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Semiclassical Green functions and Lagrangian intersection.
Applications to the propagation of Bessel beams in non-homogeneous media

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Abstract: We study semi-classical asymptotics for problems with localized right-hand sides by considering a Hamiltonian $H(x,p)$ positively homogeneous of degree $m \geq 1$ on $T^*\mathbf{R}^2 \setminus 0$. The energy shell is $E = 1$, and the right-hand side f_h is microlocalized: (1) on the vertical plane $\Lambda = \{x = x_0\}$; (2) on the "cylinder" $\Lambda = \{(X,P) = \big(\varphi\omega(\psi), \omega(\psi)\big);\ \varphi \in \mathbf{R}, \omega(\psi) = (\cos\psi, \sin\psi)\}$. Most precise results are obtained in the isotropic case $H(x,p) = \frac{|p|^m}{\rho(x)}$, with ρ a smooth positive function. In case (2), Λ is the frequency set of Bessel function $J_0(\frac{|x|}{h})$, and the solution u_h of $(H(x,hD_x) - E)u_h = f_h$ when $m = 1$, already provides an insight in the structure of "Bessel beams", which arise in the theory of optical fibers.

1. Introduction

Let $M = \mathbf{R}^n$, or possibly a smooth manifold. For $m, k \in \mathbf{Z}$, we recall (see e.g. [Iv,Chapt.1]) the usual class of symbols

$$S^k_m(M \times M \times \mathbf{R}^N) = \{a(x,y,\theta;h) \in C^\infty : |\partial^\alpha_{(x,y)}\partial^\beta_\theta a(x,y,\theta;h)| \leq C_{\alpha\beta} h^k \langle\theta\rangle^{m-|\beta|}\}$$

(we use the notation $\langle\theta\rangle = (1+\theta^2)^{1/2}$), with asymptotic expansion $a(x,y,\theta;h) \sim h^k\big(a_0(x,y,\theta) + ha_1(x,y,\theta) + \cdots$. We define analogously $S^k_m(T^*M)$. Let $H(x,hD_x;h)$ be a h-PDO whose symbol H belongs to $S^0_m(T^*M)$, which is for instance the case when $H(x,p;h)$ is positively homogeneous of degree m with respect to p, and $E \neq 0$ be a non critical energy level for Hamiltonian H_0. We will denote by $z = (x,p)$ or by $z = (x,\xi)$ a point in T^*M. On the other hand, let f_h be a Lagrangian semi-classical distribution locally of the form

$$f_h(x) = (2\pi h)^{-N/2} \int_{\mathbf{R}^N} e^{i\varphi(x,\theta)/h} a(x,\theta;h)\, d\theta \tag{1.1}$$

where φ is a non-degenerate phase function in the sense of Hörmander defining the Lagrangian manifold Λ (see Sect.3), and $a \in S^k_0(M \times \mathbf{R}^N)$ for some k. For brevity we will often denote such a (normalized) integral simply by $\int^*(\cdots)$. We shall assume Hamiltonian vector field v_{H_0} be transverse to Λ. In [AnDoNaRo] we have considered the general problem of "semi-classical Green functions", which consists in solving $(H - E)u_h = f_h$, with u_h *outgoing at infinity* in the context of Maslov canonical operator. The distributions u_h and f_h are linearly related by $u_h = E_+(h)f_h$, where $E_+(h)$ is the semi-classical *outgoing parametrix*, that we shall compute in term of Maslov canonical operators. Here are some examples of f_h (expressed in a single chart):

Examples 1.1:

(1) $\Lambda = \{x = x_0\} = T^*_{x_0}M$, $x_0 \in M$ (that we call the "vertical plane") is the conormal bundle to $\{x_0\}$, so that

$$f_h(x) = \int^* e^{i(x-x_0)p/h} a(x,p;h)\, dp \tag{1.2}$$

We say simply that f_h is a "localized function" at x_0. This is the basic example since, according to [MelUhl], Λ can alway be taken locally to such a form, and $H(x, hD_x; h)$ to hD_{x_n}, where $x = (x', x_n)$.

(2) More generally, a conormal distribution $f_h(x) = \int^* e^{ix'p'/h} a(x,p';h)\, dp'$, $x = (x', x'')$, $p = (p', p'')$, with respect to $N = \{x' = 0\}$, i.e. $\Lambda = T^*_N \mathbf{R}^n = \{(x'', p')\}$. Actually f_h can be expressed with a new amplitude $a(x'', p'; h)$ not depending on x', see [Hö,Lemma 18.2.1].

(3) WKB functions in Fourier representation

$$f_h(x) = \int^* e^{i(xp+S(p))/h} a(x,p;h)\, dp$$

here $\Lambda = \{(-\partial_p S(p), p) : p \in \mathbf{R}^n\}$.

(4) Semi-classical distributions related with Bessel functions, microlocalized on

$$\Lambda = \{x = X(\varphi,\psi) = \varphi\omega(\psi), p = P(\varphi,\psi) = \omega(\psi), \varphi \in \mathbf{R}\} \tag{1.3}$$

which is called the "cylinder" ; here $\omega \in \mathbf{S}^{n-1}$ is the unit vector parametrized by ψ, see Sect. 5.

When $n = 2$, this holds in particular for Bessel function of order 0

$$f_h(x) = \sqrt{\frac{2\pi}{h}} J_0\big(\frac{|x|}{h}\big), \quad x \in \mathbf{R}^2$$

a radially symmetric solution of Helmholtz equation $-h^2\Delta v - v = 0$, and follows from the integral representation

$$J_0\big(\frac{|x|}{h}\big) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\langle\omega(\psi),x\rangle/h}\, d\psi$$

More generally, this applies to "regular" distributions on $\mathbf{R}^2$ of the form

$$f_h(x) = (2\pi h)^{-1/2}\int_0^{2\pi} e^{i\langle\omega(\psi),x\rangle/h} a(\langle\omega(\psi),x\rangle,\psi)\, d\psi \tag{1.4}$$

with an amplitude of the special type $a(\langle\omega(\psi),x\rangle,\psi)$, see [DoMaNaTu1]. They are similar to conormal distributions in Example (2) after x'-variables has been removed from the amplitude.

When $n = 3$ this holds also [DoMaNa] for semi-classical distributions of the form

$$f_h(x) = (2\pi h)^{-1}\int e^{i\langle\omega(\psi),x\rangle/h}\sin\theta\, d\theta\, d\widetilde{\psi} = (2\pi h|x|)^{-1/2} J_{1/2}\big(\frac{|x|}{h}\big), \quad x \in \mathbf{R}^3$$

with $\psi = (\widetilde{\psi}, \theta)$ the standard angular coordinates. One can also stick in an amplitude $a(\langle\omega(\psi),x\rangle,\psi)$ as in (1.4).

(5) f_h identifies with a "Bessel beam" when

$$\Lambda = \{(x,p) \in T^*\mathbf{R}^3 : \ x = \begin{pmatrix} \varphi\omega(\psi) \\ \phi \end{pmatrix}, \ p = \begin{pmatrix} \lambda(\phi)\omega(\psi) \\ \varphi\lambda'(\phi) + k \end{pmatrix}, \omega(\psi) = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}, \ \phi, \varphi \in \mathbf{R}\}$$

$k \in \mathbf{R}$ and λ is a smooth positive function. Such Lagrangian distributions, with the notation $x = (x', x_3)$, take the form

$$f_h(x) = (2\pi h)^{-1/2} e^{ikx_3/h} \int_0^{2\pi} e^{i\lambda(x_3)\langle x', \omega(\psi)\rangle/h} a(\lambda(x_3)\langle x', \omega(\psi)\rangle + kx_3, x_3, \psi)\, d\psi$$

When $a = a(\tau, \alpha, \phi) = \widetilde{a}(\alpha, \phi)$ and is even in α, we have

$$f_h(x) = \sqrt{\frac{2\pi}{h}} \widetilde{a}(|x'|, x_3) e^{ikx_3/h} J_0\big(\frac{\lambda(x_3)|x'|}{h}\big) + \mathcal{O}(h)$$

which justifies the name "Bessel beams", see [DoMaNa2].

(6) Airy-Bessel beams, which are known also as Berry-Balasz solution ([BeBa], [DoMaNa1]) of the paraxial approximation of wave equation in 3-D, with initial manifold

$$\Lambda = \{(x,p) \in T^*\mathbf{R}^3 : \ x = \begin{pmatrix} \varphi\omega(\psi) \\ \phi^2/2 \end{pmatrix}, \ p = \begin{pmatrix} \lambda\omega(\psi) \\ \phi \end{pmatrix}, \ \omega(\psi) = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}, \ \phi, \varphi \in \mathbf{R}\}$$

(7) Lagrangian distributions with a complex phase in the sense of Melin-Sjöstrand, see [MeSj] equivalently a complex germ in the sense of Maslov, which are superposition of coherent states $f_{(x_0,\xi_0)}(x;h) = \frac{1}{h^n} \exp(-\omega^2 \cdot (x - x_0)^2/2h) \exp(ix\xi_0/h)$, and Λ is a strictly positive Lagrangian manifold.

Examples (1) and (4) will be extensively studied when $n = 2$ and H_0 is positively homogeneous of degree $m \geq 1$ ($m = 1$ in Example (4)). Since f_h in Example (5) looks like a plane wave in direction x_3, one could expect Example (4) could be generalized to this case, once the eikonal coordinate has been found. Examples (6) and (7) requires some special treatment and will not be considered either.

Thus our main problem is to represent the asymptotic solution of

$$(H(x, hD_x; h) - E)u_h(x) = f_h(x), \ u_h(x) = E_+(h)f_h(x) = \int_0^\infty e^{-it(H-E)/h} f_h(x)\, dt \tag{1.5}$$

in the most explicit form. Here $e^{-it(H-E)/h}$ is the propagator, so we need consider first Cauchy problem

$$hD_t v_h + (H(x, hD_x) - E)v_h = 0, \ v_h|_{t=0} = f_h \tag{1.6}$$

For general $H(x, hD_x; h)$ we adapt the constructions of [MelUhl] to the semi-classical setting ; in the case of homogeneous Hamiltonians of degree $m \geq 1$ we present formulas more globally.

Global existence of an outgoing solution at infinity provided suitable hypotheses on Hamilton vector flow, such as Lagrangian intersection, the non-trapping (in x-space) and the non-return conditions, is quite involved. The main strategy in case of conormal distributions, has already been set up

in [MelUhl] and received a more systematic treatment in relatively recent works [Ca], [Bo], [KlCa]. Here $H(x, hD_x)$ is a semi-classical Schrödinger operator $-h^2\Delta + V(x)$, with V non trapping at energy 0, and $V(x_0) \leq 0$. On the other hand, f_h a localized function at x_0. In this case, the non-return condition is characterized by a relation on $T^*_{x_0}M$

$$\mathcal{R} = \{(p, \eta) : p^2 + V(x_0) = \eta^2 + V(x_0) = 0;\ \exists t > 0 :\ X(t, p) = x_0, P(t, p) = \eta\} \tag{1.7}$$

where $\big(X(t, p), P(t, p)\big) = \exp tv_H(x_0, p)$ is the trajectory issued from $(x_0, p) \in T^*M$. The dimension of $\mathcal{R}$ as a submanifold of $T_{x_0}M$ is taken into account. Outgoing solutions u_h, are characterized by Sommerfeld radiation condition of the form

$$\frac{x}{|x|}\nabla_x w(x) + i\sqrt{-V(x_0)}w(x) \to 0,\ |x| \to \infty, \quad n \geq 2 \tag{1.8}$$

where $w = \lim_{h\to 0} w_h$, $w_h(x) = h^{n/2}u_h(hx)$, is the unique solution of $(-\Delta + V(x_0))w = f$, and $f_h(x) = h^{-n/2}f\big(\frac{x}{h}\big)$. It relates in a non trivial way the behavior of u_h at infinity with the value of the potential V at x_0. Sommerfeld radiation condition requires careful estimates on $U_h(t) = e^{-itH/h}$, or $U_h(t)f_h$, along with a discussion according to the relative magnitude of t and h. The proof consists, roughly speaking, in testing $U_h(t)f_h$ against some fixed $\phi \in \mathcal{S}(\mathbf{R}^n)$, and then show that $\langle u_h, \phi\rangle \to \langle w, \phi\rangle$ as $h \to 0$. In particular one needs to know asymptotics of u_h in a h-dependent neighborhood of Λ.

In this paper instead, given $H(x, hD_x; h)$ and the initial Lagrangian manifold Λ, we content ourselves to present, in the sense of *formal asymptotics*, a "closed form" for the solution of (1.5) in term of Maslov canonical operator for bi-Lagrangian distributions. By formal asymptotics [Ler] we mean that, in principle, our approximate solution has no reason to be equal to $E_+(h)f_h$ mod $\mathcal{O}(h)$. In practice however numerical simulations show that Maslov canonical operator provides an excellent agreement with the "exact solution".

1.1 Lagrangian intersection and microlocal Green functions.

As in [MelUhl] our constructions make use of symbolic calculus adapted to Lagrangian intersection. So we need first to translate some notions relative to asymptotics with respect to smoothness (or "standard Pseudo-differential Calculus"), to the framework of asymptotics with respect to small parameter h (or "h-Pseudo-differential Calculus"), in particular to allow for general phase functions, without homogeneity in the momentum variable.

Let $\iota_0 : \Lambda_0 \to T^*M$ be a smooth embedded Lagrangian manifold, and $\iota_1 : \Lambda_1 \to T^*M$ be a smooth embedded Lagrangian manifold with smooth boundary $\partial\Lambda_1$ (isotropic manifold). Following [MelUhl] we say that (Λ_0, Λ_1) is an *intersecting pair* of Lagrangian manifolds iff $\Lambda_0 \cap \Lambda_1 = \partial\Lambda_1$ and the intersection is *clean*, i.e.

$$\forall z \in \partial\Lambda_1 \quad T_z\Lambda_0 \cap T_z\Lambda_1 = T_z\partial\Lambda_1$$

On the set of intersecting pairs of Lagrangian manifolds we define an equivalence relation by saying that $(\Lambda_0, \Lambda_1) \sim (\Lambda'_0, \Lambda'_1)$ iff near any $z \in \partial\Lambda_1$, $z' \in \partial\Lambda'_1$, there is a symplectic map κ such that

$\kappa(z) = z'$, and a neighborhood $V \subset T^*M$ of z such that $\kappa(\Lambda_0 \cap V) \subset \Lambda_0'$, $\kappa(\Lambda_1 \cap V) \subset \Lambda_1'$. We will call the equivalence class a *Lagrangian pair*. The following result readily extends [MelUhl, Prop.1.3] :

Lemma 1.1: *All intersecting pairs of manifolds in T^*M are locally equivalent. More precisely near each $z \in T^*M$, there exists a canonical map $\kappa : T^*M \to T^*\mathbf{R}^n$ such that $\kappa(z) = (0,0)$, $\kappa(\Lambda_0 \cap V) \subset T_0^*\mathbf{R}^n$ (vertical fiber at 0), and $\kappa(\Lambda_1 \cap V) \subset \Lambda_+^0$, Λ_+^0 being the flow-out of $T_0^*\mathbf{R}^n$ by the Hamilton vector field $v_{\xi_n} = ((0,\cdots,0,1),0)$ of ξ_n, passing through some $(0;\xi) = (0;(\xi',0))$ i.e.*

$$\Lambda_+^0 = \{(x,\xi) \in T^*\mathbf{R}^n : x = (0,x_n), \xi = (\xi',0), x_n > 0\} \tag{1.9}$$

As a warm-up, let us construct $u_h = E_+(h)f_h$ mod $\mathcal{O}(h^\infty)$ for hD_{x_n}, and f_h as in (1.2) with a compactly supported. By a gauge transformation and a shift of the support of $\xi_n \mapsto a(x,\xi)$ in (1.1) we may assume $E = 0$. So we just need to compute a primitive of $f_h(x)$. Let $T > 0$ and $\theta_T \in C^\infty(\mathbf{R}_+)$ vanishing near $+\infty$ and $\theta_T(t) = 1$ for $t \le T$. We consider

$$u(x,h) = \frac{i}{h}\int_0^\infty \theta_T(t)\,dt \int^* e^{i(x'\xi'+(x_n-t)\xi_n)/h} a(x',x_n-t,\xi;h)\,d\xi \tag{1.10}$$

Provided $x_n \le T/2$ (say) integration by parts and a non-stationary phase argument in variables (t,ξ_n) show that

$$hD_{x_n}u_h(x) = f_h(x) + \mathcal{O}(h^\infty) \tag{1.11}$$

Note that by [Hö,Lemma 18.2.1] we could already assume $a = a(\xi;h)$. This will be crucial for the *compatibility condition* (see below).

1.2 Hypothesis and main results.

Our main goal is to represent (1.5) somewhat like (1.10) as a superposition

$$u_h(x) = \frac{i}{h}\int_0^\infty dt\big[K^h_{\Lambda_t}b\big](x) \tag{1.20}$$

where $[K^h_{\Lambda_t}b](x)$ is Maslov canonical operator associated with $\Lambda_t = \exp tv_{H_0}(\Lambda)$, and b an amplitude depending linearly from the amplitude a defining f_h. We may also to replace $\int_0^\infty dt\big[K^h_{\Lambda_t}b\big]$ by some "bi-canonical operator" $[K^h_{\Lambda,\Lambda_+}(\sigma,\sigma^+)]$ acting on pairs of symbols (σ,σ^+) depending linearly on b, resp. the *boundary-part* and the *wave-part* symbol of u_h, which we call a *bi-Lagrangian* (semi-classical) distribution.

We shall generally consider intersecting pairs (Λ,Λ_+), $\Lambda_+ \cap \Lambda = \partial\Lambda_+$, where

$$\Lambda_+ = \{(x,\xi) \in T^*M, \exists t \ge 0, \exists z = (y,\eta) \in \Lambda \cap \Sigma_E, (x,\xi) = \exp tv_{H_0}(z)\} \tag{1.21}$$

is the flow-out of Λ by v_{H_0} in $\Sigma_E = \{H(x,p = E\}$, provided v_{H_0} is transverse to Λ along $\partial\Lambda_+ = \Lambda \cap \Sigma_E$. For Example (4) however, there may be points on Λ_+ where transverse intersection fails (we call them *glancing points* by analogy with the problem of diffraction by obstacles, see Sect.5), so we miss some

informations on u_h nearby. We always assume that there is no finite motion on Σ_E, which allows to argue as in (1.10), namely

$$|z(t)| \to \infty \text{ as } t \to \infty \tag{1.22}$$

for any trajectory $z(t) = (X(t), P(t))$ issued from $\partial\Lambda_+$ at $t = 0$. For simplicity we restrict to the case where $\partial\Lambda_+$ is a compact, isotropic submanifold without boundary, which is certainly the case when $H_0(x, p)$ is elliptic. This restriction however is not essential [MelUhl] and our results should carry to the case where $H(x, hD_x; h)$ is the wave operator.

The non-return set $\mathcal{R}$ in (1.7) is irrelevant if we content ourselves with the asymptotics of $u_h(x)$ microlocally in a compact set outside $\partial\Lambda_+$. For instance if $\Lambda = T^*_{x_0}M$ we shall compute $u_h(x)$, locally uniformly in any compact set $K \subset M\setminus\{x_0\}$, as $h \to 0$. A first improvement would consist in removing only a h^δ-neighborhood of x_0 for some $0 < \delta < 1$, this we have sketched in [AnDoNaRo1,Thm.2].

For simplicity, we shall ignore throughout Maslov indices. Corrections to the asymptotics due to Maslov indices follow for instance easily from [Ar], [So], [Iv], [BaWe, Sect.4], [DoNaSh], [DoRo], [EstHagHedLitt].

Remark 1: Using Maupertuis-Jacobi correspondence we can transfer, by diffeormorphism, results related to an operator $H(x, hD_x)$ at a non critical energy surface $H(x, p) = E$, to another operator $\mathcal{H}(x, hD_x)$ with the same energy surface $\mathcal{H}(x, p) = \mathcal{E}$. For instance we can pass from Hamiltonian (1.25) below at energy E to Helmholtz Hamiltonian $\mathcal{H}(x, p) = p^2 + V(x)$ at energy $\mathcal{E} = 0$ if we choose $V(x) = -(E\rho(x))^{2/m} < 0$. In both cases, $\Lambda = T_{x_0}M$. Then we can pass to the water-waves operator, see e.g. [DoRo], [DoMiRo], [ReMiKaDo].

• **General case**.

In Sect.3, we translate the setting of classical Pseudo-differential Calculus elaborated in [MelUhl] to the semi-classical one, and make some statements more precise. We assume Lagrangian intersection, i.e. v_H is nowhere tangent to Λ. So by Lemma 1.1, near any $z \in \partial\Lambda_+$ we are reduced microlocally to the case where $\Lambda = T^*_0\mathbf{R}^n$, $\Lambda_+ = \Lambda^0_+$ and H_0 is the "model Hamiltonian" ξ_n. In Proposition 3.3 we show that $H(x; hD_x; h) - E$ can be taken microlocally near $\partial\Lambda_+$ to its normal form hD_{x_n} by conjugating with a h-FIO. To avoid difficulties with the non-return condition) we will assume $\Lambda = T^*_{x_0}M$. We have

Theorem 1.2: *Let $H(x, p; h) \in S^0_m(T^*M)$. Assume $\Lambda = T_{x_0}M$ and $\Sigma_E = \{H_0(x, p) = E\}$, where E is a non critical energy level, intersect transversally along the compact isotropic manifold $\partial\Lambda_+ = \partial\Lambda_+(E)$. Assume $|z(t)| \to \infty$ as $t \to \infty$ for all initial conditions $(X(0), P(0)) \in \partial\Lambda_+$. Let also f_h a semi-classical distribution microlocalized on Λ. Then there is $\varepsilon_0 > 0$ such that the equation $(H(x, hD_x) - E)u_h(x) = f_h(x) + \mathcal{O}(h^2)$ can be solved in the form (1.20) for $0 < |x - x_0| < \varepsilon_0$, locally uniformly for $h > 0$ small enough.*

Moreover there are symbols (σ, σ^+), such that in local coordinates $\sigma(\xi; h) = \frac{1}{\xi_n}b(\xi; h)$, $\sigma^+(\xi; h) = 2i\pi b(\xi', 0; h)$ verify the compatibility condition

$$\lim_{\xi_n \to 0} \xi_n \sigma(\xi', \xi_n; h) = \sigma^+(\xi', 0; h) \tag{1.23}$$

and such that u_h can be represented by Maslov "bi-canonical operator", see (3.33) as

$$u_h(x) = [K^h_{\Lambda,\Lambda_+}(\sigma,\sigma^+)](x)$$

Moreover we have the commutation relation

$$H(x,hD_x;h)\big[K^h_{\Lambda,\Lambda_+}(\sigma,\sigma^+)\big](x;h) = \big[K^h_{\Lambda,\Lambda_+}((H-E)\sigma,0)\big](x;h) + \mathcal{O}(h^2) = f_h + \mathcal{O}(h^2) \tag{1.24}$$

We stress that this representation is not uniform in x near x_0, see however [AnDoNaRo1, Theorem 1]. The remainder term $\mathcal{O}(h^2)$ in (1.24) can be improved by considering higher order transport equations and using the compatibility condition (1.23).

• Hamiltonian H_0 is homogeneous of degree m with respect to p and Λ is the "vertical plane $T_{x_0}M$.

The previous result is not very useful from the point of vue of applications, since it does not provide a "close form" for u_h. Much more information is available when Hamiltonian H_0 is homogeneous with respect to p, due to the relations $\langle P(t,\psi),\dot{X}(t,\psi)\rangle = mE$ and $\langle P(t,\psi),X_\psi(t,\psi)\rangle = 0$ (Huygens principle). We have chosen coordinates (λ,ψ) on Λ, ψ being a $n-1$-vector of coordinates along $\partial\Lambda_+$, and λ a "radial coordinate" related with the energy parameter τ in $H = E-\tau$. Lagrangian intersection always holds on Λ, for if $\partial_p H(x_0,p) = 0$, Euler identity gives $0 = \langle \partial_p H(x_0,p),p\rangle = mE$ which contradicts $E \neq 0$.

Note that such a symbol is not suitable for Pseudo-differential Calculus when m is not an even integer, because of the singularity at $p = 0$, but this is harmless if $(x_0,0) \notin \mathrm{WF}_h(f_h)$.

A particular case of these Hamiltonians is the "conformal metric" given by

$$H(x,p) = |p|^m \frac{1}{\rho(x)} \tag{1.25}$$

where ρ be a smooth positive function on M, $m \geq 1$. We shall pay a great attention to this Hamiltonian, for which computations are most explicit.

Recall that a *focal point* for a Lagrangian embedding $\iota : \widetilde{\Lambda} \to T^*M$ (where $\widetilde{\Lambda}$ is either Λ, $\Lambda_t = \exp tv_H(\Lambda)$, or Λ_+) is a point $z \in \widetilde{\Lambda}$ such that $d\pi_x : \widetilde{\Lambda} \to M$ has rank $< n$. The set of focal points is denoted by $\mathcal{F}(\widetilde{\Lambda})$. Recall also that there exists a covering of $\widetilde{\Lambda}$ by canonical charts U where $\operatorname{rank} d\pi_x(z) \geq k$ for all $z \in U$. These U for which $k = n$ are called *regular* charts, and those for which $k < n$ *singular* charts.

At least for $0 < |x - x_0|$ small enough i.e. $0 < t$ small enough, using at most 2 canonical charts (depending if $X_\psi(t,\psi) = 0$ or not) we can construct v_h solution of Cauchy problem (1.6), and hence u_h by integration with respect to t. To this end, we introduce eikonal coordinates (see Sect.2), and present the solution of Hamilton-Jacobi equation parametrizing Λ_+ as a linear function in x (see (4.5)):

$$\Phi(x,t,\psi,\lambda) = mEt + \lambda\langle P(t,\psi), x - X(t,\psi)\rangle$$

The "θ-variables" are thus (t,ψ,λ). Note that by Hamilton-Jacobi construction we recall in Sect.2, Φ is defined unambiguously only on Λ_+. Another, but less convenient choice for Φ is given in (4.4).

This yields (see (3.8) and (4.14)) an (inverse) density on Λ_+, given by $F\big[\Phi, d\mu_+\big] = mE\det(P, P_\psi)$, and non vanishing precisely when $\Phi(t,x,\psi,\lambda)$ is a non-degenerate phase function, see Propositions 4.1 and 4.4. This holds at least for small t. For larger t, the non-vanishing of $mE\det(P,P_\psi)$ will only be assumed.

Theorem 1.3: *Let $n=2$, and $H_0(x,p)$ be positively homogenenous of degree $m \geq 1$. Let $r_0 > 0$ such that*

$$\forall t > 0,\ \forall \psi \in [0, 2\pi]:\ \big[|X(t,\psi)| < r_0 \implies \det(P(t,\psi), P_\psi(t,\psi)) > 0\big]$$

(this holds for $t > 0$ small). Then there is $\varepsilon_0 > 0$ such that the equation $(H(x, hD_x) - E)u_h(x) = f_h(x) + \mathcal{O}(h^{3/2})$ can be solved in the form

$$u_h(x) = \frac{i}{h}\int_0^\infty dt \int e^{i\Phi(x,t,\psi,\lambda)/h} b(x,t,\psi,\lambda)\, d\psi\, d\lambda \tag{1.26}$$

for $0 < |x - x_0| < \varepsilon_{x_0}$, locally uniformly for $h > 0$ small enough.

Moreover we can decompose

$$\int_0^\infty dt\big[K^h_{\Lambda_t} b\big](x) = \int_0^\infty dt\big[K^h_{\Lambda_t}(\chi_1 b)\big](x) + \int_0^\infty dt\big[K^h_{\Lambda_t}(\chi_2 b)\big](x) \tag{1.27}$$

where $\chi_1 + \chi_2 = 1$ is a partition of unity subordinated to a chart U_1 where $X_\psi \neq 0$ and a (singular) chart U_2 where $X_\psi = 0$, $\chi_2 \equiv 1$ near $|X_\psi| \leq \frac{1}{3}$, and $\chi_1 \equiv 1$ near $|X_\psi| \geq \frac{2}{3}$.

We do not attempt here to formulate the result in terms of bi-canonical Maslov operator as in Theorem 1.2. In Sect.2.3 we discuss (somewhat informally) the case where x has several pre-images under $\pi_x : \Lambda_+ \to M$.

Remark 2: An alternative but similar way for solving (1.6) is to conjugate $hD_t + H(x, hD_x) - E$ by Weyl operator

$$\mathcal{T}_h(t,\psi)(x, hD_x) = \exp\big[i\big(\langle P(t,\psi), x\rangle - \langle X(t,\psi), hD_x\rangle\big)/h\big]$$

See also [Iv,Sect.2.3], [GraZa], [Rob].

By constructing (1.5)-(1.6) we mean also determining the Lagrangian singularities of u_h. They are revealed when reducing the number of "θ-variables" in the oscillating integral $\int_0^\infty dt\big[K^h_{\Lambda_t} b\big](x)$. We get virtually any kind of Lagrangian singularity, but because of homogeneity of H with respect to p, it is convenient to introduce another classification in T^*M, which clarifies the construction of u_h. This plays also a role when Lagrangian intersection is not trnasverse, see Sect.5.

Definition 1.4: *Let H be positively homogeneous of degree m with respect to p. We call a point $z = (x,p)$ such that $-\partial_x H(z) \neq 0$ an ordinary point if $\langle -\partial_x H(z), p\rangle \neq 0$, and a special point otherwise. If $-\partial_x H(z) = 0$ we call z a residual point.*

Examples 1.2:

(1) For Tricomi Hamiltonian, $H(x,p) = x_2 p_1^2 + p_2^2$, the residual points are those for $p_1 = 0$, the special points those for $p_1 \neq 0$ but $p_2 = 0$, and the ordinary points those for $p_1 p_2 \neq 0$. Tricomi operator is used as a model for diffraction. This model extends to the 3-D case as $H(x,p) = -p_2^2 + p_1 p_3 + x_2 p_1^2$.

(2) For Métivier Hamiltonian, $H(x,p) = p_1^2 + (x_1^2 + x_2^2)p_2^2$, the residual points are given by $p_2 = 0$ or $x = 0$, the special points by $p_2 \neq 0$ and $x \neq 0$, but $\langle x, p\rangle = 0$, and the ordinary points by $\langle x, p\rangle p_2 \neq 0$. Métivier operator provides a counter-example for analytic hypo-ellipticity.

(3) Let H be the "conformal metric" given by (1.25). The residual points are the critical points of ρ ; at a special point, $\langle \nabla\rho, p\rangle = 0$, i.e. v_H is tangent to the level curves of ρ. This is our main example here.

We denote by $\mathcal{S}(\Lambda_+)$ the set of special points on Λ_+. We shall partition points $z \in \Lambda_+$ according to the following values: (1) z is a focal (or non-focal) point; (2) z is a special (or ordinary, or residual) point. Thus each canonical chart splits again into ordinary, special or residual points. Assume $n = 2$ for simplicity.

Let $x = X(t,\psi)$, and $\langle -\partial_x H(x,p), \partial_p H(x,p)\rangle \neq 0$ at $(x,p) = z(t) = (X(t,\psi), P(t,\psi))$. Then

$$\partial_t \Phi(t,x,\psi,\lambda = 1) = 0 \implies \partial_t^2 \Phi(t,x,\psi,\lambda = 1) \neq 0 \tag{1.31}$$

so that we can perform asymptotic stationary phase in t to simplify (1.20) at $x = X(t,\psi)$. This holds when H is of the form (1.25) and $z(t)$ is an ordinary point. Likewise, if several values of parameters t contribute, we sum over such t's. Then $\Phi(x,t,\psi,\lambda = 1)$ reduces to a phase function $\Psi(x,\psi)$ and we can further reduce the number of variables in a standard way, according to the fact that $z(t)$ is a focal point or not. When $\langle -\partial_x H(x,p), \partial_p H(x,p)\rangle = 0$ but $-\partial_x H(x,p) \neq 0$, the situation looks like (1.10) and we have

$$\partial_t \Phi(t,x,\psi,\lambda) = \partial_\lambda \Phi(t,x,\psi,\lambda) = 0 \implies \det \Phi''_{(t,\lambda),(t,\lambda)}(t,x,\psi,\lambda) \neq 0 \tag{1.32}$$

where λ is constrained to be equal to 1 on the critical set C_Φ, so that we can perform asymptotic stationary phase with respect to (t,λ). Then $\pi_x : \Lambda \to T^*\mathbf{R}^n$ has rank 1 if $z(t)$ is a special point or rank 2 otherwise, see Proposition 4.3(ii). Note that Λ_+ never turns vertical, since $\partial_p H(x,p) \neq 0$ on $H = E$.

In this generality, we only succeed (see Proposition 4.3) to describe the contribution to (1.20) (by asymptotic stationary phase) of short times t ("near field"), i.e. so long as $F\big[\Phi, d\mu_+\big] = mE\det(P, P_\psi) \neq 0$, But this is actually sufficient to compute u_h microlocally near Λ, when $x \neq x_0$.

• **H_0 is the "conformal metric" $H_0(x,p) = \frac{|p|^m}{\rho(x)}$ and Λ is the "vertical plane".**

Using that $P(t,\psi)$ is parallel to $\partial_p H(x,p)$ we get more complete results in this case. If ρ is bounded, a sufficient condition for (1.22) is that energy E is non trapping, see [GeSj]. In that case, for all $R > 0$, there is $g \in C^\infty(T^*M)$ (so called *escape function*) such that $v_H g(x,p) \geq 0$ and $v_H g(x,p) = 1$ for $x \in B(0,R)$. The following stronger condition excludes natural potentials having a limit as $|x| \to \infty$, as shows the example $\rho(x) = \rho_0 + \langle x\rangle^{-\varepsilon}$. However it turns out to be convenient from the point of vue of Definition 1.4.

Definition 1.5 We say that ρ has the *defocussing condition* iff

$$G(\rho)(x,p) = \langle \nabla^2\rho(x)\cdot p, p\rangle + \frac{|\nabla\rho(x)|^2}{m\rho(x)}|p|^2 > 0,\ \forall (x,p) \in \{H_0 = E\} \tag{1.35}$$

In particular if ρ has a critical point, this is a non degenerate minimum. Under defocussing condition (1.35), if $z(s)$ is a special point along some bicharacteristic γ issued from Λ, then for all $t > s$, $z(t) \in \gamma$ is an ordinary point. We expect that (1.35) is related to the non-trapping condition (1.22), and provides an information on the "return set" in (1.7), e.g. $\mathcal{R} = \emptyset$. It also implies that special and residual points are "exceptional" compared to ordinary points, in the same way singular points are "exceptional" with respect to regular points. This allows a natural subdivision of canonical charts into ordinary and special (residual) points, so we can speak of a regular-ordinary chart, or regular-special chart and so on. The main results are summarized in Proposition 4.8.

Remark 1.1: In case (1.25) with $m = 1$ and $f_h = 0$ (scattering problem), the asymptotic solution of $H(x, hD_x)u_h = Eu_h$ has been constructed in [DoMaNaTu1], Example 6, in term of Bessel functions.

• **Λ is the "cylinder"** (1.3)**, and** $m = 1$.

To the former "θ-variables" (t, ψ, λ), one has now to add φ as a parameter. Note that Λ has a Lagrangian singularity at $\varphi = 0$. This is the most technical part of the paper, and the results are only partial, because we are ignoring glancing points.

In case of hamiltonian (1.25), the initial condition constraints somehow ρ to be radial on $\partial\Lambda_+$, namely $\nabla\rho\big(\varphi\omega(\psi)\big), \omega^\perp(\psi)\rangle = 0$. Thus $z = (x, p) \in \partial\Lambda_+$ is a special point iff $\nabla\rho(\varphi\omega(\psi)) = 0$ but $\varphi \neq 0$. Because of this constraint, implying a relation between φ and energy parameter τ, we are led to assume $m = 1$.

Due to possible glancing points, we cannot formulate a global result in term of Maslov canonical operator as in Theorem 1.3. In full generality, for an Hamiltonian positively homogeneous of degree 1 in the p variables, we cannot go much further than computing the phase and density on Λ_+, so results are most complete in case of the conformal metric. This simplifies further in case of a radially symmetric conformal metric. We refer to Sect.5 for detailed statements

1.3 Outline of the paper.

In Sect.2 we first construct eikonal coordinates on Λ_+ when $H_0(x, p)$ is positively homogeneous of degree m, and Λ is either the vertical plane, or the cylinder. Then we discuss some well-known facts about the extension of the solution of Cauchy problem (1.6) for large t. In particular we examine the case where there are several branches of Λ_+ lying over x, i.e. $\pi_x(X(t, \psi), P(t, \psi)) = x$, leading to Van-Vleck formula. Following [CdV], [GuSt] we then focus to the case when H_0 defines a metric, according to $m = 1$ (Finsler metric, or Randers symbol), or $m > 1$. Finally we make some general remarks concerning Maupertuis-Jacobi correspondence $(\mathcal{H}, \mathcal{E}) \sim (H, E)$, leaving open the problem of constructing simultaneously Maslov canonical operator (1.20) relative to both Hamiltonians.

In Sect.3 we prove Theorem 1.2. We start to recall some basic facts on Maslov theory. Then we sketch its generalisation to bi-Lagrangian distributions, following mainly [MelUhl], where H is taken microlocally to its normal form hD_{x_n} on a non critical energy surface. We make also the results of [MelUhl] more precise and adapted to asymptotics with respect to the small parameter h. Thus we can construct $u_h(x)$ microlocally outside $\partial\Lambda_+$, and locally uniformly with respect to h. We end up by computing explicitely u_h when $n = 2$, $H = -h^2\Delta$ and f_h is compactly supported, and verify that u_h can be written as the sum of 2 terms, microlocally supported on Λ and Λ_+ respectively.

We start in Sect.4 to recall from [DoMaNa2], [DoNaSh] the matrix $\widetilde{n}\times\widetilde{n}$ matrix $\mathcal{M}(\widetilde{\phi},\widetilde{\psi})$ defined on a local chart of a Lagrangian manifold $\widetilde{\Lambda}$, whose determinant turns out to be the (inverse) density on $\widetilde{\Lambda}$. It will be most useful in Sect.5. Then we define the phase function from which compute directly the (inverse) density $F[\Phi, d\mu_+]|_{C_\Phi}$ on Λ_+. Lateron we restrict to the 2-D case. In Sect.4.3, assuming this density is non zero, or equivalently, that Φ is a non-degenerate phase function in the sense of Hörmander, we investigate some configurations of Λ_+ in T^*M (according to $X_\psi = 0$ or $X_\psi \neq 0$) and describe more closely the corresponding Lagrangian singularities (focal points) in the chart where $X_\psi \neq 0$. We relate focal points with ordinary, special or residual points. In Sect.4.4 we complete the 1:st order asymptotics by considering the transport equations, and prove Theorem 1.3. In Sect.4.5 we specialize further to the case of the "conformal metric", using also the defocussing condition (1.25), which allows a more complete description of focal points.

Sect.5 is the most technical and sketchy part, since we do not take glancing point into account. In Sect.5.1 we give necessary and sufficient for a point of the "cylinder" (1.3) be glancing with respect to v_{H_0}. In particular we show that a glancing point at $t = 0$ is also a special point. Then we describe the parametrization of $\partial\Lambda_+(\tau)$ provided this is a close manifold without boundary. We compute the matrix $\mathcal{M}(t, \varphi, \psi, \tau)$ we introduced already in Sect.4, and show that we should take $m = 1$ for its determinant identifies with the density on Λ_+. All computations should be carried in the extended phase-space $T^*(M \times \mathbf{R}_t)$.

In Appendix we prove the density is non vanishing near focal points in the case of the "conformal metric".

1.4 Some open problems.

• Semi-classical structure of the Green function outside a h^δ-neighborhood of Λ.

• Other types of initial Lagrangian manifolds, e.g. Bessel or Airy-Bessel beams.

• Structure of the Green function near residual points, in particular glancing points, where Lagrangian intersection fails to be transverse.

• Hyperbolic equations ($\partial\Lambda_+$ non compact)

• Case of multiple characteristics, involving Lagrangian manifolds with boundary $\partial\Lambda_+$ and corner $c\Lambda_+$ [MelUhl].

• Complex phases as in Remark 4.2.

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2. Hamiltonians and phase functions

In this Sect. we consider integral manifolds for positively homogeneous Hamiltonians on $T^*M\setminus 0$, which is the first step in constructing semi-classical Green kernels. We discuss first general facts (eikonal coordinates, Hamilton-Jacobi equation), but more specific points will be discussed in Sect. 4 and 5. Basic references are [Ar], [Hö], [GuSt].

2.1 Eikonal coordinates

We shall deal with elliptic positively homogeneous Hamiltonians of degree $m \geq 1$ with respect to p on the cotangent bundle $T^*M \setminus 0$ ($M = \mathbf{R}^n$ for simplicity), and eventually restrict to "conformal metrics" to get most explicit results. In this Sect. we write H for H_0.

Examples 2.1:

(1) $m = 2$ if $H(x,p)$ is a geodesic flow associated with a Riemannian metric $ds^2 = g_{ij}(x)\, dx^i \otimes dx^j$. In the Riemannian case, when $E = 1$, geodesics are parametrized by arc-length.

(2) $m = 1$ if $H(x,p)$ is a "Randers symbol", associated with a Finsler metric [Tay], [DoRo2] and reference therein.

(3) H is of the form (1.15) with $m \geq 1$. Hamilton equations $(\dot{x}, \dot{p}) = v_H(x,p)$ then read

$$\dot{x} = \partial_p H = m|p|^{m-1} \frac{1}{\rho(x)} \frac{p}{|p|}, \ \dot{p} = -\partial_x H = |p|^m \frac{\nabla \rho(x)}{\rho(x)^2} \tag{2.2}$$

Our most complete results hold for such Hamiltonians with $n = 2$.

First we recall some general facts about canonical coordinates near Lagrangian manifolds, see [DoMaNaTu1], [DoNaSh]. Let $\iota : \widetilde{\Lambda}_+ \to T^*M$ be a smooth embedded Lagrangian manifold. The 1-form $p\, dx$ is closed on $\widetilde{\Lambda}$, so is locally exact, and $p\, dx = dS$ on any simply connected domain U (so called canonical chart). Such a S is called an *eikonal* (or action) and is defined up to a constant. If $p\, dx = dS \neq 0$ on U, S can thus be chosen as a coordinate on U.

• *Case of the "vertical plane"*. Here Λ_+ is the flow-out of H with initial data on $\Lambda = T^*_{x_0}M$, see Example 1.1. Let also $\psi \in \mathbf{R}^{n-1}$ be smooth coordinates on $\partial\Lambda_+$, which we complete by τ, the dual coordinate of t, so that $\partial\Lambda_+$ is given in Λ by $\tau = 0$, and in Λ_+ by $t = 0$.

In the special case $H(x,p) = \frac{|p|^m}{\rho(x)}$, we have $P(\psi, \tau) = |P|_\tau \omega(\psi)$, with

$$|P|_\tau = (H\rho(0))^{1/m} = \big((E - \tau)\rho(0)\big)^{1/m} \tag{2.3}$$

Sections are defined as follows: for small τ, let $\Lambda_+(\tau)$ be the Lagrangian manifold in the energy shell $\tau + H(x,p) = E$ issued from Λ at $t = 0$. We consider the isotropic manifold $\partial\Lambda_+(\tau) = \Lambda \cap \Lambda_+(\tau)$, viewing $\Lambda_+(\tau)$ as a manifold with boundary. When $\tau = 0$, we simply write $\Lambda_+(0) = \Lambda_+$.

Let assume that $\iota : \Lambda_+ \to T^*M$ is an embedding. For $t \geq 0$, let $\Lambda_t = \exp tv_H(\Lambda)$, we have the group property $\Lambda_{t+t'} = \exp tv_H(\Lambda_{t'})$ for all $t, t' \geq 0$. We define in a similar way the family of isotropic manifolds $\partial\Lambda_t(\tau) = \exp t\partial\Lambda_+(\tau) = \Lambda_t \cap \Lambda_+(\tau)$.

We compute the eikonal S on $\Lambda_+(\tau)$ by integrating along a path in Λ (where $dx = 0$), connecting the base point (say $(x_0, 0)$) to $(x_0, P(\psi)) \in \partial\Lambda_+(\tau)$, $\psi \in \mathbf{R}^{n-1}$, followed by the integral curve $x = X(t,\psi), p = P(t,\psi)$ of v_H starting at $(x_0, P(\psi))$, where $dS = p\, dx|_{\Lambda_+(\tau)} = \langle P(t,\psi), dX(t,\psi)\rangle$. Because $\Lambda_+(\tau)$ is Lagrangian, S doesn't depend on the choice of the base point. Since $S(x_0, \psi) =$ Const. $= S_0$ on Λ, we get

$$\begin{aligned} S(t,\psi) &= S(x_0,\psi) + \int_{(0,\psi)}^{(t,\psi)} p\, dx|_{\Lambda_+} = S(x_0,\psi) + \int_{(0,\psi)}^{(t,\psi)} P(s,\psi,\tau)\, dX(s,\psi,\tau) \\ &= S_0 + \int_0^t \langle P(s,\psi,\tau), \dot{X}(s,\psi,\tau)\rangle\, ds \end{aligned} \tag{2.4}$$

By Hamilton equations and Euler identity $\langle P(s,\psi,\tau),\dot X(s,\psi,\tau)\rangle = \langle P,\partial_p H\rangle = mH = m(E-\tau)$. Now $S(t,\psi) = S_0 + m(E-\tau)t = S_0 + mHt$ is the action on Λ_+, and the eikonal coordinate is just $S(t,\psi) = mHt$ up to a constant S_0. Restricting to $\Lambda_+(\tau)$ we get $dS = d\big(m(E-\tau)t\big) = m\,(E-\tau)dt$, or

$$m\,(E-\tau)dt = \langle P(t,\psi,\tau), dX(t,\psi,\tau)\rangle = \langle P(t,\psi,\tau),\dot X(t,\psi,\tau)\rangle\,dt + \langle P(t,\psi,\tau), X_\psi(t,\psi,\tau)\rangle\,d\psi \tag{2.5}$$

and it follows (Huygens' principle) that

$$\langle P(t,\psi,\tau),\dot X(t,\psi,\tau)\rangle = mH, \quad \langle P(t,\psi,\tau),\partial_\psi X(t,\psi,\tau)\rangle = 0 \tag{2.6}$$

We will denote for short $(X,P) = \big(X(t,\psi,\tau),P(t,\psi,\tau)\big)$ or $X(t,\psi),P(t,\psi)$ when $\tau = 0$. This is called the *leading front*.

• *Case of the "cylinder"*. Here $\Lambda_+(\tau)$ is the flow-out of H at energy $E-\tau$ with initial data on (1.3). Computing the action we find $p\,dx|_\Lambda = d\varphi$. Along $\Lambda_+(\tau)$ we have as in (2.4)

$$\begin{aligned} S(t,\varphi,\psi,\tau) &= S(0,\varphi,\psi,\tau) + \int_{(0,\varphi,\psi)}^{(t,\varphi,\psi)} p\,dx|_{\Lambda_+} = S(0,\varphi,\psi,\tau) + \int_{(0,\varphi,\psi)}^{(t,\varphi,\psi)} P(s,\varphi,\psi,\tau)\,dX(s,\varphi,\psi,\tau) = \\ &= S_0 + \varphi + \int_0^t \langle P(s,\varphi,\psi,\tau),\dot X(s,\varphi,\psi,\tau)\rangle\,ds \end{aligned}$$

As before, $\langle P,\dot X\rangle = mH$, so on $\Lambda_+(\tau)$

$$S(t,\varphi,\psi,\tau) = \varphi + m(E-\tau)t + S_0 \tag{2.10}$$

Identifying the differential of S we get (omitting again variables)

$$\langle P,\dot X\rangle = mH,\ \langle P,\partial_\psi X\rangle = 0,\ \langle P,\partial_\varphi X\rangle = 1 \tag{2.11}$$

At this point it is important to notice that φ is a variable on Λ, but only a parameter on $\Lambda_+(\tau)$ because $\partial\Lambda_+(\tau)$ is $n-1$-dimensional. In case (1.15), as well as in the case of other special geometries, ψ turns out to be a variable both on Λ and $\Lambda_+(\tau)$. However, φ,ψ are again independent variables on $\Lambda_t = \exp tv_H(\Lambda)$. In Sect.5 we develop a slightly different point of vue, extending the phase-space to $T^*(M\times\mathbf{R})$, which amount to introduce variable τ as the dual variable of t, and change $\langle P(t,\psi,\tau), dX(t,\psi,\tau)\rangle$ to $\langle P(t,\psi,\tau), dX(t,\psi,\tau)\rangle + \tau\,dt$ accordingly. This enables to treat initially φ as a variable, which is eventually constrained by a relation between τ,ψ,φ (in case of (1.15) this relation connects only τ with φ).

2.2 Hamilton-Jacobi equation for small t and phase functions

Because of focal points in general we cannot readily find a phase-function $\Psi(x)$ such that $H(x,\partial_x\Psi(x)) = E$, so we obtain it as a critical value (with respect to an auxiliary time-variable t as in Hamilton equations) of a phase $\Phi(x,t)$. To do this we solve Hamilton-Jacobi equation (HJ) in

the extended phase space $T^*(M \times \mathbf{R})$, which is the suitable framework to vary (t, τ) as well (τ being set eventually to 0). So we look for a phase function $\Phi(x, t)$ satisfying

$$\partial_t \Phi + H(x, \partial_x \Phi) = E, \ \Phi|_{t=0} = \phi \tag{2.12}$$

with given ϕ (to be chosen lateron), and prescribed $\partial_t \Phi(x_0, 0) = \tau_0, \partial_x \Phi(x_0, 0) = \eta_0$ satifying $\tau_0 + H(x_0, \eta_0) = E$. By Hamilton Eq., $\tau = \tau_0$ is a constant of the motion. It is well-known (Hamilton-Jacobi theory, see [Hö,Thm6.4.5]) that (2.12) as a unique solution for small t. This is the generating function of the Lagrangian manifold the extended phase-space

$$\widetilde{\Lambda}_+ = \{p = \partial_x \Phi(x, t), \ \tau = \partial_t \Phi(x, t), \ x, t \in M \times \mathbf{R}_+\} \subset T^*(M \times \mathbf{R}) \tag{2.13}$$

constructed along the integral curves of v_H starting at $t = 0$ from the Lagrangian manifold Λ_ϕ in T^*M given by $p = \partial_x \phi$. Its section at fixed t, τ is the Lagrangian manifold

$$\Lambda_{\Phi,t,\tau} = \{p = \partial_x \Phi(x, t), \ x \in M \times \mathbf{R}\} \subset T^*M$$

which is simply the flow out $\Lambda_t(\tau)$ of $\{p = \phi'(x)\}$ in $H(x, p) = E - \tau$ at time t. We choose the initial condition to be the standard pseudo-differential phase function of the form $\phi(x) = x\eta$. Here η is a parameter, we choose so that $\phi'|_\Lambda = P(\widetilde{\psi})$ where $\widetilde{\psi}$ are coordinates on Λ, that could be taken of the form $\widetilde{\psi} = (\psi, \tau)$, ψ being coordinates on $\partial\Lambda_+$. Actually our construction of Φ in Sect.4.2 or 5.2 will not directly rely on Hamilton-Jacobi theory, but rather on the eikonal coordinate (2.4) or (2.10).

The phase $\Phi(x, t)$ has the property (for a general Hamiltonian) that along each of these curves

$$\Phi(x(t), t) = \phi(x) + \int_0^t \left[\langle \frac{\partial H}{\partial p}(x(s), p(s)), p(s)\rangle + \tau(s))\right] ds \tag{2.17}$$

which is $\phi(x) + (mH + \tau)t$ for H positively homogeneous of degree m. This is also the action $\int_{(0,x_0)}^{(t,x)} L(q(s), \dot{q}(s))\, ds = \int_{(0,x_0)}^{(t,x)} \langle p, dq\rangle - H\, dt$, where the integral is computed along an integral curve of v_H from x_0 to x, see e.g. [Ar2,Sect.46].

2.3 The phase functions "in the large" and the semi-classical Cauchy problem

We discuss the case of the vertical plane $\Lambda = T^*_{x_0}M$, which reduces to standard variational problems in the space variable. See also [LiYau] for advanced results on the parabolic Schrödinger equation.

Let θ parametrize the initial condition ϕ in (2.12). In our case we can take $\theta = (\psi, \lambda)$ as local coordinates on Λ near $\partial\Lambda_+$ (eventually t will be added to the "θ"-variables since we require $\tau = \partial_t \Phi = 0$). Assume that for $\tau = 0$ the initial surface, $\partial\Lambda_+ \subset \Sigma_E$ is compact and of the form $(x, p) = (X(\psi) = 0, P(\psi) = \eta))$, $\psi \in \mathbf{S}^{n-1}$. We consider the map $(t, \psi) \mapsto x = \pi_x \exp tv_H(x_0, P(\psi))$, or which is the same, $(t, \eta) \mapsto x = \pi_x \exp tv_H(x_0, \eta)) = \mathrm{Exp}^t_{x_0} \eta$.

So far we have described the phase function when "moving along" Λ_+ for small t. Thus the critical point of $(t, \theta) \mapsto \Phi(x, t, \theta)$, is such that $x = X(t, \theta)$. A "dual" point of vue is to fix x and find the set of (t, θ) with $x = X(t, \theta)$ with $(x, t, \theta) \in C_\Phi$. In the Riemannian case ($m = 2$) this is related

to the problem of geodesic completeness, which holds locally. Namely if $|x|$ is small enough, there is a unique (t,η) such that $x = \mathrm{Exp}_{x_0}^t \eta$. This holds globally if the Riemaniann manifold ($m = 2$) is geodesically convex. Otherwise, the "inverse map" $x \mapsto (t,\eta)$ may be multivalued.

It is well known [CdV, pdf p.132] that the global geodesic convexity can be relaxed (locally) to a non-degeneracy condition. Namely, let H be associated with a Lagrangian $\mathcal{L}(x,\dot{x})$ strictly convex with respect to $\dot{x}$, in particular if H is positively homogeneous of degree $m > 1$ with respect to p.

Let $\mathcal{K} \subset \mathbf{R}_p^n$ be a compact set, which will be identified with the support of $a(p;h)$ in (1.2) (after eliminating x). For fixed (x,t) we make the generic assumption :

(2.18) *For all $\eta \in \mathcal{K}$ such that $x = \mathrm{Exp}_{x_0}^t \eta$, the map $\mathbf{R}^n \to \mathbf{R}^n$, $\xi \mapsto \mathrm{Exp}_{x_0}^t \xi$ is a local diffeomorphism near η : in other terms, x_0 and x are not conjugated along any trajectory that links them together within time t, with initial momentum ξ.*

The set of such (x,t) is an open set $\Omega_{x,t} \subset \mathbf{R}^{n+1}$, and by Sard's Theorem, its complement has Lebesgue measure 0.

Fixing (x,t), (2.18) implies by Morse theory that $\eta \mapsto x = \mathrm{Exp}_{x_0}^t \eta$ has a discrete set of pre-images $\eta \in \mathcal{K}$.

Fixing x, consider now the pre-images of $(t,\eta) \mapsto x = \mathrm{Exp}_{x_0}^t \eta$. It can happen that the integral manifold of v_H has several sheets over x, so several values of t can contribute to the same $x = X(t,\psi)$. However, under the non-trapping condition $|X(t,\psi)| \to \infty$ as $t \to \infty$, there is again, generically, a finite number of such t_j. Namely, it suffices that (2.18) holds with a time T such that for $t \geq T$, $X(t,\psi)$ will never coincide again with x. Moreover these t_j are non-degenerate critical points of $t \mapsto \Phi(x,t,\eta)$. This holds in particular when x_0 and x are connected by (possibly several) minimal geodesics for the Riemannian metric associated to H, each indexed by some η_α.

At least for small t we seek for a solution of (1.6) of the form

$$v_h(t,x) = \int^* e^{i\Phi(t,x,p)/h} b(t,x,p;h)\, dp \tag{2.20}$$

where the phase function Φ is as above with initial condition $\Phi|_{t=0} = xp$; the symbol $b(t,x,p;h)$ verifies $b(0,x,p;h) = a(x,p;h)$, and solves some transport equations along the integral curves of v_H. One may address the problem of a semi-classical "close form" of (2.20), i.e. of performing the integration with respect to p, so that the final expression is of WKB type. Under Assumption (2.18) the answer to this problem is given by Van Vleck formula [CdV, pdf p.132] which gives u_h as a finite sum

$$\sum_\alpha \frac{A(\eta_\alpha)}{\sqrt{\mathrm{Jac}_{x_0}(\eta_\alpha)}} e^{i\Phi(t,x,\eta_\alpha)/h} e^{-i\pi\,\mathrm{ind}(\gamma_\alpha)/2} \tag{2.21}$$

(at leading order in h) indexed by all $\eta_\alpha \in \mathrm{supp}\, A$ such that $\mathrm{Exp}_{x_0}^t \eta = x$. Here A is the principal part of a (where we have eliminated the dependence in x), $\mathrm{Jac}_{x_0}(\eta_\alpha)$ is the Jacobian of $\mathrm{Exp}_{x_0}^t$ at η_α and $\mathrm{ind}(\gamma_\alpha)$ Morse index of the integral curve $s \mapsto \exp s v_H(x_0,\eta_\alpha)$, $s \in [0,t]$. In other words, under Assumption (2.18) it suffices to use only non singular charts on Λ_+ over x, and the solution is expressed in term of finitely many oscillating functions.

When Assumption (2.18) is not met, i.e. $x = x_*$ is conjugated to x_0, then there is at least one focal point (x_*, p_*) over x_* in Λ_+. The construction of the canonical operator (see Sect.3) necessarily uses a singular chart in a neighborhood of (x_*, p_*), and the solution in a neighborhood of x_* involves not only simple oscillating functions corresponding to nonsingular charts as in (2.21) (if any) but also an integral of an oscillating function over some of the momenta (or "θ-variables"). The total number of singular and nonsingular charts over x_* however remains finite, and so (generically) only a finite sum of integrals and simple oscillating functions contribute (one summand per each chart). See Sect.3 for more details.

2.4 Distances and generating functions

When Λ is the vertical plane, the phase function Φ is related to the "distance" to x_0 for the "metric" implied by H_0, which is of special interest. We make here some general remarks, mainly following [CdV], [GuSt].

In Sect.4-5 we shall discuss how to parametrize, by a non degenerate phase function, the flow of v_H out of some Lagrangian plane, when H is positively homegeneous of degree m. It includes the case $m = 1$ which plays an important role because of Finsler metrics. So we begin with a general discussion on corresponding symplectic maps.

Let H be a positively homogeneous Hamiltonian of degree m with respect to p, defined on $T^*M\backslash 0$, and $\Gamma \subset T^*M \setminus 0 \times T^*M \setminus 0$ be the graph of $\exp v_H$ (time-1 flow). Recall from [GuSt,Chap.5, formula (5.6) and Thm 5.4.1] that

$$(\exp v_H)^*(p\,dx) - p\,dx = (m-1)dH \tag{2.29}$$

Integrating over a path $\gamma_E \subset \{H = E\}$, we recover the fact that $\int_{\gamma_E} (\exp v_H)^*(p\,dx) = \int_{\gamma_E} p\,dx$.

So when $m = 1$, not only the 2-form, but also the 1-form $p\,dx$ are preserved by v_H. In this case, v_H is actually the lift of a vector field on M. When $m > 1$, formula (2.29) gives a generating function for Γ under the following assumption (which however might never be verified when $m \neq 2$) :

(2.30) *Let $\pi_{M\times M} : T^*(M \times M) \to M \times M$ be the natural projection, and assume $\pi_{M\times M} : \Gamma \to M \times M$ is a diffeomorphism, i.e. for all $(x, y) \in M \times M$, there is a unique $\xi \in T^*_x M$ such that $y = \pi_M \exp v_H(x, \xi)$.*

Then we say that Γ is *horizontal*. In case of a geodesic flow ($m = 2$) (2.30) holds true when M is geodesically convex. Provided (2.30), Γ has a generating function χ, i.e. $d\chi = \mathrm{pr}_2^*(p\,dx) - \mathrm{pr}_1^*(p\,dx)$, where $\mathrm{pr}_i : T^*(M \times M) = T^*M \times T^*M \to T^*M$ is the projection onto the i:th factor, and $\mathrm{pr}_1 \circ (\pi_{M\times M}|_\Gamma)^{-1}$ is a diffeomorphism $M \times M \to T^*M$. Moreover we can then represent χ as

$$\chi = \left(\mathrm{pr}_1 \circ (\pi_{M\times M}|_\Gamma)^{-1}\right)^*(m-1)H \tag{2.31}$$

In case of the geodesic flow ($m = 2$) $\chi(x, y) = \frac{1}{2}\,\mathrm{dist}(x, y)^2$. Formula (2.29) is related to *exact symplectic twist maps* as follows. An exact symplectic twist map [Ar], [Ka], [GuSt], [CdV7] $F : T^*M \to T^*M$ is a symplectic map with a generating function $S_1 : M \times M \to \mathbf{R}$, $(x, X) \mapsto S_1(x, X)$ which satisfies

$$F^*(p\,dx) - p\,dx = P\,dX - p\,dx = dS_1(x, X) \tag{2.32}$$

(p,x) and (P,X) are related by $p=-\partial_x S_1, P=\partial_X S_1$. In notation S_1 the subscript 1 refers to time-1 flow. In case $H(x,p)=p^2$ (flat metric on $\mathbf{R}^n$), comparing (2.32) with (2.29), i.e. $dS_1=dH$, we get $S_1(x,X)=\frac{1}{4}(x-X)^2$, and more generally, if $H(x,p)=|p|^m$, with $m>1$, $S_1(x,X)=\left(\frac{m-1}{m}\right)^{m/(m-1)}|x-X|^{m/(m-1)}$.

Again, S_1 is not well defined when $m=1$. More generally $F(x,y)$ coincides with $\chi(x,y)$ above for the geodesic flow.

For Hamilton-Jacobi (HJ) equation we have the following Proposition, extending (2.15) for large t. Assume H is associated with a Lagrangian convex with respect to $\dot{x}$. Let $x_0,y_0\in M$ be non conjugate points along an extremal curve $\gamma_0(t)$ such that $x_0=\gamma_0(0)$ and $y_0=\gamma(t_0)$, and $(x_0,\xi_0),(y_0,\eta_0)$ the corresponding points in T^*M.

Proposition 2.2 [CdV,Thm 14, pdf p.45]: *Let (t_0,x_0,y_0) be as above. Then for any (x,y) close to (x_0,y_0), and t close to t_0, there is a unique extremal curve γ such that $x=\gamma(0)$ and $y=\gamma(t)$. Let $\widetilde{S}(t,x,y)$ be the action along these curves (minimizing the Lagrangian action) This is a generating function for the Hamiltonian flow near (x_0,ξ_0), verifying HJ equation*

$$\partial_t\widetilde{S}+H(y,\partial_y\widetilde{S})=0 \tag{2.33}$$

This is verified in the Riemannian case $\widetilde{S}(t,x,y)=\frac{F(x,y)}{2(1+t)}=\frac{\mathrm{dist}^2(x,y)}{2(1+t)}$ where F is the exact symplectic twist map considered above, and can be identified with the phase in the Heat kernel. We can check (2.33) trivially when $H=\frac{1}{2}p^2$, $\widetilde{S}(0,x,y)=\frac{1}{2}(x-y)^2$, see [Ar,p.255]. This holds also under assumption (2.30). Clearly under Hypothesis (2.18), (2.33) extends (2.12) for large times.

So far we have assumed some convexity of H with respect to p. The case $m=1$ (Finsler metric and Randers symbols) is investigated in [Ta] : it turns out that similar results hold when the *square* of Finsler metric or Randers symbol enjoys some convexity property, so for a "conformal metric" the case $m=1$ makes no difference. In Sect.5 we shall require $m=1$, but Φ is no longer associated with a distance on M.

2.5 Maupertuis-Jacobi correspondence

Geodesic flows are often used in Maupertuis-Jacobi correspondence. Let M be a smooth manifold, and $\mathcal{H}_0,H_0\in C^\infty(T^*M)$ two Hamiltonians, possessing a common regular energy surface $\Sigma=\{\mathcal{H}_\prime=\mathcal{E}\}=\{H_0=E\}$. Then $\mathcal{H}_0,H_0$ have the same integral curves $(\mathcal{X}(\tau),\mathcal{P}(\tau))=(X(t),P(t))$ on Σ, up to a reparametrization of time. Hamiltonian vector fields are related by $v_{\mathcal{H}_0}=\mathcal{G}(x,p)v_{H_0}$, parametrizations by $dt=\mathcal{G}(\tau)d\tau$ for some smooth (positive) function $\mathcal{G}$. We say that $(\mathcal{H}_0,\mathcal{E})$ and (H,E) satisfy Maupertuis-Jacobi correspondence and write $(\mathcal{H},\mathcal{E})\sim(H,E)$.

Example 2.3: $H(x,p)=p^2+V$ at energy E and $\mathcal{H}_E(x,p)=\frac{p^2}{E-V(x)}$ at energy 1 satisfy Maupertuis-Jacobi correspondence. Take in particular $H(x,p)=\frac{p^2}{1+x^2}$ and $\mathcal{H}(x,p)=p^2-x^2$, we have $(H,1)\sim(\mathcal{H},1)$ and $P(t,\psi)=\omega(\psi)\cosh f(t)$, $X(t,\psi)=\omega(\psi)\sinh f(t)$, where f satisfies the ODE $f'(t)=\frac{2}{\cosh^2 f(t)}, f(0)=0$. Integrating, we find $f+\frac{1}{2}\sinh(2f)=4t$, so $f(t)>0$ for all $t>0$. Another example with a frequency vector is $H(x,p)=\frac{p^2}{1+\nu^2\cdot x^2}$.

Example 2.4: $H(x,p) = |p|^m \frac{1}{\rho(x)}$ above; $\mathcal{H}$ is for instance the water-wave Hamiltonian $\mathcal{H}(x,p) = |p|(1+\mu(x)p^2)\tanh(D(x)|p|)$, see [DoRo], [DoMiRo], [ReDoKaMi].

Semi-classical Green functions for $(\mathcal{H},\mathcal{E})$ and (H,E), i.e. solutions of (1.4) with different Hamiltonians are mapped diffeomorphically to each other. It could be interesting to compare the construction of corresponding Maslov canonical operators at each step (phase function, density and so on...)

3. Maslov canonical operators and bi-Lagrangian distributions

Here we prove Theorem 1.2. Our purpose is to describe the solution globally, including unfolding of Lagrangian singularities. Among many references to the subject we make use in particular of [M], [Hö], [Du], [Iv], [BaWe], [CdV], [GuSt], [GrSj], [DoZh], [DoNaSh], [DoRo].

First we recall the asymptotic stationary phase formula for a quadratic phase function [Hö,Lemma 7.7.3]. Let A be a symmetric non-degenerate matrix, then

$$\int e^{i\langle Ax,x\rangle/2h}u(x)\,dx = \big(\det\big(A/(2i\pi h)\big)\big)^{-1/2}\sum_0^{k-1}(h/(2i))^j\langle A^{-1}D,D\rangle^j u(0)/j! + \mathcal{O}(h^k) \tag{3.1}$$

Since we shall ignore for simplicity Maslov indices, this formula has the advantage of hiding phase factors like $e^{-i\pi n/4}$, which we could restore by choosing an appropriate branch of the square root in the complex plane. A similar formula [Hö,Theorem 7.7.5] holds for $\langle Ax,x\rangle/2$ replaced by f with a non-degenerate critical point at x_0 and Hessian matrix A.

3.1. Lagrange immersions and non-degenerate phase functions:

A smooth function $\Phi : (x,\theta) \mapsto \Phi(x,\theta)$, $\theta \in \mathbf{R}^N$, defined near (x_0,θ_0) with $\xi_0 = \partial_x\Phi(x_0,\theta_0)$ is called a non-degenerate phase function in the sense of Hörmander iff the $(n+N)\times N$ matrix $(\Phi''_{\theta x}, \Phi''_{\theta\theta})$ has rank N on the critical set

$$C_\Phi = \{(x,\theta)\in M\times\mathbf{R}^N : \frac{\partial\Phi}{\partial\theta}(x,\theta) = 0\} \tag{3.3}$$

Then

$$\iota_\Phi : C_\Phi \to \Lambda_\Phi = \{(x,\Phi'_x(x,\theta) : (x,\theta)\in C_\Phi\} \tag{3.4}$$

is a local Lagrangian embedding (diffeomorphism).

It is easy to prove [Iv,(1.2.7)] that

$$N - \operatorname{rank}\Phi''_{\theta\theta} = n - \operatorname{rank} d\pi_\Lambda(\iota_\Phi(x,\theta)) \tag{3.5}$$

Let also $\pi_{\Lambda_\Phi} : \Lambda_\Phi \to M$ (or simply π_x) be the natural projection. If $k = \operatorname{rank} d\pi_\Lambda(\iota_\Phi(x_0,\theta_0))$, we say that Λ_Φ has rank k in a neighbhd U of (x_0,ξ_0), and call U a *local chart* of rank $(\geq)k$ near (x_0,ξ_0). If $k=n$, U is called a "regular" chart, and Λ is called "projectable" or "horizontal" on U. On the other extreme, if $k=0$, U is called a "maximally singular" chart, and Λ is called "vertical" on U.

If at some $z=(x,\xi)\in T^*M$, $T_z\Lambda_\Phi$ is transverse to the vertical plane $V_z = \{(0,\delta\xi)\}$ (i.e. z is a regular point) then (3.5) shows that $\Phi''_{\theta\theta}$ is of maximal rank N.

When $k < n$ we start to add some extra variables: namely there exists a partition of variables $x = (x', x'')$ such that the $(N+n-k) \times (N+n-k)$ matrix

$$\mathrm{Hess}_{(x'',\theta)}(\Phi) = \begin{pmatrix} \Phi''_{x''x''} & \Phi''_{x''\theta} \\ \Phi''_{\theta x''} & \Phi''_{\theta\theta} \end{pmatrix} \tag{3.6}$$

is non degenerate. So the map $(x'', \theta) \mapsto \Phi(x, \theta) - x''\xi''$ has a non-degenerate critical point $\theta_c = \theta(x', \xi''), x''_c = x''(x', \xi'')$ with the critical value $S(x', \xi'') = \Phi(x', x''_c, \theta) - x'\xi''_c$. The projection $\widetilde{\pi} : \Lambda_\Phi \to T^*_x \mathbf{R}^n, (x, \xi) \mapsto \widetilde{\pi}(x, \xi) = (x', \xi'')$ becomes of maximal rank n. Hence Λ near x is parametrized by $S(x', \xi'')$.

Remark 3.1: The above non-degeneracy condition on Φ is equivalent to (non-degeneracy in the sense of Hörmander) $d_{x,\theta}\Phi(x_0, \theta_0) \neq 0$, and $d_{(x,\theta)}\partial_{\theta_1}\Phi, \cdots, d_{(x,\theta)}\partial_{\theta_N}\Phi$ are linearly independent on the critical set C_Φ. The property stated above means that, if Φ is non-degenerate in the sense of Hörmander, then it is always possible to find coordinates such that $\pi_\xi : (x, \xi) \mapsto \xi$ has rank n. Actually, there are coordinates such that $(x, \theta) \mapsto \Phi(x, \theta) - x\xi$ has a non-degenerate critical point, so that

$$\mathrm{Hess}_{x,\theta}(\Phi) = \begin{pmatrix} \Phi''_{xx} & \Phi''_{x\theta} \\ \Phi''_{\theta x} & \Phi''_{\theta\theta} \end{pmatrix} \tag{3.7}$$

is non degenerate and Λ_Φ is of the form $\Lambda_\Phi = \{(-\phi'(\xi), \xi)\}$, see [Hö,Proposition 25.1.5]. This follows from the fact that while $\pi_x : (x, \xi) \mapsto x$ is invariantly defined under diffeomorphisms in M, this is not the case for the horizontal projection $\pi_\xi : (x, \xi) \mapsto \xi$. For the generating function Φ of Λ_+ constructed in Sect.4, $\mathrm{Hess}_{x,\theta}(\Phi)$ is actually degenerate in the "natural" coordinates of the problem, while $\mathrm{Hess}_{x'',\theta}(\Phi)$ is not, see Remark 4.2.

Let us recall the expression for the (inverse) density on C_Φ. Let $y = (y_1, \cdots, y_n)$ be some local coordinates on C_ϕ and $|dy|$ corresponding Lebesgue measure. Then the non vanishing, real function

$$F[\Phi, d\mu_+] = \frac{dy \wedge d\Phi'_\theta}{dx \wedge d\theta} = \frac{dy \wedge d\Phi'_{\theta_1} \wedge \cdots \wedge d\Phi'_{\theta_N}}{dx \wedge d\theta_1 \wedge \cdots \wedge d\theta_N} \tag{3.8}$$

is well-defined near C_Φ as the quotient of two volume forms, see [HöIV,p.14], [NaStSh], [DoNaSh, (2.8)]. Restricted to C_ϕ, its absolute value defines the (inverse) density on C_ϕ. Computed on the complexified tangent space to C_Φ, the variations of the argument of $F[\Phi]$ can define also the variations of Maslov index (see [DoNaSh]), which we shall ignore in this paper. We can also write the absolute value of (3.8) on C_Φ in the form, see [GrSj, Sect.11]

$$|F[\Phi, d\mu_+]|_{C_\Phi} = \left| \det \begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial \theta} \\ \frac{\partial^2 \Phi}{\partial x \partial \theta} & \frac{\partial^2 \Phi}{\partial \theta^2} \end{pmatrix} \right|_{C_\Phi}$$

It is actually independent of the choice of coordinates on C_Φ but it does depend on the choice of local coordinates x.

3.2 Maslov canonical operator acting on Lagrangian distributions.

Let u_h be a semi-classical Lagrangian distribution (or oscillatory integral) i.e. locally

$$u_h(x) = \int^* e^{i\Phi(x,\theta)/h} b(x,\theta;h)\, d\theta \tag{3.9}$$

where $\Phi(x,\theta)$ is a non-degenerate phase function in the sense above, and $b(x,\theta;h) = b_0(x,\theta) + hb_1(x,\theta) + \cdots$ an amplitude. With u_h we associate as in (3.4) the Lagrangian submanifold Λ_Φ.

It is proved in [Hö,Proposition 25.1.5] that, using that (3.7) is non degenerate, we can choose local coordinates $\xi \in \mathbf{R}^n$ on Λ_Φ, take h-Fourier transform $\mathcal{F}_{h,x\to\xi} u_h(\xi) = (2\pi h)^{-n/2} \int e^{-ix\xi/h} u(x)\, dx$ and expand by stationary phase. The half-density in the local chart (C_Φ, ι_Φ) is the given by $\sqrt{d\mu_\Phi} = |\det \Phi''|^{-1/2} |d\xi|^{1/2}$ (denoting $\Phi'' = \mathrm{Hess}_{x,\theta}(\Phi)$ for short), and the (oscillating) principal symbol of u_h in Λ_Φ by

$$e^{i\phi(\xi)/h} A_0(\xi) = e^{i\phi(\xi)/h} e^{i\pi \operatorname{sgn} \Phi''/4} b_0(x(\xi),\theta(\xi)) \sqrt{d\mu_\Phi} \tag{3.11}$$

Here ϕ is the "reduced phase function" such that $\Lambda = \{(-\phi'(\xi),\xi)\}$. Alternatively when (3.6) is non degenerate, one can express the (oscillating) principal symbol of u_h taking partial Fourier transform

$$v_h(x',\xi'';h) = \mathcal{F}_{h,x''\to\xi''} u_h(x',\xi'') = (2\pi h)^{(k-n)/2} \int e^{-ix''\xi''/h}\, dx'' \int e^{i\Phi(x',x'',\theta)/h} b(x',x'',\theta;h)\, d\theta$$

leading again to an expression of WKB type as in (3.11), and locally

$$\Lambda_\Phi = \Lambda_S = \{(x,\xi) : x'' = -\partial_{\xi''} S(x',\xi''), \xi' = \partial_{x'} S(x',\xi'')\}$$

Thus we obtained a *reduced phase functions*, with least possible number of variables θ, i.e. at most n. When $k = n$, then u_h assumes simply a WKB form in x variables.

Conversely, let $\iota : \Lambda \to T^*M$ be a Lagrangian immersion. We know ([Hö, Theorem 21.2.16]) that there exists a covering of Λ by canonical charts U, such that Λ is parametrized in each U by a non-degenerate phase function. The Lagrangian immersions ι and (3.4) have the same image on U and C_Φ is a submanifold of dimension n. In particular, $\iota_\Phi : C_\Phi \to \Lambda$ is a diffeomorphism onto its image. These phases can be chosen coherently, and define a class of "reduced phase functions" ϕ, parametrizing ι locally. This gives the fibre bundle of phases $\mathcal{L}_h$, including Maslov indices, equipped with transition functions. We are also given local smooth half-densities $|d\mu_\Phi|^{1/2}$ on Λ, defining the fibre bundle of half-densities $\Omega^{1/2}$, equipped with transition functions. The collection of these objects make a fibre bundle $\Omega^{1/2} \otimes \mathcal{L}_h$ over Λ. A section of $\Omega^{1/2} \otimes \mathcal{L}_h$ will be written as $\left[K^h_{(\Lambda,\mu)} a\right](x;h)$, where $K^h_{(\Lambda,\mu)}$ is called Maslov canonical operator. At leading order $\left[K^h_{(\Lambda,\mu)} a\right]$ reduces to its oscillating symbol (3.11). The set of such Lagrangian distributions microlocally supported on Λ will be denoted by $I(M;\Lambda)$.

We apply Maslov canonical operator for constructing solutions to homogeneous equation

$$(H(x,hD_x;h) - E)u_h = 0$$

microlocally supported on some Λ_+ in the characteristic foliation of Σ_E. Here $H(x,hD_x;h)$ is a h-PDO with principal symbol $H(x,p)$, and we assume for short $E = 0$. If u_h is a Lagrangian distribution locally of the form (3.9), the same holds for Hu_h.

The phase Φ (with time t in Hamilton equations as one of the θ-parameters) is determined by HJ equation (2.15), with initial data on $t=0$, which gives (locally) the Lagrangian embedding (3.4) with image Λ_+. In particular, $C_\Phi \subset \{\partial_t \Phi = 0\}$. We prescribe the amplitude such that $b|_{t=0} = a$. The construction of $\big[K^h_{(\Lambda_+,\mu)} a\big]$ goes as follows. The amplitude of $H(x, hD_x; h)u_h(x)$ has leading term

$$H_0\big(x, \partial_x \Phi(x,\theta)\big) b_0(x,\theta)$$

Moreover if $H(x,p;h)$ has sub-principal symbol $H_1(x,p)$, and $H_0(x,p)=0$ on Λ_+, $H(x,hD_x;h)u_h(x)$ has principal symbol

$$h\big(\frac{1}{i}\mathcal{L}_{v_{H_0}}|_{\Lambda_+} + H_1\big) b_0 |d\mu_\Phi|^{1/2} \tag{3.12}$$

where $\mathcal{L}_{v_{H_0}}$ denotes Lie derivative along v_{H_0} acting on half-densities $u|dy|^{1/2}$ as

$$\mathcal{L}_v(u|dy|^{1/2}) = \big(v_j(y)\frac{\partial u}{\partial y_j} + \frac{1}{2}(\operatorname{div} v(y))u(y)\big)|dy|^{1/2} \tag{3.13}$$

in local coordinates. For Schrödinger operator $H(x,hD_x) = -h^2\Delta + V(x)$, Λ_+ is "horizontal", and $\mathcal{L}_{v_{H_0}}|_{\Lambda_+}\big(b_0|d\mu_\Phi|^{1/2}\big)$ takes the form $\big(\sum_j \Phi'_j(x)\frac{\partial b_0}{\partial x_j} + \frac{1}{2}\Delta\Phi(x)b_0(x)\big)|dx|^{1/2}$, and a similar expression when Λ_+ is "vertical", see e.g. [DoRo,(b.14)]. Then (3.9) solves $Hu_h = 0 \bmod \mathcal{O}(h^2)$.

Note that on Λ_+, $\mathcal{L}_{v_{H_0}} = d/dt$. Provided Φ is a non-degenerate phase-function, (3.12) admits a global solution, computed on each canonical chart. For instance on a regular chart, this is just WKB construction. In a totally singular chart instead, we solve (3.12) in Fourier representation, and more generally in the mixed representation.

The function $b_0(x(\xi),\theta(\xi))$ is smooth in ξ, but of course when expressed in x-variable, singularities may occur do to singular Jacobians at focal points.

The fact that u_h solves $Hu_h = 0 \bmod \mathcal{O}(h^2)$ is also expressed by the *commutation relation*

$$H(x,hD_x;h)\big[K^h_{(\Lambda_+,\mu)}a\big](x;h) = \big[K^h_{(\Lambda_+,\mu)} h\big(\frac{db_0}{dt} + iH_1 b_0\big)\big](x;h) + \mathcal{O}(h^2) = \mathcal{O}(h^2) \tag{3.14}$$

3.3 Bi-Lagrangian distributions.

We want to describe solutions of the inhomogeneous equation $(H-E)u_h = f_h$, i.e. oscillatory integrals microlocally supported on $\Lambda \cup \Lambda^+$, by generalizing Maslov canonical operator to *bi-Lagrangian distributions*. This allows to represent u_h in a punctured neighborhood of $\partial\Lambda_+$. Semi-classical distributions used in [MelUhl] (adapted to the semi-classical case), are locally of the form

$$u_h(x) = \frac{i}{h}\int_0^\infty \theta_T(t)\,dt \int^* e^{i(x'\xi' + (x_n - t)\xi_n)/h} b(t,x,\xi;h)\,d\xi \tag{3.17}$$

where b is an amplitude, and $\theta_T(t)$ a cut-off as in (1.10), which we omit for simplicity, since our analysis is local in x. Maslov canonical operator extended to to bi-Lagrangian distributions allows to split u_h, modulo a remainder term localized on $\partial\Lambda_+$, into 2 terms, supported respectively on Λ

and Λ_+. This representation being invariant under local FIO's (see Proposition 3.3 below), we are reduced to decompose (3.17) this way.

As a warm-up, let us compute the semi-classical wave-front set $\mathrm{WF}_h\, u_h$ in (3.17). Fix $\overline{z} = (\overline{x}, \overline{\xi}) \in T^*M$, $\overline{\xi} = (\overline{\xi}', \overline{\xi}_n)$. It is well known that $\mathrm{WF}_h(u_h)$ is characterized by the following property: $\overline{z} \notin \mathrm{WF}_h(u_h)$ iff there exists $\chi \in C_0^\infty(T^*M)$ equal to 1 near $\overline{z}$, such that

$$\chi(x, hD_x)u_h(x) = (2\pi h)^{-1} \int\int e^{i(x-y)\eta/h} \chi(y,\eta) u_h(y)\, dy\, d\eta = \mathcal{O}(h^\infty) \tag{3.18}$$

We have the standard

Proposition 3.1: *Let u_h be as in (3.18). Then*

$$\mathrm{WF}_h(u_h) \subset \{x_n \geq 0\} \cap \big(\{x_n = 0\} \cup \{x' = 0\} \cup \{\xi_n = 0\}\big) \tag{3.19}$$

If moreover $b(t,x,\xi;h)$ verifies the transport equation $\partial_t b + \partial_{x_n} b = 0$ (and hence $hD_{x_n} u_h = f_h$), we get the sharper estimate $\mathrm{WF}_h(u_h) \subset \Lambda^0 \cup \Lambda^0_+$, $\Lambda^0 = T_0\mathbf{R}^n$, which is the conormal bundle of the manifold with boundary $x' = 0, x_n \geq 0$. We recall from Lemma 1.1 that $\Lambda^0_+ \subset \mathrm{Char}\, hD_{x_n}$ is the flow out of Λ^0 by v_{ξ_n} in $\xi_n = 0$.

Proof: Since we work locally in x, we can safely omit the cut-off θ_T in (3.17). Let $\Phi_t(x,\xi,y,\eta) = y'\xi' + (y_n - t)\xi_n + (x-y)\eta$ be the phase-function in (3.17).

(**i**) If $\overline{x}_n < 0$, we choose χ such that $\chi(y,\eta) = \widetilde{\chi}(y',\eta)\chi_n(y_n)$, with $\chi_n(\overline{x}_n) \neq 0$. It follows that $\eta_n \mapsto \Phi_t$ is non stationary in $\mathrm{supp}\,\chi_n$, so $\mathrm{WF}_h\, u_h \subset \{x_n \geq 0\}$.

(**ii**) Assume $\overline{\xi}_n \neq 0$ and choose χ such that $\chi = \widetilde{\chi}(y,\eta')\chi_{2n}(\eta_n)$, with $\chi_{2n}(\overline{\xi}_n) \neq 0$. Let ε be so small that $|\xi_n - \eta_n| > \delta > 0$ on $|\xi_n| < \varepsilon$ and $\eta_n \in \mathrm{supp}\,\chi_{2n}$, we split χ according to χ^ε and $\widehat{\chi}^\varepsilon = \chi - \chi^\varepsilon$, where $\chi^\varepsilon(x, hD_x)u_h(x) = (2\pi h)^{-n} \int dy \int_{|\xi_n| \geq \varepsilon} \int e^{i(x-y)\eta/h} \chi(y,\eta) u_h(y)\, d\eta$.

We have $hD_t e^{i\Phi_t/h} = -\xi_n e^{i\Phi_t/h}$, so that integrating by parts N times with respect to t we get $\chi^\varepsilon(x, hD_x)u_h(x) = \big(\frac{h}{i}\big)^N A^\varepsilon_N(x;h) + B^\varepsilon_N(x;h)$ where

$$\begin{aligned} A^\varepsilon_N(x;h) &= (2\pi h)^{-n} \int \chi(y,\eta)\, dy\, d\eta \int_0^\infty dt \int^*_{|\xi_n| \geq \varepsilon} e^{i\Phi_t/h} \frac{1}{\xi_n^{n+1}} \partial_t^{N+1} b(t,y,\xi;h)\, d\xi \\ B^\varepsilon_N(x;h) &= (2\pi h)^{-n} \int \chi(y,\eta)\, dy\, d\eta \int^*_{|\xi_n| \geq \varepsilon} e^{i\Phi_0/h} \frac{1}{\xi_n} \sum_{j=0}^N \big(\frac{h}{i\xi_n}\big)^j \partial_t^j b(0,y,\xi;h)\, d\xi \end{aligned} \tag{3.21}$$

Now $(y,\xi) \mapsto \Phi_0(x,\xi,y,\eta) = y\xi + (x-y)\eta$ has a non-degenerate critical point at $\xi = \eta, y = 0$. Assume $\overline{x} \neq 0$, i.e. we choose χ such that $0 \notin \pi_x(\mathrm{supp}\,\chi)$; so Φ_0 is not stationary and $B^\varepsilon_N(x;h) = \mathcal{O}(h^\infty)$. Hence $\chi^\varepsilon(x, hD_x)u_h(x) = \mathcal{O}(h^N)$ for any N.

Consider next the contribution $\widehat{\chi}^\varepsilon(x, hD_x)u_h(x)$ of $|\xi_n| < \varepsilon$ to $\chi(x, hD_x)u_h(x)$. The map

$$y_n \mapsto \Phi_t(x,\xi,y,\eta)$$

has a critical point at $\xi_n = \eta_n$. Since $|\xi_n - \eta_n| > \delta > 0$ when $\eta_n \in \mathrm{supp}\,\chi_{2n}$, $y_n \mapsto \Phi_t(x,\xi,y,\eta)$ is non stationary and $\widehat{\chi}^\varepsilon u_h(x) = \mathcal{O}(h^\infty)$. Altogether, $\chi(x, hD_x)u_h(x) = \chi^\varepsilon u_h(x) + \widehat{\chi}^\varepsilon u_h(x) = \mathcal{O}(h^\infty)$ so $\overline{z} \notin \mathrm{WF}_h\, u_h$ if $\overline{\xi}_n \neq 0$. In particular $\mathrm{WF}_h\, u_h \subset \{x_n \geq 0\} \cap \big(\{\xi_n = 0\} \cup \{x = 0\}\big)$.

(**iii**) Assume next $\overline{x}_n > 0$ and take as above $\chi(y,\eta) = \widetilde{\chi}(y',\eta)\chi_n(y_n)$, with $\chi_n(\overline{x}_n) \neq 0$. Then $(t,\xi_n,y,\eta) \mapsto \Phi_t$ is critical at $t = y_n = x_n, \xi_n = 0, y' = x', \eta' = x'$, and this is a non-degenerate critical point. So when $x_n > 0$, asymptotic stationary phase (3.1) shows that

$$\chi(x,hD_x)u_h(x) = 2i\pi \int^* e^{ix'\xi'/h}\widetilde{\chi}(x',\xi')\big[b(x_n,x,(\xi',0);h) + \frac{h}{i}\frac{\partial^2 b_0}{\partial t\partial\xi_n})(x_n,x,(\xi',0);h)\big]\,d\xi' \tag{3.22}$$

In particular, using (i), we see that $\mathrm{WF}_h(u_h) \subset \{x_n \geq 0\} \cap \big(\{x_n = 0\} \cup \{x' = 0\}\big)$, which altogether proves (3.19).

For the last statement of Proposition 3.1, apply hD_{x_n} to u_h and integrate (3.17) by parts once with respect to t. We find

$$hD_{x_n}u_h(x) = \int^* e^{ix\xi/h}b(0,x,\xi;h)\,d\xi + \int_0^\infty dt \int^* e^{i\Phi_t/h}(\partial_t + \partial_{x_n})b(t,x,\xi;h)\,d\xi \tag{3.23}$$

so if $b(t,x,\xi;h)$ verifies the transport equation, $hD_{x_n}u_h(x) = g_h(x) = \int^* e^{ix\xi/h}b(0,x,\xi;h)\,d\xi$ and since $\mathrm{WF}_h\, g_h \subset T_0^*\mathbf{R}^n$, the last (sharper) estimate on $\mathrm{WF}_h\, u_h$ follows from the well-known property $\mathrm{WF}_h\, u_h \subset \mathrm{WF}_h\, hD_{x_n}u_h \cup \mathrm{Char}\, hD_{x_n}$. ♣

We say that $u_h(x)$ is a *bi-Lagrangian* (semi-classical) *distribution* on the intersecting pair (Λ^0, Λ^0_+).

It can be shown that a h-FIO A whose canonical map preserves the Lagrangian intersection preserves also bi-Lagrangian distributions of the form (3.18). Namely, let (Λ, Λ_+) be a Lagrangian pair in the sense of Lemma 1.1, defined near $(y_0,\eta_0) \in \partial\Lambda'_+$, and u_h be of the form (3.18). We recall the following result:

Proposition 3.2 [MelUhl,Prop.3.2]: *Let $(\Lambda', \Lambda'_+) \sim (\Lambda, \Lambda_+)$ be Lagrangian pairs in the sense of Lemma 1.1. Let A be a h-FIO of the form*

$$Av(x;h) = (2\pi h)^{-(n+N)/2}\int e^{i\phi(x,y,\theta)/h}c(x,y,\theta)v(y;h)\,dy\,d\theta$$

associated with the canonical transformation κ_A, $\kappa_A(y_0,\eta_0) = (x_0,\xi_0)$ with graph

$$\Lambda_A = \{(x,\phi'_x(x,y,\theta),y;\phi'_y(x,y,\theta) : \phi'_\theta(x,y,\theta) = 0\}$$

such that (locally) $\Lambda_A \circ \Lambda' = \Lambda$, $\Lambda_A \circ \Lambda'_+ = \Lambda_+$ and the compositions are transversal [HöIV,p.19&44]. Here have denoted as usual $\Lambda'_A = \{(x,\xi;y,-\eta) : (x,\xi;y,\eta) \in \Lambda_A\}$. Let u_h be defined near (y_0,η_0) on the Lagrangian pair (Λ', Λ'_+) by (3.17). Then Au_h defined near (x_0,ξ_0) on the Lagrangian pair (Λ, Λ_+) is again of the form (3.17).

The proof essentially reduces to show that Au_h, after a change of (t,θ) variables, can be rewritten as an integral of the form (3.17), i.e. with the same phase Φ_t, and a new amplitude $b'(t,x,\xi;h)$. So we can define the class $I(M,\Lambda,\Lambda_+)$ of bi-Lagrangian distributions supported on the Lagrangian pair $(\Lambda,\Lambda_+) \sim (\Lambda',\Lambda'_+)$, all of which take locally the form (3.17).

Consider now the inhomogeneous equation $H(x,hD_x;h)u_h = f_h$, where

$$f_h(x) = \int^* e^{ix\xi/h}a(x,\xi;h)\,d\xi$$

is conormal to $\Lambda^0 = T_0^*M$. When $H(x, hD_x; h) = hD_{x_n}$, u_h as in (3.17) solves $H(x, hD_x; h)u_h = f_h$ whenever b solves the transport equation, i.e. $b(t, x, \xi; h) = a(x', x_n - t, \xi; h)$, so that (3.23) simplifies to $hD_{x_n}u_h(x) = f_h(x)$ mod $\mathcal{O}(h^\infty)$. In the general case, we can show that we can take $H(x, hD_x;)$ to its normal form hD_{x_n} by conjugating with a h-FIO. Namely we have

Proposition 3.3: *Let the energy surface $H_0 = E$ be non critical, and v_{H_0} be transverse to Λ at $(x_0, \xi_0) \in \partial\Lambda_+$, where Λ_+ is the flow out of v_H from Λ. Then there is a h-FIO A, as in Proposition 3.2, defined microlocally near $\big((x_0, \xi_0), (0, 0)\big)$, quantizing the canonical transformation of Lemma 1.1 such that $A^{-1}H(x, hD_x)A = hD_{y_n}$.*

3.4 Compatibility condition and symbolic calculus. Maslov canonical operator for bi-Lagrangian distributions.

Here we prove Theorem 1.2. By Propositions 3.2 and 3.3 it suffices to consider the Lagrangian pair (Λ^0, Λ^0_+), with $\Lambda^0 = T_0^*\mathbf{R}^n$ (which we denote again Λ for short). When u_h solves $\big(H(x, hD_x; h) - E)u_h(x) = f_h$ mod $\mathcal{O}(h^N)$ we want to define the "boundary-part" σ and "wave-part" σ^+ symbols of u_h satisfying the *compatibility condition*. See also [MelUhl,Sect.4] for a more intrinsic discussion on (σ, σ^+).

Our argument is similar to [MelUhl,Prop.2.3], and relies on the fact that if $\chi(x, hD_x)$ is a h-PDO with $\mathrm{WF}_h(\chi(x, hD_x)) \cap \Lambda = \mathrm{WF}'_h(\chi(x, hD_x)) \circ \Lambda = \emptyset$, then $\chi(x, hD_x)u_h$ is a Lagrangian distribution supported on Λ^0_+, while if $\chi(x, hD_x)$ is a h-PDO with $\mathrm{WF}_h(\chi(x, hD_x) \cap \Lambda^0_+ = \emptyset$, then $\chi(x, hD_x)u_h$ is a Lagrangian distribution supported on Λ.

We proceed in two steps. In Step 1, we construct a family $(\chi_\delta(x, hD_x))_{\delta>0}$ such that $\chi_\delta(y, \eta) \to 1$ pointwise and $\chi_\delta(x, hD_x)u_h$ is supported on Λ^0. This will give the *boundary-part symbol*. In Step 2, we construct a family $(\chi_\delta(x, hD_x))_{\delta>0}$ such that $\chi^+_\delta(y, \eta) \to 1$ pointwise and $\chi^+_\delta(x, hD_x)u_h$ is supported on Λ^0_+. This will give the *wave-part symbol*. The *compatibility condition* results in comparing these 2 symbols.

• The boundary-part symbol.

We check first the compatibility condition for $N = 1$. Consider $\chi_\delta \in C^\infty(\mathbf{R}^{2n})$ of the form $\chi_\delta(y, \eta) = \chi'_\delta(y, \eta')\chi^\delta_{2n}(\eta_n)$ with $|\eta_n| \geq \delta$ on $\operatorname{supp}\chi^\delta_{2n}(\eta_n)$, and $\chi^\delta_n(\eta_n) \to 1$ pointwise for $\eta_n \neq 0$, as $\delta \to 0$. Let

$$q(y, \xi; h) = \frac{1}{\xi_n}\big[b(0, y, \xi; h) + \frac{h}{i\xi_n}\partial_t b_0(0, y, \xi) + \mathcal{O}(h^2)\big]$$

From the part (ii) of the proof of Proposition 3.1 for $N = 1$, taking $\varepsilon < \delta$ so that the contribution of $|\xi_n| < \varepsilon$ to $\chi_\delta(x, hD_h)u_h(x)$ is $\mathcal{O}(h^\infty)$, we know that

$$\chi_\delta(x, hD_x)u_h(x) = B_1^{\delta,\varepsilon}(x; h) + \mathcal{O}(h^2) \tag{3.26}$$

We may assume χ_δ is such that $\frac{\partial\chi_\delta}{\partial y}(0, \eta) = 0$, hence computing

$$B_1^{\delta,\varepsilon}(x; h) = \int e^{ix\eta/h}\, d\eta \int^*_{|\xi_n|>\varepsilon} e^{iy(\xi-\eta)/h} q(y, \xi; h)\chi_\delta(y, \eta)\, dy\, d\xi$$

by asymptotic stationary phase in (y,ξ) with (3.1), where $A = A_\delta = \frac{\partial^2 \Phi_0}{\partial(y,\xi)^2}$, $\det A_\delta = (-1)^n$, gives

$$
\begin{aligned}
B_1^{\delta,\varepsilon}(x;h) &= (\det A_\delta)^{-1/2}(2i\pi h)^n \int d\eta\, e^{ix\eta/h}\chi_\delta(0,\eta) \\
&\quad \big[q(0,\eta;h) - \frac{h}{i\eta_n}\frac{\partial^2 q_0}{\partial y'\partial\xi'}(0,0,\eta) - \frac{h}{i\eta_n}\frac{\partial^2 b_0}{\partial y_n \partial \xi_n}(0,0,\eta) + \frac{h}{i\eta_n^2}\frac{\partial b_0}{\partial t}(0,0,\eta) + \mathcal{O}(h^2)\big] \\
&= (\det A_\delta)^{-1/2}(2i\pi h)^n \int d\eta\, e^{ix\eta/h}\chi_\delta(0,\eta) \\
&\quad \big[\frac{1}{\eta_n} b(0,0,\eta;h) + \frac{h}{i\eta_n^2}\big(\partial_t b_0(0,0,\eta) + \partial_{y_n} b_0(0,0,\eta)\big) - \frac{h}{i\eta_n}\frac{\partial^2 b_0}{\partial y \partial \xi}(0,0,\eta) + \mathcal{O}(h^2)\big]
\end{aligned}
\tag{3.27}
$$

The oscillating integral u_h solves $hD_{x_n}u_h = f_h$ iff b satisfies the transport equation

$$
\partial_t b(0,x,\xi;h) + \partial_{x_n} b(0,x,\xi;h) = 0
$$

i.e. $b(t,x,\xi;h) = b(x', x_n - t, \xi; h)$. Then (3.27) reduces to

$$
B_1^{\delta,\varepsilon}(x;h) = \int e^{ix\eta/h}\chi_\delta(0,\eta)\frac{1}{\eta_n}\big[b(0,0,\eta;h) + ih\frac{\partial^2 b_0}{\partial y \partial \xi}(0,0,\eta)\big]\, d\eta + \mathcal{O}(h^2) \tag{3.28}
$$

So for $\eta_n \neq 0$ we define the *boundary symbol* of u_h as

$$
\sigma(\eta;h) = \frac{1}{\eta_n}\big[b(0,0,\eta;h) + ih\frac{\partial^2 b_0}{\partial y \partial \xi}(0,0,\eta)\big] \tag{3.29}
$$

• The wave-part symbol.

Consider next $\chi_\delta^+ \in C^\infty(\mathbf{R}^{2n}$ of the form $\chi_\delta^+(y,\eta) = \chi_\delta'^+(y',\eta)\chi_n^\delta(y_n)$ with $y_n \geq \delta$ on $\operatorname{supp}\chi_n^\delta$, and $\chi_n^\delta(\eta_n) \to 1$ pointwise for $\eta_n > 0$, as $\delta \to 0$. As in the part (iii) in Proposition 3.1, we perform the integration

$$
\chi_\delta^+(x,hD_x)u_h(x) = \frac{i}{h}\int_0^\infty dt \int \chi_\delta^+(y,\eta)\, dy\, d\eta \int^* e^{i\Phi_t(x,\xi,y,\eta)/h} b(t,y,\xi;h)\, d\xi
$$

by asymptotic stationary phase (3.1) with respect to (t,ξ_n,y,η). Here the Hessian is $A = A_\delta^+ = \frac{\partial^2\Phi_t}{\partial(t,\xi_n,y,\eta)^2}$, $\det A_\delta^+ = (-1)^{n+1}$, and the critical value of the phase is $\Phi_c = x'\xi'$. We find

$$
\begin{aligned}
\chi_\delta^+(x,hD_h)u_h(x) &= \frac{i}{h}(\det A_\delta^+)^{-1/2}(2i\pi h)^{n+1}\int^* e^{ix'\xi'/h}\chi_\delta^+(x;\xi',0) \\
&\quad \big[b(x_n,x,\xi',0;h) - \frac{2h}{i}\frac{\partial^2 b_0}{\partial t \partial \xi_n}(x_n,x,\xi',0)\big]\, d\xi' + \mathcal{O}(h^2)
\end{aligned}
\tag{3.30}
$$

So we define the *wave-part symbol* of u_h by letting $\delta \to 0$ as

$$
\sigma^+(\xi;h) = \frac{i}{h}(\det A_\delta^+)^{-1/2}(2i\pi h)^{n+1}\big[b(x_n,x,\xi',0;h) - \frac{2h}{i}\frac{\partial^2 b_0}{\partial t \partial \xi_n}(x_n,x,\xi',0)\big] \tag{3.31}
$$

and we are to compare (3.29) with (3.31).

From [Hö,Lemma 18.2.1] and its proof, we know that if $v_h(x) = \int e^{ix\xi/h} a(x,\xi;h)\,d\xi$, then we also have $v_h(x) = \int e^{ix\xi/h}\widetilde{a}(\xi;h)\,d\xi$, with a symbol $\widetilde{a}(\xi;h) \sim \sum_j h^j \langle -iD_x, D_\xi\rangle a(x,\xi)/j!$ independent of x. Applying this to amplitude $b(x', x_n - t, \xi)$, the terms $\frac{\partial^2 b_0}{\partial y \partial \xi}(0,0,\eta)$ and $\frac{\partial^2 b_0}{\partial t \partial \xi_n}(x_n, x, \xi', 0)$ resp. in (3.28) and (3.30) disappear, $\sigma^+(\xi;h)$ is continuous up to $\partial\Lambda_+$, and we end up with the compatibility condition (1.23) mod $\mathcal{O}(h^2)$ between the wave-part and boundary-part symbols on $\partial\Lambda_+$.

It is clear, following again the proof of Proposition 3.1, that (3.26) carries by induction mod $\mathcal{O}(h^N)$, all N, when replacing $B_1^{\delta,\varepsilon}(x;h)$ by $B_N^{\delta,\varepsilon}(x;h)$.

This allow to define coherently the (bi-)symbol $(\sigma,\sigma^+) = (\sigma(u_h), \sigma^+(u_h))$ computed as above in local coordinates, and thus by analogy with Sect.3.2, an "effective" Maslov canonical operator $K^h_{\Lambda,\Lambda_+^0}(\sigma,\sigma^+)$. The commutation formula for bi-Lagrangian distributions takes the form

(3.33)
$$\begin{aligned} H(x,hD_x;h)\big[K^h_{\Lambda,\Lambda_+}(\sigma,\sigma^+)\big](x;h) &= \big[K^h_{\Lambda,\Lambda_+}\big((H-E)\sigma, h\big(\frac{1}{i}\mathcal{L}_{v_{H_0}}|_{\Lambda_+} + H_1\big)\sigma^+\big](x;h) + \mathcal{O}(h^2) \\ &= f_h + \mathcal{O}(h^2) \end{aligned}$$

which gives (1.24) once the transport equation has been solved as in (3.12). This brings the proof of Theorem 1.2 to an end. ♣

3.5 The constant coefficient case.

In general it is difficult to obtain a decomposition of u_h adapted to the splitting $u_h = u_h^0 + u_h^1$ where $\mathrm{WF}_h(u_h^0) \subset \Lambda$ and $\mathrm{WF}_h(u_h^1) \subset \Lambda_+^0$.

Here we compute u_h explicitely in the 2-D case for Helmholtz operator $-h^2\Delta - E$, but f with compact support. Let f also be radially symmetric; its Fourier transform $g = \mathcal{F}_1 f$ is again of the form $g(p) = g(|p|) = g(r)$ and extends holomorphically to $\mathbf{C}^2$. For $E = k^2$, $k > 0$, we rewrite

$$u_h(x) = (2\pi h)^{-n} \int e^{ix\xi/h} \frac{\mathcal{F}_1 f(\xi)}{\xi^2 - E - i0}\,d\xi$$

as $u_h(x) = u(x) = u_0(x) + u_1(x)$ with

$$\begin{aligned} u_0(x) &= \frac{k + i\,\varepsilon}{(2\pi h)^2} \int_0^{2\pi} d\theta \int_0^\infty \exp[i|x|r\cos\theta/h] \frac{g(r)}{r^2 - (k+i\,\varepsilon)^2}\,dr \\ u_1(x) &= \frac{1}{(2\pi h)^2} \int_0^{2\pi} d\theta \int_0^\infty \exp[i|x|r\cos\theta/h] \frac{g(r)}{r + k + i\,\varepsilon}\,dr \end{aligned}$$

To compute u_0 we use contour integrals. When $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, we shift the contour of integration to the positive imaginary axis and get by the residues formula

$$\begin{aligned} &\int_0^\infty \exp[i|x|r\cos\theta/h] \frac{g(r)}{r^2 - (k+i\,\varepsilon)^2}\,dr + \int_0^\infty \exp[-|x|r\cos\theta/h] \frac{g(ir)}{r^2 + (k+i\,\varepsilon)^2}\,idr = \\ &\quad 2i\pi \frac{g(k+i\,\varepsilon)}{2(k+i\,\varepsilon)} \exp[i|x|(k+i\,\varepsilon)\cos\theta/h] \end{aligned} \tag{3.35}$$

while for $\theta \in]\frac{\pi}{2}, \frac{3\pi}{2}[$,

$$\int_0^\infty \exp[i|x|r\cos\theta/h] \frac{g(r)}{r^2 - (k+i\,\varepsilon)^2}\,dr - \int_0^\infty \exp[|x|r\cos\theta/h] \frac{g(-ir)}{r^2 + (k+i\,\varepsilon)^2}\,idr = 0 \tag{3.36}$$

Summing up (3.35) and (3.36), integrating over $\theta \in]0, 2\pi[$ and letting $\varepsilon \to 0$, we obtain

$$u_0(x) = \frac{i\pi g(k)}{(2\pi h)^2} \int_{-\pi/2}^{\pi/2} \exp[i|x|k\cos\theta/h]\, d\theta +$$
$$\int_0^\infty \frac{dr}{r^2+k^2} \big[\int_{-\pi/2}^{\pi/2} g(ir) - \int_{\pi/2}^{3\pi/2} g(-ir)\big] \exp[-|x||\cos\theta|/h]\, d\theta$$

Since $g(ir) = g(-ir)$, the latter integral vanishes, so we end up with

$$u_0(x) = \frac{i\pi g(k)}{(2\pi h)^2} \int_{-\pi/2}^{\pi/2} \exp[i|x|k\cos\theta/h]\, d\theta$$

It is readily seen that

$$\mathrm{WF}_h\, u_0 \subset \{x=0\} \cup \{(x, k\frac{x}{|x|}), x \neq 0\} = \Lambda \cup \Lambda_+$$

Consider now u_1. We let $\varepsilon \to 0$ and set $\widetilde{g}(r) = \frac{g(r)}{r(r+k)}$. Since $\widetilde{g}(r)\sqrt{r} \in L^1(\mathbf{R}_+)$, we have $u_1(x) = H_0(\widetilde{g})(\frac{|x|}{h})$, where H_0 denotes Hankel transform of order 0. Let $\chi \in C_0^\infty(\mathbf{R}^2)$ be radially symmetric, and equal to 1 near 0, since $\mathrm{WF}_h\, f_h = \{x=0\}$, we have

$$g = \mathcal{F}_h(\chi f_h) + \mathcal{O}(h^\infty) = (2\pi h)^{-2}\mathcal{F}_h(\chi) * g + \mathcal{O}(h^\infty)$$

so in the expression for u_1 we may replace mod $\mathcal{O}(h^\infty)$, $\widetilde{g}(r)$ by a constant times $\widehat{g}(r) = \frac{(\mathcal{F}_h(\chi)*g)(r)}{r(r+m)}$ (see [Bad] for 2-D convolution and Fourier transform in polar coordinates). To estimate $\mathrm{WF}_h\, u_1$, we compute again the Fourier transform of $(1-\widetilde{\chi})\widehat{g}$ where $\widetilde{\chi}$ is a cut-off equal to 1 near 0, and we find it is again $\mathcal{O}(h^\infty)$ if $\chi \equiv 1$ on $\operatorname{supp}\widetilde{\chi}$. This shows that $\mathrm{WF}_h\, u_1 \subset \{x=0\}$.

Note that this Example makes use of Bessel function $J_0(\frac{|x|}{h})$, we shall return to such "localized functions" in Sect.5.

4. f_h is supported microlocally on the "vertical plane"

Here we shall construct objects written globally in a suitable coordinate system, using the phase functions in Sect.2. This system consists only of coordinates on Λ and of time parameter t in Hamilton equations.

Consider the case where H_0 is positively homogeneous of degree $m \geq 1$ with respect to p and f_h is microlocally concentrated on the vertical plane $\Lambda = \{x=0\}$, e.g. $f_h(x) = h^{-n} f(\frac{x}{h})$, with $f \in \mathcal{S}(\mathbf{R}^n)$ (Schwartz space).

4.1 Some non-degeneracy condition.

Recall first from [DoNaSh, Lemma 6] the following result. Let $\widetilde{\iota} : \widetilde{\Lambda} \to T^*\widetilde{M}$ be a Lagrangian embedding of dimension $\widetilde{n}$, $U \subset \widetilde{\Lambda}$ a connected simply connected open set,

$$(\widetilde{\phi}, \widetilde{\psi}) = (\phi_1, \cdots, \phi_k, \psi_1, \cdots, \psi_{\widetilde{n}-k})$$

local coordinates on U. Here we don't assume that k is the rank of $\pi_x : \widetilde{\Lambda} \to \widetilde{M}$. Thus $\widetilde{\Lambda}$ is defined by $x = X(\widetilde{\phi}, \widetilde{\psi}), p = P(\widetilde{\phi}, \widetilde{\psi})$ in the chart U. Let $\Pi(\widetilde{\phi}, \widetilde{\psi})$ be a smooth $\widetilde{n} \times k$ matrix function defined in U such that:

$$\begin{aligned} &\Pi^*(\widetilde{\phi}, \widetilde{\psi}) X_{\widetilde{\phi}}(\widetilde{\phi}, \widetilde{\psi}) = \mathrm{Id}_{k\times k} \\ &\kappa : (\widetilde{\phi}, \widetilde{\psi}) \mapsto (X(\widetilde{\phi}, \widetilde{\psi}), \widetilde{\psi}) \text{ is an embedding} \end{aligned} \tag{4.1}$$

Then there is a neighbhd V of $\kappa(U)$ such that the system

$$\Pi^*(\widetilde{\phi}, \widetilde{\psi})\big(x - X(\widetilde{\phi}, \widetilde{\psi})\big) = 0, \quad (x, \widetilde{\psi}) \in V$$

has a unique smooth solution $\widetilde{\phi} = \widetilde{\phi}(x, \widetilde{\psi})$ satisfying the condition $X(\widetilde{\phi}(x, \widetilde{\psi}), \widetilde{\psi}) = x$, when $(x, \widetilde{\psi}) \in \kappa(U)$.

For $(\widetilde{\phi}, \widetilde{\psi}) \in U$, define the $\widetilde{n} \times \widetilde{n}$ matrix

$$\mathcal{M}(\widetilde{\phi}, \widetilde{\psi}) = \big(\Pi(\widetilde{\phi}, \widetilde{\psi}); P_{\widetilde{\psi}}(\widetilde{\phi}, \widetilde{\psi}) - P_{\widetilde{\phi}}(\widetilde{\phi}, \widetilde{\psi})\Pi^*(\widetilde{\phi}, \widetilde{\psi}) X_{\widetilde{\psi}}(\widetilde{\phi}, \widetilde{\psi})\big)$$

As we shall see, invertibility of $\mathcal{M}(\widetilde{\phi}, \widetilde{\psi})$ plays an important role [DoMaNa2], [DoNaSh].

Consider now our special setting where H is positively homogeneous of degree m, Λ is the vertical plane, and recall $\langle P(t, \psi, \tau), \dot{X}(t, \psi, \tau)\rangle = mH$ from (2.6). Here τ small enough is taken as a parameter, everything depends smoothly on τ and $\Lambda_+(0) = \Lambda_+$. So $\Pi(t, \psi, \tau) = \frac{1}{mH} P(t, \psi, \tau)$ is a left inverse of $\dot{X}$: $\Pi^* \dot{X} = \frac{1}{mH}\langle P, \dot{X}\rangle = 1$. Further, the map $\Lambda_+ \to \mathbf{R}^n$, $(t, \psi) \mapsto (X(t, \psi, \tau), \psi)$ is clearly an embedding This fulfills conditions (4.1) above for $\widetilde{\Lambda} = \Lambda_+(\tau)$, with $\widetilde{n} = n$, $k = 1$ and $\widetilde{\phi} = t$ and $\widetilde{\psi} = \psi$. So the system

$$\Pi^*(t, \psi, \tau)(x - X(t, \psi, \tau)) = \langle P, x - X(t, \psi, \tau)\rangle = 0 \tag{4.2}$$

has a unique solution $t = t_0(x, \psi, \tau)$ satisfying the condition $X(t_0(x, \psi, \tau), \psi, \tau) = x$, and this solution is a smooth function. When $\tau = 0$ we omit it from the notations, and write for instance $X(t, \psi)$ for $X(t, \psi, 0)$.

Moreover by (2.6) again, the matrix

$$\mathcal{M}(t, \psi, \tau) = \big(\Pi(t, \psi, \tau), P_\psi(t, \psi, \tau) - \dot{P}\frac{1}{mH}\,{}^t P\, X_\psi\big) = (\frac{1}{mH} P, P_\psi) \tag{4.3}$$

has determinant $\frac{1}{mH}\det(P, P_\psi)$. As we shall see, it turns out that $\det \mathcal{M}$ gives the invariant (inverse) density on Λ_+.

Example 4.1: Let us compute $\det(P, P_\psi)$ at $t = 0$ for a geodesic flow $H(x, p)$, on the energy shell $E = 1$ when $n = 2$ or $n = 3$. When $n = 2$, up to a change of x coordinates such that at $x = 0$, the metric $H(0, p)$ takes the diagonal form $H(x, p) = \frac{p_1^2}{a_1^2} + \frac{p_2^2}{a_2^2}$ (elliptic polarization), and $P = (a_1 \cos\psi, a_2 \sin\psi)$ for some $a_1, a_2 > 0$. Hence $\det(P, P_\psi) = a_1 a_2$ at $x = 0$, and for small $|t|$ we have $\det(P, P_\psi) > 0$. For $n = 3$, $H(0, p) = \frac{p_1^2}{a_1^2} + \frac{p_2^2}{a_2^2} + \frac{p_3^2}{a_3^2}$, for some $a_1, a_2, a_> 0$ and in spherical coordinates (ψ_1, ψ_2) where $0 < \psi_1 < \pi$, we find $\det(P, P_\psi) = a_1 a_2 a_3 \sin\psi_1 > 0$ for small t and away from the poles $(0, 0, \pm 1)$.

Example 4.2: When $H(x,p) = \frac{|p|^m}{\rho(x)}$, recall from (2.3) that $P(\psi,\tau) = |P|_\tau \omega(\psi)$ at $t=0$. Since $\det(\omega(\psi), \omega^\perp(\psi)) = 1$, again we have $\det(P, P_\psi) \neq 0$ for small t.

4.2 Construction of the phase function and half-density, general case.

We first construct by HJ theory a generating function Φ of Λ_+ that verifies the initial condition $\Phi|_{t=0} = \langle x, \omega(\psi)\rangle$. Our approach consists in looking for a parametric form of the phase, depending on the initial data through the "front variables" $(X(t,\psi,\tau), P(t,\psi,\tau))$ only.

The most natural Ansatz (recall $\tau + H = E$), would be

$$\Phi_0(x,t,\psi,\tau) = mEt + \langle P(t,\psi,\tau), x - X(t,\psi,\tau)\rangle \tag{4.4}$$

with initial condition $\Phi|_{t=0} = \langle p, x\rangle$, $p = P(\psi,\tau)$ arbitrary. The "θ variables" in Hörmander's definition are then (ψ,τ).

In the simplest example where $n=1$, $\tau + H(x,p) = \tau + p = E$, $\Phi_0 = Et + p(x-t)$ (there are no variable ψ, and $X(t) = t$ is independent of τ). This is actually a parametrization of $\exp tv_H(z)$, $z \in T^*M$, for $t \in \mathbf{R}$ (positive and negative values). But its drawback is to depend on τ (that has eventually to bet set to 0) in a complicated way, when taking variations with respect to parameters.

The second one consists [DoMaNaTu2] in choosing a new coordinate $\lambda = \lambda(\tau)$, $\lambda(0) = 1$, on Λ completing the ψ variables, such that $\partial\Lambda_+$ is given by $\lambda = 1$. We could think of λ as a Lagrange multiplier. We define

$$\Phi(x,t,\psi,\lambda) = mEt + \lambda\langle P(t,\psi), x - X(t,\psi)\rangle \tag{4.5}$$

where now (X,P) are evaluated on Λ_+ (and not on $\Lambda_+(\tau)$). The "θ variables" in Hörmander's definition are then (ψ,λ). In the Example above, $\Phi = Et + \lambda p(x-t)$. The critical value of Φ with respect to θ, is viewed either as a function on the critical set $\widetilde{C}_\Phi$, with the Lagrangian embedding

$$\widetilde{C}_\Phi = \{(x,t,\theta) : \partial_\theta \Phi = 0\} \to \widetilde{\Lambda}_+ \tag{4.6}$$

or on

$$C_\Phi = \{(x,t,\theta) : \partial_\theta \Phi = \partial_t \Phi = 0\} \to \Lambda_+ \tag{4.7}$$

In both cases, (2.12) holds precisely on the critical set.

Moreover, such a phase function is also most suitable in the situation of Sect.5. Eikonal equation (2.12) verified at second order on C_Φ reads

$$\partial_t \Phi + H(x, \partial_x \Phi) - E = \mathcal{O}(|x - X(t,\psi), \lambda - 1|^2) \tag{4.8}$$

Variables τ and λ are diffeomorphically mapped onto each other. In case (1.25) this goes as follows : comparing (4.6) with (4.7) at $t=0$, we get $P(\psi,\tau) = \lambda P(\psi)$, so by (2.3)

$$\lambda = \big(1 - \frac{\tau}{E}\big)^{1/m} \tag{4.9}$$

A similar correspondence holds in Example 4.1.

Proposition 4.1: *Let $H(x,p)$ be positively homogeneous of degree $m \geq 1$ with respect to p on $T^*M \setminus 0$, and $\det(P, P_\psi) \neq 0$ at some point (t', ψ'). Then $\Phi(x,t,\psi,\lambda)$ given in (4.5) is a non-degenerate phase function defining Λ_+ near (t', ψ'), and solves HJ Eq. (4.8), hence (2.5), with initial condition $\Phi|_{t=0} = \langle x, p\rangle$. The positive invariant (inverse) density on Λ_+ we recall from (3.8) is given by*

$$F[\Phi, d\mu_+]|_{C_\Phi} = mE \det(P, P_\psi) \neq 0 \tag{4.10}$$

The critical set C_Φ is then determined by $x = X(t,\psi)$ (which can be inverted as $t = t_1(x,\psi)$) and $\lambda = 1$. It coincides with the set $\kappa(U)$ defined after (4.1).

Recall that in case of Hamiltonian (1.25), the condition $\det(P, P_\psi) \neq 0$ holds at $t = 0$, since $\det(P, P_\psi) = |P|^2$ there, see Example 4.2. Thus Λ_+ is parametrized by $\Phi(x,t,\psi,\lambda)$ for small t. This holds also in case of Example 4.1.

Proof: We have using (2.6), evaluated at $\tau = 0$

$$\partial_t \Phi = \dot{\Phi} = mE + \lambda\langle \dot{P}, x - X(t,\psi)\rangle - \lambda\langle P, \dot{X}\rangle = mE(1-\lambda) + \lambda\langle \dot{P}(t,\psi), x - X(t,\psi)\rangle \tag{4.11}$$

so $\partial_t \Phi = 0$ and $\partial_x \Phi = P(t,\psi)$ along $x = X(t,\psi)$ when $\lambda = 1$. We are left to show that Φ is non degenerate phase function, with (ψ, λ) as "θ-parameters". From (4.5)

$$\begin{aligned} \partial_\psi \Phi &= \lambda\langle \partial_\psi P(t,\psi), x - X(t,\psi)\rangle \\ \partial_\lambda \Phi &= \langle P(t,\psi), x - X(t,\psi)\rangle \end{aligned} \tag{4.12}$$

Let us add t to the "θ-variables", and consider the variational system $\partial_t \Phi = \partial_\psi \Phi = \partial_\lambda \Phi = 0$, which determines the critical set C_Φ. Last 2 equations $\partial_\psi \Phi = 0, \partial_\lambda \Phi = 0$ give an homogeneous linear system in $x - X(t,\psi)$ with determinant $\det(P, P_\psi)$.

So for (t,ψ) near (t', ψ'), the phase is critical with respect to (ψ, λ) precisely for $\lambda = 1$ and $x = X(t,\psi)$, in particular it is critical along $\partial\Lambda_+$ when $\lambda = 1$. Recall t_0 from (4.2). By the discussion above and [DoNaSh, Lemma 6], we find that $\langle P(t,\psi), x - X(t,\psi,\tau)\rangle = 0$ when $\lambda = 1$ has a unique solution $t = t_1(x,\psi) = t_0(x,\psi,\tau = 0)$ satisfying the condition: if $(x;\psi,\lambda = 1) \in C_\Phi$, then $X\big(t_1(x,\psi),\psi\big) = x$. Moreover t_1 is the critical point of $t \mapsto \Phi$ when $\lambda = 1$.

Condition $\det(P, P_\psi) \neq 0$ actually ensures that Φ is a non degenerate phase function, i.e. the vectors $\big(d\partial_t\Phi, d\partial_\psi\Phi, d\partial_\lambda\Phi\big)$ are linearly independent on the set $x = X(t,\psi)$. Namely, look at the variational system and use (4.11) and (4.12) to compute on C_Φ the differentials

$$\begin{aligned} d\dot{\Phi} &= -mE\, d\lambda + \lambda\langle \dot{P}(t,\psi), dx - dX(t,\psi)\rangle \\ d(\partial_\psi \Phi) &= \lambda\langle P_\psi(t,\psi), dx - dX(t,\psi)\rangle \\ d(\partial_\lambda \Phi) &= \langle P, dx - dX(t,\psi)\rangle \end{aligned} \tag{4.13}$$

Introduce the Jacobian (3.8)

$$F[\Phi, d\mu_+]|_{C_\Phi} = \frac{dt \wedge d\psi \wedge d\dot{\Phi} \wedge d(\partial_\lambda\Phi) \wedge d(\partial_\psi\Phi)}{dx \wedge dt \wedge d\psi \wedge d\lambda} \tag{4.14}$$

Here dx is the volume form. Substituting (4.13) into $\omega = dt \wedge d\psi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi)$ we get

$$\omega = -mE\, d\psi \wedge dt \wedge d\lambda \wedge \langle P, dx - dX(t,\psi)\rangle \wedge \langle P_\psi, dx - dX(t,\psi)\rangle +$$
$$d\psi \wedge dt \wedge \langle \dot{P}, dx - dX(t,\psi)\rangle \wedge \langle P, dx - dX(t,\psi)\rangle \wedge \langle P_\psi, dx - dX(t,\psi)\rangle$$

Writing $dX = \dot{X}\, dt + X_\psi\, d\psi$, we check that the second term vanishes, so we are left with

$$\omega = -mE\, \det(P, P_\psi)\, dt \wedge d\psi \wedge d\lambda \wedge dx$$

which gives (4.10). So if $\det(P, P_\psi) \neq 0$, Φ is a non-degenerate phase function, and (4.10) the invariant (inverse) density on Λ_+. ♣

4.3 More or less "θ-variables", general case.

We investigate some configurations of Λ_+, and describe the corresponding Lagrangian singularities. Consider first the critical points of the phase. Let (see Proposition A.1)

$$a = \langle \dot{P}, X_\psi\rangle,\ c = \langle P_\psi, X_\psi\rangle,\ d = \langle \dot{P}, \dot{X}\rangle,\ \alpha = \det(P, P_\psi),\ \beta = \det(P, \dot{P}),\ \gamma = \det(\dot{P}, P_\psi) \tag{4.16}$$

At the critical point

$$-\operatorname{Hess}_{(t,\psi,\lambda)} \Phi = \begin{pmatrix} \lambda\langle \dot{P}, \dot{X}\rangle & \lambda\langle \dot{P}, X_\psi\rangle & \langle P, \dot{X}\rangle \\ \lambda\langle \dot{P}, X_\psi\rangle & \lambda\langle P_\psi, X_\psi\rangle & 0 \\ \langle P, \dot{X}\rangle & 0 & 0 \end{pmatrix} \tag{4.17}$$

thus $\det \operatorname{Hess}_{(t,\psi,\lambda)} \Phi = (mE)^2 \langle P_\psi, X_\psi\rangle$.

When $T_z\Lambda_+$ is not transverse to the vertical plane $V_z = \{(0, \delta p)\} \subset T_z^* M$, we know from Sect.3.2 that we need to express u_h in Fourier representation. This will be needed in Sect.4.5 to derive the commutation formula at some $z \in \Lambda_+$.

Assume $\alpha = \det(P, P_\psi) \neq 0$ (which holds near $t = 0$). Since $\langle P, X_\psi\rangle = 0$, $\langle P_\psi, X_\psi\rangle = 0$ implies $X_\psi = 0$, so at such a point, we need to add new "θ-variables" to $\Phi(x, t, \psi, \lambda)$. If $z \in \Lambda_+$ is such that $X_\psi = 0$, $T_z\Lambda_+$ is not transverse to the vertical plane: indeed $\dim(V_z \cap T_z\Lambda_+) = 1$, for $\langle P, \dot{X}\rangle = 1$ and $\dot{X} \neq 0$.

We proceed as in Sect.3. Consider the embedding $C_\Phi \to \Lambda_+$ as in (4.7). Let x', x'' be a partition of x, we introduce a partial Legendre transformation as in Sect.3, implement the latter equations for the critical point by $\dot{\Phi} = 0$, and compute the Hessian

$$\mathcal{H}(x'', \xi') = \operatorname{Hess}_{(x', t, \psi, \lambda)} \big(\Phi(x, t, \psi, \lambda) - x'\xi'\big)$$

First we try $x' = x_2, x'' = x_1$ so that

$$\mathcal{H}(x_1, \xi_2) = \begin{pmatrix} 0 & \lambda\dot{P}_2 & \lambda\partial_\psi P_2 & P_2 \\ \lambda\dot{P}_2 & -\lambda\langle \dot{P}, \dot{X}\rangle & -\lambda\langle \dot{P}, X_\psi\rangle & -mE \\ \lambda\partial_\psi P_2 & -\lambda\langle \dot{P}, X_\psi\rangle & -\lambda\langle P_\psi, X_\psi\rangle & 0 \\ P_2 & -mE & 0 & 0 \end{pmatrix}$$

and we find, with notations (4.16)

$$\lambda^{-2}\det\mathcal{H}(x_1,\xi_2) = P_2^2(a^2-cd) + 2mEcP_2\dot{P}_2 - mEaP_2^2 - mEaP_2\partial_\psi P_2 + mE(\partial_\psi P_2)^2 \tag{4.20}$$

Similarly, choosing $x' = x_1, x'' = x_2$, we get the same expression with P_2 replaced by P_1. Now if $X_\psi = 0$, then $a = c = 0$, and since $P_\psi \neq 0$ (we assume here $\det(P, P_\psi) \neq 0$, there is a partition of variables x', x'' such that $\det\mathcal{H}(x',\xi'') \neq 0$. Actually variables (x',ξ'') are implicit in the expression of $\mathcal{H}(x',\xi'')$, but fixing (x',x'') on the critical set determines the front variables $(X(t,\psi), P(t,\psi)$.)

Remark 4.2: Compute instead $\mathcal{H}(\xi) = \mathrm{Hess}_{x,t,\psi,\lambda}\big(\Phi(x,t,\psi,\lambda) - x\xi\big)$ at the critical point. We have

$$\mathcal{H}(\xi) = \begin{pmatrix} 0 & 0 & \lambda\dot{P}_1 & \lambda\partial_\psi P_1 & P_1 \\ 0 & 0 & \lambda\dot{P}_2 & \lambda\partial_\psi P_2 & P_2 \\ \lambda\dot{P}_1 & \lambda\dot{P}_2 & -\lambda\langle\dot{P},\dot{X}\rangle & -\lambda\langle\dot{P},X_\psi\rangle & -mE \\ \lambda\partial_\psi P_1 & \lambda\partial_\psi P_2 & -\lambda\langle\dot{P},X_\psi\rangle & -\lambda\langle P_\psi,X_\psi\rangle & 0 \\ P_1 & P_2 & -mE & 0 & 0 \end{pmatrix} \tag{4.21}$$

and $-\lambda^{-3}\det\mathcal{H}(\xi) = \beta^2 c + \alpha^2 d - 2\alpha\beta a - 2mE\alpha\gamma$. When $a = c = 0$, vanishing of the determinant (4.21) reduces to

$$-\lambda^{-3}\det\mathcal{H}(\xi) = \det{}^2(P,P_\psi)\langle -\partial_x H, \partial_p H\rangle - 2mE\det(P,P_\psi)\langle -\partial_x H, P\rangle$$

In the particular case of Hamiltonian (1.15), at $t = 0$, using $\det(P,P_\psi) = |P|^2$, we find

$$\lambda^{-3}|P|^{-1}\det\mathcal{H}(\xi) = \frac{mE^2}{\rho(0)}\langle P, \nabla\rho(0)\rangle$$

which vanishes at a special or residual point. See Remark 3.1.

Remark 4.3: Consider instead the embedding $\widetilde{C}_\Phi \to \widetilde{\Lambda}_+$ as in (4.6), and compute the critical points of $(x,\psi,\lambda) \mapsto \Phi(x,t,\psi,\lambda) - x\xi$. We add $\lambda P(t,\psi) - \xi = 0$ to the previous equations $\partial_\psi\Phi = 0, \partial_\lambda\Phi = 0$. and compute the Hessian

$$\mathcal{H}_0(t,\xi) = \mathrm{Hess}_{(x,\psi,\lambda)}\big(\Phi(x,t,\psi,\lambda) - x\xi\big)$$

at $x = X(t,\psi)$, namely

$$\mathcal{H}_0(t,\xi) = \lambda\begin{pmatrix} 0 & 0 & \partial_\psi P_1 & P_1 \\ 0 & 0 & \partial_\psi P_2 & P_2 \\ \partial_\psi P_1 & \partial_\psi P_2 & -\langle P_\psi, X_\psi\rangle & 0 \\ P_1 & P_2 & 0 & 0 \end{pmatrix}$$

so that $\mathcal{H}_0(t,\xi) = \lambda^4\det(P,P_\psi)$ is non-degenerate, but as a function on $\widetilde{C}_\Phi$ (extended phase-space) instead of C_Φ.

On the other hand, in order to investigate Lagrangian singularities of Λ_+, we need to eliminate some of the "θ-variables". For short we will do it only in the case where Λ_+ is tranverse to the vertical fiber of T^*M.

So let $z(t) = \big(X(t,\psi), P(t,\psi)\big)$ be such that $T_{z(t)}\Lambda_+$ is transverse to the vertical plane $V_{z(t)}$ (namely $X_\psi \neq 0$), or $t = 0$ but $\det \mathcal{H}(\xi) \neq 0$, see (4.21); we parametrize Λ_+ with $\Phi(x,t,\psi,\lambda)$, and $\Phi_{\theta\theta} \neq 0$. When $z(t)$ is a focal point, we discuss according to the case $z(t)$ is a special point or not.

Proposition 4.3: *Let $n = 2$ for simplicity. Let $z \in \Lambda_+$ (possibly on $\partial\Lambda_+$) and assume Φ is a non-degenerate phase function near z (which holds true when $z \in \partial\Lambda_+$ except for exceptional points where $X_\psi = 0$). We have:*
(i) Let $z \in \Lambda_+$ such that $\langle -\partial_x H(z), \partial_p H(z)\rangle = \langle \dot X, \dot P\rangle \neq 0$. Then near z the rank of $d\pi|_{\Lambda_+}$ is 1 when $c = 0$ (i.e. $X_\psi = 0$), and 2 when $c \neq 0$.
(ii) Let $z_0 = \big(X(t_0,\psi_0), P(t_0,\psi_0)\big) \in \Lambda_+$ be a special point for some (t_0,ψ_0). Then the rank of $d\pi|_{\Lambda_+}(z)$ is 1 or 2. When the rank is 1, the tangent space of the caustics at $X(t_0,\psi_0)$ takes the form

$$\frac{\partial x_1}{\partial \psi}\frac{\partial \lambda}{\partial x_1} + \frac{\partial x_2}{\partial \psi}\frac{\partial \lambda}{\partial x_2} = 0 \tag{4.22}$$

where $\frac{\partial\lambda}{\partial x}(x,\psi) \neq 0$.
(iii) Let $z_0 = \big(X(t_0,\psi_0), P(t_0,\psi_0)\big) \in \Lambda_+$ be a residual point for some (t_0,ψ_0), i.e. $\dot P(t_0,\psi_0) = 0$. Then the eikonal is $mE\,dt = \langle P, dx\rangle \neq 0$ at z_0.

Proof: **(i)** On C_Φ we have $\partial_t^2\Phi = -\langle \dot P, \dot X\rangle \neq 0$, so implicit function theorem shows that (for small t) $\dot\Phi = 0$ is equivalent to $t = t(\psi,\lambda)$. Since we have eliminated t, the "θ-parameters" are now (ψ,λ), and we set

$$\Psi(x,\psi,\lambda) = \Phi(x, t(x,\psi,\lambda), \psi, \lambda)$$

Differentiating the relation $\partial_t\Phi = 0$, we get that on C_Φ and for $\lambda = 1$

$$\frac{\partial t}{\partial \psi} = -\frac{\langle P_\psi, \dot X\rangle}{\langle \dot P, \dot X\rangle}, \quad \frac{\partial t}{\partial \lambda} = -\frac{mE}{\langle \dot P, \dot X\rangle}$$

and a straightforward computation using (2.6) yields

$$\Psi''_{\theta\theta} = \Psi''_{(\psi,\lambda)} = \begin{pmatrix} \frac{\langle \dot P, X_\psi\rangle^2}{\langle \dot P, \dot X\rangle} - \langle P_\psi, X_\psi\rangle & mE\frac{\langle \dot P, X_\psi\rangle}{\langle \dot P, \dot X\rangle} \\ * & \frac{(mE)^2}{\langle \dot P, \dot X\rangle} \end{pmatrix} \tag{4.23}$$

Applying (3.5) to the non-degenerate phase function Φ with $N = n = 2$, we find that the rank of $d\pi|_{\Lambda_+}(z)$ is 1 when $c = 0$ or 2 when $c \neq 0$.

(ii) We could attempt to solve $\partial_t\Phi(x,t,\psi,\lambda) = \partial_\psi\Phi(x,t,\psi,\lambda) = 0$ but already for $t = 0$, the determinant of the Hessian of Φ with respect to (t,ψ) vanishes on Λ_+. We can solve instead (locally) $\Phi'_{(t,\lambda)}(x,t,\psi,\lambda) = (\partial_t\Phi, \partial_\lambda\Phi) = 0$. Namely since $\frac{\partial^2\Phi}{\partial t\partial\lambda} = -mE \neq 0$, the implicit function theorem gives $(t,\lambda) = \big(t(x,\psi), \lambda(x,\psi)\big)$.

We want to keep $\lambda(x,\psi) = 1$. Differentiating $\Phi'_{(t,\lambda)} = 0$ along Λ_+ with respect to x and ψ we find, using (2.6) and Hamilton equations

$$\begin{aligned} &\langle \dot P, \dot X\rangle\frac{\partial t}{\partial x} + mE\frac{\partial \lambda}{\partial x} = {}^t\dot P, \quad mE\frac{\partial t}{\partial x} = {}^tP \\ &\langle \dot P, \dot X\rangle\frac{\partial t}{\partial \psi} + mE\frac{\partial \lambda}{\partial \psi} = -\langle \dot P, X_\psi\rangle = -a, \quad mE\frac{\partial t}{\partial \psi} = 0 \end{aligned} \tag{4.24}$$

Assume $\frac{\partial\lambda}{\partial x} = 0$ at z_0. This implies $\dot P = \frac{\langle \dot P, \dot X\rangle}{mE} P$, i.e. $\partial_x H + \frac{1}{mE}\langle -\partial_x H, \dot X\rangle P = 0$. Taking scalar product with $P \neq 0$ we find $\langle \partial_x H, P\rangle + \frac{|P|^2}{mE}\langle -\partial_x H, \dot X\rangle P = 0$, and since z is a special point, $\langle -\partial_x H, \dot X\rangle P = 0$. It follows that $\partial_x H = 0$ which is a contradiction (z is not a residual point).

Now we need $\lambda = \lambda(x,\psi) = 1$; since $\partial_x \lambda \neq 0$, the implicit functions theorem shows that (possibly after renumbering the coordinates) that $x_2 = x_2(x_1,\psi)$. By second line (4.24) we have $\frac{\partial t}{\partial \psi} = 0$, and $-mE\frac{\partial\lambda}{\partial\psi} = a = \langle \dot P, X_\psi\rangle$.

• Assume $a = 0$.

Since we have eliminated t, λ, the "θ-parameter" is simply ψ, and we set

$$\Psi(x_1,\psi) = \Phi(x_1, x_2(x_1,\psi), t_2(x_1, x_2(x_1,\psi)), \psi, \lambda(x_1, x_2(x_1,\psi),\psi)$$

By (3.5) with $N = 1, n = 2$, it follows that $\operatorname{rank} d\pi_x = 2$ if $\partial^2_\psi \Psi(x_1,\psi)|_{x_1=X_1} \neq 0$, and $\operatorname{rank} d\pi_x = 1$ if $\partial^2_\psi \Psi(x_1,\psi)|_{x_1=X_1} = 0$ (X_1 being evaluated at $(t,\psi) = (t_0,\psi_0)$. In the latter case, differentiating $\lambda = \lambda(x,\psi) = 1$ gives $\frac{\partial\lambda}{\partial\psi} + \frac{\partial\lambda}{\partial x}\frac{\partial x}{\partial\psi} = 0$. Since $\frac{\partial\lambda}{\partial\psi} = 0$ at point z, (4.22) easily follows.

• Assume $a \neq 0$. From $\lambda(x,\psi) = 1$, we get $\psi = \psi(x)$ by implicit function theorem, so we have eliminated all "θ-variables" and $\operatorname{rank} d\pi_x = 2$.

(iii) We consider a residual point as a limit of special points, for which $\frac{\partial\lambda}{\partial x} = 0$. Since $a = 0$, we have $\frac{\partial\lambda}{\partial\psi} = \frac{\partial\lambda}{\partial x} = 0$ at z_0, and $\lambda(x,\psi) = 1 + \mathcal{O}(|x - X(t_0,\psi_0), \psi - \psi_0|^2)$. Then (4.24) reduces to $mE\frac{\partial t}{\partial x} = {}^tP$ at z_0, which can be cast in the form $dt = \langle P, dx\rangle \neq 0$. ♣.

For residual points Proposition 4.3 tells nothing however about $\operatorname{rank} d\pi_x(z_0)$. For instance, if $\dot P = 0$, hence $\langle \dot P, \dot X\rangle = 0$ and $\partial_t\Phi = \partial_t^2\Phi = 0$, we could have $\partial_t^3\Phi \neq 0$, and we have a cusp described by Pearcy functions (see e.g. [DoMaNaTu1,App.2]). Alternatively we could think of Hamiltonian p^2 for which $\operatorname{rank} d\pi_x(z_0) = n$ is maximal, or of Hamiltonian $H(x,p) = p_1$ for which $\operatorname{rank} d\pi_x(z_0) = 1$. It tells nothing either about ordinary points, see however Lemma 4.6 when H is of the form (1.15).

Remark 4.2: As mentionned in Example 1.1(7), we can also consider complex quadratic phase functions. For instance replace (4.5) by

$$\Phi(x,t,\psi,\lambda) = mEt + \lambda\langle x - X(t,\psi), B(t,\psi)(x - X(t,\psi))\rangle + i\lambda\mu\langle x - X(t,\psi), B(t,\psi)(x - X(t,\psi))\rangle$$

with $B(t,\psi)$ positive definite. It has the same critical point as the real phase when $\lambda = 1$, with the initial condition $\Phi_{t=0} = \langle x, \omega(\psi)\rangle + i\mu\langle x, B(0,\psi)x\rangle^2$.

4.4 Construction of $E_+(h)f_h$ and the commutation formula.

Here we prove Theorem 1.3. First we look for a solution $v(t,x;h)$ to the Cauchy problem (1.6) that can be expressed as an oscillatory integral $\int e^{i\Phi(x,t,\psi,\lambda)/h} b(x,t,\psi,\lambda)\, d\psi\, d\lambda$, see (1.26).

Assume for simplicity $H(x, hD_x; h)$ has no sub-principal symbol: $H_1(x,p) = 0$. Then it is well known that the principal term $b = b_0$ of the amplitude restricted to C_Φ, since $mE\det(P, P_\psi)$ is the (inverse) density, is of the form

$$b_0 = \big(mE\det(P,P_\psi)\big)^{-1/2} a(\psi,\lambda) \tag{4.25}$$

with a independent of t. Since P, P_ψ are linearly independent, we look for

$$b(x,t,\psi,\lambda) = \big(mE\det(P,P_\psi)\big)^{-1/2} a(\psi,\lambda) + \langle FP(t,\psi) + GP_\psi(t,\psi), x - X(t,\psi)\rangle + \mathcal{O}(|x - X(t,\psi)|^2)$$

where we can determine functions $F = F(t,\psi,\lambda), G = G(t,\psi,\lambda)$ from second derivatives of H by taking variations (already in (4.14)). Set $\widetilde{b} = \langle FP + GP_\psi, x - X(t,\psi)\rangle$, it is readily seen that

$$\partial_t \widetilde{b} + \langle \partial_p H, \partial_x \widetilde{b}\rangle = 0 \text{ on } C_\Phi \tag{4.45}$$

Let v_h solves Cauchy problem (1.6), and $u_h(x) = \frac{i}{h}\int_0^\infty v(t,x;h)\,dt$ (after sticking in a cut-off $\theta_T(t)$ as in (1.10)). We start with computing $(H-E)u_h$ and assume the general case of H homogeneous of degree m in the p variables, and $\det(P,P_\psi) \neq 0$. For simplicity we present the calculations as if H were a differential operator, see [Du]. We assume also the sub-principal symbol of H (as a h-PDO) vanishes.

• Let first $x \in M$ be such that $x = X(t,\psi)$ with $X_\psi \neq 0$, so that $T_z\Lambda_+$ is transverse to the vertical plane. We use representation (1.26). Applying $H - E$ to (1.26), since Φ is just linear in x, we get first

$$e^{-i\Phi/h}(H-E)e^{i\Phi/h}b = (H(x,\partial_x\Phi) - E)b + \frac{h}{i}\langle \partial_p H(x,\partial_x\Phi), \partial_x b\rangle + \mathcal{O}(h^2)$$

By (4.8) we have, integrating by parts

$$\begin{aligned}(H-E)u_h(x) = \int e^{i\Phi/h} b\,d\psi\,d\lambda|_{t=0} + \int_0^\infty dt \int e^{i\Phi/h}\big(\partial_t b + \langle \partial_p H(x,\partial_x\Phi), \partial_x b\rangle\big)\,d\psi\,d\lambda + \\ \frac{i}{h}\int_0^\infty dt \int e^{i\Phi/h}\mathcal{O}(|x - X(t,\psi), \lambda - 1|^2) b\,d\psi\,d\lambda + \mathcal{O}(h^2)\end{aligned} \tag{4.27}$$

To the second integral we apply asymptotic stationary phase [Hö,Theorem 7.7.5]; denote $c(x,t,\psi,\lambda) = \mathcal{O}(|x - X(t,\psi), \lambda - 1|^2) b(x,t,\psi,\lambda)$, and by Φ_c the critical value of $(t,\psi,\lambda) \mapsto \Phi(x,t,\psi,\lambda)$ with non critical degenerate point $\eta(x) = (t = t(x), \psi = \psi(x), \lambda = 1)$ (see (4.17)) we have

$$\begin{aligned}&\frac{i}{h}\int_0^\infty dt \int e^{i\Phi/h} e^{i\Phi/h} c(x,t,\psi,\lambda;h)\,d\psi\,d\lambda = \\ &e^{i\Phi_c/h}\big(\det(\Phi''/2i\pi h)\big)^{-1/2}\big(c(x,\eta(x);h) + \frac{h}{i}\langle \Phi''(x,\eta(x))^{-1}D, D\rangle c(x,\eta(x)) + \mathcal{O}(h^2)\end{aligned}$$

where D denotes the gradient with respect to the 3 variables t,ψ,λ (of course we still assumed $n = 2$). We have $c(x,\eta(x);h) = 0$, but the next term $\langle \Phi''(x,\eta(x))^{-1}D, D\rangle c(x,\eta(x))$ may not vanish because of the partial derivative $\frac{\partial^2 c}{\partial \lambda^2}$, as shows (4.17). So

$$\frac{i}{h}\int_0^\infty dt \int e^{i\Phi/h} e^{i\Phi/h} c(x,t,\psi,\lambda;h)\,d\psi\,d\lambda = \mathcal{O}(h^{3/2})$$

We consider next the first integral in (4.27). Because of (4.25) and (4.26) which implies

$$\partial_t b + \langle \partial_p H(x,\partial_x\Phi), \partial_x b\rangle = \mathcal{O}(|x - X(t,\psi)|)$$

we apply asymptotic stationary phase as before and obtain

$$\int_0^\infty dt \int e^{i\Phi/h}\big(\partial_t b + \langle \partial_p H(x,\partial_x\Phi),\partial_x b\rangle\big)\, d\psi\, d\lambda = \mathcal{O}(h^{5/2})$$

Collecting these estimates in (4.27) yields

$$(H-E)u_h(x) = \int e^{i\Phi/h} b\, d\psi\, d\lambda|_{t=0} + \mathcal{O}(h^{3/2})$$

Since we can choose b_0 in (4.25) to be equal to the amplitude defining f_h, the RHS is just f_h mod $\mathcal{O}(h^{3/2})$.

Remark 4.3: Note the loss of $h^{1/2}$ with respect to the remainder term $\mathcal{O}(h^2)$ when solving the homogeneous equation $(H-E)u_h = 0$, see the discussion after (3.13).

• Take next $x \in M$ near $X(t,\psi)$ with $X_\psi = 0$, by the discussion after (4.21), up to a permutation of x_1 and x_2, we may consider in the mixed representation $H(-hD_{p_1}, x_2, \xi_1, hD_{x_2}; h)$. We try as new phase function

$$\Phi_1(x_2,p_1,t,\psi,\lambda) = mEt - \lambda X_1(t,\psi)p_1 + \lambda P_2(t,\psi)\big(x_2 - X_2(t,\psi)\big)$$

so that the eikonal equation reads

$$\partial_t\Phi_1 + H(-\partial_{p_1}\Phi_1, x_2, p_1, \partial_{x_2}\Phi_1)) = \mathcal{O}(|x_2 - X_2(t,\psi), p_1 - P_1(t,\psi), \lambda - 1|^2)$$

Transport equations are derived similarly. Using again (1.26) we can present

$$(H(-hD_{p_1}, x_2, \xi_1, hD_{x_2}; h) - E)\mathcal{F}^h_{x_1\to p_1} u_h(p_1,x_2)$$

in the form

$$(H(-hD_{p_1}, x_2, \xi_1, hD_{x_2}; h) - E)\frac{i}{h}\int_0^\infty dt \int e^{i\Phi(x_2,p_1,t,\psi,\lambda)/h} b(x_2,p_1,t,\psi,\lambda)\, d\psi\, d\lambda$$

which we compute as before by asymptotic stationary phase. Theorem 1.3 easily follows ♣

4.5 Reduced parametrizations of Λ_+ in case of the "conformal metric", $n = 2$.

In case of the conformal metric we can make the results more precise (at least for $n = 2$), due to fact that $\dot{X}$ is parallel to P. First information is related with the density. By Proposition 4.1, Φ is a non-degenerate phase function parametrizing Λ_+ iff $\det(P, P_\psi) \neq 0$, see (4.10). This certainly holds for small t. We want to allow for larger values of t (the far field). We have no proof that (4.10) is valid everywhere on Λ_+. See however [DoMaNaTu1], Example 6, in case case $m = 1$, and ρ is radially symmetric. In general, this property is related with parametrization of Lagrangian submanifolds, see [Hö,Thm 21.2.16]. In case of the conformal metric, Lemma A.2 readily implies :

Proposition 4.4: *Let $H(x,p)$ be as in (1.25), $n = 2$. Then at least near focal points, representation (4.5) defines a non degenerate phase function parametrizing Λ_+, and the (inverse) density on Λ_+ is $mE\det(P, P_\psi) \neq 0$.*

This holds also when Λ is the "cylinder", see Proposition 5.4 below.

Next information is related to eliminating extra "θ-variables" in the phase function and determining the rank of $\pi_x : \Lambda_+ \to M$. For simplicity we consider only the position representation of u_h, i.e. the case $X_\psi \neq 0$. As in Proposition 4.3, we proceed to find the critical value of $t \mapsto \Phi(x,t,\psi,\lambda)$ when $z(t) = \big(X(t,\psi), P(t,\psi)\big)$ is an ordinary point (which is equivalent to $\langle \dot{X}, \dot{P}\rangle \neq 0$ in case of Hamiltonian (1.25). In a simple scenario there would be at most one special or residual point on each bicharacteristic. Definition 1.5 provides such a scenario. Recall $\partial_t \Phi(x,t,\psi,\lambda=1) = \langle \dot{P}(t,\psi), x - X(t,\psi)\rangle = 0$ This holds on Λ_+, i.e. for $x = X(t,\psi)$. Taking second derivative at critical point gives

$$\partial_t^2 \Phi(X(t,\psi),t,\psi,\lambda=1) = -\frac{m|P(t,\psi)|^{2m-2}}{\rho(X(t,\psi))^3}\langle \nabla\rho(X(t,\psi)), P(t,\psi)\rangle \tag{4.30}$$

so we have to take into account the set of ψ such that $\langle \nabla\rho(X(t,\psi)), P(t,\psi)\rangle = 0$, i.e. of the special or residual points. Consider $f(t,\psi) = \langle \nabla\rho(X(t,\psi)), P(t,\psi)\rangle$, so that $f = 0$ iff $(X(t,\psi)), P(t,\psi)$ is special or residual. Using Hamilton equations, we find

$$\partial_t f(t,\psi) = \frac{m|P(t,\psi)|^{m-2}}{\rho(X(t,\psi))}\big[\langle \nabla^2\rho(X(t,\psi)\cdot P(t,\psi), P(t,\psi)\rangle + \frac{|\nabla\rho(X(t,\psi))|^2}{m\rho(X(t,\psi))}|P(t,\psi)|^2\big] \tag{4.31}$$

Let $z(s,\psi)$ be a special (or residual) point for some $s \geq 0$, then whenever $\partial_t\Phi(t,x,\theta) = 0$ at some $t > s$ (this occurs when the bicharacteristic $t \mapsto z(t)$ projects again on x), $z(t)$ is no longer special (or residual). This holds under assumption (1.35), namely $\partial_t f(t,\psi) > 0$, and $t \mapsto \partial_t^2\Phi(X(t,\psi),t,\psi,\lambda=1)$ is strictly decaying on Λ_+.

• *Ordinary critical points.* They correspond to non degenerate critical points of $t \mapsto \Phi(x,t,\psi,\lambda=1)$.

Lemma 4.6: *Assume (1.35), $n = 2$ (no condition on $\nabla\rho$ is required here). Let $I_\psi = \{t : z(t,\psi) \notin \mathcal{S}(\Lambda_+)\}$ (we have already evaluated $\lambda = 1$) Then I_ψ is an interval, and*

$$\forall t \in I_\psi,\ x = X(t,\psi) \iff t = t_1(x,\psi) \quad on \quad C_\Phi \tag{4.33}$$

where t_1 is a smooth function. Moreover $\pi_x : \Lambda_+ \to M$ at every ordinary critical point has same rank as the symmetric matrix (4.23) i.e. $d\pi_x$ has rank 1 ($\langle P_\psi, X_\psi\rangle = 0$) or 2 ($\langle P_\psi, X_\psi\rangle \neq 0$).

Proof: Note that when $t = 0$, $f(0,\psi) = \langle \nabla\rho(0), P(\psi)\rangle$. So when $f(0,\psi) > 0$, 0 is non-degenerate critical point of $t \mapsto \Phi(x,t,\psi,\lambda)$, and the implicit function theorem shows that (4.33) holds. Since $t \mapsto f(t,\psi)$ is increasing, this holds for all t in the maximal interval of definition of the integral curve starting at $(0, P(\psi))$. When $f(0,\psi) < 0$ instead, (4.33) holds on an interval ending at some s such that $f(s,\psi) = 0$. Let us compute the rank of $\pi_x : \Lambda_+ \to M$ at an ordinary point. Let $U = \{(t,\psi) : t \in I_\psi,\ z(t) = (X(t,\psi), P(t,\psi)) \notin \mathcal{S}(\Lambda_t)\}$, then the same computation as in Proposition 4.3 shows that U is a canonical chart rank 1 or 2, which gives the Lemma. ♣

• *Special and residual critical points.* Near the end point s of I_ψ we can solve (locally) as in Proposition 4.3, $\partial_t\Phi(x,t,\psi,\lambda) = \partial_\lambda\Phi(x,t,\psi,\lambda) = 0$ which gives $t = t(x,\psi)$ and $\lambda = \lambda(x,\psi)$. Namely, the Hessian of Φ with respect to (t,λ) at $(s,1)$ has determinant $-\big(\frac{\partial^2\Phi}{\partial t\partial\lambda}\big)^2 = -(mE)^2 < 0$ on Λ_+. So

if $(X(s,\psi),P(s,\psi))$ is a special point then $(s,1)$ is a non degenerate point of $(t,\lambda)\mapsto\Phi(x,t,\psi,\lambda)$. Integrating Hamilton equations also for $t<0$ gives the Lagrangian manifold $\Lambda_-\cup\Lambda_+$. So there is no loss of generality in assuming the special point is at $s=0$. The following Lemma strengthens Proposition 4.3 in case $X_\psi\neq 0$.

Lemma 4.7: *Let $n=2$ and H be as in (1.25). Assume* $\det(P,P_\psi)\neq 0$.

(i) Assume $z(s)=(X(s,\psi),P(s,\psi))\in\Lambda_+$ be a special point (hence $\nabla\rho(x(s))\neq 0$). If $X_\psi=0$, then $\operatorname{rank} d\pi_x(z(s))=1$ *as in Proposition 4.3 (i). If $X_\psi\neq 0$, then $ac\neq 0$ so that* $\operatorname{rank} d\pi_x(z)=2$. *Near $z(s)$, Λ_+ is given by $t=t(x)$, $\psi=\psi(x)$, and $\frac{\partial t}{\partial x}\neq 0$, $\frac{\partial\psi}{\partial x}\neq 0$. The constraint $\lambda=1$ takes the form*

$$\frac{\partial\lambda}{\partial\psi}+\Big|\frac{\partial\psi}{\partial x}\Big|^{-2}\langle\frac{\partial\lambda}{\partial x},\frac{\partial\psi}{\partial x}\rangle=0 \tag{4.41}$$

(ii) Assume $z(s)=(X(s,\psi),P(s,\psi))\in\Lambda_+$ be a residual point (i.e. $\nabla\rho(x(s))=0$). If $X_\psi\neq 0$, then $c\neq 0$ and $\operatorname{rank} d\pi_x(z)=2$.

Proof: As in Proposition 4.3, the relations $\partial_t\Phi=\partial_\lambda\Phi=0$ being given by $(t,\lambda)=\big(t(x,\psi),\lambda(x,\psi)\big)$ we use (4.24). Since $X_\psi\neq 0$, $\langle P,X_\psi\rangle=0$ and $\det(P,P_\psi)>0$ we have $c\neq 0$.

(i) By the same geometric argument we have $\dot P\neq 0$ by (2.2), and since $\langle\nabla\rho,P\rangle=0$, the relation $a=\langle\dot P,X_\psi\rangle=0$ would contradict $X_\psi\neq 0$. So by second line (4.24), $-mE\partial_\psi\lambda=\langle\dot P,X_\psi\rangle\neq 0$, or $\partial_\psi\lambda\neq 0$. Now we need $\lambda=\lambda(x,\psi)=1$; since $\partial_\psi\lambda\neq 0$, the implicit functions theorem shows that $\psi=\psi(x)$. Then we have

$$x=X(t,\psi)\iff t=t(x),\psi=\psi(x)\text{ on }\partial_t\Phi(x,t,\psi,\lambda=1)=\partial_\lambda\Phi(x,t,\psi,\lambda=1)=0 \tag{4.42}$$

Differentiating $\lambda=\lambda(x,\psi)$ gives

$$\frac{\partial\lambda}{\partial x}+\frac{\partial\lambda}{\partial\psi}\frac{\partial\psi}{\partial x}=0$$

and together with the first equation (4.24)

$$\langle P_\psi,\dot X\rangle^t\big(\frac{\partial\psi}{\partial x}\big)=mE\dot P-\langle\dot P,\dot X\rangle P$$

Let us show that $\frac{\partial\psi}{\partial x}\neq 0$. Otherwise, we would have $\langle P,\dot X\rangle\dot P=\langle\dot P,\dot X\rangle P$, and since we know that $\langle\nabla\rho(X(s,\psi)),P(s,\psi)\rangle=0$, this would contradict the fact that $\dot P$ is parallel to $\nabla\rho(X(s,\psi)$. Moreover $\frac{\partial t}{\partial\psi}=0$, $\frac{\partial t}{\partial x}=\frac{1}{mH}{}^tP\neq 0$, which readily gives (4.41). To compute the rank of π_x at a special point, we are left to compute second derivative of the critical value, namely $-\partial_\psi^2\Psi=ac\neq 0$, so we conclude as in Proposition 4.3 that $\operatorname{rank}\pi|_{\Lambda_+}$ is 2.

(ii) Thinking of a residual point as the limit of special points, (4.24) shows that $\frac{\partial\lambda}{\partial x}=\frac{\partial\lambda}{\partial\psi}=0$, and (4.24) reduces to $mE\frac{\partial t}{\partial x}={}^tP,\frac{\partial t}{\partial\psi}=0$. Note that on C_Φ, (4.17) gives $\det\operatorname{Hess}_{(t,\psi,\lambda)}\Phi=(mE)^2\langle P_\psi,X_\psi\rangle\neq 0$ if $X_\psi\neq 0$, so by the implicit functions theorem

$$(t,\psi,\lambda)=\big(t_2(x),\psi_2(x),\lambda_2(x)\big)$$

Let us check again that $\lambda_2(x) = 1$: Differentiating $\Phi'_{t,\psi,\lambda}(x,t,\psi,\lambda) = 0$ with respect to x, λ gives the triangular system

$$\langle P_\psi, \dot{X}\rangle^t(\frac{\partial t_2}{\partial x}) + \langle P_\psi, X_\psi\rangle^t(\frac{\partial \psi_2}{\partial x}) = P_\psi$$
$$\langle \dot{P}, \dot{X}\rangle^t(\frac{\partial t_2}{\partial x}) + \langle \dot{P}, X_\psi\rangle^t(\frac{\partial \psi_2}{\partial x}) + mH^t(\frac{\partial \lambda_2}{\partial x}) = \dot{P}$$
$$mH^t(\frac{\partial t_2}{\partial x}) = P$$

Since $\dot{P} = 0$, using $a = 0$ (see (4.16)) this reduces to

$$c^t(\frac{\partial \psi_2}{\partial x}) = P_\psi, \quad \frac{\partial \lambda_2}{\partial x} = 0, \quad mH^t(\frac{\partial t_2}{\partial x}) = P$$

and in particular $\lambda = \lambda_2(x) = 1$. There are no "$\theta$-parameters" left and so $\operatorname{rank} d\pi_x(z(s) = 2$. Then $\det(P, P_\psi) \neq 0$ implies $\det(\frac{\partial \psi_2}{\partial x}, \frac{\partial t_2}{\partial x}) \neq 0$. ♣

Note that if $X_\psi = 0$ at a focal point of Λ_+, then $P_\psi \neq 0$ (otherwise this would violate property (3) of Proposition A.1).

From Lemma 4.7, the set of focal points which are also special points is $\mathcal{S}(\Lambda_t) \cap \mathcal{F}(\Lambda_t) = \{\psi : \langle P_\psi, X_\psi\rangle = 0\}$. In Example 2.3, we find $\langle P_\psi, X_\psi\rangle = \frac{1}{2}\sinh(2f(t))$ >0 for all $t > 0$ and vanishes at $t = 0$.

Now to find the canonical charts for the phase functions, we use a connectedness argument. Assume (1.35), and let s be the supremum of I_ψ, we have $f(s,\psi) = 0$. Since $G(\rho)(s,\psi) > 0$, we have $f(t,\psi) > 0$ for all $t > s$, so all points $z(t) = (X(t,\psi), P(t,\psi))$ for $t > s$ are ordinary points. So far we proved (except for the case $X_\psi = 0$ which can be handled similarly by replacing Φ by its Legendre transformation):

Proposition 4.8: *Let $H(x,p) = \frac{|p|^m}{\rho(x)}$, $n = 2$. Then there exists globally a smooth solution Ψ of HJ equation $H(x, \partial_x\Phi) = E$. Let $C_\Phi = \{(t,x,\psi), t > 0, \psi \in \mathbf{S}^1 : \partial_t\Phi = \partial_\lambda\Phi = \partial_\psi\Phi = 0\}$. Then the embedding*

$$\iota_\Phi : C_\Phi \to T^*\mathbf{R}^2, \ (t,x,\psi,\lambda) \mapsto (x, \partial_x\Phi(x,t,\psi,\lambda = 1))$$

such that $\iota_\Phi(C_\Phi) \subset \Lambda_+$ consists in charts of rank 1 or 2 [the rank is never 0 since $p \neq 0$ in the energy shell $H = E$]. Under the defocussing condition (1.35) these charts can intersect the set of special points $\mathcal{S}$ only along a line.

Remark 4.9: The canonical charts in Λ_+ where $\Phi = \Phi(x)$, i.e. of WKB type are of course of maximal rank 2, in particular there is a WKB solution near a special point z such that $\langle X_\psi, P_\psi\rangle \neq 0$.

5. f_h is supported microlocally on the "cylinder"

We recall Λ from (1.3). When $n = 2$ this is the wave-front set of Bessel function $f_h(x;h) = J_0(\frac{|x|}{h})$. The new difficulty lies in $\partial\Lambda_+$, since φ and ψ are not independent variables on the energy surfaces $H = E - \tau$.

5.1 Non degeneracy conditions

Let us check first the Lagrangian intersection.

Definition 5.1: *The point $z \in \Lambda$ is called glancing if $v_H(z) \in T_z\Lambda$. We denote by $\mathcal{G}(\Lambda)$ the set of glancing points on Λ.*

So (Λ, Λ_+) is an intersecting pair whenever $\mathcal{G}(\Lambda) = \emptyset$. Hamiltonian flow preserves the set of glancing points, i.e. $\exp tv_H(\mathcal{G}(\Lambda)) = \mathcal{G}(\Lambda_t)$.

Proposition 5.2: *Let H be homogeneous of degree m, and Λ be the "cylinder". With the notation above, $z \in \partial\Lambda_+$ is a glancing point iff*

$$\partial_p H + \varphi\partial_x H = mH\omega(\psi), \quad \langle -\partial_x H, \omega(\psi)\rangle = 0, \quad H(z) = E \tag{5.2}$$

(in particular z is a special point). More generally ($n = 2$ for simplicity) assume $z(t) \in \Lambda_t$ is a glancing point, (φ, ψ) are local coordinates on Λ_t near $z(t)$, and (P, P_ψ) is a basis of $\mathbf{R}^2$ at $z(t) = (X(t, \varphi, \psi), P(t, \varphi, \psi))$. Let

$$R = R(t, \varphi, \psi) = |X_\psi|^2\big(\langle -\partial_x H, P_\psi\rangle - mH\langle P_\varphi, P_\psi\rangle\big) - |P_\psi|^2\big(\langle \partial_p H, X_\psi\rangle - mH\langle X_\varphi, X_\psi\rangle\big)$$

Then $H(z(t)) = E$, and

$$\begin{aligned} &\langle P, P_\psi\rangle \neq 0 \implies \langle -\partial_x H - mHP_\varphi, P_\psi^\perp\rangle = 0 \text{ and } R = 0 \\ &\langle P, P_\psi\rangle = 0 \implies \langle -\partial_x H, P\rangle = mH\langle P, P_\varphi\rangle \text{ and } R = 0 \end{aligned} \tag{5.2$_t$}$$

Proof: We complete $\omega(\psi)$ in $\mathbf{S}^{n-1}$ into a (direct) orthonormal basis $\omega^\perp(\psi) = \big(\omega_1(\psi), \cdots, \omega_{n-1}(\psi)\big)$ of $\mathbf{R}^n$, and denote by $\omega^\perp(\psi)\delta\psi = \omega_1(\psi)\delta\psi_1 + \cdots + \omega_{n-1}(\psi)\delta\psi_{n-1}$ a section of $T\mathbf{S}^{n-1}$, $\delta\psi_j \in \mathbf{R}$. The tangent space $T_z\Lambda$ has the parametric equations

$$\delta X = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi, \quad \delta P = \omega^\perp(\psi)\delta\psi, \quad (\delta\varphi, \delta\psi) \in \mathbf{R}^n$$

so $v_H \in T_z\Lambda$ iff there exist $(\delta\varphi, \delta\psi)$ such that

$$\partial_p H = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi, \quad -\partial_x H = \omega^\perp(\psi)\delta\psi$$

Taking scalar products with $\omega(\psi), \omega^\perp(\psi)$, and using Euler identity, we get $\delta\varphi = \langle \partial_p H, P(\psi)\rangle = mH$, $\delta\psi = \langle -\partial_x H, \omega^\perp(\psi)\rangle$. Since $(\omega(\psi), \omega^\perp(\psi))$ form a basis of $\mathbf{R}^n$, relations (5.2) are necessary and sufficient for $v_H \in T_z\Lambda_+ \cap T_z\Lambda$.

Let now $t > 0$, $(\delta X, \delta P) \in T_{z(t)}\Lambda(t)$ is given by $\delta X = X_\varphi\delta\varphi + X_\psi\delta\psi$, $\delta P = P_\varphi\delta\varphi + P_\psi\delta\psi$, so if $v_H \in T_{z(t)}\Lambda(t)$, due to Euler identity and (2.11) we still have $\delta\varphi = mH$ and

$$\partial_p H = mHX_\varphi + X_\psi\delta\psi, \quad -\partial_x H = mHP_\varphi + P_\psi\delta\psi$$

Taking scalar product of the first equation by X_ψ gives

$$|X_\psi|^2\delta\psi = \big(\langle \partial_p H, X_\psi\rangle - mH\langle X_\varphi, X_\psi\rangle\big)$$

Taking scalar product of the second equation with P_ψ and P gives

$$\delta\psi = |P_\psi|^{-2}\big(\langle -\partial_x H, P_\psi\rangle - mH\langle P, P_\varphi\rangle\big), \ \langle -\partial_x H, P\rangle = mH\langle P, P_\varphi\rangle + \langle P, P_\psi\rangle\delta\psi$$

Discussing according to the fact P is orthogonal to P_ψ or not readily gives $(5.2)_t$. ♣

Since the glancing property is invariant under the Hamiltonian flow, (5.2) and $(5.2)_t$ are actually equivalent.

Example 5.1: When $H = p^2$, all points are glancing. When $H = \frac{|p|^m}{\rho(x)}$, $z(0)$ is a glancing point iff

$$\text{either}: \ \varphi \neq 0 \text{ and } \nabla\rho = 0, \ \text{or}: \ \varphi = 0 \text{ and } \langle \nabla\rho(0), \omega(\psi)\rangle = 0 \tag{5.3}$$

Second condition means that if $z(0) = (0, \omega(\psi))$ is a special point. Assuming (1.25) it follows that if $z(0)$ is a glancing point, $z(t)$ will be glancing but never special at later $t > 0$.

Example 5.2: Consider $H(x,p) = \langle \mu, p\rangle$, where $\mu \in \mathbf{R}^2$ (constant coefficients case), then (5.2) will always occur at some point ψ. Namely, $X(t,\psi) = \mu t + \varphi\omega(\psi)$, $P(t,\psi) = \omega(\psi)$. The system $x = X(t,\varphi,\psi)$ has a unique solution $(t_1(x,\psi), \varphi_1(x,\psi), \psi)$ provided $\langle \mu, \omega^\perp\rangle \neq 0$, i.e. v_H is transverse to Λ, and $\psi = \psi_0$ belongs to $\{\tau = 0\}$, i.e. $\langle \mu, \omega(\psi_0)\rangle = E$. It is given by

$$t = \frac{\langle x, \omega^\perp(\psi_0)\rangle}{\langle \mu, \omega^\perp(\psi_0)\rangle}, \quad \varphi = \frac{1}{\langle \mu, \omega^\perp(\psi_0)\rangle} \det \begin{pmatrix} \langle x, \omega(\psi_0)\rangle & \langle x, \omega^\perp(\psi_0)\rangle \\ \langle \mu, \omega(\psi_0)\rangle & \langle \mu, \omega^\perp(\psi_0)\rangle \end{pmatrix}$$

In particular $\varphi_1(x, \psi = \psi_0) = 0$ if the determinant of Gram matrix vanishes, i.e. when x is parallel to μ.

Example 5.3: Assume again $M = \mathbf{R}^2$. For $a, b \in \mathbf{R}$, let $\rho(x) = 1 + ax_1^2 + bx_2^2$. Consider Hamiltonian on $T^*\mathbf{R}^2$ of the form

$$H(x,p) = \frac{\langle \mu, p\rangle}{\rho(x)}$$

with $\mu = (1,0)$. A computation shows that, with $x_1 = \varphi\cos\psi, x_2 = \varphi\sin\psi$

$$\begin{aligned} \langle -\partial_x H, \omega(\psi)\rangle &= \frac{2\varphi\cos\psi}{\rho^2(x)}(a\cos^2\psi + b\sin^2\psi) \\ -\rho^2(x)\langle -\partial_x H, \omega(\psi)\rangle &= \big(\rho(x) + 2\varphi^2\cos^2\psi(a-b)\big)\sin\psi \end{aligned}$$

so choosing $a > b > 0$ and $E > 1$, we see that $\mathcal{G}(\Lambda) = \emptyset$. Otherwise, there may be a unique glancing point near $x = 0$. Such a non-transversality is called a *kiss* in [ElGr].

Example 5.4: There are in general glancing points when $\Lambda = T^*_{\{x_n=0\}}\mathbf{R}^n$ and H is totally characteristic at $x_n = 0$, see [Mel].

Let us first describe $\partial\Lambda_+(\tau)$ in case (1.15). The intersection of Λ with the energy surface $H = E - \tau$, is given by the implicit equation $H(\varphi\omega(\psi), \omega(\psi)) = E - \tau$, which usually defines a smooth 1-D isotropic submanifold. To avoid boundaries, we assume $\partial\Lambda_+(\tau)$ is a closed isotropic manifold of dimension $n-1$, i.e. $\partial\Lambda_+(\tau)$ is defined by a map

$$\mathbf{R} \times \mathbf{T}^{n-1} \to \partial\Lambda_+(\tau) \ : \ (\tau, \Theta) \mapsto (\varphi, \psi) = \big(\varphi(\tau,\Theta), \psi(\tau,\Theta)\big) \tag{5.4}$$

verifying

$$\rho\big(\varphi(\tau,\Theta)\,\omega \circ \psi(\tau,\Theta)\big) = \frac{1}{E-\tau} \tag{5.5}$$

It is clear in this case that Λ_+ can be parametrized by (t,ψ). Recall from Proposition A.1 that $\langle \dot{X}, P_\psi\rangle = \langle \dot{P}, X_\psi\rangle$. So taking the limit $t\to 0_+$ readily implies

$$\varphi\langle \nabla\rho(\varphi\omega(\psi),\omega^\perp(\psi)\rangle = 0 \tag{5.6}$$

on $\partial\Lambda_+$. By Proposition 5.2, if $z\in\partial\Lambda_+$ is ordinary, and $\varphi\neq 0$, then z is not glancing, and by (5.6) $\langle\nabla\rho(\varphi\omega(\psi),\omega(\psi)\rangle = \pm|\nabla\rho|\neq 0$. Taking derivative of (5.5) with respect to τ,Θ we get

$$\partial_\tau\varphi(\tau,\Theta) = \pm|\nabla\rho|^{-1}(E-\tau)^{-2},\ \partial_\Theta\varphi(\tau,\Theta)=0$$

So $\varphi=\varphi(\tau)$, which can be inverted by implicit functions theorem, so $\tau=\tau(\varphi)$ and it follows that we can take $\psi(\tau,\Theta)=\Theta$, i.e. $\Theta=\psi$. So in case (1.15) Λ_+ can be parametrized by (t,ψ), and $\varphi=\varphi(\tau)$.

In the general case, we replace (5.5) by

$$H\big(\varphi(\tau,\Theta)\omega\circ\psi(\tau,\Theta),\omega\circ\psi(\tau,\Theta)\big) = E-\tau \tag{5.8}$$

Assume Λ_+ can be again parametrized by (t,ψ), then we still have $\langle -\partial_x H, X_\psi\rangle = \langle\partial_p H, P_\psi\rangle$. If $z\in\partial\Lambda_+$ is an ordinary point, then differentiating (5.8) with respect to τ,Θ gives

$$\partial_\tau\varphi = \frac{1+\varphi(\partial_\tau\psi-1)\langle\partial_x H,\omega^\perp(\psi)\rangle}{\langle -\partial_x H,\omega(\psi)\rangle},\ \partial_\Theta\varphi = \frac{\varphi(\partial_\Theta\psi-1)\langle\partial_x H,\omega^\perp(\psi)\rangle}{\langle -\partial_x H,\omega(\psi)\rangle} \tag{5.9}$$

which simplifies again if $\Theta=\psi$, so that $\partial_\tau\psi=0$.

Next we discuss the properties of Hamiltonian flow issued from Λ using eikonal coordinates as in Sect.4.1. Consider Hamiltonian $\tau+H(x,p)$ on $T^*(\mathbf{R}^n\times\mathbf{R}_+)$ and the Lagrangian manifold $\widetilde{\Lambda}_+=\bigcup_\tau\Lambda_+(\tau)$ in the extended phase-space, i.e.

$$\begin{aligned}\widetilde{\Lambda}_+ = \{(x,p;t,\tau):\tau+H(x,p)=E, z(t)=\big(X(t,\varphi,\psi,\tau),P(t,\varphi,\psi,\tau)\big),\\ z(0)\in\partial\Lambda_+(\tau), t\geq 0\}\subset T^*(M\times\mathbf{R}_+)\end{aligned} \tag{5.10}$$

Assume that

$$\widetilde{\iota}:\widetilde{\Lambda}_+\to T^*(M\times\mathbf{R}_+)$$

is a Lagrangian embedding (this holds true if we take t,τ small enough). Let $\widetilde{X}=\binom{t}{X(t,\varphi,\psi,\tau)}$, and $\widetilde{P}=\binom{\tau}{P(t,\varphi,\psi,\tau)}$ in the extended phase-space. The action on $\widetilde{\Lambda}_+$ is of the form

$$\langle\widetilde{P},d\widetilde{X}\rangle = \langle P(t,\varphi,\psi,\tau),dX(t,\varphi,\psi,\tau)\rangle+\tau\,dt \tag{5.11}$$

Here φ is still considered as a variable.

With the notations of (4.1)-(4.2), let $\widetilde{n}=n+1$, $k=2$, $\widetilde{\psi}=\psi$, $\widetilde{\phi}=(t,\varphi)$. By the preceding discussion, we may assume $\tau=\tau(\varphi,\psi)$, which simplifies further to $\tau=\tau(\varphi)$ in case of Hamiltonian (1.15). Recall $S(t,\varphi,\psi,\tau)=\varphi+m(E-\tau)t+S_0$ from (2.10). In the extended phase-space, we set $\widetilde{S}(t,\varphi,\psi,\tau)=S(t,\varphi,\psi,\tau)+t\tau$, one has also to differentiate with respect to τ, so that $dS+\tau\,dt+t\,d\tau = \langle P,dX\rangle+\tau\,dt$, or

$$d\varphi+m(E-\tau)\,dt+(1-m)t\,d\tau = \langle P,\dot{X}\rangle\,dt+X_\varphi\,d\varphi+X\psi\,d\psi+X_\tau\,d\tau$$

using $d\tau = \tau_\varphi\, d\varphi + \tau_\psi\, d\psi$ we get by identification (still with the notation $y_x = \frac{\partial y}{\partial x}$)

$$\begin{aligned} &\langle P, X_\varphi\rangle + \tau_\varphi \langle P, X_\tau\rangle = 1 + (1-m)t\tau_\varphi \\ &\langle P, X_\psi\rangle + \tau_\psi \langle P, X_\tau\rangle = (1-m)t\tau_\psi \\ &\langle P, \dot X\rangle = mH \end{aligned} \tag{5.12}$$

We look for a "left inverse" of

$$(\dot{\widetilde X}, \widetilde X_\varphi) = \begin{pmatrix} 1 & 0 \\ \dot X & X_\varphi + X_\tau \tau_\varphi \end{pmatrix}$$

We try $\widetilde\Pi_1 = \begin{pmatrix} 1 & -mH \\ 0 & P \end{pmatrix}$, which gives, using (5.12)

$$\widetilde\Pi_1^* \begin{pmatrix} 1 & 0 \\ \dot X & X_\varphi + X_\tau \tau_\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + (1-m)t\tau_\varphi \end{pmatrix}$$

so we choose

$$\widetilde\Pi = \widetilde\Pi_1 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-(m-1)t\tau_\varphi} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha m H \\ 0 & \alpha P \end{pmatrix},\ \alpha = \alpha(t,\varphi,\psi) = \big(1 + (1-m)\tau_\varphi\big)^{-1} \tag{5.13}$$

which is well defined when $m = 1$ or whenever $1 - (m-1)t\tau_\varphi > 0$. By [DoNaSh,Lemma 6] (which doesn't assume $\widetilde k$ to be the rank of $\widetilde\Lambda_+$), we know in particular that the equation

$$\widetilde\Pi^* \begin{pmatrix} 0 \\ x - X(t,\varphi,\psi,\tau) \end{pmatrix} = 0 \Longleftrightarrow \langle P(t,\varphi,\psi,\tau), x - X(t,\varphi,\psi,\tau)\rangle = 0 \tag{5.14}$$

has a unique solution $t = t_1(x,\psi,\tau)$, $\varphi = \varphi_1(x,\psi,\tau)$. To meet the initial condition, we have seen that $\tau = \tau(\varphi,\psi)$, which can be inverted as $\varphi = \varphi(\tau,\psi)$. So we interprete φ as a parameter we have to adjust so to stay on the energy surface, say $\tau = 0$, in the same way (Proposition 4.3(ii)) that λ was a function of (x,ψ), so that $\lambda(x,\psi) = 1$. So we can omit τ from the notations, its role being played by φ. Thus as in [DoNaSh,Lemma 6] we can prove the following result, at least for small $t > 0$.

Proposition 5.3: *There is a open set $U \subset \mathbf{R}^{n+1}$, $(t,\varphi,\psi) \in U$, such that*

$$\kappa : U \mapsto \mathbf{R}^{n+1}_{t,x} \times \mathbf{R}^{n-1}, \quad (t,\varphi,\psi) \mapsto \big(\widetilde X(t,\varphi,\psi), \psi\big) \tag{5.15}$$

is an embedding. Then there is a neighbhd V of $C = \kappa(U)$, such that

$$\langle P(t,\varphi,\psi), x - X(t,\varphi,\psi)\rangle = 0,\ (t,x,\psi) \in V \tag{5.16}$$

has a unique solution $t = t_1(x,\psi)$, $\varphi = \varphi_1(x,\psi)$ satisfying the condition

$$(t,x,\psi) \in C \implies x = X(t_1(x,\psi), \varphi_1(x,\psi), \psi) \tag{5.17}$$

Consider now $\mathcal{M}(t,\varphi,\psi,\tau) = \big(\widetilde\Pi; \widetilde P_{\widetilde\psi} - \widetilde P_{\widetilde\phi} \widetilde\Pi^* \widetilde X_{\widetilde\psi}\big)$ as in Sect.4.1. In [DoNaSh, Lemma 9], $\mathcal{M}$ is constructed in a canonical chart of rank $\widetilde k$ for the Lagrangian embedding (without boundary) $\iota : \Lambda \to T^*M$, so that $\mathcal{M}$ becomes an invertible matrix. Here we consider instead the expression of

$\mathcal{M}$ as an Ansatz, and check that $\det \mathcal{M}$ defines the (inverse) density on Λ_+. With α as in (5.13) we have

$$\mathcal{M} = \begin{pmatrix} 1 & -mH\alpha & \tau_\psi - \alpha(m-1)\tau_\varphi\tau_\psi t \\ 0 & \alpha P & P_\psi + \tau_\psi P_\tau + \alpha(m-1)\tau_\psi t(P_\varphi + \tau_\varphi P_\tau) \end{pmatrix}$$

so that

$$\det \mathcal{M} = \alpha(t,\varphi,\psi)\det\big(P, P_\psi + \alpha\tau_\psi(P_\tau + (m-1)tP_\varphi)\big) \tag{5.19}$$

Note that when $m = 1$ (which implies that $\tau + H(x,p)$ is homogeneous of degree 1 as an Hamiltonian on $T^*(M \times \mathbf{R})$), $\det \mathcal{M} = \det(P, d_\psi P)$, where

$$d_\psi P = P_\psi + \tau_\psi P_\tau \tag{5.20}$$

On the other hand when $\tau_\psi = 0$, $\det \mathcal{M} = \alpha(t,\varphi,\psi)\det(P, P_\psi)$ is non zero for small t. By the discussion above, this holds for Hamiltonian (1.15), so for $m = 1$ we recover the same (inverse) density on Λ_+ as in Sect.4. Moreover $\det \mathcal{M} = \det(P, P_\psi) > 0$ with the condition of Proposition 4.4 that carries to this case.

5.2 Construction of the phase function in the extended phase-space

From now on we consider a general Hamiltonian $H(x,p)$ positively homogeneous of degree $m = 1$, so that $\det \mathcal{M} = \det(P, d_\psi P)$ can be interpreted as the (inverse) density on Λ_+ when $\det(P, d_\psi P) \neq 0$.

Consider HJ equation for the phase function parametrizing Λ_+ as in (2.12)

$$\partial_t \Phi + H(x, \partial_x \Phi) = E, \;\; \Phi|_{t=0} = \langle x, \omega(\psi)\rangle \tag{5.25}$$

We find as before an integral manifold $\Lambda_{\Phi,t} = \{p = \partial_x \Phi(t,x)\} \subset T^*\mathbf{R}^n$. As before we let Φ depend on additional "θ-variables". As in (4.5), we could make the choice

$$\Phi_0(x,t,\psi,\varphi,\tau) = m(E-\tau)t + \varphi + \langle P(t,\varphi,\psi,\tau), x - X(t,\varphi,\psi,\tau)\rangle$$

as in [DoNaSh, formula (2.40)], which allows for some uniformity in τ. But as in Sect.4, we add λ to the "θ"-variables, and put instead

$$\Phi(x,t,\psi,\varphi,\lambda,\tau) = mEt + \varphi + \lambda\langle P(t,\varphi,\psi,\tau), x - X(t,\varphi,\psi,\tau)\rangle \tag{5.26}$$

where the integral curve (X,P) will be eventually computed on $\tau = 0$. Remember here $\tau = \tau(\varphi,\psi)$. We check that for $\lambda = 1$

$$\Phi|_{t=0} = \int P\,dX + \langle \omega(\psi), x\rangle - \varphi = \varphi + \langle \omega(\psi), x\rangle - \varphi = \langle \omega(\psi), x\rangle$$

so $\Phi|_{t=0} = \langle \omega(\psi), x\rangle$ satisfies the initial condition. The dependence on λ in the phase thus becomes linear, but then we need to restrict to $\tau = 0$. Variables $(x,t,\varphi,\psi,\lambda)$ will be used in various combinations, reflecting the role of the boundary $\partial\Lambda_+$, or more generally $\Lambda_+ \cap \Lambda_t$. In the sequel we keep

$m > 1$ as a free parameter, but this really makes sense when $m = 1$, in order to interprete $\det(P, d_\psi P)$ as an (inverse) density on Λ_+.

Taking partial derivatives in (5.26) with respect to variables t, ψ, λ, with $\partial_t \int \langle P, dX \rangle = mE$, $\partial_\varphi \int \langle P, dX \rangle = 1$, $\partial_t \int \langle P, dX \rangle = 0$, we find

$$\begin{aligned} \partial_t \Phi &= (1-\lambda)mE + \lambda \langle \dot{P}, x - X \rangle \\ \partial_\psi \Phi &= \lambda \langle P_\psi, x - X \rangle \\ \partial_\lambda \Phi &= \langle P, x - X \rangle \end{aligned} \tag{5.27}$$

To this set of equations we could add

$$\partial_\varphi \Phi = 1 - \lambda + \lambda \langle P_\varphi, x - X \rangle$$

but actually we will not use this equation.

The critical point $x = X(t, \varphi, \psi)$ is uniquely determined (for small t) : namely the determinant of the 2×2 system for the last 2 Eq. is given by $\det(P, P_\psi)$, and this is non zero at $t = 0$. So when $\lambda = 1$ and $x = X(t, \varphi, \psi)$ belongs to the critical set C_Φ. Taking differential on C_Φ gives

$$\begin{aligned} d\partial_t \Phi &= -E d\lambda + \langle \dot{P}, dx - dX \rangle \\ d\partial_\psi \Phi &= \langle P_\psi, dx - dX \rangle \\ d\partial_\lambda \Phi &= \langle P, dx - dX \rangle \end{aligned} \tag{5.28}$$

which, again, we could implement by

$$d\partial_\varphi \Phi = -d\lambda + \langle P_\varphi, dx - dX \rangle$$

It is natural to expect the (inverse) density $F[\Phi, d\mu_+]|_{C_\Phi}$ as in (4.10) (up to a constant factor) to be $\det(P, d_\psi P)$ computed above. Namely we have:

Proposition 5.4: *Let $H(x, p)$ be positively homogeneous of degree 1 with respect to p on $T^*M \setminus 0$. Then $\Phi(x, t, \varphi, \psi, \lambda, \tau)$ given in (5.26) solves HJ Eq. (2.12). We have*

$$F[\Phi, d\mu_+]|_{C_\Phi} = \frac{dt \wedge d\psi \wedge d\varphi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi)}{dx \wedge dt \wedge d\psi \wedge d\varphi \wedge d\lambda} = E \det(P, d_\psi P) \tag{5.29}$$

Then the critical set C_Φ is determined as in Proposition 5.2 by $x = X(t, \varphi, \psi)$ (which can be inverted as $t = t_1(x, \psi), \varphi = \varphi_1(x, \psi)$) and $\lambda = 1$. It coincides with the set $\kappa(U)$ defined in Proposition 5.3. When $\det(P, d_\psi P) \neq 0$, this is (inverse) density on Λ_+, and Φ is a non-degenerate phase function defining Λ_+.

Proof: We assume $n = 2$ for simplicity and start to compute $F[\Phi, d\mu_+]|_{C_\Phi}$. We expand the $2n+2$-form

$$\omega = dt \wedge d\psi \wedge d\varphi \wedge d\dot{\Phi} \wedge d(\partial_\lambda \Phi) \wedge d(\partial_\psi \Phi) = \omega_1 + \omega_2$$

where

$$\begin{aligned} \omega_1 &= -mE dt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge \langle d_\psi P, dx - dX \rangle \wedge \langle P, dx - dX \rangle \\ \omega_2 &= dt \wedge d\psi \wedge d\varphi \wedge \langle \dot{P}, dx - dX \rangle \wedge \langle d_\psi P, dx - dX \rangle \wedge \langle P, dx - dX \rangle \end{aligned}$$

Using (5.28) a short calculation gives $\omega_2 = 0$, while only the term $-Edt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge \langle d_\psi P, dx\rangle \wedge \langle P, dx\rangle$ contributes in ω_1, so $\omega_1 = -mE \det(d_\psi P, P) dt \wedge d\psi \wedge d\varphi \wedge d\lambda \wedge dx$. So we get (5.29), and the Proposition follows as in Proposition 4.1. See also [DoNaSh, Lemma 10]. ♣

Assuming $\varphi = \varphi(\tau)$ as for Hamiltonian (1.15) and fixing $\tau = 0$, there is no longer dependence on φ. Thus we can proceed as in Sect.4, define ordinary and special points precisely as in Definition 2.1, and extend Proposition 4.3 to this case. The only requirement is that the density computed in Proposition 5.3 is non vanishing. In the same way, Lemmas 4.6 and 4.7 for Hamiltonian (1.15) extend to this case.

It may be more convenient to determine simulteneously the critical point (t, ψ) instead of (t, λ). Namely consider the system $\binom{\partial_t \Phi}{\partial_\psi \Phi} = 0$, i.e. $\Phi'_{(t,\psi)} = 0$. We have

$$\Phi''_{t,\psi} = \begin{pmatrix} \frac{\partial^2 \Phi}{\partial t^2} & \frac{\partial^2 \Phi}{\partial t \partial \psi} \\ \frac{\partial^2 \Phi}{\partial \psi \partial t} & \frac{\partial^2 \Phi}{\partial \psi^2} \end{pmatrix} = \begin{pmatrix} \langle \dot{P}, \dot{X}\rangle & \langle P_\psi, \dot{X}\rangle \\ \langle P_\psi, \dot{X}\rangle & \langle P_\psi, X_\psi\rangle \end{pmatrix} \tag{5.31}$$

where $\langle P_\psi, \dot{X}\rangle = \langle X_\psi, \dot{P}\rangle$. Let

$$D(z(t)) = \det \Phi''_{t,\psi} \tag{5.32}$$

(for short we omit other variables in the notation $z(t)$) and

$$\Omega(t) = \{\psi : z(t) = \big(X(t, \varphi, \psi), P(t, \varphi, \psi)\big),\ D(z(t)) \neq 0\}$$

For $t = 0$, we have $\det D(z(0)) = \varphi\langle -\partial_x H, \partial_p H\rangle - \langle \partial_p H, \omega(\psi)^\perp\rangle^2$, so for Hamiltonian (1.15) this is non vanishing when $z(0)$ is an ordinary point where $\varphi \neq 0$, and $D(z(0)) = 0$ if $z(0)$ is glancing. So when $D(z(t)) \neq 0$, implicit function theorem shows that $\Phi'_{(t,\psi)} = 0$ is equivalent to $t = t_0(x, \varphi), \psi = \psi_0(x, \varphi)$. We conjecture that $D(z(t)) = 0$ iff $z(0)$ is a glancing point.

In other words, for the vertical plane (see Proposition 4.3(ii)) we express the two variables (t, λ) as a function of the $2n - 1$ variables (x, ψ), while for the cylinder we express the n variables (t, ψ) as a function of the $n + 1$ variables (x, φ) (when $n = 2$ there is the same number of variables in the source and target spaces).

Remark 5.1: Assume $\det(\dot{X}, X_\psi) \neq 0$ (i.e. $z(t)$ is not a focal point), then $\Phi''_{t,\psi}$ is just Gram matrix of $(\dot{P}, P_\psi)$ in the basis $\dot{X}, X_\psi$, and $\det(\dot{P}, P_\psi) = 0$ iff $D(z(t)) = 0$. We can interchange of course the role of the pairs of vectors.

Again it is instructive to look at the model Hamiltonian $H = \langle p, \mu\rangle$, where $X(t, \psi) = \mu t + \varphi\omega(\psi)$, $P(t, \psi) = \omega(\psi)$, and $\Phi(t, \varphi, \psi, \lambda) = Et + (1 - \lambda)\varphi + \lambda\langle \omega(\psi), x - \mu t\rangle$ is independent of φ when $\lambda = 1$. Since $\langle \mu, \omega(\psi)\rangle = E$, we have $\dot{\Phi} = E - \lambda\langle \mu, \omega(\psi)\rangle = (1 - \lambda)E$, and $\partial_\psi \Phi = \lambda\langle \omega^\perp(\psi), x - \mu t\rangle$. So on the critical set $x = X(t, \varphi, \psi)$, $\lambda = 1$, we have $\dot{\Phi} = \partial_\psi \Phi = 0$; assuming Lagrangian intersection, the Hessian of Φ with respect to (t, ψ) has determinant $D(z(t)) = -\big(\frac{\partial^2 \Phi}{\partial t \partial \psi}\big)^2 = -\langle \mu, \omega^\perp(\psi)\rangle^2$. So if $\langle \mu, \omega^\perp(\psi)\rangle \neq 0$, $\psi = \psi_0$ such that $\langle \mu, \psi_0\rangle = E$, i.e. $z(t)$ is non glancing, then $D(z(t)) \neq 0$ is a constant. We notice also that the time component of the critical point is precisely $t_1(x, \psi)$ computed in Proposition 5.2.

5.3 Reduced parametrizations of Λ_+ in case of the "conformal metric".

As in Sect.4 we could retrieve global informations in case of Hamiltonian (1.15) when $m = 1$, due to the fact that the (inverse) density $\det(P, P_\psi)$ on Λ_+ is globally positive. But we content ourselves to find the critical point and the critical value of $(t, \psi) \mapsto \Phi(x, t, \varphi, \psi, \lambda = 1)$, whenever $D(z(t)) \neq 0$ (i.e. away from special points and $\varphi = 0$). This will provide the microlocal structure of (1.4) near those points of $\partial\Lambda_+$.

Take polar coordinates on M of the form (r, θ) such that $(r, \theta) = (\varphi, \psi)$ parametrize a point on Λ near $\partial\Lambda_+$. We make the identification $x = (r, \theta)$.

• *Ordinary points.* We show that near an ordinary point $z(0) \in \partial\Lambda_+$, $\operatorname{rank} d\pi_x(z(0)) = 2$. Namely we have:

Proposition 5.5: *Let $\Phi'_{t,\psi}(x, t, \varphi, \psi, \lambda = 1) = (\partial_t\Phi, \partial_\psi\Phi)$, and assume $D(z(0)) \neq 0$ as in (5.32) i.e. $z(0)$ is an ordinary point with $\varphi \neq 0$. Then for x sufficiently close to $\varphi\omega(\psi)$, the system $\Phi'_{t,\psi}(x, t, \psi, \lambda = 1) = 0$ is equivalent to $t = t(x), \psi = \psi(x)$, and $\varphi = \varphi(x)$ when x is sufficiently close to $\varphi\omega(\psi)$, $(\varphi, \psi) \in \Omega(z(0))$. The critical value of Φ takes the form*

$$\Psi(r, \theta; \varphi, \psi) = r\cos(\theta - \psi(x)) + \mathcal{O}(|r - \varphi(x), \theta - \psi(x)|) = r + \mathcal{O}(|r - \varphi(x), \theta - \psi(x)|) \tag{5.34}$$

In particular $\operatorname{rank} d\pi_x(z(0)) = 2$.

Proof: Since $D(z(0)) \neq 0$, implicit function theorem shows that $\Phi'_{t,\psi} = 0$ (we omitted $\lambda = 1$) has a unique solution $t = t_0(x, \varphi), \psi = \psi_0(x, \varphi)$, i.e. $t = t_0(r, \theta, \varphi), \psi = \psi_0(r, \theta, \varphi)$ Differentiating $\Phi'_{t,\psi} = 0$ along Λ_+ with respect to r, θ and φ we find

$$\begin{aligned}
&\langle \dot{P}, \dot{X}\rangle \frac{\partial t_0}{\partial r} + \langle \dot{P}, X_\psi\rangle \frac{\partial \psi_0}{\partial r} = \langle \dot{P}, \omega(\theta)\rangle \\
&\langle P_\psi, \dot{X}\rangle \frac{\partial t_0}{\partial r} + \langle P_\psi, X_\psi\rangle \frac{\partial \psi_0}{\partial r} = \langle P_\psi, \omega(\theta)\rangle \\
&\langle \dot{P}, \dot{X}\rangle \frac{1}{r}\frac{\partial t_0}{\partial \theta} + \langle \dot{P}, X_\psi\rangle \frac{1}{r}\frac{\partial \psi_0}{\partial r} = \langle \dot{P}, \omega^\perp(\theta)\rangle \\
&\langle P_\psi, \dot{X}\rangle \frac{1}{r}\frac{\partial t_0}{\partial r} + \langle P_\psi, X_\psi\rangle \frac{1}{r}\frac{\partial \psi_0}{\partial r} = \langle P_\psi, \omega^\perp(\theta)\rangle \\
&\langle \dot{P}, \dot{X}\rangle \frac{\partial t_0}{\partial \varphi} + \langle \dot{P}, X_\psi\rangle \frac{\partial \psi_0}{\partial \varphi} = -\langle \dot{P}, X_\varphi\rangle \\
&\langle P_\psi, \dot{X}\rangle \frac{\partial t_0}{\partial \varphi} + \langle P_\psi, X_\psi\rangle \frac{\partial \psi_0}{\partial \varphi} = -\langle P_\psi, X_\varphi\rangle
\end{aligned} \tag{5.35}$$

Using the relation $\langle \dot{P}, X_\psi\rangle = \langle P_\psi, \dot{X}\rangle$, the 3 sub-systems have determinant

$$D(z(t)) = \det\begin{pmatrix} \langle \dot{P}, \dot{X}\rangle & \langle P_\psi, \dot{X}\rangle \\ \langle P_\psi, \dot{X}\rangle & \langle P_\psi, X_\psi\rangle \end{pmatrix} \neq 0$$

So for $(\varphi, \psi) \in \Omega(z(s))$, (5.35) has the unique solution with the condition

$$t_0(\varphi, \psi, \varphi) = 0, \quad \psi_0(\varphi, \psi, \varphi) = \psi, \quad (\varphi, \psi) \in \Omega(0) \tag{5.36}$$

On the other hand we know from Proposition 5.2 that on C_Φ, i.e. when $x = X(t,\varphi,\psi)$, we have $\varphi = \varphi_1(x,\psi)$ and $t = t_1(x,\psi)$. This gives :

$$g(x,\psi) = \psi - \psi_0\big(x, \varphi_1(x,\psi)\big) = 0$$

Compute $\partial_\psi g(x,\psi) = 1 - \frac{\partial\psi_0}{\partial\varphi}\frac{\partial\varphi_1}{\partial\psi}$ at $x = \varphi\omega(\psi)$. Combining Proposition 5.3 and [DoNaSh,Lemma 7] (which still doesn't assume $\tilde{k} = 2$ to be the rank of $\widetilde{\Lambda}_+$), we get

$$\partial_\psi \begin{pmatrix} t_1 \\ \varphi_1 \end{pmatrix}(x,\psi) = -\begin{pmatrix} 1 & 0 \\ -H & {}^tP \end{pmatrix}\begin{pmatrix} 0 \\ X_\psi \end{pmatrix} + \mathcal{O}\big(\big|x - X\big(t_1(x,\psi),\varphi_1(x,\psi),\psi\big)\big|\big)$$

Since $\langle P, X_\psi\rangle = 0$, we get in particular

$$\frac{\partial\varphi_1}{\partial\psi}(x,\psi) = \mathcal{O}\big(\big|x - X\big(t_1(x,\psi),\varphi_1(x,\psi),\psi\big)\big|\big) = o(1)$$

Thus $\partial_\psi g(x,\psi) \neq 0$ along C_Φ and implicit function theorem gives $\psi = \psi_1(x)$. Sustituting into $t = t_1(x,\psi) = t_0(x,\varphi)$ we get also $t = t_1(x,\psi_1(x)) = t_0(x,\varphi_1(x,\psi_1(x))$. Substituting into (5.26) (where we have assumed $m = 1$) gives the critical value (for $\lambda = 1$) where all "θ-variables" have been eliminated

$$\Psi(r,\theta) = Et_1(t,\theta) + \varphi_1(r,\theta) + \langle P(t_1,\varphi_1,\psi_1), r\omega(\theta) - X(t_1,\varphi_1,\psi_1)\rangle \tag{5.37}$$

which we expand around $(r,\theta) = (\varphi,\psi)$ using (5.36). With

$$t_1(t,\theta) = o(1),\ \varphi_1(t,\theta) = \varphi + o(1),\ P(t_1,\varphi_1,\psi_1) = \omega(\psi) + o(1),\ X(t_1,\varphi_1,\psi_1) = \varphi\omega(\psi) + o(1)$$

substituting into (5.37) we find $\Psi(r,\theta) = r\cos(\theta-\psi) + o(1)$, where

$$o(1) = \mathcal{O}\big(\big|x - X\big(t_1(x,\psi),\varphi_1(x,\psi),\psi\big)\big|\big) = \mathcal{O}(|r-\varphi, \theta-\psi|)$$

This proves (5.34) (no "θ-variables"). ♣

• *Special points.* We know that a special point at $t = 0$ is also a critical point of ρ. We assume $\varphi \neq 0$, so it is not glancing (or residual).

Proposition 5.6: *Let $\Phi'_{t,\psi,\lambda}(x,t,\varphi,\psi,\lambda) = (\partial_t\Phi, \partial_\psi\Phi, \partial_\lambda\Phi)$, and assume $z(0)$ is an special point i.e. $\nabla\rho(\varphi\omega(\psi) = 0$, but $\varphi \neq 0$. Then for x sufficiently close to $\varphi\omega(\psi)$, the system $\Phi'_{t,\psi,\lambda}(x,t,\psi,\lambda) = 0$ is equivalent to $t = t_2(x,\varphi), \psi = \psi_2(x,\varphi)$, The critical value of Φ takes the form*

$$\Psi(x,t,\varphi,\psi,\lambda = 1) = mHt_2(x,\varphi) + \varphi + \langle P(t,\varphi,\psi), x - X(t,\varphi,\psi)\rangle|_{t=t_2(x,\varphi),\psi=\psi_2(x,\varphi)} \tag{5.39}$$

In particular $\operatorname{rank} d\pi_x(z(0)) = 1$.

Proof: We have as in (4.17)

$$\det\frac{\partial^2\Phi}{\partial(t,\psi,\lambda)}|_{t=0} = -(mH)^2\langle P_\psi, X_\psi\rangle|_{t=0} = -(mH)^2\varphi$$

this is non zero for non zero φ, so by implicit functions theorem $(t,\psi,\lambda)=\big(t_2(x,\varphi),\psi_2(x,\varphi),\lambda_2(x,\varphi)\big)$. Differentiating $\Phi'_{t,\psi,\lambda}(x,t,\varphi,\psi,\lambda)=0$ with respect to x,φ gives the triangular systems

$$\begin{aligned}
&\langle P_\psi,\dot X\rangle^t\big(\frac{\partial t_2}{\partial x}\big)+\langle P_\psi,X_\psi\rangle^t\big(\frac{\partial\psi_2}{\partial x}\big)=P_\psi\\
&\langle \dot P,\dot X\rangle^t\big(\frac{\partial t_2}{\partial x}\big)+\langle \dot P,X_\psi\rangle^t\big(\frac{\partial\psi_2}{\partial x}\big)+mH^t\big(\frac{\partial\lambda_2}{\partial x}\big)=\dot P\\
&mH^t\big(\frac{\partial t_2}{\partial x}\big)=P
\end{aligned}\tag{5.40}$$

and

$$\begin{aligned}
&\langle \dot P,\dot X\rangle\frac{\partial t_2}{\partial\varphi}+\langle \dot P,X_\psi\rangle\frac{\partial\psi_2}{\partial\varphi}+mH\frac{\partial\lambda_2}{\partial\varphi}=-\langle \dot P,X_\varphi\rangle\\
&\langle P_\psi,\dot X\rangle\frac{\partial t_2}{\partial\varphi}+\langle P_\psi,X_\psi\rangle\frac{\partial\psi_2}{\partial\varphi}=-\langle P_\psi,X_\varphi\rangle\\
&mH\frac{\partial t_2}{\partial\varphi}=-1
\end{aligned}\tag{5.41}$$

We first solve (5.41), using $a=b$, this gives $\frac{\partial\psi_2}{\partial\varphi}=-\frac{\langle P_\psi,X_\varphi\rangle}{c}$, and $\frac{\partial\lambda_2}{\partial\varphi}=0$, as expected since $\lambda=1$ and $\partial\Lambda_+(\tau)$ is now parametrized by φ. Then (5.40), still evaluated at $t=0$, gives $\frac{\partial t_2}{\partial x}=\frac{P}{mH}$, $\frac{\partial\psi_2}{\partial x}=\frac{P_\psi}{c}$ and again $\frac{\partial\lambda_2}{\partial x}=0$. Now we put as before $\Psi(x,\varphi)=\Phi(x,t,\varphi,\psi,\lambda=1)|_{t=t_2(x,\varphi),\psi=\psi_2(x,\varphi)}$, with Φ given in (5.39). Taking derivative with respect to φ gives (we evaluate at $t=0$)

$$\partial_\varphi\Psi(x,\varphi)=\frac{\partial\psi_2}{\partial\varphi}\langle\omega^\perp(\psi),x-X(t_2(x,\varphi),\varphi,\psi_2(x,\varphi)\rangle\tag{5.43}$$

and differentiating again (5.43) with respect to φ

$$-\partial_\varphi^2\Psi(x,\varphi)=\frac{\langle P_\psi,X_\varphi\rangle^2}{c}=0$$

So (3.5) ensures that $\operatorname{rank} d\psi_x(z(0))=1$. ♣

Note that the 3:rd Eqn's in (5.40) and (5.41) yield the condition that $P(t_2(x,\varphi),\varphi,\psi_2(x,\varphi))$ be independent of φ, and t_2,ψ_2 satisfy the integral equation

$$mHt_2(x,\varphi)+\varphi=\int^x\langle P,dx\rangle$$

P being evaluated along $t=t_2(x,\varphi),\psi=\psi_2(x,\varphi)$, which gives the eikonal ; cf. also Proposition 4.3 (iii).

Of course, these methods elaborated for computing the critical values of Φ do not extend to a glancing point $z(t)$ ($\varphi=0$), where all second derivatives of Φ vanish. We already notice that, since $P_\psi\neq 0$ at $t=0$, the first Eq. (5.40) is singular in the limit $\varphi\to 0$, so there is no hope to recover the variational system for (5.40) and (5.41), differentiating $\Phi'_{t,\psi,\lambda)}=0$ once again along Λ_+ with respect to x and φ.

5.4 Construction of Maslov canonical operator.

It goes the same way as in Sect. 4., using Propositions 5.5 and 5.6 instead of Proposition 4.3, and Lemmas 4.6 & 4.7 in the discussion according to ordinary or special points. For simplicity we work with the "conformal metric" (1.15) where we recall $m = 1$. To avoid glancing points, we will assume that $\varphi \neq 0$ on $\mathrm{WF}_h\, f_h$, i.e. $\mathrm{WF}_h\, f_h \cap T_0^* M = \emptyset$.

We look for u_h of the form

$$u_h(x) = \int_0^\infty \int e^{i\Phi(x,t,\varphi,\psi,\lambda,\tau)/h} a(x,t,\varphi,\psi,\lambda,\tau)\, d\mu(\psi,\varphi,\lambda)\, dt$$

where Φ as in (5.26), and $d\mu(\psi,\varphi,\lambda)$ are defined in some charts: in a neighbhd of an ordinary point $z(t)$, we take $d\mu(\psi,\lambda) = \mu_1(\psi,\lambda)\, d\psi\, d\lambda$, and consider $\Phi(x,t,\varphi,\psi,\lambda,\tau)$ as $\Phi(x,t,\psi,\lambda)$ with fixed φ, as in Proposition 5.5. In a neighbhd of a special or residual point $z(t)$ instead, we take $d\mu(\psi,\varphi) = \mu_2(\psi,\varphi)\, d\psi\, d\varphi$, and think of $\Phi(x,t,\varphi,\psi,\lambda,\tau)$ as $\Phi(x,t,\varphi,\psi)$ with $\lambda = 1$, as in Proposition 5.6. In both cases, we omit τ from the notations, since it is related to λ and φ as before. These phases an densities can be chosen coherently, and define a "new" Maslov bundle, subordinated to the additional partition between ordinary/ special or residual points.

In an ordinary chart, we proceed as in Sect.4.

5.5 Radially symmetric "conformal metric"

We choose $m = 1$ for simplicity, and $\rho(x)$ radially symmetric, i.e. $\rho = \rho(|x|)$ with ρ a smooth function on $\mathbf{R}_+$.

We have $\rho(|x|) = \rho(\varphi)$ (choosing the branch $\varphi > 0$), $\varphi = \varphi_0 > 0$ on $L = \Lambda \cap \{H = E\}$, and $p\, dx = d\varphi$ on Λ. The hypothesis $\varphi_0 > 0$ may be relaxed. We assume also $\rho'(\varphi_0) \neq 0$.

The phase function is now given by (5.16) with $\varphi = \varphi_0$. The variational system is obtained directly in the $x - X$ variables as

$$\begin{aligned} d\partial_t \Phi &= -mH\, d\lambda + (dx - dX)^* \partial_t P \\ d\partial_\psi \Phi &= (dx - dX)^* P \partial_\psi P \\ d\partial_\lambda \Phi &= (dx - dX)^* P \end{aligned}$$

We check the linear independance. Let

$$d(\partial_t \Phi)\alpha + \langle d(\partial_\psi)\Phi, \beta\rangle + d(\partial_\lambda \Phi)\gamma = 0$$

Identifying the term in $d\lambda$ we find $\alpha = 0$, then $(dx - dX)^*(\langle \partial_\psi P, \beta\rangle + \gamma P) = 0$, which readily implies $\beta = \gamma = 0$, for small t since $P|_{t=0} = \omega(\psi)$ and $\partial_\psi P|_{t=0} = \omega(\psi)^\perp$.

In a second step, we seek for a reduced generating function for Λ_+ by eliminating t when $t > 0$ by stationary phase. This follows easily from implicit functions theorem and the value at $t = 0$

$$\partial_t^2 \Phi|_{x=X(\psi)} = -\langle \partial_t P, \partial_\psi X\rangle = \mathrm{Const.}\langle \omega(\psi), \omega(\psi)\rangle \neq 0$$

(without any further assumption).

It follows that u_h has only "wave-part" component, until the Hamilton flow meets $\varphi = 0$ where the analysis has slightly to be changed to take into account the singularity of $\rho(|x|)$ at this point.

We complete this Example by the integration of motion (at least in 2-D), since the system is integrable (with H as in (1.15)). Introduce polar coordinates (r,θ) and (s,ψ) on $T^*\mathbf{R}^2$ (same parameter ψ as in the definition of Λ), so that

$$x = r\,\omega(\theta),\ p = s\,\omega(\psi) \tag{5.50}$$

and

$$p\,dx = s\cos(\psi-\theta)\,dr + rs\sin(\psi-\theta)\,d\theta \tag{5.51}$$

On $H = \frac{s}{\rho(r)}$, Hamilton equations are given by

$$\dot r = \frac{1}{\rho(r)},\ \dot s = \frac{\rho'(r)}{\rho(r)^2}s$$

that is

$$\frac{\dot s}{s} = \frac{\rho'(r)}{\rho(r)^2} = \frac{\rho'(r)}{\rho(r)}\dot r$$

Integrating these equations we find $s(t) = s_0\rho(r(t))$ so

$$P(t,\psi) = s_0\frac{\rho(r(t))}{\rho(r_0)}\omega(\psi) \tag{4.52}$$

If the trajectory leaves Λ at $r(0) = r_0 = \varphi_0$, then $s_0 = 1$, $\frac{1}{\rho(r_0)} = E$ (which fixes r_0). The first Eq. (4.51) has separated variables $\rho(r)\,dr = dt$, so if $N(r) = \frac{1}{E} + \int_{r_0}^r \rho(r')\,dr'$, we get $N(r) = T$, which is solved implicitely by $r = R(t)$. Finally, $\theta = \text{Const.}$, $\psi = \text{Const.}$, so the solution of Hamilton equations with data on $L = \partial\Lambda_+$ are simply

$$\begin{aligned} &\theta = \psi \\ &x = X(t,r_0,\psi) = R(t)\omega(\psi) \\ &p = P(t,r_0,\psi) = \frac{\rho(R(t))}{\rho(r_0)}\omega(\psi) \end{aligned} \tag{5.53}$$

These are the parametric equations of $L_t = g^t(L)$. We are left to investigate the "collision" on $r = 0$, where the solution becomes singular, and the "boundary part" comes into play.

5.6 Example in the constant coefficient case

To close this section we consider as in Remark 3.1, $H = -h^2\Delta$, $n = 2$ and

$$f_h(x) = J_0\big(\frac{|x|}{h}\big) = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{i\langle x,\omega(\psi)\rangle}\,d\psi = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{i|x|\sin\psi}\,d\psi$$

The outgoing solution is given by the oscillating integral

$$u_h(x) = (4\pi^2 h)^{-1/2}\int e^{i\Phi(x,y,\psi)/h}\big(\xi^2 - E - i0\big)^{-1}\,d\psi\,dy\,d\xi$$

which we compute (formally) by stationary phase in (y,ξ). The critical point is given by $y=x$, $\xi=\frac{x}{|x|}\sin\psi$, and

$$u_h(x)=h^{1/2}\int_{-\pi}^{\pi}e^{i|x|\sin\psi/h}\frac{1}{\sin^2\psi-E-i0}\,d\psi+\mathcal{O}(h^{3/2}) \tag{5.61}$$

and the leading term can be simply evaluated by contour integrals. The phase of course is the same as in Bessel function. We can also consider more general f_h and stick in an amplitude of the form (see [DoMaNaTu])

$$A(x,\psi)=\frac{1}{2}\big(a(|x|,\psi)+a(-|x|,\psi)\big)+\frac{\langle x,\omega(\psi)\rangle}{2|x|}\big(a(|x|,\psi)-a(-|x|,\psi)\big)$$

so that

$$f_h(x)=\big(\frac{i}{2\pi h}\big)^{1/2}\int_{-\pi}^{\pi}e^{i\langle x,\omega(\psi)\rangle}A(x,\psi)\,d\psi$$

As in (5.61) we get

$$u_h(x)=h^{1/2}\int_{-\pi}^{\pi}e^{i|x|\sin\psi/h}\frac{A(x,\psi)}{\sin^2\psi-E-i0}\,d\psi+\mathcal{O}(h) \tag{5.62}$$

When A is independent of ψ, this can lead to significant simplifications.

Appendix *Lagrange immersions and global half-densities*

Recall first some well-known properties of Lagrangian immersions (see e.g. [DoZh], [DoNaSh]) :

Proposition A.1: *Let $\iota:\Lambda\to T^*M$, be a Lagrangian immersion, parametrized on a canonical chart U by $\varphi\mapsto z=\iota(\varphi)=(X(\varphi),P(\varphi))$. Introduce the Jacobian matrices $B(z)=\frac{\partial P}{\partial\varphi}$, $C(z)=\frac{\partial X}{\partial\varphi}$. Then:*

(1) the matrix $(B(z),C(z))$ is of rank n.

(2) the matrix ${}^tC(z)B(z)$ is symmetric.

(3) $C(z)\pm iB(z)$ is non degenerate.

The symmetry of ${}^tC(z)B(z)$ expresses for instance in the situation of Sect.4 as the symmetry of Gram matrix

$$\begin{pmatrix}\langle\dot X,\dot P\rangle & \langle\dot X,P_\psi\rangle\\ \langle X_\psi,\dot P\rangle & \langle X_\psi,P_\psi\rangle\end{pmatrix} \tag{A.1}$$

We consider the rank of projections $\pi_x:\Lambda_+\to M$. It is equal to the rank of $\pi_{x,t}:\widetilde\Lambda_+\to M\times\mathbf{R}_t$.

In general we call *focal point* a point $z\in\Lambda$ where $\pi_*:T\Lambda\to TM$ is singular, and *caustics* the projection $\mathcal{C}$ of the set of focal points onto M. Assume $n=2$, and let z be a focal point, so $C(z)$ cannot be of rank 2, and by property (1) above either $B(z)$ is of rank 2 (and $C(z)$ has rank at most 1, since the projection $\pi:\Lambda_+\mapsto M$ is not a diffeomorphism at z) or both $C(z)$ and $B(z)$ are of rank 1.

Assume now $H(x,p)=\frac{|p|^m}{\rho(x)}$, $n=2$. Let $\Lambda=\Lambda_+$ be an integral manifold of v_H in the energy shell $H(x,p)=E$, and $U\subset\Lambda_+$ be a canonical chart parametrized by $\varphi=(t,\psi)$, i.e. $z=(X(t,\psi),P(t,\psi))$

verifies $\dot X = \partial_x H(X,P), \dot P = -\partial_p H(X,P)$, such that $U \to T^*M$ is an immersion (not necessarily an embedding). Recall $\langle P, \dot X\rangle = mH$, $\langle P, X_\psi\rangle = 0$. Actually $(X(t,\psi), P(t,\psi)$ may depend on additional parameters, as in Sect.5, but here only t,ψ matter. Consider the quantity $\det(P, P_\psi)$.

Lemma A.2: *Let $H(x,p) = \frac{|p|^m}{\rho(x)}$, $n=2$. Assume that at some point $z = \big(X(t,\psi), P(t,\psi)\big)$, we have* $\det(P,P_\psi) = 0$.

1) If $\nabla\rho(X(t,\psi)) \neq 0$, then either $|P(t,\psi)| = 1$, or $P_\psi(t,\psi) = 0$. In the latter case $C(z) = (\dot X, X_\psi)$ has rank 2, i.e. π_x is regular at z.

2) If ρ has a critical point at some $x_0 = X(t,\psi)$, then π_x is regular at $z = (x_0, P(t,\psi))$

Proof: We apply Proposition A.1 to the Lagrangian immersion $\iota : U \to T^*M$, $\varphi = (t,\psi)$. Assume $\det(P,P_\psi) = 0$ at some point $z = (X(t,\psi), P(t,\psi))$. Then either $P_\psi = 0$, or by Hamilton equations (3.2), there is $\alpha \in \mathbf{R}$ such that $\dot X = \alpha P_\psi$. Let first $\nabla\rho \neq 0$.

(i) Let $P_\psi = 0$. In this case $X_\psi \neq 0$, for otherwise this would contradict property (3) of Proposition A.1. The symmetry of ${}^tC(z)B(z)$ shows that $\langle \dot P, X_\psi\rangle = 0$, or $\langle \nabla\rho(X), X_\psi\rangle = 0$. Since $\langle P, X_\psi\rangle = 0$, we find that $\dot X, P, \dot P, \nabla\rho(X)$ are parallel, and all orthogonal to X_ψ. In particular, $C(z) = (\dot X, X_\psi)$ is of rank 2.

(ii) Let $P_\psi = 0$. By property (2) of Proposition A.1, the matrix

$$ {}^tC(z)B(z) = \begin{pmatrix} * & \dot X_1\partial_\psi P_1 + \dot X_2 \partial_\psi P_2 \\ \dot P_1 \partial_\psi X_1 + \dot P_2 \partial_\psi X_2 & * \end{pmatrix} $$

has to be symmetric. If $\dot X = \alpha P_\psi$, this implies $\alpha = \langle \dot P, \partial_\psi X\rangle |P_\psi|^{-2}$, and hence differentiating the dispersion relation (2.2)

$$ \dot X = \langle \dot P, \partial_\psi X\rangle |P_\psi|^{-2} P_\psi = \frac{|P|^m}{\rho(X)^2}\langle \nabla\rho(X), X_\psi\rangle |P_\psi|^{-2} P_\psi \tag{A.5} $$

• Assume $B(z)$ is of rank 2, and $C(z)$ of rank at most 1. We know that $\langle P(t,\psi), \dot X(t,\psi)\rangle = m(E-\tau) \neq 0$, $\dot X \neq 0$ parallel to $P \neq 0$, and $\langle P(t,\psi), \partial_\psi X(t,\psi)\rangle = 0$.

Assume $X_\psi \neq 0$. Since $C(z)$ is of rank at most 1, X_ψ parallel to $\dot X$, which is itself parallel to P. So P is parallel to X_ψ, which contradicts $\langle P, X_\psi\rangle = 0$. Hence $X_\psi = 0$. By first equality (A.6) we have $\langle P, P_\psi\rangle = 0$. If $\det(P,P_\psi) = 0$, then we would have P both orthogonal to X_ψ, and parallel to $P_\psi \neq 0$. But $B(z)$ is of rank 2, which is a contradiction.

• So by property (3) of Proposition A.1 we must have $B(z)$ and $C(z)$ of rank 1. So either $\dot X = \lambda X_\psi$ and $\dot P = \mu P_\psi$ for some $\lambda, \mu \in \mathbf{R}$, or $X_\psi = 0, \dot P = \mu P_\psi$, or $P_\psi = 0, \dot X = \lambda X_\psi$, or $X_\psi = P_\psi = 0$.

Examine the first case: Identifying the off-diagonal terms of ${}^tB(z)C(z)$, which is symmetric by property (2) of Proposition A.1, we find that either $\lambda = \mu$ or $\frac{\partial X}{\partial\psi} \perp \frac{\partial P}{\partial\psi}$. But $\lambda \neq \mu$ since otherwise the complex matrices $C(z) \pm iB(z)$ would be degenerate, which violates property (3) of Proposition 3.2. So $\frac{\partial X}{\partial\psi}$ and $\frac{\partial X}{\partial t}$ are colinear, and orthogonal to both $\frac{\partial P}{\partial\psi}$ and $\frac{\partial P}{\partial t}$. Assume first $\lambda\mu \neq 0$. Using again (A.5), we find that $P \perp P_\psi$, since $P_\psi \neq 0$, we find that $\det(P,P_\psi) \neq 0$. Assume then $\lambda\mu = 0$. If $\lambda = 0, \mu \neq 0$, we would have $\dot X = 0$, which is impossible. Let now $\lambda \neq 0$, then $B + iC = (P_\psi + iX_\psi, i\lambda X_\psi)$ has rank 2, so has (P_ψ, X_ψ). So by (A.6) P, $\frac{\partial X}{\partial\psi}$ and $\frac{\partial X}{\partial t}$ are colinear, and orthogonal to both $\frac{\partial P}{\partial\psi}$ and $\nabla\rho$. As before, this implies $\det(P,P_\psi) \neq 0$.

Examine the second case $X_\psi = 0, \dot{P} = \mu P_\psi$. Writing that ${}^tB(z)C(z)$ is symmetric, we find $\dot{X} \perp P_\psi$. Since $B(z) + iC(z)$ has rank 2, $P_\psi \neq 0$. Then P and $\dot{X}$ are parallel, and both orthogonal to $\nabla\rho$, P_ψ and $\dot{P}$. As before, we find $\det(P, P_\psi) \neq 0$.

The last two cases are similar. So $\det(P, P_\psi) \neq 0$ at any focal point where $\nabla\rho \neq 0$. .

(iii) Let at last, $\nabla\rho(x_0) = 0$. The first situation above cannot hold, since this would imply $\dot{X} = 0$ by (A.6), hence $P = 0$. Hence $\dot{P} = P_\psi = 0$ which implies again $C(z)$ of rank 2 by Proposition A.1. ♣

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