



# Smooth traveling-wave solutions to the inviscid surface quasi-geostrophic equations

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# Smooth traveling-wave solutions to the inviscid surface quasi-geostrophic equations

Ludovic Godard-Cadillac<sup>1</sup>

October 18, 2020

## Abstract

In a recent article by Gravejat and Smets [6], it is built smooth solutions to the inviscid surface quasi-geostrophic equation that have the form of a traveling wave. In this article we work back on their construction to provide solution to a more general class of quasi-geostrophic equation where the half-laplacian is replaced by any fractional laplacian.

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## 1 Presentation of the problem

### 1.1 The quasi-geostrophic equations

We consider the general transport equation for the vorticity of an incompressible fluid in dimension 2.

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad (1)$$

where  $\theta : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is called the *active scalar* and  $v : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}^2$  is the *velocity* of the fluid. This equation tells that the active scalar is transported by the induced velocity. Since this

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velocity  $v$  is divergence free (incompressibility condition), it is convenient to relate  $v$  and  $\theta$  through a stream function  $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ . The generalized inviscid surface quasi-geostrophic equation corresponds to a stream function that verifies

$$v = \nabla^\perp \psi \quad \text{and} \quad (-\Delta)^s \psi = \theta, \quad (2)$$

with  $s \in ]0, 1[$  and  $\perp$  denotes the rotation in the plane of angle  $\frac{\pi}{2}$ . The equations (1)-(2) are the generalized inviscid surface quasi-geostrophic equations. In the particular case  $s \rightarrow 1$  we obtain the well-known 2D Euler equation written in term of vorticity and stream function. Another important case is  $s = \frac{1}{2}$  which correspond to the work made in [6] that we generalize here. This case is the standard surface quasi-geostrophic equation that first appeared as a limit model in the context of geophysical flows [8][11]. These equations are used to model a fluid in a rotation frame with stratified density and velocity and submitted to Brunt-Väisälä thermal oscillations. This models leads to (1)-(2) using the Cafferelli-Silverstre theory on fractional laplace operator [2]. The case of the exponent  $s = \frac{1}{2}$  corresponds to the case of a Brunt-Väisälä frequency  $N$  that does not depend on the height. Other exponents for the fractional Laplace operator corresponds to different profiles for  $N$ . These equations has been intensely investigated since the work of Constantin, Majda and Tabak [4] on the case  $s = \frac{1}{2}$  where they pointed out the mathematical links that arises between (SQG- $\frac{1}{2}$ ) and the Euler equation in dimension 3. Besides stationary solution, given by a radially symmetric rearrangement on the active scalar, the only two known examples of global smooth solutions where built by Castro, Córdoba and Gómez-Serrano [3] on the one hand and by Gravejat and Smets [6] on the other hand with two different techniques. The article of Castro, Córdoba and Gómez-Serrano also provides a wide bibliography related on SQG and its Cauchy problem. In this work we generalize the result and the construction provided by [6] to the more general equations (1)-(2) with a fixed  $s \in ]0, 1[$ . The idea consists in looking for solutions that have the form of traveling waves with a positive speed  $c$  in direction  $z$ . In short, solutions of the form

$$\theta(r, z, t) = \Theta(r, z - ct), \quad v(r, z, t) = V(r, z - ct), \quad \psi(r, z, t) = \Psi(r, z - ct). \quad (3)$$

We inject this form of solution in (1)-(2) and we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}(\Theta(r, z - ct)) + V(r, z - ct) \cdot \nabla \Theta(r, z - ct) \\ &= -ce_z \cdot \nabla \Theta(r, z - ct) + V(r, z - ct) \cdot \nabla \Theta(r, z - ct) \\ &= -ce_z \cdot \nabla \Theta(r, z - ct) + \nabla^\perp \Psi(r, z - ct) \cdot \nabla \Theta(r, z - ct), \end{aligned} \quad (4)$$

where  $(e_r, e_z)$  denotes the canonical basis of  $\mathbb{R}^2$ . This leads to the orthogonality condition

$$\left( \nabla \psi - ce_r \right)^\perp \cdot \nabla \Theta = 0, \quad (5)$$

with the remark that  $e_r^\perp = e_z$ . In other words, the two vectors  $\nabla \Theta$  and  $\nabla \Psi - ce_r$  must be collinear. Following an idea from Arnold [1], Condition (5) is immediately verified if  $\Theta$  has the form

$$\Theta(r, z) = f\left(\Psi(r, z) - cr - k\right) \quad (6)$$

because in this case

$$\nabla \Theta(r, z) = f'(\Psi(r, z) - cr - k) \cdot (\nabla \Psi(r, z) - ce_r) \quad (7)$$

which gives (5). We now make consider the ansatz of a symmetry relatively to the  $z$ -axis that takes the form

$$\Psi(-r, z) = -\Psi(r, z). \quad (8)$$

This implies that  $\Theta(-r, z) = -\Theta(r, z)$  and if we denote  $V = (V_1, V_2)$  the two components of the velocity profile, then  $V_1(-r, z) = -V_1(r, z)$  and  $V_2(-r, z) = V_2(r, z)$ . More precisely, we impose the following ansatz

$$\Theta(r, z) = \begin{cases} f(\Psi(r, z) - cr - k) & \text{if } r \geq 0, \\ -f(-\Psi(r, z) + cr - k) & \text{otherwise} \end{cases} \quad (9)$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function supported in  $\mathbb{R}_+$  (to avoid a singularity at  $x = 0$ ) with the condition  $k > 0$ . Using the stream equations (2) we obtain

$$(-\Delta)^s \Psi(r, z) = \begin{cases} f(\Psi(r, z) - cr - k) & \text{if } r \geq 0, \\ -f(-\Psi(r, z) + cr - k) & \text{otherwise} \end{cases} \quad (10)$$

## 1.2 Variational formulation

The studied equation is variational and its solutions are the critical points of

$$E(\Psi) := \frac{1}{2} \int_{\mathbb{R}^2} \Psi(-\Delta)^s \Psi - \int_{\mathbb{H}} F(\Psi - cr - k) + \int_{\mathbb{H}^c} F(\Psi + cr - k), \quad (11)$$

where  $\mathbb{H} := \{x = (r, z) \in \mathbb{R}^2 : r \geq 0\}$  and  $F(\xi) := \int_0^\xi f(\xi') d\xi'$ . We are going to build a critical point of  $E$  using the technique of the Nehari manifold (defined later). For that purpose, since the choice of  $f$  is free, we are imposing on this function the following properties

$$\bullet \quad f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}), \quad f|_{\mathbb{R}_-} = 0 \quad \text{and} \quad f|_{\mathbb{R}_+^*} > 0, \quad (12)$$

$$\bullet \quad \exists \nu \in \left]1, \frac{1+s}{1-s}\right[, \quad \forall \xi \geq 0, \quad f(\xi) \leq C\xi^\nu, \quad (13)$$

$$\bullet \quad \exists \mu \in ]1, \nu[, \quad \forall \xi \geq 0, \quad \mu f(\xi) \leq \xi f'(\xi). \quad (14)$$

This last hypothesis on the variations of  $f$  is equivalent to the hypothesis that the function

$$\xi \mapsto \frac{f(\xi)}{\xi^\mu}. \quad (15)$$

is non-decreasing on  $\mathbb{R}_+$ . In particular and since  $\mu > 1$ ,

$$\forall \xi_0 \geq 0, \quad \xi \in \mathbb{R}_+ \mapsto \frac{f(\xi)}{\xi + \xi_0} \quad (16)$$

is increasing and diverging at infinity. Examples of functions that satisfies these three hypothesis (12)(13)(14) are the functions

$$\xi \mapsto \xi^\nu e^{-\frac{1}{\xi}} \mathbb{1}_{\mathbb{R}_+}(\xi), \quad (17)$$

with  $\nu \in [\mu, \nu]$ . Given the hypothesis (12) and (13), the functional  $E$  is well-defined on the Hilbert space

$$X^s := L^{\frac{2}{1-s}} \cap \dot{H}^s(\mathbb{R}^2) \quad (18)$$

with the scalar product induced by  $\dot{H}^s$  given by

$$\langle \Phi, \Psi \rangle_{X^s} := p.v. \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\Phi(x) - \Phi(y))(\Psi(x) - \Psi(y))}{|x - y|^{2(1+s)}} dx dy \quad (19)$$

where *p.v.* refers to the principal value of the singularity of the kernel  $(x, y) \mapsto 1/|x - y|^{2(1+s)}$ . For further work, we make use of the notations  $x = (r_x, z_x)$  and  $y = (r_y, z_y)$  to distinguish the coordinates of  $x$  and the coordinates of  $y$ . We recall here that the Gagliardo half-norms defining the spaces  $\dot{W}^{s,p}$  are given in general by

$$|\Phi|_{\dot{W}^{s,p}}^p := p.v. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Phi(x) - \Phi(y)|^p}{|x - y|^{d+sp}} dr dz. \quad (20)$$

For the rest of the work we refer  $E$  as being the “*energy*” of the problem although this energy does not correspond to a physical energy. We remark that it is invariant by the action of the group of symmetry generated by (8). We denote by  $X_{sym}^s$  the subspace of  $X^s$  made with the functions that are left invariant by the action of this symmetry group.

$$X_{sym}^s := \{\Psi \in X^s : \forall (r, z) \in \mathbb{R}^2, \Psi(-r, z) = -\Psi(r, z)\}. \quad (21)$$

It follows from the Palais principle of symmetric criticality [7] that any critical point of  $E$  on  $X^s$  actual belongs to  $X_{sym}^s$ . We can therefore restrict our investigations to the subspace  $X_{sym}^s$ , inside which the energy can be rewritten

$$E(\Psi) = \frac{1}{2} \|\Psi\|_{X^s}^2 - 2V(\Psi) \quad (22)$$

with

$$V(\Psi) := \int_{\mathbb{H}} F(\Psi - cr - k). \quad (23)$$

### 1.3 Nehari Manifold and presentation of the main result

The Nehari manifold associated to the energy  $E$  is defined by

$$\mathcal{N} = \{\Psi \in X_{sym}^s \setminus \{0\} : E'(\Psi)(\Psi) = 0\}, \quad (24)$$

so that  $\Psi \in \mathcal{N}$  implies

$$\int_{\mathbb{R}^2} \Psi(-\Delta)^s \Psi - 2 \int_{\mathbb{H}} f(\Psi - cr - k) \Psi = 0. \quad (25)$$

It is proven after that the Nehari manifold  $\mathcal{N}$  is a sub-manifold of  $X_{sym}^s$  non empty, of regularity  $\mathcal{C}^1$  without boundary. The main result of this article is the following theorem.

**Theorem 1.1.** *Let  $c$  and  $k$  positive. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  verifying (12), (13) and (14).*

*Then the energy  $E$  admits a minimizer  $\Psi \neq 0$  on  $\mathcal{N}$ . As a consequence there exist a non-trivial smooth solution  $\Theta$  to the inviscid quasi-geostrophic equations (1)(1) which has the form*

$$\Theta(r, z, t) = \Theta(r, z - ct) = f(\Psi(r, z - ct) - cr - k), \quad (26)$$

*for all  $(r, z) \in \mathbb{H}$  and that satisfies the symmetries  $\Theta(r, z) = -\Theta(-r, z) = \Theta(r, -z)$ , for all  $(r, z) \in \mathbb{R}^2$ . Moreover, The restriction of  $\Theta$  to  $\mathbb{H}$  is non-negative, compactly supported and non-increasing relatively to the variable  $|z|$ .*

## 2 Strategy of proof and main lemmas

We regroup in this section the main Lemmas involved in the proof of Theorem 1.1 and how they follow one another. The detailed proof of these different lemmas are provided in Section 3.

### 2.1 Properties of the Nehari Manifold and minimizing sequences

We are interested in the minimization problem

$$\alpha := \inf \{E(\Psi) : \Psi \in \mathcal{N}\}. \quad (27)$$

Since the function  $f$  worth 0 on  $\mathbb{R}_-$  then a given function  $\Psi$  cannot belong to  $\mathcal{N}$  if  $\Psi \leq 0$  on  $\mathbb{H}$ . Indeed, this would imply that

$$\int_{\mathbb{H}} f(\Psi - cr - k) \Psi = 0 \quad (28)$$

and then  $\|\Psi\|_{X^s} = 0$ . The only function on  $X_{sym}^s$  such that this quantity worth 0 is the null function which has been excluded from the definition of the Nehari manifold). We have the following description of the Nehari manifold.

**Lemma 2.1.** *The set  $\mathcal{N}$  is a  $\mathcal{C}^1$  non-empty sub-manifold of  $X_{sym}^s$ . For every  $\Psi \in X_{sym}^s$  such that  $\mathcal{L}^2(\text{supp}(\Psi_+) \cap \mathbb{H})$  is non zero<sup>1</sup>, there exist a unique  $t_\Psi > 0$  such that  $t_\Psi \Psi \in \mathcal{N}$ . The value of this  $t_\Psi$  is characterized by*

$$E(t_\Psi \Psi) = \max\{E(t\Psi) : t > 0\}. \quad (29)$$

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<sup>1</sup>The notation  $\mathcal{L}^d$  refers to the  $d$ -dimensional Lebesgue measure, ie. the lebesgue measure on  $\mathbb{R}^d$ . The function  $\Psi_+ := \max\{\Psi, 0\}$  is the positive part of  $\Psi$ .

Moreover, any local minimizer of  $E$  on  $\mathcal{N}$  is a smooth non-trivial solution of (10). We also have that

$$\beta := \inf \{ \|\Psi\|_{X^s}^2 : \Psi \in \mathcal{N} \} > 0. \quad (30)$$

and for every  $\Psi \in \mathcal{N}$ ,

$$\|\Psi\|_{X^s}^2 \leq \left(1 + \frac{1}{\mu}\right) E(\Psi). \quad (31)$$

Remark that this last assertion implies that  $\alpha$  is positive. This proposition also implies that any minimizing sequence of  $E$  on  $\mathcal{N}$  is a bounded sequence.

**Definition 2.1** (Polarization). We now define the *polarization* of a function  $\Psi \in X^s$  by

$$\forall x = (r, z) \in \mathbb{R}^2, \quad \Psi^\dagger(X) := \begin{cases} \max \{ \Psi(x), \Psi(\sigma(x)) \} & \text{if } r > 0, \\ \min \{ \Psi(x), \Psi(\sigma(x)) \} & \text{if } r < 0, \end{cases} \quad (32)$$

where  $\sigma$  denotes the linear map  $x = (r, z) \in \mathbb{R}^2 \mapsto (-r, z)$ . In the particular case  $\Psi \in X_{sym}^s$ , we obtain  $\Psi|_{\mathbb{H}}^\dagger \geq 0$  and  $\Psi|_{\mathbb{H}^c}^\dagger \leq 0$ . For more details about polarization, see for instance [9].

**Lemma 2.2** (Polarization inequality). For all  $\Psi \in \mathcal{N}$ ,

$$E(t_{\Psi^\dagger} \Psi^\dagger) \leq E(\Psi) \quad (33)$$

and this inequality is strict when  $\Psi \neq \Psi^\dagger$ .

Denote with a  $\dagger$  the image of a given set by the polarization. This lemma tells that if  $(\Psi_n)$  is a minimizing sequence for  $E$  on  $\mathcal{N}$  then so is  $t_{\Psi^\dagger} \Psi^\dagger$  because by definition of  $\Psi \mapsto t_\Psi$  the function  $t_{\Psi^\dagger} \Psi^\dagger$  belongs to  $\mathcal{N}$ . Thus, the minimizer, if it exists, belongs to  $\mathcal{N}^\dagger$ . It is then possible to restrict the investigations to  $X_{sym}^{s, \dagger}$ .

**Definition 2.2** (Steiner rearrangement). We define the *Steiner rearrangement* of  $\Psi \in X_{sym}^{s, \dagger}$ , noted  $\Psi^\sharp$ , as being the function of  $X_{sym}^{s, \dagger}$  which super-level sets on  $\mathbb{H}$  are given for all  $\nu > 0$  by

$$\{\Psi^\sharp \geq \nu\} := \bigcup_{r \in \mathbb{R}^+} \{r\} \times \left[ -\frac{\zeta_\Psi(r)}{2}, +\frac{\zeta_\Psi(r)}{2} \right] \quad (34)$$

with

$$\zeta_\Psi(r) := L^1\{z \in \mathbb{R} : \Psi(r, z) \geq \nu\}. \quad (35)$$

We extend this definition on  $\mathbb{H}^c$  by symmetry to ensure that  $\Psi^\sharp \in X_{sym}^{s, \dagger}$ .

**Lemma 2.3** (Steiner inequality). For all  $\Psi \in \mathcal{N}^\dagger$ ,

$$E(t_{\Psi^\sharp} \Psi^\sharp) \leq E(\Psi) \quad (36)$$

and the equality holds if and only if  $\Psi = \Psi^\sharp$  up to a translation on the  $z$  axis.

Then, if  $(\Psi_n)$  is a minimizing sequence for  $E$  on  $\mathcal{N}^\dagger$  then so is  $t_{\Psi^\sharp} \Psi^\sharp$ . Thus, similarly as before it is possible to restrict the investigations to  $X_{sym}^{s, \sharp}$ .

## 2.2 Existence of the solution of the minimizaing problem

Let  $(\Psi_n) \in \mathcal{N}^\sharp$  a minimizing sequence. We already know that such a sequence is bounded as a consequence of Lemma (2.1). To start with, we establish the following compactness result.

**Lemma 2.4** (compactness). Let  $c$  and  $k$  be positive. Define the map

$$T : \Psi \in X^s \mapsto \begin{cases} (\Psi - cr - k)_+ & \text{on } \mathbb{H}, \\ -(\Psi - cr + k)_- & \text{on } \mathbb{H}^c. \end{cases} \quad (37)$$

Then  $T$  maps  $X_{sym}^s$  into himself and maps bounded sets into bounded sets. Moreover, the map  $T \circ \sharp \circ \dagger$  is a compact map from  $X_{sym}^s$  into  $L_{sym}^p(\mathbb{R}^2)$ , with  $1 \leq p < \frac{2}{1-s}$ .

Up to an extraction we can suppose that the minimizing sequence  $\Psi_n \rightarrow \Psi^*$  weakly in  $X_{sym}^{s, \sharp}$  and that  $(\Psi_n - cr - k)_+ \rightarrow (\Psi^* - cr - k)_+$  strongly in  $L^p(\mathbb{H})$  for all  $p < \frac{2}{1-s}$ .

**Lemma 2.5** (convergence). The convergence of  $\Psi_n$  towards  $\Psi^*$  in  $X^{s, \sharp}$  is a strong convergence.

This implies that  $\Psi^*$  is solution to the studied minimization problem.

## 2.3 Properties of the solution

We finally define  $\Theta^*$  from  $\Psi^*$  according to formula (9). Since  $T(\Psi^*) \in L^p(\mathbb{R}^2)$  for all  $p \in [1, \frac{2}{1-s}]$  then  $\Theta^* \in L^q$  for all  $q \in [1, \frac{2}{\nu(1-s)}]$  as a consequence of (13). We have the following regularity result

**Lemma 2.6** (regularity). *The functions  $\Psi^*$  and  $\Theta^*$  are  $\mathcal{C}^\infty$ .*

We can also establish a result on the decay of  $\Psi^*$  at infinity.

**Lemma 2.7** (decay estimate). *There exists a constant  $C > 0$  such that for all  $r \in \mathbb{R}^2$ ,*

$$|\Psi^*(x)| \leq \frac{C}{1 + |x|^{2(1-s)}}, \quad (38)$$

With the positive cut-off level  $k > 0$  appearing in the definition of  $T$ , this proposition implies in particular that  $\Theta^*$  is compactly supported.

## 3 Proofs of the lemmas

### 3.1 Proofs of the lemmas of section 2.1

#### 3.1.1 Proof of Lemma 2.1

Let  $\Psi \in X_{sym}^s$  with  $\mathcal{L}^2(\text{supp}(\Psi_+) \cap \mathbb{H}) \neq 0$ . For any  $t > 0$ , we define

$$g(t) := \frac{E'(t\Psi)(t\Psi)}{t^2} = \frac{1}{2} \|\Psi\|_{X^s}^2 - 2t \int_{\mathbb{H}} f(t\Psi_+ - cr - k) \Psi_+. \quad (39)$$

We observe that the integral above can be rewritten

$$g(t) = \frac{1}{2} \|\Psi\|_{X^s}^2 - 2 \int_{\mathbb{H}} \frac{f(t\Psi_+(r, z) - cr - k)}{t\Psi_+(r, z) - cr - k + (cr + k)} (\Psi_+)^2(r, z) dr dz. \quad (40)$$

Since we have  $\mathcal{L}^2(\text{supp}(\Psi_+) \cap \mathbb{H}) \neq 0$ , our remark on the variations  $\xi \mapsto f(\xi)/(\xi + \xi_0)$ , consequence of (14), indicates that  $t \mapsto g(t)$  is decreasing and  $g(t) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Indeed, one have to apply this property of  $f$  to (40) with  $\xi = t\Psi_+(r, z) - cr - k$  and  $\xi_0 = cr + k$  and then integrate on  $\mathbb{H}$  against the non-negative weight  $(\Psi_+)^2$ . We use Hypothesis (13) to write on  $\mathbb{H}$

$$0 \leq \frac{1}{t} f(t\Psi_+ - cr - k) \Psi_+ \leq \frac{1}{t} f(t\Psi_+) \Psi_+ \leq Ct^{\nu-1} (\Psi_+)^{\nu+1}. \quad (41)$$

Since  $\nu > 1$ ,

$$g(t) \rightarrow \frac{1}{2} \|\Psi\|_{X^2}^2 > 0. \quad (42)$$

Since  $f$  is smooth then  $g$  is continuous, and then the function  $g$  admits a unique root on  $\mathbb{R}_+^*$ . The characterization (29) comes from the fact that

$$tg(t) = \frac{d}{dt} E(t\Psi). \quad (43)$$

The estimate (31) is obtained, for  $\Psi \in \mathcal{N}$ , as follows

$$\begin{aligned} E(\Psi) &= E(\Psi) - \frac{1}{\mu+1} E'(\Psi)(\Psi) \\ &= \frac{\mu}{2(\mu+1)} \|\Psi\|_{X^s}^2 + \frac{2}{\mu+1} \int_{\mathbb{H}} \left[ f(\Psi - cr - k) \Psi(r, z) - (\mu+1) F(\Psi - cr - k) \right] dr dz \\ &\geq \frac{\mu}{2(\mu+1)} \|\Psi\|_{X^s}^2, \end{aligned} \quad (44)$$

The last inequality comes from the integration on  $[0, r]$  of hypothesis (14) that gives  $\mu F(t) \leq tf(t) - F(t)$ . The fact that  $\beta$  is not zero is obtained using (13) and the Sobolev embedding

$$\|\Psi\|_{X^s}^2 = \int_{\mathbb{H}} f(\Psi_+ - cr - k)\Psi_+ \leq 4K \int_{\mathbb{H}} (\Psi_+)^{\frac{2}{1-s}} = 4K \|\Psi_+\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}} \leq C \|\Psi\|_{X^s}^{\frac{2}{1-s}}. \quad (45)$$

Concerning the regularity of  $\mathcal{N}$ , it is a consequence of the implicit functions theorem applied to  $\Xi : (s, \Psi) \mapsto E'(s\Psi)(\Psi)$  defined on the open set  $\mathbb{R}_+^* \times X_{sym}^s \setminus \{0\}$ . The hypothesis of the theorem are verified because for  $\Psi \in \mathcal{N}$  we have:

$$\partial_1 \Xi(1, \Psi) = t_{\Psi}^2 g'(t\Psi) < 0. \quad (46)$$

It remains to prove that any minimizer of  $E$  on  $\mathcal{N}$  is a critical point for  $E$  defined on the whole space. We first remark that a minimizer of  $E$  on  $N$  is a minimizer of  $\Psi \mapsto E(t\Psi\Psi)$  on  $X_{sym}^s$ . Then, using the definition of the Nehari manifold and the fact that we have  $\Psi \in \mathcal{N}$  implies  $t_{\Psi} = 1$ , we conclude

$$\forall h \in X_{sym}^s, E'(\Psi)(h) = E'(t\Psi)(\Psi)[t'_{\Psi}(h)\Psi + t_{\Psi}h] = 0. \quad (47)$$

□

### 3.2 Proof of Lemma 2.2

We first recall that  $\Psi \in \mathcal{N}$  implies that  $\mathcal{L}^2(\text{supp}(\Psi_+) \cap \mathbb{H}) \neq 0$ . Using the characterization (29) we get  $E(\Psi) \geq E(t_{\Psi^\dagger}\Psi)$ . Using the fact that  $\Psi(r, z) = -\Psi(-r, z)$ , we conclude that here the polarization consists in switching the two values of  $\Psi(r, z)$  and  $\Psi(-r, z)$  if and only if we have  $\Psi(r, z) \leq 0 \leq \Psi(-r, z)$ . Therefore since  $F$  is zero on  $\mathbb{R}_-$  and is positive on  $\mathbb{R}_+$ , we obtain

$$V(t_{\Psi^\dagger}\Psi) \leq V(t_{\Psi^\dagger}\Psi^\dagger). \quad (48)$$

To finish the proof of this lemma, we have to establish

$$\|\Psi^\dagger\|_{X^s} \leq \|\Psi\|_{X^s} \quad (49)$$

and that this inequality is strict if and only if  $\Psi^\dagger \neq \Psi$ . Actually the fact that the polarization decreases the  $W^{s,p}(\mathbb{R}^d)$  half-norms (20) is a general result so that we can establish it in the general case. By definition of the principal values, we have

$$\iint_{|x-y| \geq \varepsilon} \frac{|\Psi(x) - \Psi(y)|^p}{|x-y|^{d+sp}} dx dy \longrightarrow |u|_{W^{s,p}}^p \text{ as } \varepsilon \rightarrow 0. \quad (50)$$

We then establish the inequality for any fixed  $\varepsilon > 0$ . First, the integral is split as follows,

$$\begin{aligned} & \iint_{|x-y| \geq \varepsilon} \frac{|\Psi(x) - \Psi(y)|^p}{|x-y|^{d+sp}} dx dy \\ &= \iint_{\mathbb{H}^2 \setminus \{|x-y| < \varepsilon\}} \left( \frac{1}{|x-y|^{d+sp}} \left( |\Psi(x) - \Psi(y)|^p + |\Psi \circ \sigma(x) - \Psi \circ \sigma(y)|^p \right) \right. \\ & \quad \left. + \frac{1}{|x - \sigma(y)|^{d+sp}} \left( |\Psi(x) - \Psi \circ \sigma(y)|^p + |\Psi \circ \sigma(x) - \Psi(y)|^p \right) \right) dx dy \end{aligned} \quad (51)$$

Let  $x, y \in \mathbb{H}$ . Observe that

$$|x-y|^{d+sp} < |x - \sigma(y)|^{d+sp}. \quad (52)$$

**Case 1:**  $\Psi(x) \geq \Psi \circ \sigma(x)$  and  $\Psi(y) \geq \Psi \circ \sigma(y)$ . In this case, with the definition of the polarization,  $\Psi(x) = \Psi^\dagger(x)$  and  $\Psi(y) = \Psi^\dagger(y)$ . Then when we integrate on the couples  $(x, y)$  that belongs to Case 1, the associated term in the integral (51) is not modified by the polarization.

**Case 2:**  $\Psi(x) \geq \Psi \circ \sigma(x)$  and  $\Psi(y) < \Psi \circ \sigma(y)$ . By computing its derivative, we obtain that the function

$$u_{\beta,s} : \alpha \in \mathbb{R} \mapsto |\alpha + \beta|^p - |\alpha + s|^p \quad (53)$$



is non-decreasing when  $\beta > s$ . Indeed we have (with  $p \geq 1$ )

$$u'_{\beta,s}(\alpha) = p(\alpha + \beta)|\alpha + \beta|^{p-2} - p(\alpha + s)|\alpha + s|^{p-2} \quad (54)$$

which is non-negative because  $y \mapsto y|y|^{p-2}$  is an non-decreasing function. We now use this property of  $u_{\beta,s}$  with  $\alpha_1 := \Psi(x) \geq \Psi \circ \sigma(x) =: \alpha_2$  and with  $\beta := -u(y) > s := -u \circ \sigma(y)$ . We obtain

$$|\Psi(x) - \Psi(y)|^p + |\Psi \circ \sigma(x) - \Psi \circ \sigma(y)|^p > |\Psi \circ \sigma(x) - \Psi(y)|^p + |\Psi(x) - \Psi \circ \sigma(y)|^p. \quad (55)$$

If we combine this with (52) we get

$$\begin{aligned} & \frac{1}{|x - y|^{d+sp}} \left( |\Psi(x) - \Psi(y)|^p + |\Psi \circ \sigma(x) - \Psi \circ \sigma(y)|^p \right) \\ & + \frac{1}{|x - \sigma(y)|^{d+sp}} \left( |\Psi \circ \sigma(x) - \Psi(y)|^p + |\Psi(x) - \Psi \circ \sigma(y)|^p \right) \\ & > \frac{1}{|x - y|^{d+sp}} \left( |\Psi \circ \sigma(x) - \Psi(y)|^p + |\Psi(x) - \Psi \circ \sigma(y)|^p \right) \\ & + \frac{1}{|x - \sigma(y)|^{d+sp}} \left( |\Psi(x) - \Psi(y)|^p + |\Psi \circ \sigma(x) - \Psi \circ \sigma(y)|^p \right) \\ & = \frac{1}{|x - y|^{d+sp}} \left( |\Psi^\dagger(x) - \Psi^\dagger(y)|^p + |\Psi^\dagger \circ \sigma(x) - \Psi^\dagger \circ \sigma(y)|^p \right) \\ & + \frac{1}{|x - \sigma(y)|^{d+sp}} \left( |\Psi^\dagger \circ \sigma(x) - \Psi^\dagger(y)|^p + |\Psi^\dagger(x) - \Psi^\dagger \circ \sigma(y)|^p \right). \end{aligned} \quad (56)$$

**Case 3:**  $\Psi(x) < \Psi \circ \sigma(x)$  and  $\Psi(y) < \Psi \circ \sigma(y)$ . In this case we have both  $\Psi(x)$  and  $\Psi(y)$  that are swapped with respectively  $\Psi \circ \sigma(x)$  and  $\Psi \circ \sigma(y)$ . Then this case is the same as case 1 and the term associated to Case 3 in the integral (51) is not modified by the polarization.

**Case 4:**  $\Psi(x) < \Psi \circ \sigma(x)$  and  $\Psi(y) \geq \Psi \circ \sigma(y)$ . This case is the same as Case 2.

Gathering these four cases we obtain that for any  $\varepsilon > 0$ ,

$$\iint_{|x-y| \geq \varepsilon} \frac{|\Psi(x) - \Psi(y)|^p}{|x - y|^{d+sp}} dx dy \geq \iint_{|x-y| \geq \varepsilon} \frac{|\Psi^\dagger(x) - \Psi^\dagger(y)|^p}{|x - y|^{d+sp}} dx dy. \quad (57)$$

Concerning the cases of equality, we obtained from Cases 2 and 4 that if

$$\mathcal{L}^2 \left( \left\{ (x, y) \in \mathbb{H}^2 : \Psi(x) = \Psi^\dagger(x) \quad \text{and} \quad \Psi(y) \neq \Psi^\dagger(y) \right\} \cap |x - y| \geq \varepsilon \right) > 0 \quad (58)$$

then the inequality(57) is actually strict. We now observe that the above set is of measure zero for every  $\varepsilon > 0$  if and only if we have either  $\Psi = \Psi^\dagger$  or  $\Psi = \Psi^\dagger \circ \sigma$ . But this last case is not possible when  $\Psi \in \mathcal{N}$  and then the only case of equality in our case is  $\Psi = \Psi^\dagger$ .  $\square$

### 3.3 Proof of Lemma 2.3

Arguing similarly as the previous proof, we only have to prove that

$$\forall \Psi \in X_{sym}^{s,\dagger}, \quad E(\Psi^\sharp) \leq E(\Psi). \quad (59)$$

Since the Steiner rearrangement only involves rearrangements of the super-level sets perpendicularly to the  $r$ -axis, we get

$$V(\Psi^\sharp) = V(\Psi). \quad (60)$$

To conclude we have to establish that.

$$\|\Psi^\sharp\|_{X^s} \leq \|\Psi\|_{X^s}. \quad (61)$$

To start with, we suppose that  $\Psi$  is smooth and compactly supported. In this case

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\Psi(x) - \Psi(y)|^2}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy \longrightarrow \|\Psi\|_{X^s}, \quad (62)$$

as  $\varepsilon \rightarrow 0^+$ . Since the considered functions are  $\mathcal{C}^\infty$ , it is possible to develop the square above and write

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\Psi(x) - \Psi(y)|^2}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy \\ &= 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Psi(x)^2}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy - 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Psi(x)\Psi(y)}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy. \end{aligned} \quad (63)$$

The first integral in the right-hand side of the above inequality is not modified by rearrangement of the super-level sets of the function  $\Psi$ . Concerning the second integral, using the fact that  $\Psi(r, z) = -\Psi(-r, z)$  we get

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(r_x, z_x)\Psi(r_y, z_y)}{((r_x - r_y)^2 + (z_x - z_y)^2 + \varepsilon^2)^{1+s}} dz_x dz_y dr_x dr_y \\ &= 2 \int_0^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(r_x, z_x)\Psi(r_y, z_y)}{((r_x - r_y)^2 + (z_x - z_y)^2 + \varepsilon^2)^{1+s}} dz_x dz_y dr_x dr_y \\ &- 2 \int_0^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(r_x, z_x)\Psi(r_y, z_y)}{((r_x + r_y)^2 + (z_x - z_y)^2 + \varepsilon^2)^{1+s}} dz_x dz_y dr_x dr_y \end{aligned} \quad (64)$$

But the function

$$\Upsilon_{r_x, r_y} : u \mapsto \frac{1}{((r_x - r_y)^2 + u^2 + \varepsilon^2)^{1+s}} - \frac{1}{((r_x + r_y)^2 + u^2 + \varepsilon^2)^{1+s}} \quad (65)$$

is non-negative and radially decreasing on  $\mathbb{R}$ . Moreover, for  $r_x, r_y \geq 0$  the functions  $z_x \mapsto \Psi(r_x, z_x)$  and  $z_y \mapsto \Psi(r_y, z_y)$  are both non-negative  $\mathbb{R}$ . Thus, using the Riesz rearrangement inequality, we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(r_x, z_x)\Psi(r_y, z_y)\Upsilon_{r_x, r_y}(z_x - z_y) dz_x dz_y \\ & \leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi^\sharp(r_x, z_x)\Psi^\sharp(r_y, z_y)\Upsilon_{r_x, r_y}(z_x - z_y) dz_x dz_y. \end{aligned} \quad (66)$$

We now inject this inequality back into (64) and get

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Psi(x)\Psi(y)}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Psi^\sharp(x)\Psi^\sharp(y)}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy. \quad (67)$$

We use this estimate in (63), we take the limit and we conclude by density of the smooth compactly supported functions.  $\square$

Remark: It was not possible to use directly the Riesz rearrangement inequality to the second integral appearing in (63) because this inequality is only true for non-negative functions.  $\square$

### 3.4 Proofs of the lemmas of section 2.2

#### 3.4.1 Proof of Lemma 2.4

- **Step 1 :  $T$  maps  $X_{sym}^{s, \dagger}$  into itself and maps bounded sets into bounded sets.**

First, if  $\Psi$  satisfies the symmetry property then so does  $T(\Psi)$ . Define the set<sup>2</sup>

$$\Omega(\Psi) := \{(r, z) \in \mathbb{H} : T(\Psi)(r, z) > 0\}, \quad (68)$$

where  $T(\Psi)(r, z) := (\Psi(r, z) - cr - k)_+$ . By definition of  $T$  and of  $\Omega$

$$\mathcal{L}^2(\Omega) = \int_{\Omega} 1 \leq \int_{\Omega} \left( \frac{\Psi(r, z)}{cr + k} \right)^{\frac{2}{1-s}} dr dz \leq \int_{\Omega} \left( \frac{\Psi(r, z)}{k} \right)^{\frac{2}{1-s}} dr dz \leq \frac{1}{k^{\frac{2}{1-s}}} \int_{\mathbb{H}} \Psi^{\frac{2}{1-s}}. \quad (69)$$

<sup>2</sup> The adherence of this set is the support of the function  $\Theta$ . This corresponds physically speaking to the vorticity zone.

Using a Sobolev inequality above leads to

$$\mathcal{L}^2(\Omega) \leq \frac{C_s}{k^{\frac{2}{1-s}}} \|\Psi\|_{X^s}^{\frac{2}{1-s}}. \quad (70)$$

The computation of the double integral defining the  $\dot{H}^s$  half-norm 20 is done separating the integrals on  $\mathbb{R}^2$  on two between  $\Omega$  and  $\Omega^c$ . On  $\Omega^c$  the quantity  $\Psi(r, z) - cr - k$  is non-positive and then  $T(\Psi)(r, z) = 0$ . Therefore,

$$\int_{\Omega^c} \int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^2(1+s)} dx dy = 0. \quad (71)$$

Concerning the integral on  $\Omega \times \Omega$ , using the notation  $x = (r_x, z_x)$  and  $y = (r_y, z_y)$ ,

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy &= \int_{\Omega} \int_{\Omega} \frac{|(\Psi(x) - cr_x - k) - (\Psi(y) - cr_y - k)|^2}{|x - y|^{2(1+s)}} dr dz \\ &\leq \int_{\Omega} \int_{\Omega} \frac{|\Psi(x) - \Psi(y)|^2 + c^2|r_x - r_y|^2}{|x - y|^{2(1+s)}} dr dz. \end{aligned} \quad (72)$$

Denote with an  $*$  the radially decreasing rearrangement and  $R_{\Omega} > 0$  the radius such that

$$\mathcal{L}^2(\Omega) = \mathcal{L}^2(\mathcal{B}(0, R_{\Omega})). \quad (73)$$

To simplify the notations we simply note this ball  $B(\Omega)$ . By the Riesz rearrangement inequality we have

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|r_x - r_y|^2}{|x - y|^{2(1+s)}} dx dy &\leq \int_{\Omega} \int_{\Omega} \frac{dx dy}{|x - y|^{2s}} \leq \int_{B(\Omega)} \int_{B(\Omega)} \frac{dx dy}{|x - y|^{2s}} \\ &\leq \int_{B(\Omega)} \int_{B(\Omega)} \frac{dx}{|x|^{2s}} dy = \frac{\pi^s}{1-s} \mathcal{L}^2(\Omega)^{2-s}. \end{aligned} \quad (74)$$

Using now (70) we get

$$\int_{\Omega} \int_{\Omega} \frac{|r_x - r_y|^2}{|r - z|^{2(1+s)}} dr dz \leq C_s \|\Psi\|_{X^s}^{2\frac{2-s}{1-s}}. \quad (75)$$

Concerning the last term,

$$\int_{\Omega} \int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy = \int_{\Omega} |T(\Psi)(x)|^2 \int_{\Omega^c} \frac{dy}{|x - y|^{2(1+s)}} dx. \quad (76)$$

For all  $x \in \Omega$  we define  $\Lambda(x) := \frac{\Psi(x) - cr_x - k}{2c}$  and  $\mathcal{O}_x := \{y \in \Omega^c : r_y \geq r_x + \Lambda(x)\}$ . Then,

$$\int_{\mathcal{O}_x} \frac{dy}{|x - y|^{2(1+s)}} \leq \int_{\mathcal{O}_x} \frac{dy}{|r_x - r_y|^{2(1+s)}} \leq \int_{B(x, \Lambda(x))^c} \frac{dy}{|x - y|^{2(1+s)}} = \frac{\pi^s}{\Lambda(x)^{2s}}. \quad (77)$$

Using the fact that  $x \in \Omega$ , that  $y \in \Omega^c$  and that  $y \in \mathcal{O}_x$  in this order, we obtain

$$\begin{aligned} 0 \leq T(\Psi)(x) &= \Psi(x) - cr_x - k \leq \Psi(x) - \Psi(y) + c(r_x - r_y) \\ &\leq \Psi(x) - \Psi(y) + \Psi(x) - \frac{cr_x - k}{2} = \Psi(x) - \Psi(y) + \frac{1}{2}T(\Psi)(x). \end{aligned} \quad (78)$$

Therefore

$$|T(\Psi)(x)| \leq 2|\Psi(x) - \Psi(y)| \quad (79)$$

Combing (76), (77) and (79) leads to

$$\begin{aligned} &\int_{\Omega} \int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy \\ &\leq 4 \int_{\Omega} \int_{\Omega^c \setminus \mathcal{O}_x} \frac{|\Psi(x) - \Psi(y)|^2}{|x - y|^{2(1+s)}} dx dy + \int_{\Omega} |T(\Psi)(x)|^2 \frac{\pi}{s\Lambda(x)^{2s}} dx \\ &\leq 4 \int_{\Omega} \int_{\Omega^c} \frac{|\Psi(x) - \Psi(y)|^2}{|x - y|^{2(1+s)}} dx dy + \frac{\pi}{s} \int_{\Omega} |T(\Psi)(x)|^{2(1-s)} dx. \end{aligned} \quad (80)$$

Now, to estimate the last term of the above inequality, we use the fact that  $T(\Psi) \leq \Psi$  and then the Hölder inequality gives

$$\int_{\Omega} |T(\Psi)(x)|^{2(1-s)} dx \leq \int_{\Omega} |\Psi(x)|^{2(1-s)} dx \leq \mathcal{L}^2(\Omega)^{s(2-s)} \|\Psi\|_{L^{\frac{2}{1-s}}}^{2(1-s)}. \quad (81)$$

We continue this estimate using (70) and a Sobolev embedding,

$$\leq C \|\Psi\|_{X^s}^{2s(1+\frac{1}{1-s})} \|\Psi\|_{L^{\frac{2}{1-s}}}^{2(1-s)} \leq C \|\Psi\|_{X^s}^{\frac{2}{1-s}}. \quad (82)$$

Thus, gathering all these estimates we obtain.

$$\|T(\Psi)\|_{X^s}^2 := \int_{\Omega} \int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy \leq C \|\Psi\|_{X^s}^2 \left(1 + \|\Psi\|_{X^s}^{\frac{2s}{1-s}}\right). \quad (83)$$

Therefore,  $T$  does map  $X_{sym}^{s,\dagger}$  into itself and maps bounded subsets of  $X_{sym}^{s,\dagger}$  into bounded subsets.

• **Step 2 :  $T \circ \sharp \circ \dagger$  defined on  $X_{sym}^s$  is a compact operator for the  $L^p$  topology.** Set the convention that  $\{|z| \geq R\}$  designates the set  $\{(r, z) \in \mathbb{H} : |z| \geq R\}$ . Let  $\kappa > 0$  and  $R \geq 0$ . For all  $r \in \Omega^\sharp$  we define

$$\mathcal{U}_x := \mathcal{B}(x, \kappa) \cap (\Omega^\sharp)^c. \quad (84)$$

Then,

$$\begin{aligned} \int_{\{|z| \geq R\}} |T(\Psi^\sharp)(x)|^2 dx &= \int_{\{|z| \geq R\}} \frac{1}{\mathcal{L}^2(\mathcal{U}_x)} \int_{\mathcal{U}_x} |T(\Psi^\sharp)(x)|^2 dy dx \\ &\leq \int_{\{|z| \geq R\}} \frac{1}{\mathcal{L}^2(\mathcal{U}_x)} \int_{\mathcal{U}_x} |T(\Psi^\sharp)(x) - T(\Psi^\sharp)(y)|^2 dy dx \\ &\leq \int_{\{|z| \geq R\}} \frac{\kappa^{2(1+s)}}{\mathcal{L}^2(\mathcal{U}_x)} \int_{\mathcal{U}_x} \frac{|T(\Psi^\sharp)(x) - T(\Psi^\sharp)(y)|^2}{|x - y|^{2(1+s)}} dy dx. \end{aligned} \quad (85)$$

Denote by  $P_{\mathbb{R}}$  the projection on  $\mathbb{R} \times 0$  (that is identified to  $\mathbb{R}$ ). As a consequence of the Steiner symmetrization, with (69),

$$2R \mathcal{L}^1 \left( P_{\mathbb{R}} \left( (\Omega^\sharp)^c \cap \{|z| \geq R\} \right) \right) \leq \mathcal{L}^2(\Omega^\sharp) \leq \frac{1}{k^{\frac{2}{1-s}}} \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}. \quad (86)$$

Since  $|z_x| \geq R - \kappa$  then using again the Steiner symmetry of  $\Omega^\sharp$ , gives that  $\mathcal{U}_x$  contains the ball  $B(r, \kappa)$  minus the rectangle centered at  $x$ , of width

$$\mathcal{L}^1 \left( P_{\mathbb{R}} \left( (\Omega^\sharp)^c \cap \{|z| \geq R - \kappa\} \right) \right)$$

height  $2\kappa$ . Then, with (86),

$$\mathcal{L}^2(\mathcal{U}_x) \geq \pi \kappa^2 - \frac{\kappa}{(R - \kappa) k^{\frac{2}{1-s}}} \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}. \quad (87)$$

The choice of  $\kappa$  is free and then we choose to fix it equal to  $C/R$  with

$$C := \frac{4}{\pi k^{\frac{2}{1-s}}} \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}.$$

Since Choose now  $R$  such that  $R \geq \sqrt{2C}$ . Then in this case the inequality (87) becomes

$$\mathcal{L}^2(\mathcal{U}_x) \geq \frac{\pi C^2}{2R^2}. \quad (88)$$

Combining the estimate above with (85), leads to the following estimate

$$\int_{\{|z| \geq R\}} |T(\Psi^\sharp)(x)|^2 dx \leq \left( \frac{4}{\pi R} \right)^{2s} \left( \frac{\|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}}{k} \right)^{\frac{4s}{1-s}} \|T(\Psi^\sharp)\|_{X^s}^2. \quad (89)$$

On the other hand, using the Hölder inequality,

$$\begin{aligned} \int_{\{r \geq R\}} |T(\Psi^\#)|^2 &= \int_{\{r \geq R\}} |T(\Psi^\#)|^2 \mathbb{1}_{(\Omega^\#)^c} \leq \left( \int_{\{r \geq R\}} |T(\Psi^\#)|^{\frac{2}{1-s}} \right)^{1-s} \left( \int_{\{r \geq R\}} \mathbb{1}_{(\Omega^\#)^c} \right)^s \\ &= cL^2 \left( (\Omega^\#)^c \cap \{r \geq R\} \right)^s \|\Psi^\#\|_{L^{\frac{2}{1-s}}}^2, \end{aligned} \quad (90)$$

where by convention  $\{r \geq R\}$  designates the set  $\{(r, z) \in \mathbb{H} : r \geq R\}$ . Moreover

$$\begin{aligned} \mathcal{L}^2((\Omega^\#)^c \cap \{r \geq R\}) &= \int_{(\Omega^\#)^c \cap \{r \geq R\}} 1 \\ &\leq \int_{(\Omega^\#)^c \cap \{r \geq R\}} \left( \frac{\Psi^\#}{cr + k} \right)^{\frac{2}{1-s}} \leq \left( \frac{1}{cR + k} \right)^{\frac{2}{1-s}} \|\Psi^\#\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}. \end{aligned} \quad (91)$$

Combining (90) and (91) leads to

$$\int_{\{r \geq R\}} |T(\Psi^\#)|^2 \leq \left( \frac{1}{cR + k} \right)^{\frac{2s}{1-s}} \|\Psi^\#\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}} \quad (92)$$

The two decay estimates (89) et (92) and the Rellich-Kondrachov compactness theorem (applied at the local level) give the result.  $\square$

### 3.4.2 Proof of Lemma 2.5

It follows from the definition of  $\mathcal{N}$  and of Lemma 2.1 that

$$\int_{\mathbb{H}} f(\Psi_n - cr - k) \Psi_n = \frac{1}{2} \int_{\mathbb{R}} \Psi_n (-\Delta)^s \Psi_n = \frac{1}{2} \|\Psi_n\|_{X^s}^2 \geq \frac{\beta}{2} > 0. \quad (93)$$

By the previous lemma, up to a subsequence when  $n \rightarrow +\infty$ ,

$$\int_{\mathbb{H}} f(\Psi_n - cr - k) \Psi_n \longrightarrow \int_{\mathbb{H}} f(\Psi^\star - cr - k) \Psi^\star \geq \frac{\beta}{2}. \quad (94)$$

In particular  $(\Psi^\star - cr - k) \neq 0$  on  $\mathbb{H}$ . By proposition 2.1, there exists  $t^\star > 0$  such that  $t^\star \Psi^\star \in \mathcal{N}$ . With the characterization of  $t^\star$  and since  $\Psi_n \in \mathcal{N}$ ,

$$E(\Psi_n) = E(t_{\Psi_n} \Psi_n) \geq E(t^\star \Psi_n). \quad (95)$$

Thus,

$$\alpha = \lim_{n \rightarrow +\infty} E(\Psi_n) \geq \liminf_{n \rightarrow +\infty} E(t^\star \Psi_n) \geq E(t^\star \Psi^\star) \geq \alpha. \quad (96)$$

Therefore all these inequalities are equalities and  $\|\Psi_n\|_{X^s}^2 \rightarrow \|\Psi^\star\|_{X^s}^2$ . Since the space  $X^s$  is strictly convex, this gives that  $\Psi_n$  converges towards  $\Psi^\star$  strongly in  $X^s$ .  $\square$

## 3.5 Proofs of the lemmas of section 2.3

### 3.5.1 Proof of Proposition 2.6

We already know that  $T(\Psi^\star) \in L^{\frac{2}{1-s}}(\mathbb{R}^2)$ . Since the support of  $T(\Psi^\star)$  has finite measure,  $T(\Psi^\star) \in L^1(\mathbb{R}^2)$ . Define  $\Theta^\star$  from  $\Psi^\star$  using formula (26). Hypothesis (13) implies

$$\forall q \in \left[1, \frac{2}{\nu(1-s)}\right], \quad \Theta^\star \in L^q(\mathbb{R}^2). \quad (97)$$

Define now the function  $\tilde{\Psi}$  given by the following representation formula,

$$\tilde{\Psi}(x) = K_s \int_{\mathbb{R}^2} \frac{\Theta^\star(y)}{|x - y|^{2(1-s)}} dy. \quad (98)$$

where  $K_s$  is some renormalization constant. It follows from the weighted inequalities for singular integrals [10, §5] that  $\tilde{\Psi} \in \dot{W}^{2s,q}(\mathbb{R}^2)$ , for all  $q \in [1, \frac{2}{\nu(1-s)}]$ . Moreover, by the Hardy-Littlewood-Sobolev convolution inequality,  $\tilde{\Psi} \in L^q(\mathbb{R}^2), \forall q \in [\frac{1}{1-s}, \frac{2}{\nu-s(2+\nu)}]$ . By standard interpolation,  $\tilde{\Psi} \in X_{sym}^s$ . Now, let  $\varphi \in X_{sym}^s$  be a test function. Using the spectral properties of the Sobolev spaces [5] gives (up to multiplicative renormalization constants),

$$\begin{aligned} \langle \tilde{\Psi}, \varphi \rangle_{X^s} &= \int_{\mathbb{R}^2} |\xi|^{2s} \mathcal{F}[\tilde{\Psi}](\xi) \mathcal{F}[\varphi](\xi) d\xi \\ &= \int_{\mathbb{R}^2} |\xi|^{2s} \mathcal{F}\left[\Theta^* * \frac{1}{|\cdot|^{2(1-s)}}\right](\xi) \mathcal{F}[\varphi](\xi) d\xi \\ &= \int_{\mathbb{R}^2} \mathcal{F}[\Theta^*](\xi) \mathcal{F}[\varphi](\xi) d\xi = \langle \Theta^*, \varphi \rangle_{L^2}, \end{aligned} \quad (99)$$

where  $\mathcal{F}[\cdot]$  designates the Fourier transform. Moreover, since  $\Psi^*$  is a critical point of  $E$ , then  $\langle \Psi^*, \varphi \rangle_{X^s} = \langle \Theta^*, \varphi \rangle_{L^2}$ , which implies  $\tilde{\Psi} = \Psi^*$ . The regularity known for  $\Theta^*$  allows to conclude that  $\Psi^*$  is bounded and uniformly continuous. Seen the definitions, to conclude that  $\Psi^*$  is smooth by a bootstrap argument there remain to study possible discontinuities on  $r = 0$ . Nevertheless, it follows from the symmetry property of  $\Psi^*$  and its uniform continuity that  $T(\Psi^*)$  worth 0 at a distance uniformly positive from  $r = 0$ , meaning on a strip  $] -\delta, \delta[ \times \mathbb{R}_+$ . Therefore so is the case for  $\Theta^*$  and then the smoothness of  $\Psi^*$  is proved.  $\square$

### 3.5.2 Proof of Proposition 2.7

Let  $r \in R^2$  such that  $|r| \geq 1$ . We separate the integral (98) into two,

$$\Psi^*(x) = K_s \int_{|x-y| \leq \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy + K_s \int_{|x-y| > \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy \quad (100)$$

Concerning the first integral, we choose  $\eta \in ]1-s, \frac{1}{s+1}[$ . This interval is non-empty and included in  $]0, 1[$ . We use the Hölder inequality and Hypothesis (13) and then we are led to

$$\int_{|x-y| \leq \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy \leq \left( \int_{|\zeta| \leq \frac{|x|}{2}} \frac{d\zeta}{|\zeta|^{\frac{2(1-s)}{\eta}}} \right)^\eta \left( \int_{|x-y| \leq \frac{|x|}{2}} |T(\Psi^*)|^{\frac{\nu}{1-\eta}} \right)^{1-\eta}. \quad (101)$$

Using again the estimates (89) et (92),

$$\int_{|x-y| \leq \frac{|x|}{2}} |T(\Psi^*)|^2(y) dy \leq \frac{C}{(1+|x|)^{-2s}}. \quad (102)$$

Knowing that  $\frac{\nu}{1-\eta} \geq 2$  the above estimate used in (101) leads to (the constant that depends only on  $\eta$ )

$$\int_{|x-y| \leq \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy \leq C(\eta) (|x|^2 + 1)^{\eta(s+1)-1}. \quad (103)$$

The second integral in (100) can be estimated using directly the hypothesis on the function  $f$ ,

$$\int_{|x-y| > \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy \leq \left( \frac{2}{|x|} \right)^{2(1-s)} \int_{\mathbb{R}^2} |T(\Psi^*)|^\nu \leq \frac{C}{|x|^{2(1-s)}}. \quad (104)$$

By choosing  $\eta \in ]1-s, \frac{1}{s+1}[$  such that  $\eta \geq \frac{s}{s+1}$ , the estimates (103) et (104) give

$$\Psi^*(x) \leq \frac{C}{1+|x|^{2(1-s)}}. \quad (105)$$

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