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Institut Montpelliérain Alexander Grothendieck, Univ Montpellier, CNRS, Montpellier, France;
viviane.durand-guerrier@umontpellier.fr

There is research-based evidence that logical analysis of language is a relevant tool for revealing ambiguities in a given language. We discuss in this paper in which respect logical analysis of mathematical statements might provide a common reference allowing identifying and dealing with unavoidable ambiguities that might occur in multilingual educational contexts.

Keywords: Didactic of mathematics, logical analysis of language, multilingual context.

Introduction

A main problem in multilingual educational context concerns the possible misunderstandings that can emerge as a consequence of differing grammatical structures between the language of instruction and the vernacular languages of students, as claimed by Edmonds-Wharten, Trinick, and Durand-Guerrier (2016):

> The impacts of features of grammatical structures on mathematical thinking are still underresearched. We have shown that languages express mathematical ideas in diverse ways. These different ways of exploring mathematical ideas provide an opportunity to enrich the mathematical experience of learners in multilingual contexts. They can also introduce ambiguities or misunderstandings between teachers and students and impede the process of mathematical learning." (p. 41)

In a first section, I will briefly provide arguments that logical analysis of mathematical statements in predicate calculus is a relevant tool for mathematics education (Barrier, Durand-Guerrier, & Mesnil, 2018; Durand-Guerrier, 2008; Selden & Selden, 1995), in particular for what concerns proof and proving (Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012; Epp, 2003). In a second section, I will focus on negation, showing first some features of negation in French likely to introduce ambiguities or misunderstandings in class (Durand-Guerrier, 2016), and giving then the main results of two studies, one in Tunisia, the other one in Cameroon, supporting the hypothesis of an impact of differing grammatical structures. In the third section, I will illustrate on the expression "deux à deux" (pairwise) that logical analysis provides a tool for overcoming possible grammatical misunderstandings in mathematics education.

Predicate Calculus - A logical reference for mathematics education

In mathematics classes, it is often assumed that every statement is either true of false. In a logical perspective, this refers to propositional calculus, in which the basic unit is the propositional variables, that are interpreted either by singular statement such as "119 is a prime number" (false statement) or "π is an irrational number" (true statement), or by close statements such as "All prime numbers except 2 are odd)" (true statement) or "There is a rational number whose square is 2" (false statement). However, we have evidenced in our research that this logical reference is not sufficient for the needs

of mathematics education. Indeed, object properties and relationships are fundamental categories involved in mathematics activities, and issues of quantification play an essential role (Dawkins & Roh, 2019; Durand-Guerrier & Arsac, 2005; Durand-Guerrier, 2008). Classical first order logic (predicate calculus) allows analysing mathematical statement precisely, e.g. those proposed above that we recall here:

1. 119 is a prime number
2. π is an irrational number
3. All prime numbers except 2 are odd
4. There is a rational number whose squared is 2

In statements 1 and 3, the property 'to be a prime number' is involved. It is modelled by a one place predicate $P(x)$, where P is a letter for predicate, and x a letter for a free variable (a place holder). In the considered interpretation, this property applies to natural numbers. Assigning a natural number to the free variable provides a singular statement that is either true of false, depending on the assigned object such as in statement 1. In sentence 3, there is no assignation; a universal quantifier bounds the free variable; the quantification domain is mentioned (the set of prime natural numbers deprived of 2). Such sentences are closed sentences and are either true of false. For sentence 2 and 4, the analysis is rather similar, considering an existential quantifier.

In addition, in first order logic, relationships are modelled by two (or more) places predicate. For examples, the relation 'is divisible by' on the set of integers is modelled by a two places predicate $P(x, y)$; while the relationship 'is the GCD of … and of …' is modelled by a three places predicate. With a two places predicate, there are four possibilities to provide a close statement (a proposition) – 1. for all x, for all y $P(x, y)$ (AA statements); 2. For all x, there exists y $P(x, y)$ (AE statements); 3. There exists x such that for all x $P(x, y)$ (EA statements); 4. There exists x, there exists y such that $P(x, y)$ (EE statements). As evidenced by Dubinsky and Yiparaki (2000), in natural language, the interpretation of AE and EA statements are interpreted according to the context. For example, the statement "There exists a mother for each child" (apparently EA) will be interpreted as "For each child there exists a mother" (AE), due to the context. In mathematics, such flexibility in interpretation is not possible. For example, it is necessary to clearly distinguish between "There exists a number greater than all other numbers" (EA), which is false in the set of natural numbers, and "For every number, there exists a number that is greater" (AE) which is true in this set. Chellougui (2009) showed that Tunisian first year university's students consider that EA and AE statements have the same interpretation, even in a mathematical context. The number of possibilities for providing a close statement with quantifiers is increasing with the number of places in the predicate: 2^3 for a three places predicate, 2^4 for a four place predicate and so on.

The complexity of the logical structure of quantified mathematical statements is still increasing as soon as logical connectors in particular negation and implication, are involved. In such cases, for a correct interpretation, it is necessary to identify the respective scopes of connectors and quantifiers.

Let us consider the following example (from Njomgang Ngansop & Durand-Guerrier, 2011):

For any function f from the set $\mathbb{R}$ of real number into itself, for any a in $\mathbb{R}$, if for any sequence u with values in $\mathbb{R}$ converging to a, $f \circ u$ converge to $f(a)$, then f is continuous at a. (1)

It is possible to formalize the statement as below, showing the complexity of its logical structure.

$$\forall f\ \forall a\ [(\forall u\ (F(u,a) \Rightarrow G(f,u,a))) \Rightarrow H(f,a)]\ (2)$$

where $F(u,a)$ formalizes the relation 'u converges to a', $G(f,u,a)$ formalizes 'the composition of f with u converges to $f(a)$', and H formalizes 'f is continuous in a.' The domain of quantification are: the set of functions from $\mathbb{R}$ to $\mathbb{R}$ for the variable f, the set $\mathbb{R}$ of real number for the variable a, the set of sequences with values in $\mathbb{R}$ for the variable u. It is to notice that there are three universal quantifiers, among two are in heading positions, the scope being indicated by the brackets, while the third one is in the antecedent of the external implication; the scope of this third quantifier being indicated by the parenthesis.

Being able to unpack the logic of mathematical statements is a core competence for proof and proving in mathematics education (Selden & Selden, 1995), in particular the logical structure of a statement is likely to orient the way to engage a proof. Coming back to example (1) above, as the antecedent of the external implication is a universal conditional statement, it is difficult to think of a direct proof consisting in considering a function and a real satisfying this antecedent. Opposite, the consequent of this external implication is an atomic formula. For this reason, the classical proof of this statement is a proof by contraposition. The proof by contraposition relies on the following logical theorem in first order logic (a statement true for all interpretation of its letters in any non empty universe, Quine, 1950)

$$\forall x\ \forall y\ (P(x,y) \Rightarrow Q(x,y)) \ \Leftrightarrow\ \forall x\ \forall y\ (\neg Q(x,y) \Rightarrow \neg\ P(x,y))\ (3)$$

A consequence of this logical theorem is that a quantified conditional statement has the same truth value as its contrapositive whatever the interpretation and the domain of quantification, so that a proof of the contrapositive counts as a proof of the statement.

The logical form of the contrapositive of the initial statement is

$$\forall f\ \forall a\ [\neg H(f,a) \Rightarrow \neg(\forall u\ (F(u,a) \Rightarrow G(f,u,a)))]\ (4)$$

Using the rules for negating a conditional statement in the formal system, we get

$$\forall f\ \forall a\ [(\neg H(f,a)) \Rightarrow (\exists u\ (F(u,a) \wedge \neg G(f,u,a)))]\ (5)$$

and finally, the contrapositive in the vernacular language:

For any function f from the set $\mathbb{R}$ of real number into itself, for any a in $\mathbb{R}$, if f is not continuous at a, then there exist a sequence u such that u converges to a and $f \circ u$ does not converge to a.

In this section we have provided examples supporting the claim that classical first order logic (predicate calculus) provides relevant tools to analyse mathematical statements, which is a core competence in proof and proving. In the last example, we have shown a methodology consisting in unpacking the logical structure of the statement in order to formalize it in the language of predicate calculus, then providing the contrapositive of this formalized statement and then give the contrapositive in the vernacular language. This last step is necessary because in order to engage in a mathematical proof, it is necessary to consider objects, properties and relations involved. Considering this, it is reasonable to conjecture that such work would be difficult for students learning mathematics

in a non-native language. In the following section, I will present some features of negation in French and I will briefly report on two studies on negation in a bilingual context in the Francophone area.

Negation in French – impact in multilingual context

In French, the sentences in the form "For all *A*, *A* is not *B*" are ambiguous. In the linguistic norms, they must be interpreted as "some *A* are (is) *B*, and some are (is) not", or in a more formal way "There exists some A that are (is) not B". However, in everyday contexts it might be used with a different interpretation, namely, no *A* is *B*. For example, in a very cold winter in Lyon (France), the Public Transportation Company had widespread the following message: "Aujourd'hui, tous les bus ne circulent pas" (Today, all buses do not circulate). It seems that a number of people called to ask which buses circulated. In any case, three hours later, they changed their message to "Aujourd'hui, aucun bus ne circule" (Today, no bus is circulating). This non-standard interpretation is rather common in oral discourse, including radio and broadcast. In addition, it is noticeable that this linguistic norm does not respect the fundamental rule that "replacing a term by an equivalent expression should preserve the truth value of the statement". Here "equivalent expression" means to be satisfied/not satisfied by exactly the same elements of a given domain. For example, in the domain of natural numbers 'to be odd' is equivalent to 'not to be even.'

Let us consider now the following statement:

"Tous les diviseurs de 12 sont pairs" (All the divisors of 12 are even) - False

Its standard negation in French

"Tous les diviseurs de 12 ne sont pas pairs" (All the divisors of 12 are not even) - True

Then change "ne sont pas pairs" (are not even) in "sont impairs" (are odd)

"Tous les diviseurs de 12 sont impairs" (All divisors of 12 are odd) - False

In addition, the negation of the universal statement in French, according to the linguistic norm, is not congruent with the logical structure. Indeed, a word-for-word formalization would lead to the formalized statement "$\forall x\ \neg P(x)$" which is not the negation of "$\forall x\ P(x)$"

We might anticipate that such ambiguities inherent to the French grammar could be source of difficulties for non-francophone natives studying mathematics in French, and this especially as teachers are generally not aware of this. This has been confirmed by two studies, one In Tunisia, the other in Cameroon. I briefly summarize below the results.

In his PhD (Ben Kilani, 2005), Ben Kilani studied the differing grammatical structures between Arabic, French and predicate calculus. He showed that French and Arabic were not congruent for what concerns the negation of universal statements, while Arabic is congruent with predicate calculus. Indeed, in Arabic, when the negation is on the predicate, the scope of the negation is the predicate, not the sentence. The experimental results show that for most students, the French universal statements with negation on the predicate were not interpreted as the negation of the sentence, in coherence with the standard interpretation in Arabic and, as already said, in logic. He also showed that nobody took care of this: neither the language teachers (Arabic or French), nor the mathematics teachers, where in Tunisia, mathematics is first taught in Arabic from grade 1 to grade 9 (Ecole de

base), and then in French at secondary school (Durand-Guerrier & Ben Kilani, 2004; Durand-Guerrier, Dias, & Ben Kilani, 2006).

Following the work by Ben Kilani, Njomgang Ngansop and Durand-Guerrier (2011) examined the impact of the grammatical structure of Ewondo in the teaching and learning of logical concepts. There are two educational systems in Cameroon, one is Anglophone, the other one is Francophone. The study concerns the latter, and among the great number of languages, the Ewondo. For what concerns negation, as for Arabic, the grammatical structure of Ewondo is differing from the French grammatical structure. The results of the exploratory study support the conjecture of an impact on the teaching and learning of mathematics.

Logical analysis as a tool for overcoming grammatical misunderstandings in mathematics education

The previous examples on negation highlight the impact of differing grammar on the interpretation of statements. It also emphasizes the fact that formalization in predicate calculus is a means to identify possible differing interpretations given by interlocutors. Summarizing briefly, we could claim that formalizing is choosing an interpretation.

Beyond negation, there are other mathematical expressions likely to introduce ambiguities in mathematical discourse. For example, in French, we often use in definitions involving a binary relation the expression “deux à deux” (in English: pairwise) as in the following examples.

> *Definition 1* (plane Euclidean geometry): A non-degenerated plan quadrilateral is a parallelogram *iff* its opposite sides are pairwise parallels.
>
> *Definition 2* (set theory): A partition of a given set E is a finite collection of non empty sub-sets of E that are pairwise disjoint and such that their union is exactly the set E.
>
> *Definition 3* (probability): Events are pairwise independent *iff* the occurrence of one event does not affect the probability of the other events.
>
> *Definition 4* (plane Euclidean geometry): Two triangles are similar *iff* the corresponding angles are pairwise congruent.
>
> *Definition 5* (space Euclidean geometry): A polyhedron is regular *iff* it is convex, its faces are regular polygons that are pairwise superimposable, and there is the same number of faces meeting at each vertex.
>
> *Theorem* (complex numbers): Given a polynomial with real coefficients, the non-real complex zeros of this polynomial, if any, are pairwise conjugate.

In a teacher training session, we had asked participants to determine all the regular polyhedral (definition 5), with the possibility of using materials to build such polyhedral, or of doing drawings or of making patterns. They were working in small groups. During the session, one of the groups of participants discussed the meaning of the expression “deux à deux” (pairwise). The discussion was: does it means *whenever you consider two faces, they are congruent (superimposable)* (interpretation 1) or does it mean *whenever you consider a face, you can find another one (different from the initial) that is congruent (superimposable)* (meaning 2)? The members of the group did not manage to reach

an agreement, until the researcher came and gave the correct interpretation in this context (interpretation 1). It is noticeable that in this case, it was not possible to refer to the empirical possibility of the realization, because both interpretations allow it. At a first glance, and considering the strong willing of univocity in mathematical language, one could think that this was the standard interpretation in mathematics. However, the four other definitions and the theorem given above show that both interpretation are currently used in mathematics. Indeed, in definitions 2 and 3, pairwise refers to interpretation 1, while in definitions 1 and 4 and in the theorem, pairwise refers to interpretation 2.[2]

Once I presented and discussed this example in a seminar, a PhD student attending the seminar told me that, when he arrived in France for preparing a master degree, he encountered this expression in a topology course, and that it took him time before he understood that he misinterpreted it. His insufficient knowledge of the mathematical context did not permit him to choose immediately the right interpretation. It seems rather clear that such misinterpretation is likely to impact the understanding of the concept at stake, and consequently the learning. A path to overcome such ambiguities is to express the corresponding expressions in the formalized language of predicate calculus.

Given a binary relation, it is modelled in predicate calculus by a two-places predicate. Let us name it S (x, y). The two interpretations refer to the two following formalized expressions

Interpretation 1 - $\forall x\ \forall y\ S(x, y)$

Interpretation 2 - $\forall x\ \exists y\ (x \neq y \wedge S(x, y))$

Although the aim of mathematical language is to avoid ambiguities, such examples show that this is not always possible. Indeed, formalizing these statements has lead to considering two interpretations. In such cases, to formalize is to choose an interpretation.

It seems rather clear that in multilingual contexts, being able to deal explicitly with such ambiguities would open paths for remediation. As we have seen, in case of non-grammatical congruence, translating from one language to the other might change the interpretation, introducing misunderstandings likely to impede the learning process.

Conclusion and perspective

In this paper I have evidenced that logical analysis is likely to shed light on possible mis-understandings in mathematics education due to differing grammatical structures in multilingual educational context. There are two future lines of research. The first one consists in considering a greater variety of languages. The second one consists in testing empirically the following hypothesis: I hypothesize that, given a mathematical statement with possible problematic interpretation in multilingual contexts, asking students to first formalize the statement in predicate calculus, and then move to their own preferred language might help them to overcome misunderstandings resulting of

[2] For a wider presentation and discussion in English, see Durand-Guerrier (in press).

differing grammatical structures between the language of instruction and their preferred language. And, additionally, that this would contribute to a better understanding of the concept at stake.

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