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To cite this version:

Soufian Abja, Slawomir Dinew, Guillaume Olive. Uniform estimates for concave homogeneous complex degenerate elliptic equations comparable to the Monge-Ampère equation. 2020. hal-02973743v2

HAL Id: hal-02973743

https://hal.science/hal-02973743v2

Preprint submitted on 4 Nov 2020

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Uniform estimates for concave homogeneous complex degenerate elliptic equations comparable to the Monge-Ampère equation

Soufian Abja* Sławomir Dinew* Guillaume Olive*

November 3, 2020

Abstract

We prove sharp uniform estimates for strong supersolutions of a large class of fully nonlinear degenerate elliptic complex equations. Our findings rely on ideas of Kuo and Trudinger who dealt with degenerate linear equations in the real setting. We also exploit the pluripotential theory for the complex Monge-Ampère operator as well as suitably tailored theory of L^p-viscosity subsolutions.

Keywords. nonlinear degenerate elliptic equations, viscosity solutions, a priori estimates

AMS Subject Classifications. 35J70, 35D40, 32U15

1 Introduction

Various *maximum principles* play pivotal role in the study of elliptic second order equations. In the case of a uniformly elliptic equation[1]

$$F(D^2u) = f,$$

the basic version of the maximum principle says that $f > 0$ implies that a (suitably regular) solution u does not achieve strict local maximum. More quantitative versions are also widely studied in the literature and we refer to [PS07] for the details. One of the cornerstones in this field is the Alexandrov-Bakelman-Pucci estimate which yields a uniform bound on $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ ($\Omega \subset \mathbb{R}^n$ is a bounded domain) solving

$$F(D^2u) \leq f,$$

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[1]We shall use the sign convention so that $F(D^2u) = \Delta u$ in the case of the Laplace operator.

with F being a uniformly elliptic second order operator and $f \in C^0(\overline{\Omega})$. If $u \geq 0$ on $\partial\Omega$ it reads

$$\sup_{\Omega}(-u) \leq C(n, \Omega, F) \|f_+\|_{L^n(\Omega)},$$

where $f_+ = \max(f, 0)$ denotes the positive part of f. In this article, we shall denote by $C(n), C(n, \Omega)$, etc. a positive number that may change from line to line but that depends only on the quantities indicated between the brackets.

Various generalizations to *strong* solutions[2] in $W^{2,n}(\Omega)$ (see [CW98]) or to viscosity solutions (see e.g. [CC95, Theorem 3.2]) are plentiful in the literature.

When the equation *fails* to be uniformly elliptic maximum principle still holds under reasonable minimal conditions, see [RS64] and references therein. When it comes to quantitative estimates for *degenerate elliptic equations* the available results are considerably more restrictive. In [KT07], the authors established the following estimate generalizing the Alexandrov-Bakelman-Pucci estimate:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let*

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) u_{ij},$$

be a second order linear operator with the coefficient matrix $A = (a^{ij})_{1 \leq i,j \leq n}$ being symmetric and positive definite. We assume that $\rho_k^*(A) > 0$ for some $k \in \{1, \ldots, n\}$, where

$$\rho_k^*(A) = \inf\left\{ \frac{1}{n}\mathrm{Trace}\,(AM) \;\middle|\; \sigma_k(\lambda(M)) \geq \binom{n}{k}, \quad \sigma_\ell(\lambda(M)) > 0, \quad \forall \ell \in \{1, \cdots, k\} \right\},$$

where σ_ℓ is the ℓ-th elementary symmetric polynomial and $\lambda(M) \in \mathbb{R}^n$ is the vector of eigenvalues of M. Let $q \geq 1$ be such that

$$\begin{cases} q = k & \text{if } k > n/2, \\ q > n/2 & \text{if } k \leq n/2, \end{cases}$$

and f be such that $f/\rho_k^*(A) \in L^q(\Omega)$. Then, for any function $u \in W^{2,q}_{\mathrm{loc}}(\Omega) \cap C^0(\overline{\Omega})$ that satisfies

$$\begin{cases} Lu \leq f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \end{cases}$$

we have

$$\sup_{\Omega}(-u) \leq C(n, \Omega, q) \left\| \frac{f}{\rho_k^*(A)} \right\|_{L^q(\Omega)}.$$

[2]We recall that a strong solution is a function that belongs to $W^{2,r}_{\mathrm{loc}}(\Omega)$ for some $r \geq 1$ and that satisfies the corresponding equation almost everywhere.

In this note we shall investigate the complex analogues of Theorem 1.1 and their generalizations to nonlinear complex equations. In such a setting it is well-known that direct application of real tools, such as Theorem 1.1, leads to non-optimal bounds in terms of the exponent q (see [Wan12, DD20]). Instead, building on a fundamental theorem of Kołodziej [Koł98] we are able to establish a fairly sharp Alexandrov-Bakelman-Pucci type estimates.

1.1 Assumptions on the class of nonlinear equations

Let us now detail the class of nonlinear complex elliptic equations that is considered all along this article.

From now on, $\Omega \subset \mathbb{C}^n$ is a bounded domain. The Euclidean norm of $z \in \mathbb{C}^n$ will be denoted by $\|z\|$. The open ball of center $z \in \mathbb{C}^n$ and radius $R > 0$ will be denoted by $B_R(z)$.

Let $\mathbb{H}^n$ denote the set of all Hermitian $n \times n$ matrices and let us introduce the classical cone

$$\mathcal{C}_n = \{A \in \mathbb{H}^n \quad | \quad A > 0\}.$$

In what follows we shall be interested in families $(\Gamma(z))_{z\in\Omega} \subset \mathbb{H}^n$ of open convex cones subject to the following condition:

$$\mathcal{C}_n \subset \Gamma(z), \quad \forall z \in \Omega. \tag{1}$$

Examples will shortly be presented in Section 1.3 below.

For the rest of this article, $(\Gamma(z))_{z\in\Omega} \subset \mathbb{H}^n$ is now a fixed family of open convex cones satisfying (1). Let us then introduce the set

$$\Sigma = \{(z, A) \quad | \quad z \in \Omega, \quad A \in \Gamma(z)\}.$$

This set is clearly nonempty as it contains (z, Id) for any $z \in \Omega$ by (1).

All along this work, we shall consider operators

$$G : \Sigma \longrightarrow \mathbb{R},$$

(and $F = G^k$, $k > 0$, for nonnegative G) subject to the following conditions:

(a) **Regularity:** Σ is measurable subset of $\Omega \times \mathbb{H}^n$ and G is a measurable function on Σ. Furthermore, for a.e. $z \in \Omega$, we have $A \mapsto G(z, A) \in C^1(\Gamma(z))$.

(b) **Homogeneity:** for a.e. $z \in \Omega$, the function $A \in \Gamma(z) \longmapsto G(z, A)$ is positively homogeneous of degree 1 (that is $G(z, \alpha A) = \alpha G(z, A)$ for every $\alpha > 0$ and $A \in \Gamma(z)$).

(c) **Concavity:** for a.e. $z \in \Omega$, the function $A \in \Gamma(z) \longmapsto G(z, A)$ is concave.

(d) **Comparison:** for a.e. $z \in \Omega$, we have

$$G(z, P) \geq (\det(P))^{\frac{1}{n}}, \quad \forall P \in \mathcal{C}_n. \tag{2}$$

Once again, some examples will be presented in Section 1.3 below.

The key assumption is (d), which will allow a comparison with the Monge-Ampère equation. Note that we do not require any regularity with respect to z.

Remark 1.2. The assumptions (a), (b), (c) and (d) are stable by finite convex combination. More precisely, if $G_1, \dots, G_\ell$ are operators satisfying these assumptions, then so is $G(z, A) = \sum_{i=1}^{\ell} \alpha_i(z) G_i(z, A)$ for any measurable functions $\alpha_i : \Omega \to \mathbb{R}$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{\ell} \alpha_i = 1$.

In what follows, we will use the standard notation

$$G^{i\bar{j}}(z, A) = \frac{\partial G}{\partial a_{i\bar{j}}}(z, A), \quad 1 \leq i, j \leq n.$$

Remark 1.3. Inspired from an argument of [CNS84, p. 269], we see that:

- Thanks to the homogeneity assumption (b), the concavity assumption (c) is equivalent to the following property, which will play the role of a substitute to [KT07, Proposition 2.1]:

  (c') For every $A \in \Gamma(z)$ and $B = (B_{i\bar{j}})_{1 \leq i,j \leq n} \in \Gamma(z)$, we have

  $$\sum_{i,j=1}^{n} G^{i\bar{j}}(z, A) B_{i\bar{j}} \geq G(z, B). \tag{3}$$

  Indeed, introducing $DG(A)B = \sum_{i,j=1}^{n} G^{i\bar{j}}(z, A) B_{i\bar{j}}$, the concavity is equivalent to the inequality $DG(A)B \geq G(z, B) - G(z, A) + DG(A)A$ and we have the identity $DG(A)A = G(z, A)$ (obtained by differentiating $G(z, \alpha A) = \alpha G(z, A)$ with respect to α and taking $\alpha = 1$).

- Thanks to the assumptions (b) and (c), G satisfies (d) if, and only if,

  (d') For every $A \in \Gamma(z)$ and $P \in \mathcal{C}_n$, we have

  $$G(z, A + P) \geq G(z, A) + (\det(P))^{\frac{1}{n}}.$$

  Indeed, the concavity inequality $G(z, A + P) - G(z, A) \geq DG(A + P)P$, (c') and (d) imply (d'); the converse is proved taking $A = P$ in (d') and using the homogeneity.

  In particular, note that our assumptions guarantee that G is elliptic.

1.2 Main result and comments

From now on, $D^2u = \left(u_{i\bar{j}}\right)_{1\le i,j\le n}$ denotes the complex Hessian of u, where we use the standard notations u_j and $u_{\bar{j}}$ to denote, respectively, $\frac{\partial u}{\partial z_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}\right)$ and $\frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j}\right)$.

The main result of the present paper is the following:

Theorem 1.4. *Let $(\Gamma(z))_{z\in\Omega} \subset \mathbb{H}^n$ be a family of open convex cones satisfying* (1) *and let G be an operator satisfying (a), (b), (c) and (d). Let*

$$r > n, \quad p > n. \tag{4}$$

Let $g \in L^p(\Omega)$. Then, for any function $u \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C^0(\overline{\Omega})$ with $(D^2u)(z) \in \Gamma(z)$ for a.e. $z \in \Omega$ and that satisfies

$$\begin{cases} G(z, D^2u) \le g & \text{in } \Omega, \\ u \ge 0 & \text{on } \partial\Omega, \end{cases}$$

we have

$$\sup_{\Omega}(-u) \le C(n, \operatorname{diam}\Omega, r, p)\, \|g_+\|_{L^p(\Omega)}, \tag{5}$$

where $g_+ = \max(g, 0)$ denotes the positive part of g.

Note that this result can be in particular applied to linear equations. Then (for suitable G) it can be seen as a complex counterpart of Theorem 1.1.

For nonlinear equations with operators which are positively homogeneous of degree different from 1, we have the following immediate consequence:

Corollary 1.5. *Let $(\Gamma(z))_{z\in\Omega} \subset \mathbb{H}^n$ be a family of open convex cones satisfying* (1)*. Let $F : \Sigma \to \mathbb{R}$ be a nonnegative function such that, for some $k, \delta > 0$, $G = (F^{1/k})/\delta$ satisfies (a), (b), (c) and (d). Let*

$$r > n, \quad p \ge 1, \quad p > \frac{n}{k}. \tag{6}$$

Let $f \in L^p(\Omega)$ with $f \ge 0$ in Ω. Then, for any function $u \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C^0(\overline{\Omega})$ with $(D^2u)(z) \in \Gamma(z)$ for a.e. $z \in \Omega$ and that satisfies

$$\begin{cases} F(z, D^2u) \le f & \text{in } \Omega, \\ u \ge 0 & \text{on } \partial\Omega, \end{cases}$$

we have

$$\sup_{\Omega}(-u) \le C(n, \operatorname{diam}\Omega, r, p, k, \delta)\, \|f\|^{\frac{1}{k}}_{L^p(\Omega)}.$$

This establishes Kołodziej type uniform bounds ([Koł98]) for a large class of nonlinear equations. Note however that Corollary 1.5 is a uniform estimate for strong solutions, whereas in the case of the Monge-Ampère equation, the result from [Koł98] is valid for more general solutions. We also recall that for the Monge-Ampère equation the condition (6) for p, which becomes $p > 1$, is optimal.

Remark 1.6. All our arguments apply verbatim also to L^p-viscosity solutions once a comparison principle is established[3]. It is worth emphasizing that such a comparison principle is lacking even for general uniformly elliptic equations, see e.g. [CCKS96]. We refer to [JS05] for the up-to-date partial results on the comparison principle for uniformly elliptic equations and to [ABT07] for analogous discussion in the special case of the real Monge-Ampère equation.

Remark 1.7. We wish to point out that the methods from [KT07] are also applicable in the complex setting but yield estimates dependent on $\|f\|_{L^{2p}(\Omega)}$. The improvement in the exponent should be compared with the result in [Esc93] who improved thel regularity assumptions in the setting of uniformly elliptic equations.

Remark 1.8. In case we only know that $(D^2u)(z) \in \overline{\Gamma(z)}$ for a.e. $z \in \Omega$, Theorem 1.4 and its proof remain unchanged provided that there exists a set $N \subset \mathbb{R}$ such that, for a.e. $z \in \Omega$, $G(z, \cdot)$ is C^1 in a neighborhood of $\left\{A \in \overline{\Gamma(z)}, \quad G(z, A) \notin N\right\}$ and $G(z, (D^2u)(z)) \notin N$.

Our proof is based on L^p-viscosity techniques suitably coupled with basic results from pluripotential theory. Roughly speaking we produce a pluripotential subsolution to our equation and then show that it is also an L^p-viscosity barrier. This coupled with a maximum principle for L^p-viscosity subsolutions yields the claim. Our findings in fact show that uniform estimates for L^p-viscosity solutions to a large class of Hessian type equations can be deduced through pluripotential theoretic tools even if a pluripotential theory cannot be developed for a particular equation (see [Din20b] for a discussion of such a phenomenon).

The rest of this paper is organized as follows. In Section 1.3 we give some examples of Hessian equations that are covered by Theorem 1.4 or Corollary 1.5. In Section 2 we introduce the notion of $W^{2,r}/L^p$-viscosity subsolutions for general elliptic equations. In particular, Section 2.2 is devoted to examples explaining the differences from standard L^p-viscosity theory in the absence of uniform ellipticity. In Section 3 we show that pluripotential subsolutions to the complex Monge-Ampère equation with L^p right-hand side are also $W^{2,r}/L^p$-viscosity subsolutions when $r > n$. In Section 4 we prove a basic $W^{2,r}/L^p$-viscosity maximum principle. The final Section 5 is devoted to the proof of our main result Theorem 1.4.

1.3 Some examples for Hessian equations

In this work, our operators F are not necessarily *Hessian* but all our examples below will be. For this reason, let us first recall the notion of *Hessian type operators* on Ω.

In this section, β will be a fixed smooth positive Hermitian $(1,1)$-form. We recall that, given a smooth real $(1,1)$-form α on Ω (not necessarily positive) the *eigenvalues* of α with respect to β at a point z are the solutions λ to the equation

$$(\alpha(z) - \lambda\beta(z))^n = 0.$$

[3]The comparison principle states that a subsolution is majorized by a supersolution once this holds on the boundary.

These eigenvalues are real and they will be denoted by $\lambda_1(z,\alpha) \leq \ldots \leq \lambda_n(z,\alpha)$ and arranged in

$$\lambda(z,\alpha) = (\lambda_1(z,\alpha), \cdots, \lambda_n(z,\alpha)).$$

We recall that the eigenvalues are continuous functions of z which are furthermore smooth off the branching locus. Equivalently, writing $\beta = i\sum_{j,k=1}^n B_{j\bar{k}} dz_j \wedge d\bar{z}_k$ and $\alpha = i\sum_{j,k=1}^n A_{j\bar{k}} dz_j \wedge d\bar{z}_k$, where $A = (A_{j\bar{k}})_{1\leq j,k\leq n}$ and $B = (B_{j\bar{k}})_{1\leq j,k\leq n}$ are Hermitian matrices (d is the exterior derivative), the eigenvalues are solutions to $\det(A(z) - \lambda B(z)) = 0$.

Remark 1.9. The choice $\beta = i\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ (equivalently $B(z) = \mathrm{Id}$) yields the standard eigenvalues of a real $(1,1)$-form. In applications however, especially when studying locally equations on complex manifolds, it is more natural to work with a z-dependent background form (see e.g. [DK14, p. 230]).

If $M \in \mathbb{H}^n$, we simply denote by $\lambda_j(z,M)$ the eigenvalues of the corresponding form $\alpha = i\sum_{j,k=1}^n M_{j\bar{k}} dz_j \wedge d\bar{z}_k$ with respect to β at the point z. When B does not depend on z, we shall simply write $\lambda_j(M)$ and $\lambda(M)$.

Let us now give a precise statement of what we call an Hessian operator:

Definition 1.10. Assume that the family $(\Gamma(z))_{z\in\Omega} \subset \mathbb{H}^n$ of open convex cones satisfies (1) and

$$\forall z \in \Omega,\ \forall A, \tilde{A} \in \mathbb{H}^n, \quad \Big(A \in \Gamma(z) \text{ and } \lambda(z,\tilde{A}) = \lambda(z,A)\Big) \Longrightarrow \tilde{A} \in \Gamma(z). \tag{7}$$

Then, a function $F : \Sigma \longrightarrow \mathbb{R}$ is said to be a Hessian operator if there is exists a function $\hat{F} : \hat{\Sigma} \longrightarrow \mathbb{R}$, where $\hat{\Sigma} = \{(z, \lambda(z,A)) \quad | \quad (z,A) \in \Sigma\}$, so that, for every $(z,A) \in \Sigma$, we have

$$F(z,A) = \hat{F}(z, \lambda(z,A)).$$

When $\hat{F}$ does not depend on z, we shall simply write $\hat{F}(\lambda(z,A))$.

In the case $B(z) = \mathrm{Id}$, the property (7) can be equivalently rephrased as the $\mathcal{O}(n)$-invariance ($\mathcal{O}(n)$ denotes the orthogonal group): we have $A \in \Gamma(z)$ if, and only if,

$$O^* A O \in \Gamma(z), \quad \forall O \in \mathcal{O}(n), \tag{8}$$

where O^* denotes the Hermitian transposed matrix of O. This property (8) appears for instance in [HL18, p. 778], where it is called "ST-Invariance" (it stands for Spherical Transitivity).

Before finally presenting some examples, we recall the definition of the basic cones associated to Hessian equations. For $1 \leq m \leq n$, the cones $\Gamma_m \subset \mathbb{R}^n$ are defined as follows

$$\Gamma_m = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \quad | \quad \sigma_q(\lambda_1, \ldots, \lambda_n) > 0, \quad \forall q \in \{1, \cdots, m\}\},$$

where

$$\sigma_q(\lambda_1, \ldots, \lambda_n) = \sum_{1\leq i_1 < \cdots < i_q \leq n} \lambda_{i_1} \cdots \lambda_{i_q}.$$

The cones $\lambda^{-1}(\Gamma_k)$ clearly satisfy the $\mathcal{O}(n)$-invariance (8) and it can be shown that they are convex (see e.g. [Bło05, Section 2]).

Example 1.11. *Examples of equations covered by our framework of Section 1.1 and that are Hessian equations include:*

1) *The complex Monge-Ampère equation:*

$$\Gamma(z) = \lambda^{-1}(\Gamma_n), \quad \hat{F}(\lambda_1, \dots, \lambda_n) = \prod_{i=1}^{n} \lambda_i.$$

Here, the degree of homogeneity of F is $k = n$, the concavity is well known and the comparison (d) is trivial.

2) *The complex m-Hessian operator: for $1 \leq m \leq n$,*

$$\Gamma(z) = \lambda^{-1}(\Gamma_m), \quad \hat{F} = \sigma_m.$$

Here, the degree is $k = m$, the concavity follows from Gårding's inequality, and the comparison (d) follows from Maclaurin's inequality (see e.g. [Bło05, Section 2]).

3) *The complex m-Monge-Ampère operator (c.f. [HL18, Din20b, Din20a]): for $1 \leq m \leq n$,*

$$\Gamma(z) = \lambda^{-1}\left(\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \quad | \quad \lambda_{i_1} + \dots + \lambda_{i_m} > 0, \quad \forall 1 \leq i_1 < \dots < i_m \leq n\}\right),$$
$$\hat{F}(\lambda_1, \dots, \lambda_n) = \prod_{1 \leq i_1 < \dots < i_m \leq n} (\lambda_{i_1} + \dots + \lambda_{i_m}).$$

Here, the degree is $k = \binom{n}{m}$, the operator is concave and the comparison (d) holds (see e.g. [Din20a, Section 1.6] and [AO20]).

4) *$B(z) = \mathrm{Id}$ and, for $a \in [0, 1]$,*

$$\Gamma(z) = \lambda^{-1}(\Gamma_{2-a}), \quad \Gamma_{2-a} = \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \quad \middle| \quad \lambda_1 + a\,\lambda_2 > 0, \quad \lambda_2 + a\,\lambda_1 > 0\right\},$$
$$\hat{F}(\lambda_1, \lambda_2) = (1-a)^2 \lambda_1 \lambda_2 + a(\lambda_1 + \lambda_2)^2,$$

(the cones Γ_{2-a} interpolate between Γ_1 and Γ_2). Here, the degree is $k = 2$, the concavity and the comparison (d) are easily checked.

5) *More generally, for any polynomial P hyperbolic with respect to $v \in \mathbb{R}^n$ (cf. [Går59, CNS85, HL18]) of degree k, the operator $\hat{F}(\lambda_1, \dots, \lambda_n) = P(\lambda_1, \dots, \lambda_n)$ defined on the component of $P \neq 0$ in $\mathbb{R}^n$ containing v, satisfies (c). Whether it satisfies the comparison (d) or not may depend on P.*

6) *An example of an operator meeting all the conditions above* **except** *(d) is the Hessian quotient operator given by*

$$\Gamma(z) = \lambda^{-1}(\Gamma_m), \quad \hat{F}(\lambda_1, \dots, \lambda_n) = \frac{\sigma_m(\lambda_1, \dots, \lambda_n)}{\sigma_\ell(\lambda_1, \dots, \lambda_n)},$$

for $1 \leq \ell < m \leq n$. Here, the degree is $k = m - \ell$ and the operator is concave.

2 $W^{2,r}/L^p$-viscosity subsolutions

As our analysis will be based on the pluripotential theory for the complex Monge-Ampère equation, we need to recall that plurisubharmonic solutions to this equation, even for regular right-hand side data, need not possess sufficient Sobolev regularity (see [BT82, Bło99, Koł05, DD20] and Section 2.2 below). Hence they are *not* strong solutions in general and as such cannot be tested directly for other types of operators. On the other hand, as discussed in [Din20b], a general Hessian operator, even one satisfying the hypotheses above, may fail to have properties necessary to develop its own pluripotential theory. Hence in order to accommodate pluripotential (sub)solutions it is necessary to develop another theory of weak solutions. As it turns out a good choice is the L^p-viscosity theory which we shall briefly sketch below. We refer for instance to [CCKS96] for more background.

In all this section, we only need to assume that $A \mapsto F(z, A)$ is elliptic in $\Gamma(z)$ for a.e. $z \in \Omega$.

2.1 Definition and remarks

Below we introduce the notion of a $W^{2,r}/L^p$-viscosity subsolution associated to an operator F. It is inspired from [CCKS96, Definition 2.1] but also contains notable differences, see Remark 2.2 below.

First of all, when dealing with constrained elliptic equations (i.e. $\Gamma(z) \neq \mathbb{H}^n$) it is a standard and useful procedure in viscosity theory (see e.g. [CIL92, ABT07]) to extend the operator F by $-\infty$ outside the cone i.e.

$$F(z, A) = -\infty, \quad \forall z \in \Omega, \forall A \in \mathbb{H}^n \backslash \Gamma(z). \tag{9}$$

Definition 2.1. Let $F : \Sigma \longrightarrow \mathbb{R}$ satisfy (9), $r, p \geq 1$ and $f \in L^p(\Omega)$. An upper semicontinuous function $u : \Omega \longrightarrow \mathbb{R}$ is said to be a $W^{2,r}/L^p$-viscosity solution of

$$F(z, D^2u) \geq f \quad \text{ in } \Omega,$$

if, for every lower semicontinuous function $\varphi : \Omega \longrightarrow \mathbb{R}$ with $\varphi \in W^{2,r}_{\text{loc}}(\Omega)$, for every $\varepsilon > 0$ and nonempty open subset $U \subset \Omega$, if

$$F(z, (D^2\varphi)(z)) \leq f(z) - \varepsilon \quad \text{ a.e. } z \in U,$$

then $u - \varphi$ cannot have a **strict** local maximum in U.

We will also say that u is a $W^{2,r}/L^p$-viscosity subsolution.

Remark 2.2. Let us comment this new definition:

- The functions φ in Definition 2.1 are called test functions. If the class of $W^{2,r}_{\mathrm{loc}}$ testing functions above is exchanged by the class of C^2 tests we get the notion of C-viscosity (sub)solutions. These are much better studied and we refer to [CIL92, CC95] for the general theory of C-viscosity for uniformly elliptic equations. For complex equations, C-viscosity has been considered first in [EGZ11] and then in [Wan12, Zer13] for the complex Monge-Ampère equation and in [DDT19] for degenerate Hessian equations.

- The main difference with the definition introduced in [CCKS96, Definition 2.1] is that our operators are not uniformly elliptic and this is why we require in addition the condition that the maximum is strict. We refer to Section 2.2 for some instructive examples.

- In Theorem 1.4 we assume that $r > n$, for which we have the Sobolev embedding $W^{2,r}_{\mathrm{loc}}(\Omega) \subset C^0(\Omega)$. However, there is no need to require such a condition in Definition 2.1 since every quantity makes sense as it is introduced (compare with [CCKS96, Definition 2.1]).

- If F is a linear operator of type $F(z, D^2u) = \sum_{i,j=1}^n a^{i\bar{j}} u_{i\bar{j}}$ with $(a^{i\bar{j}})_{1\le i,j\le n} \ge 0$, contrary to the uniformly elliptic case (see [CCKS96]) we do **not** assume that the matrix entries are essentially bounded, and hence F is in general **not** expected to send $W^{2,p}$ functions u to L^p.

Finally, the following observation will be used in the proof of the main theorem.

Remark 2.3. If $A \mapsto G(z, A)$ is linear and if u_1 is a $W^{2,r}/L^p$-viscosity solution to $G(z, D^2u_1) \ge g_1$ in Ω and u_2 is a $W^{2,r}/L^p$-viscosity solution to $G(z, D^2u_2) \ge g_2$ in Ω with $u_2 \in W^{2,r}_{\mathrm{loc}}(\Omega)$, then $u = u_1 + u_2$ is a $W^{2,r}/L^p$-viscosity solution to $G(z, D^2u) \ge g_1 + g_2$ in Ω. This follows from the definition as we can subtract u_2 from any testing function $\varphi \in W^{2,r}_{\mathrm{loc}}(\Omega)$ and the resulting function is testing for u_1.

2.2 The strict maximum condition

In this section, we explain why we have required that the maximum is strict in Definition 2.1. We discuss this for the complex Monge-Ampère equation and its linearization.

Recall once again that equations of Monge-Ampère type locally admit *singular* solutions no matter how smooth the right-hand side is. The following example from [Bło99] is modeled on real Pogorelov type singular convex functions:

Example 2.4. *Let $n \ge 2$. Let u be the function defined for every $z = (z_1, z') \in \mathbb{C}^n$ by*

$$u(z) = \|z'\|^{2\left(1-\frac{1}{n}\right)} \left(1 + |z_1|^2\right). \tag{10}$$

It is smooth off $\{z' = 0\}$ and $u \in W^{2,r}_{\mathrm{loc}}(\mathbb{C}^n)$ for any $1 \le r < n(n-1)$ with $(D^2u)(z) \in \mathcal{C}_n$ for every $z \in \mathbb{C}^n \setminus \{z' = 0\}$. A computation shows that it is a strong solution to

$$\det(D^2u) = f \quad \text{in } \mathbb{C}^n, \quad f(z) = \left(1 - \frac{1}{n}\right)^n (1 + |z_1|^2)^{n-2}. \tag{11}$$

It is also easily seen that u is a C-viscosity solution, the only problem is at $\{z' = 0\}$ and there are clearly no C^2-smooth differential tests from above, while C^2 differential tests from below have to vanish along $\{z' = 0\}$ and thus have vanishing Monge-Ampère operator [Zer13].

When it comes to $W^{2,r}/L^p$-viscosity things are substantially subtler. We have built the following example:

Example 2.5. *Let $n \geq 3$ and $R > 0$ be fixed. Let u and f be the functions of Example 2.4. Consider the function φ defined for every $z = (z_1, z') \in \mathbb{C}^n$ by*

$$\varphi(z) = \|z'\|^{2\left(1-\frac{1}{n}\right)} (1 + R^2 - \|z'\|^2). \tag{12}$$

We have $\varphi \in W^{2,r}_{\mathrm{loc}}(B_R(0))$ for every $n < r < n(n-1)$ and $(D^2\varphi)(z) \in \mathcal{C}_n$ for every $z \in B_R(0) \setminus \{z' = 0\}$. As φ is a function of $n-1$ variables, it is a strong solution to the equation

$$\det(D^2\varphi) = 0 \quad \text{in } B_R(0).$$

Besides, we can always find $\varepsilon > 0$ small enough so that $0 < f - \varepsilon$. On the other hand, we clearly have

$$u - \varphi \leq 0 = (u - \varphi)_{|\{z'=0\}}, \quad \text{in } B_R(0),$$

so that $u - \varphi$ has no strict maximum in $B_R(0)$.

In conclusion, if we do not require the strict maximum condition in Definition 2.1, then the function u is a strong solution to $\det(D^2u) = f$ but it would not be a $W^{2,r}/L^p$-viscosity solution to $\det(D^2u) \geq f$, whatever $p \geq 1$ is (even if $r > n$). Note that this has nothing to do with the regularity of the right-hand side since $f \in C^\infty(\mathbb{C}^n)$.

The previous example shows that imposing *strict* local maxima in the definition of $W^{2,r}/L^p$-viscosity solutions is essential for Hessian type equations. It turns out that the same example works for the *linearized equation* provided r is taken sufficiently small.

Example 2.6. *Let u, f, φ be the same functions as in Example 2.5 but consider this time the following linear operator:*

$$L_u h = \sum_{i,j=1}^{n} a^{i\bar{j}}(z) h_{i\bar{j}}, \quad a^{i\bar{j}}(z) = \frac{\partial \det}{\partial a_{i\bar{j}}}((D^2u)(z)).$$

By homogeneity, we immediately have $L_u u = n \det(D^2u) = nf$. On the other hand, for sufficiently small $R, \varepsilon > 0$, we will show that

$$L_u \varphi \leq nf - \varepsilon \quad \text{in } B_R(0). \tag{13}$$

Therefore, even for linear equations, the strict maximum condition that we introduced in Definition 2.1 is needed if we at least want that strong solutions are $W^{2,r}/L^p$-viscosity solutions.

We provide the details of the inequality (13) *for* $n = 3$ *only, the general case is analogous. An explicit computation of the* $a^{i\bar{j}}$ *and the fact that* φ *does not depend on* z_1 *yield*

$$\sum_{i,j=1}^{3} a^{i\bar{j}}(z)\varphi_{i\bar{j}} = (u_{1\bar{1}}u_{3\bar{3}} - |u_{1\bar{3}}|^2)\varphi_{2\bar{2}} + (u_{1\bar{1}}u_{2\bar{2}} - |u_{1\bar{2}}|^2)\varphi_{3\bar{3}} + (u_{3\bar{1}}u_{1\bar{2}} - u_{1\bar{1}}u_{3\bar{2}})\varphi_{2\bar{3}} + (u_{2\bar{1}}u_{1\bar{3}} - u_{1\bar{1}}u_{2\bar{3}})\varphi_{3\bar{2}}.$$

From the expression (10) *of* u *we have, for* $z' \neq 0$,

$$\begin{aligned}
u_{1\bar{1}}u_{3\bar{3}} - |u_{1\bar{3}}|^2 &= \frac{2}{3}\left(1 + |z_1|^2\right)\|z'\|^{\frac{2}{3}} - \left(\frac{2}{9} + \frac{2}{3}|z_1|^2\right)|z_3|^2\,\|z'\|^{-\frac{4}{3}}\,,\\
u_{1\bar{1}}u_{2\bar{2}} - |u_{1\bar{2}}|^2 &= \frac{2}{3}\left(1 + |z_1|^2\right)\|z'\|^{\frac{2}{3}} - \left(\frac{2}{9} + \frac{2}{3}|z_1|^2\right)|z_2|^2\,\|z'\|^{-\frac{4}{3}}\,,\\
u_{3\bar{1}}u_{1\bar{2}} - u_{1\bar{1}}u_{3\bar{2}} &= \left(\frac{2}{9} + \frac{2}{3}|z_1|^2\right)\bar{z}_3 z_2\,\|z'\|^{-\frac{4}{3}}\,,\\
u_{2\bar{1}}u_{1\bar{3}} - u_{1\bar{1}}u_{2\bar{3}} &= \left(\frac{2}{9} + \frac{2}{3}|z_1|^2\right)\bar{z}_2 z_3\,\|z'\|^{-\frac{4}{3}}\,.
\end{aligned}$$

From the expression (12) *of* φ *we have, for* $z' \neq 0$,

$$\begin{aligned}
\varphi_{2\bar{2}} &= -\frac{5}{3}\|z'\|^{\frac{4}{3}} + \left(-\frac{10}{9}|z_2|^2 + \frac{2}{3}(1+R^2)\right)\|z'\|^{-\frac{2}{3}} - \frac{2}{9}(1+R^2)|z_2|^2\,\|z'\|^{-\frac{8}{3}}\,,\\
\varphi_{3\bar{3}} &= -\frac{5}{3}\|z'\|^{\frac{4}{3}} + \left(-\frac{10}{9}|z_3|^2 + \frac{2}{3}(1+R^2)\right)\|z'\|^{-\frac{2}{3}} - \frac{2}{9}(1+R^2)|z_3|^2\,\|z'\|^{-\frac{8}{3}}\,,\\
\varphi_{2\bar{3}} &= -\frac{10}{9}\bar{z}_2 z_3\,\|z'\|^{-\frac{2}{3}} - \frac{2}{9}(1+R^2)\bar{z}_2 z_3\,\|z'\|^{-\frac{8}{3}}\,,\\
\varphi_{3\bar{2}} &= -\frac{10}{9}\bar{z}_3 z_2\,\|z'\|^{-\frac{2}{3}} - \frac{2}{9}(1+R^2)\bar{z}_3 z_2\,\|z'\|^{-\frac{8}{3}}\,.
\end{aligned}$$

Therefore, for $z' \neq 0$,

$$\begin{aligned}
L_u\varphi &= -\left(\frac{70}{27} + \frac{50}{27}|z_1|^2\right)\|z'\|^2 + \frac{16}{27}(1+R^2) + \frac{8}{27}(1+R^2)|z_1|^2\\
&\leq \frac{16}{27}(1+R^2) + \frac{8}{27}(1+R^2)|z_1|^2.
\end{aligned}$$

As $3f = \frac{24}{27} + \frac{24}{27}|z_1|^2$ *(recall* (11)*), we see that* $L_u\varphi < 3f$ *for any* $R < \frac{1}{\sqrt{2}}$, *so that* (13) *holds provided* ε *is taken small enough (since* $L_u\varphi$ *and* f *are continuous on the closure of* $B_R(0)$*).*

3 $W^{2,r}/L^p$-viscosity subsolutions and pluripotential theory

All along this section, we denote the open unit ball by $B_1 = B_1(0)$.

Let $F_{\mathrm{MA}}(A) = \det A$ for $A \in \mathcal{C}_n$ and $F_{\mathrm{MA}}(A) = -\infty$ otherwise. The goal of this section is to show the following result:

Theorem 3.1. *Let*

$$r > n, \quad q > 1, \tag{14}$$

and let $g \in L^q(B_1)$ with $g \geq 0$. Then, there exists a $W^{2,r}/L^q$-viscosity solution $\rho \in C^0(\overline{B_1})$ of

$$F_{\mathrm{MA}}(D^2\rho) \geq g \quad \text{in } B_1,$$

with $\rho = 0$ on ∂B_1 and which satisfies

$$\sup_{B_1}(-\rho) \leq C(n,q)\,\|g\|_{L^q(B_1)}^{\frac{1}{n}}.$$

3.1 Background on pluripotential theory

First of all, let us discuss some pluripotential tools. One of the main problems which pluripotential theory handles is the solvability complex Monge-Ampère equation. For every function $\rho \in \mathrm{PSH}(B_1) \cap C^2(B_1)$, the complex Monge-Ampère operator is given as follows:

$$(dd^c\,\rho)^n = 4^n n! \det\left(D^2\rho\right) dV.$$

Here dV denotes the Lebesgue measure in $\mathbb{C}^n$, $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, $\mathrm{PSH}(B_1)$ is the set of plurisubharmonic functions in B_1 i.e. the set of upper semicontinuous, locally integrable functions ρ such that $dd^c\,\rho \geq 0$. The operator $(dd^c\,\rho)^n$ is well defined on bounded plurisubharmonic functions, as follows from the work [BT76] (see also [BT82]).

Below we consider the Dirichlet problem associated to the Monge–Ampère operator in the ball B_1. It reads:

$$\begin{cases} (dd^c\,\rho)^n = g dV & \text{in } B_1, \\ \rho = \psi & \text{on } \partial B_1, \end{cases} \tag{15}$$

where $g \in L^q(B_1)$ $(q > 1)$ and $\psi \in C^0(\partial B_1)$. We recall now the following fundamental result from [Koł98]:

Theorem 3.2. *Let $g \in L^q(B_1)$ $(q > 1)$ with $g \geq 0$. There exists a unique solution $\rho \in \mathrm{PSH}(B_1) \cap C^0(\overline{B_1})$ of (15) with $\psi = 0$, and it satisfies*

$$\sup_{B_1}(-\rho) \leq C(n,q)\,\|g\|_{L^q(B_1)}^{\frac{1}{n}}.$$

Consequently, we see that Theorem 3.1 will be a straightforward consequence of this result if we manage to establish the following:

Proposition 3.3. *Assume (14) and let $g \in L^q(B_1)$ with $g \geq 0$. If $\rho \in \mathrm{PSH}(B_1) \cap C^0(\overline{B_1})$ is a solution to $(dd^c\,\rho)^n \geq 4^n n! g dV$ in B_1 in the sense of currents, then ρ is a $W^{2,r}/L^q$-viscosity solution to $F_{\mathrm{MA}}(\rho) \geq g$ in B_1.*

The proof of this proposition is the purpose of the next section.

3.2 Comparison theorem for non PSH functions

In this part, B is any fixed open ball. In order to prove Propostion 3.3, we will use the following comparison theorem between PSH and $W^{2,r}$ functions, which is adapted from [RT77, Theorem 5.1] to our complex setting:

Theorem 3.4. *Let $v \in W^{2,r}(B)$ with $r > n$. Let $u \in \mathrm{PSH}(B) \cap C^0(\overline{B})$ satisfy*

$$(dd^c u)^n \geq ((dd^c v)^n)^+ \qquad \text{in } B,$$

in the sense of currents, where $((dd^c v)^n)^+$ is by definition equal to $(dd^c v)^n$ if the current $dd^c v \geq 0$ (i.e. $D^2 v \in \overline{\mathcal{C}_n}$) and is equal to zero otherwise. Then, we have

$$\max_{z \in B} (u(z) - v(z)) = \max_{z \in \partial B} (u(z) - v(z)). \tag{16}$$

We would to emphasize that an essential difference in the proof of this result below compared with the one of [RT77, Theorem 5.1] is that the complex Monge-Ampère operator, contrary to the real one, is not continuous with respect to the weak convergence (see e.g. [Kli91, Section 3.8]). We will circumvent this by exploiting the stability properties of the Monge-Ampère operator. This in particular is one of the reasons for the assumption $r > n$ in this section.

Before proving Theorem 3.4, let us show how it leads to Proposition 3.3:

Proof of Proposition 3.3. Let $\rho \in \mathrm{PSH}(B_1) \cap C^0(\overline{B_1})$ be a solution to $(dd^c \rho)^n \geq 4^n n! g dV$ in B_1 in the sense of currents. Let $\varphi \in W^{2,r}_{\mathrm{loc}}(B_1)$ and a nonempty open subset $U \subset B_1$ (φ is continuous since $r > n$). Assume that $u - \varphi$ has a strict maximum, say at z_0 and in some small ball $\overline{B_R(z_0)} \subset B_1$. Assume by contradiction that

$$F_{\mathrm{MA}}((D^2\varphi)(z)) \leq g(z) \qquad \text{a.e. } z \in B_R(z_0). \tag{17}$$

If we show that this implies $(dd^c \rho)^n \geq ((dd^c \varphi)^n)^+$ in $B_R(z_0)$, then Theorem 3.4 gives a contradiction. We have to distinguish two cases. If the current $dd^c \varphi$ is not nonnegative, then $((dd^c \varphi)^n)^+ = 0$ by definition and the desired inequality holds since $(dd^c \rho)^n \geq 4^n n! g dV$ and $g \geq 0$. Assume then that the current $dd^c \varphi$ is nonnegative, that is $D^2\varphi \in \overline{\mathcal{C}_n}$. We have $((dd^c \varphi)^n)^+ = 4^n n! \det(D^2\varphi)\, dV$ by definition and the desired inequality will be proved if we show that $g \geq \det(D^2\varphi)$ in $B_R(z_0)$. On the measurable subset $\{z \in B_R(z_0) \quad | \quad (D^2\varphi)(z) \in \mathcal{C}_n\}$ we have $F_{\mathrm{MA}}(D^2\varphi) = \det(D^2\varphi)$, so that the previous inequality holds thanks to (17). On the remaining set we have $D^2\varphi \in \partial\mathcal{C}_n = \{0\}$, so that previous inequality holds as well thanks to $g \geq 0$.

□

For the proof of Theorem 3.4, we need to recall two results. The first one is the following comparison principle that follows from [BT82, Theorem 4.1] and which is by now a basic tool in pluripotential theory.

Theorem 3.5. *Let $u, v \in \mathrm{PSH}(B) \cap C^0(\overline{B})$ satisfy*

$$(dd^c u)^n \geq (dd^c v)^n \quad \text{in } B,$$

in the sense of currents. Then, the same equality as in (16) *holds.*

The second result that will be needed is [Sib77, Théorème 1] coupled with [Bło96, Theorem 3.9]:

Theorem 3.6. *Let $v \in C^2(B) \cap C^0(\overline{B})$. Let $u \in \mathrm{PSH}(B) \cap C^0(\overline{B})$ satisfy*

$$(dd^c u)^n \geq (dd^c v)^n \quad \text{in } B,$$

in the sense of currents. Then, the same equality as in (16) *holds.*

Note that Sibony's theorem states (in modern terminology) that solutions of $(dd^c u)^n \geq f dV$ (for continuous f) in the sense of currents are C-viscosity subsolutions (see the proof of Proposition 3.3).

We have now all the ingredients to prove the desired comparison theorem.

Proof of Theorem 3.4. We adapt the proof of [RT77, Theorem 5.1]. First of all, we note that there exists a sequence $(v_j)_j \subset C^\infty(\mathbb{C}^n)$ such that

$$v_j \to v \text{ in } W^{2,r}(B), \tag{18}$$

$$v_j \to v \text{ in } C^0(\overline{B}). \tag{19}$$

Indeed, since B is a ball, there exists an extension operator $E : W^{2,r}(B) \to W^{2,r}(\mathbb{C}^n)$. Let then $\varphi \in C_c^\infty(\mathbb{C}^n)$ be a cut-off function which is equal to 1 in B and 0 outside an open set ω such that $\omega \supset \overline{B}$. Thus, $\hat{v} = \varphi E v \in W_0^{2,r}(\omega)$ and there exists a sequence $(\hat{v}_j)_j \subset C_c^\infty(\omega)$ such that $\hat{v}_j \to \hat{v}$ in $W^{2,r}(\omega)$. We define v_j as the extension of $\hat{v}_j$ by zero outside ω, which then satisfies (18). The convergence (19) is consequence of (18) by the Sobolev embedding $W^{2,r}(B) \subset C^0(\overline{B})$ since $r > n$.

Let us now introduce the functions

$$g_j = \begin{cases} 4^n n! \det(D^2 v_j), & \text{if } dd^c v_j \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad g = \begin{cases} 4^n n! \det(D^2 v), & \text{if } dd^c v \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to (18), we have

$$g_j \to g \text{ in } L^{r/n}(B). \tag{20}$$

Since $g_j, g \geq 0$, from [BT82] and [Koł98] there exist $w_j, w \in \mathrm{PSH}(B) \cap C^0(\overline{B})$, respective unique solutions to

$$\begin{cases} (dd^c w_j)^n = g_j dV, & \text{in } B, \\ w_j = v_j, & \text{on } \partial B, \end{cases} \qquad \begin{cases} (dd^c w)^n = g dV, & \text{in } B, \\ w = v, & \text{on } \partial B. \end{cases}$$

Moreover, Kolodziej's stability theorem (contained in the proof of [Koł96, Theorem 3]) states that

$$\|w_j - w\|_{L^\infty(B)} \leq \|v_j - v\|_{L^\infty(\partial B)} + C(n,r)\,\|g_j - g\|^{\frac{1}{n}}_{L^{\frac{r}{n}}(B)}.$$

It follows that $w_j \to w$ in $C^0(\overline{B})$ thanks to (19) and (20).

Since $(dd^c\, w_j)^n \geq (dd^c\, v_j)^n$ with $w_j = v_j$ on ∂B and $v_j \in C^2(B)$, we can apply Theorem 3.6 to obtain that

$$w_j \leq v_j \quad \text{in } \overline{B}.$$

It follows that

$$\begin{aligned} w - v &= w - w_j + w_j - v_j + v_j - v \\ &\leq w - w_j + v_j - v. \end{aligned}$$

Passing to the limit, we deduce that

$$w - v \leq 0 \quad \text{in } \overline{B}. \tag{21}$$

To complete the proof, note that for $z \in B$,

$$\begin{aligned} u(z) - v(z) &= u(z) - w(z) + w(z) - v(z) \\ &\leq u(z) - w(z) \quad \text{(by (21))}, \\ &\leq \max_{z\in\partial B}(u(z) - w(z)) \quad \text{(by Theorem 3.5 since } (dd^c\, u)^n \geq gdV = (dd^c\, w)^n), \\ &= \max_{z\in\partial B}(u(z) - v(z)) \quad \text{(since } w = v \text{ on } \partial B). \end{aligned}$$

□

4 Maximum principle for $W^{2,r}/L^p$-viscosity subsolutions

The starting point of any viscosity theory is the maximum principle. Below we prove a version adapted to our setting. Similar result for smoother functions can be found in [RS64, Theorem 1].

Theorem 4.1. *Let $a^{i\bar{j}} : \Omega \longrightarrow \mathbb{C}$ $(1 \leq i, j \leq n)$ be such that the coefficient matrix $(a^{i\bar{j}}(z))_{1\leq i,j\leq n}$ is Hermitian and nonnegative for a.e. $z \in \Omega$. Assume that there exists $M > 0$ such that, for a.e. $z \in \Omega$,*

$$\sum_{i=1}^{n} a^{i\bar{i}}(z) \geq M. \tag{22}$$

Let $u \in C^0(\overline{\Omega})$ be a $W^{2,r}/L^p$-viscosity solution $(r, p \geq 1)$ of

$$\sum_{i,j=1}^{n} a^{i\bar{j}}(z) u_{i\bar{j}} \geq 0 \quad \text{in } \Omega. \tag{23}$$

Then, $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.

Proof. Suppose on contrary that the value $\max_{\overline{\Omega}} u$ is reached only in Ω and let

$$K = \left\{ z \in \overline{\Omega} \quad \middle| \quad u(z) = \max_{\overline{\Omega}} u \right\}.$$

Clearly, K is compact and, by assumption, $K \subset \Omega$. Our goal will be to construct a strictly concave polynomial barrier at (some) point of K. This coupled with the properties of our linear operator would lead to a contradiction.

Assume first that K is reduced to a point, or more generally that it has an isolated point z_0. Let then $R > 0$ be fixed such that $B_R(z_0) \subset \Omega$ and such that no other maximum lies in $B_R(z_0)$. Set $h(z) = -\varepsilon \|z - z_0\|^2 + u(z_0)$. For $\varepsilon > 0$ small enough, the maximum of the difference $u - h$ is necessarily achieved inside $B_R(z_0)$, say at some point z_1. Then, for the smooth function $\varphi(z) = h(z) + \frac{\varepsilon}{2} \|z - z_1\|^2$, the difference $u - \varphi$ has a strict maximum in $B_R(z_0)$. As $\varphi_{i\bar{j}}(z) = -\frac{\varepsilon}{2}\delta_{ij}$, we have

$$\sum_{i,j=1}^{n} a^{i\bar{j}}(z)\varphi_{i\bar{j}}(z) = -\frac{\varepsilon}{2}\sum_{i=1}^{n} a^{i\bar{i}}(z) \leq -\frac{\varepsilon}{2}M,$$

and hence φ is a valid test function, which contradicts the fact that u is a $W^{2,r}/L^p$-viscosity solution of (23) (see Definition 2.1).

In general the set K need not contain isolated points and we proceed differently. Pick a point $x_0 \in K$ and inflate the balls centered at x_0 up until the whole set K is inside. If K is not reduced to a point, then $R = \max_{z \in K} \|z - x_0\|$ is positive and we have $K \subset \overline{B_R(x_0)}$ and $K \cap \partial B_R(x_0) \neq \emptyset$. Let then $z_0 \in K \cap \partial B_R(x_0)$. Rotating and shifting coordinates if necessary, we may assume that $x_0 = (0, \cdots, 0, R)$, $K \subset \left\{ \|z'\|^2 + |z_n - R|^2 \leq R^2 \right\}$ and $z_0 = (0, \cdots, 0) \in \partial K$ (observe that the assumptions of the theorem are invariant under such transformations). Note that any point $z = (z', x_n + iy_n) \in K$ has to satisfy

$$x_n \geq \frac{y_n^2 + \|z'\|^2}{2R}.$$

Then fixing a constant $C > 2R$, for any $\eta > 0$ the set

$$\begin{aligned} \hat{U}_\eta = \left\{ z \in \mathbb{C}^n \quad \middle| \quad \mathfrak{Re}\, z_n = -\eta, \quad (\mathfrak{Im}\, z_n)^2 + \|z'\|^2 \leq C\eta \right\} \\ \cup \left\{ z \in \mathbb{C}^n \quad \middle| \quad |\mathfrak{Re}\, z_n| \leq \eta, \quad (\mathfrak{Im}\, z_n)^2 + \|z'\|^2 = C\eta \right\}, \end{aligned}$$

is disjoint from K. Since $\hat{U}_\eta$ is compact, it follows that $\max_{\hat{U}_\eta} u < \max_{\overline{\Omega}} u = u(0)$, so that there is a $\delta > 0$, such that

$$u|_{\hat{U}_\eta} < u(0) - 2\delta. \tag{24}$$

Since $0 \in K \subset \Omega$, there is a ball $B_r(0) \subset \Omega$ and for η small enough so that $\eta^2 + C\eta < r^2$, the domain

$$U_\eta = \left\{ z \in \mathbb{C}^n \quad \middle| \quad |\mathfrak{Re}\, z_n| < \eta, \quad (\mathfrak{Im}\, z_n)^2 + \|z'\|^2 < C\eta \right\},$$

is such that $\overline{U_\eta} \subset \Omega$. Fix such an η and set $U = U_\eta$.

Consider now the quadratic polynomial

$$h(z) = -\frac{\delta}{4C\eta}(\|z'\|^2 + (\mathfrak{Im}\, z_n)^2) - \frac{\delta}{3\eta^2}(\mathfrak{Re}\, z_n - \eta)^2 + \frac{\delta}{3} + u(0).$$

By continuity, $u-h$ has a maximum on $\overline{U}$. Let us show that this value is necessarily reached in U. On $\overline{U}$, we have

$$h(z) \geq -\frac{\delta}{4} - \frac{\delta}{3\eta^2}(\mathfrak{Re}\, z_n - \eta)^2 + \frac{\delta}{3} + u(0).$$

On $\hat{U}_\eta$, using (24) we obtain

$$h(z) \geq -\frac{5\delta}{4} + u(0) > u(z).$$

On the remaining part of ∂U, i.e. on the piece where $\mathfrak{Re}\, z_n = \eta$, we simply use that $0 \in K$:

$$h(z) \geq \frac{\delta}{12} + u(0) > u(0) \geq u(z).$$

In summary, $u - h < 0$ on ∂U. Since $0 \in U$ with $u(0) - h(0) = 0$, the maximum of $u - h$ has to be reached into U. We also have $h_{i\bar{j}}(z) = -\varepsilon_i \delta_{ij}$ for some $\varepsilon_i > 0$.

The conclusion is now as before: define $\varphi(z) = h(z) + \frac{\varepsilon}{2}\|z - z_1\|^2$, where z_1 is a point of maximum of $u - h$ in U, so that the difference $u - \varphi$ now has a strict maximum in U; taking $\varepsilon < \varepsilon_i$ and using $a^{i\bar{i}} \geq 0$, we have

$$\sum_{i,j=1}^{n} a^{i\bar{j}}(z)\varphi_{i\bar{j}}(z) = -\sum_{i=1}^{n} a^{i\bar{i}}(z)\,(\varepsilon_i - \varepsilon) \leq -M \min_{1\leq i\leq n}(\varepsilon_i - \varepsilon)\,,$$

and hence φ is a valid test function, which contradicts the fact that u is a $W^{2,r}/L^p$-viscosity solution of (23). □

5 Proof of the main result

In this section we finally prove Theorem 1.4. All along this section, fix a sufficiently large ball containing Ω. Without loss of generality, we assume that it is $B_1 = B_1(0)$. We then extend g by zero in $B_1 \backslash \Omega$ (still denoted by g).

Let then $\rho \in C^0(\overline{B_1})$ be the corresponding $W^{2,r}/L^{p/n}$-viscosity solution of

$$F_{\mathrm{MA}}(D^2\rho) \geq (g_+)^n \quad \text{in } B_1,$$

provided by Theorem 3.1 with $q = p/n$, whose hypotheses (14) are satisfied thanks to our assumption (4). It also satisfies $\rho = 0$ on ∂B_1 and the Kołodziej's estimate

$$\sup_{B_1}(-\rho) \leq C(n,q)\,\|g_+\|_{L^p(B_1)}\,. \tag{25}$$

We will show that $-\rho$ is an upper barrier for $-u$ in Ω. The conclusion will then follow from (25). Let L_u be the linearization of G about D^2u:

$$L_u h = \sum_{i,j=1}^{n} G^{i\bar{j}}(z, (D^2u)(z)) h_{i\bar{j}}.$$

First of all, we have the following lemma:

Lemma 5.1. *The function ρ is a $W^{2,r}/L^p$-viscosity solution of*

$$L_u \rho \geq g_+ \quad \text{in } B_1.$$

Proof. Assume not. Then, there exist $\varphi \in W^{2,r}_{\text{loc}}(B_1)$, $\varepsilon > 0$ and an open subset $U \subset B_1$ such that, in U,

$$L_u \varphi \leq g_+ - \varepsilon, \tag{26}$$

and $\rho - \varphi$ has a strict local maximum in U. Since ρ is a $W^{2,r}/L^{p/n}$-viscosity solution of $F_{\text{MA}}(D^2\rho) \geq (g_+)^n$ in B_1, by very definition we obtain that, for every $\eta > 0$, the set

$$V_\eta = \left\{ z \in U \quad \middle| \quad F_{\text{MA}}(D^2\varphi(z)) > (g_+(z))^n - \eta \right\},$$

has to be of positive measure. In particular, note that $D^2\varphi \in \mathcal{C}_n$ a.e. in V_η since F_{MA} is equal to $-\infty$ outside $\mathcal{C}_n$ by definition. In this set V_η, we have

$$\begin{aligned} L_u\varphi &\geq G\left(z, D^2\varphi\right) \quad \text{(by (3))}, \\ &\geq \left(\det(D^2\varphi)\right)^{\frac{1}{n}} \quad \text{(by (2) since } D^2\varphi \in \mathcal{C}_n), \\ &\geq g_+ - \eta^{\frac{1}{n}} \quad \text{(by definition of } V_\eta \text{ and subadditivity of } x \longmapsto x^{\frac{1}{n}}). \end{aligned}$$

As $\varepsilon > 0$ is fixed, by taking $\eta > 0$ small enough we reach a contradiction with (26).

□

We can now prove our main result:

Proof of Theorem 1.4. By homogeneity, we have

$$L_u u = G(z, D^2u).$$

Therefore,

$$L_u(-u) = -G(z, D^2u) \geq -g \geq -g_+.$$

From Lemma 5.1 and Remark 2.3 we see that $\rho - u$ is then a $W^{2,r}/L^p$-viscosity solution to

$$L_u\left(\rho - u\right) \geq 0 \quad \text{in } \Omega.$$

We can check that $a^{i\bar{j}}(z) = G^{i\bar{j}}(z, (D^2u)(z))$ satisfy the assumptions of Theorem 4.1. Indeed, the condition (22) follows from (3) (with $B = \text{Id}$) and the fact that $G(z, \text{Id}) \geq 1$ (by (2)); the nonnegativity of $(a^{i\bar{j}}(z))_{1\leq i,j\leq n}$ follows from the ellipticity of $A \mapsto G(z, A)$. Therefore, using this maximum principle, we obtain that (recall that $\rho \leq 0$ in $\overline{B_1}$ and $u \geq 0$ on $\partial\Omega$)

$$\rho - u \leq 0 \quad \text{in } \Omega.$$

The desired estimate (5) then follows from Kołodziej's estimate (25) of ρ. □

Acknowledgements

The first and third named authors were partially supported by the Polish National Science Centre Grant 2017/26/E/ST1/00955. The second named author was partially supported by the Polish National Science Centre Grant 2017/27/B/ST1/01145.

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