Output Feedback H_∞ Control Design For Polynomial T–S Systems: A Novel SOS Approach

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***Abstract*—This paper deals with the design of a static output feedback (SOF) H_∞ controller for a polynomial Takagi-Sugeno (T–S) system in the continuous-time setting. The sought SOF controller must guarantee asymptotic stability and an H_∞ performance level of the closed-loop system. Sufficient conditions for the existence of such SOF controller are derived, in the form of sum-of-squares (SOS) by dint of a polynomial Lyapunov function (PLF). These conditions do not include neither an iterative algorithm nor an equality constraint which leads to a more tractable solution. The proposed gain is obtained by means of less conservative conditions than existing ones. This is illustrated through some numerical examples which demonstrate, at the same time, the applicability of the suggested design approach.**

***Index Terms*—Polynomial Takagi-Sugeno (T–S) systems, Static Output Feedback (SOF), H_∞ controllers, Sum-Of-Squares (SOS)**

I. Introduction

Takagi-Sugeno (T–S) fuzzy models [1] are used for modeling systems with nonlinearities since they have the capability to encapsulate their dynamics by a set of linear models interpolated by some membership functions. Various strategies for T–S fuzzy systems control design have been explored in the literature in both continuous and discrete-time settings [2]–[5].

On another note, the static output feedback (SOF) control strategy for T–S fuzzy systems is still a challenging issue in control theory due to its practical simplicity when compared to dynamic output feedback control [6]–[13]. Particularly, a numerical procedure based on an equality constraint was established in [6]. Recently, in [11], sufficient conditions for the existence of a SOF controller have been obtained using the concept of the decay rate in the quadratic Lyapunov function.

Alternatively, polynomial systems can be used to deal with a wider class of non-linear systems. In addition, methods based on the sum-of-squares (SOS) make it possible to investigate the stability of polynomial systems for a larger range of problems than LMI-based approaches [14]–[16]. Based on the polynomial fuzzy model, the stability analysis of the polynomial system is invetigated through a polynomial Lyapunov method. Moreover, including an equality constraint, the problem of SOF controllers was solved by means of some bilinear matrix inequalities (BMIs) [13]. An iterative approach based on SOS decomposition was introduced in [17], [18] to solve H_∞ SOF controller design problem for polynomial systems. Unfortunately, the conservativeness of the underlying conditions increases with regards to the complexity of the nonlinear system considered. The main objective of the current paper is to re-investigate the SOF control design for polynomial T–S fuzzy systems without constraints on the system state-space matrices which results in reducing the conservatism of some existing results.

Hence, the present paper proposes new sufficient conditions for the SOF H_∞ control design of continuous-time polynomial systems. These conditions ensure that the $\mathcal{L}_2$ gain from the disturbance input to the controlled output is less than a prescribed value. The major benefit of the proposed method is that it avoids the optimization under BMI constraints while minimizing the conservatism of existing methods.

The remainder of this paper is organized as follows: Section II presents a system description and some preliminaries. The main results are presented in the Section III, whereas the Section IV provides some examples to show the validity and benefits of the suggested methods. The conclusion takes place in Section V.

II. Problem statement

We consider the continuous polynomial T–S fuzzy mode which is described by the following fuzzy model

Plant Rule i: IF $\sigma_1(t)$ is μ_{i1} AND ...AND $\sigma_s(t)$ is μ_{is} THEN

$$\begin{cases} \dot{x}(t) = A_i(x(t))\tilde{x}(x(t)) + B_{1i}(x(t))\omega(t) + B_{2i}(x(t))u(t) \\ z(t) = C_{1i}(x(t))\tilde{x}(x(t)) + D_{11i}(x(t))\omega(t) \\ \qquad + D_{12i}(x(t))u(t) \\ y(t) = C_{2i}(x(t))\tilde{x}(x(t)) + D_{21i}(x(t))\omega(t) \end{cases} \tag{1}$$

where $\sigma(t) = [\sigma_1(t) \quad \sigma_2(t) \ldots \sigma_s(t)]$ are known premise variables, μ_{ij} ($i = 1, 2, ..., r$; $j = 1, 2, ..., s$) are fuzzy sets, r is the number of If-Then rules, $A_i(x(t))$, $B_{1i}(x(t))$, $B_{2i}(x(t))$, $C_{1i}(x(t))$, $D_{1i}(x(t))$, $D_{12i}(x(t))$, $C_{2i}(x(t))$ and $D_{21i}(x(t))$ are polynomial matrices in $x(t)$ with appropriate dimensions. $x(t) \in \mathbb{R}^{n_x}$ is the state vector; $y(t) \in \mathbb{R}^{n_y}$ denotes the measurement output, $u(t) \in \mathbb{R}^{n_u}$ is the control input,

$z(t) \in \mathbb{R}^{n_z}$ is the controlled output, $\omega(t) \in \mathbb{R}^{n_w}$ denotes the disturbance, that belongs to $\mathcal{L}_2[0,\infty)$.

The defuzzification process of model (1) can be represented as

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \eta_i(\sigma(t))[A_i(x(t))\tilde{x}(x(t)) + B_{1i}(x(t))\omega(t) \\ \quad + B_{2i}(x(t))u(t)] \\ z(t) = \sum_{i=1}^{r} \eta_i(\sigma(t))[C_{1i}(x(t))\tilde{x}(x(t)) + D_{11i}(x(t))\omega(t) \\ \quad + D_{12i}(x(t))u(t)] \\ y(t) = \sum_{i=1}^{r} \eta_i(\sigma(t))[C_{2i}(x(t))\tilde{x}(x(t)) + D_{21i}(x(t))\omega(t)] \end{cases} \tag{2}$$

where

$$\eta_i(\sigma(t)) = \frac{w_i(\sigma(t))}{\sum_{i=1}^{r} w_i(\sigma(t))},\ w_i(\sigma(t)) = \prod_{j=1}^{s} \mu_{ij}(\sigma_j(t))$$

$\mu_{ij}(\sigma_j(t))$ denotes the grade of membership of $\sigma_j(t)$ in μ_{ij} and $w_i(\sigma(t))$ represents the weight of the i^{th} rule.

It is clear that fuzzy weighting functions $\eta_i(\sigma(t))$ satisfy

$$\begin{cases} \sum_{i=1}^{r} \eta_i(\sigma(t)) = 1 \\ 0 \le \eta_i(\sigma(t)) \le 1 \end{cases} \tag{3}$$

According to the concept of parallel distributed compensation (PDC) [15], the polynomial SOF controller is described as follows:

Controller Rule i : IF $\sigma_1(t)$ is μ_{i1} AND ...AND $\sigma_s(t)$ is μ_{is} THEN

$$u(t) = F_i(x(t))y(t) \tag{4}$$

with $i = 1, 2, ...r$, and $F_i(x(t))$ is the polynomial matrices of appropriate dimensions to be determined. The overall controller can be represented by

$$\begin{aligned} u(t) &= \sum_{i=1}^{r} \eta_i(\sigma(t))F_i(x(t))y(t) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} \eta_i\eta_j F_i(x(t))[C_{2j}(x(t))\tilde{x}(x(t)) \\ &\quad + D_{21j}(x(t))\omega(t)] \end{aligned} \tag{5}$$

Combining (5) and (2), the closed-loop system is given by

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} \eta_i\eta_j\eta_l[\bar{A}_{ijl}(x(t))\tilde{x}(x(t)) \\ \qquad\qquad + \bar{B}_{ijl}(x(t))\omega(t)] \\ z(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} \eta_i\eta_j\eta_l[\bar{C}_{ijl}(x(t)))\tilde{x}(x(t)) \\ \qquad\qquad + \bar{D}_{ijl}(x(t))\omega(t))] \end{cases} \tag{6}$$

where

$$\eta_i = \eta_i(\sigma(t)),\ \ \eta_j = \eta_j(\sigma(t)),\ \ \eta_l = \eta_l(\sigma(t)).$$

and

$$\begin{aligned} \bar{A}_{ijl}(x(t)) &= A_i(x(t)) + B_{2i}(x(t))F_j(x(t))C_{2l}(x(t)) \\ \bar{B}_{ijl}(x(t)) &= B_{1i}(x(t)) + B_{2i}(x(t))F_j(x(t))D_{21l}(x(t)) \\ \bar{C}_{ijl}(x(t)) &= C_{1i}(x(t)) + D_{12i}(x(t))F_j(x(t))C_{2l}(x(t)) \\ \bar{D}_{ijl}(x(t)) &= D_{11i}(x(t)) + D_{12i}(x(t))F_j(x(t))D_{21l}(x(t)) \end{aligned}$$

The objective of SOF H_∞ polynomial control is then to find polynomial gains $F_i(x(t))$ $i = 1, \ldots, r$ such that the system (6) is asymptotically stable, and the output $z(t)$ satisfies the following condition (under zero initial conditions):

$$\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty \omega^T(t)\omega(t)dt \tag{7}$$

Remark 1: In [15], [16], the stabilization conditions were investigated by the SOS approach, for the polynomial T-S fuzzy system. These results have focused on state feedback control problems. Whereas, in this paper, we consider the SOF H_∞ controller case.

The following lemmas will be used in the sequel.

Lemma 1: [19] For matrices $\mathbb{T}$, Λ, L, and Ξ with appropriate dimensions and a scalar β, the inequality

$$\mathbb{T} + \Xi^T\Lambda^T + \Lambda\Xi < 0 \tag{8}$$

is fulfilled if the following condition holds:

$$\begin{bmatrix} \mathbb{T} & \bullet \\ \beta\Lambda^T + L\Xi & -\beta L - \beta L^T \end{bmatrix} < 0$$

with $\bullet$ abbreviates the off-diagonal block of the symmetric matrix represented block-wise.

Proposition 1: [20] Let $g(x)$ be a polynomial in $x \in \mathbb{R}^n$ of degree $2d$. Let $W(x)$ be a column vector whose entries are all monomials in x with degree no greater than d. Then, $g(x)$ is said to be SOS if and only if there exists a positive semi-definite matrix Q such that

$$g(x) = W(x)^T Q W(x) \tag{9}$$

Definition 1: [21] A multivariate polynomial $g(x)$, for $x \in \mathbb{R}^N$ is a SOS if there exist polynomials $g_i(x)$, $i = 1, \ldots, n$ such that

$$g(x) = \sum_{i=1}^{n} g_i^2(x) \tag{10}$$

This implies $g(x) \ge 0$ for any $x \in \mathbb{R}^n$.

Remark 2: In [17] and [18], iterative approaches based on SOS decomposition have been presented to determine the SOF controller. Authors in [22] consider only the stability of the autonomous polynomial systems by SOS approach. In [22], the SOS method is used to design the SOF controller for polynomial systems. The stabilization of polynomial systems based on SOS is investigated in [15], [16]. We emphasize in the current paper, that none of the above references have dealt with, the H_∞ norm performance.

III. Main result

In this section, the objective is to design a SOF H_∞ controller, that stabilizes the polynomial T–S fuzzy system (6). To simplify the notations, function arguments may be omitted when their meaning is straightforward. Hence, in the sequel x is used instead of $x(t)$ and $\tilde{x}(x)$ instead of $\tilde{x}(x(t))$. In addition, $A_i^k(x)$ indicates the k^{th} row of A_i, $K = \{k_1,\ k_2, ...,\ k_m\}$ indicates the row indices of $B_i(x)$ whose corresponding row is zero, and $\hat{x} = (x_{k_1}, x_{k_2}, .., x_{k_m})$.

Theorem 1: Let $\beta, \lambda > 0$ be some given scalars. System (6) is asymptotically stable with H_∞ performance γ, if there exist symmetric polynomial matrices $P(\hat{x})$, and polynomial matrices $N_i(x)$, $G(x)$ and $L(x)$, such that (11)-(14) hold where $\varepsilon_i(x)$, $\varepsilon_{ij}(x)$, and $\varepsilon_{ijl}(x)$, are non-negative polynomials such that $\varepsilon_i(x) > 0$, for $x \neq 0$, $\varepsilon_{ij}(x) \geq 0$, $\varepsilon_{ijl}(x) \geq 0$, for all x, $i, j, l = 1, 2, \ldots, r$.

$$\upsilon_1^T (P(\hat{x}) - \varepsilon_1(x)I)\upsilon_1 \ \ is \ \ SOS \tag{11}$$

$$-\upsilon_2^T \Big(\phi_{iii}(x) + \varepsilon_i(x)I\Big)\upsilon_2 \ \ is \ \ SOS \qquad i = 1, 2, \ldots, r \tag{12}$$

$$\begin{aligned}&-\upsilon_2^T \Big(\phi_{iij}(x) + \phi_{iji}(x) + \phi_{jii}(x) + \varepsilon_{ij}(x)I\Big)\upsilon_2 \ \ is \ \ SOS \\ &\qquad 1 \leq i \neq j \leq r\end{aligned} \tag{13}$$

$$\begin{aligned}&-\upsilon_2^T \Big(\phi_{ijl}(x) + \phi_{ilj}(x) + \phi_{jil}(x) + \phi_{jli}(x) + \phi_{lij}(x) \\ &\quad + \phi_{lji}(x) + \varepsilon_{ijl}(x)I\Big)\upsilon_2 \ \ is \ \ SOS \ \ 1 \leq i \neq j \neq l \leq r\end{aligned} \tag{14}$$

where

$$\phi_{ijl}(x) = \begin{bmatrix} \Upsilon_{ijl} & \bullet \\ \Gamma_{ijl} & R \end{bmatrix}$$

$$\Upsilon_{ijl} = \begin{bmatrix} \Psi_{ijl} & \bullet & \bullet \\ \Upsilon_{21ijl} & -\gamma^2 I & \bullet \\ \Upsilon_{31ijl} & \Upsilon_{32ijl} & I - G(x) - G^T(x) \end{bmatrix} \tag{15}$$

$$\begin{aligned}\Psi_{ijl} &= A_i^T(x)T^T(x)P(\hat{x}) + P(\hat{x})T(x)A_i(x) \\ &\quad + \sum_{k\in K} \frac{\partial\ P(\hat{x})}{\partial x_k} A_i^k(x)\tilde{x}(x) + T(x)B_{2i}(x)N_j(x)C_{2l}(x) \\ &\quad + C_{2l}^T(x)N_j^T(x)B_{2i}^T(x)T^T(x) \\ \Upsilon_{21ijl} &= B_{1i}^T(x)T^T(x)P(\hat{x}) + D_{21l}^T(x)N_j^T(x)B_{2i}^T(x)T^T(x) \\ \Upsilon_{31ijl} &= G(x)C_{1i}(x) + \lambda D_{12i}(x)N_j(x)C_{2l}(x) \\ \Upsilon_{32ijl} &= G(x)D_{11i}(x) + \lambda D_{12i}(x)N_j(x)D_{21l}(x)\end{aligned}$$

$$\Gamma_{ijl} = \begin{bmatrix} \Gamma_{11ijl} & \Gamma_{12jl} & \Gamma_{13i} \end{bmatrix}$$

$$\begin{aligned}\Gamma_{11ijl} &= \beta(B_{2i}^T(x)T^T(x)P(\hat{x}) - L^T(x)B_{2i}^T(x)T^T(x)) \\ &\quad + N_j(x)C_{2l}(x) \\ \Gamma_{12jl} &= N_j(x)D_{21l}(x) \\ \Gamma_{13i} &= \beta(D_{12i}^T(x)G^T(x) - \lambda L^T(x)D_{12i}^T(x)) \\ R &= -\beta L(x) - \beta L^T(x)\end{aligned}$$

Proof 1: Since the conditions in (12)-(14) hold, we can write

$$\begin{aligned}&\sum_{i=1}^{r} \eta_i^3 \phi_{iii}(x) + \sum_{i=1}^{r} \sum_{j=1, i\neq j}^{r} \eta_i^2 \eta_j (\phi_{iij}(x) + \phi_{iji}(x) + \phi_{jii}(x)) \\ &+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} \sum_{l=j+1}^{r} \eta_i\eta_j\eta_l(\phi_{ijl}(x) + \phi_{ilj}(x) + \phi_{jil}(x) + \\ &\phi_{jli}(x) + \phi_{lij}(x) + \phi_{lji}(x)) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} \eta_i\eta_j\eta_l\phi_{ijl}(x) < 0\end{aligned}$$

which is satisfied if

$$\phi_{ijl}(x) = \begin{bmatrix} \Upsilon_{ijl} & \bullet \\ \Gamma_{ijl} & R \end{bmatrix} < 0 \tag{16}$$

Lemma 1 along with

$$\Lambda_i(x) = \begin{bmatrix} P(\hat{x})T(x)B_{2i}(x) - T(x)B_{2i}(x)L(x) \\ 0 \\ G(x)D_{12i}(x) - \lambda D_{12i}(x)L(x) \end{bmatrix}$$

$$\Xi_{jl}(x) = L^{-1}(x) \begin{bmatrix} N_j(x)C_{2l}(x) & N_j(x)D_{21l}(x) & 0 \end{bmatrix}$$

induce that the inequality (16) leads to

$$\varphi_{ijl}(x) = \Upsilon_{ijl} + \Lambda_i(x)\Xi_{jl}(x) + \Xi_{jl}^T(x)\Lambda_i^T(x) < 0 \tag{17}$$

The latter inequality (17) can be rewritten as:

$$\varphi_{ijl}(x) = \Upsilon_{ijl} + \begin{bmatrix} \bar{\phi}_{11ijl} & \bullet & \bullet \\ \Upsilon_{21ijl} & 0 & \bullet \\ \Upsilon_{31ijl} & \Upsilon_{32ijl} & 0 \end{bmatrix} < 0 \tag{18}$$

where

$$\begin{aligned}\bar{\phi}_{11ijl} &= [P(\hat{x})T(x)B_{2i}(x) - T(x)B_{2i}(x)L(x)]L^{-1}(x) \\ &\quad \times N_j(x)C_{2l}(x) + C_{2l}^T(x)N_j^T(x)L^{-T}(x)[B_{2i}^T(x)T^T(x) \\ &\quad \times P(\hat{x}) - L^T(x)B_{2i}^T(x)T^T(x)] \\ \Upsilon_{21ijl} &= D_{2l}^T(x)N_j^T(x)L^{-T}(x)[B_{2i}^T(x)T^T(x)P(\hat{x}) \\ &\quad - L^T(x)B_{2i}^T(x)T^T(x)] \\ \Upsilon_{31ijl} &= [G(x)D_{12i}(x) - \lambda D_{12i}(x)L(x)] \\ &\quad L^{-1}(x)N_j(x)C_{2l}(x) \\ \Upsilon_{32ijl} &= [G(x)D_{12i}(x) - \lambda D_{12i}(x)L(x)] \\ &\quad L^{-1}(x)N_j(x)D_{21l}(x)\end{aligned}$$

Using the change of variable $F_j(x) = L^{-1}(x)N_j(x)$ and substituting (15) into (18) we obtain:

$$\varphi_{ijl}(x) = \begin{bmatrix} \phi_{11ijl} & \bullet & \bullet \\ \bar{B}_{ijl}^T(x)T^T(x)P(\hat{x}) & -\gamma^2 I & \bullet \\ G(x)\bar{C}_{ijl}(x) & G(x)\bar{D}_{ijl}(x) & \phi_{33} \end{bmatrix} < 0 \tag{19}$$

where

$$\begin{aligned}\phi_{11ijl} &= \bar{A}_{ijl}^T(x)T^T(x)P(\hat{x}) + P(\hat{x})T(x)\bar{A}_{ijl}(x) \\ &\quad + \sum_{k\in K} \frac{\partial\ P(\hat{x})}{\partial x_k} A_i^k(x)\tilde{x}(x) \\ \phi_{33} &= I - G(x) - G(x)^T\end{aligned}$$

Since $-(V-Q)Q^{-1}(V-Q)^T \leq 0$, $Q > 0$ implies that $-VQ^{-1}V^T \leq -V - V^T + Q$, multiplying (19) by $diag\{I, I, G^{-1}(x)\}$ on the left and its transpose on the right leads to

$$\bar{\varphi}_{ijl}(x) = \begin{bmatrix} \phi_{11ijl} & \bullet & \bullet \\ \bar{B}_{ijl}^T(x)T^T(x)P(\hat{x}) & -\gamma^2 I & \bullet \\ \bar{C}_{ijl}(x) & \bar{D}_{ijl}(x) & -I \end{bmatrix} < 0 \tag{20}$$

We introduce the polynomial Lyapunov function, as in [15], represented by

$$V(x) = \tilde{x}^T(x)P(\hat{x})\tilde{x}(x) \tag{21}$$

Taking the time derivative of $V(x)$ yields

$$\begin{aligned} \dot{V}(x) &= \dot{\tilde{x}}^T(x)P(\hat{x})\tilde{x}(x) + \tilde{x}^T(x)P(\hat{x})\dot{\tilde{x}}(x) \\ &+ \tilde{x}^T(x)\dot{P}(\hat{x})\tilde{x}(x) < 0 \end{aligned} \tag{22}$$

Computing $\dot{P}(\hat{x})$ as in [16], and using (6), we obtain,

$$\dot{V}(x) + z^T(x)z(x) - \gamma^2\omega^T(x)\omega(x) = \zeta^T(t)M(x)\zeta(t) \tag{23}$$

where

$$\begin{aligned} \zeta(t) &= \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix}, \quad M(x) = \begin{bmatrix} \phi_{11ijl} & \bullet \\ \bar{B}_{ijl}^T(x)T^T(x)P(\hat{x}) & -\gamma^2 I \end{bmatrix} \\ &\times \begin{bmatrix} \bar{C}_{ijl}(x)^T \\ \bar{D}_{ijl}(x)^T \end{bmatrix} \begin{bmatrix} \bar{C}_{ijl}(x) & \bar{D}_{ijl}(x) \end{bmatrix} \end{aligned} \tag{24}$$

Since (20) holds, by schur complement we obtain that $M(x) < 0$ which implies that,

$$\dot{V}(x) + z^T(x)z(x) - \gamma^2\omega^T(x)\omega(x) < 0 \tag{25}$$

Integrating both sides of the inequality (23) from 0 to ∞ yields

$$\begin{aligned} &V(x(\infty)) - V(x(0)) + \\ &\int_0^\infty (z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t))dt < 0 \end{aligned}$$

Thus, the zero initial condition leads to

$$\int_0^\infty (z^T(t)z(t))dt < \gamma^2 \int_0^\infty (\omega^T(t)\omega(t))dt$$

This ends the proof.

IV. Computer simulations

Example 1: This example presents the design of a SOF H_∞ controller for a system borrowed from [23]. This example was solved using the homogeneous polynomial matrices of arbitrary and independent degrees.

TABLE I
VALUES OF γ FOR EXAMPLE (1)

Methods	γ
[24] g=1	3.67
[24] g=2	1.01
[23] C1-T2 g=1	0.89
[23] C1-T2 g=2	0.87
[23] C1-T2 g=6	0.77
[23] C1-T2 g=10	0.71
Theorem 1, 2d=0	1.59
Theorem 1, 2d=2	0.70

Consider the two-rules T–S fuzzy system of the form (2), with the same data as in [23]:

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, \; B_{21} = \begin{bmatrix} 0 \\ 10 \end{bmatrix},$$
$$B_{22} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \; B_{11} = \begin{bmatrix} 0 \\ -0.25 \end{bmatrix}, \; B_{12} = B_{11},$$
$$C_{11} = \begin{bmatrix} 2 & -10 \\ 5 & -1 \end{bmatrix}, \; C_{12} = \begin{bmatrix} -3 & 20 \\ -7 & -2 \end{bmatrix}, \; D_{121} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$
$$D_{122} = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix}, \; D_{111} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, \; D_{112} = \begin{bmatrix} 0.35 \\ 0.5 \end{bmatrix},$$
$$C_{21} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; C_{22} = C_{21}, \; D_{211} = 0.1, \; D_{212} = D_{211}$$

The fuzzy membership functions considered are

$$\begin{aligned} \eta_1(\sigma(t)) &= (1 + sin(x_1(t)))/2 \\ \eta_2(\sigma(t)) &= 1 - \eta_1(\sigma(t)) \end{aligned}$$

Based on the SOS conditions in Theorem 1, we obtain an H_∞ attenuation level of $\gamma_{min} = 1.59$ for $(2d = 0)$ with the following gains $F_1 = 0.5873$, $F_2 = -0.8347$. For $(2d = 2)$, we obtain $\gamma_{min} = 0.70$ and $F_1 = -0.12713x_1(t)^2 + 5.7446 \times 10^{-4}x_2(t)^2$, $F_2 = 0.18319x_1(t)^2 + 9.9959 \times 10^{-4}x_2(t)^2$.

Table I lists the values of γ_{min} obtained from [23], [24] and Theorem 1 in this paper for different degrees. It can be easily seen that the results obtained with Theorem 1 with $(2d = 2)$ outperform those based on C1-T2 in [23] with ($g = 1$, $g = 2$, $g = 6$, $g = 10$). It can also be seen that increasing the degree of the polynomial Lyapunov function from $2d = 0$ to $2d = 2$ reduces significantly the norm bound γ_{min} and hence improves the obtained performances.

In comparison with the results in [23], Theorem 1 advantages are twofold: the SOF H_∞ controller is obtained without any iterative procedure. It achieves better results (with (2d=2), $\gamma = 0.7$ is obtained), while the approach proposed in [23] achieves a norm bound of $\gamma = 0.71$ for a degree $g \geq 10$. Simulation results are presented in Fig. 1. The state trajectories $x_1(t)$ and $x_2(t)$ are obtained for the initial condition $x(0) = [1 \;\; 1.4]^T$ and $\omega(t) = 0$. It can be seen in Fig. 1 that when no disturbance affects the system (i.e., $\omega(t) = 0$) the closed-loop is asymptotically stable.

Example 2: Let us consider a nonlinear mass-spring-damper mechanical system borrowed for instance from [9] with the following dynamic equation

$$M\ddot{x}(t) + c_1\dot{x}(t) + c_2 x(t) = (1 + c_3\dot{x}^3(t))u(t) + \omega(t) \tag{26}$$

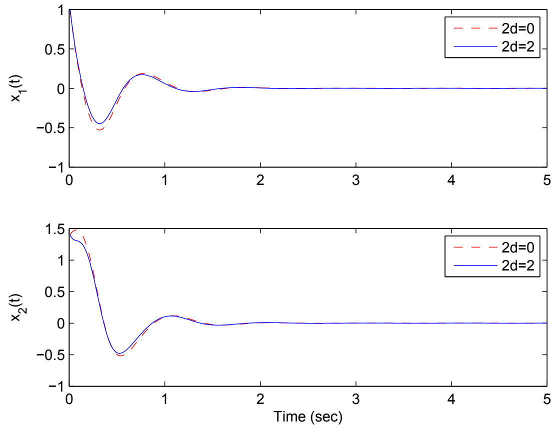

Fig. 1. Evolution of the closed-loop states in Example (1)

where M, $u(t)$ and $\omega(t)$ are the mass, the force, and the disturbance respectively. The parameters of the mechanical system are set to $M = 1$, $c_1 = 1$, $c_2 = 1.155$, $c_3 = 0.13$. Following the lines in [9], the states chosen are the velocity and the position that is $x_1(t) = \dot{x}(t)$ and $x_2(t) = x(t)$. We define also $z(t)$ and $y(t)$ as follows

$$\begin{aligned} z(t) &= \begin{bmatrix} 2x_2(t) \\ 2u(t) \end{bmatrix}, \\ y(t) &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ 2x_1(t) + x_2(t) \end{bmatrix}. \end{aligned} \tag{27}$$

If $x_1(t) \in [-1.5 \;\; 1.5]$, then the nonlinear mass-spring-damper (26) can be described by the following T–S model

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{2} \eta_1(x_1)(t)\{A_i x(t) + B_{1i}\omega(t) + B_{2i}u(t)\} \\ z(t) &= \sum_{i=1}^{2} \eta_1(x_1)(t)\{C_{1i}x(t) + D_{11i}\omega(t) + D_{12i}u(t)\} \\ y(t) &= \sum_{i=1}^{2} \eta_1(x_1)(t)\{C_{2i}x(t) + D_{21i}\omega(t)\} \end{aligned}$$

where

$$A_1 = \begin{bmatrix} -1 & -1.155 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & -1.155 \\ 1 & 0 \end{bmatrix},$$

$$B_{11} = B_{12}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_{21} = \begin{bmatrix} 1.4387 \\ 0 \end{bmatrix}, B_{22} = \begin{bmatrix} 0.5613 \\ 0 \end{bmatrix},$$

$$C_{11} = C_{12} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, D_{121} = D_{122} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$C_{21} = C_{22}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, D_{111} = D_{112} = D_{211} = D_{212} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where the membership functions are

$$\eta_1(x_1(t)) = 0.5 + \frac{x_1^3(t)}{6.75}, \qquad \eta_2(x_1(t)) = 1 - \eta_1(x_1(t))$$

The obtained H_∞ norm bounds and the associated SOF gains are summarized in Table II. The H_∞ norm bound

TABLE II

VALUES OF THE NORM BOUND γ AND THE OBTAINED GAINS F_i FOR EXAMPLE 2,

Methods	γ	F_i
Theorem 1 in [9]	3	$F_1 = [0.093 \;\; -0.095]$ $F_2 = [0.2309 \;\; -0.2479]$
Theorem 1	2.0838	$F_1 = [-0.07515 \;\; -0.01693]$ $F_2 = [-0.06447 \;\; -0.01077]$

obtained with the proposed approach is smaller than the one obtained with the LMI based conditions in [9]. It can also be noted that a simpler controller of degree $2d = 0$ can be used for practical implementation without a significant loss of performance.

Figures 2 and 3 show the evolutions of the state response and the control input of the closed-loop system respectively, from an initial condition $x(0) = [-1 \;\; -1.4]^T$ and $\omega(t) = 0$ when the gains obtained by means of the proposed result are used. The plot of the ratio $\rho(t) = \sqrt{\frac{\int_0^t z^s(s)z(s)ds}{\int_0^t \omega^T(s)\omega(s)ds}}$ is shown in Fig. 4, from the initial condition $x(0) = [0; 0]^T$ when

$$\omega(t) = \begin{cases} 2 & 0 \le t \le 1 \\ (2sin(2\pi t))/t & t > 1 \end{cases}$$

It is clear that this attenuation estimation is smaller than the guaranteed value of $\gamma_{min} = 2.0838$. In summary from the results in Figs. 2-4, it is possible to conclude that the closed-loop system is asymptotically stable and provides the required level of disturbance attenuation. This shows that the proposed method is effective to real control problems.

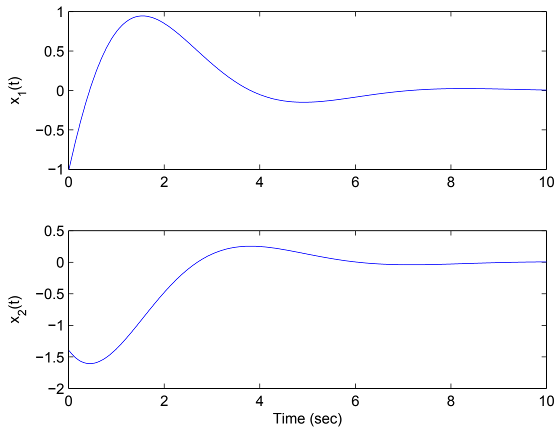

Fig. 2. Evolution of the closed loop states for Example (2)

V. CONCLUSION

This paper has presented a static output feedback controller that achieves asymptotic stability and optimizes the $\mathcal{L}_2-$ gain for a class of polynomial T–S systems. Using polynomial Lyapunov functions, the SOF H_∞ controller is constructed by means of some less conservative conditions than existing ones.

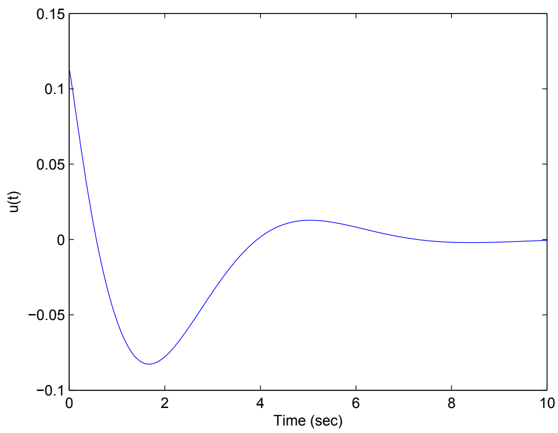

Fig. 3. Evolution of the control inputs for Example (2)

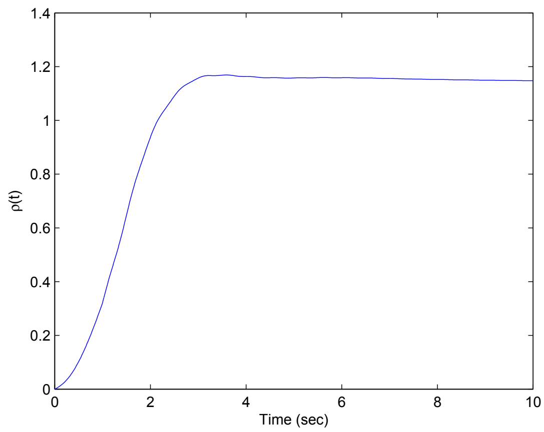

Fig. 4. The ratio $\rho(t) = \sqrt{\frac{\int_0^t z^T(s)z(s)ds}{\int_0^t \omega^T(s)\omega(s)ds}}$ for Example (2)

It must be emphasized that these conditions include neither equality constraints nor iterative algorithms which leads to a tractable solution. Some numerical examples have been given to demonstrate the advantages and the applicability of the proposed approach.

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