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# On the number of edges of comparability and incomparability graphs

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**Abstract.** Since the 1990s, as we know, comparability and incomparability graphs find use in experimental sciences, particularly in chemistry and physical chemistry as well as in statistical physics, to characterize molecular properties, order isomers or specify the topology of certain structures such as chemical structures. In this mostly historical article, readable by a non-specialist, we recall first some basic notions of graph theory. Then, we define the notions of comparability and incomparability graphs of a poset, summarize the recent advances on the question and compute the number of edges of these graphs, using in particular Boolean representation of posets. We also give a special method in the case of Boolean lattices.

**Key words.** Graphs, complementary graphs, partial orders, lattices, Boolean lattices, comparability and incomparability graphs.

## 1 Introduction

For the past thirty years or more, advances in experimental sciences have shown the importance of graph theory and the interest of studying particular graphs such as interval graphs, comparability graphs (see [Gilmore 64]; [Olariu 92]), and incomparability graphs which, like interval graphs, have applications in mathematical chemistry (see [Dehmer 14]), physical chemistry (see [Todeschini 02]) or statistical physics (see [Zhu 04]). We find indeed applications in the study of molecular properties (see [Bonchev 90<sup>1</sup>];[Bonchev 90<sup>2</sup>]) electronic spectra of condensed benzenoid hydrocarbons (see [Bonchev 83], in the ordering of isomers (see [Bonchev 84]), in the recording

of invariant vertexes and topological indices of chemical graphs based on information of distances (see [Balaban 91]), also in chemoinformatics and molecular biology (see [Bajorath 04]). In this article, after having recalled some elements of the vocabulary of graph theory, we review the history of some studies and offer some thoughts on the number of edges of incomparability graphs.

## 2 Some recalls on Graphs

**Definition 2.1** (graph, multigraph,  $p$ -graph). A graph  $G(X, U)$ , with  $X$  a set of vertices and  $U$  a set of edges (or arcs) which connect them (see [Berge 70], 3), is the pair made up:

1. By a set  $X = \{x_1, x_2, \dots, x_n\}$  where the  $x_i$  are "vertices";
2. By a family  $U = \{u_1, u_2, \dots, u_m\}$  where the  $u_j$  are elements of the Cartesian product:

$$X \times X = \{(x, y) \mid x \in X, y \in Y\}.$$

When two vertices are connected by more than one edge, we speak of *multigraph* and, more precisely, if  $(x, y)$  occurs  $p$  times in  $G$ , then  $G(X, U)$  is a  $p$ -graph.

**Definition 2.2.** A multigraph  $G(X, U)$  is a *simple graph* if:

1. It has no loop;
2. Between 2 vertices, there is never more than one arc (or edge) to connect them.

**Definition 2.3** (adjacency matrix of a graph). We recall that the adjacency matrix of a graph is a matrix  $M_{i,j}$  ( $i = 1, \dots, n; j = 1, \dots, n$ ) with coefficients 0 or 1, where each row corresponds to a vertex of  $G$ , where each column corresponds to a vertex of  $G$  and where:

$$M_{ij} = \begin{cases} 1 & \text{if and only if } (i, j) \in U; \\ 0 & \text{otherwise.} \end{cases}$$

In all this article, we are dealing with undirected simple graphs.

Among the graphs, some present regularities: for example, the same "degree", that is to say the same number of edges or arcs pertaining to each vertex. They are called "regular" graphs. Others are not like that: they are "irregular", even "highly

irregular" and are today the object of numerous studies and of many conjectures (see for example [Chartland 16], 1-16).

We will leave aside this question here to focus on complementarity graphs (a question often neglected in graph theory<sup>1</sup>). We will go on by studying the comparability and incomparability graphs on some ordered structures.

### 3 Complementarity graphs

**Definition 3.1** (Complementary graph of a simple graph). (See [Gondran 79], 4) Given a simple graph  $G = [X, U]$ , the complementary graph  $G' = [X, \bar{U}]$  has the same set of vertices as  $G$  and as edges, the complementary edges of those of  $U$ . We have:

$$(i, j) \in U \Rightarrow (i, j) \notin \bar{U};$$

$$(i, j) \notin U \Rightarrow (i, j) \in \bar{U}.$$

Fig. 1 shows a simple graph  $G$  and its complementary graph  $\bar{G}$ .

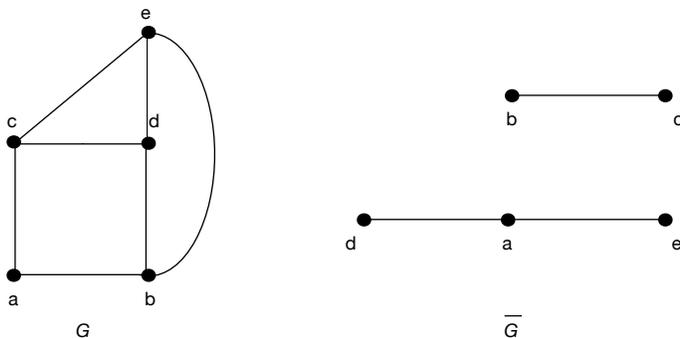


Figure 1: Graph  $G$  and its complementary  $\bar{G}$

As we can see, the complementary  $\bar{G}$  of the graph  $G$  is disconnected. In general, as we know, the complementary graph of a disconnected graph is connected but the

<sup>1</sup>We have to note that certain books, however reputed to be "bibles" in graph theory, devoted very few lines to these questions, and in particular to comparability and incomparability graphs, the properties of which are often left to prove to the reader (see [Diestel 06], 127, 136 and 242).

converse is not true.

**Definition 3.2** (Complementary Matrix of a simple graph). We call *complementary matrix* of a simple graph  $G$  the matrix  $M'$  associated to the complementary graph  $\overline{G}$  of  $G$ .

**Proposition 3.1.** *The complementary matrix is obtained as the difference between the inverse matrix  $\overline{M}$  of the adjacency matrix of the graph  $G$  and the unit matrix  $I_n$  if  $G$  has  $n$  vertices.*

*Proof.* Consider  $\overline{M}$ , the matrix where the 1s of  $M$  have been changed into 0s, and conversely, the 0s have been changed into 1s. A consequence is necessarily that  $\overline{M}$  is a matrix of a loop graph because the 0s of the diagonal of  $M$  have been replaced by 1s, which means that  $a$  is in relation to itself,  $b$  in relation to itself, etc. If we want to obtain as complement of  $G$  a simple graph  $\overline{G}$  without loops, we need to find a matrix  $M'$  with zero diagonal. In other words, we must subtract 1s from 1s on the diagonal of  $M$  and, for that, we must ask:

$$M' = \overline{M} - I_n,$$

where  $I_n$  is the identity matrix with  $n$  rows and  $n$  columns, whose diagonal terms are all equal to 1 and non-diagonal terms equal to 0.  $\square$

Knowing the number of edges of  $G[X, U]$ , it is easy to calculate the number of edges of its complementary graph.

**Proposition 3.2.** *If  $|X| = n$ ,  $|\overline{U}| = \binom{n}{2} - |U|$ .*

*Proof.* To each edge of a graph corresponds a pair of vertices. So the total number of all possible edges of a graph is the total number of all possible combinations of 2 elements taken from  $n$ , say  $\binom{n}{2} = N$ . To get the cardinal of  $\overline{U}$ , it suffices to subtract from this number  $N$  the cardinal  $|U|$  of the set of edges of  $G[X, U]$ , since  $|U| + |\overline{U}| = N = \binom{n}{2}$ .  $\square$

## 4 Order relation, ordered set

**Definition 4.1.** (Order relation - see [Kaufmann 78], 5-7) A binary internal relation  $\leq$  on a nonempty set  $E$  is an *order relation* if for all  $x, y$  and  $z$  elements of  $E$ , we have:

1.  $x \leq x$  (reflexivity);
2.  $(x \leq y \text{ and } y \leq x) \Rightarrow x = y$  (antisymmetry);
3.  $(x \leq y \text{ and } y \leq z) \Rightarrow x \leq z$  (transitivity).

If we replace  $\leq$  by  $<$ , then the relation becomes a "strict order" relation. In this case, it is irreflexive. It reads "x is less than y", or "y is greater than x" and we have:

$$x < y \text{ if and only if } x \leq y \text{ and } x \neq y.$$

Recall that some order relations are *total*, i.e. two elements of the set  $E$  are always *comparable*, in other words, for all  $x, y$  of  $E$ :

$$x \leq y \text{ or } y \leq x.$$

We then say that  $\leq$  is a relation of "total order", and that the set  $E$  is *totally ordered* by this relation.

**Definition 4.2** (Ordered set - see [Harzheim 05]). An ordered set is a set provided with an order relation.

**Definition 4.3** (Partial order - partially ordered set (or poset)). An order relation on  $E$  is said to be "partial" if it is not total, and  $E$  is then a *partially ordered set* (or *poset*).

**Definition 4.4** (Negation of an order relation). The negation of a binary relation  $R$  defined on a set  $E$  is the relation associated with the complementary graph of the relation  $R$  in  $E \times E$ . It is usually written  $\not R$ . In other words, two elements are related by  $\not R$  if and only if they are not related by  $R$ .

To say that an order is total is to say that its negation is the strict reverse order. This means there is an equivalence for an order  $\leq$  between:

- $\leq$  is total;

- $x \not\leq y \iff y < x$ .

On the other hand, as soon as there are two distinct elements not comparable within an order, the negation of the order relation cannot be an order (strict or not), because it is not antisymmetric. The negation of a non-total order is therefore never an order.

## 5 Graphs associated with a partially ordered set

**Definition 5.1** (Comparable and incomparable elements). Let  $P$  be a partially ordered set, endowed with the relation  $<$ , reflexive, antisymmetric, transitive. We say that two elements  $x$  and  $y$  of  $P$  are *comparable* if  $x < y$  or  $y < x$  (we can also replace  $<$  by  $\leq$ ). Otherwise, that is to say when two elements of  $P$  are *incomparable*, we have between them a relation  $\parallel$  called “relation of incomparability”, reflexive, symmetric and transitive.

**Definition 5.2** (Comparability and incomparability graphs). When  $P$  is a partially ordered set (poset), we can thus associate to  $P$  two graphs. One is called *comparability graph* of  $P$  and the other *incomparability graph* of  $P$ . These two graphs have the elements of  $P$  as a set of vertices. In the comparability graph, two distinct elements  $x$  and  $y$  of  $P$  are adjacent when they are comparable in  $P$ . Likewise,  $x$  and  $y$  are adjacent in the incomparability graph when they are incomparable in  $P$ .

We will focus more specifically on the latter case. Before that, let us introduce the following definitions:

**Definition 5.3** (Clique). A clique  $C$  in an undirected simple graph  $G = (X, U)$  is a subset of the vertices,  $C \subseteq X$ , such that every two distinct vertices are adjacent. This is equivalent to the condition that the subgraph of  $G$  induced by  $C$  is a complete graph.

**Definition 5.4** (Coloring a graph). The *coloring* of the vertices (resp. edges) of a graph  $G = (X, U)$  corresponds to the assignment of a color to each of its vertices (resp. edges) so that two adjacent vertices (resp. edges) do not carry the same color.

**Definition 5.5** (Chromatic number of a graph). A graph is said to be  $p$ -chromatic if its vertices admit coloring in  $p$  colors. We call *chromatic number*  $\chi(G)$  (resp.

*chromatic index*  $q(G)$ ) the minimum number of distinct colors needed to color the vertices (resp. edges) of the graph  $G$ .

**Definition 5.6** (Perfect graph - see [Berge 84]). A perfect graph is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph (i.e the clique that has the maximum number of vertices – or "clique number"). Equivalently, stated in symbolic terms, an arbitrary graph  $G = (X, U)$  is perfect if and only if :

$$\text{for all } C \subseteq X \text{ we have } \chi(G[C]) = \omega(G[C]).$$

In 1950, Dilworth (see [Dilworth 50]) was able to prove that incomparability graphs were perfect graphs, so that the chromatic number of an incomparability graph is equal to its number of cliques (an analogous result for the comparability graphs had previously been proved by Erdős and Szekeres (see [Erdős 35])). In 1967, Gallai (see [Gallai 67]) gave a characterization of incomparability graphs in terms of minimal induced prohibited subgraphs. We also know that the incomparability graphs can be recognized in a polynomial time (see [Golombic 80]). In 1983, Golombic, Rotem and Urrutia (see [Golombic 83] and Lov'asz (see [Lov'asz 83]) proved that any incomparability graph was also a string graph (on all of this, see [Fox 12]). But this does not allow us to precise the structure of these graphs. We will show that, in some cases, we can directly compute the number of edges of their incomparability associated graphs.

It will be found in Fig. 2 an example of poset (A), its comparability graph (B) and its incomparability graph (C).

## 6 Complementarity graph and incomparability graph of a poset

Let  $P$  a poset and  $D(X, U)$  the Hasse diagram of  $P$ . The set of edges  $U$  is formed by the explicit order relations in  $P$ , that is:

$$U = \{(x, y) \in X \mid x \leq y\}.$$

Let now  $G(P)$  the comparability graph of the poset  $P$ . Let  $E$  be the set of edges of  $G(P)$ .  $E$  is formed by all the ordered pairs of elements of  $P$ , and we have :

$$E = \{(x, y) \in P \mid x \leq y\}.$$

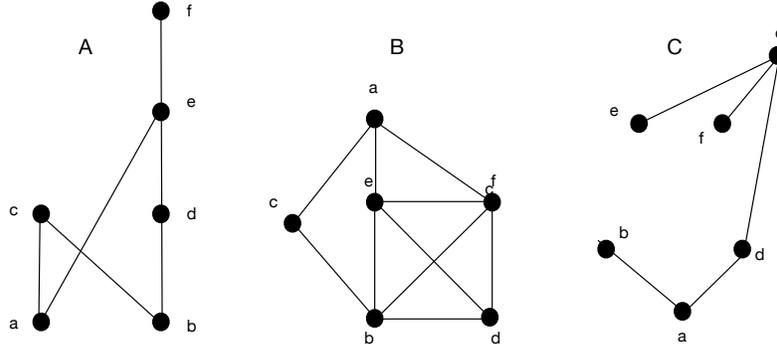


Figure 2: Poset, comparability graph, incomparability graph

Hence we can deduce that  $E - U$  is the set of non-explicit order relations between the elements of  $P$ .

Let now  $D(X, \bar{U})$  the complementary graph of  $D(X, U)$ , the Hasse diagram of  $P$ . We can now deduce the following proposition:

**Proposition 6.1.** *The number  $N$  of edges of  $I(P)$ , the incomparability graph of  $P$ , is given by the difference between the cardinal of  $\bar{U}$  and the cardinal of the set of non-explicit order relations between the elements of  $P$ . We have:*

$$N = |\bar{U}| - |E - U| = |\bar{U}| + |U| - |E| = \binom{n}{2} - |E|.$$

In other words, knowing the number of edges of the comparability graph of a poset  $P$ , we can get easily the number of edges of its incomparability graph. And, of course, conversely, knowing the number of edges  $N$  of the incomparability graph of a poset  $P$ , we can get easily the number of edges of its comparability graph, since  $|E| = \binom{n}{2} - N$ .

#### Examples

In the case of the poset of Fig. 2, we have  $U = 6$ ,  $|\bar{U}| = 9$ ,  $|E| = 10$ . Hence  $|E - U| = 10 - 6 = 4$  and  $N = |\bar{U}| - |E - U| = 9 - 4 = 5$ . We obtain this result more quickly by simply writing  $N = \binom{6}{2} - 10 = 15 - 10 = 5$ .

## 7 Boolean representation of a poset

In fact, we can calculate in a very simple way the number of edges of a poset incomparability graph. There are indeed Boolean representations of posets (which can also be applied to lattices, simplicial complexes and matroids (see [Rhodes 15]))<sup>2</sup>. We just define the Boolean matrix  $M(P) = m_{xu}$  of a poset  $P$  by setting:

$$m_{xu} = \begin{cases} 0 & \text{if } x \geq u; \\ 1 & \text{otherwise.} \end{cases}$$

Then we have just to compute in the matrix the 0s which are not on the diagonal. By this method, we get directly the cardinal of the comparability graph of  $P$ . Hence we can easily deduce the cardinal of its incomparability graph. In the case of the poset of Fig. 2, the matrix is given in table 1:

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>
<i>a</i>	0	1	1	1	1	1
<i>b</i>	1	0	1	1	1	1
<i>c</i>	0	0	0	1	1	1
<i>d</i>	1	0	1	0	1	1
<i>e</i>	0	0	1	0	0	1
<i>f</i>	0	0	1	0	0	0

Table 1: Comparability matrix of  $P$

Of course we cannot permute the 0s and the 1s of this matrix to get the matrix of the incomparability graph: this one has not the same number of vertices. Besides, it would make no sense to do so. The following example will suffice to show it: Let  $T = \{0, a, b, 1\}$  with  $0 \leq a, b \leq 1$ . Since we do not have  $a \geq b$  (as  $a$  and  $b$  are incomparable),  $m_{ab} = m_{ba} = 1$ . But by passing to the matrix  $M_c$  complementary to  $M$ , we necessarily obtain  $m_{ab} = m_{ba} = 0$ , which means that we have both  $a \geq b$  and

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<sup>2</sup>There is an alternative possibility: following Stanley (see [Stanley 70], [Stanley 88], [Stanley 04]) and more recent searchers (see [Wu-Shizen 12]), we shall say that the incidence matrix  $\mathbf{m}_P$ , which encodes full information about the poset  $P$ , is the indicator function of  $<$  and may be described as follows :

$$\mathbf{m}_P = \begin{cases} 1 & \text{if } x < y; \\ 0 & \text{otherwise.} \end{cases}$$

$b \geq a$ . But this results in  $a = b$ , which is wrong. So, a problem could be: how to get directly the number of edges of an incomparability graph  $I(P)$ , without knowing the number of edges of the comparability graph of the poset  $P$ . We will see that, in the case of Boolean lattices, this is perfectly possible.

## 8 Case of lattices

As we know (see [Birkhoff 40]), a lattice  $(T, \leq)$  is a partially ordered set (poset) in which every element has one and the same supremum (least upper bound) and one and the same infimum (greatest lower bound). Fig. 3 gives a typical example.

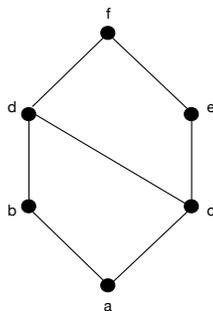


Figure 3: A classic example of a lattice

Suppose we want to know explicitly the matrix of the incomparability graph of this lattice and how it is connected to the matrix of its complementary graph. We could proceed as follows: we draw the adjacency matrix of the Hasse diagram of the lattice, then the matrix of its complementarity graph. The matrix of the incomparability graph can be deduced very easily.

The adjacency matrix  $M(T)$  of the Hasse diagram of  $T$ , the lattice of Fig. 3 is given in Table 2.

The matrix  $M_c(T)$ , complementary to  $M(T)$ , appears in Table 3.

Fig. 4 shows the complementary graph (A) and the incomparability graph (B) of the lattice  $T$ . As we can see, the incomparability graph of  $T$  has 6 vertices and only 3 edges.

Table 4 gives the matrix  $M'(T)$  of the incomparability graph of  $T$ :

$$M(T) = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 1 & 0 & 0 \\ c & 1 & 0 & 0 & 1 & 1 & 0 \\ d & 0 & 1 & 1 & 0 & 0 & 1 \\ e & 0 & 0 & 1 & 0 & 0 & 1 \\ f & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Table 2: Adjacency matrix of  $T$

$$M_c(T) = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 0 & 0 & 1 & 1 & 1 \\ b & 0 & 0 & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 0 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 1 & 0 \\ e & 1 & 1 & 0 & 1 & 0 & 0 \\ f & 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

Table 3: Adjacency matrix of the complement of  $T$

Note that  $M'(T)$  can be obtained from  $M_c(T)$  by deleting the first and last row, and the first and last column.

We may observe that two non-isomorphic lattices can have the same incomparability graph (see Fig. 5).

As Wasadikar and Survase (see [Wasadikar 12]) point out, there are many papers that interlink lattice theory and graph theory. Filipov (see [Filipov 80]) was undoubtedly one of the first to study the comparability graphs of partially ordered sets by defining the adjacency between two elements of a poset by the comparability relation:  $a$  and  $b$  are adjacent if  $a \leq b$  or  $b \leq a$ . In the same order of ideas, Duffus and Rival (see [Duffus 77]) define the overlap graph of a poset by the overlap relation:  $a$  and  $b$  are adjacent if  $a \prec b$  or  $b \prec a$ . Articles by Gedenova (see [Gedenova 80]), Bollobas and Rival (see [Bollobas 79]) discuss the properties of overlap graphs derived from lattices. In [Nimborkar 10], Nimborkar, Wasadikar and Pawar introduced graphs on a lattice  $L$  with  $0$ , and defined adjacency of two elements  $x, y \in L$  by the relation:  $x \wedge y = 0$ . More recently, Bostjan Bresar et.al. (see [Bresar 08]) have themselves introduced cover graphs which are incomparability graphs of posets. In particular, they defined a graph in which the set of edges is the union of the set of edges of the cover graph and of the corresponding incomparability graph. In short, a lot of works

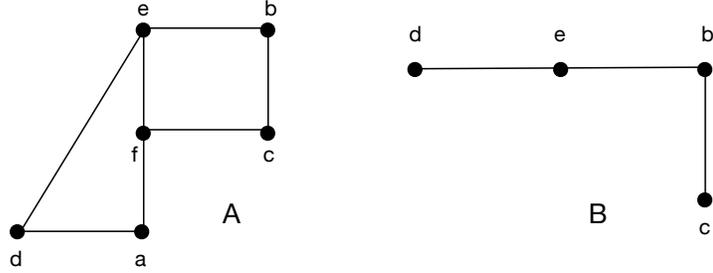


Figure 4: Complementary graph (A) and incomparability graph (B) of the Hasse diagram of the lattice  $T$

$$M'(T) = \begin{array}{c|cccc} & b & c & d & e \\ \hline b & 0 & 1 & 0 & 1 \\ c & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 1 \\ e & 1 & 0 & 1 & 0 \end{array}$$

Table 4: Adjacency matrix of the incomparability graph of  $T$

have focused on these questions since the 1980s.

In their paper, Wasadikar and Survase draw all graphs with a number of edges  $n \leq 4$  which are incomparability graphs associated with lattices and give some simple properties of these graphs without relating them to particular lattices. On the contrary, we will show that certain types of lattices make it possible to define additional properties, in particular Boolean lattices, which make it possible to compute directly the number of edges of the incomparability graph associated with them.

## 8.1 Case of Boolean lattices

As we know, Boolean structures (in conjunction with order theory) are of particular importance (see [Ponasse 79]; [Rhodes 15]) in many fields. The Boolean lattices  $T_n$  are of the form  $\{0, 1\}^n$ . Consider for example the lattice  $T_3$ , isomorphic to  $\{0, 1\}^3$ , shown in Fig. 6.

The adjacency matrix of the associated graph is that of Table 5.

The complementarity matrix of the graph associated to  $T_3$  is represented on Table

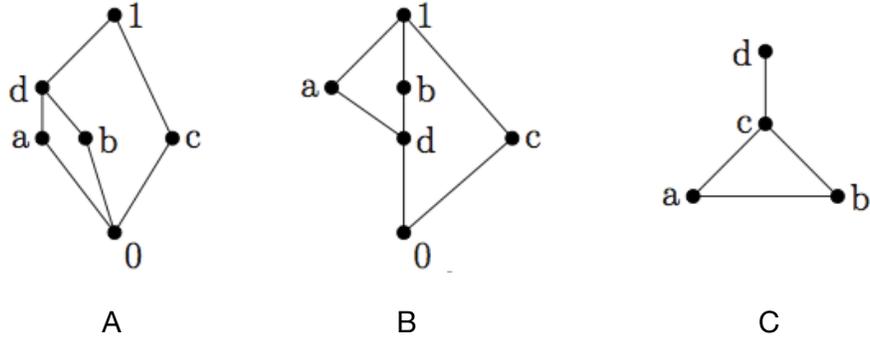


Figure 5: Non-isomorphic  $A$  and  $B$  lattices having the same incomparability graph  $C$

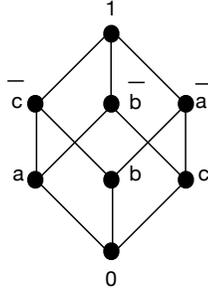


Figure 6: Hasse diagram of  $T_3$

6.

Hence the associated graph, which can take several configurations (see Fig. 7).

We can go to the incomparability graph as before, by removing the vertices which are in an order relation (in this case, 0 and 1), as well as the related edges. The graph being regular of degree 4, there would in principle be 8 edges to remove, but 0 and 1 being connected by an order relation, they have one edge in common. We must therefore remove  $8 - 1 = 7$  edges. Since the complementary graph of  $T_3$  has 8 vertices and 16 edges, the incomparability graph will therefore have 6 vertices and  $16 - 7 = 9$  edges (see Fig. 8).

At the matrix level, this amounts to removing the first and the last row, as well as the first and the last column of  $M_c(T_3)$  and replacing this matrix by a matrix  $M'(T_3)$  with 6 rows and 6 columns (see Table 7).

	0	a	c	$\bar{b}$	b	$\bar{c}$	$\bar{a}$	1
0	0	1	1	0	1	0	0	0
a	1	0	0	1	0	1	0	0
c	1	0	0	1	0	0	1	0
$\bar{b}$	0	1	1	0	0	0	0	1
b	1	0	0	0	0	1	1	0
$\bar{c}$	0	1	0	0	1	0	0	1
$\bar{a}$	0	0	1	0	1	0	0	1
1	0	0	0	1	0	1	1	0

Table 5: Adjacency matrix of  $T_3$

	0	a	c	$\bar{b}$	b	$\bar{c}$	$\bar{a}$	1
0	0	0	0	1	0	1	1	1
a	0	0	1	0	1	0	1	1
c	0	1	0	0	1	1	0	1
$\bar{b}$	1	0	0	0	1	1	1	0
b	0	1	1	1	0	0	0	1
$\bar{c}$	1	0	1	1	0	0	1	0
$\bar{a}$	1	1	0	1	0	1	0	0
1	1	1	1	0	1	0	0	0

Table 6: Adjacency matrix of the complementary graph of  $T_3$

## 8.2 Generalization: Boolean lattices as $n$ -cubes

In the case of Boolean lattices  $T_n$ , the Hasse diagrams associated with the  $T_n$  are  $n$ -cubes. So we may observe that it is possible to directly compute the edges of Boolean lattice incomparability graphs, without using the previous method or the comparability graph boolean representation.

**Proposition 8.1.** *It is well known that the number  $|U|$  of edges of a  $n$ -cube with  $2^n$  vertices is given by the formula:*

$$|U| = n \times 2^{n-1}.$$

**Proposition 8.2.** *The number  $|U'|$  of edges of the incomparability graph of a Boolean lattice  $T_n$  with  $n$  vertices is obtained by subtracting from the cardinal  $|U|$  (number of edges of the complementary graph of  $T_n$ ), the total sum of the  $n$ -faces ( $n \geq 2$ ) of the corresponding  $n$ -cube.*

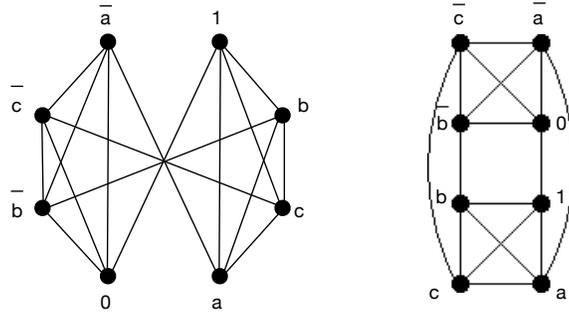


Figure 7: Complementary graph of  $T_3$

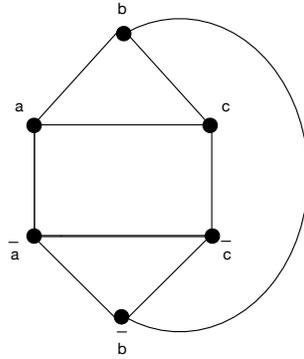


Figure 8: Incomparability graph of  $T_3$

*Proof.* Let  $T_n$  be a Boolean lattice with  $2^n$  elements, and  $|\overline{U}|$  the number of edges of its complementary graph. To find the number  $|U'|$  of edges of the incomparability graph of  $T_n$ , we must remove from the graph  $G(X, \overline{U})$ , complementary to  $T_n$ , the ordered pairs of non-adjacent vertices in  $T_n$ . These ones are in bijection with the successive  $n$ -faces of the  $n$ -cube associated with  $T_n$ . Let  $k$  be the dimension of the faces. Given a fixed  $n$ , for each value of  $k$ , the number of these faces is  $N_{n,k} = 2^{n-k} \binom{n}{k}$  (see [Coxeter 73], 122; [Sommerville 29], 29). We then have:

$$|U'| = |\overline{U}| - \sum N_{n,k}.$$

□

Some special formulas for computing the number of faces of a  $n$ -cube:

- 2-sided:  $n(n-1)2^{n-3}$ ;

	$a$	$c$	$\bar{b}$	$b$	$\bar{c}$	$\bar{a}$
$a$	0	1	0	1	0	1
$c$	1	0	0	1	1	0
$\bar{b}$	0	0	0	1	1	1
$b$	1	1	1	0	0	0
$\bar{c}$	0	1	1	0	0	1
$\bar{a}$	1	0	1	0	1	0

Table 7: Adjacency matrix of the incomparability graph of  $T_3$

- 3-sides:  $n(n-1)\frac{n-2}{3}2^{n-4}$ ;
- 4-faces:  $n(n-1)(n-2)\frac{n-3}{3}2^{n-7}$ ;

In the general case, it suffices to calculate  $N_{n,k}$ .

*Examples:*

For the lattice  $T_3$  (3-cube), we have, by the above formulas, 6 2-faces and 1 3-face, which gives  $6+1=7$ . As  $|\overline{U}|=16$ , we check that we get  $|U'|=16-7=9$ .

For the lattice  $T_4$  (4-cube), we have, by the same formulas, 24 2-faces, 8 3-faces and 1 4-faces, which gives  $24+8+1=33$ . Knowing that here  $|\overline{U}|=88$ , we get  $|U'|=88-33=55$ .

For the lattice  $T_5$  (5-cube), we would obtain: 80 2-faces, 40 3-faces, 10 4-faces and 1 5-face, which gives:  $80+40+10+1=131$ . Knowing that here  $|\overline{U}|=416$ , we get  $|U'|=416-131=285$ .

Finally, for the lattice  $T_6$  (6-cube), we would obtain: 240 2-faces, 160 3-faces, 60 4-faces and 12 5-faces and 1 6-faces, which gives:  $240+160+60+12+1=473$ . Knowing that here  $|\overline{U}|=1824$ , we get  $|U'|=1824-473=1351$ .

Here is (see Table 8) a summary of the characteristics of the  $T_n$ , the complementary graphs and the corresponding incomparability graphs (appear successively in the matrix, the lattices  $T_n$ , the corresponding product Boolean sets  $B^n$ , the number of vertices  $|X|$  of the associated graphs, the number of edges  $|U|$ , the number of edges  $\overline{U}$  of the complementary graphs, finally, the number of vertices  $|X'|$  and the number of edges  $U'$  of the corresponding incomparability graphs.

Note finally that we can always get the number of edges of a comparability graph in another way. If we call  $V$  the number of edges corresponding to non explicit order relations in a poset  $P$ , it is clear that the number of edges  $|E|$  of the comparability

$T_n$	$B^n$	$ X $	$ U $	$ \bar{U} $	$ X' $	$ U' $
$T_2$	$\{0, 1\}^2$	4	4	2	2	1
$T_3$	$\{0, 1\}^3$	8	12	16	6	9
$T_4$	$\{0, 1\}^4$	16	32	88	14	55
$T_5$	$\{0, 1\}^5$	32	80	416	30	285
$T_6$	$\{0, 1\}^6$	64	192	1824	62	1351

Table 8: Characteristics of some lattices, complementarity graphs and incomparability graphs

graph of  $P$  is:

$$|E| = |U| + |V|.$$

And we have also, of course (see prop. 6.1):

$$|U| + |V| = \binom{n}{2} - N.$$

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