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Disentangling discourse practices and language means for developing conceptual understanding: The case of pre-algebraic equivalence

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Explaining meanings has been shown to be the crucial discourse practice for students' conceptual development. However, more research is required to disentangle what kind of language means are involved in different mathematical topics. The paper reports on a Design Research project aiming at developing language learners' conceptual understanding for pre-algebraic equivalence. The qualitative analysis of the first design experiments with fifth graders shows that activities of connecting representations can foster students' learning, but language support is required for deepening a meaning-related transformational approach to what we call restructuring equivalence.

Keywords: discourse practice, explaining meanings, pre-algebra, equivalence of expressions

Background: Explaining meanings of equivalence of expressions

Discourse practices as key language demands for students' processes of meaning making

For supporting (monolingual and multilingual) language learners in developing conceptual understanding for mathematical concepts, engaging all students in rich discourse practices has been shown to be crucial (Moschkovich, 2015; Setati, 2005). Thereby, the attribute “rich” is not interpreted by “the more students speak, the better it is”, but with respect to the quality of the discourse practices: The learning opportunity gap for language learners occurs when they mainly participate in procedural rather than conceptual talk. Various empirical studies have started to identify typical discourse practices and their relevance for students' processes of meaning making (e.g., Erath, Prediger, Quasthoff, & Heller, 2018; Moschkovich, 2015; Prediger & Zindel, 2017). The main distinction is illustrated by the task in Figure 1: *reporting the procedure* of transforming one expression into an equivalent expression is much less demanding with respect to language than *explaining the meaning* of equivalence or *justifying why* the transformational rules guarantee the transformation into equivalent expressions. Although empirical studies have generally shown the relevance of the three discourse practices (e.g. Erath et al., 2018; Moschkovich, 2015; Setati, 2005), little is known about what students need to learn to engage in each of them for a specific topic such as equivalence of algebraic expressions. In order to close this research gap, this paper reports on a study in topic-specific Design Research methodology that aims at developing language-responsive teaching-learning arrangements and at analyzing the students' learning pathways, based upon the existing topic-specific state of research as presented in the next section.

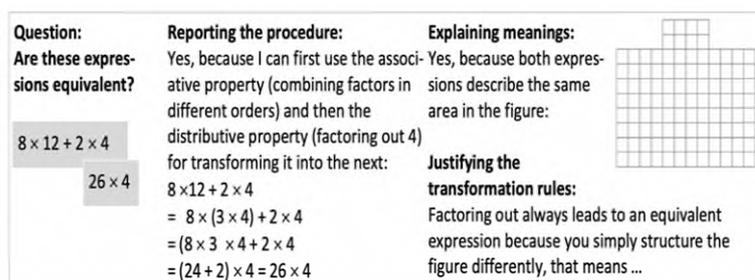


Figure 1. Three discourse practices with equivalent expressions in a pre-algebraic context

Findings of algebra education research on procedures and concepts for equivalence

The research overview by Bush and Karp (2013) reveals that students have multiple difficulties and misconceptions in algebra, among them procedural difficulties with transformations between algebraic expressions, and the understanding of equality as equivalence of expressions instead of “results”. For understanding equality, three different characterizations of equivalence of expressions have been identified as relevant (Kieran, 2004; Knuth, Stephens, McNeil, & Alibali, 2006; Zwetschler & Prediger, 2013):

- (1) In the *operational approach*, two expressions are said to be equivalent if they have the same value; when containing with variables, then same value for each evaluated number (*result equivalence*).
- (2) In the *relational approach*, two expressions are said to be equivalent if they describe the same constellation, i.e. when they can be related to the same everyday situation or the same geometric figure (*description equivalence*).
- (3) In the *formal transformational approach*, two expressions are said to be equivalent if they can be transformed according to formal transformation rules (*transformation equivalence*).

Many instructional approaches in existing textbooks have been criticized to promote only the formal transformational approach without providing learning opportunities for meaning making (Kieran, 2004; Knuth et al., 2006). When meaning making is promoted, students’ strong operational approach (which is often focused in arithmetic: an equal sign only signifies “result”) often hinders the development of a relational approach with a description equivalence (Kieran, 2004; Zwetschler & Prediger, 2013). For developing a conceptual understanding, however, the relational approach is crucial and must be tightly connected to the transformational approach (Knuth et al., 2006).

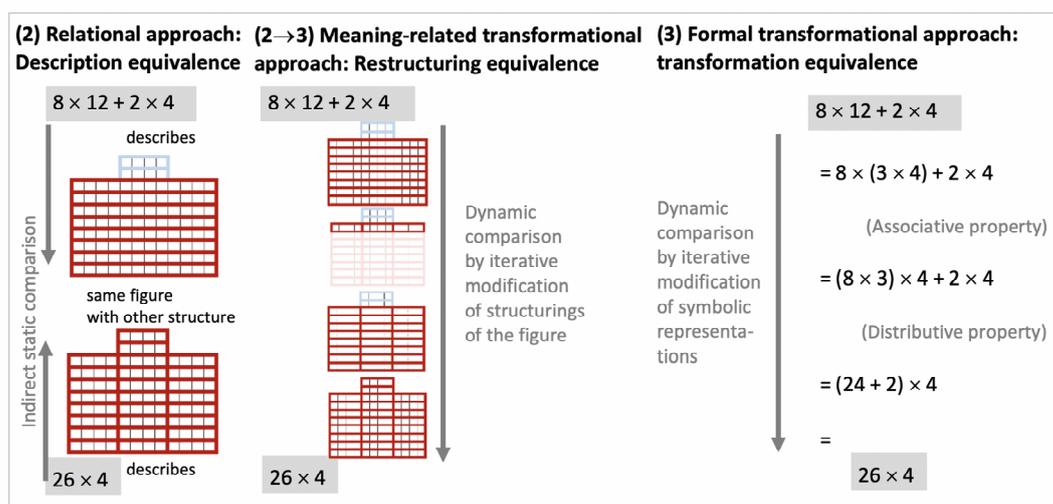


Figure 2. Restructuring equivalence: Bridging between static meaning related relational approach and dynamic formal transformational approach for comparison of expressions

This connection is not trivial: The relational approach in the description equivalence requires an indirect but *static* comparison whereas the transformational approach refers to a *dynamic* comparison. Most students can learn to explain the meaning of description equivalence, but many of them (especially, but not only language learners) struggle with connecting the two approaches in the discourse practice of justifying the rules. Figure 2 illustrates the suggestion we make in this article to

overcome this gap between the meaning-related static relational approach and the dynamic transformational approach: We introduce a fourth approach between (2) and (3) and explore empirically how to support students to overcome the conceptual and language-related challenges:

(2→3) In a *meaning-related transformational approach*, two expressions are characterized to be equivalent when we can explain how the structuring related to one expression can be modified into the structuring of the second expression (*restructuring equivalence*).

Design principles for the teaching-learning arrangement on restructuring equivalence

This new bridging approach was implemented in a language-responsive teaching-learning arrangement that follows three language-responsive design principles (cf. Prediger & Wessel, 2013):

- (DP1) engaging students in rich discourse practices;
- (DP2) connecting multiple representations and language registers;
- (DP3) macro scaffolding, i.e. providing discursive and lexical language learning opportunities for each step of the conceptual learning trajectory.

Figure 3 shows the core task for establishing the meaning-related transformational approach and promoting students' mental and discursive construction of the conception of restructuring equivalence. The example in Figure 3 indicates an important language means required for explaining how the expressions fit the structuring of the picture: Multiplicative language means for expressing unitizing, e.g. two groups of 4 or two fours.

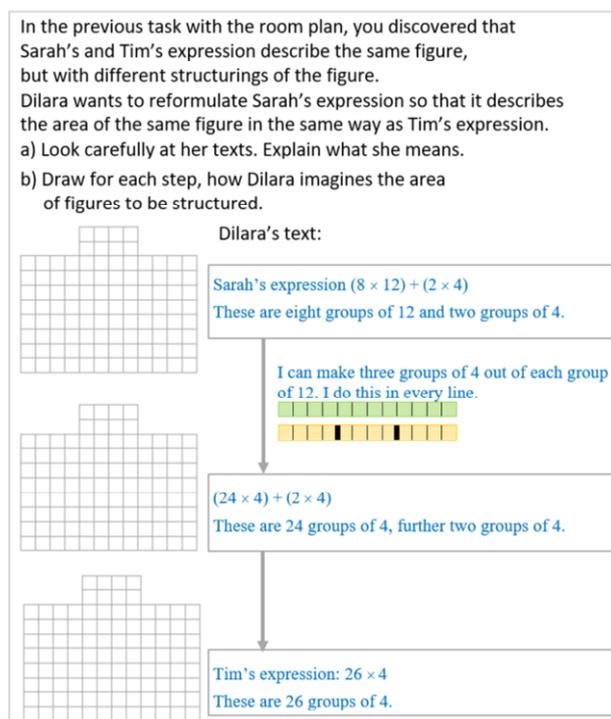


Figure 3. Core task for establishing restructuring equivalence

Research question

The Design Research project pursues the following research questions: *How can the bridging approach of restructuring equivalence support students in constructing meanings for transformations, and which language demands occur for students when involved in discourse practices of explaining meanings and describing meaning-related transformations?*

Methodology of the case study

Research context. The qualitative case study presented in this paper is embedded in the project *MuM pre-algebra*. It follows a Design Research methodology (Gravemeijer & Cobb, 2006) and aims at developing a language-responsive teaching-learning arrangement for fostering language learners' conceptual understanding of pre-algebraic equivalence in Grade 5 and at generating theoretical contributions to unpack the conceptual and language-related learning pathways.

Methods of data gathering. Conducting design experiments in laboratory settings is the central method for data gathering. So far, two design experiment cycles have been conducted with seven

pairs of fifth graders (10-11 years old) with varying language proficiency. In total, 27.5 hours of video were recorded and partly transcribed. The case studies analyzed for this paper focus on two pairs chosen for illustrating contrasting phenomena for the research question in view.

Methods of data analysis. The transcripts are analyzed qualitatively in three steps: In Step 1, the students' utterances are coded according to their conceptions-in-action on equivalence of expressions, structurings of figures and the match between them. In Step 2, students' utterances are disentangled with respect to the activated language means in static and dynamic views. In Step 3, all inventoried language means and coded conceptions are systematized in order to generate hypotheses about typical learning pathways.

Empirical insights: Students' language in handling restructuring equivalence

Episode 1: Mira and Victoria's dynamic language

Episode 1 illustrates nicely the claim that restructuring equivalence (2→3) can provide a bridging approach between the static description equivalence (2) and the dynamic symbolic transformation equivalence (3).

Mira and Victoria (11 years old) work on the first step of the task in Figure 3, trying to construct meaning for substituting 12 by 3×4 (and later 8×3 by 24). They do not talk about factorizing or substituting in a symbolic representation, but use different language means for expressing Dilara's restructurings in a meaning-related way (Dilara is erroneously taken as a male by the girls):

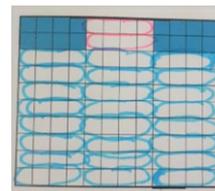
15 Mira He has divided these rows here [hints to the rows of 12 in the first figure] into three.

...

17 Victoria He has broken this apart [hints forward and backward between the expression 26×4 and the groups of 4 in their figure]

18 Mira He has always divided into three.

19 Victoria Yes, thus, and these are three rows, then [7 seconds break].
Actually, for example, he has cut it here and then together [hints to the area with eight groups of four] and here, he has also cut it.



Mira and Victoria investigate the existing figures and seem to understand the undertaken restructurings. In Turns #17 and #19, Victoria refers to the concrete activities, but without making these references very explicit, a typical phenomenon in students' everyday language. Mira uses a more explicit and concise language ("divided into 3", #15, similarly in #18). Both students adopt a dynamic language of describing the changes as concrete activities, rather than a static language of expressing description equivalence.

Beyond the case of Mira and Victoria, Figure 4 provides an overview of language means activated by several students to explain the equivalence of both expressions. Students' utterances substantially vary with respect to their degree of explicitness. Whereas some expressions could also be used for

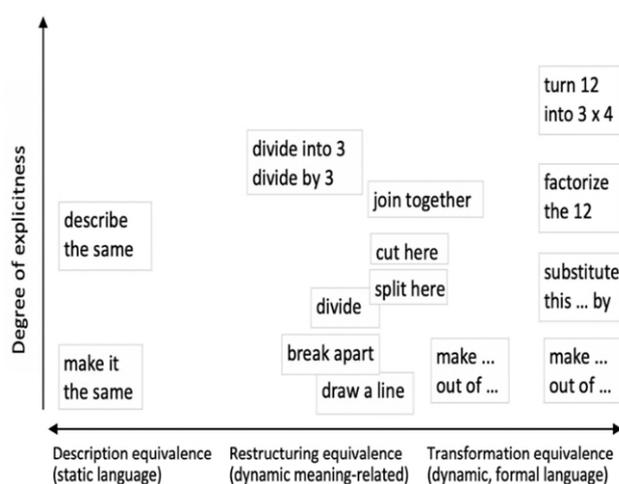


Figure 4: Ordering students' language

the formal transformation in the symbolic representations, nearly none of their words refer to a static view. Hence, the restructuring equivalence seems to have indeed the potential to bridge the gap between the meaning-related but static approach and the formal but dynamic approach. According to the current state of analysis for seven pairs of students, the task in Figure 3 seems to support students in developing a language for explaining the meanings of transformations and justify their match to the description equivalence.

Episode 2: Jessica's and Annica's struggle with seeing and expressing structures

Even if the general design decision of enriching the learning trajectory (by asking students to explain the restructurings in a meaning-related approach to equivalence) seems to have a great didactical potential for fostering students' learning pathways, many students still struggle with seeing and expressing structures. Episode 2 with Jessica and Annica can provide an insight into typical challenges which still occur on this pathway.

Jessica and Annica (11 years old) also work on the first step of the task in Figure 3, trying to construct meaning for substituting 8×12 by 24×4 . They have mastered the first substep of factorizing 12 into 3×4 and now try to construct meanings of transforming $8 \times (3 \times 4)$ into $(8 \times 3) \times 4$. The girls struggle to articulate that they wonder why there are exactly 24 groups of four.

75 Teacher What – What has Dilara changed, now? [*hints forward and backward between their first and second figure*]

76 Jessica She has – ehh – divided the eight groups of 12 into 24 four-groups

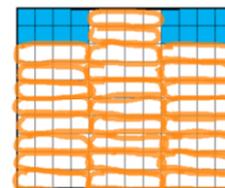
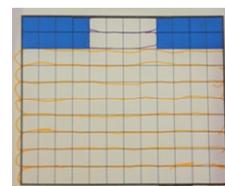
77 Annica groups of 4, [...] nothing said

78 Teacher Why are these **24** groups of 4?

79 Annica Because, she has written this

...

82 Annica One, two – wait – [*starts counting the groups of 4 in the second figure*]
1, 2, 3, 4, 5, 6, 7, 8, [*tips on the figure without counting 9 aloud*]
10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22. Oh, that does not fit!

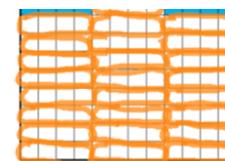


Annica does not seem to recognize the underlying structure in the figures. She counts the groups of four without using any structure and does not come to the right result (#82) and does not notice her counting mistake. This activity is an indicator for her lack of seeing structure: she treats the figures in a quasi-empirical way. Jessica can go beyond Annica's quasi-empirical ideas:

97 Jessica Let us see, in each row, there are three, aren't they? Then, you only need to count down here [*gestures down a column of fours*]

98 Annica And then, times two?

99 Jessica Eh, yes. That means, it goes eight times here [*hints to the first column of fours*].
This way down, then you must calculate 3×8 .



...

102 Annica Three times four. [*three seconds break*] Why three times eight?

103 Teacher [*To Jessica*] Explain it. Annica has not yet understood. Explain again, longer and more precisely. Try to use the language of groups. Perhaps, this works.

104 Jessica Ok. [*two seconds break*] How shall I explain that?

In contrast to Annica, Jessica discovers the underlying structure of 8×3 (#97 & # 99), but she struggles with finding a meaning-related language to explain her ideas (#104). The analysis of her struggle reveals empirical insights into the particular challenges of the transformation from $8 \times (3 \times$

4) to $(8 \times 3) \times 4$: When thinking in the graphical figure, students have to reinterpret the number of groups and the size of each group. In the first structuring, there are eight groups, each group consists of three groups of four. Therefore, the nested structure is in the *size* of the eight groups. Using the associative property means reinterpreting this structure: In $(8 \times 3) \times 4$, the *number* of groups is described as a structure of eight times three groups, while the size of each group is four.

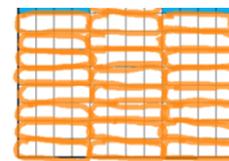
Annica's challenge in explaining this step can be located in recognizing the structure. For Jessica, the challenge is to verbalize the restructuring. The language of grouping (which we use successfully to speak in meaning-related ways about multiplication) reaches limitations as the nested structure of three factors must be further explicated:

Symbolic expression	English verbalization	German verbalization
$8 \times (3 \times 4)$	eight groups of (three groups of four)	acht Gruppen mit drei 4ern
$(8 \times 3) \times 4$	(eight groups of three), consisting of four elements each	acht Dreier mit je vier Elementen

The more complicated nested mathematical structure results in a more nested language with unclear references. Therefore, the grammatical structure gets more complicated, too.

However, with the support of the teacher, Jessica overcomes the challenges:

- 108 Jessica Look, here are always three, aren't they?
In – three groups in each row [*hints to a row in 2nd figure*]
- 109 Annica Yes.
- 110 Jessica Though, then you simply have to – you
- 111 Annica Oh, I am clever. Because there are three [*hints to a row with three fours*]. Now three, and if – well, you must calculate three times eight, because there are three [*hints to a row with three fours*] and there, eight downwards [*hints to a complete column*]
- 112 Jessica Yes



Jessica overcomes the obstacle of explaining the structure of nested groups by excluding the *size* of four of each group in her utterance, she merely focuses on the *number* of groups. Thus, she can explain this by classifying it as number and size of the groups again. She marks the reinterpretation in her language when she signifies “groups” as new elements and highlights that by stressing “three groups” (#108). She also co-uses “groups” and “rows”. By this she can distinguish the external structure of 8×3 (eight rows, three in each row) and the internal structure: each element is a group (of four) itself. Although Annica does not articulate the nested structure explicitly, her explanation with deictic means shows that she has discovered the new structure. Therefore, she finally understood the restructuring, even if the utterance in #111 might also hint to a further challenge in her multiplicative understanding, talking only about rows and columns but not groups of rows.

Whereas the associative property is often treated as trivial in the symbolic representation, its verbalization has shown to be a major challenge for students' conceptual understanding and for their participation in meaning-related discourse practices, particularly in justifying why a symbolic transformation is valid.

Discussion and Outlook

Main results of the case study

Summarizing the observations, we conclude that the empirical insights add plausibility to the assumption that the meaning-related transformational approach of restructuring equivalence can indeed reveal a bridging approach between the well-established approaches of static description equivalence and dynamic transformation equivalence. However, restructuring turns out to be much more challenging than anticipated, especially the meaning-related explanation of the associative property. Figure 5 summarizes the substeps and shows how they are expressed in the different representations.

Transformation	Formal steps	Meaning-related verbalization of the steps	Graphical representation of the steps	Meaning-related verbalization of graphical transformation
12 is splitted into 3×4 Associative property	8×12 $+ 2 \times 4$	Eight rows of groups of 12 and two rows of groups of 4		I cut every group of 12 into three groups of 4 (splitting) I see the structure differently: In the step above, I see each row and then count eight of them. In the step below, I first see the number of rows and the number of groups in a row. Then I see the group size.
	$8 \times (3 \times 4)$ $+ 2 \times 4$	Eight rows of (three groups of 4) and two rows of groups of 4 Unfolded into: Eight rows. And each row consists of three groups of 4		
	$(8 \times 3) \times 4$ $+ 2 \times 4$	Eight rows of three, each of the three is a group of 4 and two rows of groups of 4 Unfolded into: Eight rows of three groups each. And each group is a group of 4.		

Figure 5: Summary of representations for the transformation steps

The first substep can easily be expressed as concrete actions in the graphical representation, the analysis shows that students have multiple language means for articulating it (cut, split, make ... out of ..., etc.), and some of these means can also be referred to the symbolic representation. The second substep, applying the associative property, does not correspond to restructuring in the graphical representation, but to another look at the same structure by flexibly re-unitizing. Here, the meaning-related language for the units require a further unfolding into a stepwise unitizing. Unfolding the unitizing is challenging, but (at least in the presented case of Annica and Jessica) it is successful in order to enable them to express the restructuring step (even if they do it in a less condensed way than presented in Figure 5).

Limitations of the study and future steps of Design Research

Even if the presented results of our Design Research study provide an interesting insight into the didactical potential and affordances of restructuring equivalence, future research is required to overcome the current methodological limitations of the study. The major limitation is its contextualization within the specific tasks in the teaching-learning arrangements. We can assume that other tasks for restructuring expressions graphically can extend the list of language demands. Additionally, we have not yet investigated whether the bridging function from description to transformation equivalence can be transferred to other contexts and situations.

So far, the presented potentials and affordances indicate specific challenges in the verbalization and the requirement of further unfolding the language for which not all students might be prepared. However, supporting the students' language unfolding processes might be the key to providing access to algebra for more students. In further design experiment cycles, we will try to overcome some of these limitations and thereby extend the scope of the results. We will specifically investigate in which way the bridging construct can really strengthen the students' abilities to see structures also in the formal representation of the symbolic expression.

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