 a^0$$

In order that (3.11) solves a PDE such that  $(H - E)u_h = f_h \bmod \mathcal{O}(h^2)$ ,  $\Phi$  needs to satisfy HJ equation, and  $a_0$  a transport equation. If  $H(x, p; h) = H_0(x, p) + hH_1(x, p) + \dots$ , the principal symbol of (3.11) is given by  $\frac{1}{i} \mathcal{L}_{v_{H_0}} a + H_1 a \bmod \mathcal{O}(h)$ , which we require to vanish  $\bmod \mathcal{O}(h)$  on  $C_\Phi$ ; this gives

$$(3.17) \quad \left( \frac{1}{i} \mathcal{L}_{v_{H_0}}|_\Lambda + H_1 \right) a_0(x(\xi), \theta(\xi)) \sqrt{d\mu_\varphi} = 0$$

Note that on  $\Lambda_+$ , we can replace Lie derivative  $\mathcal{L}_{v_{H_0}}$  by  $d/dt$ . Provided  $\Phi$  is a non-degenerate phase-function, (3.17) admits a global solution, computed on each canonical chart. For instance on a regular chart, this is just WKB construction. In a totally singular chart instead, we solve (3.17) in Fourier representation, the example of Helmholtz (Schrödinger) operator is given e.g. in [DoRo].

The function  $a_0(x(\xi), \theta(\xi))$  is smooth in  $\xi$ , but of course when expressed in  $x$ -variable, singularities may occur due to singular Jacobians at focal points. For the sake of simplicity, we have discarded Maslov indices, see e.g. [Iv], [BaWe, Sect.4], [DoNaSh] for a complete description.

### 3.3 Maslov canonical operator for bi-Lagrangian distributions and symbolic calculus.

We want to describe oscillatory integrals microlocally supported near a Lagrangian intersection, i.e. on  $\Lambda \cup \Lambda^+$ . As in [MelUhl], we extend Maslov canonical operator by introducing amplitudes (or symbols) verifying the “compatibility” condition. Semi-classical distributions used in [MelUhl] (adapted to the semi-classical case), are locally of the form (see (1.8))

$$(3.18) \quad u(x; h) = \int_0^\infty dt \int^* e^{i(x'\xi' + (x_n - t)\xi_n)/h} b(t, x, \xi) d\xi$$

where  $b$  is a classical symbol. We say that  $u_h(x) = u(x; h)$  is a *bi-Lagrangian distribution* on the intersecting pair  $(\Lambda, \Lambda_+^0)$ .

Recall  $\text{WF}_h(u_h) \subset \Lambda \cup \Lambda_+^0$ , which is the conormal bundle of the manifold with boundary  $x' = 0, x_n \geq 0$ .

Let  $I(M, \Lambda, \Lambda_+^0)$  be the set of bi-Lagrangian distributions. Using in particular Lemma 1.1 it can be shown that it is stable under  $h$ -FIO's preserving the Lagrangian intersection, so there is no loss of generality here in assuming  $\Lambda^+ = \Lambda_+^0$ .

Ignoring non-vanishing densities and Maslov contributions, we find that, if  $u_h$  is a bi-Lagrangian distribution on  $(\Lambda, \Lambda_+^0)$ , then the principal symbol of  $u_h$  as a Lagrangian distribution near  $\Lambda_+^0 \setminus L_0$ ,  $L_0 = \partial\Lambda_+^0$ , is given by  $\sigma^+(u_h) = a(x_n, (0, x_n), (\xi', 0))$ , while the principal symbol of  $u_h$  as a Lagrangian distribution near  $\Lambda \setminus L_0$  equals  $\sigma(u_h) = -ia(0, 0, \xi)/\xi_n$  on  $\Lambda \setminus L_0$ . Observe that  $\sigma^+$  is smooth up to  $L_0$  and  $\xi_n \sigma(u_h) = -i\sigma^+(u_h)$  on  $L_0$ , which we call the *compatibility condition*.

The space of principal symbols that satisfy the compatibility condition is stable under the action of  $h$ -PDO, and denoted by  $S(\Lambda, \Lambda_+^0, \Omega^{1/2} \otimes \mathcal{L}_h)$ . As in [MelUhl] one can show that

$$(3.20) \quad I(M, \Lambda, \Lambda_+^0) \rightarrow S(\Lambda, \Lambda_+^0, \Omega^{1/2} \otimes \mathcal{L}_h), \quad u_h \mapsto \tilde{\sigma}(u_h) = \tilde{\sigma}(u_h)$$

is surjective. This allows to define the “restriction operator”  $R : \tilde{\sigma}(u_h) \mapsto \sigma^+(u)|_{L_0}$ . Thus there is a well-defined Maslov bundle over  $\Lambda, \Lambda_+^0$ .

At this stage, we prefer to write  $\tilde{\sigma}(b)$  instead of  $\tilde{\sigma}(u_h)$ , since  $\tilde{\sigma}(u_h)$  depends only on the amplitude.

Adapting [MelUhl, Prop.2.3] to asymptotics with respect to  $h$ , we can show that if  $u$  is a bi-Lagrangian distribution on  $(\Lambda, \Lambda^+)$ , and  $B$  is a  $h$ -PDO with  $\text{WF}_h(B) \cap \Lambda = \text{WF}'_h(B) \circ \Lambda = \emptyset$ , then  $Bu$  is a Lagrangian distribution supported on  $\Lambda_+^0$ , while if  $B$  is a  $h$ -PDO with  $\text{WF}_h(B) \cap \Lambda_+^0 = \emptyset$ , then  $Bu$  is a Lagrangian distribution supported on  $\Lambda$ . The main result on symbolic calculus for bi-Lagrangian distributions [MelUhl, Prop.5.4] is an extension of (3.15)-(3.16), which we rephrase as follows:

**Proposition 3.3:** *Let  $u_h \in I(M, \Lambda, \Lambda_+^0)$  be a bi-Lagrangian distribution with symbol*

$$\tilde{\sigma}(u_h) \in S(\Lambda, \Lambda_+^0, \Omega^{1/2} \otimes \mathcal{L}_h)$$

(i) Let  $B(x, hD_x; h)$  be a  $h$ -PDO with principal symbol  $B(x, p)$ . Then  $Bu_h \in I(M, \Lambda, \Lambda_+^0)$  is again a bi-Lagrangian distribution with symbol  $B(x, p)\tilde{\sigma}(u_h) \in S(\Lambda, \Lambda_+^0, \Omega^{1/2} \otimes \mathcal{L}_h)$ , identified with  $(B|_{\Lambda}\sigma(u_h), B|_{\Lambda_+^0}\sigma^+(u_h))$ .

(ii) Assume moreover  $B(x, p)$  vanishes on  $\Lambda_+^0$  and  $B(x, hD_x; h)$  has sub-principal symbol  $B_1$ . Then  $Bu_h = v_h + hw_h$ , where  $v_h \in I(M, \Lambda)$  and  $w_h \in I(M, \Lambda, \Lambda_+^0)$ . Their principal symbols (with the identification above) are given by

$$(3.23) \quad \begin{aligned} \tilde{\sigma}(v_h) &= B(x, p)\tilde{\sigma}(u_h)|_{\Lambda} \approx (B(x, p)\sigma(u_h), 0) \\ \tilde{\sigma}(w_h)|_{\Lambda_+^0} &\approx \left(\frac{1}{i}\{B, \tilde{\sigma}(u_h)\} + B_1\tilde{\sigma}(u_h)\right)|_{\Lambda_+^0} \end{aligned}$$

From this Proposition, it is easy to recover (at least locally) as in [UhlMel, Prop.6.6], the solution  $u_h$  of  $H(x, hD_x)u_h = f_h$  as a bi-Lagrangian distribution  $u_h \in I(M, \Lambda, \Lambda_+^0)$ .

Let us try to formulate these ideas more globally, replacing  $\Lambda_+^0$  by  $\Lambda_+$ . Consider the fibration  $\Lambda_+ = \bigcup_{t>0} \Lambda_t$ . Define as in (3.18) (forgetting about densities and Maslov indices)

$$u_h(x) = \int_0^\infty dt [K_{\Lambda_t}^h b](t, x; h)$$

where  $b$  is an amplitude, whose principal symbol in a canonical chart takes the form  $B_0(\xi)$  as in (3.12).

Let

$$[K_{\Lambda, \Lambda_+}^h b](x; h) = \int_0^\infty [K_{\Lambda_t}^h b](t, x; h)$$

Then the map  $b \mapsto [K_{\Lambda, \Lambda_+}^h b]$  is linear, and defines Maslov canonical operator for bi-Lagrangian distributions at first order in  $h$  as

$$(3.25) \quad [K_{\Lambda, \Lambda_+}^h b](x; h) = [K_{\Lambda, \Lambda_+}^h \tilde{\sigma}(B_0)](x; h)$$

where in (3.21), we have changed  $\tilde{\sigma}(u_h)$  to  $\tilde{\sigma}(b)$  according to our convention.

Let us write (1.5) in terms of Maslov canonical operator for bi-Lagrangian distributions. At first order in  $h$  we can write the initial data as  $f_h = [K_{\Lambda}^h A_0]$ . We construct the solution according to the standard scheme (HJ and transport equations) which determines an amplitude  $b$ , depending linearly on  $a$ . We write symbolically, according to (3.25)

$$(3.26) \quad u_h(x) = [K_{\Lambda, \Lambda_+}^h \tilde{\sigma}(B_0)](x; h)$$

where  $B_0$  depends linearly on  $A_0$ . Actually we shall determine  $B_0$  by constructing, at first order, the pair  $(\sigma(u_h), \sigma^+(u_h))$ .

#### 3.4 The constant coefficient case.

In general it is difficult to obtain a decomposition of  $u_h$  adapted to the splitting (3.21), i.e.  $u_h = u_h^0 + u_h^1$  where  $\text{WF}_h(u_h^0) \subset \Lambda$  and  $\text{WF}_h(u_h^1) \subset \Lambda_+^0$ . A “weak form” of this property is given in [MelUh, Prop.2.3]. In [AnDoNaRo3] we consider Helmholtz operator with constant coefficient in 2-D, when  $f$  is radially symmetric and has compact support (instead of  $\mathcal{F}f$ ); its Fourier transform

$g = \mathcal{F}_1 f$  is again of the form  $g(p) = g(|p|) = g(r)$  and extends holomorphically to  $\mathbf{C}^2$ , so that we can compute explicitly  $u_h$  by contour integrals when  $E = k^2$ ,  $k > 0$ . As a result (after taking  $\varepsilon \rightarrow 0$ ),

$$u_h^0(x) = \frac{i\pi g(k)}{(2\pi h)^2} \int_{-\pi/2}^{\pi/2} \exp[i|x|k \cos \theta/h] d\theta$$

with  $\text{WF}_h u_h^0 \subset \{x = 0\} \cup \{(x, k\frac{x}{|x|}), x \neq 0\} = \Lambda \cup \Lambda_+$ . Consider now  $u_h^1$ , after taking  $\varepsilon \rightarrow 0$ ,

$$u_h^1(x) = \frac{1}{(2\pi h)^2} \int \exp[i|x|r \cos \theta/h] \frac{g(r)}{r+k+i\varepsilon} dr d\theta$$

We set  $\tilde{g}(r) = \frac{g(r)}{r(r+k)}$ , and assume for simplicity  $\tilde{g}(r)\sqrt{r} \in L^1(\mathbf{R}_+)$ , we have  $u_1(x) = H_0(\tilde{g})(\frac{|x|}{h})$ , where  $H_0$  denotes Hankel transform of order 0. Let  $\chi \in C_0^\infty(\mathbf{R}^2)$  be radially symmetric, and equal to 1 near 0, since  $\text{WF}_h f_h = \{x = 0\}$ , we have

$$g = \mathcal{F}_h(\chi f_h) + \mathcal{O}(h^\infty) = (2\pi h)^{-2} \mathcal{F}_h(\chi) * g + \mathcal{O}(h^\infty)$$

so in the expression for  $u_1$  we may replace mod  $\mathcal{O}(h^\infty)$ ,  $\tilde{g}(r)$  by a constant times  $\hat{g}(r) = \frac{(\mathcal{F}_h(\chi)*g)(r)}{r(r+m)}$  (see [Bad] for 2-D convolution and Fourier transform in polar coordinates). To estimate  $\text{WF}_h u_1$ , we compute again the Fourier transform of  $(1 - \tilde{\chi})\hat{g}$  where  $\tilde{\chi}$  is a cut-off equal to 1 near 0, and we find it is again  $\mathcal{O}(h^\infty)$  if  $\chi \equiv 1$  on  $\text{supp } \tilde{\chi}$ . This shows that  $\text{WF}_h u_1 \subset \{x = 0\}$ .

Note that this Example makes use of Bessel function  $J_0(\frac{|x|}{h})$ , we shall return to such “localized functions” in Sect.5.

#### 4. $f_h$ is supported microlocally on the vertical plane

Consider the case where  $H_0$  is positively homogeneous of degree  $m \geq 1$  with respect to  $p$  and  $f_h$  is microlocally concentrated on the vertical plane  $\Lambda = \{x = 0\}$ , e.g.  $f_h(x) = h^{-n}f(\frac{x}{h})$ , with  $\mathcal{F}_1 f \in \mathcal{S}(\mathbf{R}^n)$ .

##### 4.1 Some non-degeneracy condition.

Recall first from [DoNaSh, Lemma 6] the following result. Let  $\tilde{\iota} : \tilde{\Lambda} \rightarrow T^*\tilde{M}$  be a Lagrangian embedding of dimension  $\tilde{n}$ ,  $U \subset \tilde{\Lambda}$  a connected simply connected open set,

$$(\tilde{\phi}, \tilde{\psi}) = (\phi_1, \dots, \phi_k, \psi_1, \dots, \psi_{\tilde{n}-k})$$

local coordinates on  $U$  where  $k$  is the rank of  $\tilde{\Lambda}$ . Thus  $\tilde{\Lambda}$  is defined by  $x = X(\tilde{\phi}, \tilde{\psi}), p = P(\tilde{\phi}, \tilde{\psi})$  in the chart  $U$ . Let  $\Pi(\tilde{\phi}, \tilde{\psi})$  be a smooth  $\tilde{n} \times k$  matrix function defined in  $U$  such that:

$$(4.1) \quad \Pi^*(\tilde{\phi}, \tilde{\psi})X_{\tilde{\psi}}(\tilde{\phi}, \tilde{\psi}) = \text{Id}_{k \times k}$$

$$(4.2) \quad \kappa : (\tilde{\phi}, \tilde{\psi}) \mapsto (X(\tilde{\phi}, \tilde{\psi}), \tilde{\psi}) \text{ is an embedding}$$

Then there is a neighbhd  $V$  of  $\kappa(U)$  such that the system

$$(4.3) \quad \Pi^*(\tilde{\phi}, \tilde{\psi})(x - X(\tilde{\phi}, \tilde{\psi})) = 0, \quad (x, \tilde{\psi}) \in V$$

has a unique smooth solution  $\tilde{\phi} = \tilde{\phi}(x, \tilde{\psi})$  satisfying the condition  $X(\tilde{\phi}(x, \tilde{\psi}), \tilde{\psi}) = x$ , when  $(x, \tilde{\psi}) \in \kappa(U)$ .

For  $(\tilde{\phi}, \tilde{\psi}) \in U$ , define the  $\tilde{n} \times \tilde{n}$  matrix

$$(4.4) \quad M(\tilde{\phi}, \tilde{\psi}) = (\Pi(\tilde{\phi}, \tilde{\psi}); P_{\tilde{\psi}}(\tilde{\phi}, \tilde{\psi}) - P_{\tilde{\phi}}(\tilde{\phi}, \tilde{\psi})\Pi^*(\tilde{\phi}, \tilde{\psi})X_{\tilde{\psi}}(\tilde{\phi}, \tilde{\psi}))$$

As we shall see, invertibility of  $M(\tilde{\phi}, \tilde{\psi})$  plays an important role [DoMaNa], [DoNaSh].

Consider now our special setting where  $H$  is positively homogeneous of degree  $m$ ,  $\Lambda$  is the vertical plane, and recall  $\langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau) \rangle = mH$  from (2.6). Here  $\tau$  is taken as a parameter, everything depends smoothly on  $\tau$  and  $\Lambda_+(0) = \Lambda_+$ . So  $\Pi(t, \psi, \tau) = \frac{1}{mH}P(t, \psi, \tau)$  is a left inverse of  $\dot{X}$ :  $\Pi^*\dot{X} = \frac{1}{mH}\langle P, \dot{X} \rangle = 1$ . Further, the map  $\Lambda_+ \rightarrow \mathbf{R}^n$ ,  $(t, \psi) \mapsto (X(t, \psi, \tau), \psi)$  is clearly an embedding. This fulfills conditions (4.1),(4.2) above for  $\tilde{\Lambda} = \Lambda_+(\tau)$ , with  $\tilde{n} = n$ ,  $k = 1$  and  $\tilde{\phi} = t$  and  $\tilde{\psi} = \psi$ . So the system  $\Pi^*(t, \psi, \tau)(x - X(t, \psi, \tau)) = \langle P, x - X(t, \psi, \tau) \rangle = 0$  has a unique solution  $t = t_0(x, \psi, \tau)$  satisfying the condition  $X(t_0(x, \psi, \tau), \psi, \tau) = x$ , and this solution is a smooth function.

Moreover by (2.6) again, the matrix

$$(4.5) \quad M(t, \psi) = (\Pi(t, \psi), P_{\psi}(t, \psi) - \dot{P}^t P X_{\psi}) = (\frac{1}{mH}P, P_{\psi})$$

has determinant  $\frac{1}{mH} \det(P, P_{\psi})$ . As we shall see, the invariant (inverse) density on  $\Lambda_+$  is proportional to  $\det(P, P_{\psi})$ , so we need to know if  $\det(P, P_{\psi}) \neq 0$ , at least for small  $t$ .

*Example 4.1:* Let us compute  $\det(P, P_\psi)$  at  $t = 0$  for a geodesic flow  $H(x, p)$ , on the energy shell  $E = 1$  when  $n = 2$  or  $n = 3$ . When  $n = 2$ , up to a change of  $x$  coordinates such that at  $x = 0$ , the metric  $H(0, p)$  takes the diagonal form  $H(x, p) = \frac{p_1^2}{a_1^2} + \frac{p_2^2}{a_2^2}$  (elliptic polarization), and  $P = (a_1 \cos \psi, a_2 \sin \psi)$ . Hence  $\det(P, P_\psi) = a_1 a_2$  at  $x = 0$ , and for small  $|t|$  we have  $\det(P, P_\psi) > 0$ . For  $n = 3$ ,  $H(0, p) = \frac{p_1^2}{a_1^2} + \frac{p_2^2}{a_2^2} + \frac{p_3^2}{a_3^2}$ , and in spherical coordinates  $(\psi_1, \psi_2)$  where  $0 < \psi_1 < \pi$ , we find  $\det(P, P_\psi) = a_1 a_2 a_3 \sin \psi_1 > 0$  for small  $t$  and away from the poles  $(0, 0, \pm 1)$ .

*Example 4.2:* When  $H(x, p) = \frac{|p|^m}{\rho(x)}$ , recall from (2.3) that  $P(\psi, \tau) = |P|_\tau \omega(\psi)$  at  $t = 0$ . Since  $\det(\omega(\psi), \omega^\perp(\psi)) = 1$ , again we have  $\det(P, P_\psi) \neq 0$  for small  $t$ .

#### 4.2 Construction of the phase function and half-density.

First we construct a generating function  $\Phi$  of  $\Lambda_+$  using HJ theory in eikonal coordinates, and verifying  $\Phi|_{t=0} = \langle x, \omega(\psi) \rangle$ . There are several ways to do that. The first one gives

$$(4.7) \quad \Phi_0(x, t, \psi, \tau) = mEt + \langle P(t, \psi, \tau), x - X(t, \psi, \tau) \rangle$$

with initial condition  $\Phi(x, 0) = \langle p, x \rangle$ , where again  $\tau + H = E$ . The “ $\theta$  variables” in Hörmander’s definition are then  $(\psi, \tau)$ . In the simple example where  $n = 1$ ,  $\tau + H(x, p) = \tau + p = E$ , (there are no variable  $\psi$ ),  $\Phi_0 = Et + p(x - t)$ .

The second one consists [DoMaNaTu2] in choosing a new coordinate  $\lambda = \lambda(\tau)$ ,  $\lambda(0) = 1$ , on  $\Lambda$  completing the  $\psi$  variables, such that  $\partial\Lambda_+$  is given by  $\lambda = 1$ ; it will play the role of a “Lagrange multiplier”. We define

$$(4.8) \quad \Phi(x, t, \psi, \lambda) = mEt + \lambda \langle P(t, \psi), x - X(t, \psi) \rangle$$

where now  $(X, P)$  are evaluated on  $\Lambda_+$  (and not on  $\Lambda_+(\tau)$ ). The “ $\theta$  variables” in Hörmander’s definition are then  $(\psi, \lambda)$ . In the Example above,  $\Phi = Et + \lambda p(x - t)$ .

It is more convenient to work with  $\Phi$  than with  $\Phi_0$ . Variables  $\tau$  and  $\lambda$  are diffeomorphically mapped onto each other. In case (1.15) this goes as follows as follows : comparing (4.7) with (4.8) at  $t = 0$ , we get  $P(\psi, \tau) = \lambda P(\psi)$ , with  $P(\psi, \tau)$  as in (2.3). But  $|P|_\tau^m = \rho(0)(E - \tau)$ , while  $|P|^m = \rho(0)E$ , hence

$$(4.9) \quad \lambda = \left(1 - \frac{\tau}{E}\right)^{1/m}$$

A similar scaling holds in Example 4.1.

**Proposition 4.1:** *Let  $H(x, p)$  be positively homogeneous of degree  $m \geq 1$  with respect to  $p$  on  $T^*M \setminus 0$ , and  $\det(P, P_\psi) > 0$ . Then  $\Phi(x, t, \psi, \lambda)$  given in (4.8) is a non-degenerate phase function defining  $\Lambda_+$ , and solves HJ Eq. (2.5), with initial condition  $\Phi|_{t=0} = \langle x, p \rangle$ . The positive invariant (inverse) density on  $\Lambda_+$  is given by*

$$(4.10) \quad F[\Phi, d\mu_+]|_{C_\Phi} = mE \det(P, P_\psi) > 0$$

The critical set  $C_\Phi$  is then determined by  $x = X(t, \psi)$  (which can be inverted as  $t = t(x, \psi)$ ) and  $\lambda = 1$ . It coincides with the set  $\kappa(U)$  defined in (4.2)-(4.3).

*Proof:* We have using (2.6)

$$(4.11) \quad \partial_t \Phi = \dot{\Phi} = mE + \lambda \langle \dot{P}, x - X(t, \psi) \rangle - \lambda \langle P, \dot{X} \rangle = mE(1 - \lambda) + \lambda \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle$$

so  $\partial_t \Phi = 0$  and  $\partial_x \Phi = P(t, \psi)$  along  $x = X(t, \psi)$  when  $\lambda = 1$ . We are left to show that  $\Phi$  is non degenerate phase function, with  $(\psi, \lambda)$  as “ $\theta$ -parameter”s. From (4.8)

$$(4.12) \quad \begin{aligned} \partial_\psi \Phi &= \lambda \langle \partial_\psi P(t, \psi), x - X(t, \psi) \rangle \\ \partial_\lambda \Phi &= \langle P(t, \psi), x - X(t, \psi) \rangle \end{aligned}$$

Let us add  $t$  to the “ $\theta$ -variables”, and consider the variational system  $\partial_t \Phi = \partial_\psi \Phi = \partial_\lambda \Phi = 0$ , which determines the critical set  $C_\Phi$ . Last 2 equations  $\partial_\psi \Phi = 0, \partial_\lambda \Phi = 0$  give an homogeneous linear system in  $x - X(t, \psi)$  with determinant  $\det(P, P_\psi)$ .

So at least for small  $t$  the phase is critical with respect to  $(\psi, \lambda)$  precisely for  $\lambda = 1$  and  $x = X(t, \psi)$ , in particular it is critical along  $\partial\Lambda_+$  when  $\lambda = 1$ . By the discussion above and [DoNaSh, Lemma 6], we find that  $\langle P(t, \psi), x - X(t, \psi, \tau) \rangle = 0$  when  $\lambda = 1$  has a unique solution  $t = t_1(x, \psi) = t_0(x, \psi, \tau = 0)$  satisfying the condition: if  $(x; \psi, \lambda = 1) \in C_\Phi$ , then  $X(t_1(x, \psi), \psi) = x$ . We recall  $t_0$  is defined in the discussion after (4.4). Moreover  $t_1$  is the critical point of  $t \mapsto \Phi$  when  $\lambda = 1$ .

Condition  $\det(P, P_\psi) > 0$  (which holds at  $t = 0$ ) actually ensures that  $\Phi$  is a non degenerate phase function, i.e. the vectors  $(d\partial_t \Phi, d\partial_\psi \Phi, d\partial_\lambda \Phi)$  are linearly independent on the set  $x = X(t, \psi)$ . Namely, look at the variational system and compute on  $C_\Phi$  the differentials

$$(4.13) \quad \begin{aligned} d\dot{\Phi} &= -mE d\lambda + \lambda \langle \dot{P}(t, \psi), dx - dX(t, \psi) \rangle \\ d(\partial_\psi \Phi) &= \lambda \langle P_\psi(t, \psi), dx - dX(t, \psi) \rangle \\ d(\partial_\lambda \Phi) &= \langle P, dx - dX(t, \psi) \rangle \end{aligned}$$

Introduce the Jacobian (see e.g. [NaStSh], [DoNaSh] and references therein)

$$(4.14) \quad F[\Phi, d\mu_+]|_{C_\Phi} = \frac{dt \wedge d\psi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi)}{dx \wedge dt \wedge d\psi \wedge d\lambda}$$

Here  $dx$  is the volume form. Substituting (4.13) into  $\omega = dt \wedge d\psi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi)$  we get

$$\begin{aligned} \omega &= -mE d\psi \wedge dt \wedge d\lambda \wedge \langle P, dx - dX(t, \psi) \rangle \wedge \langle P_\psi, dx - dX(t, \psi) \rangle + \\ & d\psi \wedge dt \wedge \langle \dot{P}, dx - dX(t, \psi) \rangle \wedge \langle P, dx - dX(t, \psi) \rangle \wedge \langle P_\psi, dx - dX(t, \psi) \rangle \end{aligned}$$

Writing  $dX = \dot{X} dt + X_\psi d\psi$ , we check that the second term vanishes, so we are left with

$$\omega = -mE dt \wedge d\psi \wedge d\lambda \wedge dx \det(P, P_\psi)$$

which gives (4.10). So if  $\det(P, P_\psi) > 0$ ,  $\Phi$  is a non-degenerate phase function, and (4.10) the invariant (inverse) density on  $\Lambda_+$ . ♣

*Remark 4.1:* Let  $y$  be coordinates on  $C_\Phi = \{(x, t, \psi, \lambda) : \frac{\partial \Phi}{\partial(t, \psi, \lambda)} = 0\}$  we can also write (4.14) in the form [GrSj, Sect.11]

$$F[\Phi, d\mu_+]|_{C_\Phi} = \det \left( \begin{array}{cc} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial(t, \psi, \lambda)} \\ \frac{\partial^2 \Phi}{\partial x \partial(t, \psi, \lambda)} & \frac{\partial^2 \Phi}{\partial(t, \psi, \lambda)^2} \end{array} \right) |_{C_\Phi}$$

This formula makes use of the implicit relation between  $x$  and  $(t, \psi)$  such that  $x = X(t, \psi)$  when  $\lambda = 1$ . It is actually independent of the choice of coordinates on  $C_\Phi$  but it does depend on the choice of local coordinates  $x$ .

*Remark 4.2:* As mentionned in the last Example 1.1, we can also consider complex phase functions. For instance replace (4.8) by

$$\Phi(x, t, \psi, \lambda) = mEt + \lambda \langle P(t, \psi), x - X(t, \psi) \rangle + i \frac{\lambda \mu}{2} \langle P(t, \psi), x - X(t, \psi) \rangle^2$$

As in (4.11)-(4.12) we get

$$\begin{aligned} \partial_t \Phi &= mE(1 - \lambda - i\mu\lambda) + \langle \dot{P}, x - X \rangle + i\lambda\mu \langle P, x - X \rangle \langle \dot{P}, x - X \rangle \\ \partial_\psi \Phi &= \lambda \langle \partial_\psi P, x - X \rangle + i\lambda\mu \langle P, x - X \rangle \langle \partial_\psi P, x - X \rangle \\ \partial_\lambda \Phi &= \langle P, x - X \rangle + i \frac{\mu}{2} \langle P, x - X \rangle^2 \end{aligned}$$

so it has the same critical point as the real phase when  $\lambda = 1$ , with the initial condition  $\Phi_{t=0} = \langle x, \omega(\psi) \rangle + i \frac{\mu}{2} \langle x, \omega(\psi) \rangle^2$ .

Now we look for the critical points of the phase, and describe the Lagrangian singularities. Let (see Proposition A.1)

$$(4.16) \quad a = \langle \dot{P}, X_\psi \rangle = b = \langle P_\psi, \dot{X} \rangle, \quad c = \langle P_\psi, X_\psi \rangle$$

**Definition 4.2:** Let  $H$  be positively homogeneous of degree  $m$ . We call a point  $z = (x, p)$  an ordinary point if  $\langle p, -\partial_x H \rangle \neq 0$ , and a special point otherwise.

In the following we investigate some configurations of  $\Lambda_+$  near  $t = 0$ .

**Proposition 4.3:** Assume  $n = 2$ . We have:

- (i) Let  $p$  such that  $\langle -\partial_x H(0, p), \partial_p H(0, p) \rangle \neq 0$ . Then near  $(0, p)$ , and small  $t$ , the rank of  $\pi|_{\Lambda_+}$  is 1 or 2.
- (ii) Let  $p$  such that  $-\partial_x H(0, p) \neq 0$  and  $\langle -\partial_x H(0, p), p \rangle = 0$ , i.e.  $z = (0, p)$  is a special point. Then the rank of  $\pi|_{\Lambda_+}$  is 1.

In case (ii)  $X_\psi(0, \psi_0) = 0$  so  $z = (0, p)$  is a focal point. This Proposition doesn't cover the case  $-\partial_x H(0, p) = 0$ , where the Lagrangian singularity is again of standard type.

*Proof:* (i) On  $C_\Phi$  we have  $\partial_t^2 \Phi = -\langle \dot{P}, \dot{X} \rangle \neq 0$ , so implicit function theorem shows that (for small  $t$ )  $\dot{\Phi} = 0$  is equivalent to  $t = t(\psi, \lambda)$ . Since we have eliminated  $t$ , the “ $\theta$ -parameters” are now  $(\psi, \lambda)$ , and we set

$$\Psi(x, \psi, \lambda) = \Phi(x, t(x, \psi, \lambda), \psi, \lambda)$$

Differentiating the relation  $\partial_t \Phi = 0$ , we get that on  $C_\Phi$  and  $\lambda = 1$

$$(4.17) \quad \frac{\partial t}{\partial \psi} = -\frac{\langle P_\psi, \dot{X} \rangle}{\langle \dot{P}, \dot{X} \rangle}, \quad \frac{\partial t}{\partial \lambda} = -\frac{\langle P, \dot{X} \rangle}{\langle \dot{P}, \dot{X} \rangle}$$

and a straightforward computation yields

$$(4.18) \quad -\Psi''_{\theta\theta} = \left( \frac{\partial^2 \Phi}{\partial t^2} \right)^{-1} \left( \begin{array}{c} \langle \dot{P}, X_\psi \rangle^2 + \langle P_\psi, X_\psi \rangle \frac{\partial^2 \Phi}{\partial t^2} \\ * \\ mE \langle P_\psi, \dot{X} \rangle - mE \langle \dot{P}, X_\psi \rangle + \langle \dot{P}, X_\psi \rangle \langle P_\psi, \dot{X} \rangle \\ (mH)^2 \end{array} \right)$$

which simplifies due to (4.16). The lower right term is  $\partial_\lambda^2 \Phi|_{\lambda=1}$ , so  $\lambda = 1$  is a non-degenerate critical point. Applying (3.9) with  $N = n = 2$ , we find that the rank of  $\pi|_{\Lambda_+}$  is 1 or 2.

(ii) For small  $t$ , the system  $F(x, t, \psi, \lambda) = \left( \frac{\partial_t \Phi}{\partial \lambda} \right) = 0$  is equivalent to  $t = t(x, \psi)$ ,  $\lambda = \lambda(x, \psi)$ . We can eliminate variables  $(t, \lambda)$  by stationary phase as in (1.8) by noticing that  $\frac{\partial^2 \Phi}{\partial t \partial \lambda} = -mE \neq 0$ . The implicit function theorem gives  $(t, \lambda) = (t(x, \psi), \lambda(x, \psi))$ .

We want to keep  $\lambda(x, \psi) = 1$ . Differentiating  $F = 0$  along  $\Lambda_+$  with respect to  $x$  and  $\psi$  we find, using Hamilton equations

$$(4.20) \quad \begin{aligned} \langle \dot{P}, \dot{X} \rangle \frac{\partial t}{\partial x} + mE \frac{\partial \lambda}{\partial x} &= {}^t \dot{P}, & mE \frac{\partial t}{\partial x} &= {}^t P \\ \langle \dot{P}, \dot{X} \rangle \frac{\partial t}{\partial \psi} + mE \frac{\partial \lambda}{\partial \psi} &= -\langle \dot{P}, X_\psi \rangle = -a, & mE \frac{\partial t}{\partial \psi} &= 0 \end{aligned}$$

Assume  $\frac{\partial \lambda}{\partial x} = 0$ . This implies  $\dot{P} = \frac{\langle \dot{P}, \dot{X} \rangle}{mE} P$ , i.e.  $\partial_x H + \frac{1}{mE} \langle -\partial_x H, \partial_p H \rangle P = 0$ . When  $-\partial_x H(0, p) \neq 0$  and  $\langle -\partial_x H(0, p), p \rangle = 0$ , Euler relation  $\langle \partial_p H, p \rangle = mE$  gives a contradiction. Now we need  $\lambda = \lambda(x, \psi) = 1$ ; since  $\partial_x \lambda \neq 0$ , the implicit functions theorem shows that  $x = x(\psi)$ . Then by second line (4.20) we have  $\frac{\partial t}{\partial \psi} = 0$ , and  $-mE \frac{\partial \lambda}{\partial \psi} = a = \langle \dot{P}, X_\psi \rangle = 0$  at  $t = 0$  where  $X_\psi = 0$ . Next, differentiating  $\lambda = \lambda(x, \psi) = 1$  gives

$$\frac{\partial \lambda}{\partial \psi} + \frac{\partial \lambda}{\partial x} \frac{\partial x}{\partial \psi} = 0$$

so  $\frac{\partial x}{\partial \psi} = 0$  at  $t = 0$  and by Taylor expansion around  $(0, \psi_0)$   $X(t, \psi) = \partial_p H(0, p)t + \mathcal{O}(t^2)$  ( $\mathcal{O}(t^2)$  depends on  $\psi$ ), while if  $P(0, \psi_0) = p$ , then  $P(t, \psi) = p - \partial_x H(0, p)t + P_\psi(0, \psi_0)(\psi - \psi_0) + \mathcal{O}(|t, \psi - \psi_0|^2)$ , with  $P_\psi(0, \psi_0) \neq 0$ . It follows (as expected when  $n = 2$ ) that  $\Lambda_+$  has a simple fold, and the rank of  $\pi|_{\Lambda_+}$  is 1. ♣

#### 4.3 Reduced parametrizations of $\Lambda_+$ in case of the conformal metric, $n = 2$ .

We have shown so far that (4.8) is a non-degenerate phase function parametrizing  $\Lambda_+$  for small  $t$ . We want to allow for larger values of  $t$  (the far field). Actually Proposition 4.1 and its proof show that  $\Phi(x, t, \psi, \lambda)$  is a non-degenerate phase function parametrizing  $\Lambda_+$  so long as  $\det(P, P_\psi) > 0$ . We have a positive result in the special case of a conformal “metric”, namely :

**Proposition 4.4:** *Let  $H(x, p)$  be as in (1.15),  $n = 2$ . Then representation (4.8) defines globally a non-degenerate phase function parametrizing  $\Lambda_+$ , and the (inverse) density on  $\Lambda_+$  is  $mE \det(P, P_\psi) > 0$ .*

*Proof:* By general Hamilton-Jacobi theory we recalled in Sect.4.2, there exists a well-defined density on the regular part of  $\Lambda_+$ , which must coincide with (4.10). So we are left to show that (4.10) holds near focal points: but this follows from Lemmas A.2-3 [we have no direct proof that (4.10) is valid everywhere on  $\Lambda_+$ ]. ♣

Next, in order to simplify (1.5), we need to evaluate  $\Phi$  at the critical point  $t = t_1(x, \psi)$ . We have to do this in various canonical charts, introducing additional Jacobians and Maslov indices.

Considering  $t$  as an extra “ $\theta$ -variable”, we solve for  $t$  in  $\tau = \partial_t \Phi = 0$ , so (2.5) simplifies to  $H(x, \partial_x \Phi) = E$  (Hamilton-Jacobi on  $\Lambda_+ = \Lambda_+(0)$ ). We need  $\lambda = 1$  to satisfy the boundary condition in (4.8), so by (4.11)-(4.12)

$$(4.29) \quad \partial_t \Phi(x, t, \psi, \lambda = 1) = \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle = 0$$

This holds on  $\Lambda_+$ , i.e. for  $x = X(t, \psi)$ . From now on, we assume  $H$  to be of the form (1.15). We have

$$(4.30) \quad \partial_t^2 \Phi(X(t, \psi), t, \psi, \lambda = 1) = -\frac{m|P(t, \psi)|^{2m-2}}{\rho(X(t, \psi))^3} \langle \nabla \rho(X(t, \psi)), P(t, \psi) \rangle$$

so we have to take into account the set of  $\psi$  such that  $\langle \nabla \rho(X(t, \psi)), P(t, \psi) \rangle = 0$ , i.e. of the special points. When  $\partial_t^2 \Phi(X(t, \psi), t, \psi, \lambda = 1) \neq 0$  we can apply asymptotic stationary phase with respect to  $t$ .

More generally let  $H$  be of the form (1.15) and consider  $f(t, \psi) = \langle \nabla \rho(X(t, \psi)), P(t, \psi) \rangle$ ,  $f = 0$  on  $\mathcal{S}$ . Using Hamilton equations, we find

$$(4.31) \quad \partial_t f(t, \psi) = \frac{m|P(t, \psi)|^{m-2}}{\rho(X(t, \psi))} [\langle \nabla^2 \rho(X(t, \psi)) \cdot P(t, \psi), P(t, \psi) \rangle + \frac{|\nabla \rho(X(t, \psi))|^2}{m\rho(X(t, \psi))} |P(t, \psi)|^2]$$

We shall partition  $\Lambda_+$  according to the following values: (1)  $z \in \mathcal{F} \cup \mathcal{F}^c$  is a focal (or non-focal) point; (2)  $z \in \mathcal{S} \cup \mathcal{S}^c$  is a special (or ordinary) point. Let  $I_\psi = \{t : (t, \psi) \notin \mathcal{S}\}$ . To simplify statements, we present a condition such that there is at most one special point on each bicharacteristic : Let  $z(s)$  be a special point for some  $s \geq 0$ , then whenever  $\partial_t \Phi(t, x, \theta) = 0$  at some  $t > s$  (this occurs when the bicharacteristic  $t \mapsto z(t)$  projects again on  $x$ ),  $z(t)$  is no longer special. To ensure this scenario we need some convexity on  $\rho$ , or strong “defocussing” condition.

$$(4.32) \quad G(\rho)(x, p) = \langle \nabla^2 \rho(x) \cdot p, p \rangle + \frac{|\nabla \rho(x)|^2}{m\rho(x)} |p|^2 > 0$$

In particular the only critical point of  $\rho$  is a non degenerate minimum. This excludes also natural potentials having a limit as  $|x| \rightarrow \infty$ , since the second term in (4.32) cannot rescue the lack of convexity at infinity, as shows the example  $\rho(x) = \rho_0 + \langle x \rangle^{-\varepsilon}$ . By assumption (4.32),  $\partial_t f(t, \psi) > 0$ , and  $t \mapsto \partial_t^2 \Phi(X(t, \psi), t, \psi, \lambda = 1)$  is strictly decaying on  $\Lambda_+$ .

• *Ordinary critical points.* They correspond to non degenerate critical points of  $t \mapsto \Phi(x, t, \psi, \lambda = 1)$ .

**Lemma 4.6:** *Assume (4.32) (no condition on  $\nabla \rho$  is required here). Then  $I_\psi$  is an interval, and*

$$(4.33) \quad \forall t \in I_\psi, x = X(t, \psi) \iff t = t_1(x, \psi) \quad \text{on } C_\Phi$$

where  $t_1$  is a smooth function. Moreover  $\pi_x : \Lambda_+ \rightarrow M$  at every ordinary critical point has same rank as the symmetric matrix

$$(4.34) \quad \begin{pmatrix} a^2 + \frac{\partial^2 \Phi}{\partial t^2} c & -a + b + \frac{ab}{mE} \\ * & 1 \end{pmatrix} = \begin{pmatrix} a^2 + \frac{\partial^2 \Phi}{\partial t^2} c & \frac{a^2}{mE} \\ * & 1 \end{pmatrix}$$

i.e.  $\pi_x$  has rank 1 or 2.

*Proof:* Note that when  $t = 0$ ,  $f(0, \psi) = \langle \nabla \rho(0), P(\psi) \rangle$ . So when  $f(0, \psi) > 0$ , 0 is non-degenerate critical point of  $t \mapsto \Phi(x, t, \psi, \lambda)$ , and the implicit function theorem shows that (4.33) holds. Since  $t \mapsto f(t, \psi)$  is increasing, this holds for all  $t$  in the maximal interval of definition of the integral curve starting at  $(0, P(\psi))$ . When  $f(0, \psi) < 0$  instead, (4.33) holds on an interval ending at some  $s$  such that  $f(s, \psi) = 0$ .

Let us compute the rank of  $\pi : \Lambda_+ \rightarrow M$  at an ordinary point. Let  $U = \{(t, \psi) : t \in I_\psi, (0, P(\psi)) \text{ is an ordinary point}\}$ , and show that  $U$  is a canonical chart of non zero rank. This is precisely the same computation we gave in (4.18). ♣

• *Special critical points.* Near the end point  $s$  of  $I_\psi$  we could attempt to solve  $\partial_t \Phi(x, t, \psi, \lambda) = \partial_\psi \Phi(x, t, \psi, \lambda) = 0$  but already for  $t = 0$ , the determinant of the Hessian of  $\Phi$  with respect to  $(t, \psi)$  vanishes on  $\Lambda_+$ . But we can solve (locally)

$$\partial_t \Phi(x, t, \psi, \lambda) = \partial_\lambda \Phi(x, t, \psi, \lambda) = 0$$

which gives  $t = t(x, \psi)$  and  $\lambda = \lambda(x, \psi)$ . Namely, the Hessian of  $\Phi$  with respect to  $(t, \lambda)$  at  $(s, 1)$  has determinant  $-\left(\frac{\partial^2 \Phi}{\partial t \partial \lambda}\right)^2$ , and by (4.11)-(4.12)  $\frac{\partial^2 \Phi}{\partial t \partial \lambda} = -mE < 0$  on  $\Lambda_+$ . So if  $(X(s, \psi), P(s, \psi))$  is a special point then  $(s, 1)$  is a non degenerate point of  $(t, \lambda) \mapsto \Phi(x, t, \psi, \lambda)$ . Integrating Hamilton equations also for  $t < 0$  gives the Lagrangian manifold  $\Lambda_- \cup \Lambda_+$ . So there is no loss of generality in assuming the special point is at  $s = 0$ .

**Lemma 4.7:** *Assume  $z = z(s) \in \Lambda_+$  is a special point,  $z(s) = (X(s, \psi), P(s, \psi))$ , with  $\nabla \rho(z(s)) \neq 0$ . Let  $F(x, t, \psi, \lambda) = \left(\frac{\partial_t \Phi}{\partial_\lambda \Phi}\right)$ . Then the system  $F(x, t, \psi, \lambda) = 0$  is equivalent to  $t = t(x, \psi)$ ,  $\lambda = \lambda(x, \psi)$ .*

(i) *Assume  $z(s)$  is not a focal point (the rank of  $\pi : \Lambda_+ \rightarrow M$  is equal to 2), i.e.  $X_\psi \neq 0$ . Then  $c = \langle P_\psi, X_\psi \rangle \neq 0$ , and  $a = \langle \dot{P}, X_\psi \rangle \neq 0$ . Restricting to  $\lambda = 1$  gives  $t = t(x)$ ,  $\psi = \psi(x)$ . We have  $\frac{\partial t}{\partial x} \neq 0$ ,  $\frac{\partial \psi}{\partial x} \neq 0$ .*

(ii) *Assume  $z(s)$  is a focal point. The statement is as in Prop.4.3(ii).*

*Proof:* We are left to prove (i). Implicit function theorem gives  $(t, \lambda) = (t(x, \psi), \lambda(x, \psi))$ , so we use (4.20).

(i) We know from (2.6) that  $\langle P, X_\psi \rangle = 0$ , if  $\langle P_\psi, X_\psi \rangle = 0$  then  $P$  and  $P_\psi$  would be parallel, which contradicts Proposition 4.4. In the same way, by (2.2),  $\dot{P} \neq 0$ , and since  $\langle \nabla \rho, P \rangle = 0$ ,  $\langle \dot{P}, X_\psi \rangle = 0$  would contradict  $X_\psi \neq 0$ . So by second line (4.20),  $-mE \partial_\psi \lambda = \langle \dot{P}, X_\psi \rangle \neq 0$ , or  $\partial_\psi \lambda \neq 0$ . Now we need  $\lambda = \lambda(x, \psi) = 1$ ; since  $\partial_\psi \lambda \neq 0$ , the implicit functions theorem shows that  $\psi = \psi(x)$ . Then we have

$$(4.42) \quad x = X(t, \psi) \iff t = t(x), \psi = \psi(x) \text{ on } \partial_t \Phi(x, t, \psi, \lambda = 1) = \partial_\lambda \Phi(x, t, \psi, \lambda = 1) = 0$$

Differentiating  $\lambda = \lambda(x, \psi)$  gives

$$\frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial \psi} \frac{\partial \psi}{\partial x} = 0$$

and together with the first equation (4.20)

$$\langle P, \dot{X} \rangle^t \left( \frac{\partial \psi}{\partial x} \right) = mE\dot{P} - \langle \dot{P}, \dot{X} \rangle P$$

Let us show that  $\frac{\partial \psi}{\partial x} \neq 0$ . Otherwise, we would have  $\langle P, \dot{X} \rangle \dot{P} = \langle \dot{P}, \dot{X} \rangle P$ , since we know already that  $\langle \nabla \rho(X(s, \psi)), P(s, \psi) \rangle = 0$ , and  $\dot{P}$  is parallel to  $\nabla \rho(X(s, \psi))$ , this is a contradiction.

Moreover  $\frac{\partial t}{\partial \psi} = 0$ ,  $\frac{\partial t}{\partial x} = \frac{1}{mH} {}^t P \neq 0$ , and

$$\frac{\partial \lambda}{\partial \psi} = - \left| \frac{\partial \psi}{\partial x} \right|^{-2} \left\langle \frac{\partial \lambda}{\partial x}, \frac{\partial \psi}{\partial x} \right\rangle$$

To compute the rank of  $\pi$  at a special point, we notice that since we have eliminated  $(t, \lambda)$ , the “ $\theta$ -parameter” is  $\psi$ . We set

$$\Psi(x, \psi) = \Phi(x, t(x, \psi), \psi, \lambda(t(x, \psi), \psi))$$

Differentiating  $F = 0$  along  $\Lambda_+$  with respect to  $\psi$ , we find by a straightforward computation

$$(4.43) \quad -\partial_\psi^2 \Psi = \langle P_\psi, X_\psi \rangle \neq 0$$

We apply (3.9) with  $N = 1, n = 2$ , so the rank of  $\pi|_{\Lambda_+}$  is 2. ♣

*Remark 4.7:* Note that if  $X_\psi = 0$ , then  $P_\psi \neq 0$  (otherwise this would violate property (3) of Proposition A.1), and  $X_\psi = 0$  iff  $\langle \dot{P}, X_\psi \rangle = 0$  at a special point. From Lemma 4.7, the set of focal points which are also special points is  $\mathcal{S} \cap \mathcal{F} = \{(s, \psi) : \langle P_\psi, X_\psi \rangle = 0\}$ . In Example 2.4, we find  $\langle P_\psi, X_\psi \rangle = \frac{1}{2} \sinh(2f(t)) > 0$  for all  $t > 0$  and vanishes at  $t = 0$ .

*Remark 4.8:* The case of a non degenerate the critical point  $\nabla \rho(X(s, \psi)) = 0$  is again standard.

Now to find the canonical charts for the phase functions, we use a connectedness argument. Assume (4.32), and let  $s$  be the supremum of  $I_\psi$ , we have  $f(s, \psi) = 0$ . Since  $G(\rho)(s, \psi) > 0$ , we have  $f(t, \psi) > 0$  for all  $t > s$ , so all points  $z(t) = (X(t, \psi), P(t, \psi))$  for  $t > s$  are ordinary points.

So far we proved:

**Proposition 4.8:** *Let  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ . Then there exists globally a smooth solution  $\Psi$  of HJ equation  $H(x, \partial_x \Phi) = E$ . Let  $C_\Phi = \{(t, x, \psi), \psi \in \mathbf{S}^1 : \partial_t \Phi = \partial_\lambda \Phi = \partial_\psi \Phi = 0\}$ . Then the embedding*

$$\iota_\Phi : C_\Phi \rightarrow T^*\mathbf{R}^2, (t, x, \psi, \lambda) \mapsto (x, \partial_x \Phi(x, t, \psi, \lambda = 1))$$

*such that  $\iota_\Phi(C_\Phi) \subset \Lambda_+$  consists in charts of rank 1 or 2 [the rank is never 0 since  $p \neq 0$  in the energy shell  $H = E$ ]. Under hypothesis (4.32) these charts can intersect the set of special points  $\mathcal{S}$  only along a line.*

Note that on  $C_\Phi$  (this holds for general  $H$  positively homogeneous of degree  $m$  on  $T^*M \setminus 0$ )

$$-\text{Hess}_{(t, \psi, \lambda)} \Phi = \begin{pmatrix} \langle \dot{P}, \dot{X} \rangle & \langle \dot{P}, X_\psi \rangle & \langle P, \dot{X} \rangle \\ \langle \dot{P}, X_\psi \rangle & \langle P_\psi, X_\psi \rangle & 0 \\ \langle P, \dot{X} \rangle & 0 & 0 \end{pmatrix}$$

and

$$(4.44) \quad \det \text{Hess}_{(t,\psi,\lambda)} \Phi = (mE)^2 \langle P_\psi, X_\psi \rangle$$

*Remark 4.9:* The canonical charts in  $\Lambda_+$  where  $\Phi = \Phi(x)$ , i.e. of WKB type are of course of maximal rank 2, in particular there is a WKB solution near a special point  $z$  such that  $\langle X_\psi, P_\psi \rangle \neq 0$ .

#### 4.4 Construction of Maslov canonical operator for bi-Lagrangian distributions.

Assume already we have constructed the amplitude  $a(x, t, \psi, \lambda)$ : thus we got a solution  $v(t, x; h)$  to the Cauchy problem that can be expressed as an oscillatory integral  $\int e^{i\Phi(x,t,\psi,\lambda)/h} \tilde{a}(x, t, \psi, \lambda) d\psi d\lambda$  with a suitable  $\tilde{a}$ . If we content to the first order approximation of the outgoing solution to

$$H(x, hD_x; h)u(x; h) = f_h(x)$$

we need to compute  $u(x; h) = \int_0^\infty v(t, x; h) dt$  i.e.

$$(4.45) \quad u(x; h) = \int_0^\infty \int e^{i\Phi(x,t,\psi,\lambda)/h} a(x, t, \psi, \lambda) d\psi d\lambda dt$$

(lateron we will choose  $a$  independent of  $x$ ). We shall rely on the symbolic calculus set up in Sect.3, but proceed with a specific choice of symbols.

Given  $f_h$ , we construct  $(\sigma, \sigma^+)$ , and a bi-Lagrangian distribution  $u$  solution of  $H(x, hD_x; h)u = f_h \text{ mod } \mathcal{O}(h)$ . It is obtained by choosing in (4.45) a suitable symbol  $a$  “interpolating” between  $\sigma$  and  $\sigma^+$ . Since our previous constructions are global, we shall obtain a generalization of Maslov canonical operator associated with the pair  $(\Lambda, \Lambda_+)$ .

Let  $\sigma : p \mapsto \sigma(p)$  be a function on  $\Lambda \setminus \partial\Lambda_+(\tau)$  and  $\sigma^+ : (t, \psi) \mapsto \sigma^+(t, \psi)$  a function on  $\Lambda_+$ , that we consider as the restriction of  $\tilde{\sigma}^+ : (t, \psi, \lambda) \mapsto \tilde{\sigma}^+(t, \psi, \lambda)$  as a function on  $\tilde{\Lambda}_+$  in the extended phase-space. We introduce densities as in (4.10) but ignore Maslov contributions. Let  $\delta(\psi, \lambda) d\psi d\lambda$  and

$$\delta^+(t, \psi) dt d\psi = (F[\Phi, d\mu_+]|_{C_\Phi})^{-1} dt d\psi = (mH \det(P, P_\psi))^{-1} dt d\psi$$

be densities on  $\Lambda$  and  $\Lambda_+$  respectively. We shall actually deal with the square root of these densities (half-densities). We can extend  $\delta^+(t, \psi) dt d\psi$  as a density  $\delta^+(t, \psi, \tau) dt d\psi$  or  $\delta^+(t, \psi, \lambda) dt d\psi$  on  $\Lambda_+(\tau)$ , where we recall  $\lambda$  and  $\tau$  are related by (4.9) in case of Hamiltonian (1.15). Now we take  $\tilde{\sigma}^+$  the solution of transport equation (3.17) with initial value  $\tilde{\sigma}^+(0, \psi, \lambda) = 1$ , i.e.

$$\left(\frac{1}{i} \frac{d}{dt} + H_1\right) a_0(t, \psi, \lambda) \sqrt{\delta^+(t, \psi, \lambda)} = 0$$

Let  $a(t, \psi, \lambda)$  be a continuous function, smooth outside  $\partial\Lambda_+$ , such that

$$(4.50) \quad a(0, \psi, \lambda) = (\delta(\psi, \lambda))^{1/2} (\lambda - 1) \sigma(P(\psi, \lambda)), \quad a(t, \psi, \lambda = 1) = (\delta^+(t, \psi))^{1/2} \sigma^+(t, \psi)$$

The existence of  $a$  is ensured by the *compatibility condition*

$$(4.51) \quad \lim_{\lambda \rightarrow 1} (\delta(\psi, \lambda))^{1/2} (\lambda - 1) \sigma(P(\psi, \lambda)) = (\delta^+(0, \psi))^{1/2} \sigma^+(0, \psi)$$

(so  $\sigma(P(\psi, \lambda))$  may blow up as  $\lambda \rightarrow 1$ ), and we can choose

$$(4.52) \quad a(t, \psi, \lambda) = (\delta(\psi, \lambda))^{1/2}(\lambda - 1)\sigma(P(\psi, \lambda)) + (\delta^+(t, \psi))^{1/2}\sigma^+(t, \psi) - (\delta^+(0, \psi))^{1/2}\sigma^+(0, \psi)$$

which verifies

$$\lim_{(t, \lambda) \rightarrow (0, 1)} a(t, \psi, \lambda) = (\delta^+(0, \psi))^{1/2}\sigma^+(0, \psi)$$

In the special case (1.15) and  $n = 2$ , (2.3) shows that in polar coordinates we can choose the density on  $\Lambda$  of the form  $dp = \lambda d\lambda d\psi$  (Lebesgue measure). With these choices, (4.51) writes

$$(4.53) \quad \lim_{\lambda \rightarrow 1} (\lambda - 1)\sigma(P(\psi, \lambda)) = \sigma^+(0, \psi)$$

**Definition 4.11:** *With  $a$  as in (4.52), and ignoring for the moment Maslov contributions, we call Maslov canonical operator for the pair  $(\sigma, \sigma^+)$*

$$(4.54) \quad \mathcal{K}_+(\sigma, \sigma^+)(x; h) = \int_0^\infty dt \int e^{i\Phi(x, t, \psi, \lambda)/h} a(t, \psi, \lambda) d\psi d\lambda$$

We shall simplify expressions (4.54) using stationary phase, but this will be possible only when  $t \neq 0$ , i.e.  $x \neq 0$ . Consider again  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ , where results are most complete.

First we prove consistency of Definition 4.11 relative to standard Maslov canonical operators.

**Proposition 4.12:** *Let  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ .*

(i) *If  $\sigma = 0$  and  $\text{supp } \sigma^+ \cap \partial\Lambda_+ = \emptyset$ , then (up to a constant factor)  $\mathcal{K}_+(0, \sigma^+) = \mathcal{K}_{\Lambda_+}\sigma^+ + \mathcal{O}(h)$ , where  $\mathcal{K}_{\Lambda_+}$  is the standard canonical operator on  $\Lambda_+ \setminus \partial\Lambda_+$ .*

(ii) *If  $\sigma^+ = 0$  and  $\text{singsupp } \sigma \cap \partial\Lambda_+ = \emptyset$  (singular support), then (up to a constant factor)  $\mathcal{K}_+(\sigma, 0) = \mathcal{K}_\Lambda\sigma + \mathcal{O}(h)$ , where  $\mathcal{K}_\Lambda$  is the standard canonical operator on  $\Lambda$ .*

*Moreover the expressions for  $\mathcal{K}_+(0, \sigma^+)$  and  $\mathcal{K}_+(\sigma, 0)$  can be simplified by standard stationary phase methods.*

*Proof:*

(i) *Wave part  $\mathcal{K}_{\Lambda_+}(A)$ .* By (4.52) we have  $a(t, \psi, \lambda) = \sigma^+(t, \psi)$ . To find the pre-images of  $(t, \psi) \mapsto x = X(t, \psi)$ , we need to discuss according to the case  $\pi_x : \Lambda \rightarrow M$  is of maximum rank or not. For small  $x$ , local geodesic completeness (at least when  $m = 2$ ) shows that  $(t, \psi) \mapsto x$  is 1-1. Otherwise, under Assumption (2.18) (i.e. if  $x$  and  $x_0 = 0$  are not conjugated along any trajectory with initial momentum  $\eta = P(\psi) \in \text{supp } A \subset \Lambda$ , in particular if  $x$  is not a focal point), then  $x = X(t, \psi)$  for a finite number (locally constant near such an  $x$ ) of  $(t, \psi)$ . We discuss according to  $(t, \eta)$  is ordinary or not.

•  $X(t, \psi)$  is a special, non focal point (i.e.  $\langle \nabla\rho(X(t, \psi), P(t, \psi)) \rangle = 0$ , but  $X_\psi \neq 0$ ). There we apply Lemma 4.7, which gives the critical point  $(t, \psi, \lambda) = (t(x), \psi(x), 1)$ , and by (4.44)

$$(\delta^+(t, \psi))^{1/2} = \det \text{Hess}_{(t, \psi, \lambda)} \Phi(x, t(x), \psi(x), 1) = (mH)^2 \langle X_\psi, P_\psi \rangle \neq 0$$

By stationary phase formula

$$(4.55) \quad \mathcal{K}_+(\sigma, \sigma^+)(x; h) = \frac{1}{mE} e^{i\Phi(x, t(x), \psi(x), 1)/h} \left( \det \left( \frac{\langle X_\psi, P_\psi \rangle}{2i\pi h} \right) \right)^{-1/2} (\delta^+\sigma^+)_{1/2}(t(x), \psi(x)) + \mathcal{O}(h)$$

i.e.  $\mathcal{K}_+(\sigma, \sigma^+)(x; h)$  is of WKB type.

- $X(t, \psi)$  is ordinary, non focal, we apply Lemma 4.6 instead. First we perform stationary phase in  $t$ , at  $t = t(x, \psi)$ , which is a non-degenerate point, this introduces in the density the additional factor

$$\delta_1^+(t(x, \psi), \psi) = \left(\frac{\partial_t^2 \Phi}{2i\pi h}\right)^{-1} = \left(\frac{\langle \dot{P}, \dot{X} \rangle}{2i\pi h}\right)^{-1}$$

Next we perform stationary phase in  $\lambda$  at  $\lambda = 1$  which is a non-degenerate point, multiplying the density by

$$\delta_2^+(t(x, \psi), \psi) = \left(\frac{\partial_\lambda^2 \Phi}{2i\pi h}\right)^{-1} = \left(\frac{1}{mE} \frac{\langle \dot{P}, \dot{X} \rangle}{2i\pi h}\right)^{-1}$$

As a result

$$(4.58) \quad \mathcal{K}_+(\sigma, \sigma^+)(x; h) = \frac{2\pi h}{mE} \int e^{i\Phi(x, t(x, \psi), \psi, 1)/h} (\delta_1^+(t(x, \psi), \psi) \delta_2^+(t(x, \psi), \psi))^{1/2} \sigma^+(t(x, \psi), \psi) d\psi + \mathcal{O}(h)$$

which can be further simplified by looking at the stationary point of  $\psi \mapsto \Phi(x, t(x, \psi), \psi, 1)$ . We start from (4.8) where we substitute  $t(x, \psi)$  for  $t$ , and denote  $\Psi = \Phi|_{t=t(x, \psi), \lambda=1}$  this gives, using  $\langle P, X_\psi \rangle = \langle \dot{X}, X_\psi \rangle = 0$

$$\frac{\partial \Psi}{\partial \psi} = (mE - \langle \dot{P}, x - X \rangle) \frac{\partial t}{\partial \psi} + \langle P_\psi, x - X \rangle$$

We expand around  $C_\Phi$  (or which is the same, around  $\Lambda_+$ ), and use (4.17). Thus if  $\langle \dot{P}, X_\psi \rangle \neq 0$  at the ordinary, non focal point, (such identities hold on  $C_\Phi$  but they are valid at first order near this set) the phase is non stationary, which gives  $\mathcal{K}_+(\sigma, \sigma^+)(x; h) = \mathcal{O}(h^\infty)$ . Otherwise, we can find  $\psi = \psi(x)$  such that  $\frac{\partial \Psi}{\partial \psi} = 0$  at this point. Since we assumed it is non focal (i.e. the matrix (4.34) is regular),  $\psi = \psi(x)$  is a non degenerate critical point and we can expand the integral by stationary phase, which gives again a WKB form.

Actually we have to take into account all the pre-images of  $X(t, \psi)$ . In case of a non-focal point, Assumption (2.18) ensures that  $\mathcal{K}_+(\sigma, \sigma^+)(x; h)$  can be written as a finite sum (over all pre-images  $\eta_\alpha = P(\psi_\alpha), t = t_\alpha$ ) of WKB expansions of the form  $\mathcal{K}_+(\sigma_\alpha, \sigma_\alpha^+)(x; h)$  above.

- $X(t, \psi)$  is a focal point. We argue as before, using Lemma 4.6 or 4.7 according to the focal point is ordinary or not. To add all contributions of the oscillating integrals we use the discussion in Sect.2.4.

(ii) *Boundary part*  $\mathcal{K}_\Lambda(A)$ . By (4.52)  $a(t, \psi, \lambda) = (\delta(\psi, \lambda))^{1/2} (\lambda - 1) \sigma(P(\psi, \lambda))$ . We have

$$\frac{h}{i} \frac{d}{dt} e^{i\Phi(x, t, \psi, \lambda)/h} = e^{i\Phi(x, t, \psi, \lambda)/h} [mE(1 - \lambda) + \lambda \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle]$$

so integrating over  $t$  (after inserting a cut-off in  $t$ ), we get

$$\begin{aligned} \mathcal{K}_+(\sigma_0, 0) &= \frac{h}{imE} \int e^{i\Phi(x, 0, \psi, \lambda)/h} (\delta(\psi, \lambda))^{1/2} \sigma(P(\psi, \lambda)) d\psi d\lambda + \\ &\frac{1}{mE} \int_0^t dt \int e^{i\Phi(x, 0, \psi, \lambda)/h} \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle \delta(\psi, \lambda) \sigma(P(\psi, \lambda)) d\psi \lambda d\lambda \end{aligned}$$

To the second integral we apply stationary phase, so using  $x = X(t, \psi)$  on  $C_\psi$ , we find this term is  $\mathcal{O}(h^2)$ . We are left with the integral over  $\Lambda$ , where we change variables  $(\psi, \lambda)$  to  $p$ . Using the initial condition  $\Phi(x, 0, \psi, \lambda) = \langle x, P(\psi, \lambda) \rangle$ , we find this term is nothing but  $\int e^{ipx/h} \sigma(p) dp = \mathcal{K}_\Lambda \sigma(x; h)$ .

So we have found reduced representations of  $\mathcal{K}(\sigma, \sigma^+)(x; h)$ . ♣

*Remark 4.7:* By linearity, if  $(\text{singsupp } \sigma \cup \text{supp } \sigma^+) \cap \partial\Lambda_+ = \emptyset$  we obtain also an asymptotics for  $\mathcal{K}_+(\sigma, \sigma^+)$ . We needn't assume  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$  if we content with asymptotics for small  $x$ .

Next we apply symbolic calculus on  $I(M, \Lambda, \Lambda^+)$  adapted from [MelUhl, Prop.5.4& Prop.6.6] in Sect.3, and take advantage of using symbols of the form (4.52). This yields the main result of this section:

**Proposition 4.13** (commutation formula): We have

$$H(x, hD_x; h)\mathcal{K}_+(\sigma, \sigma^+) = \mathcal{K}_+(H(x, hD_x; h)\sigma; \frac{d\sigma^+}{dt} + iH_1\sigma^+) + \mathcal{O}(h^2)$$

and the solution of  $H(x, hD_x; h)u_h = f_h = \int e^{ixp/h} A(p) dp$  is given at first order in  $h$  by

$$\mathcal{K}_+\left(\frac{A(p)}{\lambda - 1}; \sigma^+\right)(x; h)$$

where  $\sigma^+$  solves  $\frac{d\sigma^+}{dt} + iH_1\sigma^+ = 0$ .

*Proof:* We apply the  $h$ -PDO  $H(x, hD_x)$  under the integral sign in (4.54). By eikonal equation (2.15)

$$\begin{aligned} (H(x, hD_x) - E)e^{i\Phi(x,t,\psi,\lambda)/h} &= e^{i\Phi(x,t,\psi,\lambda)/h} (H(x, \partial_x \Phi) - E) = \\ &= -e^{i\Phi(x,t,\psi,\lambda)/h} \frac{\partial \Phi}{\partial t} = -hD_t e^{i\Phi(x,t,\psi,\lambda)/h} \end{aligned}$$

all equalities being understood mod  $\mathcal{O}(h)$ , whatever the quantization rule is. Let  $A(p)$  be the symbol of  $f_h$  on  $\Lambda$ , we put (we can make  $\sigma$  depend on  $\lambda$  also)

$$\sigma(P(\psi, \lambda), \lambda) = \frac{A(P(\psi, \lambda))}{\lambda - 1}$$

Integrating by parts in  $t$ , the first term in  $H(x, hD_x)\mathcal{K}_+(\sigma, \sigma^+)(x; h)$  is (change of variables)

$$-\frac{h}{i} \int e^{i\Phi(x,0,\psi,\lambda)/h} A(P(\psi, \lambda)) d\psi d\lambda = -\frac{h}{i} \int e^{ixp/h} A(p) dp$$

**5.  $f_h$  is supported microlocally on the “cylinder”  $\Lambda = \{X = \varphi\omega(\psi), P = \omega(\psi)\}$**

Here we assume

$$(5.1) \quad \Lambda = \{x = X(\varphi, \psi) = \varphi\omega(\psi), p = P(\varphi, \psi) = \omega(\psi), \varphi \in \mathbf{R}, \omega \in \mathbf{S}^{n-1}\}$$

When  $n = 2$  this is the wave-front set of Bessel function  $f_h(x; h) = J_0(\frac{|x|}{h})$ ; such functions arise in the wave beam theory (see [Ki], [DoMaNa] and references therein), so we call  $f_h$  a “Bessel beam”. As before,  $\psi$  are the usual angles parametrizing  $\omega \in \mathbf{S}^{n-1}$ . Let again  $n = 2$ . The intersection of  $\Lambda$  with the energy surface  $H = E$ , is given by the implicit equation  $H(\varphi\omega(\psi), \omega(\psi)) = E$ , which usually defines a smooth 1-D isotropic submanifold. To avoid boundaries, we assume  $\partial\Lambda_+$  to be a closed curve. When  $H$  is of the form (1.15), relation  $\langle \dot{P}, X_\psi \rangle = \langle \dot{X}, P_\psi \rangle = 0$  shows that  $\varphi \langle \nabla \rho(\varphi\omega(\psi)), \omega^\perp(\psi) \rangle = 0$  at  $\partial\Lambda_+$ .

*5.1 Non degeneracy conditions*

Let us check first the Lagrangian intersection. We say that  $z \in \partial\Lambda_+$  is a *glancing point* when  $v_H(z) \in T_z\Lambda$ ; so  $(\Lambda, \Lambda_+)$  is an intersecting pair whenever the set of glancing points is empty.

We complete  $\omega(\psi)$  in  $\mathbf{S}^{n-1}$  into a (direct) orthonormal basis  $\omega^\perp(\psi) = (\omega_1(\psi), \dots, \omega_{n-1}(\psi))$  of  $\mathbf{R}^n$ .

**Proposition 5.1:** *Let  $H$  be homogeneous of degree  $m$ , and  $\Lambda$  be the cylinder. With the notation above,  $z \in \partial\Lambda_+$  is a glancing point iff*

$$(5.2) \quad \langle \partial_p H + \varphi \partial_x H, \omega^\perp(\psi) \rangle = 0, \quad \langle -\partial_x H, \omega(\psi) \rangle = 0, \quad H(z) = E$$

More generally assume  $z(t) \in \partial\Lambda_+(t)$  is a glancing point and  $P_\psi \neq 0$ . Then  $H(z(t)) = E$ , and

$$(5.2)_t \quad \begin{aligned} &\text{either } \langle P, P_\psi \rangle \neq 0 \text{ and } \langle -\partial_x H, P_\psi^\perp \rangle = 0 \text{ or} \\ &\langle P, P_\psi \rangle = 0 \text{ and } \langle -\partial_x H, P \rangle = mH \langle P, P_\varphi \rangle \end{aligned}$$

*Proof:* We denote by  $\omega^\perp(\psi)\delta\psi = (\omega_1(\psi)\delta\psi_1, \dots, \omega_{n-1}(\psi)\delta\psi_{n-1})$  a smooth section of  $T\mathbf{S}^{n-1}$ ,  $\delta\psi_j \in \mathbf{R}$ . The tangent space  $T_z\Lambda$  has the parametric equations

$$\delta X = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi, \quad \delta P = \omega^\perp(\psi)\delta\psi, \quad (\delta\varphi, \delta\psi) \in \mathbf{R}^n$$

so  $v_H \in T_z\Lambda$  iff there exist  $(\delta\psi, \delta\varphi)$  such that

$$\partial_p H = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi, \quad -\partial_x H = \omega^\perp(\psi)\delta\psi$$

Taking scalar products with  $\omega(\psi), \omega^\perp(\psi)$ , and using Euler identity, we get  $\delta\varphi = \langle \partial_p H, P(\psi) \rangle = mH$ ,  $\delta\psi = \langle -\partial_x H, \omega^\perp(\psi) \rangle$ . Since  $(\omega(\psi), \omega^\perp(\psi))$  form a basis of  $\mathbf{R}^n$ , relations (5.2) are necessary and sufficient for  $v_H \in T_z\Lambda_+ \cap T_z\Lambda = T_z\partial\Lambda_+$ .

Let now  $t > 0$ ,  $v_H \in T_{z(t)}\Lambda(t)$ , from (2.6) we still have  $\delta\varphi = mH$ , so

$$\partial_p H = mHX_\varphi + X_\psi\delta\psi, \quad -\partial_x H = mHP_\varphi + P_\psi\delta\psi$$

We know  $\det(P, P_\psi) \neq 0$ , so  $P, P_\psi$  form a basis of  $\mathbf{R}^n$ , assume  $n = 2$  for simplicity. Taking scalar product of the second equation with  $P_\psi$  and  $P$  gives

$$\delta\psi = |P_\psi|^{-2}(\langle -\partial_x H, P_\psi \rangle - mH\langle P, P_\psi \rangle), \quad \langle -\partial_x H, P \rangle = mH\langle P, P_\psi \rangle + \langle P, P_\psi \rangle$$

Discussing according to the fact  $P$  is orthogonal to  $P_\psi$  or not readily gives (5.2)<sub>t</sub>. ♣

When  $H = p^2$ , all points are glancing.

Since the glancing property is invariant under the Hamiltonian flow, (5.2) and (5.2)<sub>t</sub> are actually equivalent. Recall  $\langle \dot{X}, P_\psi \rangle = \langle \dot{P}, X_\psi \rangle$  from Proposition A.1. For Hamiltonian (1.15), this implies  $\nabla\rho(\varphi\omega(\psi), \omega^\perp)$  at  $t = 0$ , which is actually a condition on  $\rho$ . So first condition (5.2) is always fulfilled. Second condition means that if  $z(0)$  is a special point, then  $\nabla\rho(z(0)) = 0$  and  $z(0)$  is a glancing point, and so is  $z(t)$  for all  $t$ .

*Example 5.1:* Consider  $H(x, p) = \langle \mu, p \rangle$ , where  $\mu \in \mathbf{R}^2$  (constant coefficients case), then  $X(t, \psi) = \lambda t + \varphi\omega(\psi)$ ,  $P(t, \psi) = \omega(\psi)$ . The system  $x = X(t, \varphi, \psi)$  has a unique solution  $(t_1(x, \psi), \varphi_1(x, \psi), \psi)$  provided  $\langle \mu, \omega^\perp \rangle \neq 0$ , i.e.  $v_H$  is transverse to  $\Lambda$ . So in the constant coefficient case, (5.2) will always occur at some point  $\psi$ .

*Example 5.2:* For  $a, b \in \mathbf{R}$  we set  $\langle x \rangle_{a,b} = (1 + ax_1^2 + bx_2^2)^{1/2}$ . Consider Hamiltonian on  $T^*\mathbf{R}^2$  of the form

$$H(x, p) = \langle \mu, p \rangle \langle x \rangle_{a,b}^\alpha$$

with constant  $\mu, \alpha$ , and initial manifold  $\Lambda = \{X = \varphi\omega(\psi), P = \omega(\psi)\}$ . A computation shows that, with  $x_1 = \varphi \cos \psi$ ,  $x_2 = \varphi \sin \psi$

$$\langle -\partial_x H, \omega(\psi) \rangle = \alpha \langle \mu, p \rangle \langle x \rangle_{a,b}^{\alpha-2} (a \cos^2 \psi + b \sin^2 \psi) \varphi$$

When  $ab > 0$ ,  $\langle -\partial_x H, \omega(\psi) \rangle = 0$  implies  $\langle \mu, p \rangle \varphi = 0$ , but if  $ab < 0$ , this condition specifies  $\tan^2 \psi = -\frac{a}{b}$ . Inserting this value into first equation (6.2), choosing  $\mu = (1, 0)$ , we find  $\varphi = -1/(2b)$ . So there is a unique glancing point. Such a non-transversality is called a *kiss* in [ElGr].

When  $a > 0, b > 0$ , Example 5.2 gives a Lagrangian intersection, and if moreover  $\alpha < 0$ , condition (4.32) holds. We discuss the properties of Hamiltonian flow issued from  $\Lambda$  using eikonal coordinates as in Sect.4.1.

Consider Hamiltonian  $\tau + H(x, p)$  on  $T^*\mathbf{R}^n \times \mathbf{R}_+$  and the Lagrangian manifold  $\tilde{\Lambda}_+ = \bigcup_\tau \Lambda_+(\tau)$  in the extended phase-space, i.e.

$$(5.4) \quad \begin{aligned} \tilde{\Lambda}_+ &= \{(x, p; t, \tau) : \tau + H(x, p) = E, z(t) = (X(t, \varphi, \psi, \tau), P(t, \varphi, \psi, \tau)), \\ & z(0) \in \partial\Lambda_+(\tau), t \geq 0\} \subset T^*(M \times \mathbf{R}_+) \end{aligned}$$

Assume that

$$\tilde{\iota} : \tilde{\Lambda}_+ \rightarrow T^*(M \times \mathbf{R}_+)$$

is a Lagrangian embedding (this holds true if we take  $t, \tau$  small enough).

Let  $\tilde{n} = n + 1$ ,  $k = 2$ ,  $\tilde{\psi} = \psi$ ,  $\tilde{\phi} = (t, \varphi)$ ,  $\tilde{X} = (X(t, \varphi, \psi, \tau))^t$ , and  $\tilde{P} = (P(t, \varphi, \psi, \tau))^\tau$  in the extended phase-space. We need a “left inverse” for

$$(\dot{\tilde{X}}, \tilde{X}_\varphi) = \begin{pmatrix} 1 & 0 \\ \dot{X} & X_\varphi \end{pmatrix}$$

of the form

$$\tilde{\Pi} = \begin{pmatrix} 1 & -mH \\ 0 & P \end{pmatrix}$$

This fulfills (4.1) because of (2.11). By [DoNaSh, Lemma 6], we know in particular that the equation

$$\tilde{\Pi}^* \begin{pmatrix} 0 \\ x - X(t, \varphi, \psi, \tau) \end{pmatrix} = 0 \iff \langle P(t, \varphi, \psi), x - X(t, \varphi, \psi, \tau) \rangle = 0$$

has a unique solution  $t = t_1(x, \psi, \tau)$ ,  $\varphi = \varphi_1(x, \psi, \tau)$ . But we need to restrict to the energy surface  $\{\tau = 0\}$ . From now on we omit to write  $\tau$  when it is set to 0. Already in Example 5.1, Eq.  $x = X(t, \psi) = \mu t + \varphi \omega(\psi)$  can be solved for  $t, \varphi$  iff  $\langle \mu, \omega^\perp(\psi) \rangle \neq 0$ , where  $\psi = \psi_0$  belongs to  $\{\tau = 0\}$ , i.e.  $\langle \mu, \omega(\psi_0) \rangle = E$ . By the discussion above, this means that  $v_H$  should be transverse to  $\Lambda$  at  $\partial\Lambda_+$ . Then

$$t = \frac{\langle x, \omega^\perp(\psi_0) \rangle}{\langle \mu, \omega^\perp(\psi_0) \rangle}, \quad \varphi = \frac{1}{\langle \mu, \omega^\perp(\psi_0) \rangle} \det \begin{pmatrix} \langle x, \omega(\psi_0) \rangle & \langle x, \omega^\perp(\psi_0) \rangle \\ \langle \mu, \omega(\psi_0) \rangle & \langle \mu, \omega^\perp(\psi_0) \rangle \end{pmatrix}$$

are uniquely determined. In particular  $\varphi_1(x, \psi = \psi_0) = 0$  if the determinant of Gram matrix vanishes, i.e. when  $x$  is parallel to  $\mu$ .

As in [DoNaSh, Lemma 6] we can prove the following

**Proposition 5.2:** *(Assume  $(\Lambda, \Lambda_+)$  is an intersecting pair at  $\partial\Lambda_+$  ?) and there is a open set  $U \subset \mathbf{R}^{n+1}$ ,  $(t, \varphi, \psi) \in U$ , such that*

$$(5.5) \quad \kappa : U \mapsto \mathbf{R}_x^n \times \mathbf{R}^{n-1}, \quad (t, \varphi, \psi) \mapsto (X(t, \varphi, \psi), \psi)$$

*is an embedding (in particular,  $\varphi \neq 0$  on  $U$ ). Then there is a neighbhd  $V$  of  $C = \kappa(U)$ , such that Eq.*

$$(5.6) \quad \langle P(t, \varphi, \psi), x - X(t, \varphi, \psi) \rangle = 0, \quad (t, x, \psi) \in V$$

*has a unique solution  $t = t_1(x, \psi)$ ,  $\varphi = \varphi_1(x, \psi)$  satisfying the condition*

$$(5.7) \quad (t, x, \psi) \in C \implies x = X(t_1(x, \psi), \varphi_1(x, \psi), \psi)$$

Consider then matrix of the form (4.4)

$$(5.12) \quad M(t, \varphi, \psi) = (\tilde{\Pi}; \tilde{P}_{\tilde{\psi}} - \tilde{P}_\varphi \tilde{\Pi}^* \tilde{X}_{\tilde{\psi}}) = (\tilde{\Pi}; \tilde{P}_\psi)$$

Then  $M(t, \varphi, \psi) = \begin{pmatrix} 1 & -mH & 0 \\ 0 & P & P_\psi \end{pmatrix}$  has determinant  $\det(P, P_\psi)$  which is non zero for small  $t$ . So we recover the same quantity giving the density on  $\Lambda_+$  as in Sect.4, but we only conjecture that it is non vanishing everywhere, see Hypothesis (5.20) below.

## 5.2 Construction of the phase function in the extended phase-space

Consider HJ equation for the phase function parametrizing  $\Lambda_+$  as in (2.5)

$$(5.15) \quad \partial_t \Phi + H(x, \partial_x \Phi) = E, \quad \Phi|_{t=0} = \langle x, \omega(\psi) \rangle$$

We find as before an integral manifold  $\Lambda_{\Phi,t} = \{p = \partial_x \Phi(t, x)\} \subset T^*\mathbf{R}^n$ . As before we let  $\Phi$  depend on additional "θ"-variables, but contrary to (4.7), we could not make the choice

$$mEt + \varphi + \langle P(t, \varphi, \psi, \tau), x - X(t, \varphi, \psi, \tau) \rangle$$

We add  $\lambda$  to the "θ"-variables, and put instead

$$(5.16) \quad \Phi(x, t, \psi, \varphi, \lambda) = mEt + \varphi + \lambda \langle P(t, \varphi, \psi), x - X(t, \varphi, \psi) \rangle$$

where the integral curve  $(X, P)$  is computed on  $\tau = 0$ . We check that for  $\lambda = 1$

$$\Phi|_{t=0} = \int P dX + \langle \omega(\psi), x \rangle - \varphi = \varphi + \langle \omega(\psi), x \rangle - \varphi = \langle \omega(\psi), x \rangle$$

so  $\Phi|_{t=0} = \langle \omega(\psi), x \rangle$  satisfies the initial condition. Variables  $(x, t, \varphi, \psi, \lambda)$  will be used in various situations.

Taking partial derivatives in (5.16) with respect to variables  $t, \varphi, \psi, \lambda$ , with  $\partial_t \int \langle P, dX \rangle = mE$ ,  $\partial_\varphi \int \langle P, dX \rangle = 1$ ,  $\partial_t \int \langle P, dX \rangle = 0$ , we find

$$(5.17) \quad \begin{aligned} \partial_t \Phi &= (1 - \lambda)mE + \lambda \langle \dot{P}, x - X \rangle \\ \partial_\varphi \Phi &= 1 - \lambda + \lambda \langle P_\varphi, x - X \rangle \\ \partial_\psi \Phi &= \lambda \langle P_\psi, x - X \rangle \\ \partial_\lambda \Phi &= \langle P, x - X \rangle \end{aligned}$$

The critical point  $x = X(t, \varphi, \psi)$  is uniquely determined (for small  $t$ ): namely the determinant of the  $2 \times 2$  system for the last 2 Eq. is given by  $\det(P, P_\psi)$ , and this is non zero at  $t = 0$ . So when  $\lambda = 1$  and  $x = X(t, \varphi, \psi)$  belongs to the critical set  $C_\Phi$ . Taking differential on  $C_\Phi$  gives

$$(5.18) \quad \begin{aligned} d\partial_t \Phi &= -Ed\lambda + \langle \dot{P}, dx - dX \rangle \\ d\partial_\varphi \Phi &= -d\lambda + \langle P_\varphi, dx - dX \rangle \\ d\partial_\psi \Phi &= \langle P_\psi, dx - dX \rangle \\ d\partial_\lambda \Phi &= \langle P, dx - dX \rangle \end{aligned}$$

Since matrices  $M(\tilde{\phi}, \tilde{\psi})$  as in (4.4) for both the "vertical plane" and the "cylinder" have determinant  $\det(P_\psi, P)$ , and since  $\det(P_\psi, P)$  gives the (inverse) density  $F[\Phi, d\mu_+]|_{C_\Phi}$  (up to a constant factor) for the "vertical plane", it is natural to seek for a density again of the form (4.10) for the cylinder, inserting an additional  $d\varphi$ . This is indeed the case, by an argument similar to [DoNaSh, Lemma 10] but where the density is defined another way.

**Proposition 5.3:** *Let  $H(x, p)$  be positively homogeneous of degree  $m \geq 1$  with respect to  $p$  on  $T^*M \setminus 0$ , and  $\det(P, P_\psi) > 0$ . Then  $\Phi(x, t, \varphi, \psi, \lambda)$  given in (5.16) is a non-degenerate phase function defining  $\Lambda_+$ , and solves HJ Eq. (2.15). The invariant (inverse) density on  $\Lambda_+$  is given by*

$$(5.19) \quad F[\Phi, d\mu_+]|_{C_\Phi} = \frac{dt \wedge d\psi \wedge d\varphi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi)}{dx \wedge dt \wedge d\psi \wedge d\varphi \wedge d\lambda} = mE \det(P, P_\psi)$$

*Assume moreover Lagrangian intersection. Then the critical set  $C_\Phi$  is determined as in (5.6)-(5.7) by  $x = X(t, \varphi, \psi, \tau = 0)$  (which can be inverted as  $t = t_1(x, \psi, \tau = 0)$ ,  $\varphi = \varphi_1(x, \psi, \tau = 0)$ ) and  $\lambda = 1$ . It coincides with the set  $\kappa(U)$  defined in Prop.5.2.*

*Proof:* We assume  $n = 2$  for simplicity and start to compute  $F[\Phi, d\mu_+]|_{C_\Phi}$  at  $t = 0$ , i.e. on  $(x, p) \in \Lambda$ . We expand the  $2n + 2$ -form

$$\omega = dt \wedge d\psi \wedge d\varphi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi) = \omega_1 + \omega_2$$

where

$$\begin{aligned} \omega_1 &= -mEdt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge \langle P_\psi, dx - dX \rangle \wedge \langle P, dx - dX \rangle \\ \omega_2 &= dt \wedge d\psi \wedge d\varphi \langle \dot{P}, dx - dX \rangle \wedge \langle P_\psi, dx - dX \rangle \wedge \langle P, dx - dX \rangle \end{aligned}$$

Using (5.8) a short calculation gives  $\omega_2 = 0$ , while only the term  $-mEdt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge \langle P_\psi, dx \rangle \wedge \langle P, dx \rangle$  contributes in  $\omega_1$ , so  $\omega_1 = -mE \det(P_\psi, P) dt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge dx$ . So we get (5.19), and the Proposition follows as in Proposition 4.1. See also [DoNaSh, Lemma 10]. ♣

To (4.16) we add, from Proposition A.1,  $d = \langle X_\varphi, \dot{P} \rangle = \langle \dot{X}, P_\varphi \rangle$  and notice that  $a = b = d = 0$  at  $t = 0$ .

### 5.3 Reduced parametrizations of $\Lambda_+$ in case of the “conformal metric”.

In case of the cylinder we only conjecture the statement of Proposition 4.2 relative to the density in case of a “conformal metric”. So we need assume that representation (5.16) is valid “globally” and defines a non degenerate phase function parametrizing  $\Lambda_+$ . In particular the density on  $\Lambda_+$  is (up to a constant factor) is  $\det(P, P_\psi)$ , and

$$(5.20) \quad \det(P, P_\psi) > 0$$

holds on the domain where we want to make sense of (1.5). In order to simplify (1.5), as in Sect.4 we need to find the critical points of  $\Phi$ .

• *Ordinary points.* We proceed as in Proposition 4.3, and Lemma 4.6, by eliminating  $t$ . The “ $\theta$ -parameters” are now  $(\varphi, \psi, \lambda)$ . We denote by  $\mathcal{S}$  the set of special points on  $\Lambda_+$ , and  $I_{\varphi, \psi} = \{t : (t, \varphi, \psi) \notin \mathcal{S}\}$  (we have already evaluated  $\lambda = 1$ ). By a straightforward computation we have

**Lemma 5.4:** *Assume (4.32). Then  $I_{\varphi, \psi}$  is an interval,  $0 \in I_{\varphi, \psi}$ , and*

$$(4.21) \quad \forall t \in I_{\varphi, \psi}, x = X(t, \varphi, \psi) \iff t = t_2(x, \varphi, \psi) \quad \text{on } C_\Phi$$

Moreover  $\pi_x : \Lambda_+ \rightarrow M$  at every ordinary point has same rank as the symmetric matrix (for simplicity we have made the computations when  $m = 2$ )

$$(5.22) \quad -\Psi''_{\theta\theta} = \begin{pmatrix} -\frac{d^2}{\langle \dot{P}, \dot{X} \rangle} - \langle P_\varphi, X_\varphi \rangle & -\frac{d^2}{\langle \dot{P}, \dot{X} \rangle} + \frac{2ad}{\langle \dot{P}, \dot{X} \rangle} & \frac{a^2}{\langle \dot{P}, \dot{X} \rangle} - 1 \\ * & \frac{a^2}{\langle \dot{P}, \dot{X} \rangle} - c & \frac{a^2}{\langle \dot{P}, \dot{X} \rangle} \\ * & * & \frac{(2E)^2}{\langle \dot{P}, \dot{X} \rangle} \end{pmatrix}$$

i.e.  $\pi_x$  has rank 1 or 2 or 3. In particular, if  $z(0)$  is an ordinary point, then  $-\Psi''_{\theta\theta}$  is singular (of rank 2) iff  $\varphi$  verifies the implicit equation

$$\varphi = E^2 \frac{\rho(\varphi\omega(\psi)) \langle \nabla\rho(\varphi\omega(\psi)), \omega(\psi) \rangle}{2(1 + E^2\rho^2(\varphi\omega(\psi)))} = 0$$

It may be more convenient to solve directly the system  $\Phi'_{(t,\psi)} = 0$ . We have

$$\Phi''_{t,\psi} = \begin{pmatrix} \frac{\partial^2 \Phi}{\partial t^2} & \frac{\partial^2 \Phi}{\partial t \partial \psi} \\ \frac{\partial^2 \Phi}{\partial \psi \partial t} & \frac{\partial^2 \Phi}{\partial \psi^2} \end{pmatrix} = \begin{pmatrix} \langle \dot{P}, \dot{X} \rangle & \langle P_\psi, \dot{X} \rangle \\ \langle P_\psi, \dot{X} \rangle & \langle P_\psi, X_\psi \rangle \end{pmatrix}$$

where  $\langle P_\psi, \dot{X} \rangle = \langle X_\psi, \dot{P} \rangle$ . Let

$$(5.23) \quad D(z(t)) = \det \Phi''_{t,\psi}$$

(for short we omit other variables) and  $\Omega(t) = \{(\varphi, \psi) : z(t) = (X(t, \varphi, \psi), P(t, \varphi, \psi)), D(z(t)) \neq 0\}$ .

*Remark 5.3:* Assume  $\det(\dot{X}, X_\psi) \neq 0$  (i.e.  $z(t)$  is not a focal point), then  $\Phi''_{t,\psi}$  is just Gram matrix of  $(\dot{P}, P_\psi)$  in the basis  $\dot{X}, X_\psi$ , and  $\det(\dot{P}, P_\psi) = 0$  iff  $D(z(t)) = 0$ . We can interchange of course the role of the pairs of vectors.

Again it is instructive to look at the model Hamiltonian  $H = \langle p, \mu \rangle$ , where  $X(t, \psi) = \mu t + \varphi\omega(\psi)$ ,  $P(t, \psi) = \omega(\psi)$ , and  $\Phi(t, \varphi, \psi, \lambda) = Et + (1 - \lambda)\varphi + \lambda\langle \omega(\psi), x - \mu t \rangle$  is independent of  $\varphi$  when  $\lambda = 1$ . Since  $\langle \mu, \omega(\psi) \rangle = E$ , we have  $\dot{\Phi} = E - \lambda\langle \mu, \omega(\psi) \rangle = (1 - \lambda)E$ , and  $\partial_\psi \Phi = \lambda\langle \omega^\perp(\psi), x - \mu t \rangle$ . So on the critical set  $x = X(t, \varphi, \psi)$ ,  $\lambda = 1$ , we have  $\dot{\Phi} = \partial_\psi \Phi = 0$ ; assuming Lagrangian intersection, the Hessian of  $\Phi$  with respect to  $(t, \psi)$  has determinant  $D(t) = -\left(\frac{\partial^2 \Phi}{\partial t \partial \psi}\right)^2 = -\langle \mu, \omega^\perp(\psi) \rangle^2$ . So if  $\langle \mu, \omega^\perp(\psi) \rangle \neq 0$ ,  $\psi = \psi_0$  such that  $\langle \mu, \psi_0 \rangle = E$ , i.e.  $z(t)$  is non glancing, then  $D(t) \neq 0$  for all  $t \geq 0$ . We notice also that the time component of the critical point is precisely  $t_1(x, \psi)$  computed in Proposition 5.2.

For Hamiltonian (1.15) instead  $D(z(0)) = \frac{m\varphi}{\rho(\varphi\omega(\psi))^3} \langle \nabla\rho, \omega \rangle$  is non vanishing if  $\varphi \neq 0$  and  $z(0)$  is an ordinary point, i.e. a point (necessarily non glancing) where

$$\langle \nabla\rho(\varphi\omega(\psi)), \omega^\perp(\psi) \rangle = 0, \quad \langle \nabla\rho(\varphi\omega(\psi)), \omega(\psi) \rangle \neq 0$$

We restrict for simplicity to Hamiltonian (1.15), and describe the Lagrangian singularities for  $\Phi$  as in Sect.4. Variables  $(t, \psi)$  will play the role of  $(t, \lambda)$  in Lemma 4.7.

**Lemma 5.5:** Let  $F(x, t, \varphi, \psi, \lambda = 1) = \left(\frac{\partial_t \Phi}{\partial_\psi \Phi}\right)$ , and  $s$  such that  $D(s) \neq 0$  as in (5.23). Then for  $t$  close enough to  $s$ , the system  $F(x, t, \psi, \lambda = 1) = 0$  is equivalent to  $t = t_1(x, \psi)$ ,  $\varphi = \varphi_1(x, \psi)$ , where  $t_1, \varphi_1$  are defined in Prop. 5.2. This gives  $t = t(x), \psi = \psi(x)$ , and  $\varphi = \varphi(x)$  when  $x$  is sufficiently close to  $\varphi\omega(\psi)$ ,  $(\varphi, \psi) \in \Omega(z(s))$ .

*Proof:* Since  $D(s) \neq 0$ , implicit function theorem shows that  $F(x, t, \varphi, \psi) = 0$  (we omitted  $\lambda = 1$ ) has a unique solution  $t = t_0(x, \varphi), \psi = \psi_0(x, \varphi)$ . Differentiating  $F = 0$  along  $\Lambda_+$  with respect to  $x$  and  $\varphi$  we find

$$(5.30) \quad \begin{aligned} \langle \dot{P}, \dot{X} \rangle \partial_x t_0 + \langle \dot{P}, X_\psi \rangle \partial_x \psi_0 &= {}^t \dot{P} \\ \langle P_\psi, \dot{X} \rangle \partial_x t_0 + \langle P_\psi, X_\psi \rangle \partial_x \psi_0 &= {}^t P_\psi \\ \langle \dot{P}, \dot{X} \rangle \frac{\partial t_0}{\partial \varphi} + \langle \dot{P}, X_\psi \rangle \frac{\partial \psi_0}{\partial \varphi} &= -\langle \dot{P}, X_\psi \rangle \\ \langle P_\psi, \dot{X} \rangle \frac{\partial t_0}{\partial \varphi} + \langle P_\psi, X_\psi \rangle \frac{\partial \psi_0}{\partial \varphi} &= -\langle P_\psi, X_\psi \rangle \end{aligned}$$

Using the relation  $\langle \dot{P}, X_\psi \rangle = \langle P_\psi, \dot{X} \rangle$ , the 2 sub-systems have determinant

$$(5.31) \quad D(t) = \det \begin{pmatrix} \langle \dot{P}, \dot{X} \rangle & \langle P_\psi, \dot{X} \rangle \\ \langle \dot{P}, \dot{X}_\psi \rangle & \langle P_\psi, X_\psi \rangle \end{pmatrix} \neq 0$$

So for  $(\varphi, \psi) \in \Omega(z(s))$ , (5.30) has the unique solution with the condition

$$t_0(\varphi\omega(\psi), \varphi) = 0, \quad \psi_0(\varphi\omega(\psi), \varphi) = \psi, \quad (\varphi, \psi) \in \Omega(0)$$

On the other hand we know from Proposition 5.2 that on  $C_\Phi$ , i.e. when  $x = X(t, \varphi, \psi)$ , we have  $\varphi = \varphi_1(x, \psi)$  and  $t = t_1(x, \psi)$ . This gives :

$$(5.33) \quad g(x, \psi) = \psi - \psi_0(x, \varphi_1(x, \psi)) = 0$$

Compute  $\partial_\psi g(x, \psi) = 1 - \frac{\partial \psi_0}{\partial \varphi} \frac{\partial \varphi_1}{\partial \psi}$  at  $x = \varphi\omega(\psi)$ . We know  $\frac{\partial \psi_0}{\partial \varphi}$  from (5.32). Combining Proposition 5.2 and [DoNaSh, Lemma 7] with  $\langle P, X_\psi \rangle = 0$ , we get  $\frac{\partial \varphi_1}{\partial \psi} = 0$  along  $C_\Phi$ . Thus  $\partial_\psi g(x, \psi) \neq 0$  and implicit function theorem gives  $\psi = \psi_1(x)$ . Substituting into  $t = t_1(x, \psi) = t_0(x, \varphi)$  we get also  $t = t_1(x, \psi_1(x)) = t_0(x, \varphi_1(x, \psi_1(x)))$ . ♣

We are left, as in Lemma 4.7, to describe the Lagrangian singularity at  $t = t(x), \psi = \psi(x)$ , and  $\varphi = \varphi(x)$ , so we set

$$(5.36) \quad \Psi(x) = \Phi(x, t_1(x, \psi_1(x)), \varphi_1(x, \psi_1(x)), \psi_1(x))$$

and look at the rank of  $\Psi''(x)$ .

*Remark 5.4:* Following the flow, an ordinary point at  $s$  may become special at a later time  $t$  (only once). But then  $\nabla \rho(x(t)) = 0$ ,  $x(t)$  is a critical point, so  $t = +\infty$ . When (4.32) holds, this critical point is necessarily non degenerate.

- *Glancing points.* Consider small  $t$ . We have

$$\det \frac{\partial^2 \Phi}{\partial(t, \psi, \lambda)} \Big|_{t=0} = -(mH)^2 \langle P_\psi, X_\psi \rangle \Big|_{t=0} = -(mH)^2 \varphi$$

this is non zero for non zero  $\varphi$ , so by implicit functions theorem  $(t, \psi, \lambda) = (t_2(x, \varphi), \psi_2(x, \varphi), \lambda_2(x, \varphi))$ .

Computing the variations

$$\nabla_{x, \varphi} \dot{\Phi} = 0, \quad \nabla_{x, \varphi} \partial_\psi \Phi = 0, \quad \nabla_{x, \varphi} \partial_\lambda \Phi = 0$$

we are led in particular to

$$(5.40) \quad \begin{aligned} mH \frac{\partial t_2}{\partial \varphi} + 1 &= 0, \quad mH \frac{\partial t_2}{\partial x} = {}^t P \\ mHc \frac{\partial \psi_2}{\partial \varphi} &= \langle P_\psi, \dot{X} \rangle - mH \langle P_\psi, X_\varphi \rangle, \quad c \frac{\partial \psi_2}{\partial x} = {}^t P_\psi - \langle P_\psi, \dot{X} \rangle {}^t P \end{aligned}$$

Introduce some polar coordinates  $x = (r, \theta)$  such that  $r = \varphi$  and  $\theta = \psi$  on  $\Lambda$ , we find by integration

$$(5.41) \quad \begin{aligned} mHt_2(r, \theta, \varphi) &= r \langle \omega(\psi), \omega(\theta) \rangle - \varphi \\ \psi_2(r, \theta, \varphi) &= r \frac{\langle \omega^\perp(\psi), \omega(\theta) \rangle}{\langle P_\psi, X_\psi \rangle} + \psi \end{aligned}$$

Replacing in  $\Phi$  we find

$$(5.42) \quad \Phi(t_2(x, \varphi), \varphi, \psi_2(x, \varphi)) = r \cos(\psi - \theta) + \langle P(t_2, \varphi, \psi_2), x - X(t_2, \varphi, \psi_2) \rangle + \dots$$

#### 5.4 Construction of Maslov canonical operator.

It goes the same way as in Sect. 4., using Lemmas 5.4 and 5.5 instead of Lemmas 4.6 and 4.7 in the discussion according to ordinary or special points. However it is more difficult to justify the expression  $\mathcal{K}_+(\sigma, \sigma^+)(x; h)$  as a finite sum (over all pre-images  $\eta_\alpha = P(\psi_\alpha), t = t_\alpha$ ) of different canonical charts. The main reason is that  $\Phi$  cannot be interpreted as a distance, allowing for computation of extremals : thus there seems to be no natural equivalent of Assumption (2.18).

#### 5.5 Radially symmetric “conformal metric”

We choose  $m = 1$  for simplicity, and  $\rho(x)$  radially symmetric, i.e.  $\rho = \rho(|x|)$  with  $\rho$  a smooth function on  $\mathbf{R}_+$ .

We have  $\rho(|x|) = \rho(\varphi)$  (choosing the branch  $\varphi > 0$ ),  $\varphi = \varphi_0 > 0$  on  $L = \Lambda \cap \{H = E\}$ , and  $p dx = d\varphi$  on  $\Lambda$ . The hypothesis  $\varphi_0 > 0$  may be relaxed. We assume also  $\rho'(\varphi_0) \neq 0$ .

The phase function is now given by (5.16) with  $\varphi = \varphi_0$ . The variational system is obtained directly in the  $x - X$  variables as

$$\begin{aligned} d\partial_t \Phi &= -mH d\lambda + (dx - dX)^* \partial_t P \\ d\partial_\psi \Phi &= (dx - dX)^* P \partial_\psi P \\ d\partial_\lambda \Phi &= (dx - dX)^* P \end{aligned}$$

We check the linear independence. Let

$$d(\partial_t \Phi)\alpha + \langle d(\partial_\psi)\Phi, \beta \rangle + d(\partial_\lambda \Phi)\gamma = 0$$

Identifying the term in  $d\lambda$  we find  $\alpha = 0$ , then  $(dx - dX)^*(\langle \partial_\psi P, \beta \rangle + \gamma P) = 0$ , which readily implies  $\beta = \gamma = 0$ , for small  $t$  since  $P|_{t=0} = \omega(\psi)$  and  $\partial_\psi P|_{t=0} = \omega(\psi)^\perp$ .

In a second step, we seek for a reduced generating function for  $\Lambda_+$  by eliminating  $t$  when  $t > 0$  by stationary phase. This follows easily from implicit functions theorem and the value at  $t = 0$

$$\partial_t^2 \Phi|_{x=X(\psi)} = -\langle \partial_t P, \partial_\psi X \rangle = \text{Const.} \langle \omega(\psi), \omega(\psi) \rangle \neq 0$$

(without any further assumption).

It follows that  $u_h$  has only “wave-part” component, until the Hamilton flow meets  $\varphi = 0$  where the analysis has slightly to be changed to take into account the singularity of  $\rho(|x|)$  at this point.

We complete this Example by the integration of motion (at least in 2-D), since the system is integrable (with  $H$  as in (1.15)). Introduce polar coordinates  $(r, \theta)$  and  $(s, \psi)$  on  $T^*\mathbf{R}^2$  (same parameter  $\psi$  as in the definition of  $\Lambda$ ), so that

$$(5.50) \quad x = r\omega(\theta), \quad p = s\omega(\psi)$$

and

$$(5.51) \quad p dx = s \cos(\psi - \theta) dr + rs \sin(\psi - \theta) d\theta$$

On  $H = \frac{s}{\rho(r)}$ , Hamilton equations are given by

$$\dot{r} = \frac{1}{\rho(r)}, \quad \dot{s} = \frac{\rho'(r)}{\rho(r)^2} s$$

that is

$$\frac{\dot{s}}{s} = \frac{\rho'(r)}{\rho(r)^2} = \frac{\rho'(r)}{\rho(r)} \dot{r}$$

Integrating these equations we find  $s(t) = s_0 \rho(r(t))$  so

$$(4.52) \quad P(t, \psi) = s_0 \frac{\rho(r(t))}{\rho(r_0)} \omega(\psi)$$

If the trajectory leaves  $\Lambda$  at  $r(0) = r_0 = \varphi_0$ , then  $s_0 = 1$ ,  $\frac{1}{\rho(r_0)} = E$  (which fixes  $r_0$ ). The first Eq. (4.51) has separated variables  $\rho(r) dr = dt$ , so if  $N(r) = \frac{1}{E} + \int_{r_0}^r \rho(r') dr'$ , we get  $N(r) = T$ , which is solved implicitly by  $r = R(t)$ . Finally,  $\theta = \text{Const.}$ ,  $\psi = \text{Const.}$ , so the solution of Hamilton equations with data on  $L = \partial\Lambda_+$  are simply

$$(5.53) \quad \begin{aligned} \theta &= \psi \\ x &= X(t, r_0, \psi) = R(t)\omega(\psi) \\ p &= P(t, r_0, \psi) = \frac{\rho(R(t))}{\rho(r_0)} \omega(\psi) \end{aligned}$$

These are the parametric equations of  $L_t = g^t(L)$ . We are left to investigate the “collision” on  $r = 0$ , where the solution becomes singular, and the “boundary part” comes into play.

### 5.6 Example in the constant coefficient case

To close this section we consider as in Remark 3.1,  $H = -h^2\Delta$ ,  $n = 2$  and

$$f_h(x) = J_0\left(\frac{|x|}{h}\right) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i\langle x, \omega(\psi) \rangle} d\psi = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i|x|\sin\psi} d\psi$$

The outgoing solution is given by the oscillating integral

$$u_h(x) = (4\pi^2 h)^{-1/2} \int e^{i\Phi(x, y, \psi)/h} (\xi^2 - E - i0)^{-1} d\psi dy d\xi$$

which we compute (formally) by stationary phase in  $(y, \xi)$ . The critical point is given by  $y = x$ ,  $\xi = \frac{x}{|x|} \sin\psi$ , and

$$(5.61) \quad u_h(x) = h^{1/2} \int_{-\pi}^{\pi} e^{i|x|\sin\psi/h} \frac{1}{\sin^2\psi - E - i0} d\psi + \mathcal{O}(h^{3/2})$$

and the leading term can be simply evaluated by contour integrals. Obviously energy  $E = 1$  plays a special role. The phase of course is the same as in Bessel function. We can also consider more general  $f_h$  and stick in an amplitude of the form (see [DoMaNaTu])

$$A(x, \psi) = \frac{1}{2} (a(|x|, \psi) + a(-|x|, \psi)) + \frac{\langle x, \omega(\psi) \rangle}{2|x|} (a(|x|, \psi) - a(-|x|, \psi))$$

so that

$$f_h(x) = \left(\frac{i}{2\pi h}\right)^{1/2} \int_{-\pi}^{\pi} e^{i\langle x, \omega(\psi) \rangle} A(x, \psi) d\psi$$

As in (5.61) we get

$$(5.62) \quad u_h(x) = h^{1/2} \int_{-\pi}^{\pi} e^{i|x|\sin\psi/h} \frac{A(x, \psi)}{\sin^2\psi - E - i0} d\psi + \mathcal{O}(h)$$

When  $A$  is independent of  $\psi$ , this can lead to significant simplifications.

## Appendix Lagrange immersions and global half-densities

Recall first some well-known properties of Lagrangian immersions (see e.g. [DoZh], [DoNaSh]) :

**Proposition A.1:** *Let  $\iota : \Lambda \rightarrow T^*M$ , be a Lagrangian immersion, parametrized on a canonical chart  $U$  by  $\varphi \mapsto z = \iota(\varphi) = (X(\varphi), P(\varphi))$ . Introduce the Jacobian matrices  $B(z) = \frac{\partial P}{\partial \varphi}$ ,  $C(z) = \frac{\partial X}{\partial \varphi}$ . Then:*

- (1) *the matrix  $(B(z), C(z))$  is of rank  $n$ .*
- (2) *the matrix  ${}^tC(z)B(z)$  is symmetric.*
- (3)  *$C(z) \pm iB(z)$  is non degenerate.*

The symmetry of  ${}^tC(z)B(z)$  expresses for instance in the situation of Sect.4 (resp.Sect.5) as the symmetry of Gram matrices

$$(A.1) \quad \begin{pmatrix} \langle \dot{X}, \dot{P} \rangle & \langle \dot{X}, P_\psi \rangle \\ \langle X_\psi, \dot{P} \rangle & \langle X_\psi, P_\psi \rangle \end{pmatrix}, \quad \begin{pmatrix} \langle \dot{X}, \dot{P} \rangle & \langle \dot{X}, P_\varphi \rangle & \langle \dot{X}, P_\psi \rangle \\ \langle X_\varphi, \dot{P} \rangle & \langle X_\varphi, P_\varphi \rangle & \langle X_\varphi, P_\psi \rangle \\ \langle X_\psi, \dot{P} \rangle & \langle X_\psi, P_\varphi \rangle & \langle X_\psi, P_\psi \rangle \end{pmatrix}$$

We consider the rank of projections  $\pi_x : \Lambda_+ \rightarrow M$ . It is equal to the rank of  $\pi_{x,t} : \tilde{\Lambda}_+ \rightarrow M \times \mathbf{R}_t$ .

In general we call *focal point* a point  $z \in \Lambda$  where  $\pi_* : T\Lambda \rightarrow TM$  is singular, and *caustics* the projection  $\mathcal{C}$  of the set of focal points onto  $M$ . Assume  $n = 2$ , and let  $z$  be a focal point, so  $C(z)$  cannot be of rank 2, and by property (1) above either  $B(z)$  is of rank 2 (and  $C(z)$  has rank at most 1, since the projection  $\pi : \Lambda_+ \mapsto M$  is not a diffeomorphism at  $z$ ) or both  $C(z)$  and  $B(z)$  are of rank 1.

Assume now  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ . Let  $\Lambda = \Lambda_+$  be an integral manifold of  $v_H$  in the energy shell  $H(x, p) = E$ , and  $U \subset \Lambda_+$  be a canonical chart parametrized by  $\varphi = (t, \psi)$ , i.e.  $z = (X(t, \psi), P(t, \psi))$  verifies  $\dot{X} = \partial_x H(X, P)$ ,  $\dot{P} = -\partial_p H(X, P)$ , such that  $U \rightarrow T^*M$  is an immersion (not necessarily an embedding). Recall  $\langle P, \dot{X} \rangle = mH$ ,  $\langle P, X_\psi \rangle = 0$ . Actually  $(X(t, \psi), P(t, \psi))$  may depend on additional parameters, as in Sect.5, but here only  $t, \psi$  matter. Consider the quantity  $\det(P, P_\psi)$ .

**Lemma A.2:** *Let  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$ . Assume that at some point  $z = (X(t, \psi), P(t, \psi))$ , we have  $\det(P, P_\psi) = 0$ .*

1) *If  $\nabla \rho(X(t, \psi)) \neq 0$ , then either  $|P(t, \psi)| = 1$ , or  $P_\psi(t, \psi) = 0$ . In the latter case  $C(z) = (\dot{X}, X_\psi)$  has rank 2, i.e.  $\pi_x$  is regular at  $z$ .*

2) *If  $\rho$  has a critical point at some  $x_0 = X(t, \psi)$ , then  $\pi_x$  is regular at  $z = (x_0, P(t, \psi))$*

*Proof:* We apply Proposition A.1 to the Lagrangian immersion  $\iota : U \rightarrow T^*M$ ,  $\varphi = (t, \psi)$ . Assume  $\det(P, P_\psi) = 0$  at some point  $z = (X(t, \psi), P(t, \psi))$ . Then either  $P_\psi = 0$ , or by Hamilton equations (3.2), there is  $\alpha \in \mathbf{R}$  such that  $\dot{X} = \alpha P_\psi$ . Let first  $\nabla \rho \neq 0$ .

• We examine the case  $\dot{X} = \alpha P_\psi$ . By property (2) of Proposition A.1, the matrix

$${}^tC(z)B(z) = \begin{pmatrix} * & \dot{X}_1 \partial_\psi P_1 + \dot{X}_2 \partial_\psi P_2 \\ \dot{P}_1 \partial_\psi X_1 + \dot{P}_2 \partial_\psi X_2 & * \end{pmatrix}$$

has to be symmetric. If  $\dot{X} = \alpha P_\psi$ , this implies  $\alpha = \langle \dot{P}, \partial_\psi X \rangle |P_\psi|^{-2}$ , hence by (2.2)

$$(A.5) \quad \dot{X} = \langle \dot{P}, \partial_\psi X \rangle |P_\psi|^{-2} P_\psi = \frac{|P|^m}{\rho(X)^2} \langle \nabla \rho(X), X_\psi \rangle |P_\psi|^{-2} P_\psi$$

Along  $\Lambda_+$  we differentiate the relation  $|p|^m = (E - \tau)\rho(x)$ , and get (with  $H = E - \tau$ )

$$(A.6) \quad m|P|^{m-1}\langle P, P_\psi \rangle = (E - \tau)\langle \nabla\rho(X), X_\psi \rangle, \quad m|P|^{m-1}\langle P, \dot{P} \rangle = (E - \tau)\langle \nabla\rho(X), \dot{X} \rangle$$

Now the relation  $\det(P, P_\psi) = 0$  shows that  $\langle P, P_\psi \rangle = \pm|P||P_\psi|$ , and substituting in (A.5) (using again (3.2)) we find  $|P| = 1$ .

• We are left with the case  $P_\psi = 0$ . In this case  $X_\psi \neq 0$ , for otherwise this would contradict property (3) of Proposition 3.2. The symmetry of  ${}^tC(z)B(z)$  shows that  $\langle \dot{P}, X_\psi \rangle = 0$ , or by (3.2)  $\langle \nabla\rho(X), X_\psi \rangle = 0$ . Since  $\langle P, X_\psi \rangle = 0$ , we find that  $\dot{X}, P, \dot{P}, \nabla\rho(X)$  are parallel, and all orthogonal to  $X_\psi$ . In particular,  $C(z) = (\dot{X}, X_\psi)$  is of rank 2.

Assume now  $\nabla\rho(x_0) = 0$ . The first situation above cannot hold, since this would imply  $\dot{X} = 0$  by (A.6), hence  $P = 0$ . Hence  $\dot{P} = P_\psi = 0$  which implies again  $C(z)$  of rank 2 by Proposition A.1. ♣

*Remark A.1:* Assume  $\rho$  is bounded, and  $E$  chosen in such a way that

$$(A.7) \quad \frac{1}{\rho(x)} \neq E \text{ for all } x$$

Then  $\det(P, P_\psi) = 0$  at  $z$  implies  $C(z)$  has rank 2, i.e.  $z$  is not a focal point.

Actually, condition (A.7) is not necessary, as shows the following:

**Lemma A.3:** *Let  $H(x, p) = \frac{|p|^m}{\rho(x)}$ ,  $n = 2$  and  $\nabla\rho \neq 0$  everywhere. Then condition  $\det(P, P_\psi) > 0$  holds at any focal point. In particular,  $P_\psi$  can vanish only at a regular point.*

*Proof:* Assume instead  $\det(P, P_\psi) = 0$ . Again we apply Proposition A.1

• Assume  $B(z)$  is of rank 2, and  $C(z)$  of rank at most 1. We know that  $\langle P(t, \psi), \dot{X}(t, \psi) \rangle = m(E - \tau) \neq 0$ ,  $\dot{X} \neq 0$  parallel to  $P \neq 0$ , and  $\langle P(t, \psi), \partial_\psi X(t, \psi) \rangle = 0$ .

Assume  $X_\psi \neq 0$ . Since  $C(z)$  is of rank at most 1,  $X_\psi$  parallel to  $\dot{X}$ , which is itself parallel to  $P$ . So  $P$  is parallel to  $X_\psi$ , which contradicts  $\langle P, X_\psi \rangle = 0$ . Hence  $X_\psi = 0$ . By first equality (A.6) we have  $\langle P, P_\psi \rangle = 0$ . If  $\det(P, P_\psi) = 0$ , then we would have  $P$  both orthogonal to  $X_\psi$ , and parallel to  $P_\psi \neq 0$ . But  $B(z)$  is of rank 2, which is a contradiction.

• So by property (3) of Proposition A.1 we must have  $B(z)$  and  $C(z)$  of rank 1. So either  $\dot{X} = \lambda X_\psi$  and  $\dot{P} = \mu P_\psi$  for some  $\lambda, \mu \in \mathbf{R}$ , or  $X_\psi = 0, \dot{P} = \mu P_\psi$ , or  $P_\psi = 0, \dot{X} = \lambda X_\psi$ , or  $X_\psi = P_\psi = 0$ .

Examine the first case: Identifying the off-diagonal terms of  ${}^tB(z)C(z)$ , which is symmetric by property (2) of Proposition A.1, we find that either  $\lambda = \mu$  or  $\frac{\partial X}{\partial \psi} \perp \frac{\partial P}{\partial \psi}$ . But  $\lambda \neq \mu$  since otherwise the complex matrices  $C(z) \pm iB(z)$  would be degenerate, which violates property (3) of Proposition 3.2. So  $\frac{\partial X}{\partial \psi}$  and  $\frac{\partial X}{\partial t}$  are colinear, and orthogonal to both  $\frac{\partial P}{\partial \psi}$  and  $\frac{\partial P}{\partial t}$ . Assume first  $\lambda\mu \neq 0$ . Using again (A.6), we find that  $P \perp P_\psi$ , since  $P_\psi \neq 0$ , we find that  $\det(P, P_\psi) \neq 0$ . Assume then  $\lambda\mu = 0$ . If  $\lambda = 0, \mu \neq 0$ , we would have  $\dot{X} = 0$ , which is impossible. Let now  $\lambda \neq 0$ , then  $B + iC = (P_\psi + iX_\psi, i\lambda X_\psi)$  has rank 2, so has  $(P_\psi, X_\psi)$ . So by (A.6)  $P, \frac{\partial X}{\partial \psi}$  and  $\frac{\partial X}{\partial t}$  are colinear, and orthogonal to both  $\frac{\partial P}{\partial \psi}$  and  $\nabla\rho$ . As before, this implies  $\det(P, P_\psi) \neq 0$ .

Examine the second case  $X_\psi = 0, \dot{P} = \mu P_\psi$ . Writing that  ${}^tB(z)C(z)$  is symmetric, we find  $\dot{X} \perp P_\psi$ . Since  $B(z) + iC(z)$  has rank 2,  $P_\psi \neq 0$ . Then  $P$  and  $\dot{X}$  are parallel, and both orthogonal to  $\nabla\rho, P_\psi$  and  $\dot{P}$ . As before, we find  $\det(P, P_\psi) \neq 0$ .

The last two cases are similar. So  $\det(P, P_\psi) \neq 0$  at any focal point. ♣

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