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This paper presents the ongoing refinement of a construct (a Rational Mathematical Template, RMT) emerging from observations of effective classroom discussions that are aimed at producing a definition, statement of a theorem, proof for a given statement, mathematical model for a physical situation, etc., as key elements of mathematical rationality. An RMT concerning one of these elements provides students with guidelines for how to manage the related process of production in a conscious way and how to check the validity of the product. In order to introduce the RMT construct, the paper refers to two episodes of definition production. A brief discussion of the analogies and differences with other constructs follows. The paper ends with an outline of possible developments of the refinement of the RMT construct and related educational issues.

Keywords: Habermas' rationality, routines, rituals, schemata, rational mathematical templates.

Introduction

The motivation for the study reported in this paper came from recent developments in our work on the implementation of Habermas' construct of rationality in mathematics education and from recent research advances regarding routines and rituals which resulted in a special issue of *Educational Studies in Mathematics* (2019, vol. 101[2]). The advances that are of interest for us concern the relationships between "ritual-enabling and exploration-requiring opportunities to learn" (Nachlieli & Tabach, 2019, p. 253) and the elaboration by Lavie, Steiner, and Sfard (2019) regarding the notion of routine as "repetition-generated patterns of our actions" (p. 153); see more in the next section.

In our recent research work, we have considered what happens in classroom discussions when the teacher attempts to promote students' rational behavior in their approach to theoretical mathematics, particularly as concerns defining, conjecturing, proving (in synthetic geometry and elementary arithmetic), and mathematical modeling. In order to participate productively and develop their rational behavior in those activities, students need to interiorize specific ways of organizing their thought processes related to the content at stake, which share some aspects with routines that combine ritual and exploratory aspects (hence our interest in Nachlieli and Tabach's and Lavie, Steiner, and Sfard's development of routines, rituals, and explorations). However, in spite of these shared aspects, our objective (the promotion of students' effective approach to rational behavior in theoretical mathematics) produces important differences, with implications for the construct needed to plan and analyze activities aimed at achieving our goal. The (provisional) name for this construct is the Rational Mathematical Template (RMT), and the choice of name alludes to what we mean by it. For instance, in the case (dealt with in this paper) of producing a definition and, at the same time, gradually becoming aware of what "definition" means in mathematics (a crucial condition for mathematical rationality), students gradually interiorize a specific "model" of rational behavior for

this activity. This "model" provides the students with guidelines for the exploration and production of a definition (the *teleological* component of rational behavior) and with criteria for validating and communicating the produced definition (its *epistemic* and *communicative* components). In a classroom interaction situation aimed at producing a definition, an RMT offers students an opportunity for purposeful interaction towards that common goal, according to Habermas' development of communicative rationality:

> Communicative rationality is expressed in the unifying force of speech oriented towards understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (Habermas, 1988, p. 315)

Moreover, in an RMT-led process, an affective shared dimension plays a relevant role (see Discussion). In a long-term perspective, it is the driving force that ensures a (relative) autonomy from the teacher and the purposeful development of the process.

In this paper, we will move from a brief account of theories of interest for our work to two episodes concerning two steps in the students' interiorization of what we will call the RMT of mathematical definition. Then, we will discuss the RMT's relationship to other constructs from mathematics education. Finally, in the Discussion, we will present the current status of our refinement of the RMT idea based on classroom observations and reflections by the second co-author of this paper.

Habermas' rationality, routines, and concepts

Habermas (1998) proposed the construct of rationality (and that of rational behavior) in order to account for discursive practices that are characterized by *awareness* when checking the truth of statements and the validity of reasoning according to shared criteria (*epistemic rationality*), evaluating strategies to attain the aim of the activity (*teleological rationality*), and choosing suitable communication tools to reach others in a given social context (*communicative rationality*), the three components being strictly interconnected. In the past years, researchers both in our group and outside it have attempted to adapt Habermas' construct to mathematics teacher education (one of these studies is reported in Guala & Boero, 2018) and to plan teaching aimed at developing and analyzing students' rational behaviors (see Boero & Planas, 2014 for a general account and a presentation of five studies).

Lavie, Steiner, and Sfard (2019) considered routines to be "repetition-generated patterns of our actions" (p. 153); they further elaborated this notion by distinguishing between "process-oriented routines" (called rituals) and "product-oriented routines" (called "deeds or explorations, depending on whether the routine is practical or discursive"), in order to support their claim that

> in the process of learning mathematics, the germinal routines, from which a discourse new to the learner is to emerge, are initially implemented as rituals. In the longer run, these routines are expected to undergo gradual de-ritualization until they become fully fledged explorations. (p. 153)

Exploration and ritual are "different genres of routines, corresponding to different mechanisms of utilizing our past experiences" (p. 154). The following quote highlights the difference between them in terms of awareness and autonomy:

> Deeds and explorations may be seen as acts of production: their execution is judged exclusively by the artifacts they generate. In the case of a ritual, […] the performers do not ask themselves "what is that I want to get?" as do those involved in deeds and explorations. This time, "how do I proceed?" is the only question that guides their actions. Since the resulting performance does not count as an act of production, one can only have social reasons for a ritual activity. Indeed, we perform rituals when we feel expected to do so, and in particular, when the expectation comes from those whom we see as in any way superior to ourselves. (p. 166)

Vergnaud (1990) defines mathematical concepts as triads consisting of reference situations, operational invariants, and linguistic representations. In Durand-Guerrier et al. (2012), proof is considered to be a mathematical concept according to Vergnaud's definition, and in the same perspective, definition and mathematical model can also be considered concepts.

Two episodes from the same classroom

A teacher (Marina Molinari) belonging to our research team wanted to introduce a class of 22 seventh-grade students to the process of defining and the concept of definition in mathematics. In the previous school year with the same class, the teacher had gradually developed the students' methods of productive interaction both among themselves and with the teacher in the perspective of Habermas' rational behavior (i.e., with attention to their conscious consideration of the truth of statements according to shared epistemic criteria, the effectiveness of strategies, and the quality of communication in the classroom).

The reported excerpts come from audio recordings, with supplementary information taken from a student teacher's field notes.

The first episode: How to define the diameter of a circle

Students had already encountered the term "diameter" in elementary school, where it was associated with a segment passing through the center of a circumference with extremities on that circumference. According to a classroom discussion after the second episode, they had no experience of constructing a definition or of reflecting on what "definition" means in mathematics (even if they knew, for instance, that "a right-angled triangle is a triangle with an angle of 90°"). The teacher's aim in the reported discussion was to allow the students to experience a guided defining activity and become aware of its goal. She drew a circumference (with its center) on the blackboard.

(T is the teacher; students [Sn] are numbered according to the order in which they entered the discussion)

1. T: Are you able to draw a diameter of this circumference?
2. S1: Yes, I will do this. [*S1 goes to the blackboard and draws a diameter for the drawn circumference.*]
3. T: Could you explain to a friend of yours what a diameter is?
4. S1: Yes, it is when I draw a line through the center of the circle. [*He turns to look at his classmates.*]
5. S2: A straight line with a ruler.
6. S3: A straight line through the center of the circumference.

7. T: A straight line like this? [*She draws another segment passing through the center of the circumference that is shorter than a diameter.*]

[*The discussion develops as to how to describe the construction of a diameter*.]

13. S1: Thus I should say: A diameter is a segment that I draw by choosing one point on the circumference and then by drawing a segment from that point through the center until it reaches the circumference again.

14. T: OK, this is a fine description of how to draw a diameter. [*She writes it down on the blackboard*.] But what is a diameter? How do you distinguish this segment which is not a diameter [*she draws a chord that does not pass through the center*] from this segment [*she draws a diameter*]?

15. S2: The segment must pass through the center, as S1 said.

16. T: Thus, what is a diameter?

17. S2: It is a segment that passes through the center.

18. S8: And which moves from one side of the circumference to the other side.

19. S2: A segment through the center joining two points of the circumference.

20. T: [*She writes: A diameter is, etc., as S2 said*.] What is the difference between this sentence and the previous sentence?

21. S9: The first one describes how to draw a diameter.

22. S2: While the other says what a diameter is.

23. T: [*She draws a chord that does not pass through the center*.] Is this a diameter?

24. S2: No, it does not pass through the center.

[*The teacher underlines the corresponding part of the definition with red chalk, then she draws a very short segment through the center, she asks if it is a diameter, etc.*]

The second episode: How to define prime numbers

In the same classroom, one month later, students had refined their understanding of the definitions of even numbers and adjacent angles; gradually, the teacher's support and guidance had become less concerned with the process of orienting the students towards their goal (the students' labels correspond to the students from the first episode).

1. T: Do you know what a prime number is?
2. S2: A number without divisors, like three, seven, eleven.
3. S7: I think that it is not necessary to give examples.
4. S8: The words are sufficient; like for even numbers, it was not necessary to say two, eight, ten.
5. S7: Yes, it is sufficient to say that it is a number which has no divisors.

[*The teacher writes "a prime number is a number which has no divisors" on the blackboard.*]

6. S1: But [*pause*] every number is divisible by one [*emphasis*]
7. S3: Yes, we should say [*pause*] a prime number is a number which has no divisors, except one.

[*The teacher completes the sentence this way*.]

8. T: Do you agree?

9. S2: A number [*pause*]. What happens with decimal numbers?
10. S12: One point two divided by two makes zero point six.
[*A discussion follows about how to exclude cases like that or like 3:1,5 from the definition*.]
16. S8: We need to define [*pause*] exactly the numbers that we know as prime numbers from elementary school [*pause*].
[*students produce and discuss some examples of prime numbers*.]
21. S7: We must add that we take only integer numbers that are divisible by one.
22. S4: With an integer result [*emphasis*].
23. S1: Yes, we need to be precise: divisibility means [*pause*].
24. S2: We need another definition, that of the divisibility of an integer.
25. S3: Divisibility [*pause*]. Divisibility by another integer.
26. S12: I think that we may say that a prime number is when we have a number and we find whether it has any divisors by dividing it by the numbers that are smaller than it.
27. S9: Yes, if we find that it is only divisible by one, it is a prime number.
28. T: Thus, what is a prime number for you?
29. S9: It is a number that if I divide it by all the numbers that are smaller than it, I find that its only divisor is one.
30. T: [*Writes what S9 said on the blackboard.*] What do you think about what S9 is proposing as a definition of a prime number?
31. S8: We need to explain what a prime number is, not how to ascertain if a number is a prime number. It is like in the case of a diameter. [*Pause.*]
32. S6: It does not work, I remember when we worked out the definition of a diameter that how to draw it was not a good definition
33. S15: I do not understand. Explain better what you mean
34. S2: Yes, I understand: in that case, how to draw a diameter was not a definition, while saying that a diameter is a segment through the center, etc., is a definition.
35. S8: In this case, we must say that a number is a prime number if its only divisor is one.
36. S14: Or, to be more precise: A number is a prime number if it is an integer number and its only integer divisor is one.
37. T: [*Points at the definition read by S8.*] Thus, what do we need to add to what is written on the blackboard? And why?

[*The discussion continues regarding the need to consider "integer numbers" and also accept the prime number itself as a divisor*.]

Two weeks later, after a similar activity on the definition of a circumference, this individual test was proposed (the definition of a square had not been previously discussed in class):

Are the following sentences definitions of a square? If not, explain why, for each sentence that is not a definition of a square.

- A square is a quadrilateral with four right angles

- A square is a quadrilateral with four sides of the same length
- A square is a quadrilateral with opposite parallel sides

Nineteen students out of 22 answered that none of these sentences was a definition of a square; among them, 16 students found valid justifications, based on salient features of a definition, for rejecting the three sentences: e.g., "the first sentence includes also those rectangles that are not squares", "the third definition also includes all parallelograms".

Analyzing and reflecting on the episodes: Rational Mathematical Templates

In the first discussion, the teacher leads an interaction that involves students in the guided construction of a definition; during the discussion, the teacher mediates salient features of defining as a rational behavior: epistemic criteria for judging the validity of a statement as a definition (16, 22); the need to ascertain whether the statement includes diameters, and only diameters, and the ways of doing so (production of various kinds of segments, 23); the precision of wording and communication (7).

By comparing what students do in the second discussion (and how the teacher orchestrated the discussion in the first discussion) with their behavior in the first discussion, we may observe how the process of defining developed in strict relation to the development of mathematical rationality in the specific domain of theoretical mathematics: in the second episode, the students take care of criteria that characterize a mathematical definition (3, 4, 5; 31, 34) and that allow them to decide whether a formulation satisfies them (9, 21) (*epistemic rationality*); they engage in a goal-oriented process (*teleological rationality:* 16, 33); during the discussion, the students are concerned with the formulation of the statement that is to represent the set of objects (which constitutes the semantic field of the definition and needs to include all the objects associated with the expression "prime numbers") and engage in mutual corrections about it (*communicative rationality*: 23, 36). This is the reason why we have chosen the provisional expression "Rational Mathematical Template" to name *the couplet consisting of a mathematical entity (in our case, the definition) and a rational process that is purposefully oriented to produce an instance of that mathematical entity* – for the moment, a simple provisional definition of a more complex object, as we will see in the Discussion.

Comparison with other constructs

What students have acquired is more than a concept, according to Vergnaud's definition of concepts as adapted by Durand-Guerrier et al. (2012) to the case of proof. In Episode 2, we find *reference situations* (e.g., the reference to previous activities about the definition of even numbers and diameter, interventions 4 and 31); *operational invariants* (implicit in most cases – like "theorems in action" in Vergnaud's definition, but also explicit, e.g., 3, 5, 33); and *linguistic representations* (e.g., 34). However, students develop the second discussion according to interiorized "how-to" guidelines and move towards a goal to be achieved. This is not included in a concept. Note that in the case of the RMT of definition (or theorem, or mathematical model), the "mathematical entity" component of the RMT is a concept, according to Durand-Guerrier et al. (2012).

In spite of some surface analogies (the RMT process of definition looks like an exploration routine, in situations of definition production), there are profound differences between routines and RMTs concerning the content-specificity of the definition of RMTs as couplets (mathematical entity,

process); the crucial role of awareness (related to the rationality perspective) in RMT-led activities, be they guided by the teacher or autonomously developed by students; the characterization of the RMT process as "rational", which implies a reference to the components of rationality in the analysis of RMT-led activities; the narrow interconnection between epistemic and teleological dimensions of the RMT-led activities; and the attention to communication issues during them.

Discussion: Hints for further research

In the last year, we have realized that there is more to say about the object emerging from the analysis of classroom discussions aimed at the development of rational behavior than what is represented in the provisional definition above. In particular, the second author's observations in her classes and the related reflections have led us to focus on both the affective and the cultural dimensions of how RMTs work in a "satisfactory" classroom interaction. At present, we see the RMT as a medium in the 1:1 teacher-student relationship, in the 1:n teacher-classroom relationship, and in the network relationship among the members of the classroom. It seems to us that the RMT as a medium plays a double role: both towards the epistemic dimension and towards the affective relationship. The medium role of the RMT strengthens and inter-exchanges with the medium role of the teacher in an interaction space, which looks like a game space; the participants gradually enter it with different roles. This game space, constituted/made possible by the RMT, is the space of cognition and affect, which develops students' subjectivity, identity, autonomy. In the classroom, we may interpret Habermas' communicative rationality thus: she who behaves according to communicative rationality is someone who wishes to get in touch with another person and with the Other (the Culture): thus, her driving interest is to be understood by, and to understand, the other (and the Other) in order to create a discourse – i.e., something which dynamically flows among the participants in the communication; communication is conditioned by their communicative productions, and at the same time, they are modified by it. Not only do RMTs allow communicative rationality to be exercised and connect communicative rationality with epistemic rationality (a connection already hypothesized by Habermas), but they also contribute to creating (thanks to the teacher's guidance) and maintaining a space of wellbeing for all participants under the driving force of reaching the other (and the Other).

In the last few months, teachers who collaborate with us have been engaged in designing and reporting classroom situations of the initiation and development of RMTs and of communicative rationality "in action" based on those RMTs. Those teaching experiments should contribute to answering the following questions: How should one characterize and (possibly) frame the dialectic relationship between the evolution of the RMTs and the development of the communicative rationality whose well-functioning depends on one (or more) RMTs being available to students? What about the affective dimension (motivation, interest in reaching others and being reached by others) of classroom RMT-led interactions? How does it depend on students' familiarity with RMTs, and how does the growth of students' familiarity with RMTs depend on the affective quality of interactions in the classroom? And what about the possibility that students who have experienced RMTs after the teachers' initial guidance are enabled to develop other RMTs by themselves in new situations? The last two questions are connected to another important question: How should students be initiated to RMTs? The mediating role of the teacher should play a crucial role in engaging the students in the "game" and in making the aim of the activity explicit. It is important to avoid the danger of a one-

sense transmission with the related negative contractual effects on students, because the goal is to develop their rational behavior. A participated, gradual appropriation of the RMTs (see the first episode) might avoid this; the "voices and echoes game" (Garuti & Boero, 2002), a participated cultural transmission method relying on the dialogue with the Other, might be another valid method. In the classroom context, the above perspective of the dialectic-dynamic relationship between the development of the communicative rationality space and the evolution of the RMT is aimed at establishing a constructive relationship with the Other (the Culture), intended not as an object to be received and conserved "as such" by the students, but as a means for their cultural growth, as they use and simultaneously transform it in the "intersubjectively shared lifeworld" (Habermas, 1998, p. 315). This consideration establishes a close connection with Engeström's construct of "expansive learning" (Engeström & Sannino, 2010), to be further explored.

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