



HAL
open science

Finite element approximation of a singular elliptic problem with discontinuous coefficients

Nada El Berdan, Ahmad Makki, Morgan Pierre

► **To cite this version:**

Nada El Berdan, Ahmad Makki, Morgan Pierre. Finite element approximation of a singular elliptic problem with discontinuous coefficients. 2020. hal-02971693

HAL Id: hal-02971693

<https://hal.science/hal-02971693>

Preprint submitted on 19 Oct 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FINITE ELEMENT APPROXIMATION OF A SINGULAR ELLIPTIC PROBLEM WITH DISCONTINUOUS COEFFICIENTS

NADA EL BERDAN¹, AHMAD MAKKI², AND MORGAN PIERRE²

ABSTRACT. We propose a finite element discretization of a singular elliptic problem with discontinuous coefficients. We use a “regularize then discretize” approach. We show that our method converges in 1, 2 and 3 space dimensions. We also perform numerical simulations in two space dimensions with FreeFem++, using an adaptive mesh strategy to deal with the singularity. The simulations confirm the validity of our approach.

Keywords: finite element method, singular elliptic problem, monotone operator, adaptive mesh refinement.

Mathematics Subject Classification: 65N30, 35J70, 47H05, 65N50.

1. INTRODUCTION

We consider the following singular elliptic problem,

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = \frac{c(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here, Ω is a bounded domain of \mathbb{R}^d , $\gamma > 0$, the function c satisfies

$$c \in L^\infty(\Omega), \quad c \geq 0, \quad c \neq 0, \quad (1.2)$$

and $A = (a_{ij})_{1 \leq i, j \leq d}$ is a bounded elliptic matrix, i.e.

$$a_{ij} \in L^\infty(\Omega), \quad \text{for all } 1 \leq i, j \leq d, \quad (1.3)$$

and there exists $\alpha > 0$ such that

$$A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad (1.4)$$

for all $\xi \in \mathbb{R}^d$, for almost every $x \in \Omega$.

Problem (1.1) is a generalization of the singular elliptic problem

$$\begin{cases} -\Delta u = \frac{c(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

which has been extensively studied, along with many of its variants such as (1.1) (see, e.g., [2, 7, 8, 9, 13, 14] and references therein). Questions related to existence,

¹LEBANESE UNIVERSITY, NABATIEH, LEBANON.

²LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS, UNIVERSITÉ DE POITIERS, CNRS, F-86073 POITIERS, FRANCE

uniqueness and regularity of solutions have been investigated. Such problems arise in the modeling of boundary layers [4, 5, 11].

If A , c and $\partial\Omega$ are regular enough, it is well-known that (1.1) has a unique classical solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, which cannot generally belong to $C^2(\bar{\Omega})$. For instance, the function

$$u(x) = \sqrt{1 - x^2} \quad (1.6)$$

satisfies (1.5) on the interval $(-1, 1)$ with $c(x) = 1$ and $\gamma = 3$, that is

$$-u''(x) = \frac{1}{u^3} \text{ in } (-1, 1), \quad u > 0 \text{ in } (-1, 1) \text{ and } u(-1) = u(1) = 0. \quad (1.7)$$

We note that the function defined by (1.6) has the regularity $C^{1/2}([-1, 1])$ and does not belong to $H_0^1(-1, 1)$. These two remarks have been generalized to problem (1.5). In particular, if Ω is of class C^2 , if c is Hölder continuous and bounded from below by a positive constant, a result of Gui and Lin [14] shows that

$$\text{If } 0 < \gamma < 1, \text{ then } u \in C^{1,\nu}(\bar{\Omega}), \text{ where } \nu = 1 - \gamma, \quad (1.8)$$

$$\text{If } \gamma = 1, \text{ then } u \in C^\nu(\bar{\Omega}), \text{ for all } \nu \in (0, 1), \quad (1.9)$$

$$\text{If } \gamma > 1, \text{ then } u \in C^\nu(\bar{\Omega}), \text{ where } \nu = \frac{2}{1 + \gamma}. \quad (1.10)$$

Under the same assumptions, the solution u to (1.5) belongs to $H_0^1(\Omega)$ if and only if $\gamma < 3$ (see [18] and [1, Lemma 2.1]).

In [1], Barrett and Schneider performed a very nice a priori estimate analysis for a P^1 finite element approximation of (1.5) in 1, 2 and 3 space dimensions. In particular, they obtained a priori estimates in H^1 norm for $\gamma < 3$ and L^∞ norm for all $\gamma > 0$. They also provided extensive numerical simulations in space dimension 1. They used uniform subdivisions for the computations, but they pointed out in their conclusion that adaptive meshes would be more appropriate. This is illustrated by the example (1.6) or by the result (1.10): since the C^ν regularity occurs near the boundary of Ω , it seems natural to use a mesh which is refined near the boundary.

Our purpose here is to propose a finite element approximation of problem (1.1). This is done on a “regularize then discretize” basis. We show that our method converges for $d = 1, 2$ and 3 and we also perform numerical simulations in two space dimensions with **FreeFem++**. In order to deal with the singularity, we use an adaptive mesh strategy.

Our paper is organized as follows. We first introduce in Section 2 a regularized version of problem (1.1) by using a local regularization of the nonlinearity $s \mapsto 1/s^\gamma$, as in [1]. A monotonicity argument shows that the regularized problem has a unique solution u_ε in $H_0^1(\Omega)$ (Proposition 2.2). By adapting the approach in [2, 9], we show in Section 3 that u_ε converges to a solution u to problem (1.1) in appropriate sense as ε goes to 0. In particular, when $\gamma \leq 1$, $u_\varepsilon \rightarrow u$ weakly in $H_0^1(\Omega)$ (Theorem 3.4) and when $\gamma > 1$, $u_\varepsilon \rightarrow u$ strongly in $L^2(\Omega)$ (Theorem 3.7). Then, we propose in Section 4 a discretization of the regularized problem. For a fixed ε , we prove that the solution to the discretized problem converges strongly in $H_0^1(\Omega)$ to u_ε as the mesh size goes to 0 (Theorem 4.5). The assumptions on the family of triangulations are quite general. In Section 5, several numerical simulations show the validity of our approach.

2. THE REGULARIZED PROBLEM

Throughout the paper, we assume that $\gamma > 0$, c satisfies (1.2) and A satisfies (1.3)-(1.4). In Sections 2 and 3, we only assume that Ω is a bounded domain of \mathbb{R}^d where d is any positive integer. In Section 4, we make further assumptions on Ω and d .

For the regularized problem, we work in the Sobolev space $H_0^1(\Omega)$ equipped with the Hilbertian norm

$$\|v\|_1 = \left(\int_{\Omega} |\nabla v|^2 dx \right)^{1/2},$$

which is equivalent to the standard H^1 norm. The norm in $L^2(\Omega)$ is denoted $\|\cdot\|_0$. The first eigenvalue of $-\Delta$ is

$$\lambda_1 = \inf_{v \in H_0^1(\Omega), v \neq 0} \frac{\|v\|_1^2}{\|v\|_0^2} > 0. \quad (2.1)$$

The duality product between $H_0^1(\Omega)$ and its topological dual $H^{-1}(\Omega)$ is denoted $\langle \cdot, \cdot \rangle$.

Following [1], we define the following C^1 regularization f_ε of $f(s) = s^{-\gamma}$, for each $\varepsilon \in (0, 1]$:

$$f_\varepsilon(s) = \begin{cases} f(s) & \text{if } s > \varepsilon, \\ f(\varepsilon) + f'(\varepsilon)(s - \varepsilon) & \text{if } s \leq \varepsilon. \end{cases} \quad (2.2)$$

We note that f_ε is decreasing and positive on \mathbb{R} .

Remark 2.1. Another regularization of $f(s)$ could be

$$\tilde{f}_\varepsilon(s) = f(\varepsilon + s). \quad (2.3)$$

However, as pointed out in [1], f_ε performs numerically better. The reason is that \tilde{f}_ε is a global regularization which always effects the practical computation, whereas f_ε is a local regularization which has a better computational effect, cf. Section 5.2.

The regularized version of problem (1.1) reads

$$\begin{cases} -\operatorname{div}(A(x)\nabla u_\varepsilon) = c(x)f_\varepsilon(u_\varepsilon) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

We have

Proposition 2.2. *Problem (2.4) has a unique solution $u_\varepsilon \in H_0^1(\Omega)$. Moreover, $u_\varepsilon > 0$ a.e. in Ω .*

Proof. We use a monotonicity argument. We consider the nonlinear operator

$$B_\varepsilon v = Lv - c(\cdot)f_\varepsilon(v), \text{ where } Lv = -\operatorname{div}(A\nabla v). \quad (2.5)$$

We will show that

- i) B_ε is a nonlinear continuous operator from $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$.
- ii) B_ε is strictly monotone.
- iii) B_ε is coercive, i.e.

$$\lim_{\|v\|_1 \rightarrow +\infty} \frac{\langle B_\varepsilon v, v \rangle}{\|v\|_1} \rightarrow +\infty.$$

It follows from the theory of maximal monotone operators (see, e.g. [23, Theorem 26.A]) that there exists a unique $u_\varepsilon \in H_0^1(\Omega)$ such that $B_\varepsilon u_\varepsilon = 0$. The strong maximum principle [12, Theorem 8.19] implies that $u_\varepsilon > 0$ on Ω .

We start with i). The linear operator L which is defined by duality through

$$\langle Lv, w \rangle = \int_{\Omega} A(x) \nabla v \cdot \nabla w dx$$

is continuous from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ since $A \in L^\infty(\Omega; \mathbb{R}^{d \times d})$. By definition (2.2) of f_ε , we see that f_ε is bounded on $[\varepsilon, \infty)$ and affine on $(-\infty, \varepsilon]$. Thus, there exist positive constants C_1, C_2 (which depend on ε) such that

$$|f_\varepsilon(s)| \leq C_1 |s| + C_2 \quad (s \in \mathbb{R}). \quad (2.6)$$

As a consequence, the operator $v \mapsto a(\cdot) f_\varepsilon(v)$ is continuous from $L^2(\Omega)$ into $L^2(\Omega)$ (see, e.g., [17, 16.1 Lemme]). This shows assertion i).

Next, we see that for all $v, \tilde{v} \in H_0^1(\Omega)$ with $v \neq \tilde{v}$, we have

$$\begin{aligned} \langle B_\varepsilon v - B_\varepsilon \tilde{v}, v - \tilde{v} \rangle &= \int_{\Omega} (A(x) \nabla(v - \tilde{v})) \cdot \nabla(v - \tilde{v}) \\ &\quad - \int_{\Omega} a(x) [f_\varepsilon(v) - f_\varepsilon(\tilde{v})] (v - \tilde{v}) dx \\ &\geq \alpha \|v - \tilde{v}\|_1^2 \\ &> 0, \end{aligned}$$

where in the second line, we used (1.4). This proves assertion ii).

For the coercivity of B_ε , we note that

$$-s f_\varepsilon(s) = \begin{cases} \gamma s^2 \varepsilon^{-(\gamma+1)} - s(1+\gamma) \varepsilon^{-\gamma} & \text{if } s \leq \varepsilon, \\ -s^{1-\gamma} & \text{if } s > \varepsilon, \end{cases} \quad (2.7)$$

and we consider three cases, namely $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$.

If $\gamma > 1$, then we have

$$-s f_\varepsilon(s) \geq -C_3, \quad \forall s \in \mathbb{R}, \quad (2.8)$$

for some positive constant C_3 which depends on ε . From (1.4), we deduce that

$$\begin{aligned} \langle B_\varepsilon v, v \rangle &= \int_{\Omega} A(x) \nabla v \cdot \nabla v dx - \int_{\Omega} c(x) f_\varepsilon(v(x)) v(x) dx \\ &\geq \alpha \|v\|_1^2 - \|c\|_{L^\infty(\Omega)} |\Omega| C_3, \end{aligned} \quad (2.9)$$

for all $v \in H_0^1(\Omega)$.

If $\gamma \leq 1$, the minimum value of $-s f_\varepsilon(s)$ on $(-\infty, \varepsilon]$ is attained at $s_\varepsilon = \varepsilon$. Thus,

$$-s f_\varepsilon(s) \geq -s_\varepsilon f_\varepsilon(s_\varepsilon) = -\varepsilon^{1-\gamma} \geq -1, \quad \forall s \in (-\infty, \varepsilon], \quad (2.10)$$

since $\varepsilon \in (0, 1]$. If $\gamma = 1$, the estimate (2.10) holds for all $s \in \mathbb{R}$ (see (2.7)) and we recover the estimate (2.9) but with a constant $C_3 = 1$ independent of ε .

If $\gamma < 1$, we deduce from Young's inequality when $s > \varepsilon$ and from (2.10) when $s \leq \varepsilon$ that for all $\beta > 0$, there exists $C_\beta > 0$ independent of ε such that

$$-s f_\varepsilon(s) \geq -\beta s^2 - C_\beta \quad (s \in \mathbb{R}). \quad (2.11)$$

Using (1.4) and (2.1), we obtain

$$\begin{aligned} \langle B_\varepsilon v, v \rangle &= \int_\Omega A(x) \nabla v \cdot \nabla v dx - \int_\Omega c(x) f_\varepsilon(v(x)) v(x) dx \\ &\geq \frac{\alpha}{2} \|v\|_1^2 + \frac{\alpha \lambda_1}{2} \|v\|_0^2 - \|c\|_{L^\infty(\Omega)} |\Omega| (\beta \|v\|_0^2 + C_\beta), \end{aligned}$$

for all $v \in H_0^1(\Omega)$. Thus, for $\beta > 0$ small enough, we have

$$\langle B_\varepsilon v, v \rangle \geq \frac{\alpha}{2} \|v\|_1^2 - C_4, \quad (2.12)$$

for some constant C_4 independent of ε .

Summing up, we have proved that (2.12) holds for a constant C_4 independent of $\varepsilon \in (0, 1]$ when $\gamma \leq 1$, and for a constant which depends on ε when $\gamma > 1$. This shows assertion iii) and the proof is complete. \square

3. THE LIMIT $\varepsilon \rightarrow 0$

In order to understand the behaviour of (u_ε) as ε goes to 0, we adapt the approach in [2, 9]. We first have:

Proposition 3.1. *If $0 < \varepsilon \leq \delta \leq 1$, then*

$$u_\varepsilon \geq u_\delta \text{ a.e. in } \Omega. \quad (3.1)$$

For all $\omega \subset\subset \Omega$, there exists c_ω independent of ε such that

$$u_\varepsilon(x) \geq c_\omega > 0 \text{ for a.e. } x \in \omega, \text{ for every } \varepsilon \in (0, 1]. \quad (3.2)$$

Proof. Let $0 < \varepsilon \leq \delta \leq 1$. We have, in the sense of distributions,

$$\begin{aligned} -\operatorname{div}(A(x) \nabla u_\delta) &= c(x) f_\delta(u_\delta), \\ -\operatorname{div}(A(x) \nabla u_\varepsilon) &= c(x) f_\varepsilon(u_\varepsilon), \end{aligned} \quad (3.3)$$

so that

$$-\operatorname{div}(A(x) \nabla (u_\delta - u_\varepsilon)) = c(x) [f_\delta(u_\delta) - f_\varepsilon(u_\varepsilon)]. \quad (3.4)$$

We choose $(u_\delta - u_\varepsilon)_+$ (which belongs to $H_0^1(\Omega)$) as a test function, and we note that

$$[f_\delta(u_\delta) - f_\varepsilon(u_\varepsilon)] (u_\delta - u_\varepsilon)_+ \leq 0. \quad (3.5)$$

Indeed, f_δ is nonincreasing so that

$$[f_\delta(u_\delta) - f_\delta(u_\varepsilon)] (u_\delta - u_\varepsilon)_+ \leq 0,$$

and $\delta \mapsto f_\delta(s)$ is nonincreasing for all s , so that

$$[f_\delta(u_\varepsilon) - f_\varepsilon(u_\varepsilon)] (u_\delta - u_\varepsilon)_+ \leq 0.$$

On summing these two inequalities, we obtain (3.5). Thus, the choice $(u_\delta - u_\varepsilon)_+$ as a test function in (3.4) yields

$$\alpha \|(u_\delta - u_\varepsilon)_+\|_1^2 \leq 0,$$

thanks to assumption (1.4). We conclude that $(u_\delta - u_\varepsilon)_+ = 0$, which shows (3.1).

Next, we note that for all $\varepsilon > 0$, u_ε belongs to $L^\infty(\Omega)$. This follows from [20, Théorème 4.2] since the right-hand side of (3.3) belongs to $L^\infty(\Omega)$. In particular, for $\varepsilon = 1$, there exists a constant C such that

$$\|u_1\|_{L^\infty(\Omega)} \leq C,$$

and we have

$$-\operatorname{div}(A(x)\nabla u_1) = c(x)f_1(u_1) \geq c(x)f_1(\|u_1\|_{L^\infty(\Omega)}) \geq c(x)f_1(C).$$

Since $c(x)f_1(C)$ is not identically zero, the strong maximum principle [12, Theorem 8.19] implies that $u_1 > 0$ a.e. in Ω and that (3.2) holds for $\varepsilon = 1$. On choosing $\delta = 1$ in (3.1), we obtain that (3.2) holds for all $\varepsilon \in (0, 1]$. \square

In order to pass to the limit $\varepsilon \rightarrow 0$, we need a priori estimates. We consider two cases, namely $\gamma \leq 1$, in which case uniqueness holds [9, Remark 3], and $\gamma > 1$.

3.1. The case $\gamma \leq 1$. We have:

Lemma 3.2. *If $\gamma \leq 1$, the family (u_ε) is bounded in $H_0^1(\Omega)$.*

Proof. We choose $v = u_\varepsilon$ in (2.12). Since $B_\varepsilon u_\varepsilon = 0$, this yields

$$0 \geq \frac{\alpha}{2} \|u_\varepsilon\|_1^2 - C_4.$$

The constant C_4 is independent of ε since $\gamma \leq 1$. This proves the claim. \square

We recall the following representation formula.

Lemma 3.3 (see [3]). *Let $v \in L_{loc}^1(\Omega) \cap H^{-1}(\Omega)$ such that $v \geq 0$ a.e. in Ω . Then for all $\varphi \in H_0^1(\Omega)$, we have $v\varphi \in L^1(\Omega)$ and*

$$\langle v, \varphi \rangle = \int_{\Omega} v\varphi dx. \quad (3.6)$$

Proof. Let $\varphi \in H_0^1(\Omega)$. If $\varphi \geq 0$ a.e. in Ω , the result is a consequence from [3]. In the general case, we write $\varphi = \varphi_+ - \varphi_-$ where φ_+ and φ_- are nonnegative functions in $H_0^1(\Omega)$. \square

Our convergence result reads as follows.

Theorem 3.4. *Assume that $\gamma \leq 1$. Then there exists a unique solution $u \in H_0^1(\Omega)$ to (1.1) in the sense that*

$$\forall \omega \subset\subset \Omega, \exists c_\omega > 0 \text{ such that } u(x) \geq c_\omega > 0, \text{ for a.e. } x \in \omega, \quad (3.7)$$

$$\frac{c}{u^\gamma} \in L_{loc}^\infty(\Omega) \cap H^{-1}(\Omega), \quad (3.8)$$

$$\int_{\Omega} A(x)\nabla u \cdot \nabla \varphi dx = \int_{\Omega} \frac{c(x)}{u^\gamma} \varphi dx, \quad \forall \varphi \in H_0^1(\Omega). \quad (3.9)$$

Moreover, $u_\varepsilon \rightarrow u$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, as $\varepsilon \rightarrow 0$.

We note that the right-hand side of (3.9) is well-defined because we may apply Lemma 3.3 to the nonnegative function $v = \frac{c}{u^\gamma}$.

Proof. We first prove uniqueness of u . If $\tilde{u} \in H_0^1(\Omega)$ is another solution of (1.1) in the sense of (3.7)-(3.9), then $u - \tilde{u}$ satisfies

$$\int_{\Omega} A(x)\nabla(u - \tilde{u}) \cdot \nabla \varphi dx = \int_{\Omega} c(x) \left(\frac{1}{u^\gamma} - \frac{1}{\tilde{u}^\gamma} \right) \varphi dx,$$

for all $\varphi \in H_0^1(\Omega)$. We choose $\varphi = u - \tilde{u}$ as a test function, we use that $f(s) = s^{-\gamma}$ is decreasing and assumption (1.4). This yields

$$\alpha \|u - \tilde{u}\|_1^2 \leq \int_{\Omega} A(x) \nabla(u - \tilde{u}) \cdot \nabla(u - \tilde{u}) dx \leq 0.$$

Thus, $\tilde{u} = u$ as claimed.

Next, we consider the behaviour (u_ε) as ε tends to 0. By Lemma 3.2, (u_ε) is bounded in $H_0^1(\Omega)$. Thus, up to a subsequence, (u_ε) converges weakly in $H_0^1(\Omega)$ to a function $u \in H_0^1(\Omega)$. We know by Rellich's theorem that $L^2(\Omega)$ is compactly imbedded into $H_0^1(\Omega)$, so that (up to a subsequence) $u_\varepsilon \rightarrow u$ strongly in $L^2(\Omega)$ and a.e. in Ω . Proposition 3.1 implies that (3.7) holds for u . In turn, this implies that

$u > 0$ a.e. in Ω and $\frac{c}{u^\gamma} \in L_{loc}^\infty(\Omega)$.

Now, we use that $B_\varepsilon u_\varepsilon = 0$, that is

$$\int_{\Omega} A \nabla u_\varepsilon \cdot \nabla \varphi dx = \int_{\Omega} c(x) f_\varepsilon(u_\varepsilon) \varphi dx, \quad \forall \varphi \in H_0^1(\Omega). \quad (3.10)$$

Let $\varphi \in C_c^\infty(\Omega)$ in (3.10). Thanks to (3.1), $f_\varepsilon(u_\varepsilon)$ is bounded in $L_{loc}^\infty(\Omega)$ and $f_\varepsilon(u_\varepsilon) \rightarrow f(u)$ a.e. in Ω . Since φ has compact support in Ω , we may pass to the limit in (3.10). We obtain that (3.9) holds for all $\varphi \in C_c^\infty(\Omega)$. Equation (3.10) also shows that $c(x) f_\varepsilon(u_\varepsilon)$ belongs to a bounded set of $H^{-1}(\Omega)$ since

$$\left| \int_{\Omega} c(x) f_\varepsilon(u_\varepsilon) \varphi dx \right| \leq C \|u_\varepsilon\|_1 \|\varphi\|_1 \leq C' \|\varphi\|_1,$$

for all $\varphi \in H_0^1(\Omega)$, where the constants C and C' are independent of ε . This shows that $c(x) f(u) \in H^{-1}(\Omega)$. Finally, we use that $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ and the representation formula of Brezis and Browder (Lemma 3.3), and we find that equation (3.9) holds for all $\varphi \in H_0^1(\Omega)$. The uniqueness of the limit u implies that the whole family (u_ε) converges to u , weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. This concludes the proof. \square

Remark 3.5. The results in [2, 9] show that $u^{\frac{1+\gamma}{2}} \in H_0^1(\Omega)$ and $u \in L^\infty(\Omega)$, for all $\gamma \in (0, 1]$.

3.2. The case $\gamma > 1$. By arguing as in [2, Lemma 4.1], we have:

Lemma 3.6. *Assume that $\gamma > 1$. Then $(u_\varepsilon^{\frac{1+\gamma}{2}})$ is bounded in $H_0^1(\Omega)$ and (u_ε) is bounded in $H_{loc}^1(\Omega)$.*

As a consequence, we can state the following convergence result:

Theorem 3.7. *Assume that $\gamma > 1$ and let (ε_n) be a decreasing sequence of positive real numbers which tends to 0. Then the pointwise limit u of (u_{ε_n}) solves (1.1) in the sense that*

$$\forall \omega \subset\subset \Omega, \exists c_\omega > 0 \text{ such that } u(x) \geq c_\omega > 0, \text{ for a.e. } x \in \omega, \quad (3.11)$$

$$u \in H_{loc}^1(\Omega), \quad (3.12)$$

$$\int_{\Omega} A(x) \nabla u \cdot \nabla \varphi dx = \int_{\Omega} \frac{c(x)}{u^\gamma} \varphi dx, \quad \forall \varphi \in C_c^1(\Omega), \quad (3.13)$$

$$u^{\frac{1+\gamma}{2}} \in H_0^1(\Omega) \text{ (this is the meaning of } u = 0 \text{ on } \partial\Omega). \quad (3.14)$$

Moreover, $u_{\varepsilon_n} \rightarrow u$ weakly in $H^1(\omega)$, for all $\omega \subset\subset \Omega$, $u_{\varepsilon_n}^{\frac{1+\gamma}{2}} \rightarrow u^{\frac{1+\gamma}{2}}$ weakly in $H_0^1(\Omega)$ and $u_{\varepsilon_n} \rightarrow u$ strongly in $L^2(\Omega)$.

Proof. We note that by (3.1), the sequence $(u_{\varepsilon_n})_n$ is nondecreasing so that its pointwise limit $u(x) = \lim_{n \rightarrow +\infty} u_{\varepsilon_n}(x)$ is well defined a.e. in Ω . Property (3.11) is a consequence of (3.2). Let $\varphi \in C_c^1(\Omega)$ and let

$$\omega = \{x \in \Omega : \varphi(x) \neq 0\}.$$

We note that $\omega \subset\subset \Omega$. By Lemma 3.6, (u_{ε_n}) is bounded in $H^1(\omega)$, so that, up to a subsequence, $u_{\varepsilon_n} \rightarrow u$ weakly in $H^1(\omega)$, strongly in $L^2(\omega)$ and a.e. in Ω . The uniqueness of the limit u implies that the whole sequence (u_{ε_n}) converges.

By definition of u_{ε_n} , we have

$$\int_{\Omega} A(x) \nabla u_{\varepsilon_n} \cdot \nabla \varphi dx = \int_{\Omega} c(x) f_{\varepsilon_n}(u_{\varepsilon_n}) \varphi dx. \quad (3.15)$$

Since $\omega \subset\subset \Omega$, we may pass to the limit in (3.15) and we obtain that u satisfies (3.13). Finally, we know by Lemma 3.6 that $(u_{\varepsilon_n}^{\frac{1+\gamma}{2}})$ is bounded in $H_0^1(\Omega)$. This shows that u satisfies (3.14) and that $(u_{\varepsilon_n}^{\frac{1+\gamma}{2}})$ converges weakly in $H_0^1(\Omega)$ to $u^{\frac{1+\gamma}{2}}$.

Assume now that $d \geq 3$ and let $s < 2^* = \frac{2d}{d-2}$. Thanks to the Sobolev imbeddings [12], we know that $H_0^1(\Omega)$ is compactly imbedded in $L^s(\Omega)$, so $(u_{\varepsilon_n}^{\frac{1+\gamma}{2}}) \rightarrow u^{\frac{1+\gamma}{2}}$ strongly in $L^s(\Omega)$. There exists $g \in L^s(\Omega)$ such that, up to a subsequence, $(u_{\varepsilon_n}^{\frac{1+\gamma}{2}}) \rightarrow u^{\frac{1+\gamma}{2}}$ a.e. in Ω and $0 \leq u_{\varepsilon_n}^{\frac{1+\gamma}{2}} \leq g$, for all n . Thus, $0 \leq u_{\varepsilon_n} \leq g^{\frac{2}{1+\gamma}}$ for all n , with $g^{\frac{2}{1+\gamma}} \in L^{\frac{s(1+\gamma)}{2}}(\Omega)$. The dominated convergence theorem shows that $u_{\varepsilon_n} \rightarrow u$ in $L^q(\Omega)$, with $q = \frac{s(1+\gamma)}{2} < \frac{d(1+\gamma)}{d-2}$. Since $\frac{d(1+\gamma)}{d-2} > 2$ and $s < 2^*$ is arbitrary, we may choose $q = 2$. Note that by uniqueness of the limit a.e., the whole sequence converges.

If $d = 2$, a similar argument holds for all $s < +\infty$ and we conclude similarly.

If $d = 1$, then $H_0^1(\Omega)$ is compactly imbedded in $C^0(\overline{\Omega})$, so $(u_{\varepsilon_n}^{\frac{1+\gamma}{2}}) \rightarrow u^{\frac{1+\gamma}{2}}$ strongly in $C^0(\overline{\Omega})$. In particular, there exists $M > 0$ such that $0 \leq u_{\varepsilon_n}(x) \leq M$ for all $x \in \overline{\Omega}$ and for all n . Since the function $t \mapsto t^{\frac{2}{1+\gamma}}$ is uniformly continuous on $[0, M]$, we conclude that $u_{\varepsilon_n} \rightarrow u$ strongly in $C^0(\overline{\Omega})$, and therefore in $L^2(\Omega)$. This concludes the proof. \square

4. THE FINITE ELEMENT APPROXIMATION

Throughout Section 4, we assume that $\varepsilon \in (0, 1]$ is fixed and $d = 1, 2$ or 3 . For the finite element approximation of the regularized problem (2.4), we assume that Ω is a bounded convex d -polyhedron, that is a bounded interval if $d = 1$, a convex polygon if $d = 2$ and a convex polyhedron if $d = 3$.

We consider a regular family $\{\mathcal{T}_h\}_{h>0}$ of triangulations of Ω into d -simplices and such that each triangulation covers Ω exactly. For each triangulation \mathcal{T}_h , the P^1 finite element space is

$$V_h = \{v_h \in C^0(\overline{\Omega}), : v_h \text{ is affine on every } d\text{-simplex of } \mathcal{T}_h \text{ and } v_h = 0 \text{ on } \partial\Omega\}. \quad (4.1)$$

The space V_h is a finite dimensional subspace of $V = H_0^1(\Omega)$. We denote I_h the interpolation operator at the vertices of the triangulation. It is well-known [10] that for every $v \in H^2(\Omega)$, we have

$$I_h v \rightarrow v \text{ in } V, \text{ as } h \rightarrow 0. \quad (4.2)$$

Since $H^2(\Omega)$ is dense in V , this implies

$$\inf_{v_h \in V_h} \|u - v_h\|_1 \rightarrow 0, \quad (4.3)$$

for all $u \in V$.

Remark 4.1. Similar properties hold if we assume that Ω is a bounded convex open set with C^2 boundary. In this case, the d -polyhedron Ω_h defined by the triangulation satisfies $\Omega_h \subset \Omega$ and the vertices of the boundary of Ω_h lie on the boundary of Ω . We use that $V_h \subset H_0^1(\Omega_h) \subset H_0^1(\Omega)$ by setting $v_h = 0$ on $\bar{\Omega} \setminus \Omega_h$. Properties (4.2)-(4.3) are still true (see, e.g., [19, Section 4.1]), so the results in this section are also valid in such a case.

The finite element approximation of problem (2.4) reads: find $u_h \in V_h$ such that

$$\int_{\Omega} A(x) \nabla u_h \cdot \nabla v_h dx = \int_{\Omega} c(x) f_{\varepsilon}(u_h) v_h dx, \quad \forall v_h \in V_h. \quad (4.4)$$

We have

Proposition 4.2. *Problem (4.4) has a unique solution.*

Remark 4.3. The function $u_h = u_{\varepsilon, h}$ defined in (4.4) depends also on ε . Since $\varepsilon \in (0, 1]$ is fixed throughout Section 4, we omit the subscript ε .

Proof. We use a monotonicity argument, as in the continuous case. Let $B_h : V_h \rightarrow V_h$ be the nonlinear operator defined by

$$\int_{\Omega} (B_h v_h) w_h dx = \int_{\Omega} A(x) \nabla v_h \cdot \nabla w_h - \int_{\Omega} c(x) f_{\varepsilon}(v_h) w_h dx. \quad (4.5)$$

The same arguments as in the proof of Proposition 2.2 show that B_h is continuous, strictly monotone and coercive. Thus, there exists a unique $u_h \in V_h$ such that $B_h u_h = 0$. \square

Remark 4.4. We do not know if $u_h \geq 0$ in every case. If a discrete maximum principle holds then $u_h \geq 0$, but this requires additional assumptions on the triangulation and the matrix A [6]. If $Lv = -\Delta v$ and if the triangulation is *weakly acute*, then a discrete maximum principle holds and we have $u_h \geq 0$, see [1, 16, 21]. If $Lv = -\Delta v$ and $d = 1$, the discrete maximum principle holds without any additional assumption on the mesh [22].

We recall that the parameter $\varepsilon > 0$ is fixed throughout this section. The following convergence holds:

Theorem 4.5. *We have $u_h \rightarrow u_{\varepsilon}$ strongly in V as $h \rightarrow 0$.*

Proof. We notice that

$$\int_{\Omega} (B_h v_h) w_h dx = \langle B_{\varepsilon} v_h, w_h \rangle, \quad \forall v_h, w_h \in V_h, \quad (4.6)$$

where B_h and B_ε are defined in (4.5) and (2.5), respectively. We apply this relation with $v_h = w_h = u_h$. Using $B_h u_h = 0$ and (2.12), we obtain

$$0 \geq \frac{\alpha}{2} \|u_h\|_1^2 - C_4.$$

This shows that the sequence (u_h) is bounded in V .

We may therefore assume that, up to a subsequence, u_h converges weakly in V to some function \tilde{u} . Let $\varphi \in C_c^\infty(\Omega)$. We choose $v_h = u_h$ and $w_h = I_h \varphi$ in (4.6). We find

$$0 = \langle B_\varepsilon u_h, I_h \varphi \rangle = \int_\Omega A(x) \nabla u_h \cdot \nabla (I_h \varphi) dx - \int_\Omega c(x) f_\varepsilon(u_h) I_h(\varphi) dx. \quad (4.7)$$

The quadratic term tends to $\int_\Omega A(x) \nabla \tilde{u} \cdot \nabla \varphi dx$ thanks to (4.2). Since V is compactly imbedded into $L^2(\Omega)$, we know that $u_h \rightarrow \tilde{u}$ strongly in $L^2(\Omega)$. Using the continuity of the nonlinear operator $v \mapsto c(\cdot) f_\varepsilon(v)$ from $L^2(\Omega)$ into $L^2(\Omega)$ (cf. (2.6)), we may pass to the limit in (4.7) and we find

$$0 = \langle B_\varepsilon \tilde{u}, \varphi \rangle.$$

This is true for all $\varphi \in C_c^\infty(\Omega)$. Thus, $\tilde{u} = u_\varepsilon$ and by uniqueness of the limit, the whole sequence (u_h) converges to \tilde{u} , weakly in V .

Next, we denote $g(x) = c(x) f_\varepsilon(u(x))$ and $g_h(x) = c(x) f_\varepsilon(u_h(x))$. We have already seen that $g_h \rightarrow g$ in $L^2(\Omega)$. The definition of u_h and u_ε reads

$$\begin{aligned} \int_\Omega A(x) \nabla u_h \cdot \nabla v_h dx &= \int_\Omega g_h v_h dx, \quad \forall v_h \in V_h, \\ \int_\Omega A(x) \nabla u \cdot \nabla v dx &= \int_\Omega g v dx, \quad \forall v \in V. \end{aligned}$$

We introduce the solution $\tilde{u}_h \in V_h$ to the linear problem

$$\int_\Omega A(x) \nabla \tilde{u}_h \cdot \nabla v_h dx = \int_\Omega g v_h dx, \quad \forall v_h \in V_h.$$

The assumptions on A ensure that \tilde{u}_h exists and is unique. Cea's lemma [10] and (4.3) imply that $\tilde{u}_h \rightarrow u$ strongly in V . On the other hand, we have

$$\begin{aligned} \alpha \|u_h - \tilde{u}_h\|_1^2 &\leq \int_\Omega A(x) \nabla (u_h - \tilde{u}_h) \cdot \nabla (u_h - \tilde{u}_h) dx \\ &= \int_\Omega (g_h - g)(u_h - \tilde{u}_h) dx \\ &\leq \|g_h - g\|_0 \|u_h - \tilde{u}_h\|_0, \end{aligned}$$

so

$$\|u_h - \tilde{u}_h\|_1 \leq \frac{1}{\alpha \lambda_1^{1/2}} \|g - g_h\|_0,$$

which shows that $\|u_h - \tilde{u}_h\|_1 \rightarrow 0$. On writing $u_h = (u_h - \tilde{u}_h) + \tilde{u}_h$, we obtain that $u_h \rightarrow u$ strongly in V , as claimed. \square

5. NUMERICAL SIMULATIONS

We performed numerical simulations with the **FreeFem++** software [15] in two space dimension. We used a mesh adaptation algorithm available in **FreeFem++** with the keyword **adaptmesh**. Our algorithm reads as follows, for a bounded domain Ω of \mathbb{R}^2 with smooth boundary and a given $\varepsilon > 0$.

Algorithm 5.1. *Input:* an initial mesh \mathcal{T}_h^0 which gives a triangulation of Ω_h (a polygonal approximation to Ω) and an initial guess \tilde{u}_h^0 of the solution u .

For k from 0 to k_{max} :

- (1) build the space $V_h^k \subset H_0^1(\Omega_h)$ of continuous P^1 finite element associated to \mathcal{T}_h^k , cf. (4.1);
- (2) Compute the solution $u_h^k \in V_h^k$ to the variational problem

$$\int_{\Omega} A(x) \nabla u_h^k \cdot \nabla v_h^k dx = \int_{\Omega} c(x) f_{\varepsilon}(u_h^k) v_h^k dx, \quad \forall v_h^k \in V_h^k,$$

using a Newton algorithm with the initial value \tilde{u}_h^k ;

- (3) Build a new triangulation \mathcal{T}_h^{k+1} of Ω_h adapted to the solution u_h^k ;
- (4) Compute the P^1 interpolate \tilde{u}_h^{k+1} of u_h^k on \mathcal{T}_h^{k+1} .

Output: $\mathcal{T}_h^{k_{max}}$ and $u_h^{k_{max}}$.

We recall that f_{ε} is the regularization of $f(s) = s^{-\gamma}$ given by (2.2) (except in Table 2 where we choose \tilde{f}_{ε} defined by (2.3)). For a polygonal domain Ω , we have $\Omega_h = \Omega$ but in the simulations below we have $\Omega_h \neq \Omega$ because the boundary $\partial\Omega$ is of class C^2 .

5.1. A radial test case for $A = Id$. We first consider problem (1.5) on the ring

$$\Omega = \{(x, y) \in \mathbb{R}^2 : e^{-2} < x^2 + y^2 < e^2\},$$

with

$$c(x, y) = \frac{1}{x^2 + y^2} \quad \text{and} \quad \gamma = 3. \quad (5.1)$$

Using polar coordinates, it is easy to check that the exact solution to this problem is

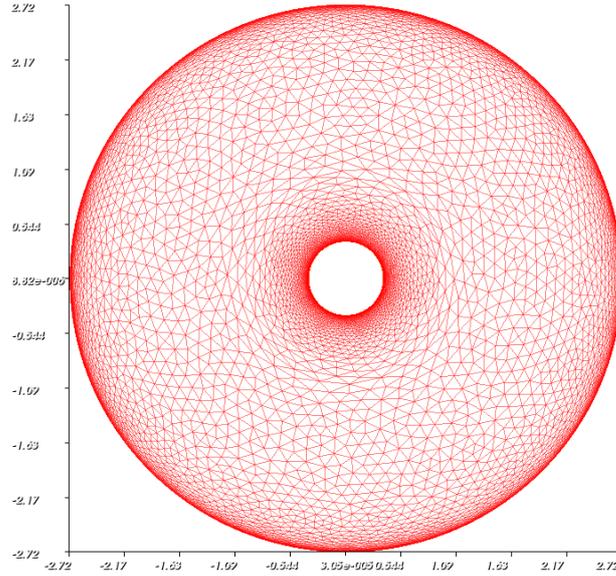
$$u(x, y) = \left(1 - \log^2(\sqrt{x^2 + y^2})\right)^{1/2}.$$

This solution belongs to $C^{\infty}(\Omega) \cap C^{1/2}(\bar{\Omega})$ (cf. (1.10)), and it does not belong to $H_0^1(\Omega)$ since $\gamma > 3$ [1, Lemma 2.1].

The exterior (interior, respectively) circle of Ω with radius e ($1/e$, resp.) is approximated by a regular polygon having 170 (46, resp.) vertices; its edges have a length of approximately 0.1 (0.05, resp.). This yields a polygonal domain Ω_h which approximates the ring Ω . Since Ω is *not* convex, $\Omega_h \not\subset \Omega$ and the approximation is nonconforming, that is $H_0^1(\Omega_h) \not\subset H_0^1(\Omega)$.

We performed Algorithm 5.1 with $k_{max} = 5$ mesh adaptations, starting with an initial triangulation \mathcal{T}_h^0 of Ω_h and the initial guess $\tilde{u}_h^0 = 0$. The **FreeFem++** command in step (3) was

`Th=adaptmesh(Th,u,err=0.005,nbvx=9e4).`

FIGURE 1. Mesh for case (5.1) and $\varepsilon = 0.1$ ($\gamma = 3$)

This command means that \mathcal{T}_h^{k+1} is built using \mathcal{T}_h^k and u_h^k with a P^1 interpolation error level of 0.005 and a maximum number of vertices equal to $9e4$.

The final mesh \mathcal{T}_h^5 obtained for $\varepsilon = 0.1$ is shown in Figure 1. It has 11084 vertices and it has been obtained by successive mesh refinements by starting from an initial mesh \mathcal{T}_h^0 with 4181 vertices. The mesh \mathcal{T}_h^k at iteration $k = 0, 1, 2, 3, 4$ and 5 contains 4181, 9356, 11758, 11366, 11097 and 11084 vertices respectively.

ε	0.4	0.2	0.1	0.05	0.025	0.0125	0.00625
L^2 error	0.18	0.053	0.014	0.0036	0.0026	0.0029	0.0031
L^2 ratio	.	3.4	3.8	3.9	1.4	0.9	0.9
L^∞ error	0.090	0.045	0.023	0.012	0.0057	0.0035	0.0048
L^∞ ratio	.	2.0	2.0	1.9	2.1	1.6	0.7
Nb vertices	6790	8877	11084	13350	17370	21121	50717
h_{min}	2.7e-03	9.3e-04	3.3e-04	1.5e-04	6.7e-05	3.9e-05	2.7e-05
h_{max}	0.28	0.26	0.31	0.29	0.28	0.29	0.28

TABLE 1. L^2 error, L^∞ error and mesh characteristics for case (5.1)

In Table 1, we present the L^2 error $\|u - u_h\|_{L^2(\Omega_h)}$ and the L^∞ error $\|u - u_h\|_{L^\infty(\Omega_h)}$ obtained for different value of ε (with $u_h = u_h^5$). The discretization parameters are the same as for the case $\varepsilon = 0.1$, which means that the initial triangulation \mathcal{T}_h^0 is the same, the initial guess is $\tilde{u}_h^0 = 0$ and the parameters in the `adaptmesh` command are the same. We checked numerically that the solution u_h is nonnegative on Ω_h , up to computer accuracy. For each ε , Table 1 gives the number of vertices in the final mesh $\mathcal{T}_h^{k_{max}}$, as well as the minimum and maximum length of the edges of this triangulation.

On writing

$$u - u_h = (u - u_\varepsilon) + (u_\varepsilon - u_{\varepsilon,h}), \quad (5.2)$$

the error $\|u - u_{\varepsilon,h}\|$ can be interpreted as an error $\|u - u_\varepsilon\|$ due to the regularization and an error $\|u_\varepsilon - u_{\varepsilon,h}\|$ due to the discretization (we write $u_h = u_{\varepsilon,h}$ to stress that u_h depends on ε , cf. Remark 4.3).

In line 5 of Table 1, we have computed for each ε the ratio of the error in the L^∞ norm for 2ε and ε , namely

$$r_{\infty,\varepsilon} = \frac{\|u - u_{2\varepsilon,h}\|_{L^\infty(\Omega_h)}}{\|u - u_{\varepsilon,h}\|_{L^\infty(\Omega_h)}}.$$

We see that for ε varying from 0.2 down to 0.025, the ratio $r_{\infty,\varepsilon}$ is very close to 2. This means that the error is of order $O(\varepsilon)$ and the error due to the regularization is dominant in (5.2). For ε between 0.025 and 0.00625, the error due to the discretization becomes dominant.

In comparison, the L^2 error decreases more rapidly than the L^∞ error for ε between 0.4 and 0.025 (cf. line 3 in Table 1, which represents the same ratio as $r_{\infty,\varepsilon}$, but in L^2 norm). This can be related to the choice of a local regularization f_ε for f : the error $u(x, y) - u_h(x, y)$ is mostly located near the boundary $\partial\Omega$ (cf. Remark 2.1).

ε	0.1	0.05	0.025	0.0125	6.25e-3	3.125e-3	1.5625e-3
L^2 error	0.44	0.23	0.11	0.057	0.027	0.012	0.0047
L^2 ratio	.	1.9	2.1	1.9	2.1	2.2	2.6
L^∞ error	0.098	0.050	0.026	0.013	0.0071	0.0042	0.0057
L^∞ ratio	.	2.0	1.9	2.0	1.8	1.7	0.7
L^2/L^∞	0.94	0.96	0.89	0.92	0.80	0.60	0.17

TABLE 2. L^2 and L^∞ error for the regularization \tilde{f}_ε and case (5.1)

5.2. About the choice of the regularization. In Table 2, we have computed the L^2 and L^∞ error with the regularization \tilde{f}_ε defined by (2.3) instead of f_ε given by (2.2). The test case is (5.1) and the algorithm and the discretization parameters are exactly as previously. We have divided the values of ε by 4 in order to have a L^∞ error which is easily comparable to the values in Table 1. We see that for ε from 0.1 down to 6.25e-3, the L^∞ error and the L^2 error are both dominated by the regularization error which is of order $O(\varepsilon)$. The L^2 error is much larger than for the regularization f_ε because we use a global regularization of f : the error $u(x, y) - u_h(x, y)$ is located everywhere on Ω and not just near the boundary. This is confirmed by the last line of Table 2, which shows for each ε the normalized ratio

$$\frac{\|u - u_{\varepsilon,h}\|_{L^2(\Omega_h)}}{|\Omega|^{1/2}\|u - u_{\varepsilon,h}\|_{L^\infty(\Omega_h)}}.$$

This ratio is close to 1 as long as the regularization error dominates.

5.3. A radial test case with regular coefficients. We consider now problem (1.1) on the ring

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2^2\},$$

with

$$A(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c(x, y) = \frac{1}{4\sqrt{x^2 + y^2}} \quad \text{and} \quad \gamma = 3. \quad (5.3)$$

The exact solution to this problem is

$$u(x, y) = v(\sqrt{x^2 + y^2}), \quad (5.4)$$

where

$$v(r) = [(r - 1)(2 - r)]^{1/2}.$$

Indeed, it is straightforward to check that

$$-v''(r) = \frac{1}{4v^3(r)} \quad \text{for } r \in (1, 2), \quad \text{and} \quad v(1) = v(2) = 0.$$

The ODE can be rewritten

$$-\frac{1}{r} \frac{\partial}{\partial r} \left(ra(r) \frac{\partial v(r)}{\partial r} \right) = \frac{1}{(4r)v^3(r)} \quad \text{for } r \in (1, 2), \quad \text{with } a(r) = \frac{1}{r}. \quad (5.5)$$

For a radial function u , this corresponds to the PDE (1.1) with the values (5.3). Since the coefficients are regular, the solution u to (1.1) is unique in the class $C^2(\Omega) \cap C^0(\bar{\Omega})$ [7]. It is therefore radial. It is easy to check that u given by (5.4) belongs to $C^\infty(\Omega) \cap C^{1/2}(\bar{\Omega})$ and that it does not belong to $H_0^1(\Omega)$.

ε	0.4	0.2	0.1	0.05	0.025	0.0125	0.00625
L^2 error	0.17	0.049	0.020	0.016	0.016	0.016	0.016
L^2 ratio	.	3.5	2.5	1.3	1.0	1.0	1.0
L^∞ error	0.098	0.046	0.023	0.012	0.0098	0.0099	0.0099
L^∞ ratio	.	2.1	2.0	1.9	1.2	1.0	1.0
Nb vertices	8652	12387	15633	19520	24442	36397	69629
h_{min}	1.0e-02	3.4e-03	1.2e-03	4.5e-4	1.8e-4	8.1e-05	4.7e-05
h_{max}	0.22	0.25	0.25	0.25	0.25	0.24	0.25

TABLE 3. L^2 error, L^∞ error and mesh characteristics for case (5.3)

For the numerical resolution of (5.3), we proceeded as previously. The interior and exterior circle of radius 1 and 2 were both approximated by a regular polygon with 251 vertices, yielding a polygonal approximation Ω_h of Ω (with $\Omega_h \not\subset \Omega$ since Ω is not convex). The sides of the small polygon had a length of approximately 0.025, whereas the sides of the large polygon had a length of approximately 0.05.

We performed Algorithm 5.1 with $k_{max} = 5$ iterations, starting with an initial triangulation of Ω_h and the initial guess $\tilde{u}_h^0 = 0$. The parameters in the `adaptmesh` command were set as previously. Table 3 is the analogue of Table 1 for this test case.

Due to the regularity of the coefficients, the results are very similar to the first test case. In particular, for ε from 0.4 down to 0.05, the L^∞ error is $O(\varepsilon)$ and the error due to the regularization is dominant. Then, for smaller values of ε , the error due to the discretization is dominant.

5.4. A numerical simulation with a discontinuous matrix. In order to illustrate our method in the case of a discontinuous elliptic matrix, we computed the numerical solution obtained on the unit disc Ω in \mathbb{R}^2 for the values

$$A(x, y) = a(x, y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c(x, y) = 1 \quad \text{and} \quad \gamma = 1.1, \quad (5.6)$$

with

$$a(x, y) = \begin{cases} 0.2 & \text{if } x^2 + y^2 < 0.2^2, \\ a_1(x, y) & \text{if } x^2 + y^2 \geq 0.2^2, \end{cases} \quad (5.7)$$

where

$$a_1(x, y) = \begin{cases} 1.05 - y & \text{if } x \geq 0 \text{ and } y \geq 0, \\ 2.10 - 2x & \text{if } x \geq 0 \text{ and } y < 0, \\ 1.05 + x & \text{if } x < 0 \text{ and } y \geq 0, \\ 2.10 + 2y & \text{if } x < 0 \text{ and } y < 0. \end{cases} \quad (5.8)$$

We note that a is discontinuous across the circle of radius 0.2 and along the x and y axis when $0.2 < |x| < 1$ or $0.2 < |y| < 1$.

We use Algorithm 5.1 with $\varepsilon = 0.05$, $k_{max} = 5$ and $\tilde{u}_h^0 = 0$. The unit circle is approximated by a regular polygon with 314 sides having a length of approximately 0.02. This yields a polygonal approximation Ω_h of Ω and an initial triangulation of Ω_h . Since Ω is convex, we have $\Omega_h \subset \Omega$ and we are in the situation described in Remark 4.1. The parameters in the `adaptmesh` command are the same as previously except for `err` whose value is set to 0.0025.

The output mesh $\mathcal{T}_h^{k_{max}}$ is shown in Figure 2. We observe that it is refined near the unit circle, due to the singularity in the PDE, but also along the discontinuities of the matrix A . The corresponding output solution $u_h^{k_{max}}$ is shown in Figure 3.

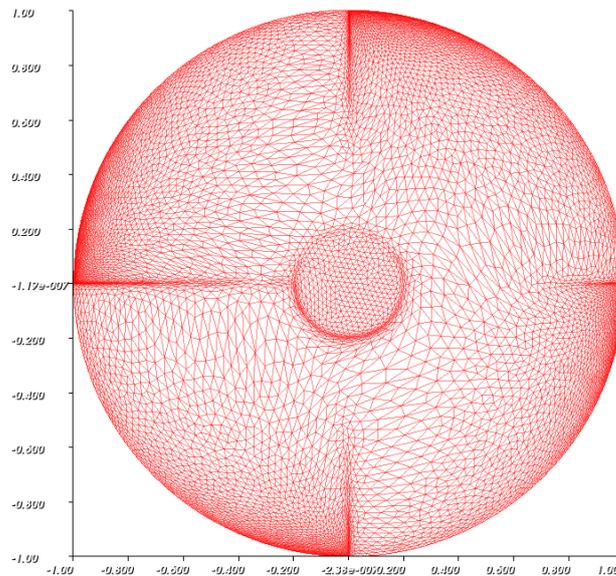


FIGURE 2. Output mesh for case (5.6) and $\varepsilon = 0.05$ ($\gamma = 1.1$)

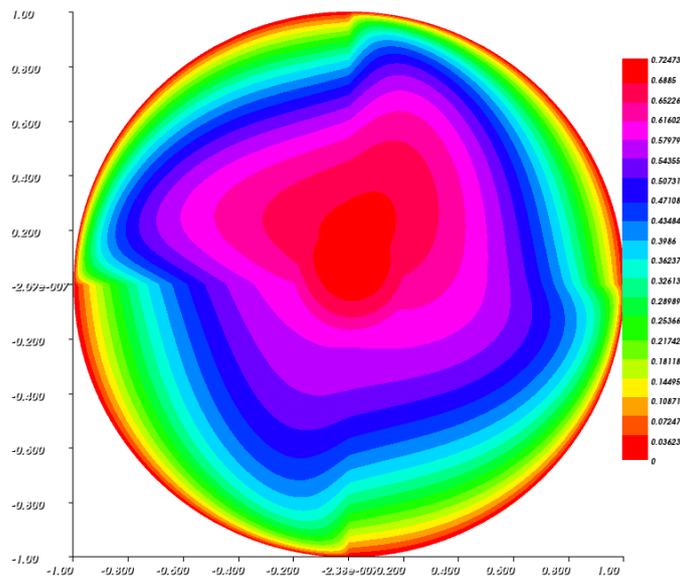


FIGURE 3. Output solution for case (5.6) and $\varepsilon = 0.05$ ($\gamma = 1.1$)

REFERENCES

- [1] J. W. Barrett and M. Schneider. Finite element approximation of a semilinear elliptic problem with a singular nonlinearity. *Numer. Math.*, 82(1):21–56, 1999.
- [2] L. Boccardo and L. Orsina. Semilinear elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations*, 37(3-4):363–380, 2010.
- [3] H. Brézis and F. Browder. A property of Sobolev spaces. *Comm. Partial Differential Equations*, 4(9):1077–1083, 1979.
- [4] A. Callegari and A. Nachman. Some singular, nonlinear differential equations arising in boundary layer theory. *J. Math. Anal. Appl.*, 64(1):96–105, 1978.
- [5] A. Charkaoui and N. Alaa. Weak periodic solution for semilinear parabolic problem with singular nonlinearities and L1 data. *Mediterr. J. Math.*, 17(108), 2020.
- [6] P. G. Ciarlet and P.-A. Raviart. Maximum principle and uniform convergence for the finite element method. *Comput. Methods Appl. Mech. Engrg.*, 2:17–31, 1973.
- [7] M. G. Crandall, P. H. Rabinowitz, and L. Tartar. On a Dirichlet problem with a singular nonlinearity. *Comm. Partial Differential Equations*, 2(2):193–222, 1977.
- [8] J. I. Díaz and J.-M. Rakotoson. Elliptic problems on the space of weighted with the distance to the boundary integrable functions revisited. In *Proceedings of the Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems*, volume 21 of *Electron. J. Differ. Equ. Conf.*, pages 45–59. Texas State Univ., San Marcos, TX, 2014.
- [9] N. El Berdan, J. I. Díaz, and J. M. Rakotoson. The uniform Hopf inequality for discontinuous coefficients and optimal regularity in BMO for singular problems. *J. Math. Anal. Appl.*, 437(1):350–379, 2016.
- [10] A. Ern and J.-L. Guermond. *Éléments finis: théorie, applications, mise en œuvre*, volume 36 of *Mathématiques & Applications (Berlin)*. Springer-Verlag, Berlin, 2002.
- [11] G.-M. Gie, M. Hamouda, C.-Y. Jung, and R. M. Temam. *Singular perturbations and boundary layers*, volume 200 of *Applied Mathematical Sciences*. Springer, Cham, 2018.
- [12] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
- [13] T. Godoy. Elliptic problems with singular nonlinearities of indefinite sign. *AIMS Mathematics*, 5(3):1779, 2020.

- [14] C. Gui and F.-H. Lin. Regularity of an elliptic problem with a singular nonlinearity. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(6):1021–1029, 1993.
- [15] F. Hecht. New development in FreeFem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.
- [16] X. Jiang and R. H. Nochetto. Effect of numerical integration for elliptic obstacle problems. *Numer. Math.*, 67(4):501–512, 1994.
- [17] O. Kavian. *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, volume 13 of *Mathématiques & Applications (Berlin)*. Springer-Verlag, Paris, 1993.
- [18] A. C. Lazer and P. J. McKenna. On a singular nonlinear elliptic boundary-value problem. *Proc. Amer. Math. Soc.*, 111(3):721–730, 1991.
- [19] M. Pierre. Convergence of exponential attractors for a finite element approximation of the Allen-Cahn equation. *Numer. Funct. Anal. Optim.*, 39(16):1755–1784, 2018.
- [20] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15(fasc. 1):189–258, 1965.
- [21] R. Vanselow. About Delaunay triangulations and discrete maximum principles for the linear conforming FEM applied to the Poisson equation. *Appl. Math.*, 46(1):13–28, 2001.
- [22] T. Vejchodský and P. Šolín. Discrete maximum principle for higher-order finite elements in 1D. *Math. Comp.*, 76(260):1833–1846, 2007.
- [23] E. Zeidler. *Nonlinear functional analysis and its applications. II/B. Nonlinear monotone operators*. Springer-Verlag, New York, 1990.