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# From adherence to slip in nanofluidics: a mathematical justification based on a drop of viscosity. The scalar case

M. Bonnivard\*, J. Olivier

October 19, 2020

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# 1 Introduction

## 1.1 Context and goals

Nanofluidics is the study of the flow of materials at the nanoscale [2]. This is the scale of flow in Carbon Nanotubes which are physical systems of great interest nowadays [16, 26]. At this scale, flows present some striking features, such as the capacity for the material to slip much more easily than one would expect [1, 24, 25]. The origin of this slip is the subject of current debate in the physics community [13, 20].

In classical fluid mechanics, the interaction between the fluid and the walls of a bounded domain is usually modeled *via* appropriate boundary conditions. Probably the three most used boundary conditions are (let us note  $u$  the velocity field of the flow)

**No-slip (or adherence):** imposing that  $u$  vanishes at the wall.

**Slip:** imposing that the normal component of  $u$  vanishes and that the tangential part of the normal stress is proportional to the tangential part of the velocity field. The inverse of the proportionality factor has the dimension of a length called the slip length.

**Perfect slip:** imposing that the normal component of  $u$  vanishes and that the tangential part of the normal stress also vanishes.

Let us point that, although one could expect the averaging procedure usually applied to obtain macroscopic flow equations to fail at the nanoscale, Stokes equations remain surprisingly efficient in nanofluidics [14, 19].

From a mathematical perspective, one successful strategy initiated in the early 2000's [9, 17] to explain the occurrence of adherence or partial slip on solid walls, consists in modeling micro-asperities on the surface and analyzing their effect on the flow by an homogenisation process, imposing only a mild non penetration boundary condition on the rugous wall, *i.e.* that the normal component of the fluid velocity vanishes. This so-called "rugosity effect" has been studied quite extensively in the last decades, which has led to a rather complete description of the asymptotic effect of rough patterns on viscous flow [7, 8, 17].

In this paper, we consider a completely different interpretation of the apparent slip length measured in nanoscopic devices, proposed in [21], where the author postulates that the source of this slip arises from a "depletion layer with reduced viscosity near the wall". This hypothesis is supported by experimental evidence [22] and Molecular Dynamics simulations [18] bringing out that the viscosity drops near the wall of the nanotube. In [21], the flow is modeled as a Stokes flow in an infinite cylindrical pipe, with no-slip boundary condition, but two viscosities: a "bulk" one at the center of the pipe and a "wall" one near the walls which is smaller than the bulk one. In this model, the fluid is supposed to adhere at the wall. Yet, by solving the equation in this simple geometry, the author was able to describe the resulting flow as if it had an effective slip length and a constant viscosity equal to the one in the bulk, computing this length in terms of the viscosity drop and the sizes of the depletion layer and tube radius. From this result we started wondering: is there a general mathematical framework to study the passing from a model with varying viscosity and no-slip to a model with constant viscosity and slip?

The question is to link two models with different equations in the same domain and different boundary conditions. Many different problems of this type are (or have been) studied with the help of asymptotic analysis in PDE: one expresses the parameters of one of the model as function of small parameter  $\varepsilon$  and by studying the behaviour of the solution as  $\varepsilon$  goes to 0, one proves that the solution converges to the second model. The now classical problem that might be the most closely related to ours, and that we already mentioned, is the rugosity effect. In that case, one attempts to link a model posed on a family of domains depending on a parameter  $\varepsilon$  and “converging” in some sense with the perfect slip boundary condition on each domain, and to explain how one obtains a positive slip length, or no-slip on the limit domain. In a certain sense, the problem that we propose to address is the opposite: whereas rugosity aims at explaining how a fluid can “slow down” because of the wall, our problem is to explain and justify mathematically the “speeding up” of the fluid caused by a drop of viscosity near the wall.

## 1.2 The model

The model we will study is as follows. Our spatial domain will be the square

$$\Omega = ]0, 1[ \times ]0, 1[$$

and is loosely thought of as representing half of a longitudinal section of nanotube. In this framework one can think of  $x_1$  to be the longitudinal coordinate along the tube while  $x_2$  is the radial coordinate away from the axis. The axis of the tube would be at  $x_2 = 0$  and the carbon layer is at  $x_2 = 1$ . In this setting, what we call the lateral boundaries  $x_1 = 0$  and  $x_1 = 1$  do not have much of an influence on the problem as the sequel will show. The depletion layer will thus be located around  $x_2 = 1$  and we will model it to have a typical size of  $\varepsilon$ . More precisely, let us introduce a function  $d \in C^1([0, 1])$  and the function  $\gamma_\varepsilon$  defined on  $[0, 1]$  by

$$\gamma_\varepsilon(x_1) = 1 - \varepsilon d(x_1). \quad (1)$$

We denote by  $\Gamma_\varepsilon$  the graph of  $\gamma_\varepsilon$ , defined by

$$\Gamma_\varepsilon = \{(x_1, \gamma_\varepsilon(x_1)), x_1 \in [0, 1]\}. \quad (2)$$

In our setting, the depletion layer in our setting is defined as

$$B_\varepsilon = \{(x_1, x_2) \in \Omega, \gamma_\varepsilon(x_1) < x_2 < 1\} \subset \Omega.$$

To simplify notation we will note

$$\Omega_\varepsilon = \Omega \setminus B_\varepsilon.$$

To the macroscopic variable  $x = (x_1, x_2) \in B_\varepsilon$ , we associate the microscopic variable  $y = (y_1, y_2)$  defined by

$$y_1 = x_1, \quad y_2 = \frac{1 - x_2}{\varepsilon}.$$

The depletion layer is then described in microscopic variable by

$$B_\varepsilon = \{(y_1, 1 - \varepsilon y_2), (y_1, y_2) \in \omega\},$$

where  $\omega$  is defined by

$$\omega = \{(y_1, y_2) \in (0, 1)^2, 0 < y_2 < d(y_1)\}.$$

In this paper we will not study the fluid problem directly, but present a toy model that we believe contains the main features of the full fluid problem. Basically, we will replace the

vectorial Stokes operator with a 2d scalar elliptic operator to present the basic ideas and tools. The extension to the full vectorial fluid problem is left for subsequent work.

Our equation is thus a stationary diffusion equation with a space varying coefficient to account for the depletion layer. To stay close to [21], the diffusion will take two constant values, 1 in  $\Omega_\varepsilon$  and a smaller value in the depletion layer  $B_\varepsilon$ . To parametrize the drop of viscosity inside the depletion later we introduce the parameter  $0 < \alpha < 2$ , such that inside  $B_\varepsilon$  the viscosity is  $\varepsilon^\alpha$ . In other words, our viscosity  $\mu_\varepsilon$  is defined by

$$\mu_\varepsilon(x_1, x_2) = \mathbf{1}_{\Omega_\varepsilon}(x_1, x_2) + \varepsilon^\alpha \mathbf{1}_{B_\varepsilon}(x_1, x_2) \quad (3)$$

where  $\mathbf{1}_{\Omega_\varepsilon}$  and  $\mathbf{1}_{B_\varepsilon}$  stand for the indicator functions of  $\Omega_\varepsilon$  and  $B_\varepsilon$  respectively. We will work with the PDE in  $\Omega$  where the unknown is  $u_\varepsilon$  and  $f$  is a right-hand side defined on  $\Omega$ :

$$-\operatorname{div}(\mu_\varepsilon \nabla u_\varepsilon) = f. \quad (4)$$

As usual this system is supplemented with boundary conditions. For the sake of simplicity we have chosen to put an homogeneous Dirichlet boundary condition at the bottom  $x_2 = 0$  and on the lateral boundary condition  $x_1 = 0$  and  $x_1 = 1$ . As mentioned before, our analysis does not rely much on the fact that we know that  $u_\varepsilon$  vanishes at the lateral boundaries and we think that other boundary conditions such as periodic ones for instance could be analysed exactly as in our method. On the other hand, the Dirichlet boundary condition on  $x_2 = 0$  is really important in our analysis and changing it to something else would need a careful look at our study to see what stays and what must be changed. So in the sequel we will always have

$$\begin{cases} u_\varepsilon = 0 & \text{at } x_2 = 0, \\ u_\varepsilon = 0 & \text{at } x_1 = 0 \text{ and } x_1 = 1. \end{cases} \quad (5)$$

Finally our main concern is with the boundary condition at  $x_2 = 1$ . We will study the system with an homogeneous Neumann boundary condition and an homogeneous Dirichlet boundary condition in the sequel.

To conclude this part, let us stress that we make the following analogy with the fluid problem: the no-slip property will be modeled by a Dirichlet boundary condition, the slip condition property will be modeled by a Robin (also called Fourier) boundary condition while the perfect slip boundary condition will be modeled by a Neumann boundary condition.

### 1.3 A 1d example

In this section we would like to give a foretaste of what is to come, by simplifying even more the model to a 1d case, and taking the external force  $f$  to be constant (equal to 1). We will not give full details of the computation but try and present the main features of our model. In this 1d model the geometry simplifies and  $B_\varepsilon$  just becomes  $]1 - \varepsilon, 1[$  and the viscosity is defined on the segment  $[0, 1]$  by

$$\mu_\varepsilon = \mathbf{1}_{]0, 1 - \varepsilon[} + \varepsilon^\alpha \mathbf{1}_{]1 - \varepsilon, 1[}. \quad (6)$$

The problem reduces here to finding  $u_\varepsilon \in H^1(0, 1)$  satisfying

$$\begin{cases} -(\mu_\varepsilon u'_\varepsilon)' = 1 & \text{in } ]0, 1[ \\ u_\varepsilon(0) = 0 \end{cases}$$

and either  $u'_\varepsilon(1) = 0$  for the Neumann case or  $u_\varepsilon(1) = 0$  in the Dirichlet case.

### 1.3.1 Neumann case

As in the paper, let us start with the Neumann case. It is easy to compute by hand the solution of this problem:

$$u_\varepsilon(x) = \begin{cases} \frac{1}{2}(1 - (1 - x)^2) & \text{for } 0 \leq x \leq 1 - \varepsilon \\ \frac{1}{2}(1 - \varepsilon^2) + \frac{1}{2\varepsilon^\alpha}(\varepsilon^2 - (1 - x)^2) & \text{for } 1 - \varepsilon \leq x \leq 1 \end{cases}$$

What is to be noticed in this case are the following facts:

- Whatever the value of  $\alpha$ , the sequence of functions  $u_\varepsilon$  converges to the function  $\bar{u}(x) = \frac{1}{2}(1 - (1 - x)^2)$ . Moreover  $\bar{u}$  is “determined” by what happens in the intervals  $]0, 1 - \varepsilon[$ . For instance  $\bar{u}$  is also the limit of the sequence of functions

$$\bar{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x) & \text{for } 0 \leq x \leq 1 - \varepsilon \\ u_\varepsilon(1 - \varepsilon) & \text{for } 1 - \varepsilon \leq x \leq 1 \end{cases}$$

This will be important in our analysis as the sequence  $(\bar{u}_\varepsilon)$  is obviously bounded in  $H^1(0, 1)$  whereas we have

$$\int_0^1 (u'_\varepsilon(x))^2 dx = \int_0^{1-\varepsilon} (u'_\varepsilon(x))^2 dx + \int_{1-\varepsilon}^1 (u'_\varepsilon(x))^2 dx = \frac{1 - \varepsilon^3}{3} + \frac{\varepsilon^3}{\varepsilon^{2\alpha}}$$

which means that  $(u_\varepsilon)$  is not necessarily bounded in  $H^1(0, 1)$ .

- The limit function  $\bar{u}$  is the solution to the system

$$\begin{cases} -u'' = 1 & \text{in } ]0, 1[ \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

This means that the limit problem seems to keep the same boundary condition at  $x = 1$ . This is coherent with our view: if the fluid is allowed to slip perfectly right from the start, then it cannot be “sped up” by getting slip as it is already perfectly slipping.

- The energy of the system is defined by  $\int_0^1 \mu_\varepsilon (u'_\varepsilon)^2$  and thus is equal to

$$\int_0^1 \mu_\varepsilon(x) (u'_\varepsilon(x))^2 dx = \frac{1 - \varepsilon^3}{3} + \frac{\varepsilon^3}{\varepsilon^\alpha}$$

We conclude from this that the energy of the system is not necessarily bounded if  $\alpha$  is too large. This explains why we will put some restrictions on the values of  $\alpha$ . In this particular example, the right-hand side  $f$  is a very simple and regular function, and even in this case we could not guarantee from qualitative properties alone on the data that the energy would be bounded. But since our goal is to study a physically sensible system, we will have to ensure that the energy remains bounded, and hence include this as an hypothesis in our framework. In the one-dimensional Neumann case, this corresponds to assuming that  $\alpha < 3$ .

- Finally one can compute the difference  $u_\varepsilon - \bar{u}$  to be

$$u_\varepsilon(x) - \bar{u}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 - \varepsilon \\ \left(\frac{1}{\varepsilon^\alpha} - 1\right) \frac{1}{2}(\varepsilon^2 - (1 - x)^2) & \text{if } 1 - \varepsilon < x < 1 \end{cases}$$

From this expression, it is easy to compute  $\int_0^1 (u'_\varepsilon - \bar{u}')^2 = \left(\frac{1}{\varepsilon^\alpha} - 1\right)^2 \frac{\varepsilon^3}{3}$  which means that  $u_\varepsilon$  does not converge strongly to  $\bar{u}$  in  $H^1(0, 1)$  if  $\alpha$  is larger than  $3/2$ . It is not surprising that the sequence does not converge in the  $H^1$  norm to its limit as it is customary in rugosity problems for instance [3, 6, 10]. So let us look at the limit in  $L^p$  norm instead. In this case we have

$$\int_0^1 (u_\varepsilon - \bar{u})^p = \left(\frac{1}{\varepsilon^\alpha} - 1\right)^p \frac{1}{2^p} \int_{1-\varepsilon}^1 (\varepsilon^2 - (1-x)^2)^p dx.$$

We change variable and introduce the microscopic variable  $y = (1-x)/\varepsilon$ , a variable that we will also use later on. We then obtain:

$$\int_0^1 (u_\varepsilon - \bar{u})^p = \left(\frac{1}{\varepsilon^\alpha} - 1\right)^p \frac{\varepsilon^{2p+1}}{2^p} \int_0^1 (1-y^2)^p dy.$$

From this we obtain the convergence of  $u_\varepsilon$  to  $\bar{u}$  in  $L^p$  as long as  $\alpha < \frac{2p+1}{p}$ . On the one hand, setting  $p = 1$  in this inequality yields  $\alpha < 3$ . This means that in the limit case  $\alpha = 3$  where the energy is still bounded, there is not even a convergence in  $L^1$  even if both  $u_\varepsilon$  and the limit  $\bar{u}$  are uniformly bounded in  $L^2$ . On the other hand, the inequality is satisfied for all  $p$  if  $\alpha \leq 2$  and in particular for  $p = 2$ . Finally, if  $\alpha > 2$  we can write  $p < \frac{1}{\alpha-2}$  which means the strength of the convergence decreases as  $\alpha$  increases.

This discussion on strong convergence in  $L^p$  shows that taking large values of  $\alpha$  can make the analysis more difficult, and gives another insight on the necessity to impose an upper bound on the value of  $\alpha$  in order to get useful information on the limit when  $\varepsilon$  goes to zero.

### 1.3.2 Dirichlet case

To end this section we would like to give some elements of the same  $1d$  model with a Dirichlet boundary condition. This time we consider an arbitrary (continuous or integrable for instance)  $f$  and try to solve:

$$\begin{cases} -(\mu_\varepsilon u'_\varepsilon)' = f & \text{in } ]0, 1[ \\ u_\varepsilon(0) = 0 \\ u_\varepsilon(1) = 0 \end{cases} \quad (7)$$

We will not go into detail but there is a kernel  $K_\varepsilon$  defined on  $[0, 1]^2$  such that the solution  $u_\varepsilon$  can be written as

$$u_\varepsilon(x) = \int_0^1 K_\varepsilon(x, s) f(s) ds, \quad (8)$$

where for  $s > x$ ,

$$K_\varepsilon(x, s) = \frac{\int_0^x \mu_\varepsilon^{-1} \int_s^1 \mu_\varepsilon^{-1}}{\int_0^1 \mu_\varepsilon^{-1}}, \quad (9)$$

and for  $x > s$ ,  $K_\varepsilon(x, s) = K_\varepsilon(s, x)$ . Using definition (3), a straightforward computation gives for  $s > x$

$$K_\varepsilon(x, s) = \begin{cases} \frac{x(1-\varepsilon-s+\varepsilon^{1-\alpha})}{1-\varepsilon+\varepsilon^{1-\alpha}} & \text{on } 0 \leq x \leq s \leq 1-\varepsilon \\ \frac{x \frac{1-s}{\varepsilon^\alpha}}{1-\varepsilon+\varepsilon^{1-\alpha}} & \text{for } s \in [1-\varepsilon, 1], x \in [0, 1-\varepsilon] \\ \left(1 - \frac{1}{1-\varepsilon+\varepsilon^{1-\alpha}} \frac{1-x}{\varepsilon^\alpha}\right) \frac{1-s}{\varepsilon^\alpha} & \text{on } 1-\varepsilon \leq x \leq s \leq 1 \end{cases}.$$

Hence, we can study the limiting process of  $u_\varepsilon$  as  $\varepsilon$  goes to 0 by studying the limiting process of  $K_\varepsilon$ . Interestingly we have three possible regimes:

**subcritical**  $\alpha < 1$ : in this regime  $K_\varepsilon$  converges (at least pointwise) to  $K_0(x, s) = x(1 - s)$  for  $0 \leq x \leq s \leq 1$

**critical**  $\alpha = 1$ : in this regime  $K_\varepsilon$  converges (at least pointwise) to  $K_1(x, s) = x(1 - \frac{s}{2})$  for  $0 \leq x \leq s \leq 1$

**overcritical**  $\alpha > 1$ : in this regime  $K_\varepsilon$  converges (at least pointwise) to  $K_2(x, s) = x$  for  $0 \leq x \leq s \leq 1$

and again  $K_i(s, x) = K_i(x, s)$ . Now define  $u_i(x) = \int_0^1 K_i(x, s)f(s)ds$ . Interestingly enough, these functions are solutions to diffusion problems with different boundary conditions namely:

$\alpha < 1$ :  $u_0$  solves a Dirichlet problem at  $x = 1$ :

$$\begin{cases} -u_0'' = f \\ u_0(0) = 0 \\ u_0(1) = 0 \end{cases} \quad (10)$$

$\alpha = 1$ :  $u_1$  solves a Robin problem at  $x = 1$ :

$$\begin{cases} -u_1'' = f \\ u_1(0) = 0 \\ u_1'(1) + u_1(1) = 0 \end{cases} \quad (11)$$

$\alpha > 1$ :  $u_2$  solves a Neumann problem at  $x = 1$ :

$$\begin{cases} -u_2'' = f \\ u_2(0) = 0 \\ u_2'(1) = 0 \end{cases} \quad (12)$$

Again these computations can be done by hand and the various convergence can be more or less strengthened depending on the value of  $\alpha$  just like in the Neumann case. What is promising is that this  $1d$  version of our toy model seems to lead exactly to the type of discussion we want to obtain. The sequel of the article is dedicated to understanding these questions in  $2d$  in a framework general enough to avoid computing solutions exactly, which is rarely possible.

## 1.4 Outline of the paper

The rest of the paper is organized as follows. In Section 2, we state our main results on the asymptotic behaviour of  $u_\varepsilon$ , solution to (4) and satisfying (5) as well as Neumann or Dirichlet boundary conditions on  $x_2 = 1$ , assuming a certain energy bound (Theorems 1 and 2). Relying on explicit computations, we also establish in Theorem 3 that this bound is automatically satisfied by  $u_\varepsilon$  in the case of a flat interface. Sections 3 and 4 are dedicated to the proof of Theorems 1 and 2; Section 3 gathers compactness results on  $u_\varepsilon$  and Section 4 characterizes the limit  $\bar{u}$  as the solution to the Poisson problem  $-\Delta \bar{u} = f$  in  $\Omega$ , with zero boundary condition on  $x_2 = 0$ ,  $x_1 = 0$ ,  $x_1 = 1$ , completed with the appropriate boundary condition on  $x_2 = 1$ , that expresses the effect of the drop of viscosity. In Section 5, we prove Theorem 3, in the case of Dirichlet boundary condition on  $x_2 = 1$ ; to lighten up the presentation, most of the explicit computations are postponed to Appendix A. Finally, Section 6 discusses possible generalizations of our results.



## 2 Main results

Let  $\alpha \in ]0, 2[$  and for any  $\varepsilon > 0$ , let  $\mu_\varepsilon$  be defined by (3).

### 2.1 The Neumann case

First, from standard elliptic theory (see [12]) we have the following result for fixed  $\varepsilon > 0$ :

**Proposition 1.** *Let  $V_N$  be defined by*

$$V_N = \{v \in H^1(\Omega), v|_{x_1=0} = v|_{x_1=1} = 0, v|_{x_2=0} = 0\}, \quad (13)$$

*which is a Hilbert space when endowed with the scalar product*

$$(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \quad \text{for any } v, w \in V_N.$$

*Let  $f$  be in  $L^2(\Omega)$ . Then there exists a unique function  $u_\varepsilon$  in  $V_N$ , solution of the following problem*

$$\begin{cases} -\operatorname{div}(\mu_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } x_1 = 0, x_1 = 1 \text{ and } x_2 = 0 \\ \partial_2 u_\varepsilon = 0 & \text{on } x_2 = 1 \end{cases} \quad (14)$$

*which means for all  $\phi \in V_N$*

$$\int_{\Omega \setminus B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi = \int_{\Omega} f \phi. \quad (15)$$

Before we can state our results let us give a definition.

**Definition 1.** We say that a family  $(u_\varepsilon)_{\varepsilon>0}$  of functions in  $H^1(\Omega)$  satisfy the *energy bound* if there exists a real constant  $C > 0$  independent of  $\varepsilon$  such that

$$\int_{\Omega \setminus B_\varepsilon} |\nabla u_\varepsilon|^2 + \varepsilon^\alpha \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 \leq C. \quad (16)$$

Now let us state our main results for the Neumann case. First the general result:

**Theorem 1.** *Let  $(u_\varepsilon)_\varepsilon$  be the family of solutions obtained from Proposition 1 and suppose that this family satisfy the energy bound (16). Then  $u_\varepsilon$  converges weakly in  $L^2(\Omega)$  toward  $\bar{u}$  the variational solution to the problem:*

$$\begin{cases} -\Delta \bar{u} = f & \text{in } \Omega \\ \bar{u} = 0 & \text{on } x_1 = 0, x_1 = 1 \text{ and } x_2 = 0 \\ \partial_2 \bar{u} = 0 & \text{on } x_2 = 1 \end{cases}$$

To us it is not standard elliptic theory that the family of solutions given by Proposition 1 satisfy automatically an energy bound (16) and we take it as an hypothesis in Theorem 1. We suspect that the validity of this hypothesis might not be true in general but depends on  $f$ . Nevertheless, this hypothesis is physically meaningful: it means that there is finite energy in the system independently of the scale at which we look at it which is a reasonable assumption for this procedure to make sense.

## 2.2 The Dirichlet case

Again from standard elliptic theory we can rely on the following theorem:

**Proposition 2.** *Let  $V_D$  be defined as*

$$V_D = H_0^1(\Omega) \quad (17)$$

*Let  $f$  be in  $L^2(\Omega)$ . Then there exists a unique function  $u_\varepsilon$  in  $V_D$  solution of the following problem*

$$\begin{cases} -\operatorname{div}(\mu_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } x_1 = 0, x_1 = 1 \text{ and on } x_2 = 0 \\ u_\varepsilon = 0 & \text{on } x_2 = 1 \end{cases} \quad (18)$$

*which means for all  $\phi \in V_D$*

$$\int_{\Omega \setminus B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi = \int_{\Omega} f \phi.$$

Our main results are thus:

**Theorem 2.** *Let  $(u_\varepsilon)_\varepsilon$  be the family of solutions obtained from Proposition 2 and suppose that this family satisfy the energy bound (16). Then  $u_\varepsilon$  converges weakly in  $L^2(\Omega)$  toward a function  $\bar{u}$  such that*

**if  $0 < \alpha < 1$ ,**  $\bar{u}$  *is the variational solution to the problem:*

$$\begin{cases} -\Delta \bar{u} = f & \text{in } \Omega \\ \bar{u} = 0 & \text{on } x_1 = 0, x_1 = 1 \text{ and on } x_2 = 0 \\ \bar{u} = 0 & \text{on } x_2 = 1 \end{cases}$$

**if  $\alpha = 1$ ,**  $\bar{u}$  *is the variational solution to the problem:*

$$\begin{cases} -\Delta \bar{u} = f & \text{in } \Omega \\ \bar{u} = 0 & \text{on } x_1 = 0, x_1 = 1 \text{ and on } x_2 = 0 \\ \partial_2 \bar{u} + \frac{1}{d(x_1)} \bar{u} = 0 & \text{on } x_2 = 1 \end{cases} \quad (19)$$

**if  $1 < \alpha < 2$ ,**  $\bar{u}$  *is the variational solution to the problem:*

$$\begin{cases} -\Delta \bar{u} = f & \text{in } \Omega \\ \bar{u} = 0 & \text{on } x_1 = 0, x_1 = 1 \text{ and on } x_2 = 0 \\ \partial_2 \bar{u} = 0 & \text{on } x_2 = 1 \end{cases} \quad (20)$$

## 2.3 The flat interface case

The previous results relied on the hypothesis that the family of solutions of (14) or (18) satisfy the energy bound (16) and we have already explained the relevance of this hypothesis in our physical context.

In the case of a flat interface ( $d(x_1) = 1$  and  $\gamma_\varepsilon(x_1) = 1 - \varepsilon$ ), however, we can use the simplified geometry to compute more finely the solutions of (14) and (18) using Fourier analysis and the tensorial nature of the geometry to check that the energy bound (16) is satisfied. This analysis is based on the following lemma:

**Lemma 1.** From basic Fourier theory (see for instance Th. II.10 p.52 and Problem 12 p. 64 in [23]) we can state that

- For any  $g$  in  $L^2(]0, 1[ \times ]a, b[)$  there exists a unique sequence  $(g_k)_{k \geq 1}$  of functions in  $L^2(a, b)$  such that  $\sum_{k \geq 1} \|g_k\|_{L^2(a, b)}^2 < +\infty$  and such that the series  $\sum_{k \geq 1} g_k(x_2) \sin(k\pi x_1)$  converges to  $g$  in  $L^2(]0, 1[ \times ]a, b[)$ . Moreover  $\|g\|_{L^2(]0, 1[ \times ]a, b[)}^2 = \frac{1}{2} \sum_{k \geq 1} \|g_k\|_{L^2(a, b)}^2$
- Moreover,  $g$  is in  $H^1(]0, 1[ \times ]a, b[)$  with zero lateral traces if, and only if  $g_k \in H^1(a, b)$  for all  $k \geq 1$  and  $\sum_{k \geq 1} \int_0^1 (k^2 \pi^2 |g_k(x_2)|^2 + |g'_k(x_2)|^2) dx_2 < +\infty$  and one has  $\|\partial_1 g\|_{L^2(]0, 1[ \times ]a, b[)}^2 = \frac{1}{2} \sum_{k \geq 1} k^2 \pi^2 \|g_k\|_{L^2(a, b)}^2$  and  $\|\partial_2 g\|_{L^2(]0, 1[ \times ]a, b[)}^2 = \frac{1}{2} \sum_{k \geq 1} \|g'_k\|_{L^2(a, b)}^2$

Then we can state:

**Theorem 3.** Assume  $\gamma_\varepsilon = 1 - \varepsilon$  and  $f$  is in  $L^2(\Omega)$ . Then by Lemma 1 there exists a family  $(f_k)_{k \geq 1}$  of functions in  $L^2(0, 1)$  such that

$$f(x_1, x_2) = \sum_{k \geq 1} f_k(x_2) \sin(k\pi x_1)$$

with equality in the space  $L^2(\Omega)$ .

1. For all  $k \geq 1$  and for all  $\varepsilon > 0$ , there exist  $c_k^-$  in  $H^2(0, \gamma_\varepsilon)$  and  $c_k^+$  in  $H^2(\gamma_\varepsilon, 1)$  solutions in the strong sense of

$$\begin{cases} -c_k^{-''} + k^2 \pi^2 c_k^- = f_k & \text{on } ]0, \gamma_\varepsilon[ \\ \varepsilon^\alpha (-c_k^{+'''} + k^2 \pi^2 c_k^+) = f_k & \text{on } ]\gamma_\varepsilon, 1[ \\ c_k^-(0) = 0 \\ c_k^-(\gamma_\varepsilon) = c_k^+(\gamma_\varepsilon) \\ c_k^{-'}(\gamma_\varepsilon) = \varepsilon^\alpha c_k^{+'}(\gamma_\varepsilon) \\ c_k^+(1) = 0 \end{cases} \quad (21)$$

2. For all  $\varepsilon > 0$ ,

$$\sum_{k \geq 1} \int_0^1 (k^2 \pi^2 |c_k^\pm(x_2)|^2 + |c_k^{\pm'}(x_2)|^2) dx_2 < +\infty \quad (22)$$

so one can define  $u_\varepsilon^-$  in  $H^1(\Omega_\varepsilon)$  and  $u_\varepsilon^+$  in  $H^1(B_\varepsilon)$  by

$$u_\varepsilon^\pm(x_1, x_2) = \sum_{k \geq 1} c_k^\pm(x_2) \sin(k\pi x_1).$$

3.  $u_\varepsilon^-$  does not belong to  $H^2(\Omega_\varepsilon)$  but it is in  $\{v \in H^1(\Omega_\varepsilon), \Delta v \in L^2(\Omega_\varepsilon)\}$  and similarly  $u_\varepsilon^+$  belongs in  $\{v \in H^1(B_\varepsilon), \Delta v \in L^2(B_\varepsilon)\}$  and we have

$$\begin{cases} -\Delta u_\varepsilon^- = f & \text{in } \Omega_\varepsilon \\ -\varepsilon^\alpha \Delta u_\varepsilon^+ = f & \text{in } B_\varepsilon \\ u_\varepsilon^-|_{x_2=0} = 0 \\ u_\varepsilon^\pm|_{x_1=0} = u_\varepsilon^\pm|_{x_1=1} = 0 \\ u_\varepsilon^-|_{x_2=\gamma_\varepsilon} = u_\varepsilon^+|_{x_2=\gamma_\varepsilon} \\ \partial_{x_2} u_\varepsilon^-|_{x_2=\gamma_\varepsilon} = \varepsilon^\alpha \partial_{x_2} u_\varepsilon^+|_{x_2=\gamma_\varepsilon} \\ u_\varepsilon^+|_{x_2=1} = 0 \end{cases}$$

4. The function  $u_\varepsilon$  defined by

$$u_\varepsilon(x) = \begin{cases} u_\varepsilon^-(x) & \text{if } x \in \Omega_\varepsilon \\ u_\varepsilon^+(x) & \text{if } x \in B_\varepsilon \end{cases}$$

is in  $H^1(\Omega)$  and is the variational solution to (18).

5. The family  $(u_\varepsilon)_\varepsilon$  satisfies (16) as long as  $\alpha \leq 2$ .

### 3 Asymptotic analysis of a sequence $(u_\varepsilon)$ satisfying the energy bound (16)

The aim of this section is to analyze the behaviour as  $\varepsilon$  goes to 0 of a family of function  $(u_\varepsilon)_\varepsilon$  in  $V$  which can be  $V_N$  (recall that it is defined by (13)) or  $V_D$  (defined by (17)) satisfying the energy bound (16).

Thus in the rest of this section,  $(u_\varepsilon)_\varepsilon$  will be a sequence of functions in  $V$  satisfying (16) but not necessarily the solution of a problem for the moment.

#### 3.1 Modifying $u_\varepsilon$ in $B_\varepsilon$

Since  $u_\varepsilon$  vanishes on  $x_2 = 0$ , by Poincaré inequality, there exists a constant  $C > 0$  such that

$$\int_{\Omega_\varepsilon} |u_\varepsilon|^2 \leq C \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2. \quad (23)$$

In this inequality, the constant  $C$  can be chosen independent of  $\varepsilon$  since the domains  $\Omega \setminus B_\varepsilon$  are uniformly bounded in the  $x_2$  direction. Combining (23) with the energy bound (16), we obtain the existence of a constant  $C > 0$  such that

$$\forall \varepsilon > 0 \quad \int_{\Omega \setminus B_\varepsilon} |u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \leq C. \quad (24)$$

Since the family of functions  $\gamma_\varepsilon$  defined by (1) is uniformly Lipschitz, there exists a family of linear extension operators

$$E_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow H^1(\Omega)$$

such that for any  $\varepsilon > 0$ ,

$$\forall w \in H^1(\Omega_\varepsilon) \quad E_\varepsilon(w)|_{\Omega_\varepsilon} = w$$

and a constant  $C > 0$  such that

$$\forall \varepsilon > 0 \quad \|E_\varepsilon\|_{\mathcal{L}(H^1(\Omega_\varepsilon), H^1(\Omega))} \leq C$$

(see for instance [11]). Throughout the paper, we will note  $\bar{u}_\varepsilon$  the function defined in  $H^1(\Omega)$  by

$$\bar{u}_\varepsilon = E_\varepsilon(u_\varepsilon). \quad (25)$$

Combining the previous bound with (24), we get that the sequence  $(\bar{u}_\varepsilon)_\varepsilon$  is bounded in  $H^1(\Omega)$ . Since  $H^1(\Omega)$  is compactly embedded in  $L^2(\Omega)$ , there exists a function  $\bar{u}$  in  $H^1(\Omega)$  such that, up to extracting a subsequence,  $\bar{u}_\varepsilon$  converges to  $\bar{u}$  weakly in  $H^1(\Omega)$  and strongly in  $L^2(\Omega)$ . Moreover, by continuity of the trace operator  $H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ , the weak limit  $\bar{u}$  also belongs to  $V$ .

### 3.2 Weak convergence of $u_\varepsilon$ to $\bar{u}$ in $L^2(\Omega)$

To prove that  $u_\varepsilon$  also converges to  $\bar{u}$  albeit only weakly in  $L^2(\Omega)$ , we rely on a density argument.

First, we establish the existence of a constant  $C > 0$  such that

$$\forall \varepsilon > 0 \quad \int_{\Omega} u_\varepsilon^2 \leq C. \quad (26)$$

Let  $\varepsilon > 0$  be fixed. By density of regular functions in  $H^1(\Omega)$ , it is not restrictive to assume that  $u_\varepsilon$  is regular. Since  $u_\varepsilon$  vanishes on  $\{x_2 = 0\}$ , integrating on vertical lines, there holds for every  $(x_1, x_2) \in B_\varepsilon$

$$\frac{1}{2} u_\varepsilon(x_1, x_2)^2 = \int_0^{\gamma_\varepsilon(x_1)} \partial_2 u_\varepsilon(x_1, s) u_\varepsilon(x_1, s) \, ds + \int_{\gamma_\varepsilon(x_1)}^{x_2} \partial_2 u_\varepsilon(x_1, s) u_\varepsilon(x_1, s) \, ds.$$

Integrating the previous relation on  $B_\varepsilon$  gives

$$\begin{aligned} \frac{1}{2} \int_{B_\varepsilon} u_\varepsilon^2 &= \int_0^1 \int_{\gamma_\varepsilon(x_1)}^1 \int_0^{\gamma_\varepsilon(x_1)} \partial_2 u_\varepsilon(x_1, s) u_\varepsilon(x_1, s) \, ds \, dx_2 \, dx_1 \\ &\quad + \int_0^1 \int_{\gamma_\varepsilon(x_1)}^1 \int_{\gamma_\varepsilon(x_1)}^{x_2} \partial_2 u_\varepsilon(x_1, s) u_\varepsilon(x_1, s) \, ds \, dx_2 \, dx_1 =: I_1 + I_2. \end{aligned}$$

The first integral can be simplified as

$$I_1 = \int_0^1 (1 - \gamma_\varepsilon(x_1)) \left( \int_0^{\gamma_\varepsilon(x_1)} \partial_2 u_\varepsilon(x_1, s) u_\varepsilon(x_1, s) \, ds \right) dx_1$$

so that by definition of  $\gamma_\varepsilon$ , Cauchy-Schwarz inequality and the energy bound (16),

$$\begin{aligned} |I_1| &\leq \varepsilon \|d\|_\infty \int_{\Omega_\varepsilon} |\partial_2 u_\varepsilon| |u_\varepsilon| \\ &\leq \varepsilon C \left( \int_{\Omega_\varepsilon} u_\varepsilon^2 \right)^{1/2}. \end{aligned}$$

By Poincaré Inequality, since  $\gamma_\varepsilon(x_1) \leq 1$  for any  $x_1 \in (0, 1)$ ,

$$\int_{\Omega_\varepsilon} u_\varepsilon^2 \leq \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \leq C. \quad (27)$$

Hence,

$$|I_1| \leq \varepsilon C.$$

Intervverting the integration order between  $x_2$  and  $s$  variables in  $I_2$  yields

$$\begin{aligned} I_2 &= \int_0^1 \int_{\gamma_\varepsilon(x_1)}^1 \int_s^1 \partial_2 u_\varepsilon(x_1, s) u_\varepsilon(x_1, s) \, dx_2 \, ds \, dx_1 \\ &= \int_0^1 \int_{\gamma_\varepsilon(x_1)}^1 (1 - s) \partial_2 u_\varepsilon(x_1, s) u_\varepsilon(x_1, s) \, ds \, dx_1. \end{aligned}$$

As a result, we get the following bound:

$$\begin{aligned} |I_2| &\leq \varepsilon \|d\|_\infty \int_{B_\varepsilon} |\partial_2 u_\varepsilon| |u_\varepsilon| \\ &\leq \varepsilon \|d\|_\infty \left( \varepsilon^\alpha \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 \right)^{1/2} \left( \frac{1}{\varepsilon^\alpha} \int_{B_\varepsilon} u_\varepsilon^2 \right)^{1/2} \\ &\leq \varepsilon C \left( \frac{1}{\varepsilon^\alpha} \int_{B_\varepsilon} u_\varepsilon^2 \right)^{1/2}, \end{aligned}$$

where we have once again used the energy bound on  $\nabla u_\varepsilon$ .

Summing up the estimates on  $I_1$  and  $I_2$ , we deduce that

$$\int_{B_\varepsilon} u_\varepsilon^2 \leq C \varepsilon \left[ 1 + \frac{1}{\varepsilon^{\alpha/2}} \left( \int_{B_\varepsilon} u_\varepsilon^2 \right)^{1/2} \right].$$

Setting  $I_\varepsilon = \left( \int_{B_\varepsilon} u_\varepsilon^2 \right)^{1/2}$ , we see that  $I_\varepsilon$  satisfies

$$I_\varepsilon^2 - C\varepsilon^{1-\frac{\alpha}{2}} I_\varepsilon - C\varepsilon \leq 0.$$

Hence, the nonnegative quantity  $I_\varepsilon$  is bounded by the only positive root of the polynomial  $X^2 - C\varepsilon^{1-\frac{\alpha}{2}} X - C\varepsilon$ , that is

$$I_\varepsilon \leq \frac{1}{2} \left( C\varepsilon^{1-\frac{\alpha}{2}} + \sqrt{C^2\varepsilon^{2-\alpha} + 4C\varepsilon} \right).$$

This proves that  $\int_{B_\varepsilon} u_\varepsilon^2$  goes to zero as  $\varepsilon$  goes to zero, which combined with (27), yields (26).

Let  $\psi \in L^2(\Omega)$  and fix  $\delta > 0$ . Let  $C_c^\infty(\Omega)$  be the space of infinitely differentiable functions with compact support in  $\Omega$ . By density, there exists  $\bar{\psi} \in C_c^\infty(\Omega)$  such that  $\int_\Omega (\psi - \bar{\psi})^2 \leq \delta^2$ . Since  $\bar{\psi}$  has a compact support in  $\Omega$ , there exists  $\varepsilon_0 > 0$  such that for any  $0 < \varepsilon < \varepsilon_0$ ,  $\bar{\psi} \equiv 0$  in  $B_\varepsilon$ , so that

$$\int_\Omega u_\varepsilon \bar{\psi} = \int_{\Omega_\varepsilon} u_\varepsilon \bar{\psi} = \int_{\Omega_\varepsilon} \bar{u}_\varepsilon \bar{\psi} = \int_\Omega \bar{u}_\varepsilon \bar{\psi}.$$

By strong convergence of  $\bar{u}_\varepsilon$  to  $\bar{u}$  in  $L^2(\Omega)$ , we can pass to the limit in the last integral and deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega u_\varepsilon \bar{\psi} = \int_\Omega \bar{u} \bar{\psi}. \quad (28)$$

Now, consider the following decomposition:

$$\int_\Omega (u_\varepsilon - \bar{u}) \psi = \int_\Omega u_\varepsilon (\psi - \bar{\psi}) + \int_\Omega (u_\varepsilon - \bar{u}) \bar{\psi} + \int_\Omega \bar{u} (\bar{\psi} - \psi).$$

By Cauchy-Schwarz inequality and (26), there exists a constant  $C > 0$  such that for any  $\varepsilon > 0$ ,  $|\int_\Omega u_\varepsilon (\psi - \bar{\psi})| + |\int_\Omega \bar{u} (\bar{\psi} - \psi)| \leq C \delta$ . Using (28), we can pass to the lim sup in the previous equality to obtain

$$\limsup_{\varepsilon \rightarrow 0} \left| \int_\Omega (u_\varepsilon - \bar{u}) \psi \right| \leq C \delta,$$

which concludes the proof since  $\delta$  is arbitrary.

## 4 Identification of the limit $\bar{u}$ when $u_\varepsilon$ is the solution of (14) or (18)

In this section we will prove Theorem 1 and Theorem 2. Based on the hypothesis of these theorems, and the results of Section 3 we already know that up to extraction, the family of solutions to (14) and (18) converge in the weak  $L^2(\Omega)$  topology to some limit  $\bar{u}$ . This section aims at characterizing this limit. We will prove that  $\bar{u}$  can be seen as the weak solution of an identified elliptic problem. This problem having a unique solution, a standard argument shows that the whole sequence  $(u_\varepsilon)_\varepsilon$  also converges to  $\bar{u}$ . For this reason, we work in the sequel of this section as if the family  $(u_\varepsilon)_\varepsilon$  converges and discard any further reference to any extraction of this sequence.

## 4.1 The Neumann case

In this paragraph, we will characterize the weak limit  $\bar{u} \in V_N$  as the unique solution to the elliptic problem (20). Let  $\phi \in V_N$  be a given test function. Using the function  $\bar{u}_\varepsilon$  defined by (25), the weak formulation (15) can be rewritten as

$$\int_{\Omega_\varepsilon} \nabla \bar{u}_\varepsilon \cdot \nabla \phi + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi = \int_{\Omega} f \phi$$

or equivalently, as

$$\int_{\Omega} \nabla \bar{u}_\varepsilon \cdot \nabla \phi - \int_{B_\varepsilon} \nabla \bar{u}_\varepsilon \cdot \nabla \phi + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi = \int_{\Omega} f \phi.$$

Let us pass to the limit in every term of the left-hand side of the previous equality.

By weak convergence of  $(\bar{u}_\varepsilon)_\varepsilon$  toward  $\bar{u}$  in  $H^1(\Omega)$ , there holds immediately

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla \bar{u}_\varepsilon \cdot \nabla \phi = \int_{\Omega} \nabla \bar{u} \cdot \nabla \phi.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \int_{B_\varepsilon} \nabla \bar{u}_\varepsilon \cdot \nabla \phi \right| &\leq \left( \int_{B_\varepsilon} |\nabla \bar{u}_\varepsilon|^2 \right)^{1/2} \left( \int_{B_\varepsilon} |\nabla \phi|^2 \right)^{1/2} \\ &\leq C \left( \int_{B_\varepsilon} |\nabla \phi|^2 \right)^{1/2} \end{aligned}$$

since  $B_\varepsilon \subset \Omega$  and  $\bar{u}_\varepsilon$  is uniformly bounded in  $H^1(\Omega)$ . Writing  $\int_{B_\varepsilon} |\nabla \phi|^2 = \int_{\Omega} \mathbf{1}_{B_\varepsilon} |\nabla \phi|^2$  and noticing that  $\mathbf{1}_{B_\varepsilon}$  converges to zero a.e. in  $\Omega$ , we conclude by Lebesgue's dominated convergence theorem that  $\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} |\nabla \phi|^2 = 0$ . Thus, by the previous inequality,

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \nabla \bar{u}_\varepsilon \cdot \nabla \phi = 0.$$

Lastly, using the energy bound (16) and Young inequality, we obtain the estimate

$$\begin{aligned} \left| \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi \right| &\leq \frac{\varepsilon^\alpha}{2} \left( \varepsilon^{\alpha/2} \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 + \varepsilon^{-\alpha/2} \int_{B_\varepsilon} |\nabla \phi|^2 \right) \\ &\leq \frac{\varepsilon^{\alpha/2}}{2} \left( \varepsilon^\alpha \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 + \int_{\Omega} |\nabla \phi|^2 \right) \\ &\leq C \varepsilon^{\alpha/2}, \end{aligned}$$

whence

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi = 0.$$

Gathering the previous computations, we obtain that

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla \phi = \int_{\Omega} f \phi,$$

which proves that  $\bar{u}$  is the variational solution to problem (20), since we have already established that  $\bar{u} \in V_N$ .

## 4.2 The Dirichlet case

One major difference between the Neumann and the Dirichlet cases is that, in the Dirichlet case, the boundary condition  $u_\varepsilon = 0$  on  $x_2 = 1$  may or may not pass to the limit as  $\varepsilon$  goes to zero, depending on the exponent  $\alpha$ .

**Case  $\alpha < 1$ .** We claim that if  $0 < \alpha < 1$ , the limit  $\bar{u}$  actually satisfies the same boundary condition

$$\bar{u} = 0 \quad \text{on } x_2 = 1,$$

in other words,  $\bar{u} \in V_D$ . The distinction between the cases  $\alpha < 1$  and  $\alpha \geq 1$  comes from the following estimate of the  $L^2$ -norm of the trace of  $u_\varepsilon$  on  $\Gamma_\varepsilon$  (defined by (2)):

$$\int_{\Gamma_\varepsilon} u_\varepsilon^2 \leq C \varepsilon^{1-\alpha}. \quad (29)$$

The previous inequality can be obtained using the energy bound, the boundary condition  $u_\varepsilon(x_1, 1) = 0$  for a.e.  $x_1 \in (0, 1)$  and integrating on vertical lines  $\{x_1\} \times (1 - \varepsilon d(x_1), 1)$ , as follows:

$$\begin{aligned} \int_{\Gamma_\varepsilon} u_\varepsilon^2 &= \int_0^1 u_\varepsilon(x_1, 1 - \varepsilon d(x_1))^2 \sqrt{1 + \varepsilon^2 d'(x_1)^2} \, dx_1 \\ &\leq \sqrt{1 + \varepsilon^2 \|d'\|_\infty} \int_0^1 u_\varepsilon(x_1, 1 - \varepsilon d(x_1))^2 \, dx_1 \\ &\leq \sqrt{1 + \varepsilon^2 \|d'\|_\infty} \int_0^1 \left( \int_{1-\varepsilon d(x_1)}^1 |\partial_2 u_\varepsilon(x_1, s)| \, ds \right)^2 \, dx_1 \\ &\leq \sqrt{1 + \varepsilon^2 \|d'\|_\infty} \varepsilon \|d\|_\infty \int_0^1 \int_{1-\varepsilon d(x_1)}^1 |\partial_2 u_\varepsilon(x_1, s)|^2 \, ds \, dx_1 \\ &\leq C \varepsilon \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 \leq C \varepsilon^{1-\alpha}. \end{aligned}$$

Applying a similar argument to the function  $\bar{u}_\varepsilon$ , which is not vanishing on  $x_2 = 1$ , we obtain the following trace inequality:

$$\int_{\{x_2=1\}} \bar{u}_\varepsilon^2 \leq C \left( \varepsilon \int_{B_\varepsilon} |\nabla \bar{u}_\varepsilon|^2 + \int_{\Gamma_\varepsilon} \bar{u}_\varepsilon^2 \right).$$

Since  $\bar{u}_\varepsilon$  is bounded in  $H^1(\Omega)$ , and  $\bar{u}_\varepsilon$  and  $u_\varepsilon$  have the same trace on  $\Gamma_\varepsilon$ , (29) implies that

$$\int_{\{x_2=1\}} \bar{u}_\varepsilon^2 \leq C (\varepsilon + \varepsilon^{1-\alpha}).$$

Hence, if  $0 < \alpha < 1$ ,  $\lim_{\varepsilon \rightarrow 0} \int_{\{x_2=1\}} \bar{u}_\varepsilon^2 = 0$ .

Denote by  $T : H^1(\Omega) \rightarrow H^{1/2}(\{x_2 = 1\})$  the trace operator. Since  $T$  is linear and continuous and  $\bar{u}_\varepsilon$  converges weakly to  $\bar{u}$  in  $H^1(\Omega)$ ,

$$T\bar{u}_\varepsilon \text{ converges weakly to } T\bar{u} \quad \text{in } H^{1/2}(\{x_2 = 1\}) \quad (30)$$

(see, for instance, [5, Theorem III.9]). By compact embedding of  $H^{1/2}(\{x_2 = 1\})$  into  $L^2(\{x_2 = 1\})$ , this convergence is strong in  $L^2(\{x_2 = 1\})$ . But  $\lim_{\varepsilon \rightarrow 0} \int_{\{x_2=1\}} \bar{u}_\varepsilon^2 = 0$  so  $T\bar{u}_\varepsilon$  converges strongly to zero in  $L^2(\{x_2 = 1\})$ , whence  $T\bar{u} = 0$ . This proves that  $\bar{u} \in V_D$  if  $0 < \alpha < 1$ .



It remains to prove that for any  $\phi \in V_D$ ,

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla \phi = \int_{\Omega} f \phi.$$

This can be achieved following the same steps as in Section 4.1, starting from the equality

$$\int_{\Omega_\varepsilon} \nabla \bar{u}_\varepsilon \cdot \nabla \phi + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi = \int_{\Omega} f \phi.$$

**Case  $\alpha \geq 1$ .** In this case we need to do a finer analysis of the behaviour of  $u_\varepsilon$  in  $B_\varepsilon$ . To this aim, we introduce a rescaled function  $v_\varepsilon$ , depending on the micro-variable  $y = (y_1, y_2) \in \omega$  and defined by

$$v_\varepsilon(y) = \varepsilon^{\frac{\alpha+1}{2}} u_\varepsilon(y_1, 1 - \varepsilon y_2) \quad \text{for a.e. } y = (y_1, y_2) \in \omega. \quad (31)$$

Notice that  $v_\varepsilon$  vanishes on  $\{y_2 = 0\}$ . Setting  $x = (x_1, x_2)$  and using the change of variables

$$y_1 = x_1, \quad y_2 = \frac{1 - x_2}{\varepsilon}, \quad (32)$$

there holds

$$\begin{aligned} \int_{\omega} |\partial_1 v_\varepsilon(y)|^2 dy &= \varepsilon^\alpha \int_{B_\varepsilon} |\partial_1 u_\varepsilon(x)|^2 dx, \\ \int_{\omega} |\partial_2 v_\varepsilon(y)|^2 dy &= \varepsilon^{\alpha+2} \int_{B_\varepsilon} |\partial_2 u_\varepsilon(x)|^2 dx. \end{aligned}$$

Hence, the energy bound (16) yields the existence of a constant  $C > 0$  such that for any  $\varepsilon > 0$ ,

$$\int_{\omega} |\nabla v_\varepsilon|^2 \leq C, \quad \int_{\omega} |\partial_2 v_\varepsilon|^2 \leq C \varepsilon^2.$$

What is mainly of interest to us is that up to extraction there exists  $w$  in  $L^2(\omega)$  such that

$$\frac{1}{\varepsilon} \partial_2 v_\varepsilon \rightharpoonup w \quad \text{weakly in } L^2(\omega). \quad (33)$$

Now we can end our proof of Theorem 2 by identifying the limit  $\bar{u}$ . Let  $\phi \in V_N$  and define  $\hat{\phi} \in H^1(\omega)$  by

$$\hat{\phi}(y_1, y_2) = - \left( 1 - \frac{y_2}{d(y_1)} \right) \phi(y_1, 1) \quad \text{for a.e. } (y_1, y_2) \in \omega. \quad (34)$$

Then for every  $\varepsilon$ , the function  $\phi_\varepsilon$  defined by

$$\phi_\varepsilon(x) = \phi(x) + \mathbf{1}_{B_\varepsilon}(x) \hat{\phi} \left( x_1, \frac{1 - x_2}{\varepsilon} \right) \quad \text{for a.e. } x = (x_1, x_2) \in \Omega$$

is in  $V_D$ , so it is an admissible test function for problem (18). Testing against this function, we obtain

$$\begin{aligned} & \int_{\Omega} f \phi + \int_{B_\varepsilon} f(x) \hat{\phi} \left( x_1, \frac{1 - x_2}{\varepsilon} \right) dx \\ &= \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi \\ & \quad + \varepsilon^\alpha \left[ \int_{B_\varepsilon} \partial_1 u_\varepsilon(x) \partial_1 \hat{\phi} \left( x_1, \frac{1 - x_2}{\varepsilon} \right) dx - \int_{B_\varepsilon} \frac{1}{\varepsilon} \partial_2 u_\varepsilon(x) \partial_2 \hat{\phi} \left( x_1, \frac{1 - x_2}{\varepsilon} \right) dx \right]. \end{aligned}$$

Using the functions  $\bar{u}_\varepsilon$  introduced in Section 3 by (25) and  $v_\varepsilon$  defined by (31), and the change of variables (32), this can be rewritten as

$$\begin{aligned}
& \int_{\Omega} f\phi + \int_{B_\varepsilon} f(x) \hat{\phi}\left(x_1, \frac{1-x_2}{\varepsilon}\right) dx \\
&= \int_{\Omega} \nabla \bar{u}_\varepsilon \cdot \nabla \phi - \int_{B_\varepsilon} \nabla \bar{u}_\varepsilon \cdot \nabla \phi + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi \\
&+ \varepsilon^{\frac{\alpha-1}{2}} \left[ \int_{B_\varepsilon} \partial_1 v_\varepsilon\left(x_1, \frac{1-x_2}{\varepsilon}\right) \partial_1 \hat{\phi}\left(x_1, \frac{1-x_2}{\varepsilon}\right) dx + \int_{B_\varepsilon} \frac{1}{\varepsilon^2} \partial_2 v_\varepsilon\left(x_1, \frac{1-x_2}{\varepsilon}\right) \partial_2 \hat{\phi}\left(x_1, \frac{1-x_2}{\varepsilon}\right) dx \right] \\
&= \int_{\Omega} \nabla \bar{u}_\varepsilon \cdot \nabla \phi - \int_{B_\varepsilon} \nabla \bar{u}_\varepsilon \cdot \nabla \phi + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi + \varepsilon^{\frac{\alpha-1}{2}} \left[ \varepsilon \int_{\omega} \partial_1 v_\varepsilon \partial_1 \hat{\phi} + \int_{\omega} \frac{1}{\varepsilon} \partial_2 v_\varepsilon \partial_2 \hat{\phi} \right].
\end{aligned} \tag{35}$$

Let us examine the limit of each term of the previous equality as  $\varepsilon$  goes to zero.

- By Cauchy-Schwarz inequality and using once again the change of variables (32),

$$\begin{aligned}
\left| \int_{B_\varepsilon} f(x) \hat{\phi}\left(x_1, \frac{1-x_2}{\varepsilon}\right) dx \right| &\leq \left( \int_{B_\varepsilon} f(x)^2 dx \right)^{1/2} \left( \int_{B_\varepsilon} \hat{\phi}\left(x_1, \frac{1-x_2}{\varepsilon}\right)^2 dx \right)^{1/2} \\
&\leq \left( \int_{\Omega} f(x)^2 dx \right)^{1/2} \left( \varepsilon \int_{\omega} \hat{\phi}(y)^2 dy \right)^{1/2} \\
&\leq \varepsilon^{1/2} \|f\|_{L^2(\Omega)} \|\hat{\phi}\|_{L^2(\omega)}.
\end{aligned}$$

Hence,

$$\left| \frac{1}{\varepsilon^{\frac{\alpha-1}{2}}} \int_{B_\varepsilon} f(x) \hat{\phi}\left(x_1, \frac{1-x_2}{\varepsilon}\right) dx \right| \leq \varepsilon^{\frac{2-\alpha}{2}} \|f\|_{L^2(\Omega)} \|\hat{\phi}\|_{L^2(\omega)},$$

and since  $0 < \alpha < 2$ , we deduce that  $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\frac{\alpha-1}{2}}} \int_{B_\varepsilon} f(x) \hat{\phi}\left(x_1, \frac{1-x_2}{\varepsilon}\right) dx = 0$ .

- By the same arguments as in Subsection 4.1, we get

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla \bar{u}_\varepsilon \cdot \nabla \phi &= \int_{\Omega} \nabla \bar{u} \cdot \nabla \phi, \\
\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \nabla \bar{u}_\varepsilon \cdot \nabla \phi &= 0, \\
\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi &= 0.
\end{aligned}$$

- Since  $\partial_1 v_\varepsilon$  is uniformly bounded in  $L^2(\omega)$ , applying Cauchy-Schwarz inequality we immediately obtain that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\omega} \partial_1 v_\varepsilon \partial_1 \hat{\phi} = 0.$$

- Let  $w \in L^2(\omega)$  be the function satisfying (33), then by definition of the weak convergence,

$$\lim_{\varepsilon \rightarrow 0} \int_{\omega} \frac{1}{\varepsilon} \partial_2 v_\varepsilon \partial_2 \hat{\phi} = \int_{\omega} w \partial_2 \hat{\phi}.$$

In the case  $\alpha > 1$ , it is enough to pass to the limit in (35) and obtain the relation

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla \phi = \int_{\Omega} f \phi.$$

This proves that  $\bar{u}$  is the solution to (20).

In the critical case  $\alpha = 1$ , in order to establish that  $\bar{u}$  is the solution to (19), we need to rewrite the limit  $\int_{\omega} w \partial_2 \hat{\phi}$  as a surface integral on  $\{x_2 = 1\}$ . To this aim, we define for a.e.  $y = (y_1, y_2) \in \omega$ , the quantity  $W(y_1, y_2)$  by

$$W(y_1, y_2) = \int_0^{y_2} w(y_1, s) \, ds. \quad (36)$$

For a.e.  $y_1 \in (0, 1)$ , the function  $W(y_1, \cdot)$  is in  $H^1(0, 1)$  as a function of  $y_2$  and vanishes at  $y_2 = 0$ . Coming back to the definition of  $\hat{\phi}$  (34), we get

$$\begin{aligned} \int_{\omega} w \partial_2 \hat{\phi} &= \int_0^1 \frac{1}{d(y_1)} \phi(y_1, 1) \left( \int_0^{d(y_1)} w(y_1, y_2) \, dy_2 \right) dy_1 \\ &= \int_0^1 \frac{1}{d(y_1)} \phi(y_1, 1) W(y_1, d(y_1)) \, dy_1. \end{aligned}$$

We claim that

$$W(y_1, d(y_1)) = \bar{u}(y_1, 1) \quad \text{for a.e. } y_1 \in (0, 1), \quad (37)$$

so that the previous inequality can be rephrased as

$$\int_{\omega} w \partial_2 \hat{\phi} = \int_0^1 \frac{1}{d(x_1)} \bar{u}(x_1, 1) \phi(x_1, 1) \, dx_1.$$

Hence, passing to the limit in (35) in the case  $\alpha = 1$  yields

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla \phi + \int_0^1 \frac{1}{d(x_1)} \bar{u}(x_1, 1) \phi(x_1, 1) \, dx_1 = \int_{\Omega} f \phi,$$

which proves that  $\bar{u}$  is the weak solution to (19).

It remains to prove relation (37). Taking  $\alpha = 1$  and applying the definition of  $v_{\varepsilon}$  (31) at  $y_2 = d(y_1)$ , there holds

$$\frac{v_{\varepsilon}(y_1, d(y_1))}{\varepsilon} = u_{\varepsilon}(y_1, 1 - \varepsilon d(y_1)) \quad \text{for a.e. } y_1 \in (0, 1).$$

We will prove that, as sequences of functions of  $y_1$ :

- (i)  $\frac{v_{\varepsilon}(y_1, d(y_1))}{\varepsilon} \rightharpoonup W(y_1, d(y_1))$  weakly in  $L^2(0, 1)$  ;
- (ii)  $u_{\varepsilon}(y_1, 1 - \varepsilon d(y_1)) \rightarrow \bar{u}(y_1, 1)$  strongly in  $L^2(0, 1)$ .

Then by uniqueness of the limit, the identity (37) follows.

To prove (i), we notice that since  $v_{\varepsilon}$  vanishes on  $y_2 = 0$ ,

$$\frac{v_{\varepsilon}(y_1, d(y_1))}{\varepsilon} = \int_0^{d(y_1)} \frac{\partial_2 v_{\varepsilon}(y_1, y_2)}{\varepsilon} \, dy_2 \quad \text{for a.e. } y_1 \in (0, 1).$$

For any  $\zeta \in L^2(0, 1)$ , there holds

$$\begin{aligned} \int_0^1 \frac{v_{\varepsilon}(y_1, d(y_1))}{\varepsilon} \zeta(y_1) \, dy_1 &= \int_0^1 \int_0^{d(y_1)} \frac{\partial_2 v_{\varepsilon}(y_1, y_2)}{\varepsilon} \, dy_2 \zeta(y_1) \, dy_1 \\ &= \int_{\omega} \frac{\partial_2 v_{\varepsilon}(y)}{\varepsilon} \zeta(y_1) \, dy. \end{aligned}$$

Using property (33), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 \frac{v_\varepsilon(y_1, d(y_1))}{\varepsilon} \zeta(y_1) dy_1 = \int_\omega w(y) \zeta(y_1) dy.$$

By definition (36),  $W(y_1, d(y_1)) = \int_0^{d(y_1)} w(y_1, y_2) dy_2$  for a.e.  $y_1 \in (0, 1)$ , so the previous limit can be rewritten as

$$\int_\omega w(y) \zeta(y_1) dy = \int_0^1 \left( \int_0^{d(y_1)} w(y_1, y_2) dy_2 \right) \zeta(y_1) dy_1 = \int_0^1 W(y_1, d(y_1)) \zeta(y_1) dy_1.$$

This proves (i).

To prove (ii), we take advantage of the fact that by construction,  $u_\varepsilon$  and  $\bar{u}_\varepsilon$  have the same trace on  $y_2 = 1 - \varepsilon d(y_1)$ , and introduce for a.e.  $y_1 \in (0, 1)$  the decomposition:

$$u_\varepsilon(y_1, 1 - \varepsilon d(y_1)) - \bar{u}(y_1, 1) = (\bar{u}_\varepsilon(y_1, 1 - \varepsilon d(y_1)) - \bar{u}_\varepsilon(y_1, 1)) + (\bar{u}_\varepsilon(y_1, 1) - \bar{u}(y_1, 1)).$$

We estimate  $\int_0^1 |u_\varepsilon(y_1, 1 - \varepsilon d(y_1)) - \bar{u}(y_1, 1)|^2 dy_1$  by

$$\begin{aligned} & \int_0^1 |u_\varepsilon(y_1, 1 - \varepsilon d(y_1)) - \bar{u}(y_1, 1)|^2 dy_1 \\ & \leq 2 \left[ \int_0^1 |\bar{u}_\varepsilon(y_1, 1 - \varepsilon d(y_1)) - \bar{u}_\varepsilon(y_1, 1)|^2 dy_1 + \int_0^1 |\bar{u}(y_1, 1) - \bar{u}_\varepsilon(y_1, 1)|^2 dy_1 \right]. \end{aligned}$$

Integrating on vertical lines and using that  $\nabla \bar{u}_\varepsilon$  is uniformly bounded in  $L^2(\Omega)$  and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \int_0^1 |\bar{u}_\varepsilon(y_1, 1 - \varepsilon d(y_1)) - \bar{u}_\varepsilon(y_1, 1)|^2 dy_1 &= \int_0^1 \left| \int_{1-\varepsilon d(y_1)}^1 \partial_2 \bar{u}_\varepsilon(y_1, s) ds \right|^2 dy_1 \\ &\leq \varepsilon \|d\|_\infty \int_0^1 \int_{1-\varepsilon d(y_1)}^1 |\partial_2 \bar{u}_\varepsilon(y_1, s)|^2 ds dy_1 \\ &\leq \varepsilon \|d\|_\infty \int_\Omega |\nabla \bar{u}_\varepsilon|^2 \leq C \varepsilon. \end{aligned}$$

As a consequence of (30), up to a subsequence,  $\bar{u}_\varepsilon(y_1, 1)$  converges strongly in  $L^2_{y_1}(0, 1)$  to  $\bar{u}(y_1, 1)$ , hence  $\lim_{\varepsilon \rightarrow 0} \int_0^1 |\bar{u}(y_1, 1) - \bar{u}_\varepsilon(y_1, 1)|^2 dy_1 = 0$ . Combining this property with the previous upper bound, we deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 |u_\varepsilon(y_1, 1 - \varepsilon d(y_1)) - \bar{u}(y_1, 1)|^2 dy_1 = 0,$$

which proves (ii).

## 5 The flat interface

This section is devoted to the proof of Theorem 3. This approach being fairly standard we will concentrate on the points that matter in our specific case.

## 5.1 Proof of Theorem 3.1

For each  $k$  the function  $f_k$  is in  $L^2(0,1)$ . Then we can use Duhamel's Formula to solve the differential equations of (21) under the form:

$$\begin{aligned} c_k^-(x) &= A_k^- \operatorname{ch}(k\pi x) + B_k^- \operatorname{sh}(k\pi x) - \frac{e^{k\pi x}}{2k\pi} \int_0^x e^{-k\pi s} f_k(s) ds + \frac{e^{-k\pi x}}{2k\pi} \int_0^x e^{k\pi s} f_k(s) ds \\ c_k^+(x) &= A_k^+ \operatorname{ch}(k\pi x) + B_k^+ \operatorname{sh}(k\pi x) - \frac{e^{k\pi x}}{2k\pi} \int_{\gamma_\varepsilon}^x e^{-k\pi s} \frac{f_k(s)}{\varepsilon^\alpha} ds + \frac{e^{-k\pi x}}{2k\pi} \int_{\gamma_\varepsilon}^x e^{k\pi s} \frac{f_k(s)}{\varepsilon^\alpha} ds \end{aligned}$$

where  $A_k^\pm$ , and  $B_k^\pm$  are 4 degrees of freedom which can be set uniquely to enforce the four remaining equations of (21).

We will see subsequently more precise formulas for  $c_k^\pm$  but on this version we can clearly see that  $c_k^\pm$  are  $H^1$  functions on their respective domains since the primitives are primitives of  $L^2$  functions (see [4] for instance). But using the differential equations it is also clear that their second derivative (in the distribution sense) is also  $L^2$  and thus  $c_k^\pm$  are in fact  $H^2$  in their respective domain and solution in the strong sense of the differential equations which concludes the proof of the first point of Theorem 3.

## 5.2 Proof of Theorem 3.3 and Theorem 3.4

We delay the proof of the second point of Theorem 3 as it is actually the main difficulty in the proof of the theorem.

From Lemma 1 we already know that  $u_\varepsilon^-$  is in  $H^1(\Omega_\varepsilon)$  and  $u_\varepsilon^+$  is in  $H^1(B_\varepsilon)$ . Taking a test function  $\phi$  in  $H^1(\Omega_\varepsilon)$  and using again Lemma 1 to define the  $\phi_k$  associated, we obtain by Parseval formula that:

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla u_\varepsilon^- \cdot \nabla \phi &= \frac{1}{2} \sum_{k \geq 1} \int_0^{\gamma_\varepsilon} \left( c_k^-(x_2) \phi_k'(x_2) + k^2 \pi^2 c_k^-(x_2) \phi_k(x_2) \right) dx_2 \\ &= \frac{1}{2} \sum_{k \geq 1} \int_0^{\gamma_\varepsilon} f_k(x_2) \phi_k(x_2) dx_2 \end{aligned}$$

by construction of  $c_k^-$  and thus reversing the Parseval formula yields:

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon^- \cdot \nabla \phi = \int_{\Omega_\varepsilon} f \phi$$

This means that we have  $-\Delta u_\varepsilon^- = f$  at least in the distributional sense in  $\Omega_\varepsilon$ , and justifies that  $u_\varepsilon^-$  is in  $\{v \in H^1(\Omega_\varepsilon), \Delta v \in L^2(\Omega_\varepsilon)\}$ . A similar argument shows that  $u_\varepsilon^+$  is in  $\{v \in H^1(B_\varepsilon), \Delta v \in L^2(B_\varepsilon)\}$ . To prove the trace equalities, we start by using Lemma 1 and the estimation of Theorem 3.2 to confirm that  $u_\varepsilon^\pm$  have zero lateral trace. Now for the property of traces on  $x_2 = 0$ ,  $x_2 = \gamma_\varepsilon$  and  $x_2 = 1$ , one can see that since for all  $k$ ,  $c_k^\pm$  are in  $H^2$  of their respective domains (from the previous paragraph), then  $c_k^\pm$  are in  $\mathcal{C}^{1,1/2}$  and thus,  $u_\varepsilon^\pm$  actually belongs to  $\mathcal{C}_{x_2}^1([0, \gamma_\varepsilon], H_0^1(0, 1))$  and  $u_\varepsilon^\pm$  in  $\mathcal{C}_{x_2}^1([\gamma_\varepsilon, 1], H_0^1(0, 1))$ . Thus the remaining trace equalities can be checked directly on the Fourier coefficients and come from the construction of the  $c_k^\pm$ .

Now we can end the proof of Theorem 3.4. First, considering that  $u_\varepsilon^\pm$  are in  $H^1$  and their traces coincide on  $x_2 = \gamma_\varepsilon$  the reasoning behind Theorem 1.7.1 p75 in [15] applies and shows that  $u_\varepsilon$  is at least  $H^1(\Omega)$  and

$$\nabla u_\varepsilon = \begin{cases} \nabla u_\varepsilon^- & \text{in } \Omega_\varepsilon \\ \nabla u_\varepsilon^+ & \text{in } B_\varepsilon \end{cases}.$$

Now let us take  $v$  in  $\mathcal{C}_c^\infty(\Omega)$ . Since  $\Omega_\varepsilon$  and  $B_\varepsilon$  are rectangles, they are rectilinear polygons. Then from [15] Lemma 1.5.3.8 p. 61 and the remark following it, we can state that

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla u_\varepsilon^- \cdot \nabla v &= - \int_{\Omega_\varepsilon} \Delta u_\varepsilon^- v + \int_{\partial\Omega_\varepsilon} \partial_\nu u_\varepsilon^- v \\ \int_{B_\varepsilon} \nabla u_\varepsilon^+ \cdot \nabla v &= - \int_{B_\varepsilon} \Delta u_\varepsilon^+ v + \int_{\partial B_\varepsilon} \partial_\nu u_\varepsilon^+ v\end{aligned}$$

where  $\partial_\nu$  stand for the normal derivative in the sense of the trace. The boundary integral are in general in the sense of the duality  $\langle \mathbf{H}^{-1/2}, \mathbf{H}^{1/2} \rangle$  but here we have enough regularity and  $\partial_\nu u_\varepsilon^\pm$  are actually in  $L^2$  of the boundary.

Given that  $v$  is 0 on  $\partial\Omega$  the previous formulas can be recast as

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla u_\varepsilon^- \cdot \nabla v &= - \int_{\Omega_\varepsilon} \Delta u_\varepsilon^- v + \int_{x_2=\gamma_\varepsilon} \partial_{x_2} u_\varepsilon^- v \\ \int_{B_\varepsilon} \nabla u_\varepsilon^+ \cdot \nabla v &= - \int_{B_\varepsilon} \Delta u_\varepsilon^+ v - \int_{x_2=\gamma_\varepsilon} \partial_{x_2} u_\varepsilon^+ v\end{aligned}$$

Consequently adding the first line to  $\varepsilon^\alpha$  times the second and using the properties of  $u_\varepsilon^\pm$  yields:

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla u_\varepsilon^- \cdot \nabla v + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon^+ \cdot \nabla v &= \int_{\Omega_\varepsilon} (-\Delta u_\varepsilon^-) v + \int_{B_\varepsilon} (-\varepsilon^\alpha \Delta u_\varepsilon^+) v + \int_{x_2=\gamma_\varepsilon} (\partial_{x_2} u_\varepsilon^- - \varepsilon^\alpha \partial_{x_2} u_\varepsilon^+) v \\ &= \int_{\Omega} f v\end{aligned}$$

and thus we have

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v + \varepsilon^\alpha \int_{B_\varepsilon} \nabla u_\varepsilon \cdot \nabla v = \int_{\Omega} f v.$$

In the Dirichlet case, by density of  $\mathcal{C}_c^\infty(\Omega)$  in  $\mathbf{H}_0^1(\Omega)$ , we conclude that  $u_\varepsilon$  is the weak solution of (18). In the Neumann case, the previous argument can be easily adapted to take into account the boundary  $x_2 = 1$  and obtain that  $u_\varepsilon$  is also the solution to (14).

### 5.3 Proof of Theorem 3.2 and Theorem 3.5

Since our goal is to check (16) for  $u_\varepsilon$ , we use its construction as a Fourier series to see that because of Parseval formula :

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 + \varepsilon^\alpha \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 &= \frac{1}{2} \sum_{k \geq 1} \int_0^{\gamma_\varepsilon} (|(c_k^-)'(x_2)|^2 + k^2 \pi^2 |c_k^-(x_2)|^2) dx_2 \\ &\quad + \frac{\varepsilon^\alpha}{2} \sum_{k \geq 1} \int_{\gamma_\varepsilon}^1 (|(c_k^+)'(x_2)|^2 + k^2 \pi^2 |c_k^+(x_2)|^2) dx_2\end{aligned}$$

Consequently, we will prove Theorem 3.2 with enough detail to show simultaneously (22) and (16) in this setting which will yield Theorem 3.

The main tool is to use Duhamel's formula to represent the solutions of (21) which will be slightly different between the Neumann case and the Dirichlet case. We will present in the sequel the rationale in the case of a Dirichlet boundary condition and most technical details will be found in the Appendix.

Firstly let us introduce  $K$  defined on  $[0, 1] \times [0, 1] \times \mathbf{R}_+^*$  by

$$K(y, s, \lambda) = \begin{cases} \frac{\text{sh}(\lambda(1-s))\text{sh}(\lambda y)}{\lambda \text{sh}(\lambda)} & \text{if } y \leq s \\ \frac{\text{sh}(\lambda(1-y))\text{sh}(\lambda s)}{\lambda \text{sh}(\lambda)} & \text{if } y \geq s \end{cases} \quad (38)$$

This function is useful because of the following Lemma:

**Lemma 2.** Let  $\lambda > 0$ . If  $\beta$  is in  $L^2(0, 1)$  the function  $v(y) = \int_0^1 K(y, s, \lambda)\beta(s)ds$  is in  $H^2(0, 1)$  and a strong solution to

$$\begin{cases} -v'' + \lambda^2 v = \beta \\ v(0) = v(1) = 0 \end{cases}$$

Moreover, we have  $v'(y) = \int_0^1 \partial_y K(y, s, \lambda)\beta(s)ds$  where  $\partial_y K$  is the almost everywhere derivative of  $K$  with respect to  $y$ .

This lemma is proved using Duhamel's formula in the Appendix. Thus we can rescale  $c_k^\pm$  to be defined on  $[0, 1]$  and using the previous lemma, solve the equations up to solutions of the homogeneous equation. Then setting the degrees of freedom with the various conditions at  $x = 0$ ,  $x = \gamma_\varepsilon$  and  $x = 1$  yields the unique solution of (21) via the formulas:

$$\begin{aligned} c_k^-(x) &= \int_0^{\gamma_\varepsilon} \left[ \gamma_\varepsilon K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)\text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right] f_k(t) dt \\ &\quad + \int_{\gamma_\varepsilon}^1 \frac{\text{sh}(k\pi(1-t))\text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha} f_k(t) dt \end{aligned} \quad (39)$$

$$\begin{aligned} c_k^+(x) &= \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{1-\alpha} K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right] f_k(t) dt \\ &\quad + \int_0^{\gamma_\varepsilon} \frac{\text{sh}(k\pi t)\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha} f_k(t) dt \end{aligned} \quad (40)$$

where  $D$  is a shorthand for

$$D = \text{ch}(k\pi\gamma_\varepsilon) \frac{\text{sh}(k\pi\varepsilon)}{\varepsilon^\alpha} + \text{sh}(k\pi\gamma_\varepsilon)\text{ch}(k\pi\varepsilon) \quad (41)$$

With these formulas we obtain via Cauchy-Schwarz and Young inequalities:

$$\begin{aligned} \int_0^{\gamma_\varepsilon} |c_k^-(x)|^2 dx &\leq 2 \left( \int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \gamma_\varepsilon K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)\text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \right) \left( \int_0^{\gamma_\varepsilon} |f_k(t)|^2 dt \right) \\ &\quad + 2 \left( \int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-t))\text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha} \right]^2 dt dx \right) \left( \int_{\gamma_\varepsilon}^1 |f_k(t)|^2 dt \right) \end{aligned} \quad (42)$$

$$\begin{aligned} \int_{\gamma_\varepsilon}^1 |c_k^+(x)|^2 dx &\leq 2 \left( \int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{1-\alpha} K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right]^2 dt dx \right) \left( \int_{\gamma_\varepsilon}^1 |f_k(t)|^2 dt \right) \\ &\quad + 2 \left( \int_{\gamma_\varepsilon}^1 \int_0^{\gamma_\varepsilon} \left[ \frac{\text{sh}(k\pi t)\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha} \right]^2 dt dx \right) \left( \int_0^{\gamma_\varepsilon} |f_k(t)|^2 dt \right) \end{aligned} \quad (43)$$

It is then a matter of estimating the double integrals in the previous inequality and sort out their dependence on  $\varepsilon$  and  $k$ . The details of the computations can again be found in appendix but we find that (using a generic numerical constant  $C$  independent of  $\varepsilon$  and  $k$  that can change

from line to line):

$$\begin{aligned} \int_{\gamma_\varepsilon}^1 \int_0^{\gamma_\varepsilon} \left[ \frac{\text{sh}(k\pi t)\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha} \right]^2 dt dx &\leq C \frac{\varepsilon\gamma_\varepsilon}{k^2\pi^2} \\ \int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{1-\alpha} K\left(\frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon\right) + \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha\text{sh}(k\pi\varepsilon)} \right]^2 dt dx &\leq C \frac{\varepsilon^{2-2\alpha}}{k^2\pi^2} \\ \int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \gamma_\varepsilon K\left(\frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon\right) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)\text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha\text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx &\leq C \frac{\gamma_\varepsilon^2}{k^2\pi^2} \end{aligned}$$

From these estimations and the fact that  $\gamma_\varepsilon = O(1)$  we infer:

$$k^2\pi^2 \int_0^{\gamma_\varepsilon} |c_k^-(x)|^2 dx \leq C(1+\varepsilon)\|f_k\|_{L^2}^2 \quad (44)$$

$$\varepsilon^\alpha k^2\pi^2 \int_{\gamma_\varepsilon}^1 |c_k^+(x)|^2 dx \leq C(\varepsilon^{2-\alpha} + \varepsilon^{\alpha+1})\|f_k\|_{L^2}^2 \quad (45)$$

Now we have to conduct a similar analysis for the first derivatives of  $c_k^\pm$ . It follows the same pattern, we first differentiate equations (39) and (40):

$$\begin{aligned} (c_k^-)'(x) &= \int_0^{\gamma_\varepsilon} \left[ \partial_y K\left(\frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon\right) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)\text{ch}(k\pi x)}{D\varepsilon^\alpha\text{sh}(k\pi\gamma_\varepsilon)} \right] f_k(t) dt \\ &\quad + \int_{\gamma_\varepsilon}^1 \frac{\text{sh}(k\pi(1-t))\text{ch}(k\pi x)}{D\varepsilon^\alpha} f_k(t) dt \end{aligned} \quad (46)$$

$$\begin{aligned} (c_k^+)'(x) &= \int_{\gamma_\varepsilon}^1 \left[ -\varepsilon^{-\alpha} \partial_y K\left(\frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon\right) - \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha\text{sh}(k\pi\varepsilon)} \right] f_k(t) dt \\ &\quad - \int_0^{\gamma_\varepsilon} \frac{\text{sh}(k\pi t)\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha} f_k(t) dt \end{aligned} \quad (47)$$

Then again using Cauchy-Schwarz and Young inequality we obtain:

$$\begin{aligned} \int_0^{\gamma_\varepsilon} |(c_k^-)'(x)|^2 dx &\leq 2 \left( \int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \partial_y K\left(\frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon\right) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)\text{ch}(k\pi x)}{D\varepsilon^\alpha\text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \right) \left( \int_0^{\gamma_\varepsilon} |f_k(t)|^2 dt \right) \\ &\quad + 2 \left( \int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-t))\text{ch}(k\pi x)}{D\varepsilon^\alpha} \right]^2 dt dx \right) \left( \int_{\gamma_\varepsilon}^1 |f_k(t)|^2 dt \right) \end{aligned} \quad (48)$$

$$\begin{aligned} \int_{\gamma_\varepsilon}^1 |(c_k^+)'(x)|^2 dx &\leq 2 \left( \int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{-\alpha} \partial_y K\left(\frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon\right) - \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha\text{sh}(k\pi\varepsilon)} \right]^2 dt dx \right) \left( \int_{\gamma_\varepsilon}^1 |f_k(t)|^2 dt \right) \\ &\quad + 2 \left( \int_{\gamma_\varepsilon}^1 \int_0^{\gamma_\varepsilon} \left[ \frac{\text{sh}(k\pi t)\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha} \right]^2 dt dx \right) \left( \int_0^{\gamma_\varepsilon} |f_k(t)|^2 dt \right) \end{aligned} \quad (49)$$



and finally using the estimation from the appendix we obtain

$$\begin{aligned}
& \int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-t))\text{ch}(k\pi x)}{D\varepsilon^\alpha} \right]^2 dt dx \leq C\varepsilon\gamma_\varepsilon \\
& \int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \partial_y K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)\text{ch}(k\pi x)}{D\varepsilon^\alpha\text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \leq C\gamma_\varepsilon^2 \\
& \int_{\gamma_\varepsilon}^1 \int_0^{\gamma_\varepsilon} \left[ \frac{\text{sh}(k\pi t)\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha} \right]^2 dt dx \leq C \frac{\gamma_\varepsilon}{\varepsilon^\alpha k\pi} \\
& \int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{-\alpha} \partial_y K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha\text{sh}(k\pi\varepsilon)} \right]^2 dt dx \leq C\varepsilon^{2-2\alpha}
\end{aligned}$$

Injecting the previous estimations in (48) and (49) and adding respectively with (44) and (45) yields

$$k^2\pi^2 \int_0^{\gamma_\varepsilon} |c_k^-(x)|^2 dx + \int_0^{\gamma_\varepsilon} |(c_k^-)'(x)|^2 dx \leq C(1+\varepsilon)\|f_k\|_{L^2}^2 \quad (50)$$

$$\varepsilon^\alpha k^2\pi^2 \int_{\gamma_\varepsilon}^1 |c_k^+(x)|^2 dx + \varepsilon^\alpha \int_{\gamma_\varepsilon}^1 |(c_k^+)'(x)|^2 dx \leq C(1+\varepsilon^{2-\alpha} + \frac{1}{k\pi} + \varepsilon^{1+\alpha})\|f_k\|_{L^2}^2 \quad (51)$$

For fixed  $\varepsilon > 0$ , this yields Theorem 3.2 because the right-hand sides are summable since  $f$  is in  $L^2(\Omega)$ .

Now we can end the proof of Theorem 3.5. From Theorem 3.4 we obtain that

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 + \varepsilon^\alpha \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{\Omega_\varepsilon} |\nabla u^+|^2 + \varepsilon^\alpha \int_{B_\varepsilon} |\nabla u^-|^2$$

Then from Lemma 1 we can state that:

$$\begin{aligned}
\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 + \varepsilon^\alpha \int_{B_\varepsilon} |\nabla u_\varepsilon|^2 &= \frac{1}{2} \sum_{k \geq 1} \left( \int_0^{\gamma_\varepsilon} k^2\pi^2 |c_k^-|^2 + \int_0^{\gamma_\varepsilon} |(c_k^-)'|^2 \right) \\
&\quad + \frac{\varepsilon^\alpha}{2} \sum_{k \geq 1} \left( \int_{\gamma_\varepsilon}^1 k^2\pi^2 |c_k^+|^2 + \int_{\gamma_\varepsilon}^1 |(c_k^+)'|^2 \right) \\
&\leq C(1+\varepsilon)\|f\|_{L^2(\Omega)}^2 + C(1+\varepsilon^{2-\alpha} + \varepsilon^{1+\alpha})\|f\|_{L^2}^2
\end{aligned}$$

Thus, as long as  $\alpha \leq 2$  the solution to (18) satisfy (16) and Theorem 3 is proved.

## 6 Further comments

In this section, we examine some possible extensions of our results, in two directions:

1. considering more general boundary conditions on lateral boundaries and at the bottom;
2. imposing a smooth transition between the bulk viscosity and the reduced viscosity near the wall, rather than a jump of viscosity.

### 6.1 Adaptation to more general boundary conditions

For the sake of clarity, we have only considered homogeneous Dirichlet boundary conditions on lateral boundaries  $x_1 = 0, x_1 = 1$  and at the bottom  $x_2 = 0$ , but our results apply to more general boundary conditions.

As regards lateral boundary conditions, our analysis can be directly adapted to more general conditions, such as nonhomogeneous Dirichlet conditions, Neumann conditions or periodic ones. As a matter of fact, the condition imposed on the lateral boundaries did not play a significant role in the asymptotic analysis of the model.

Concerning the condition on the lower boundary, the requirement that  $u_\varepsilon$  vanishes on  $x_2 = 0$  is useful from a technical point of view, since it allows us to deduce from the energy bound a uniform bound of  $u_\varepsilon$  in  $H^1$  (see Section 3.1), and gain compactness. More fundamentally, to ensure that limit problems of Robin (19) or Neumann types (20) are well-posed, one should typically impose a (not necessarily homogeneous) Dirichlet boundary condition  $u_\varepsilon = g$  on  $x_2 = 0$ , or at least on a given subset of positive measure. Introducing a suitable lift of this boundary value  $g$ , one could reduce the problem to an homogeneous one and apply the same analysis.

## 6.2 Smooth drop of viscosity near the boundary

In the model described in Section 1.2, we have considered the case of a jump of viscosity along the interface  $\Gamma_\varepsilon$  located at  $x_2 = 1 - \varepsilon d(x_1)$ . Starting from Dirichlet boundary condition on  $x_2 = 1$ , in the critical case  $\alpha = 1$ , we have obtained in the limit a friction coefficient  $1/d(x_1)$  on  $x_2 = 1$  (see system (19)). In the case where  $d(x_1) \equiv d$  is constant, this corresponds to a slip length equal to  $d$  (the bulk viscosity being normalised to 1 in  $\Omega_\varepsilon$ ).

In this paragraph, we investigate the possible effect of a smooth drop of viscosity near the boundary  $x_2 = 1$ , on the value of the slip length obtained in the limit system. To illustrate this effect, we consider the simplified one-dimensional setting of Subsection 1.3.2, introducing

- a sequence  $(\delta_\varepsilon)_{\varepsilon>0}$  such that  $0 < \delta_\varepsilon < \varepsilon$  for any  $\varepsilon > 0$ ,
- a continuous, nonnegative function  $\tau : [0, 1] \rightarrow [0, +\infty[$  such that  $\tau(0) = 0$  and  $\tau(1) = 1$ ,

and replacing definition (6) of  $\mu_\varepsilon$  by

$$\mu_\varepsilon(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 - \varepsilon \\ \left( (\varepsilon^{-\alpha} - 1) \tau \left( \frac{x - (1 - \varepsilon)}{\delta_\varepsilon} \right) + 1 \right)^{-1} & \text{if } 1 - \varepsilon \leq x \leq 1 - \varepsilon + \delta_\varepsilon \\ \varepsilon^\alpha & \text{if } 1 - \varepsilon + \delta_\varepsilon \leq x \leq 1 \end{cases} \quad (52)$$

With this new definition of  $\mu_\varepsilon$ , the viscosity transitions smoothly between the bulk value 1 and the boundary value  $\varepsilon^\alpha$ ; the transition region is of length  $\delta_\varepsilon$  and the function  $\tau$  gives the viscosity profile in the transition area. The case of a jump of viscosity considered in the rest of the paper corresponds to the limit case  $\delta_\varepsilon = 0$ .

The solution  $u_\varepsilon$  of problem (7) where  $\mu_\varepsilon$  is given by (52), can be represented by formula (8), where the kernel  $K_\varepsilon$  is defined by (9). In order to pass to the limit in  $u_\varepsilon$ , it is enough to derive the expression of  $K_\varepsilon(x, s)$  for  $0 \leq x \leq s \leq 1 - \varepsilon$ , since this expression converges to the kernel of the limit  $u$  as  $\varepsilon$  goes to zero. We perform the following computations for  $0 \leq x \leq s \leq 1 - \varepsilon$ :

$$\int_0^x \mu_\varepsilon^{-1} = \int_0^x 1 = x$$

$$\begin{aligned}
\int_s^1 \mu_\varepsilon^{-1} &= \int_s^{1-\varepsilon} 1 + \int_{1-\varepsilon}^{1-\varepsilon+\delta_\varepsilon} \left[ (\varepsilon^{-\alpha} - 1) \nu \left( \frac{y - (1-\varepsilon)}{\delta_\varepsilon} \right) + 1 \right] dy + \int_{1-\varepsilon+\delta_\varepsilon}^1 \varepsilon^{-\alpha} \\
&= 1 - \varepsilon - s + \delta_\varepsilon (\varepsilon^{-\alpha} - 1) \int_0^1 \nu(y) dy + \delta_\varepsilon + \varepsilon^{-\alpha} (\varepsilon - \delta_\varepsilon) \\
&= 1 - \varepsilon - s + \varepsilon^{1-\alpha} + \delta_\varepsilon (\varepsilon^{-\alpha} - 1) \left( \int_0^1 \nu(y) dy - 1 \right) \\
&= 1 - \varepsilon - s + \varepsilon^{1-\alpha} + \frac{\delta_\varepsilon}{\varepsilon} (\varepsilon^{1-\alpha} - \varepsilon) \left( \int_0^1 \nu(y) dy - 1 \right)
\end{aligned}$$

and in particular,

$$\int_0^1 \mu_\varepsilon^{-1} = 1 - \varepsilon + \varepsilon^{1-\alpha} + \frac{\delta_\varepsilon}{\varepsilon} (\varepsilon^{1-\alpha} - \varepsilon) \left( \int_0^1 \nu(y) dy - 1 \right).$$

If  $0 < \alpha < 1$ , we see that  $K_\varepsilon(x, s) \rightarrow x(1-s)$  so that  $u_\varepsilon$  converges to the solution  $u_0$  of problem (10), as it was the case when the viscosity was given by (6).

If  $\alpha = 1$ , up to extracting a subsequence, we can assume the existence of  $\lambda \in [0, 1]$  such that  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon / \varepsilon = \lambda$ , and define the parameter  $\theta \in ]0, 1]$  by

$$\theta = \frac{1}{2 + \lambda \left( \int_0^1 \tau(y) dy - 1 \right)}.$$

In that case,  $K_\varepsilon(x, s) \rightarrow K_\theta(x, s) := x(1-\theta s)$ , and the function  $u_\theta(x) = \int_0^1 K_\theta(x, s) f(s) ds$  satisfies

$$\begin{aligned}
u_\theta(x) &= \int_0^x s(1-\theta x) f(s) ds + \int_x^1 x(1-\theta s) f(s) ds, \\
u'_\theta(x) &= -\theta \int_0^x s f(s) ds + \int_x^1 (1-\theta s) f(s) ds, \\
u''_\theta(x) &= -f(x),
\end{aligned}$$

hence  $u_\theta(1) = (1-\theta) \int_0^1 s f(s) ds$  and  $u'_\theta(1) = -\theta \int_0^1 s f(s) ds$ . Thus,  $u_\theta$  satisfies Robin boundary condition

$$u'_\theta(1) + \frac{\theta}{1-\theta} u_\theta(1) = 0.$$

The factor  $\theta/(1-\theta)$  in the previous boundary condition can be interpreted as a friction coefficient, which means that the slip length is multiplied by a factor  $(1-\theta)/\theta$ , taking into account the transition between the bulk viscosity and the boundary viscosity through the limit  $\lambda$  and the profile  $\tau$ . Notice that in the case  $\lambda = 0$ , we retrieve the limit problem (11) since  $\theta = 1/2$ .

Finally, if  $\alpha > 1$ ,  $K_\varepsilon(x, s) \rightarrow x$  so that  $u_\varepsilon$  converges again to the solution of problem (12), with Neumann boundary conditions on  $x = 1$ .

This study of a simplified 1d model suggests that working with a smooth drop of viscosity near the boundary may result in a large panel of slip lengths. This could be an interesting lead to explore in order to interpret certain large values of slip length that are reported in the experimental fluid mechanics literature.

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## A Details of the computations in the Dirichlet case

### A.1 Proof of Lemma 2

We can start with the proof of Lemma 2. Using the definition of  $K$ , the function  $v(y) = \int_0^1 K(y, s, \lambda)\beta(s)ds$  can be written as

$$v(y) = \left( \int_0^y \frac{\text{sh}(\lambda s)}{\lambda \text{sh}(\lambda)} \beta(s) ds \right) \text{sh}(\lambda(1-y)) + \left( \int_y^1 \frac{\text{sh}(\lambda(1-s))}{\lambda \text{sh}(\lambda)} \beta(s) ds \right) \text{sh}(\lambda y)$$

From [4] we know that a function is in  $H^1(]0, 1[)$  if it is the primitive of an  $L^2(]0, 1[)$  function. With this knowledge, it is clear from the previous formula that thusly defined,  $v$  is in  $H^1(]0, 1[)$ . Moreover, setting in the formula  $y = 0$  and  $1$  yields  $v(0) = v(1) = 0$  so it is actually in  $H_0^1(]0, 1[)$ . We now derive  $v$  to obtain:

$$\begin{aligned} v'(y) &= \frac{\text{sh}(\lambda y)}{\lambda \text{sh}(\lambda)} \beta(y) \text{sh}(\lambda(1-y)) - \lambda \left( \int_0^y \frac{\text{sh}(\lambda s)}{\lambda \text{sh}(\lambda)} \beta(s) ds \right) \text{ch}(\lambda(1-y)) \\ &\quad - \frac{\text{sh}(\lambda(1-y))}{\lambda \text{sh}(\lambda)} \beta(y) \text{sh}(\lambda y) + \lambda \left( \int_y^1 \frac{\text{sh}(\lambda(1-s))}{\lambda \text{sh}(\lambda)} \beta(s) ds \right) \text{ch}(\lambda y) \\ &= -\lambda \left( \int_0^y \frac{\text{sh}(\lambda s)}{\lambda \text{sh}(\lambda)} \beta(s) ds \right) \text{ch}(\lambda(1-y)) + \lambda \left( \int_y^1 \frac{\text{sh}(\lambda(1-s))}{\lambda \text{sh}(\lambda)} \beta(s) ds \right) \text{ch}(\lambda y) \end{aligned}$$

Note that we thus have  $v'(y) = \int_0^1 \partial_y K(y, s, \lambda) \beta(s) ds$  where  $\partial_y K$  is the almost everywhere derivative of  $K$  with respect to  $y$  meaning that we are allowed to derive once under the integral sign the formula for  $v$ . Moreover we have  $v'(1) = -\int_0^1 \frac{\text{sh}(\lambda s)}{\text{sh}(\lambda)} \beta(s) ds$ .

From this formula we check that  $v'$  is in  $H^1(]0, 1[)$  and thus  $v$  is in  $H^2(]0, 1[)$  as announced. Derivating a second time yields

$$\begin{aligned} v''(y) &= -\lambda \frac{\text{sh}(\lambda y)}{\lambda \text{sh}(\lambda)} \beta(y) \text{ch}(\lambda(1-y)) + \lambda^2 \left( \int_0^y \frac{\text{sh}(\lambda s)}{\lambda \text{sh}(\lambda)} \beta(s) ds \right) \text{sh}(\lambda(1-y)) \\ &\quad - \lambda \frac{\text{sh}(\lambda(1-y))}{\lambda \text{sh}(\lambda)} \beta(y) \text{ch}(\lambda y) + \lambda^2 \left( \int_y^1 \frac{\text{sh}(\lambda(1-s))}{\lambda \text{sh}(\lambda)} \beta(s) ds \right) \text{sh}(\lambda y) \\ &= -\frac{\text{sh}(\lambda(1-y)) \text{ch}(\lambda y) + \text{sh}(\lambda y) \text{ch}(\lambda(1-y))}{\text{sh}(\lambda)} \beta(y) + \lambda^2 v(y) \\ &= -\beta(y) + \lambda^2 v(y) \end{aligned}$$

using hyperbolic trigonometry.

## A.2 Proof of (39) and (40)

Let us start from expressions (39) and (40) which we recall here for convenience:

$$\begin{aligned} c_k^-(x) &= \int_0^{\gamma_\varepsilon} \left[ \gamma_\varepsilon K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon) \text{sh}(k\pi t) \text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right] f_k(t) dt \\ &\quad + \int_{\gamma_\varepsilon}^1 \frac{\text{sh}(k\pi(1-t)) \text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha} f_k(t) dt \\ c_k^+(x) &= \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{1-\alpha} K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon) \text{sh}(k\pi(1-t)) \text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right] f_k(t) dt \\ &\quad + \int_0^{\gamma_\varepsilon} \frac{\text{sh}(k\pi t) \text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha} f_k(t) dt \end{aligned}$$

- Since  $K(0, \cdot, \cdot) = 0$  we easily check that  $c_k^-(0) = c_k^+(1) = 0$ .
- Since  $K(1, \cdot, \cdot) = 0$  and  $1 - \gamma_\varepsilon = \varepsilon$  we see that  $c_k^-(\gamma_\varepsilon) = c_k^+(\gamma_\varepsilon)$ .
- We can rewrite  $c_k^- = p_k^- + h_k^-$  and  $c_k^+ = p_k^+ + h_k^+$  where

$$\begin{aligned} p_k^-(x) &= \int_0^{\gamma_\varepsilon} \gamma_\varepsilon K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) f_k(t) dt \\ h_k^-(x) &= \left[ \int_0^{\gamma_\varepsilon} \frac{\text{sh}(k\pi\varepsilon) \text{sh}(k\pi t)}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} f_k(t) dt + \int_{\gamma_\varepsilon}^1 \frac{\text{sh}(k\pi(1-t))}{k\pi D\varepsilon^\alpha} f_k(t) dt \right] \text{sh}(k\pi x) \\ p_k^+(x) &= \int_{\gamma_\varepsilon}^1 \varepsilon^{1-\alpha} K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) f_k(t) dt \\ h_k^+(x) &= \left[ \int_{\gamma_\varepsilon}^1 \frac{\text{sh}(k\pi\gamma_\varepsilon) \text{sh}(k\pi(1-t))}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} f_k(t) dt + \int_0^{\gamma_\varepsilon} \frac{\text{sh}(k\pi t)}{k\pi D\varepsilon^\alpha} f_k(t) dt \right] \text{sh}(k\pi(1-x)) \end{aligned}$$

- It is obvious from the expressions that  $-(h_k^\pm)'' + k^2\pi^2 h_k^\pm = 0$ . Now we write

$$\begin{aligned} p_k^-(\gamma_\varepsilon y) &= \gamma_\varepsilon \int_0^{\gamma_\varepsilon} K \left( y, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) f_k(t) dt = \int_0^1 K(y, s, k\pi\gamma_\varepsilon) \gamma_\varepsilon^2 f_k(\gamma_\varepsilon s) ds \\ p_k^+(1 - \varepsilon y) &= \varepsilon^{1-\alpha} \int_{\gamma_\varepsilon}^1 K \left( y, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) f_k(t) dt = \int_0^1 K(y, s, k\pi\varepsilon) \varepsilon^{2-\alpha} f_k(1 - \varepsilon s) ds \end{aligned}$$

Thus from Lemma 2 we obtain that

$$\begin{aligned} & -\frac{d^2}{dy^2}(p_k^-(\gamma_\varepsilon y)) + (k\pi\gamma_\varepsilon)^2 p_k^-(\gamma_\varepsilon y) = \gamma_\varepsilon^2 f_k(\gamma_\varepsilon y) \\ & -\frac{d^2}{dy^2}(p_k^+(1-\varepsilon y)) + (k\pi\varepsilon)^2 p_k^+(1-\varepsilon y) = \varepsilon^{2-\alpha} f_k(1-\varepsilon y) \end{aligned}$$

which is equivalent to

$$\begin{aligned} & -\gamma_\varepsilon^2 (p_k^-)''(\gamma_\varepsilon y) + \gamma_\varepsilon^2 (k\pi)^2 p_k^-(\gamma_\varepsilon y) = \gamma_\varepsilon^2 f_k(\gamma_\varepsilon y) \\ & -\varepsilon^2 (p_k^+)''(1-\varepsilon y) + \varepsilon^2 (k\pi)^2 p_k^+(1-\varepsilon y) = \varepsilon^{2-\alpha} f_k(1-\varepsilon y) \end{aligned}$$

Simplifying and setting  $x = \gamma_\varepsilon y$  in the first equation and  $x = 1 - \varepsilon y$  in the second yields

$$\begin{aligned} & -(p_k^-)'' + (k\pi)^2 p_k^- = f_k \quad \text{on } (0, \gamma_\varepsilon), \\ & -(p_k^+)'' + (k\pi)^2 p_k^+ = \frac{1}{\varepsilon^\alpha} f_k \quad \text{on } (\gamma_\varepsilon, 1). \end{aligned}$$

Combining with the equation satisfied by  $h_k^\pm$ , we obtain

$$\begin{aligned} & -(c_k^-)'' + k^2 \pi^2 c_k^- = f_k \quad \text{on } (0, \gamma_\varepsilon), \\ & -(c_k^+)'' + k^2 \pi^2 c_k^+ = \frac{f_k}{\varepsilon^\alpha} \quad \text{on } (\gamma_\varepsilon, 1). \end{aligned}$$

- Finally derivating (recall that the proof of Lemma 2 showed that we are allowed to differentiate  $p_k^\pm$  under the integral sign) and setting  $x = \gamma_\varepsilon$  yields

$$\begin{aligned} (p_k^-)'(\gamma_\varepsilon) &= \int_0^{\gamma_\varepsilon} \partial_y K \left( 1, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) f_k(t) dt = - \int_0^{\gamma_\varepsilon} \frac{\text{sh} \left( k\pi\gamma_\varepsilon \frac{t}{\gamma_\varepsilon} \right)}{\text{sh}(k\pi\gamma_\varepsilon)} f_k(t) dt \\ (h_k^-)'(\gamma_\varepsilon) &= \left[ \int_0^{\gamma_\varepsilon} \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)}{D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} f_k(t) dt + \int_{\gamma_\varepsilon}^1 \frac{\text{sh}(k\pi(1-t))}{D\varepsilon^\alpha} f_k(t) dt \right] \text{ch}(k\pi\gamma_\varepsilon) \\ (p_k^+)'\gamma_\varepsilon) &= -\frac{1}{\varepsilon^\alpha} \int_{\gamma_\varepsilon}^1 \partial_y K \left( 1, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) f_k(t) dt = \frac{1}{\varepsilon^\alpha} \int_{\gamma_\varepsilon}^1 \frac{\text{sh} \left( k\pi\varepsilon \frac{1-t}{\varepsilon} \right)}{\text{sh}(k\pi\varepsilon)} f_k(t) dt \\ (h_k^+)'\gamma_\varepsilon) &= - \left[ \int_{\gamma_\varepsilon}^1 \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))}{D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} f_k(t) dt + \int_0^{\gamma_\varepsilon} \frac{\text{sh}(k\pi t)}{D\varepsilon^\alpha} f_k(t) dt \right] \text{ch}(k\pi\varepsilon) \end{aligned}$$

We have using (41)

$$\begin{aligned} \frac{\text{sh}(k\pi\varepsilon)\text{ch}(k\pi\gamma_\varepsilon)}{D\varepsilon^\alpha} - 1 &= \frac{\text{sh}(k\pi\varepsilon)\text{ch}(k\pi\gamma_\varepsilon) - \text{ch}(k\pi\gamma_\varepsilon)\text{sh}(k\pi\varepsilon) - \varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)\text{ch}(k\pi\varepsilon)}{D\varepsilon^\alpha} \\ &= -\frac{\text{sh}(k\pi\gamma_\varepsilon)\text{ch}(k\pi\varepsilon)}{D} \\ 1 - \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{ch}(k\pi\varepsilon)}{D} &= \frac{\text{ch}(k\pi\gamma_\varepsilon)\text{sh}(k\pi\varepsilon)}{\varepsilon^\alpha D}. \end{aligned}$$

This yields

$$\begin{aligned} (c_k^-)'(\gamma_\varepsilon) &= - \int_0^{\gamma_\varepsilon} \frac{\text{ch}(k\pi\varepsilon)\text{sh}(k\pi t)}{D} f_k(t) dt + \int_{\gamma_\varepsilon}^1 \frac{\text{sh}(k\pi(1-t))\text{ch}(k\pi\gamma_\varepsilon)}{D\varepsilon^\alpha} f_k(t) dt \\ (c_k^+)'\gamma_\varepsilon) &= - \int_0^{\gamma_\varepsilon} \frac{\text{sh}(k\pi t)\text{ch}(k\pi\varepsilon)}{D\varepsilon^\alpha} f_k(t) dt + \int_{\gamma_\varepsilon}^1 \frac{\text{ch}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))}{D(\varepsilon^\alpha)^2} f_k(t) dt \end{aligned}$$

Thus we do have  $(c_k^-)'(\gamma_\varepsilon) = \varepsilon^\alpha (c_k^+)'\gamma_\varepsilon)$ .

This ends the proof that the formulas (39) and (40) define the solutions to (21) with the Dirichlet boundary condition.

### A.3 Proof of the estimates needed for the $L^2$ norm of $c_k^\pm$

As stated in Section 5.3 we now have to compute various estimates to obtain controls over the  $L^2$  norms of  $c_k^\pm$  and  $(c_k^\pm)'$ .

We start with

$$\int_{\gamma_\varepsilon}^1 \int_0^{\gamma_\varepsilon} \left[ \frac{\text{sh}(k\pi t)\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha} \right]^2 dt dx$$

By Fubini we can then deduce:

$$\begin{aligned} \int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-x))\text{sh}(k\pi t)}{k\pi D\varepsilon^\alpha} \right]^2 dx dt &= \frac{1}{(k\pi)^2 D^2 \varepsilon^{2\alpha}} \left( \int_{\gamma_\varepsilon}^1 \text{sh}(k\pi(1-x))^2 dx \right) \left( \int_0^{\gamma_\varepsilon} \text{sh}(k\pi t)^2 dt \right) \\ &= \frac{1}{(k\pi)^2 D^2 \varepsilon^{2\alpha}} \left( \frac{\text{sh}(2k\pi\varepsilon)}{4k\pi} - \frac{\varepsilon}{2} \right) \left( \frac{\text{sh}(2k\pi\gamma_\varepsilon)}{4k\pi} - \frac{\gamma_\varepsilon}{2} \right) \end{aligned}$$

Now we use that  $\text{sh}(2\theta) = 2\text{sh}(\theta)\text{ch}(\theta)$  so we can write:

$$\begin{aligned} \int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-t))\text{sh}(k\pi x)}{k\pi D} \right]^2 dx dt &= \frac{\varepsilon\gamma_\varepsilon}{4(k\pi)^2} \frac{\text{sh}(k\pi\varepsilon)^2 \text{ch}(k\pi\gamma_\varepsilon)^2}{D^2 \varepsilon^{2\alpha}} \times \\ &\quad \left( \frac{\coth(k\pi\varepsilon)}{k\pi\varepsilon} - \frac{1}{\text{sh}(k\pi\varepsilon)^2} \right) \left( \frac{\text{th}(k\pi\gamma_\varepsilon)}{k\pi\gamma_\varepsilon} - \frac{1}{\text{ch}(k\pi\gamma_\varepsilon)^2} \right) \end{aligned}$$

It is routine to check that the functions  $\theta \mapsto \coth(\theta)/\theta - 1/\text{sh}(\theta)^2$  and  $\theta \mapsto \text{th}(\theta)/\theta - 1/\text{ch}(\theta)^2$  are bounded functions over  $\mathbf{R}$ . Moreover from (41) we see that

$$\frac{\text{sh}(k\pi\varepsilon)^2 \text{ch}(k\pi\gamma_\varepsilon)^2}{D^2 \varepsilon^{2\alpha}} \leq 1$$

Thus as announced, there is a numerical constant  $C > 0$  (independent of  $k$  and  $\varepsilon$ ) such that:

$$\int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-t))\text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha} \right]^2 dt dx \leq C \frac{\varepsilon\gamma_\varepsilon}{(k\pi)^2}.$$

Now we turn to

$$\begin{aligned} &\int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \gamma_\varepsilon K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)\text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \\ &\int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{1-\alpha} K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right]^2 dt dx \end{aligned}$$

We set  $x = \gamma_\varepsilon y$  and  $t = \gamma_\varepsilon s$  in the first integral and  $x = 1 - \varepsilon y$  and  $t = 1 - \varepsilon s$  in the second one which yields:

$$\begin{aligned} &\int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \gamma_\varepsilon K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi t)\text{sh}(k\pi x)}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \\ &= \frac{\gamma_\varepsilon^2}{(k\pi)^2} \int_0^1 \int_0^1 \left[ k\pi\gamma_\varepsilon K(y, s, k\pi\gamma_\varepsilon) + \frac{\text{sh}(k\pi\varepsilon)\text{sh}(k\pi\gamma_\varepsilon s)\text{sh}(k\pi\gamma_\varepsilon y)}{D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 ds dy \\ &\int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{1-\alpha} K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi(1-t))\text{sh}(k\pi(1-x))}{k\pi D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right]^2 dt dx \\ &= \frac{\varepsilon^2}{(k\pi)^2 \varepsilon^{2\alpha}} \int_0^1 \int_0^1 \left[ k\pi\varepsilon K(y, s, k\pi\varepsilon) + \frac{\text{sh}(k\pi\gamma_\varepsilon)\text{sh}(k\pi\varepsilon s)\text{sh}(k\pi\varepsilon y)}{D\text{sh}(k\pi\varepsilon)} \right]^2 ds dy \end{aligned}$$



We thus see that evaluating these integrals amounts to computing

$$\int_0^1 \int_0^1 \left[ \lambda K(y, s, \lambda) + M \frac{\text{sh}(\lambda s) \text{sh}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy$$

in the case  $(\lambda, M) = (k\pi\gamma_\varepsilon, \text{sh}(k\pi\varepsilon)/\varepsilon^\alpha)$  and  $(\lambda, M) = (k\pi\varepsilon, \text{sh}(k\pi\gamma_\varepsilon))$ . As we can check,  $\lambda K(y, s, \lambda) + M \frac{\text{sh}(\lambda s) \text{sh}(\lambda y)}{D \text{sh}(\lambda)}$  is invariant through exchange of  $y$  and  $s$  thus we have:

$$\begin{aligned} & \int_0^1 \int_0^1 \left[ \lambda K(y, s, \lambda) + M \frac{\text{sh}(\lambda s) \text{sh}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy \\ &= 2 \iint_{0 \leq s \leq y \leq 1} \left[ \lambda K(y, s, \lambda) + M \frac{\text{sh}(\lambda s) \text{sh}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy \\ &= 2 \iint_{0 \leq s \leq y \leq 1} \left[ \frac{\text{sh}(\lambda(1-y)) \text{sh}(\lambda s)}{\text{sh}(\lambda)} + M \frac{\text{sh}(\lambda s) \text{sh}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy \\ &= \frac{2}{\text{sh}(\lambda)^2} \int_{y=0}^1 \left[ \text{sh}(\lambda(1-y)) + \frac{M}{D} \text{sh}(\lambda y) \right]^2 \left( \int_0^y \text{sh}(\lambda s)^2 ds \right) dy \\ &= \frac{1}{\text{sh}(\lambda)^2} \int_{y=0}^1 \left[ \text{sh}(\lambda(1-y)) + \frac{M}{D} \text{sh}(\lambda y) \right]^2 \left( \frac{\text{sh}(2\lambda y)}{2\lambda} - y \right) dy. \end{aligned}$$

Expanding the square and performing the integrations yield now:

$$\begin{aligned} \int_0^1 \int_0^1 \left[ \lambda K(y, s, \lambda) + M \frac{\text{sh}(\lambda s) \text{sh}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy &= \left( \frac{M \text{ch}(\lambda)}{D} \right)^2 \left( \frac{1}{2\lambda} - \frac{1}{\text{sh}(2\lambda)} \right)^2 \\ &+ \frac{1}{4 \text{sh}(\lambda)^2} \left( 2 - \frac{\text{th}(\lambda)}{\lambda} - \frac{\text{sh}(\lambda)^2}{\lambda^2} \right) \left( 1 - \frac{M}{D} \text{ch}(\lambda) \right) \\ &+ \frac{1}{2} \left( \frac{1}{\text{th}(\lambda)} - \frac{1}{\lambda} \right) \left( \frac{1}{2\lambda} - \frac{1}{\text{sh}(2\lambda)} \right) \end{aligned}$$

First we remark that since  $\text{sh}(\lambda) \geq \lambda$  for non negative  $\lambda$  by convexity, we have

$$2 - \frac{\text{th}(\lambda)}{\lambda} - \frac{\text{sh}(\lambda)^2}{\lambda^2} \leq 1 - \frac{\text{th}(\lambda)}{\lambda}.$$

Then we claim that it is again routine to check that

$$\lambda \mapsto \left( \frac{1}{2\lambda} - \frac{1}{\text{sh}(2\lambda)} \right)^2 \quad \lambda \mapsto \frac{1}{\text{sh}(\lambda)^2} \left( 1 - \frac{\text{th}(\lambda)}{\lambda} \right) \quad \lambda \mapsto \left( \frac{1}{\text{th}(\lambda)} - \frac{1}{\lambda} \right) \left( \frac{1}{2\lambda} - \frac{1}{\text{sh}(2\lambda)} \right)$$

are bounded functions over  $\mathbf{R}$ .

Finally, when  $(\lambda, M) = (k\pi\gamma_\varepsilon, \text{sh}(k\pi\varepsilon)/\varepsilon^\alpha)$ , we note that

$$\frac{M}{D} \text{ch}(\lambda) = \frac{\frac{\text{sh}(k\pi\varepsilon)}{\varepsilon^\alpha} \text{ch}(k\pi\gamma_\varepsilon)}{\text{ch}(k\pi\gamma_\varepsilon) \frac{\text{sh}(k\pi\varepsilon)}{\varepsilon^\alpha} + \text{ch}(k\pi\varepsilon) \text{sh}(k\pi\gamma_\varepsilon)} \in [0, 1]$$

and when  $(\lambda, M) = (k\pi\varepsilon, \text{sh}(k\pi\gamma_\varepsilon))$  we have

$$\frac{M}{D} \text{ch}(\lambda) = \frac{\text{sh}(k\pi\gamma_\varepsilon) \text{ch}(k\pi\varepsilon)}{\text{ch}(k\pi\gamma_\varepsilon) \frac{\text{sh}(k\pi\varepsilon)}{\varepsilon^\alpha} + \text{ch}(k\pi\varepsilon) \text{sh}(k\pi\gamma_\varepsilon)} \in [0, 1]$$

Thus we obtain, as claimed in Section 5.3, a numerical constant  $C$  such that:

$$\begin{aligned} & \int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \gamma_\varepsilon K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon) \text{sh}(k\pi t) \text{sh}(k\pi x)}{k\pi D \varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \leq C \frac{\gamma_\varepsilon^2}{(k\pi)^2} \\ & \int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{1-\alpha} K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon) \text{sh}(k\pi(1-t)) \text{sh}(k\pi(1-x))}{k\pi D \varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right]^2 dt dx \leq C \frac{\varepsilon^2}{(k\pi)^2 \varepsilon^{2\alpha}} \end{aligned}$$

## A.4 Proof of the estimates needed for the $L^2$ norm of $(c_k^\pm)'$

Again we start with the estimation of

$$\begin{aligned} & \int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-t))\text{ch}(k\pi x)}{D\varepsilon^\alpha} \right]^2 dt dx \\ & \int_{\gamma_\varepsilon}^1 \int_0^{\gamma_\varepsilon} \left[ \frac{\text{sh}(k\pi t)\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha} \right]^2 dt dx \end{aligned}$$

Changing variables  $t' = 1 - t$  and  $x' = 1 - x$  in the first integral shows that

$$\int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-t))\text{ch}(k\pi x)}{D\varepsilon^\alpha} \right]^2 dt dx = \int_\varepsilon^1 \int_0^\varepsilon \left[ \frac{\text{sh}(k\pi t')\text{ch}(k\pi(1-x'))}{D\varepsilon^\alpha} \right]^2 dt' dx'$$

so what we really want to compute is

$$\begin{aligned} \int_a^1 \int_0^a \left[ \frac{\text{sh}(k\pi t)\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha} \right]^2 dt dx &= \frac{1}{(D\varepsilon^\alpha)^2} \left( \int_a^1 \text{ch}(k\pi(1-x))^2 dx \right) \left( \int_0^a \text{sh}(k\pi t)^2 dt \right) \\ &= \frac{1}{(D\varepsilon^\alpha)^2} \left( \frac{\text{sh}(2k\pi(1-a))}{4k\pi} + \frac{1-a}{2} \right) \left( \frac{\text{sh}(2k\pi a)}{4k\pi} - \frac{a}{2} \right) \\ &= \frac{a(1-a)}{4(D\varepsilon^\alpha)^2} \left( \frac{\text{sh}(2k\pi(1-a))}{2k\pi(1-a)} + 1 \right) \left( \frac{\text{sh}(2k\pi a)}{2k\pi a} - 1 \right) \end{aligned}$$

Thus, setting  $a = \varepsilon$  yields

$$\begin{aligned} \int_\varepsilon^1 \int_0^\varepsilon \left[ \frac{\text{sh}(k\pi t)\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha} \right]^2 dt dx &= \frac{\varepsilon\gamma_\varepsilon}{4(D\varepsilon^\alpha)^2} \left( \frac{\text{sh}(2k\pi\gamma_\varepsilon)}{2k\pi\gamma_\varepsilon} + 1 \right) \left( \frac{\text{sh}(2k\pi\varepsilon)}{2k\pi\varepsilon} - 1 \right) \\ &= \frac{\varepsilon\gamma_\varepsilon \text{sh}(k\pi\varepsilon)^2 \text{ch}(k\pi\gamma_\varepsilon)^2}{4(D\varepsilon^\alpha)^2} \times \\ &\quad \left( \frac{\text{th}(k\pi\gamma_\varepsilon)}{k\pi\gamma_\varepsilon} + \frac{1}{\text{ch}(k\pi\gamma_\varepsilon)^2} \right) \left( \frac{\text{coth}(k\pi\varepsilon)}{k\pi\varepsilon} - \frac{1}{\text{sh}(k\pi\varepsilon)^2} \right) \end{aligned}$$

One can check that  $\theta \mapsto \frac{\text{th}(\theta)}{\theta} + \frac{1}{\text{ch}(\theta)^2}$  and  $\theta \mapsto \frac{\text{coth}(\theta)}{\theta} - \frac{1}{\text{sh}(\theta)^2}$  are bounded functions over  $\mathbf{R}$ . Moreover, as  $\text{sh}(k\pi\varepsilon)\text{ch}(k\pi\gamma_\varepsilon) \leq \varepsilon^\alpha D$  we obtain a numerical constant such that

$$\int_0^{\gamma_\varepsilon} \int_{\gamma_\varepsilon}^1 \left[ \frac{\text{sh}(k\pi(1-t))\text{ch}(k\pi x)}{D\varepsilon^\alpha} \right]^2 dt dx \leq C \frac{\varepsilon\gamma_\varepsilon}{4}.$$

On the other hand, setting  $a = \gamma_\varepsilon$  yields

$$\begin{aligned} \int_{\gamma_\varepsilon}^1 \int_0^{\gamma_\varepsilon} \left[ \frac{\text{sh}(k\pi t)\text{ch}(k\pi(1-x))}{D\varepsilon^\alpha} \right]^2 dt dx &= \frac{\varepsilon\gamma_\varepsilon}{4(D\varepsilon^\alpha)^2} \left( \frac{\text{sh}(2k\pi\varepsilon)}{2k\pi\varepsilon} + 1 \right) \left( \frac{\text{sh}(2k\pi\gamma_\varepsilon)}{2k\pi\gamma_\varepsilon} - 1 \right) \\ &= \frac{\gamma_\varepsilon \text{sh}(2k\pi\varepsilon)\text{sh}(2k\pi\gamma_\varepsilon)}{8(D\varepsilon^\alpha)^2(k\pi)} \times \\ &\quad \left( \frac{1}{2k\pi\gamma_\varepsilon} - \frac{1}{\text{sh}(2k\pi\gamma_\varepsilon)} \right) \left( 1 + \frac{2k\pi\varepsilon}{\text{sh}(2k\pi\varepsilon)} \right) \end{aligned}$$

Now we have

$$\begin{aligned} (D\varepsilon^\alpha)^2 &= (\text{sh}(k\pi\varepsilon)\text{ch}(k\pi\gamma_\varepsilon))^2 + 2\varepsilon^\alpha \text{sh}(k\pi\varepsilon)\text{ch}(k\pi\varepsilon)\text{sh}(k\pi\gamma_\varepsilon)\text{ch}(k\pi\gamma_\varepsilon) + (\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)\text{ch}(k\pi\varepsilon))^2 \\ &= (\text{sh}(k\pi\varepsilon)\text{ch}(k\pi\gamma_\varepsilon))^2 + \frac{1}{2}\varepsilon^\alpha \text{sh}(2k\pi\varepsilon)\text{sh}(2k\pi\gamma_\varepsilon) + (\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)\text{ch}(k\pi\varepsilon))^2 \\ &\geq \frac{1}{2}\varepsilon^\alpha \text{sh}(2k\pi\varepsilon)\text{sh}(2k\pi\gamma_\varepsilon) \end{aligned}$$

Moreover, the functions  $\theta \mapsto \frac{1}{\theta} - \frac{1}{\text{sh}(\theta)}$  and  $\theta \mapsto 1 + \frac{\theta}{\text{sh}(\theta)}$  are bounded functions over  $\mathbf{R}$  so that we have a numerical constant  $C$  such that

$$\int_{\gamma_\varepsilon}^1 \int_0^{\gamma_\varepsilon} \left[ \frac{\text{sh}(k\pi t) \text{ch}(k\pi(1-x))}{D\varepsilon^\alpha} \right]^2 dt dx \leq C \frac{\gamma_\varepsilon}{4\varepsilon^\alpha k\pi}$$

as announced.

Finally we turn to the last to integrals to estimate:

$$\begin{aligned} & \int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \partial_y K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon) \text{sh}(k\pi t) \text{ch}(k\pi x)}{D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \\ & \int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{-\alpha} \partial_y K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon) \text{sh}(k\pi(1-t)) \text{ch}(k\pi(1-x))}{D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right]^2 \end{aligned}$$

Again, setting  $x = \gamma_\varepsilon y$  and  $t = \gamma_\varepsilon s$  in the first integral and  $x = 1 - \varepsilon y$  and  $t = 1 - \varepsilon s$  in the second one yields

$$\begin{aligned} & \int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \partial_y K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon) \text{sh}(k\pi t) \text{ch}(k\pi x)}{D\varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \\ & = \gamma_\varepsilon^2 \int_0^1 \int_0^1 \left[ \partial_y K(y, s, k\pi\gamma_\varepsilon) + \frac{\text{sh}(k\pi\varepsilon) \text{sh}(k\pi\gamma_\varepsilon s) \text{ch}(k\pi\gamma_\varepsilon y)}{\varepsilon^\alpha D \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 ds dy \\ & \int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{-\alpha} \partial_y K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon) \text{sh}(k\pi(1-t)) \text{ch}(k\pi(1-x))}{D\varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right]^2 dt dx \\ & = \frac{\varepsilon^2}{\varepsilon^{2\alpha}} \int_0^1 \int_0^1 \left[ \partial_y K(y, s, k\pi\varepsilon) + \text{sh}(k\pi\gamma_\varepsilon) \frac{\text{sh}(k\pi\varepsilon s) \text{ch}(k\pi\varepsilon y)}{D \text{sh}(k\pi\varepsilon)} \right]^2 ds dy \end{aligned}$$

and we have to compute

$$\int_0^1 \int_0^1 \left[ \partial_y K(y, s, \lambda) + M \frac{\text{sh}(\lambda s) \text{ch}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy$$

with again  $(\lambda, M) = (k\pi\gamma_\varepsilon, \text{sh}(k\pi\varepsilon)/\varepsilon^\alpha)$  and  $(\lambda, M) = (k\pi\varepsilon, \text{sh}(k\pi\gamma_\varepsilon))$ . Here a difficulty comes from the fact that we don't have symmetry with respect to the exchange of  $y$  and  $s$ .

Using (38) we can state

$$\partial_y K(y, s, \lambda) = \begin{cases} \frac{\text{sh}(\lambda(1-s)) \text{ch}(\lambda y)}{\text{sh}(\lambda)} & \text{if } y \leq s \\ -\frac{\text{ch}(\lambda(1-y)) \text{sh}(\lambda s)}{\text{sh}(\lambda)} & \text{if } y \geq s \end{cases}$$

so that

$$\begin{aligned}
& \int_0^1 \int_0^1 \left[ \partial_y K(y, s, \lambda) + M \frac{\text{sh}(\lambda s) \text{ch}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy \\
&= \iint_{y \leq s} \left[ \frac{\text{sh}(\lambda(1-s)) \text{ch}(\lambda y)}{\text{sh}(\lambda)} + M \frac{\text{sh}(\lambda s) \text{ch}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy \\
&\quad + \iint_{y \geq s} \left[ -\frac{\text{ch}(\lambda(1-y)) \text{sh}(\lambda s)}{\text{sh}(\lambda)} + M \frac{\text{sh}(\lambda s) \text{ch}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy \\
&= \frac{1}{\text{sh}(\lambda)^2} \int_{s=0}^1 \left[ \text{sh}(\lambda(1-s)) + \frac{M}{D} \text{sh}(\lambda s) \right]^2 \left( \int_{y=0}^s \text{ch}(\lambda y)^2 dy \right) ds \\
&\quad + \frac{1}{\text{sh}(\lambda)^2} \int_{y=0}^1 \left[ -\text{ch}(\lambda(1-y)) + \frac{M}{D} \text{ch}(\lambda y) \right]^2 \left( \int_{s=0}^y \text{sh}(\lambda s)^2 ds \right) dy \\
&= \frac{1}{\text{sh}(\lambda)^2} \int_{s=0}^1 \left[ \text{sh}(\lambda(1-s)) + \frac{M}{D} \text{sh}(\lambda s) \right]^2 \left( \frac{\text{sh}(2\lambda s)}{4\lambda} + \frac{s}{2} \right) ds \\
&\quad + \frac{1}{\text{sh}(\lambda)^2} \int_{y=0}^1 \left[ -\text{ch}(\lambda(1-y)) + \frac{M}{D} \text{ch}(\lambda y) \right]^2 \left( \frac{\text{sh}(2\lambda y)}{4\lambda} - \frac{y}{2} \right) dy
\end{aligned}$$

Let us call  $u$  the integration variable in both integrals. Expanding the squares, regrouping similar terms and using hyperbolic trigonometry yield:

$$\begin{aligned}
& \text{sh}(\lambda)^2 \int_0^1 \int_0^1 \left[ \partial_y K(y, s, \lambda) + M \frac{\text{sh}(\lambda s) \text{ch}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy \\
&= \int_0^1 \left( \text{ch}(2\lambda(1-u)) - 2\frac{M}{D} \text{ch}(\lambda(1-2u)) + \frac{M^2}{D^2} \text{ch}(2\lambda u) \right) \frac{\text{sh}(2\lambda u)}{4\lambda} du \\
&\quad + \int_0^1 \left( -1 + 2\frac{M}{D} \text{ch}(\lambda) - \frac{M^2}{D^2} \right) \frac{u}{2} du
\end{aligned}$$

Now performing the integration finally leads to:

$$\begin{aligned}
\int_0^1 \int_0^1 \left[ \partial_y K(y, s, \lambda) + \frac{M \text{sh}(\lambda s) \text{ch}(\lambda y)}{D \text{sh}(\lambda)} \right]^2 ds dy &= \left( \frac{M \text{ch}(\lambda)}{D} \right)^2 \left( \frac{1}{(2\lambda)^2} - \frac{1}{\text{sh}(2\lambda)^2} \right) \\
&\quad + \frac{1}{4} \left( \frac{\text{coth}(\lambda)}{\lambda} - \frac{1}{\text{sh}(\lambda)^2} \right) \\
&\quad + \frac{1}{4 \text{sh}(\lambda)^2} \frac{M \text{ch}(\lambda)}{D} \left( 2 - \frac{\text{sh}(\lambda)^2}{\lambda^2} - \frac{\text{th}(\lambda)}{\lambda} \right)
\end{aligned}$$

As in the previous section, for  $(\lambda, M) = (k\pi\gamma_\varepsilon, \text{sh}(k\pi\varepsilon)/\varepsilon^\alpha)$  and  $(\lambda, M) = (k\pi\varepsilon, \text{sh}(k\pi\gamma_\varepsilon))$  we have  $M \text{ch}(\lambda)/D \in ]0, 1[$  and moreover we know that  $\lambda \mapsto \frac{1}{(2\lambda)^2} - \frac{1}{\text{sh}(2\lambda)^2}$ ,  $\lambda \mapsto \frac{\text{coth}(\lambda)}{\lambda} - \frac{1}{\text{sh}(\lambda)^2}$  and  $\lambda \mapsto \frac{1}{\text{sh}(\lambda)^2} \left( 2 - \frac{\text{sh}(\lambda)^2}{\lambda^2} - \frac{\text{th}(\lambda)}{\lambda} \right)$  are bounded from above functions.

This allows us to conclude as claimed that there is a numerical constant such that

$$\begin{aligned}
& \int_0^{\gamma_\varepsilon} \int_0^{\gamma_\varepsilon} \left[ \partial_y K \left( \frac{x}{\gamma_\varepsilon}, \frac{t}{\gamma_\varepsilon}, k\pi\gamma_\varepsilon \right) + \frac{\text{sh}(k\pi\varepsilon) \text{sh}(k\pi t) \text{ch}(k\pi x)}{D \varepsilon^\alpha \text{sh}(k\pi\gamma_\varepsilon)} \right]^2 dt dx \leq C \gamma_\varepsilon^2 \\
& \int_{\gamma_\varepsilon}^1 \int_{\gamma_\varepsilon}^1 \left[ \varepsilon^{-\alpha} \partial_y K \left( \frac{1-x}{\varepsilon}, \frac{1-t}{\varepsilon}, k\pi\varepsilon \right) + \frac{\text{sh}(k\pi\gamma_\varepsilon) \text{sh}(k\pi(1-t)) \text{ch}(k\pi(1-x))}{D \varepsilon^\alpha \text{sh}(k\pi\varepsilon)} \right]^2 dt dx \leq C \varepsilon^{2-2\alpha}
\end{aligned}$$