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The Askey–Wilson algebra and its avatars

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Abstract: The original Askey–Wilson algebra introduced by Zhedanov encodes the bispectrality properties of the eponym polynomials. The name *Askey–Wilson algebra* is currently used to refer to a variety of related structures that appear in a large number of contexts. We review these versions, sort them out and establish the relations between them. We focus on two specific avatars. The first is a quotient of the original Zhedanov algebra; it is shown to be invariant under the Weyl group of type D_4 and to have a reflection algebra presentation. The second is a universal analogue of the first one; it is isomorphic to the Kauffman bracket skein algebra (KBSA) of the four-punctured sphere and to a subalgebra of the universal double affine Hecke algebra (C_1^\vee, C_1) . This second algebra emerges from the Racah problem of $U_q(\mathfrak{sl}_2)$ and is related via an injective homomorphism to the centralizer of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product. How the Artin braid group acts on the incarnations of this second avatar through conjugation by R -matrices (in the Racah problem) or half Dehn twists (in the diagrammatic KBSA picture) is also highlighted. Attempts at defining higher rank Askey–Wilson algebras are briefly discussed and summarized in a diagrammatic fashion.

Keywords: Askey–Wilson algebra, Kauffman bracket skein algebra, $U_q(\mathfrak{sl}_2)$ algebra, double affine Hecke algebra, centralizer, universal R -matrix, $W(D_4)$ Weyl group, half Dehn twist.

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1 Introduction

In order to provide an algebraic underpinning for the Askey–Wilson polynomials [1], Zhedanov introduced what he called the Askey–Wilson algebra [2]. We shall refer to it rather as the Zhedanov algebra. The Askey–Wilson polynomials sit at the top of the Askey classification scheme of the hypergeometric orthogonal polynomials [3] and are, consequently, of fundamental interest; their algebraic interpretation by Zhedanov hence bears commensurate importance. These q -polynomials are bispectral: in addition to verifying a three-term recurrence prescribed by Favard’s theorem for any family of orthogonal polynomials [4], they are also eigenfunctions of a q -difference operator. The Zhedanov algebra was constructed by taking these two bispectral operators as generators and identifying the relations they obey. As sometimes happens with natural constructs, related structures have emerged in a variety of contexts and have typically all been called Askey–Wilson algebras. This propensity keeps rising and it is hence timely to review the topic. This paper will provide a taxonomy and a description of the algebras that loosely go under the name of *Askey–Wilson algebras* and will characterize in some depth two avatars of particular relevance. It will also set the stage for the exploration of generalizations.

The focus of this survey will be on algebraic aspects. Before we discuss the contents in more details, let us briefly go over some of the manifestations of these Askey–Wilson algebras and the advances they have generated. Grosso modo, they have had direct applications in physical models and have also been at the heart of mathematical developments establishing useful interconnections between fields. One occurrence is in the recoupling of three irreducible representations of $U_q(\mathfrak{sl}_2)$ which is called its Racah problem. It is known that the $6j$ -symbols of this algebra are expressed in terms of q -Racah polynomials which are a finite truncation of the Askey–Wilson ones. As a rule, whenever the Askey–Wilson polynomials (or their truncated version) appear, the associated algebra will be present. In the case of the Racah problem, it is found that the intermediate Casimir elements verify Askey–Wilson relations [5, 6]. These polynomials and algebras appear in the study of the ASEP model with open boundaries [7], as martingale polynomials and quadratic harnesses in probabilistic models [8] and are connected to (a degeneration of) the Sklyanin algebra [9–11]. Quite generally, the Askey–Wilson algebras are present in the context of integrable models, through the Yang–Baxter and reflection equations [12–17], and can be viewed as truncations of the q -Onsager algebra [12]. Elements of representation theory have been investigated in [2, 6, 18–20] and another of its manifestations is as a coideal subalgebra of $U_q(\mathfrak{sl}_2)$ [21–23]. The Askey–Wilson algebras have also been cast in the framework of Howe duality using the pair $(U_q(\mathfrak{sl}_2), \mathfrak{o}_{q^{1/2}}(2n))$ [24–27]; they are special cases of the recently introduced Painlevé algebras [28] and belong to the Calabi–Yau class [29]. There is a significant connection to the field of algebraic combinatorics, as Askey–Wilson algebras are central in the classification of P - and Q - polynomial association schemes and the study of Leonard pairs and triples [30–35]. The Askey–Wilson algebras have also been shown to offer a promising platform to extend the quantum Schur–Weyl duality to arbitrary representations and have been seen in that respect to admit the Temperley–Lieb and Birman–Murakami–Wenzl algebras [36] as quotients. Askey–Wilson algebras have moreover found their way in the general framework of knot theory through their identification with the Kauffman bracket skein algebras of the four-punctured sphere $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ and other elementary surfaces [37–39]. This is also closely connected to double affine Hecke algebras (DAHA) as the Askey–Wilson algebra is related to the spherical subalgebra of the DAHA of type (C_1^\vee, C_1) [20, 28, 40–46].

This overview of the relevance of Askey–Wilson algebras in different domains motivates the present topical report. Let us make at this point a few additional remarks on the introduction

of the algebra $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ in the Askey–Wilson picture to stress that this paper also features novel results relating the Askey–Wilson algebra, the Kauffman bracket skein algebra and the braid group.

Kauffman bracket skein algebras (KBSA) have been defined independently by Turaev [47] and Bullock and Przytycki [37] in the study of knot invariants and can be seen to encompass the celebrated Jones polynomial [48, 49]. Computations in the KBSA are done through diagrammatic manipulations given by a set of rules (the skein relations). It is appreciated that this $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ algebra is closely related to the centralizer of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product. This ties in with the Temperley–Lieb algebra which admits a diagrammatic presentation [49–51] for generic q , is precisely the centralizer of $U_q(\mathfrak{sl}_2)$ in the threefold tensor product of the fundamental representations of $U_q(\mathfrak{sl}_2)$ [52] and, as already indicated, was found to be a quotient of the Askey–Wilson algebra [36].

A natural question that has arisen asks about higher rank extensions of Askey–Wilson algebras. In view of the ubiquity of the 3-generated Askey–Wilson algebras it is to be expected that such generalizations will prove quite fruitful. This question is non-trivial however since many avenues that are likely to yield different outcomes can be followed. Among those possibilities, one is to consider the algebra realized by the intermediate Casimir elements in multifold tensor products of $U_q(\mathfrak{sl}_2)$ [53–56], and another is to increase the rank of the algebra $U_q(\mathfrak{sl}_2)$ to, say, $U_q(\mathfrak{sl}_3)$ when studying the Racah problem. Augmenting the number of punctures of the sphere in the KBSA approach could also be envisaged. Making much sense is the idea to start from the multivariate Askey–Wilson polynomials [57], to work out the algebra formed by its bispectral operators [55, 58, 59] and to take things from there. This is after all how the story began. Steps have been taken in these directions but final conclusions have not been reached. Some authors have considered higher order truncations of the reflection algebra [60] understood as a quotient of the q -Onsager algebra (see also [61] for the classical limit of this result). The upshot is that there is currently no clear consensus on what the higher rank Askey–Wilson algebra is*. This is not too surprising since there are still a few loose ends in the rank one cases.

As a prelude to a solid understanding of the higher rank Askey–Wilson algebra, it is appropriate to clarify the picture for the ordinary Askey–Wilson algebras. Indeed, as these algebras have appeared in multiple instances in the literature, names, conventions and notations are quite diverse. We are here proposing a standardization and offering a number of new results. The paper will unfold as follows. The various Askey–Wilson avatars will be introduced in Section 2. They will be given names and defined in a comparative way. Emphasis will be put on two particular versions. The first is a quotient of the Zhedanov algebra which we will call the Special Zhedanov algebra. In Section 3, we will show that the Zhedanov algebra is obtained as the reflection algebra defined from particular R - and reflection matrices. In this formalism, the Special Zhedanov algebra corresponds to fixing the Sklyanin determinant to a certain value; the name *Special* is chosen in analogy with the nomenclature of Lie groups. A Weyl group $W(D_4)$ symmetry of the Special Zhedanov algebra will then be presented in Section 4, thus generalizing an analogous result for the Racah algebra. The second avatar that will be closely looked at will be called the Special Askey–Wilson algebra. It can be seen as the equivalent of the Special Zhedanov algebra where the parameters are promoted to central elements in the algebra. That this algebra is isomorphic to the Kauffman bracket skein algebra of the four-punctured sphere $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ is the object of Section 5. In Section

*Remarkably, for the $q \rightarrow 1$ and $q \rightarrow -1$ limits of the Askey–Wilson algebra, higher rank extensions have been more successfully defined respectively in [62] for the Racah algebra and in [63] for the Bannai–Ito algebra.

6, the Special Askey–Wilson algebra will further be related to the algebra \mathcal{A}_3 associated to the Racah problem of $U_q(\mathfrak{sl}_2)$ and to the centralizer \mathfrak{C}_3 of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product. An injective homomorphism of algebras between the latter two structures will be stated and its proof will be found in Appendix A. The relation between the Special Askey–Wilson algebra and the universal double affine Hecke algebra (DAHA) of type (C_1^\vee, C_1) will be discussed in Section 7. How the Artin braid group B_3 acts on both the \mathcal{A}_3 and $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ algebras, respectively through conjugation by braided R -matrices and through half Dehn twists will be highlighted in Section 8. The question of the possible higher-rank generalizations of the Askey–Wilson algebra will be addressed in Section 9. A crossing index will be introduced and used to summarize efficiently the main results of [53] and [56] and new relations for the higher rank analogues will be provided. Elements of interest for further study of the higher rank generalizations of the Special Askey–Wilson algebra will be offered in addition. Concluding remarks will end the paper.

2 Askey–Wilson algebras

2.1 A jungle of Askey–Wilson algebras

As mentioned in the above, the name *Askey–Wilson algebra* has appeared and been connected to diverse objects in a multitude of contexts. Therefore, the notations and appellations in the literature are sometimes confusing. For the sake of clarity, we start by presenting these different algebraic structures and give to them unambiguous names to distinguish them.

The Askey–Wilson algebra $\mathbf{aw}(3)$ is the unital associative algebra depending on the parameter q with generators C_{12}, C_{23}, C_{13} and central elements C_1, C_2, C_3, C_{123} obeying the \mathbb{Z}_3 -symmetric relations

$$C_{12} + \frac{[C_{23}, C_{13}]_q}{q^2 - q^{-2}} = \frac{C_1 C_2 + C_3 C_{123}}{q + q^{-1}}, \quad (2.1a)$$

$$C_{23} + \frac{[C_{13}, C_{12}]_q}{q^2 - q^{-2}} = \frac{C_2 C_3 + C_1 C_{123}}{q + q^{-1}}, \quad (2.1b)$$

$$C_{13} + \frac{[C_{12}, C_{23}]_q}{q^2 - q^{-2}} = \frac{C_3 C_1 + C_2 C_{123}}{q + q^{-1}}, \quad (2.1c)$$

where the q -commutator is defined by $[A, B]_q = qAB - q^{-1}BA$. Throughout the paper, we suppose that $q \in \mathbb{C}$ is not a root of unity. The Casimir element of this algebra is

$$\begin{aligned} \Omega := & qC_{12}C_{23}C_{13} + q^2C_{12}^2 + q^{-2}C_{23}^2 + q^2C_{13}^2 - qC_{12}(C_1C_2 + C_3C_{123}) \\ & - q^{-1}C_{23}(C_2C_3 + C_1C_{123}) - qC_{13}(C_3C_1 + C_2C_{123}). \end{aligned} \quad (2.1d)$$

Let us emphasize that this algebra $\mathbf{aw}(3)$ is not the algebra called Askey–Wilson algebra by A. Zhedanov, and denoted $AW(3)$ in [2]. In the present paper, we call the latter the *Zhedanov algebra* (see below).

From the $\mathbf{aw}(3)$ algebra, we define multiple quotients or subalgebras which appear in different contexts; these justify the importance of this algebra.

The Special Askey–Wilson algebra $\text{saw}(3)$ is the quotient of $\mathbf{aw}(3)$ by the supplementary relation

$$\Omega = (q + q^{-1})^2 - C_{123}^2 - C_1^2 - C_2^2 - C_3^2 - C_{123}C_1C_2C_3. \quad (2.2)$$

A justification of the adjective *special* is given in Section 3. This algebra is isomorphic to the Kauffman bracket skein module of the four-punctured sphere (see Section 5) and is directly associated to the centralizer of the diagonal action of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product (see Section 6).

The universal Askey–Wilson algebra Δ_q defined in [33] is the subalgebra of $\mathbf{aw}(3)$ generated by C_{12} , C_{23} , C_{13} as well as the central elements $\alpha = C_1C_2 + C_3C_{123}$, $\beta = C_2C_3 + C_1C_{123}$ and $\gamma = C_3C_1 + C_2C_{123}$. The Casimir element of Δ_q becomes

$$\Omega = qC_{12}C_{23}C_{13} + q^2C_{12}^2 + q^{-2}C_{23}^2 + q^2C_{13}^2 - qC_{12}\alpha - q^{-1}C_{23}\beta - qC_{13}\gamma. \quad (2.3)$$

An injective homomorphism of Δ_q into $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ has been studied in [6] and its finite irreducible representations have been classified in [19]. The universal Askey–Wilson algebra also intersects the theory of free Lie algebras, see e.g. [64] and [65].

The evaluated Askey–Wilson algebra $Z_q(m_1, m_2, m_3)$ is the quotient of $\mathbf{aw}(3)$ by the supplementary relations

$$C_i = q^{m_i} + q^{-m_i}, \quad i = 1, 2, 3. \quad (2.4)$$

It plays a central role in the study of the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the threefold tensor product of representations of $U_q(\mathfrak{sl}_2)$ [36].

The Zhedanov algebra $Zh_q(m_1, m_2, m_3, m_4)$ is the quotient of $\mathbf{aw}(3)$ by

$$C_i = q^{m_i} + q^{-m_i}, \quad C_{123} = q^{m_4} + q^{-m_4}, \quad i = 1, 2, 3, \quad (2.5)$$

and was first introduced by Zhedanov as the algebra encoding the bispectrality of the Askey–Wilson polynomials [2]. To be precise, in [2], an alternative equivalent presentation recalled in (4.10a)–(4.10c), (4.10e)–(4.10g) has been given. The above \mathbb{Z}_3 -symmetric presentation of $Zh_q(m_1, m_2, m_3, m_4)$ is introduced in [10]. This algebra appears to be also the proper algebraic setting to characterize the Leonard pairs [32].

The Special Zhedanov algebra $sZh_q(m_1, m_2, m_3, m_4)$ is obtained as the quotient of $\text{saw}(3)$ by relations (2.5) (see (4.10a)–(4.10h) for an alternative presentation). It appears naturally as the commutation relations of the intermediate Casimir elements acting on the multiplicity space of the decomposition of the threefold tensor product of representations of $U_q(\mathfrak{sl}_2)$ (see Section 6.3).

2.2 Miscellaneous properties

PBW basis

The Askey–Wilson algebra $\mathbf{aw}(3)$ has a Poincaré–Birkhoff–Witt (PBW) basis given explicitly by the following elements

$$C_{12}^i C_{23}^j C_{13}^k C_1^m C_2^n C_3^p C_{123}^q, \quad i, j, k, m, n, p, q \in \mathbb{N}. \quad (2.6)$$

The proof is a slight generalization of the proof of the PBW basis for the universal Askey–Wilson algebra Δ_q given in [33]. We can also obtain a PBW basis for the Special Askey–Wilson algebra $\mathbf{aw}(3)$ from the one of $\mathbf{aw}(3)$ by restricting the range of the exponent j to $\{0, 1\}$ instead of \mathbb{N} .

Calabi–Yau algebra

The Zhedanov algebra $Zh_q(m_1, m_2, m_3, m_4)$ can be derived from a Calabi–Yau potential in the following sense [66]. Let $F = \mathbb{C}[x_1, x_2, x_3]$ be a free associative algebra and view F as a graded algebra such that $\deg(x_1) = d_1$, $\deg(x_2) = d_2$ and $\deg(x_3) = d_3$ (with $0 < d_1 \leq d_2 \leq d_3$). We define $F_{cycl} = F/[F, F]$ and the map $\frac{\partial}{\partial x_j} : F_{cycl} \rightarrow F$ on cyclic words as follows

$$\frac{\partial[x_{i_1}x_{i_2}\dots x_{i_r}]}{\partial x_j} = \sum_{\{s|i_s=j\}} x_{i_s+1}x_{i_s+2}\dots x_{i_r}x_{i_1}x_{i_2}\dots x_{i_s-1} \quad (2.7)$$

and we extend it to F_{cycl} by linearity. Let $\Phi(x_1, x_2, x_3) \in F_{cycl}$ be a potential which can be decomposed as follows

$$\Phi(x_1, x_2, x_3) = \Phi^{(d)}(x_1, x_2, x_3) + \Phi^{<d}(x_1, x_2, x_3), \quad (2.8)$$

where $\Phi^{(d)}(x_1, x_2, x_3)$ is homogeneous of degree $d = d_1 + d_2 + d_3$ and $\Phi^{<d}(x_1, x_2, x_3)$ is composed of terms of degree strictly inferior to d . Then the algebra whose defining relations are given by

$$\frac{\partial\Phi}{\partial x_j} = 0, \quad j = 1, 2, 3, \quad (2.9)$$

is a Calabi–Yau algebra [29].

Now, let $x_1 = K_{12}$, $x_2 = K_{23}$, $x_3 = K_{13}$ and $\deg(x_1) = \deg(x_2) = 2$, $\deg(x_3) = 3$. Consider the potential

$$\begin{aligned} \Phi^{(7)}(x_1, x_2, x_3) &= q[x_1x_2x_3] - q^{-1}[x_1x_3x_2], \\ \Phi^{<7}(x_1, x_2, x_3) &= (q + q^{-1})([x_1x_2^2] + [x_1^2x_2]) - \xi_4[x_1] - \xi_4'[x_2] - \frac{1}{2}[x_3^2] - \xi_2[x_1x_2]. \end{aligned} \quad (2.10)$$

It is easy to see that the defining relations of $Zh_q(m_1, m_2, m_3, m_4)$ presented in (4.10a)-(4.10c) are equivalent to imposing (2.9) for the potential (2.10). In other words, $Zh_q(m_1, m_2, m_3, m_4)$ derives from the Calabi–Yau potential Φ (2.10).

3 The Zhedanov algebra as a truncated reflection algebra

In this section, we recall [12] that the defining relations of the algebra $Zh_q(m_1, m_2, m_3, m_4)$ can be equivalently encoded in a reflection equation [67]. This realization of an algebra is usually called the FRT presentation, in honor of the authors of [68]. This presentation allows one to connect the Zhedanov algebra to the reflection algebra which is intensively studied in the context of quantum integrable systems. In addition, we show that the algebra $sZh_q(m_1, m_2, m_3, m_4)$ can be also obtained naturally by setting the Sklyanin determinant to a certain value; this justifies the appellation *special* for the quotiented algebra since it is obtained by fixing the value of a determinant, as in the definition of the Special Linear group SL_n .

The cornerstone of the FRT presentation is the R -matrix. For the case of the algebra $Zh_q(m_1, m_2, m_3, m_4)$, we start with the following R -matrix

$$R(u) = \begin{pmatrix} uq - \frac{1}{uq} & 0 & 0 & 0 \\ 0 & u - \frac{1}{u} & q - \frac{1}{q} & 0 \\ 0 & q - \frac{1}{q} & u - \frac{1}{u} & 0 \\ 0 & 0 & 0 & uq - \frac{1}{uq} \end{pmatrix}. \quad (3.1)$$

This R -matrix is associated to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and is a solution of the Yang–Baxter equation

$$R_{12}(u_1/u_2)R_{13}(u_1/u_3)R_{23}(u_2/u_3) = R_{23}(u_2/u_3)R_{13}(u_1/u_3)R_{12}(u_1/u_2), \quad (3.2)$$

where $R_{12} = \sum_a R_a \otimes R^a \otimes \mathbb{1}$, $R_{23} = \sum_a \mathbb{1} \otimes R_a \otimes R^a$, $R_{13} = \sum_a R_a \otimes \mathbb{1} \otimes R^a$ if one writes $R = \sum_a R_a \otimes R^a$ and $\mathbb{1}$ as the 2×2 identity matrix. We define also the following truncated reflection matrix (see remark 3.2 below) given by

$$B(u) = \begin{pmatrix} uqC_{12} - \frac{C_{23}}{uq} + \frac{p_4/u + p'_4u}{u^2 - 1/u^2} & qu^2 + \frac{1}{qu^2} - \frac{[C_{23}, C_{12}]_q}{q^2 - 1/q^2} + \frac{p''_4}{q + 1/q} \\ -qu^2 - \frac{1}{qu^2} + \frac{[C_{12}, C_{23}]_q}{q^2 - 1/q^2} - \frac{p''_4}{q + 1/q} & uqC_{23} - \frac{C_{12}}{uq} + \frac{p_4u + p'_4/u}{u^2 - 1/u^2} \end{pmatrix}, \quad (3.3)$$

where we refer to (4.7a)–(4.7c) for the definition of p_4 , p'_4 and p''_4 .

Proposition 3.1. [12] *The set of relations obtained from the reflection equation*

$$R(u/v)B_1(u)R(uv)B_2(v) = B_2(v)R(uv)B_1(u)R(u/v), \quad (3.4)$$

where $B_1(u) = B(u) \otimes \mathbb{1}$ and $B_2(u) = \mathbb{1} \otimes B(u)$, is equivalent to the defining relations of $Zh_q(m_1, m_2, m_3, m_4)$.

Proof. We look at each matrix element of the reflection equation (3.4) and derive 16 relations. For each of them, we extract the different coefficients w.r.t. the parameter u ; this provides relations between C_{12} and C_{23} . By direct investigation, we verify that all the obtained relations are equivalent to the defining relations of $Zh_q(m_1, m_2, m_3, m_4)$. \square

Rephrasing this proposition, the Zhedanov algebra $Zh_q(m_1, m_2, m_3, m_4)$ is isomorphic to the truncated reflection algebra defined by the R -matrix (3.1) and the truncated reflection matrix (3.3).

Remark 3.2. *There exists a more general form for the reflection matrix, containing an infinite number of generators encompassed in formal series of u and $\frac{1}{u}$. The elements of the reflection matrix (3.3) can be obtained as a truncation of these formal series. The algebra defined by the general reflection matrix obeying the reflection equation (3.4) is isomorphic to the q -Onsager algebra [14]. Therefore, the Zhedanov algebra can also be seen as a quotient of the q -Onsager algebra.*

In the context of the reflection algebra it is well-known how to obtain central elements [67]. Indeed, let us define the Sklyanin determinant $\text{sdet}B(u)$ as follows

$$\text{sdet}B(u) := -\frac{1}{2} \text{tr}_{12} \left(R(1/q)B_1(u/q)R(u^2/q)B_2(u) \right). \quad (3.5)$$

We can show that the coefficients of $\text{sdet}B(u)$ commute with C_{12} and C_{23} . We recover in this way that the operator Ω given by (2.1d) commutes with C_{12} and C_{23} . The Sklyanin determinant gives solely Ω as a central element. Fixing the Sklyanin determinant to an appropriate value allows us to give a FRT presentation of $sZh_q(m_1, m_2, m_3, m_4)$:

Proposition 3.3. *The truncated reflection algebra defined by the R-matrix (3.1), the truncated reflection matrix (3.3) and quotiented by the relation*

$$\begin{aligned} \text{sdet}B(u) = & q^2(1 - q^4)^2 (u^2 + q^{-m_2 - m_4})(u^2 + q^{m_2 + m_4})(u^2 + q^{m_4 - m_2})(u^2 + q^{m_2 - m_4}) \\ & \times (u^2 + q^{-m_1 - m_3})(u^2 + q^{m_1 + m_3})(u^2 + q^{m_3 - m_1})(u^2 + q^{m_1 - m_3}), \end{aligned} \quad (3.6)$$

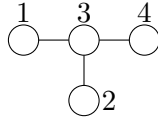
is isomorphic to $sZh_q(m_1, m_2, m_3, m_4)$.

Proof. By direct computations, we show that (3.6) is equivalent to imposing (2.2). \square

The fact that $sZh_q(m_1, m_2, m_3, m_4)$ can be defined as a truncated reflection algebra was expected, but it is a surprise that the r.h.s. of (3.6) factorizes into such a simple form.

4 A $W(D_4)$ symmetry

The algebra $sZh_q(m_1, m_2, m_3, m_4)$ has a remarkable symmetry based on the Weyl group $W(D_4)$ associated to the Lie algebra D_4 . To describe it, let us introduce a root system of type D_4 and fix a set of simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with labeling according to the following Dynkin diagram:



The Weyl group $W(D_4)$ is generated by the reflections s_i associated to the simple roots α_i which satisfy, for $1 \leq i, j \leq 4$,

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_j &= s_j s_i && \text{if } i \text{ and } j \text{ are not connected in the Dynkin diagram,} \\ s_i s_j s_i &= s_j s_i s_j && \text{if } i \text{ and } j \text{ are connected in the Dynkin diagram.} \end{aligned} \quad (4.1)$$

Its order is 192. Let us now associate the parameters m_1, m_2, m_3, m_4 with some of the roots as follows:

$$m_1 = \alpha_1, \quad m_2 = \alpha_2, \quad m_3 = \alpha_3, \quad m_4 = \Theta, \quad (4.2)$$

where Θ is the longest positive root. The explicit expression of Θ is:

$$\alpha_3 = \frac{1}{2}(m_4 - m_1 - m_2 - m_3). \quad (4.3)$$

It is elementary to calculate the actions s_i expressed in terms of the parameters:

$$s_1 : m_1 \mapsto -m_1, \quad s_2 : m_2 \mapsto -m_2, \quad s_4 : m_3 \mapsto -m_3, \quad s_3 : \begin{cases} m_1 \mapsto m_1 + \alpha_3, \\ m_2 \mapsto m_2 + \alpha_3, \\ m_3 \mapsto m_3 + \alpha_3, \\ m_4 \mapsto m_4 - \alpha_3, \end{cases} \quad (4.4)$$

where the omitted actions are trivial and the explicit expression of α_3 is given above. The action of the Weyl group is extended to any function as follows:

$$(\sigma f)(m_1, m_2, m_3, m_4) = f(\sigma(m_1), \sigma(m_2), \sigma(m_3), \sigma(m_4)) \quad (4.5)$$

for $\sigma \in W(D_4)$.

Proposition 4.1. *The Weyl group $W(D_4)$ is a symmetry of $sZh_q(m_1, m_2, m_3, m_4)$ i.e.*

$$sZh_q(m_1, m_2, m_3, m_4) = sZh_q(\sigma(m_1), \sigma(m_2), \sigma(m_3), \sigma(m_4)), \quad (4.6)$$

for any $\sigma \in W(D_4)$.

Proof. In $sZh_q(m_1, m_2, m_3, m_4)$, we remark that the only functions of m_i which appear are

$$p_4 = \chi_{m_1}\chi_{m_2} + \chi_{m_3}\chi_{m_4}, \quad (4.7a)$$

$$p'_4 = \chi_{m_2}\chi_{m_3} + \chi_{m_1}\chi_{m_4}, \quad (4.7b)$$

$$p''_4 = \chi_{m_1}\chi_{m_3} + \chi_{m_2}\chi_{m_4}, \quad (4.7c)$$

$$p_6 = \chi_{m_1}^2 + \chi_{m_2}^2 + \chi_{m_3}^2 + \chi_{m_4}^2 + \chi_{m_1}\chi_{m_2}\chi_{m_3}\chi_{m_4}, \quad (4.7d)$$

where $\chi_m = q^m + q^{-m}$. By direct computations, we can show that these functions are invariant by the transformations s_1, s_2, s_3 and s_4 given by (4.4), which concludes the proof since they generate $W(D_4)$. \square

In the study of the finite representations of the universal algebra Δ_q a $W(D_4)$ symmetry has been also investigated [69, 70].

4.1 Connection with the $W(D_4)$ symmetry in the Racah algebra

Let us perform the transformation

$$K_I = \frac{C_I - (q + q^{-1})}{(q - q^{-1})^2}, \quad (4.8)$$

with $I \in \{1, 2, 3, 123, 12, 23\}$. Note that 13 does not belong to this set. In the algebra $sZh_q(m_1, m_2, m_3, m_4)$, one gets, for $i = 1, 2, 3, 4$,

$$K_i = \frac{\chi_{m_i} - (q + q^{-1})}{(q - q^{-1})^2} = \left[\frac{m_i}{2} \right]_q^2 - \left[\frac{1}{2} \right]_q^2, \quad (4.9)$$

where the q -number is defined by $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$. The commutation relations of the algebra $sZh_q(m_1, m_2, m_3, m_4)$ become

$$[K_{12}, K_{23}]_q = K_{13}, \quad (4.10a)$$

$$[K_{23}, K_{13}]_q = (q + q^{-1})(-\{K_{12}, K_{23}\} - K_{23}^2 + \xi_2 K_{23} + \xi_4), \quad (4.10b)$$

$$[K_{13}, K_{12}]_q = (q + q^{-1})(-\{K_{12}, K_{23}\} - K_{12}^2 + \xi_2 K_{12} + \xi'_4), \quad (4.10c)$$

and the supplementary relation becomes

$$\begin{aligned} & -q \frac{q - q^{-1}}{q + q^{-1}} K_{12} K_{23} K_{13} - q K_{12} K_{23} K_{12} - q^{-1} K_{23} K_{12} K_{23} + \frac{q^2}{(q + q^{-1})^2} K_{13}^2 \\ & + \left(\frac{\xi_2}{q + q^{-1}} - 1 \right) \{K_{12}, K_{23}\} + q \xi_4 K_{12} + q^{-1} \xi'_4 K_{23} = \xi_6 - \xi_4 - \xi'_4 - \frac{\xi_2^2}{4}, \end{aligned} \quad (4.10d)$$

with

$$\xi_2 = \frac{1}{q + q^{-1}} \left(2(M_1^2 + M_2^2 + M_3^2 + M_4^2 - 1) + (q - q^{-1})^2 (M_1^2 M_3^2 + M_2^2 M_4^2) \right), \quad (4.10e)$$

$$\xi_4 = (M_1^2 - M_4^2)(M_3^2 - M_2^2), \quad (4.10f)$$

$$\xi'_4 = (M_1^2 - M_2^2)(M_3^2 - M_4^2), \quad (4.10g)$$

$$\xi_6 = (M_1^2 M_3^2 - M_2^2 M_4^2)(M_1^2 - M_2^2 + M_3^2 - M_4^2) + \frac{1}{4}(q - q^{-1})^2 (M_1^2 M_3^2 - M_2^2 M_4^2)^2, \quad (4.10h)$$

where we use the notation $M_i = \left[\frac{m_i}{2} \right]_q$. As expected, we can check that the functions ξ_2 , ξ_4 , ξ'_4 and ξ_6 are invariant under the action of the Weyl group $W(D_4)$.

The advantage of this presentation of $sZh_q(m_1, m_2, m_3, m_4)$ is that the classical limit $q \rightarrow 1$ (see Appendix A) is well-defined and provides straightforwardly the commutation relations of the Racah algebra. Thus, the description of the Weyl group $W(D_4)$ action also holds for the Racah algebra and we recover the results of [71].

Remark 4.2. *In the classical limit $q \rightarrow 1$, the functions ξ_2 , ξ_4 , ξ'_4 and ξ_6 form a basis for polynomials invariant under the action of $W(D_4)$, as expected. In the generic case ($q \in \mathbb{C}$, not a root of unity), one gets two different sets of invariant functions: $S_\xi = \{\xi_2, \xi_4, \xi'_4, \xi_6\}$ on the one hand and $S_p = \{p_4, p'_4, p''_4, p_6\}$ on the other hand. We have checked that there exists an invertible polynomial mapping between these two sets. However, only S_ξ admits a non-trivial classical limit.*

5 Kauffman bracket skein modules and algebras

Kauffman bracket skein module quantizations have been introduced in [37, 47] and further studied along our lines of interest for this paper in [39, 72, 73]. We will now recall some key definitions and results from these investigations. We shall work with an oriented 3-manifold \mathcal{M} which is a thickened surface, that is $\mathcal{M} = \Sigma_{0,n} \times I$, where $I = [0, 1]$ and $\Sigma_{0,n}$ is the n -punctured sphere.

Definition 5.1. *The quantized skein module $\text{Sk}_\theta(\mathcal{M})$ is the $\mathbb{C}[\theta^{\pm 1}]$ -module spanned by framed and unoriented links in \mathcal{M} modulo the Kauffman bracket skein relations that allow to “simplify the crossings”:*

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \theta \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} + \theta^{-1} \begin{array}{c} \diagdown \diagdown \\ \diagup \diagup \end{array}, \quad (5.1a)$$

$$\bigcirc = -(\theta^2 + \theta^{-2}), \quad (5.1b)$$

where $\theta \in \mathbb{C}$ is not a root of unity and in the framing relation (5.1b) the link should not enclose a puncture. This defines an algebra, which we will denote $\text{Sk}_\theta(\Sigma_{0,n})$, for which multiplication is given by stacking the links on top of each other in the I direction.

We shall use diagrams that correspond to the projection of the links on the surface (all the while keeping the information about the relative “height” of the links in the I direction). Let us now establish the conventions for these drawings (framed links diagrams).

The n -punctured sphere $\Sigma_{0,n}$ is equivalent to the plane with $n - 1$ punctures (denoted by the $(n - 1)$ drawn \times 's):

$$\begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \quad \times \quad \cdots \quad \times \\ \text{---} \\ \text{---} \end{array} \quad (5.2)$$

The dashed contour corresponds to the n^{th} puncture of the sphere. We will omit the contour in the subsequent diagrams but it is always understood to be there.

Framed links that enclose punctures are represented by loops drawn around the \times 's. We shall use the term "loops" to refer unambiguously to the framed links in the remainder of the paper. These loops can be homotopically deformed without crossing the holes (punctures). Remark that loops enclosing a single puncture are central elements in $\text{Sk}_\theta(\Sigma_{0,n})$. This is also true for the n^{th} puncture, which amounts to saying that the loop enclosing the $(n - 1)$ punctures \times is also central.

Let us now consider the surface $\Sigma_{0,4}$ and give names to a few loops:

$$\begin{array}{ll} \begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \times = \mathbb{A}_{12} & \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \times \times = \mathbb{A}_1 \\ \times \begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} = \mathbb{A}_{23} & \times \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \times = \mathbb{A}_2 \\ \begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} = \mathbb{A}_{13} & \times \times \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} = \mathbb{A}_3 \\ & \begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \quad \times \\ \text{---} \\ \text{---} \end{array} = \mathbb{A}_{123} \end{array} \quad (5.3)$$

Following the definition, multiplication of two loops $X \cdot Y$ means putting Y on top of X , for example:

$$\mathbb{A}_{12} \cdot \mathbb{A}_{23} = \begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \quad (5.4)$$

One would then proceed to use relations (5.1) to simplify the expressions:

$$\begin{aligned} \mathbb{A}_{12} \cdot \mathbb{A}_{23} &= \theta \left(\begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \right) - \theta^{-1} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \right) \\ &= \theta^2 \left(\begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \right) + \left(\begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \right) \\ &\quad + \left(\begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \right) + \theta^{-2} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \right) \\ &= \theta^2 \mathbb{A}_{13} + \mathbb{A}_2 \cdot \mathbb{A}_{123} + \mathbb{A}_1 \cdot \mathbb{A}_3 + \theta^{-2} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \right). \end{aligned} \quad (5.5)$$

Similarly, exchanging the order of multiplication, one obtains the same diagrams but with inverse coefficients:

$$\mathbb{A}_{23} \cdot \mathbb{A}_{12} = \theta^{-2} \mathbb{A}_{13} + \mathbb{A}_2 \cdot \mathbb{A}_{123} + \mathbb{A}_1 \cdot \mathbb{A}_3 + \theta^2 \left(\begin{array}{c} \text{---} \\ \text{---} \\ \times \quad \times \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \times \\ \text{---} \\ \text{---} \end{array} \right). \quad (5.6)$$

We see immediately that one gets

$$\theta^2 \mathbb{A}_{12} \cdot \mathbb{A}_{23} - \theta^{-2} \mathbb{A}_{23} \cdot \mathbb{A}_{12} = (\theta^4 - \theta^{-4}) \mathbb{A}_{13} + (\theta^2 - \theta^{-2})(\mathbb{A}_2 \cdot \mathbb{A}_{123} + \mathbb{A}_1 \cdot \mathbb{A}_3). \quad (5.7)$$

The skein algebra $Sk_\theta(\Sigma_{0,4})$ is directly linked to the Askey–Wilson algebra as stated in the following proposition:

Proposition 5.2. *The Special Askey–Wilson algebra $\mathbf{saw}(3)$ is isomorphic to the Kauffman bracket skein algebra $Sk_{iq^{1/2}}(\Sigma_{0,4})$. The isomorphism is given by the following invertible map:*

$$C_I \mapsto \mathbb{A}_I, \quad (5.8)$$

for $I \in \{1, 2, 3, 123, 12, 23, 13\}$.

Proof. The isomorphism is directly verified by comparing the relations of $\mathbf{saw}(3)$ and the ones of the Kauffman bracket skein algebra obtained in [37] (see also Proposition 3.1 of [73] and [38] for additional details). \square

This proposition gives a diagrammatic approach to study the algebra $\mathbf{saw}(3)$.

Let us emphasize that the previous isomorphism involves the Special Askey–Wilson algebra $\mathbf{saw}(3)$. If we replace $\mathbf{saw}(3)$ by $\mathbf{aw}(3)$ in the map of the proposition, the homomorphism would be not injective and if we instead replace $\mathbf{saw}(3)$ by Δ_q (as in [6, 74]), it would be not surjective.

One notes that the \mathbb{Z}_3 -symmetry of the $\mathbf{saw}(3)$ relations is made manifest in terms of the framed links picture, as the punctures do not have fixed positions and can be switched around.

From now on we will unambiguously refer to the drawn loops identified as the generators of $Sk_{iq^{1/2}}(\Sigma_{0,4})$ directly as their C_I counterpart following (5.8). This correspondence (5.8) leads to a natural labeling of the punctures. Indeed, consider the generators given in (5.3): the punctures enclosed in a given loop correspond precisely to the set of indices I of the corresponding generator C_I if one labels the punctures consecutively as:

$$\begin{array}{ccc} \times & \times & \times \\ 1 & 2 & 3 \end{array} \quad (5.9)$$

Remark 5.3. *We recall that one arrives to the Special Zhedanov algebra $sZh_q(m_1, m_2, m_3, m_4)$ from the Special Askey–Wilson algebra $\mathbf{saw}(3)$ by attributing a value to the central elements C_i , $i = 1, 2, 3, 123$, see (2.5). In the same way, starting from the Kauffman bracket skein algebra $Sk_{iq^{1/2}}(\Sigma_{0,4})$, one can define an evaluated Kauffman bracket skein algebra, denoted $Sk_{iq^{1/2}}(\Sigma_{0,4}; m_1, m_2, m_3, m_4)$ by attributing a value to the puncture-framing relations:*

$$\begin{array}{l} \begin{array}{c} \otimes \\ i \end{array} = q^{m_i} + q^{-m_i}, \quad i = 1, 2, 3, \\ \begin{array}{c} \otimes \\ \times \quad \times \quad \times \end{array} = q^{m_4} + q^{-m_4}. \end{array} \quad (5.10)$$

Note that the last drawing corresponds in fact to a contour enclosing the fourth puncture on the sphere, see (5.2). As a corollary of Proposition 5.2, the algebra $Sk_{iq^{1/2}}(\Sigma_{0,4}; m_1, m_2, m_3, m_4)$ is isomorphic to the Special Zhedanov algebra $sZh_q(m_1, m_2, m_3, m_4)$.

Relations (5.10) with $m_i = 1$ already appear in the definition of the skein algebra of arcs and link introduced in [75], from where we borrowed the terminology ‘puncture-framing’.

6 $U_q(\mathfrak{sl}_2)$ and its centralizer in $U_q(\mathfrak{sl}_2)^{\otimes 3}$

The goal of this section is to discuss the notion of centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$, which we denote by \mathfrak{C}_3 , and connect it with the Special Askey–Wilson algebra $\mathbf{saW}(3)$.

6.1 $U_q(\mathfrak{sl}_2)$ and its universal R -matrix

Let us fix the notation and conventions that will be used to perform the explicit calculations in $U_q(\mathfrak{sl}_2)$ (note that the results obtained will be independent of these conventions at the end). We shall first define the quasi-triangular Hopf algebra $U_q(\mathfrak{sl}_2)$, present its braided universal R -matrix and list some additional properties of interest.

$U_q(\mathfrak{sl}_2)$ is an associative algebra generated by E, F, q^H and q^{-H} obeying the defining relations

$$q^H q^{-H} = q^{-H} q^H = 1, \quad q^H E = q E q^H, \quad q^H F = q^{-1} F q^H \quad \text{and} \quad [E, F] = [2H]_q. \quad (6.1)$$

The center of this algebra is generated by the following Casimir element (denoted Λ in [54, 56])

$$Q = (q - q^{-1})^2 \left(FE + \frac{qq^{2H} + q^{-1}q^{-2H}}{(q - q^{-1})^2} \right). \quad (6.2)$$

The algebra $U_q(\mathfrak{sl}_2)$ can be endowed with a Hopf structure. In particular, its comultiplication (or coproduct) homomorphism $\Delta : U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ is given by

$$\Delta(E) = E \otimes q^{-H} + q^H \otimes E, \quad \Delta(q^H) = q^H \otimes q^H, \quad (6.3a)$$

$$\Delta(F) = F \otimes q^{-H} + q^H \otimes F, \quad \Delta(q^{-H}) = q^{-H} \otimes q^{-H}, \quad (6.3b)$$

and is coassociative

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta. \quad (6.4)$$

The quantum algebra $U_q(\mathfrak{sl}_2)$ is called quasi-triangular because in a completion of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, there exists a universal R -matrix \mathcal{R} which is invertible and satisfies

$$\Delta(x)\mathcal{R} = \mathcal{R}\Delta^{op}(x) \quad \text{for } x \in U_q(\mathfrak{sl}_2), \quad (6.5)$$

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{13}, \quad (6.6)$$

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{13}, \quad (6.7)$$

where in the Sweedler notation we write the opposite comultiplication $\Delta^{op}(x) = x_{(2)} \otimes x_{(1)}$ if $\Delta(x) = x_{(1)} \otimes x_{(2)}$. In the previous relation, we have used the notations $\mathcal{R}_{12} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ and $\mathcal{R}_{13} = \mathcal{R}^\alpha \otimes 1 \otimes \mathcal{R}_\alpha$ where $\mathcal{R} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ (the sum over repeated indices α is understood). The universal R -matrix is given explicitly by [76]

$$\mathcal{R} = q^{2(H \otimes H)} \sum_{n=0}^{\infty} \frac{(q - q^{-1})^n}{[n]_q!} q^{n(n-1)/2} (E q^H \otimes q^{-H} F)^n, \quad (6.8)$$

where $[n]_q! = [n]_q [n-1]_q \dots [2]_q [1]_q$ and, by convention, $[0]_q! = 1$.

One can also define the so-called braided universal R -matrix $\check{\mathcal{R}}$ by

$$\check{\mathcal{R}}_i = \mathcal{R}_{i,i+1} \sigma_{i,i+1} \quad (6.9)$$

where $\sigma_{i,i+1}$ acts on the i^{th} and $(i+1)^{\text{th}}$ factors of the tensor product as

$$\sigma_{i,i+1}(\cdots \otimes x_i \otimes x_{i+1} \otimes \cdots) = (\cdots \otimes x_{i+1} \otimes x_i \otimes \cdots) \sigma_{i,i+1}. \quad (6.10)$$

This braided universal R -matrix satisfies the braided Yang–Baxter equation

$$\check{R}_i \check{R}_{i+1} \check{R}_i = \check{R}_{i+1} \check{R}_i \check{R}_{i+1}. \quad (6.11)$$

6.2 An algebra generated by the intermediate Casimir elements

Let us define the following *intermediate* Casimir elements

$$\begin{aligned} Q_1 &= Q \otimes 1 \otimes 1, & Q_2 &= 1 \otimes Q \otimes 1, & Q_3 &= 1 \otimes 1 \otimes Q, \\ Q_{12} &= \Delta(Q) \otimes 1 = Q_{(1)} \otimes Q_{(2)} \otimes 1, & Q_{23} &= 1 \otimes \Delta(Q) = 1 \otimes Q_{(1)} \otimes Q_{(2)}, \\ Q_{123} &= (\Delta \otimes \text{id})\Delta(Q). \end{aligned} \quad (6.12)$$

The labeling of these intermediate Casimir elements is chosen so as to refer to the non-trivial factors in the tensor product $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

Definition 6.1. *The algebra \mathcal{A}_3 is the subalgebra of $U_q(\mathfrak{sl}_2)^{\otimes 3}$ generated by the intermediate Casimir elements $Q_1, Q_2, Q_3, Q_{12}, Q_{23}$ and Q_{123} .*

Let us define an additional intermediate Casimir element

$$Q_{13} = \check{R}_2^{-1} Q_{12} \check{R}_2 = \check{R}_1 Q_{23} \check{R}_1^{-1}. \quad (6.13)$$

It has been proven in [77] that this element is in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ (and not in its completion), that the second equality is compatible with the first one and that the following proposition holds:

Proposition 6.2. *The intermediate Casimir elements $Q_1, Q_2, Q_3, Q_{123}, Q_{12}, Q_{23}$ and Q_{13} belong to the centralizer \mathfrak{C}_3 of the diagonal action of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ defined by*

$$\mathfrak{C}_3 = \{X \in U_q(\mathfrak{sl}_2)^{\otimes 3} \mid [(\Delta \otimes \text{id})\Delta(x), X] = 0, \quad \forall x \in U_q(\mathfrak{sl}_2)\}. \quad (6.14)$$

The precise links between the Askey–Wilson algebra, the centralizer and the algebra \mathcal{A}_3 generated by the intermediate Casimir elements are given in the following proposition.

Proposition 6.3. *The algebra $\mathbf{aw}(3)$ has an homomorphic injective image in \mathfrak{C}_3 . The mapping is done as follows:*

$$C_I \mapsto Q_I, \quad \text{for } I \in \{1, 2, 3, 123, 12, 23, 13\}. \quad (6.15)$$

The algebra $\mathbf{aw}(3)$ is isomorphic to \mathcal{A}_3 .

Proof. All the relations of $\mathbf{aw}(3)$ given by (2.1) and (2.2) are easily checked in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ upon rewriting the Q_I 's in terms of the $U_q(\mathfrak{sl}_2)^{\otimes 3}$ generators. The proof of the injectivity is postponed to Appendix A. The method used in [6] to prove the injectivity of Δ_q into $U_q(\mathfrak{sl}_2)^{\otimes 3}$ seems difficult to generalize to the case treated here and we propose an alternative method based on classical invariant theory. Since the algebra \mathcal{A}_3 is the image of the map (6.15), it follows that $\mathbf{aw}(3)$ is isomorphic to \mathcal{A}_3 . \square

This realization of the Askey–Wilson algebra in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ was the motivation for adding the relation (2.2) to the “intuitive” set of relations of $\mathbf{aw}(3)$. Indeed, since relation (2.2) is obeyed by the intermediate Casimir elements, it should also be included in the algebra encoding the properties of these Casimir elements.

Corollary 6.4. *The algebra \mathcal{A}_3 is isomorphic to the Kauffman bracket skein algebra of the four-punctured sphere $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. The isomorphism is given by the following map:*

$$\phi : Q_I \mapsto \mathbb{A}_I, \quad \text{for } I \in \{1, 2, 3, 123, 12, 23, 13\}. \quad (6.16)$$

Proof. A direct consequence of the Propositions 5.2 and 6.3.

6.3 Fundamental theorems of invariant theory

In the previous section, we introduced the centralizer \mathfrak{C}_3 of the diagonal action of $U_q(\mathfrak{sl}_2)$ in the threefold tensor product and showed its connection with the Askey–Wilson algebra $\text{saw}(3)$. We now focus on similar objects in the case where we represent each factor $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ by a finite-dimensional irreducible representation.

The quantum algebra $U_q(\mathfrak{sl}_2)$ has finite irreducible representations of dimension $m = 2j + 1$ that we will denote by $M(m)$, with $m \in \mathbb{Z}_{>0}$. The name “spin- j representation” is usually used to refer to $M(m = 2j + 1)$. The representation map will be denoted by $\pi_m : U_q(\mathfrak{sl}_2) \rightarrow \text{End}(M(m))$. The representation of the Casimir element (6.2) in the space $M(m)$ is

$$\pi_m(Q) = \chi_m \mathbb{1}_m, \quad (6.17)$$

where $\chi_m = q^m + q^{-m}$ and $\mathbb{1}_m$ is the $m \times m$ identity matrix.

From now on, we fix three integers m_1, m_2 and m_3 . The threefold tensor product of irreducible representations of $U_q(\mathfrak{sl}_2)$ decomposes into the following direct sum of irreducible representations

$$M(m_1) \otimes M(m_2) \otimes M(m_3) = \bigoplus_{m_4} M(m_4) \otimes V_{m_1, m_2, m_3}^{m_4}, \quad (6.18)$$

where $V_{m_1, m_2, m_3}^{m_4}$ is called the multiplicity space. We recall that we look at cases where q is not a root of unity otherwise the previous statement would be wrong.

We now fix four integers m_1, m_2, m_3, m_4 and denote by \mathcal{Q}_I the image of Q_I in $V_{m_1, m_2, m_3}^{m_4}$ (for $I \in \{1, 2, 3, 123, 12, 23, 13\}$). We get $\mathcal{Q}_1 = \chi_{m_1}$, $\mathcal{Q}_2 = \chi_{m_2}$, $\mathcal{Q}_3 = \chi_{m_3}$ and $\mathcal{Q}_{123} = \chi_{m_4}$.

Proposition 6.5. *There exists a surjective algebra homomorphism from $sZh_q(m_1, m_2, m_3, m_4)$ to $\text{End}(V_{m_1, m_2, m_3}^{m_4})$ given by*

$$C_I \mapsto \mathcal{Q}_I, \quad \text{for } I \in \{12, 23, 13\}. \quad (6.19)$$

This proposition which provides the generators for the centralizer of the diagonal action is sometimes called in invariant theory the “first fundamental theorem”. The map in the previous proposition is not injective. The description of the kernel of this map is the subject of [36] (see also [78]) and is called the “second fundamental theorem”.

We recall that the algebra $sZh_q(m_1, m_2, m_3, m_4)$ possesses a $W(D_4)$ -symmetry. Let us remark that a similar Weyl group symmetry of type E_6 has been discovered recently [79] in the case of the centralizer of the diagonal embedding of $U(\mathfrak{sl}_3)$ in two copies of $U(\mathfrak{sl}_3)$.

7 The Double Affine Hecke Algebra (C_1^\vee, C_1)

Double affine Hecke algebras (DAHA) of type (C_1^\vee, C_1) were introduced in [80] and their connections with Askey–Wilson polynomials were first explored in [18] and [40]. Universal analogues of these DAHA were later introduced and studied in [20, 43, 44].

In this section, we present another connection between the Special Askey–Wilson algebra $\text{saw}(3)$ and a certain subalgebra of a universal DAHA of type (C_1^\vee, C_1) .

Definition 7.1. We introduce the following algebras

- The universal Double Affine Hecke Algebra of type (C_1^\vee, C_1) [43] is defined as the associative algebra \widehat{H}_q with generators $\{t_i^{\pm 1}, i = 0, \dots, 3\}$ and relations:

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad (7.1a)$$

$$t_i + t_i^{-1} \quad \text{is central}, \quad (7.1b)$$

$$t_0 t_1 t_2 t_3 = q^{-1}. \quad (7.1c)$$

The “usual” DAHA, denoted $H_q(k_0, k_1, k_2, k_3)$, is recovered when the central elements $t_i + t_i^{-1}$ have complex values $k_i + k_i^{-1}$, with $k_i \neq 0$.

- The algebra Γ_q [44] is the subalgebra of \widehat{H}_q commuting with the distinguished generator t_0 (Γ_q is the centralizer of t_0 in \widehat{H}_q):

$$\Gamma_q = \{h \in \widehat{H}_q \mid [h, t_0] = 0\}. \quad (7.2)$$

- Let \mathbf{e} be the following idempotent of $H_q(k_0, k_1, k_2, k_3)$ [73]

$$\mathbf{e} = \frac{t_0 - k_0}{k_0^{-1} - k_0}. \quad (7.3)$$

The spherical DAHA, denoted $SH_q(k_0, k_1, k_2, k_3)$ [41, 42], is defined as

$$SH_q(k_0, k_1, k_2, k_3) = \mathbf{e} H_q(k_0, k_1, k_2, k_3) \mathbf{e}. \quad (7.4)$$

The following theorems relate DAHA to the previously introduced algebraic structures.

Theorem 7.2. [44] The map $\Theta : \mathbf{saw}(3) \rightarrow \Gamma_q$ defined by

$$\begin{aligned} C_{12} &\mapsto t_1 t_0 + (t_1 t_0)^{-1}, & C_1 &\mapsto t_1 + t_1^{-1}, \\ C_{23} &\mapsto t_3 t_0 + (t_3 t_0)^{-1}, & C_2 &\mapsto t_2 + t_2^{-1}, \\ C_{13} &\mapsto t_2 t_0 + (t_2 t_0)^{-1}, & C_3 &\mapsto t_3 + t_3^{-1}, \\ & & C_{123} &\mapsto q^{-1} t_0 + q t_0^{-1}. \end{aligned} \quad (7.5)$$

is an injective algebra homomorphism.

Theorem 7.3. (Theorem 3.2 in [42]) The Special Zhedanov algebra $sZh_q(m_1, m_2, m_3, m_4)$ is isomorphic to the spherical DAHA $SH_q(k_0, k_1, k_2, k_3)$.

Remark 7.4. Spherical DAHAs have also been connected to skein algebras of higher genus. The Kauffman bracket skein algebra of the once-punctured torus $\text{Sk}_\theta(\Sigma_{1,1})$ is related to a (spherical) DAHA of type A_1 [37, 81] and the genus two skein algebra is related to a genus two spherical double affine Hecke algebra in [82].

8 Actions of the braid group

In this section, we provide two actions of the braid group: the first one on the algebra \mathcal{A}_3 and the second one on the skein algebra $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. Then, we show how these two actions are compatible and give a diagrammatic presentation of the intermediate Casimir elements of $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

We recall that the braid group on n strands B_n is generated by the elements s_1, \dots, s_{n-1} as well as their inverses $s_1^{-1}, \dots, s_{n-1}^{-1}$ satisfying

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i \quad \text{if } |i - j| \geq 2, \\ s_i^{-1} s_i &= s_i s_i^{-1} = 1. \end{aligned} \tag{8.1}$$

8.1 The braided universal R -matrix and a braid group action on \mathcal{A}_3

Let us recall that we define the generators Q_{13} as follows

$$Q_{13} = Q_{13d} = \check{\mathcal{R}}_2^{-1} Q_{12} \check{\mathcal{R}}_2 = \check{\mathcal{R}}_1 Q_{23} \check{\mathcal{R}}_1^{-1}. \tag{8.2}$$

From the result of Proposition 6.3, we know that Q_{13} satisfies

$$Q_{13} = \frac{Q_1 Q_3 + Q_2 Q_{123}}{q + q^{-1}} - \frac{[Q_{12}, Q_{23}]_q}{q^2 - q^{-2}}, \tag{8.3}$$

and is in the algebra \mathcal{A}_3 which is generated by $Q_1, Q_2, Q_3, Q_{12}, Q_{23}$ and Q_{123} . Now from (8.2) it is natural to consider the following element which is analogous to Q_{13d} :

$$Q_{13u} = \check{\mathcal{R}}_2 Q_{12} \check{\mathcal{R}}_2^{-1} = \check{\mathcal{R}}_1^{-1} Q_{23} \check{\mathcal{R}}_1. \tag{8.4}$$

It has been shown in [77] that this element is also in \mathcal{A}_3 since it can be obtained as

$$Q_{13u} = \frac{Q_1 Q_3 + Q_2 Q_{123}}{q + q^{-1}} - \frac{[Q_{23}, Q_{12}]_q}{q^2 - q^{-2}}. \tag{8.5}$$

The labels u and d added on the Casimir elements Q_{13d} and Q_{13u} stand for *up* and *down*. These names come from the form of their image in $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ given in Corollary 6.4:

$$\begin{array}{c} \text{Diagram 1} \\ \times \quad \times \quad \times \\ \text{Diagram 2} \end{array} = \phi(Q_{13u}), \tag{8.6a}$$

$$\begin{array}{c} \text{Diagram 3} \\ \times \quad \times \quad \times \\ \text{Diagram 4} \end{array} = \phi(Q_{13d}). \tag{8.6b}$$

This procedure of obtaining additional elements of \mathcal{A}_3 by conjugations of braided R -matrices can be described by an automorphism action. Let us define the following automorphisms of \mathcal{A}_3 denoted Ψ_{s_i} and $\Psi_{s_i^{-1}}$ by

$$\Psi_{s_i}(X) = \check{\mathcal{R}}_i X \check{\mathcal{R}}_i^{-1} \quad \text{and} \quad \Psi_{s_i^{-1}}(X) = \check{\mathcal{R}}_i^{-1} X \check{\mathcal{R}}_i = \Psi_{s_i}^{-1}(X), \tag{8.7}$$

for $i = 1, 2$ and $X \in \mathcal{A}_3$, The previous maps are well-defined since the images of the generators of \mathcal{A}_3 are precisely in \mathcal{A}_3 (and not in its completion). Indeed, by direct computations making use of the explicit form (6.8) of the universal R -matrix and the commutation relations of $U_q(\mathfrak{sl}_2)$, one gets

$$\begin{aligned} \Psi_{s_1}(Q_1) &= Q_2, & \Psi_{s_1}(Q_2) &= Q_1, & \Psi_{s_1}(Q_3) &= Q_3, & \Psi_{s_1}(Q_{123}) &= Q_{123}, \\ \Psi_{s_1}(Q_{12}) &= Q_{12}, & \Psi_{s_1}(Q_{23}) &= Q_{13d}. \end{aligned} \quad (8.8)$$

and

$$\begin{aligned} \Psi_{s_2}(Q_1) &= Q_1, & \Psi_{s_2}(Q_2) &= Q_3, & \Psi_{s_2}(Q_3) &= Q_2, & \Psi_{s_2}(Q_{123}) &= Q_{123}, \\ \Psi_{s_2}(Q_{12}) &= Q_{13u}, & \Psi_{s_2}(Q_{23}) &= Q_{23}. \end{aligned} \quad (8.9)$$

We obtain similarly the actions of $\Psi_{s_i^{-1}}$ on the generators of \mathcal{A}_3 .

Since the braided R -matrix satisfies the braided Yang–Baxter equation (6.11), we can show that the defining relations (8.1) of the braid group B_3 are reproduced

$$\Psi_{s_1} \circ \Psi_{s_2} \circ \Psi_{s_1} = \Psi_{s_2} \circ \Psi_{s_1} \circ \Psi_{s_2}, \quad (8.10a)$$

$$\Psi_{s_i} \circ \Psi_{s_i^{-1}} = \Psi_{s_i^{-1}} \circ \Psi_{s_i} = id. \quad (8.10b)$$

We extend the automorphisms Ψ_S to any $S \in B_3$ by

$$\Psi_S(X) = (\Psi_{g_1} \circ \Psi_{g_2} \circ \cdots \circ \Psi_{g_\ell})(X), \quad (8.11)$$

where S is decomposed as $S = g_1 g_2 \dots g_\ell$ and $g_i \in \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$. Note that the map (8.11) does not depend on the choice of the decomposition of S due to (8.10).

Remark 8.1. *The realization of the braid group given by Ψ_S is not faithful. For example, one can verify that $\Psi_{(s_1 s_2)^3} = id$. This is checked to be true on the intermediate Casimir elements by making repeated use of (8.8)-(8.9). It follows that it is also true for any polynomial in those elements. Moreover, some elements of \mathcal{A}_3 have additional stabilizers, e.g.*

$$\Psi_{s_1 s_1}(Q_1) = \check{\mathcal{R}}_1^{-1} \check{\mathcal{R}}_1^{-1} Q_1 \check{\mathcal{R}}_1 \check{\mathcal{R}}_1 = \check{\mathcal{R}}_1^{-1} Q_2 \check{\mathcal{R}}_1 = Q_1, \quad (8.12a)$$

$$\Psi_{s_2}(Q_{23}) = Q_{23}. \quad (8.12b)$$

Identifying stabilizers of the braid group action on elements of \mathcal{A}_3 is easy to do but giving an exhaustive list is harder.

Remark 8.2. *It was shown in [83] how such a braid group action translates to the $q \rightarrow -1$ limit. This limit of the Askey–Wilson algebra is referred to as the Bannai–Ito algebra. In that case, the B_3 braid group action simplifies to an action of the S_3 symmetric group. It is possible to study more generally the action of the S_n symmetric group on the higher rank Bannai–Ito algebra $B(n)$.*

8.2 Half Dehn twists and the braid group action on $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$

We now present a B_3 group action on the Kauffman bracket skein algebra $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$, denoted $\psi_S : \text{Sk}_{iq^{1/2}}(\Sigma_{0,4}) \rightarrow \text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$, with $S \in B_3$. The braid group action rotates the placement of the punctures with respect to each other.

Here is how it goes. First, the actions ψ_{s_i} and $\psi_{s_i^{-1}}$ on $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ are defined by the so-called half Dehn twists [73, 84]. The four generators of B_3 act as

$$\begin{aligned} \psi_{s_1} &= \begin{array}{c} \xrightarrow{\hspace{1cm}} \\ \times \quad \times \quad \times \\ \xleftarrow{\hspace{1cm}} \end{array}, & \psi_{s_2} &= \begin{array}{c} \xrightarrow{\hspace{1cm}} \\ \times \quad \times \quad \times \\ \xleftarrow{\hspace{1cm}} \end{array}, \\ \psi_{s_1^{-1}} &= \begin{array}{c} \xleftarrow{\hspace{1cm}} \\ \times \quad \times \quad \times \\ \xrightarrow{\hspace{1cm}} \end{array}, & \psi_{s_2^{-1}} &= \begin{array}{c} \xleftarrow{\hspace{1cm}} \\ \times \quad \times \quad \times \\ \xrightarrow{\hspace{1cm}} \end{array}, \end{aligned} \quad (8.13)$$

where any framed link gets deformed continuously without crossing the punctures as the rotations happen. For example, one gets

$$\begin{aligned} \psi_{s_2^{-1}}(\mathbb{A}_{12}) &= \psi_{s_2^{-1}} \left(\begin{array}{c} \times \quad \times \\ \text{---} \end{array} \times \right) = \begin{array}{c} \xleftarrow{\hspace{1cm}} \\ \times \quad \times \quad \times \\ \xrightarrow{\hspace{1cm}} \end{array} \\ &= \begin{array}{c} \text{---} \\ \times \quad \times \quad \times \\ \text{---} \end{array} = \mathbb{A}_{13}, \end{aligned} \quad (8.14)$$

and

$$\begin{aligned} \psi_{s_2}(\mathbb{A}_{23}) &= \psi_{s_2} \left(\times \begin{array}{c} \times \quad \times \\ \text{---} \end{array} \right) = \begin{array}{c} \xrightarrow{\hspace{1cm}} \\ \times \quad \times \quad \times \\ \xleftarrow{\hspace{1cm}} \end{array} \\ &= \begin{array}{c} \text{---} \\ \times \quad \times \quad \times \\ \text{---} \end{array} = \mathbb{A}_{23}. \end{aligned} \quad (8.15)$$

Proposition 8.3. *The actions ψ_g for $g \in \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$ are automorphisms of $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$.*

Proof. For any $X, Y \in \text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ and $g \in \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$, one understands that

$$\psi_g(X \cdot Y) = \psi_g(X) \cdot \psi_g(Y). \quad (8.16)$$

Indeed, from the way they were defined, the rotations do not add or change crossings. Thus, the Kauffman bracket relations (5.1) that one makes use of to “simplify the crossings” of a given product are unchanged under these rotations. Since the rotations are also defined in order to avoid links crossing punctures, the topological properties (such as which punctures are circled by which links) are preserved. Hence the action ψ_g is a homomorphism. Moreover ψ_g is an endomorphism because links in $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ are mapped to other links in $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$, and it is invertible, as rotations can be inverted, thus ψ_g is an automorphism. \square

Let $S = g_1 g_2 \dots g_\ell \in B_3$ be a decomposition of an element of the braid group on three strands with $g_i \in \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$. We define the automorphism ψ_S as follows:

$$\psi_S(X) = (\psi_{g_1} \circ \psi_{g_2} \circ \dots \circ \psi_{g_\ell})(X). \quad (8.17)$$

We use also the definition $\psi_1 = id$. The previous map (8.17) does not depend on the choice of the decomposition of S . Indeed, it is straightforward to check that the defining relations of the braid group (8.1) are verified on the generators. By the homomorphism property (8.16), it follows that these braid relations are verified for any element of the Kauffman bracket skein module $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$.

Remark 8.4. More visually complicated loops can always be created by further “twisting” the loops. For example,

$$\psi_{(s_1^{-1})^4}(\mathbb{A}_{23}) = \text{[Diagram: A complex loop with three punctures (marked with 'x') and a shaded interior region, representing a more complicated analog of } \mathbb{A}_{23} \text{]} \quad (8.18)$$

is a more complicated analog of \mathbb{A}_{23} . The shadow filling the inside of the loop is there to guide the eyes of the reader. These have also been studied in [85].

Remark 8.5. Let us remark that in [73], the author considers a similar braid group action by half Dehn twists on the Kauffman bracket skein algebra of the four-punctured sphere. In that paper, it is shown that the group $SL(2; \mathbb{Z})$ acts on the DAHA of type (C_1^\vee, C_1) through conjugations. Furthermore, the Artin braid group B_3 action on the Kauffman bracket skein algebra can be seen as a translation of this $SL(2; \mathbb{Z})$ action. We also note that Terwilliger had presented a B_3 action on both the universal Askey–Wilson algebra and the universal DAHA of type (C_1^\vee, C_1) [44].

8.3 Connection between both braid actions

The following proposition establishes the connections between both braid group actions presented above.

Proposition 8.6. The following diagram of isomorphisms

$$\begin{array}{ccc} \mathcal{A}_3 & \xrightarrow{\Psi_S} & \mathcal{A}_3 \\ \downarrow \phi & & \downarrow \phi \\ \text{Sk} & \xrightarrow{\psi_S} & \text{Sk} \end{array}$$

is commutative for any $S \in B_3$. Here we used the shortened notation $\text{Sk} \equiv \text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. The isomorphisms ϕ , Ψ_S and ψ_S are given in (6.16), (8.11) and (8.17), respectively.

Proof. We can show that this diagram is commutative for all the generators of \mathcal{A} and for any $S = s_i$ or $S = s_i^{-1}$. For example:

$$\phi \circ \Psi_{s_1}(Q_1) = \phi(Q_2) = \mathbb{A}_2 = \psi_{s_1}(\mathbb{A}_1) = \psi_{s_1} \circ \phi(Q_1). \quad (8.19)$$

A more complicated example is

$$\phi \circ \Psi_{s_2}(Q_{12}) = \phi(Q_{13u}) = \text{[Diagram: A loop with three punctures (marked with 'x') and a shaded interior region]} = \psi_{s_2}(\mathbb{A}_{12}) = \psi_{s_2} \circ \phi(Q_{12}). \quad (8.20)$$

Since all the maps of the diagram are homomorphisms, the commutativity of the diagram on the generators of \mathcal{A} is enough to prove the proposition for any $S \in B_3$. \square

The commutativity of this diagram allows us to identify the conjugation by the braided R -matrix for \mathcal{A}_3 as half Dehn twists around the punctures of $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. In addition, we can identify easily the elements of the algebra \mathcal{A}_3 obtained as an image by Ψ_S with a link of $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$.

9 Towards a higher rank $\text{saw}(n)$ algebra

Some natural generalizations of the different algebras have previously been introduced and studied:

- the generalized Askey–Wilson algebra $\mathbf{aw}(n)$ is the algebra generated by $\{C_I \mid I \subset \{1, 2, \dots, n\}\}$ subject to the relations introduced in Theorems 3.1 and 3.2 of [56];
- the algebra \mathcal{A}_n is the subalgebra of $U_q(\mathfrak{sl}_2)^{\otimes n}$ generated by all the intermediate Casimir elements $\{Q_I \mid I \subset \{1, 2, \dots, n\}\}$ obtained by the repeated action of the coproduct of $U_q(\mathfrak{sl}_2)$;
- the centralizer \mathfrak{C}_n is defined by

$$\mathfrak{C}_n = \{X \in U_q(\mathfrak{sl}_2)^{\otimes n} \mid [\Delta^{(n-1)}(x), X] = 0, \quad \forall x \in U_q(\mathfrak{sl}_2)\} \quad (9.1)$$

where $\Delta^{(n)} = (\Delta^{(n-1)} \otimes id)\Delta$ and $\Delta^{(1)} = \Delta$;

- the algebra $Sk_\theta(\Sigma_{0,n+1})$ is the Kauffman bracket skein algebra associated to the $(n+1)$ -punctured sphere $\Sigma_{0,n+1}$ [39]. Let us now associate to each set $I \subseteq [1; n] \equiv \{1, 2, \dots, n\}$ a ‘simple’ loop \mathbb{A}_I of $Sk_\theta(\Sigma_{0,n+1})$. We write a set I as $I = I_1 \cup I_2 \cup \dots \cup I_\ell$, where I_i are sets of consecutive integers and then we define the ‘simple’ loop \mathbb{A}_I as:

$$\mathbb{A}_I = \dots \left(\overset{I_1}{\times} \dots \overset{I_2}{\times} \dots \overset{I_\ell}{\times} \right) \dots \quad (9.2)$$

These simple loops do not bend around, unlike (8.18). They are only extending in the lower half of the plane. In particular, for $I = \{i, i+1, \dots, j\}$, a set of consecutive integers, one gets

$$\mathbb{A}_I = \left(\begin{array}{c} \times \quad \dots \quad \left(\times \dots \times \right) \quad \dots \quad \times \\ 1 \quad \quad \quad i \quad j \quad \quad \quad n \end{array} \right) = \left(\dots \left(\begin{array}{c} \times \\ I \end{array} \right) \dots \right) \quad (9.3)$$

What is lacking in the previous list is the generalization $\mathbf{saw}(n)$ of the algebra $\mathbf{saw}(3)$. Such a generalization would provide a description of the algebra \mathcal{A}_n in terms of generators and relation. We know that $\mathbf{saw}(n)$ will be a quotient of the algebra $\mathbf{aw}(n)$ by relation(s) of the type (2.2), with some Casimir elements to be determined. *We conjecture that the map ϕ_n from $\mathbf{saw}(n)$ to $Sk_\theta(\Sigma_{0,n+1})$ which sends Q_I to \mathbb{A}_I is an isomorphism* [†].

Let us mention that there also exist generalizations in the non-deformed case ($q = 1$ and $q = -1$) of the Askey–Wilson algebra: these are respectively called the ‘‘higher rank Racah algebra’’ introduced in [62] as well as the ‘‘higher rank Bannai–Ito algebra’’ introduced in [63].

In the remainder, we give different indications regarding ways to define $\mathbf{saw}(n)$.

[†]During the preparation of this paper, the authors have been informed by J. Cooke that a similar idea was pursued in an upcoming publication [39].

9.1 Punctures on a sphere and a coassociative homomorphism of Kauffman bracket skein modules

Recall we had highlighted that the punctures of the sphere were related to the tensor product factors. Additionally, a loop encircling a puncture is associated to some intermediate Casimir element with non-trivial factors in the tensor product factor corresponding to the puncture.

Further recall that the coproduct Δ acts as an algebra morphism from $U_q(\mathfrak{sl}_2)$ to $U_q(\mathfrak{sl}_2)^{\otimes 2}$. One can define an action of the coproduct on any i^{th} factor of a tensor product: for any $X \in U_q(\mathfrak{sl}_2)^{\otimes n}$, we define $\Delta_i : U_q(\mathfrak{sl}_2)^{\otimes n} \rightarrow U_q(\mathfrak{sl}_2)^{\otimes(n+1)}$ as:

$$\Delta_i(x) = \left(1^{\otimes(i-1)} \otimes \Delta \otimes 1^{\otimes(n-i)} \right) (X). \quad (9.4)$$

Now in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ some intermediate Casimir elements are related to each other by the coproduct, such as Q_1 and Q_{12} :

$$\Delta_1(Q_1) = (\Delta \otimes 1 \otimes 1)Q_1 = Q_{12} \otimes 1. \quad (9.5)$$

This relation between Q_1 and Q_{12} appears in the framed links picture as well.

More precisely, Δ_i has an analog, the δ_i morphism, which acts on a single puncture i by creating another puncture next to it. If the puncture i is enclosed in a loop, the created puncture is also enclosed in the same loop. The example (9.5) is illustrated as follows:

$$\begin{aligned} \delta_1 \mathbb{A}_1 &= \delta_1 \left(\begin{array}{ccc} \textcircled{\times} & \times & \times \end{array} \right) = \delta_1 \left(\begin{array}{ccc} \textcircled{\times} & \times & \times \end{array} \right) \\ &= \left(\begin{array}{ccc} \textcircled{\times} & \times & \times \end{array} \right) = \mathbb{A}_{12} \in \text{Sk}_\theta(\Sigma_{0,5}) \end{aligned} \quad (9.6)$$

This δ_i is a Kauffman bracket skein module coassociative algebra homomorphism. It provides embeddings of $\text{Sk}_\theta(\Sigma_{0,n}) \rightarrow \text{Sk}_\theta(\Sigma_{0,n+1})$. This can be seen as the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{A}_n & \xrightarrow{\Delta_i} & \mathcal{A}_{n+1} \\ \downarrow \phi_n & & \downarrow \phi_{n+1} \\ \text{Sk}_\theta(\Sigma_{0,n+1}) & \xrightarrow{\delta_i} & \text{Sk}_\theta(\Sigma_{0,n+2}) \end{array}$$

9.2 A crossing index

The defining algebra relations of $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ (2.1)–(2.2) (see Proposition 5.2) can be classified in three types. The relations always involve two generators, whose product, commutator or q -commutator is reexpressed in terms of other generators. Now imagine we draw both generators simultaneously in a framed links diagram (as if we were to multiply them). Some crossings will appear if the two generators don't commute.

Definition 9.1. *The crossing index is defined as the minimal number of crossings that appear in a framed link diagram no matter how the generators are drawn.*

The relations (2.1)–(2.2) can be classified in terms of the crossing index as follows:

- If the generators can be drawn simultaneously in such a way that the loops have no crossings (crossing index of 0), they will commute (for example, this is the case for any central element Q_1, Q_2, Q_3, Q_{123} multiplied with any other generator).
- If the generators can be drawn in such a way that their minimum number of crossings is two (crossing index of 2), linear q -commutation relations of **aw**(3)-type will be obtained, such as relations (2.1).
- If the generators have a crossing index of 4, such as

$$\phi(Q_{13u}Q_{13d}) = \left(\begin{array}{c} \text{---} \times \text{---} \times \text{---} \\ \text{---} \times \text{---} \times \text{---} \\ \text{---} \times \text{---} \times \text{---} \end{array} \right), \quad (9.7)$$

higher order relations of the type (2.2) will be obtained.

This crossing index proves useful for the analysis of the higher rank generalizations of **aw**(3).

9.3 The algebras **aw**(n) and $Sk_\theta(\Sigma_{0,n+1})$

As mentioned previously, the algebra **aw**(n) is generated by C_I with $I \subseteq [1; n]$ and subject to the relations of Proposition 3.1 of [56]. We can show by using the action of the morphism δ_i that we have an homomorphism from **aw**(n) to $Sk_\theta(\Sigma_{0,n+1})$. Moreover, we can show that all the relations of Proposition 3.1 of [56] correspond to the product of two simple loops with crossing index 2. We believe that the relations in [56] exhaust all possibilities of relations involving the product of simple loops with crossing index 2. We conjecture also that the above mentioned homomorphism is surjective (but it is certainly not injective, even for the case $n = 3$). The description of the kernel would involve products of links with a crossing index strictly greater than 2. The complete description of this kernel would lead to the definition of **aw**(n) and give an algebraic description of \mathcal{A}_n and $Sk_\theta(\Sigma_{0,n+1})$.

The study of **aw**(n) should be guided by the intuition gained from the framed links picture. To illustrate the type of insight we can gain, let us efficiently summarize some of the results of [53]. In this paper, the authors study the intermediate Casimir elements in $U_q(\mathfrak{sl}_2)^{\otimes 4}$ and introduce an involution I of the algebra as well as “involved” generators IQ_{13} and IQ_{24} satisfying

$$[Q_{13}, IQ_{24}] = 0, \quad \text{and} \quad [IQ_{13}, Q_{24}] = 0. \quad (9.8)$$

That these generators commute becomes evident when we rewrite (following our definitions) $IQ_{24} = Q_{24u}$, $IQ_{13} = Q_{13u}$, and then draw the corresponding links. Indeed, the products

$$\phi(Q_{13d}Q_{24u}) = \left(\begin{array}{c} \text{---} \times \text{---} \times \text{---} \\ \text{---} \times \text{---} \times \text{---} \\ \text{---} \times \text{---} \times \text{---} \end{array} \right) = \phi(Q_{24u}Q_{13d}), \quad (9.9a)$$

$$\phi(Q_{13u}Q_{24d}) = \left(\begin{array}{c} \text{---} \times \text{---} \times \text{---} \\ \text{---} \times \text{---} \times \text{---} \\ \text{---} \times \text{---} \times \text{---} \end{array} \right) = \phi(Q_{24d}Q_{13u}), \quad (9.9b)$$

have 0 crossing hence $[Q_{13d}, Q_{24u}] = 0$ and $[Q_{13u}, Q_{24d}] = 0$.

What about the product of terms like Q_{13d} and Q_{24d} ? This calculation has never appeared in the papers mentioned above because it has a crossing number of 4:

$$\phi(Q_{13d}Q_{24d}) = \text{Diagram} \quad (9.10)$$

Remarkably, this calculation can be effected in $\text{Sk}_{iq^{1/2}}(\Sigma_{0,5})$ using the conjectured morphism. One writes the Q_I in terms of \mathbb{A}_I , computes using the skein relations of $\text{Sk}_{iq^{1/2}}(\Sigma_{0,5})$, then reexpresses all \mathbb{A}_I in terms of Q_I . This yields the following results

$$\begin{aligned} Q_{13d}Q_{24d} &= q^2Q_{14d}Q_{23} + q^{-2}Q_{12}Q_{34} - q(Q_{14d}Q_2Q_3 + Q_{23}Q_1Q_4) - q^{-1}(Q_{12}Q_3Q_4 + Q_{34}Q_1Q_2) \\ &\quad - (q + q^{-1})Q_{1234} + Q_1Q_2Q_3Q_4 + Q_1Q_{234} + Q_2Q_{134d} + Q_3Q_{124d} + Q_4Q_{123} \end{aligned} \quad (9.11)$$

and

$$\begin{aligned} Q_{24d}Q_{13d} &= q^{-2}Q_{14d}Q_{23} + q^2Q_{12}Q_{34} - q^{-1}(Q_{14d}Q_2Q_3 + Q_{23}Q_1Q_4) - q(Q_{12}Q_3Q_4 + Q_{34}Q_1Q_2) \\ &\quad - (q + q^{-1})Q_{1234} + Q_1Q_2Q_3Q_4 + Q_1Q_{234} + Q_2Q_{134d} + Q_3Q_{124d} + Q_4Q_{123}. \end{aligned} \quad (9.12)$$

These have been checked to hold in $U_q(\mathfrak{sl}_2)^{\otimes 4}$.

Let us also mention that the action of the braid group B_3 can be generalized to the action of B_n on $Sk_\theta(\Sigma_{0,n+1})$ and \mathcal{A}_n . This might turn out useful for proving results in the future.

10 Conclusion

Three objectives were principally pursued in this paper. The first aimed to review the different avatars of the Askey–Wilson algebra and to clarify the relations between them. Among those algebras, we focused on two and presented novel results related to these cases; this was the second main goal. The Special Zhedanov algebra $sZh_q(m_1, m_2, m_3, m_4)$ was obtained from (a quotient of) the reflection algebra by setting the Sklyanin determinant to an appropriate value; its $W(D_4)$ symmetry was exhibited in addition. The Special Askey–Wilson algebra $\mathbf{saw}(3)$, a universal analogue of $sZh_q(m_1, m_2, m_3, m_4)$, was shown to be isomorphic to the algebra \mathcal{A}_3 that emerges from the Racah problem of $U_q(\mathfrak{sl}_2)$ and also to the Kauffman bracket skein algebra of the four-punctured sphere $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. An injective homomorphism between \mathcal{A}_3 and the centralizer \mathfrak{C}_3 of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product was stated and proved. Actions of the braid group on both $\text{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ (through half Dehn twists) and \mathcal{A}_3 (through conjugation by braided R -matrices) were illustrated and shown to be compatible. The third main objective was to discuss the generalization of $\mathbf{saw}(3)$ to $\mathbf{saw}(n)$. To that end, we emphasized the diagrammatic approach, defined a crossing index, and revisited the results of [53, 56] in a unified fashion.

Let us conclude with more remarks regarding generalizations of Askey–Wilson algebras. It would certainly be desirable to return to Zhedanov’s original quest and to determine directly from the multivariate Askey–Wilson polynomials (of Tratnik type) [57] the algebra that encapsulates their bispectral properties. Steps have been carried out [55, 58, 59] but this should be completed. A definite higher rank generalization of the Zhedanov algebra will emerge, whose quotients and central extensions could then be examined and should connect to various fields in mathematics and physics. Considering higher rank Lie algebras \mathfrak{g} instead of \mathfrak{sl}_2 is another avenue that should be explored. The centralizer of the diagonal action of $U_q(\mathfrak{g})$ in the n -fold

tensor product $U_q(\mathfrak{g})^{\otimes n}$, or the algebra generated by all the intermediate Casimir elements of \mathfrak{g} in the associated Racah problem should be studied. Connections with a generalization of $Sk_\theta(\Sigma_{0,n})$ to punctured manifolds of higher genera would be worth investigating (see also [82]). We may also wonder whether the braided universal R -matrix of $U_q(\mathfrak{g})$ plays a role in this context. Furthermore, the truncated reflection algebra presented in Section 3 provides a natural framework to obtain generalizations of Zhedanov algebras. Different possibilities are here conceivable. One could consider more general truncations of the reflection algebra. This type of generalization has been already studied in [60] and has been associated to quotients of q -Onsager algebras [‡]. Connections with centralizers and/or skein algebras remain to be examined. Another possibility with respect to truncated reflection algebras is the following. Instead of using the R -matrix associated to quantum affine algebras, one could consider the R -matrix corresponding to Yangians. In this case, a particular truncation of the reflection algebra leads to the Hahn algebra, which is a specialization of the Zhedanov algebra, see [86]. Other truncations should provide interesting generalizations of this algebra. Finally, the FRT presentation of the reflection algebra associated to higher rank Lie algebras and superalgebras is well-known. For instance, the twisted Yangians $Y^{\text{tw}}(\mathfrak{o}_n)$ and $Y^{\text{tw}}(\mathfrak{sp}_n)$ [87] and the reflection algebra $\mathcal{B}(n, \ell)$ [88] correspond to subalgebras of the Yangian of \mathfrak{sl}_n . Some q -deformations of these structures have been also studied previously [89] and are related to the quantum affine algebra of \mathfrak{sl}_n . Their truncations have yet to be scrutinized and should possess interesting features [§]. These ideas that we plan on pursuing in the near future are indications that there is much lying ahead with respect to algebras of the Askey–Wilson type and what they will reveal and lead to.

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A Classical limit and injectivity

We provide an explicit description of the classical limit of the realization of $\mathbf{saw}(3)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ in terms of polarized traces, and use it to prove the injectivity of the map from $\mathbf{saw}(3)$ to the centralizer \mathfrak{C}_3 . In this appendix, we will work with the formal series version of $U_q(\mathfrak{sl}_2)$ and

[‡]The classical limit $q \rightarrow 1$ leads to subalgebras of the loop algebra of \mathfrak{sl}_2 and to quotients of the Onsager algebra by Davis relations [61].

[§]Such an approach has been pursued in the classical limit $q \rightarrow 1$ [90] to obtain generalizations of the so-called classical Askey–Wilson algebra and are seen as subalgebras of the \mathfrak{sl}_n Onsager algebra [91].

reduce the proof of the injectivity statement to one in the universal enveloping algebra $U(\mathfrak{sl}_2)$, where we can use known results of classical invariant theory involving polarized traces.

A.1 Polarised traces in $U(\mathfrak{sl}_2)^{\otimes 3}$

The algebra $U(\mathfrak{sl}_2)$ is generated by elements e_{ij} , $i, j \in \{1, 2\}$, with the defining relations $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$ and $e_{11} + e_{22} = 0$. To join up with the notations used in the paper for $U_q(\mathfrak{sl}_2)$, we set $E = e_{12}$, $F = e_{21}$ and $H = \frac{1}{2}(e_{11} + e_{22}) = e_{11} = -e_{22}$, and the relations become:

$$[H, E] = E, \quad [H, F] = -F, \quad [E, F] = 2H. \quad (\text{A.1})$$

In a tensor product $U(\mathfrak{sl}_2)^{\otimes N}$, we denote the generators by $e_{ij}^{(a)}$, where $a \in \{1, \dots, N\}$ indicates the corresponding factor in the tensor product. The polarized traces are the following elements:

$$T^{(a_1, \dots, a_d)} = e_{i_2 i_1}^{(a_1)} e_{i_3 i_2}^{(a_2)} \dots e_{i_1 i_d}^{(a_d)}, \quad a_1, \dots, a_d \in \{1, \dots, N\}, \quad (\text{A.2})$$

where the summation over repeated indices is understood. The specific combinations of polarized traces that will appear are:

$$\begin{aligned} k_1 &:= T^{(1,1)}, & k_2 &:= T^{(2,2)}, & k_3 &:= T^{(3,3)}, & k_4 &:= k_1 + k_2 + k_3 + 2(T^{(1,2)} + T^{(2,3)} + T^{(1,3)}), \\ X &:= k_1 + k_2 + 2T^{(1,2)}, & Y &:= k_2 + k_3 + 2T^{(2,3)}, & Z &:= [X, Y]. \end{aligned} \quad (\text{A.3})$$

By direct computation, we can show that the expression of Z in terms of polarized traces is $Z = -8T^{(1,2,3)}$.

A.2 The algebra $U_\alpha(\mathfrak{sl}_2)$

In this appendix, we will work with the formal series version of the quantum group $U_q(\mathfrak{sl}_2)$. We consider a formal parameter α . The algebra $U_\alpha(\mathfrak{sl}_2)$ is, as a vector space, the space $U(\mathfrak{sl}_2)[[\alpha]]$ of all formal power series in α with coefficients in $U(\mathfrak{sl}_2)$, and the multiplication is determined by the defining relations of $U_q(\mathfrak{sl}_2)$, see section 6.1, where q is replaced by e^α and q^H is replaced by $e^{\alpha H}$. This results in the following relations deforming (A.1):

$$[H, E] = E, \quad [H, F] = -F, \quad [E, F] = \frac{e^{2\alpha H} - e^{-2\alpha H}}{e^\alpha - e^{-\alpha}}. \quad (\text{A.4})$$

Similarly, the algebra $U_\alpha(\mathfrak{sl}_2)^{\hat{\otimes} N}$ is the vector space $U(\mathfrak{sl}_2)^{\otimes N}[[\alpha]]$ of formal series with coefficients in $U(\mathfrak{sl}_2)^{\otimes N}$ and multiplication induced by the above relations in each factor. The comultiplication of $U_\alpha(\mathfrak{sl}_2)$ is naturally obtained from the comultiplication given for $U_q(\mathfrak{sl}_2)$.

Note that the limit $\alpha \rightarrow 0$ from $U_\alpha(\mathfrak{sl}_2)$ yields the algebra $U(\mathfrak{sl}_2)$ and the comultiplication becomes the diagonal embedding.

A.3 Reduction to $U(\mathfrak{sl}_2)$

We want to prove that the following elements

$$Q_{12}^i Q_{23}^j Q_{13}^k Q_1^m Q_2^n Q_3^p Q_{123}^q, \quad i, j, m, n, p, q \in \mathbb{N}, \quad k \in \{0, 1\}, \quad (\text{A.5})$$

are linearly independent in $U_\alpha(\mathfrak{sl}_2)^{\hat{\otimes} 3}$. First it is more convenient (and equivalent) to replace the generators Q_I by modified versions analogous to what was done in Section 4:

$$K_I := \frac{Q_I - (q + q^{-1})}{(q - q^{-1})^2}, \quad I \in \{1, 2, 3, 123, 12, 23\}. \quad (\text{A.6})$$

The index 13 does not belong to this set, and for this one, we set:

$$K_{13} := [K_{12}, K_{23}]_q. \quad (\text{A.7})$$

Calculating explicitly the first terms in the expansions in α , we find that the new elements K_I are well-defined in $U_\alpha(\mathfrak{sl}_2)^{\otimes 3}$, and moreover that their degree 0 coefficients are expressed in terms of polarized traces, using the notations in (A.1), as follows

$$K_i|_{\alpha=0} = \frac{1}{2}k_i \quad (i = 1, 2, 3), \quad K_{123}|_{\alpha=0} = \frac{1}{2}k_4, \quad (\text{A.8})$$

$$K_{12}|_{\alpha=0} = \frac{1}{2}X, \quad K_{23}|_{\alpha=0} = \frac{1}{2}Y, \quad K_{13}|_{\alpha=0} = \frac{1}{4}Z. \quad (\text{A.9})$$

Now, to prove that the elements of the set (A.5), with Q_I replaced by K_I , are linearly independent in $U_\alpha(\mathfrak{sl}_2)^{\otimes 3}$, it is enough to prove that their ‘‘classical limits’’ (the degree 0 coefficients) are linearly independent in $U(\mathfrak{sl}_2)^{\otimes 3}$. In view of the above calculations, it remains to show that the following set:

$$k_1^i k_2^j k_3^k k_4^m X^n Y^p Z^q, \quad i, j, k, m, n, p \in \mathbb{N}, \quad q \in \{0, 1\}, \quad (\text{A.10})$$

is linearly independent in $U(\mathfrak{sl}_2)^{\otimes 3}$.

A.4 Racah algebra and diagonal centraliser in $U(\mathfrak{sl}_2)^{\otimes 3}$

To prove that the set (A.10) is linearly independent, we use the same line of arguments as the one used in the study of the recoupling of two copies of $\mathfrak{sl}(3)$. Thus we only give here a sketch and refer for more details to [79].

It is known from classical invariant theory [92, 93] that the centralizer of the diagonal embedding of $U(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)^{\otimes 3}$ is generated by the polarised traces $T^{(i,i)}$, $T^{(k,l)}$, $T^{(1,2,3)}$, with $i = 1, 2, 3$ and $1 \leq k < l \leq 3$, and moreover that the Hilbert–Poincaré series of the centralizer is:

$$\frac{1 - t^6}{(1 - t^2)^6(1 - t^3)}. \quad (\text{A.11})$$

This series records the dimension for each degree of the centralizer, where the degree in $U(\mathfrak{sl}_2)^{\otimes 3}$ is defined by $\deg(e_{ij}^{(a)}) = 1$. From this information, we extract at once that the set $k_1, k_2, k_3, k_4, X, Y, Z$ generates the centralizer. Now, we have that these elements satisfy the classical Racah relations:

$$\begin{aligned} k_1, k_2, k_3, k_4 & \text{ commute with all generators,} \\ [X, Y] & = Z, \\ [X, Z] & = 4\{X, Y\} + 4X^2 - 4(k_1 + k_2 + k_3 + k_4)X + 4(k_1 - k_2)(k_4 - k_3), \\ [Y, Z] & = 4\{X, Y\} + 4Y^2 - 4(k_1 + k_2 + k_3 + k_4)Y + 4(k_3 - k_2)(k_4 - k_1), \end{aligned} \quad (\text{A.12})$$

together with

$$\Gamma = 8(k_1 - k_2 + k_3 - k_4)(k_1 k_3 - k_2 k_4) - 32(k_1 k_3 + k_2 k_4), \quad (\text{A.13})$$

where the element Γ is

$$\begin{aligned} \Gamma : & = Z^2 - 8(XYX + YXY) + 4(k_1 + k_2 + k_3 + k_4 - 4)\{X, Y\} \\ & - 8(k_1 - k_2)(k_4 - k_3)Y - 8(k_3 - k_2)(k_4 - k_1)X. \end{aligned} \quad (\text{A.14})$$

The relations (A.12) allow to rewrite any product in terms of ordered monomials in the generators and (A.13) allows to rewrite Z^2 . So we deduce easily that the set (A.10) is a spanning set for the centralizer. Finally, the comparison with the Hilbert–Poincaré series in (A.11) shows that this set must be linearly independent.

This concludes the proof of the injectivity of the map from $\mathbf{saW}(3)$ to $\mathfrak{C}_3 \subset U_\alpha(\mathfrak{sl}_2)^{\otimes 3}$.

Remark A.1. *Specializing the central elements k_i to $\frac{m_i^2-1}{2}$, one finds that the relations (A.12)–(A.13) are expressed in terms of the polynomials:*

$$\sum_{i=1}^4 m_i^2, \quad \begin{cases} (m_1^2 - m_2^2)(m_4^2 - m_3^2), \\ (m_3^2 - m_2^2)(m_4^2 - m_1^2), \end{cases} \quad (m_1^2 m_3^2 - m_2^2 m_4^2)(m_1^2 - m_2^2 + m_3^2 - m_4^2). \quad (\text{A.15})$$

These polynomials are invariant polynomials under the action of the Weyl group $W(D_4)$ of Section 4. This recovers explicitly the classical limit of the results in Section 4.

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