Proceedings of the Seventh ERME Topic Conference on Language in the Mathematics Classroom

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Seventh ERME Topic Conference

Language in the Mathematics Classroom

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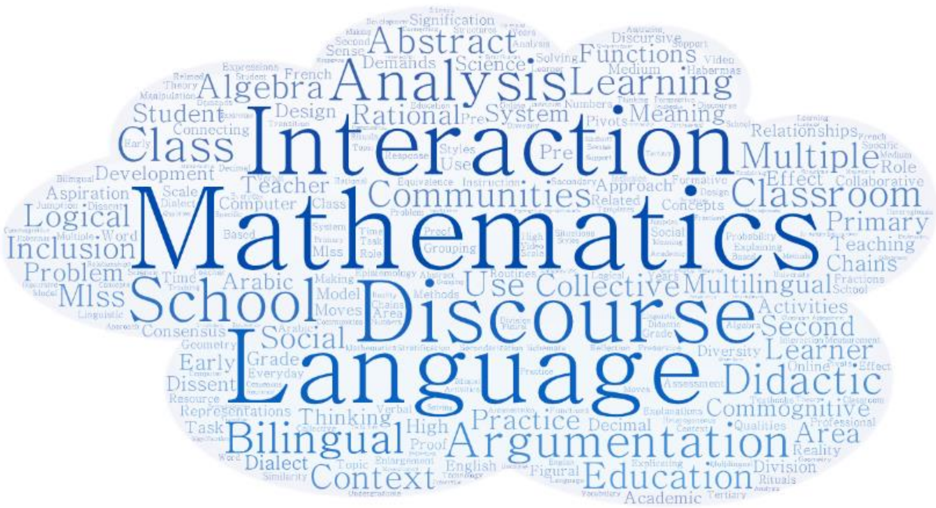

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Language in the Mathematics Classroom: An introduction to the papers and presentations within ETC 7

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Research considering the role of language in the teaching and learning of mathematics continues to grow and develop, drawing on a range of theoretical, methodological and pedagogical approaches. In this introduction, we detail the discussions had and issues raised at the 7th ERME Topic Conference as a result of the bringing together of the theoretical perspectives, foci and findings of the papers presented that are included in these proceedings. These rich discussions also raised new challenges for those researching language and mathematics and identified new possibilities for the future work of the Mathematics and Language thematic working group.

Keywords: Language and mathematics, multilingual contexts, meaning-making, interaction.

Introduction

Research into language and mathematics has seen a shift away from distinctions between the language of the teacher, the language of the students, and the language of mathematics, towards a more integrated understanding focusing on language in interaction and the role of language in meaning-making. Much of this research is now situated in classrooms, and the focus is more on the interactions between teachers, students and mathematics than the language of each as exemplified by the focus of ETC 4 (Planas & Schütte, 2018). This attention to interactions also blurs the boundaries and distinctions between the foci of different researchers, theories and the discussions between researchers. Whilst the papers in this conference have been grouped into three themes, there is considerable overlap between the themes, and we learn a great deal from discussions both within and between the themes.

The first theme is *language in multilingual contexts* where the multilingual nature of mathematics learning and teaching is the focus of the discussions. The second theme of *language for meaning-making* focuses more on the language aspects of the conceptual understanding of particular mathematical concepts such as multiplication, angle or proof. The third theme concentrates on *classroom interactions and discursive practices* such as argumentation or explaining. The research presented within these themes often intersects with each other. For example, the issue of learning the discursive practices of proving could equally well have been investigated within the third theme as in the second theme. However, as universities increasingly become multilingual, this issue could also have been investigated in the first theme. And indeed, a current question is how to make intricate logical relations accessible to students who learn mathematics in a second or third language.

The interaction between themes highlights why continuing work on language in mathematics is necessary. The papers in this conference draw from an increasing diversity of theoretical approaches, many with their origins in other fields such as sociology, linguistics and psychology. This diversity enables researchers to consider a range of contexts, issues and foci but also raises the question of how coherence can be sustained and developed as we learn more about the role of language in the learning and teaching of mathematics. Furthermore, this diversity also emphasises the challenge of communicating the research included in this conference, alongside and building upon other language-sensitive research (Planas, Farrugia, Ingram, & Schütte, 2019). However, as past research has shown, such diversity can often be the starting point for identifying further contexts, issues and foci worth investigating.

Language in multilingual contexts

A total of 6 papers and 1 poster were presented in the theme of *language in multilingual contexts*. The theme *language in multilingual contexts* has a long tradition in research into language and mathematics. The focus of the theme within ERME conferences has been on multilingual students' resources for learning mathematics (e.g. Planas, 2018) since the beginning, and on how learning and teaching practices in mathematics could allow multilingual students to better utilise their resources (e.g. Barwell, 2020; Norén, 2015).

At this second topic study conference on language and mathematics, a new issue emerged in the theme of language in multilingual contexts. This new emerging issue concerns the learning and teaching of mathematics in a university context. In the past, research on language in multilingual contexts has often focused on primary and secondary education. This new emerging issue has become relevant in the recent years because of new emerging contexts for learning and teaching at university: 1. African countries are strengthening their own cultural identities, which results in moves towards utilizing students' first language for learning and teaching of mathematics in schools. For example, Arabic and Berber languages replace French in Algeria, which was the regular academic language since colonial times. However, French continues to be the lingua franca at the university level, so that many students learn mathematics in their second (or third) language (Azrou). Similarly, to strengthen the Irish cultural identity, one Irish University utilises Irish as a language of instruction in the first year (Ní Ríordáin). 2. Many universities in Europe start to switch their education towards English as Medium of Instruction (EMI). This means that all students, international or national students alike, learn mathematics in their second (or third) language, English. Traditionally, a similar language context could be found in the US and Australia / New Zealand, where international students learn mathematics in their second (or third) language English, which was found to be challenging for students (Barton, Chan, King, Neville-Barton, & Sneddon, 2005). This situation is now becoming more common in Europe as well, for a larger group of students. 3. Universities in regions with a multilingual population teach in their respective languages of instruction (e.g. South Africa, Russia), so that many students learn mathematics in a second or third language. Related to this is the need for student teachers to learn to communicate multilingually in their future school teaching.

At the conference, the issue of multilingual mathematics learning and teaching has been investigated from various perspectives, but mainly with the perspective on cultural backgrounds and language resources to *describe* new phenomena of bilingual mathematics learning at university. With respect to the first context outlined above, Ní Ríordáin studies the teachers' use of Irish language and how they connect to representations and finds that English is used as a language to clarify meanings. Framed within the traditional perspective of language differences, Azrou investigates the different ways in which different languages allow to express conditional statements, which can impact logical reasoning. While this analysis focuses on structural aspects of different languages, it highlights potential difficulties that need to be investigated further in students' actual language use in the future. It could be hypothesized that such structural deficiencies of certain languages can be compensated by translingual practices, where multiple languages can be used as a combined resource. In a similar perspective on structural features of languages, Durand-Guerrier argues that the translation of logical statements into formal statements could build bridges to connect multiple languages. However, in the discussion it was questioned whether logical reasoning with formal statements can be independent of the specific language contexts from which they have been translated.

Within the context of university learning with a multilingual student body, Salekhova as well as Meaney and Rangnes investigate student teachers' learning. Salekhova proposes a model to rate the quality of multilingual mathematical communication in school mathematics classrooms. Meaney and Rangnes investigate student teachers' learning in a multilingual university context and find that a multilingual context offers opportunities to make implicit assumptions about mathematics learning explicit. However, if English is used as a shared language in such a context, such implicit assumptions can often go unnoticed, so that these opportunities are often not realized. Within the context of the impact of EMI on university mathematics learning, Schüler-Meyer investigates students' writing practices. As writing becomes a central medium of communication at university, challenges with writing mathematically in a multilingual context could potentially hinder mathematical understanding.

In addition to the focus on university contexts, there is a continuation in investigating multilingual mathematics learning in a secondary school context. For example, Barwell introduces the constructs of "flow of language" and of "scales", where the first is a metaphor for the fact that students, by speaking in a classroom, insert themselves into a flow of language, which has been formed by previous speakers, and will be changed by his or her utterance. Scales describe different levels of sources of meaning, where, for example, a mathematical idea of a student can be acceptable by peers (smaller scale), but not when being considered in the context of nationwide exams (bigger scale). This study illustrates how new theoretical constructs continue to highlight new and relevant phenomena in multilingual mathematics learning.

Language for meaning-making

A total of nine papers and one poster were presented in the theme of *language for meaning-making*. As mentioned in the introduction, there is a certain overlap between the three groups, so several of the papers also in this group could equally well have been presented in the other groups. A characteristic of most of the papers in this group is that the object of study is closely linked to a particular mathematical topic. In the papers and presentations, one can find examples of studies connected to counting and early number understanding (Farrugia), measurement (Chesnais & Constantin), decimal numbers and fractions (Coulange & Train), early algebra (Dohle & Prediger), multiplicative structures (Rønning), geometry (Akdoğan, Güçler, & Argün; Bolondi, Branchetti, & Giberti; Mithalal) and probability (Post & Prediger), sometimes also combinations of these topics. Contexts for the papers span from kindergarten to secondary school. Most of the papers are based on observational (classroom) studies but examples of quantitative studies are also present.

The role of language for meaning-making and for conceptual understanding has been acknowledged for a long time and a central issue is the development of a mathematical discourse necessary for competent participation in mathematical practices (Moschkovich, 2015). One dimension of competent participation is facilitating learners to transition from an everyday to a mathematical discourse. This dimension is indeed addressed by several of the papers. Mithalal's study is situated in early learning of geometry in France and is connected to the curricular requirement that children at a very early stage should use '*specific vocabulary'*. Coulange and Train, in their work with decimal numbers and fractions, are interested in what they, with reference to Bakthin, denote as a transition from first discourses to second discourses. Chesnais and Constantin discuss *implicitness* in the mathematics classroom, meaning that some elements of the mathematical practice are not made clear to the students and therefore may hinder conceptual understanding and the development of the mathematical discourse.

Mathematical topics covered in the papers in the group are predominantly topics that are central in the middle grades of compulsory school. Dohle and Prediger look into early algebra by investigating fifth graders' meaning-making when transforming expressions like $8 \times 12 + 2 \times 4$, using a variety of representations, and justifying why the transformations are valid. Representations, and connecting representations, are also important in the work by Coulange and Train on decimal numbers and fractions. They refer to the connecting of representations as constructing coherence between voices. In the paper by Rønning, the central theme is proportionality, and the challenges involved in expressing shape-preserving enlargement as a multiplicative structure.

Bolondi et al. take as a starting point Fischbein's (1993) theory of figural concepts. Using a geometrical figure from Fischbein's work they analyse students' judgement of the truth value of certain assertions connected to this figure. Their work is with students in Grade 9, and also Post and Prediger, in their work with conditional probability, frame their study among students of similar age. Their main interest is to explore what academic language demands students meet when developing conceptual understanding for conditional probability. The same age group is also addressed in Elçin et al.'s work when they look at one 16-year-old student's discursive development on the topic of reflection in relation to the teacher's discourse. The authors look at this development using Sfard's (2008) theory of commognition. Farrugia also refers to Sfard in her work on pre-school children's development of a mathematical discourse during play.

In addition to the nine papers, one poster was presented, by Hache, Dias, Millon-Fauré and Azaoui. The setting in the poster is among multilingual immigrant learners and their development of a mathematical discourse in the French language.

In a summary of the work in the group, one of the big questions for further work was formulated as follows: How do students develop from emerging (everyday, informal) discourses to new discourses (new words and new usages) and get deeper understanding? Other topics to address were phrased as to look at the discourse and meaning transformation arising from the task compared to the external discourse appropriation and to study links between students' discourse and their activity in general.

Language in interaction.

The ten papers discussed in the group fill the common theme *language in interaction* differently. However, all seek to better understand students' mathematical learning in collaborative settings or to better facilitate student communication and learning mathematics at different levels. Two themes occurred in the discussions of the presented research that highlight possibilities for future collaboration and interaction between the researchers that presented their work in the group.

First, the group expressed a need for further coordination of theories that particularly consider theories (and the corresponding tools) from outside mathematics education. Already within established theories in the context of language and interaction, different theoretical notions and their implications for analysing and interpreting student data were discussed. For instance, mediation and Habermas' (1988) construct of communicative rationality (e.g. in Boero & Turiano on the new construct of rational mathematical templates) as well as negotiation of meaning connected to a more interactionist perspective on learning mathematics (Krummheuer, 2011) were examined. The latter is elucidated in numerous studies. For example, in Bitterlich's analysis of real-world contexts and their impact on language and learning, Ludes-Adamy and Schütte's identification of dissent and consensus situational structures, or Friesen and Schütte's observation of interactional obligations for participating in deeper collective argumentations. With great interest, the group also discussed new ideas on theorising subject choice and consideration of time as suggested by Smith in the context of analysing students' accounts for choosing advanced mathematical pathways. Also Ingram and Andrews' insights from a study on working with teachers on improving students' communication skills based on the Discipline of Noticing (Mason, 2012) amplified the range of presented frameworks. Furthermore, the question of how to approach the problem of balancing theory and empirical insights (transcripts) when talking or writing about research on language in interaction was discussed (as they were in Planas et al., 2019). This first theme of networking theories is part of the ongoing discussion not only in the CERME TWG on language and mathematics but also in other working groups and sometimes even by means of a dedicated working group on networking theories in CERMEs. However, the second theme of researching language in interaction in the context of digital learning environments arose for the first time.

The second theme of discussion was initiated by a presentation on developing digital learning environments for mathematics classrooms which aim at supporting students' collaboration and language production (Albano, Coppola, Dello Iacono, & Pierri), and a presentation on a digital tool facilitating teachers to use students' written answers to organize meaningful whole class discussions (Zöchbauer & Hohenwarter). The group extensively discussed the issues of collaboration, interaction and language against the background of chances and obstacles of digital tools and environments for learning mathematics. Furthermore, the question arose as to which theories, already implemented in mathematics education research, and with a focus on language in interaction, are functional for researching classroom interaction connected to digital learning environments. Of course, the group only started this discussion and did not reach any definitive answers and solutions. However, in particular with the Corona Virus forcing teachers and students around the world to engage in various kinds of E-Learning, home-schooling etc., the question of how we can transfer our existing insights on language in interaction into teaching and learning with digital tools and environments are important areas for collaborative research in the near future. For example, it would be interesting to discuss to what extent Tewes' work on support systems for participation in collective argumentations or Zindel's study on teacher moves for promoting students' participation for language learners can contribute to developing and refining digital learning environments. Likewise, to what extent can digital tools and environments help teachers to promote and support language in mathematics classrooms? And to what extent becomes "language of the tool" another language learning goal in mathematics classrooms?

Linking the existing work on language, interaction and learning mathematics to digital learning environments and tools for teaching might be one of the challenges in the coming years not only for our subgroup but for all researchers in the area of language in mathematics education. For example, it could be fascinating to see which opportunities of relating registers and representations (as indicated by Albano et al.) arise by digital means as well as what opportunities and obstacles online group work involves compared to face-to-face interaction. Thus, a focus on online interaction as a form of interaction in mathematics education seems to be a new field for future collaboration, which could also be perfectly combined with research on meaning-making and multilingual contexts, as was in focus for the other subgroups.

Plenary sessions

Plenary talks allowed the two speakers, Candia Morgan and David Pimm to expose and present some of their consequent and inspiring research work. This gave the opportunity for people who were not as familiar with the nuance of their work to discover it further and for others to deepen their comprehension of it and to better identify the richness and significance of their work.

Candia Morgan, in her talk about "Conceptualising and researching mathematics classrooms as sites of communication" defended the potentialities of switching, in research, from looking at language and communication (including not only written and spoken language but also gestures, bodily movements or visual representations) in the classroom to considering mathematics teaching and learning as a communicative situation. David Pimm's talk about "Speaking, writing and mathematics registers: Denying "the dream of a common language" for mathematics", on his part, led him to pledge for a strict recognition of Halliday's notion of registers to avoid risking a too broad understanding of what counts as communication in mathematics. For example, he warned against conceptualizing symbolic language of mathematics as a means of communication that can be independent from spoken language.

Both talks also allowed the presenters to share ideas which could provide support for designing new research directions and support each of us in reflecting on the landscape of research on the topic of language and mathematics education, as well as on the theoretical assumptions which underpin the research field. These two themes continued throughout the discussions within each of the groups as they considered the future direction for research in multilingual contexts, language for meaning-making, and language in interaction within mathematics education.

Conclusion

The contexts in which we are researching continue to evolve. Mathematics classrooms today do not look like those of 20 years ago, and it is likely that in 20 years' time we will be saying the same thing about classrooms today. Multilingualism is now a common feature of many classrooms, lecture theatres and other educational settings. Choices or decisions (unconscious or deliberate) about which languages to use are politically and socially complex, as are the choices and decisions around the use of mathematical language, mathematical discursive practices and representations. These choices or decisions are becoming even more complex as we embrace and utilise digital technologies.

There are many challenges that we continue to face as the contexts in which we research evolve, and as research on language in mathematics education evolves and develops. As Candia Morgan reminded us, we draw on a range of theories from different disciplines that we have found useful in helping us to understand the relationships between mathematics and language, and in particular the role of language in the learning of mathematics. Yet we need to continue to revisit and evolve these theories to specifically address the concerns of mathematics education.

The focus of this ERME Topic Conference on classroom interactions also points to the growing need for teacher education to take account of language in the teaching and learning of mathematics. Teachers are increasingly working in multilingual and linguistically diverse contexts and there is now significant research that points to the role of language in meaning-making in mathematics, and in interaction. One challenge here is to not only to make the research accessible to teachers, and teacher educators, but also to make it relevant and of interest.

Research on language in the mathematics classroom has come a long way in the last few decades, and the journey continues. Opportunities like those in the ERME conferences offer us insights from different perspectives, disciplines and foci, as well as providing an opportunity to discuss and examine the issues and challenges we encounter as we move the research of the working group forward. These proceedings can only offer a glimpse of the richness of the discussions that were had, but we hope they will enable you to become as intrigued and enthused by the theoretical and empirical insights the different authors offer.

Acknowledgment

We want to thank all the participants of ETC7 for their presentations, participation in the discussions and their collaborative working together in the sessions. It provided us with a rich opportunity to reflect on the present and future lines of our research on language in the mathematics classroom.

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ST1
Language in multilingual contexts

Linguistic difficulties in the transition to the university: Learning mathematics in three structurally different languages

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This paper reports a study of the structural differences of the three languages used in Algeria: Arabic, dialect and French. These languages are used by both students and teachers at the university: French for written mathematics and the two others for oral expressions and thinking. The use of Arabic and dialect is meant to help students in learning mathematics in French as their mastery of French is weak, but the many differences concerning logical structures might be a source of difficulty for students, when they use the three languages. Some research questions are stated together with how to validate them in a further research.

Keywords: Arabic, dialect, French, undergraduate mathematics.

Introduction

Many studies in the literature show positive aspects of teaching and learning mathematics in multilingual contexts (Poisard, Nì Rìordaìn, & Le Pipec, 2019; Adler, 2001). I support this position and I think that mastering many languages can be an advantage as it allows expressing meanings, concepts and thoughts in many ways, according to different linguistic structures and words of the different languages used, which would enhance mathematical performances. However, this nice image fades in the case of teaching and learning mathematics in multilingual contexts in university (Durand-Guerrier, Kazima, Libbrecht, Njomgang Ngansop, Salekhova, Tuktamyshov, & Winsløw, 2016), where mathematics is taught in a second or a third language that students and teachers do not master very well. The situation is even worse when the different languages have 'different modes' of expressing logical relationships in mathematics, which might introduce ambiguities or misunderstanding between teachers and students and in students' mathematical activities, thus affecting mathematical learning (Edmonds-Wathen, Trinick, & Durand-Guerrier, 2016). According to the linguistic relativity hypothesis, the structure and the vocabulary of a language influences the way we think in that language (Whorf, 1956); however exploring the link between the grammatical differences of languages and the mathematical thinking process is not easy (Nì Rìordàin, 2013; Edmonds-Wathen et al., 2016).

Another motivation for the content of this paper is that literature presents many studies of teaching and learning mathematics in multi-linguistic contexts. However, specific studies investigating the mathematical difficulties of undergraduate students who are native Arabic speakers when learning mathematics in a second language, are very few (Yushau & Hafidz Omar, 2015; Yushau & Bokhari, 2005) and they are not focused on logical aspects.

The research performed in my PhD thesis that concerned some difficulties with proof and proving at the undergraduate level, together with the findings of Azrou and Khelladi (2019) revealed linguistic difficulties of Algerian students caused by the fact that they learn mathematics in a second language (French). The subsequent study (Azrou, 2019) concerned Algerian university students who enter the university and start to learn mathematics completely in French after having learnt mathematics, in all previous school levels, with verbal expressions in Arabic and written formulas in French. It investigated students' linguistic difficulties as they use three different languages: French to write and read, Arabic and dialect for discussions between teachers and students, and among students, and also for thinking. Findings from interviews showed a complete dependence of students' learning mathematics on translation to Arabic (to think) and from Arabic (to write in French). To understand mathematics when translating from one language to another, students must translate the mathematical statements from one language and express it in the other language. If this translation would be hindered by different linguistic structures of the three languages, students might meet problems when understanding and expressing mathematical statements.

An in-depth investigation of the linguistic structures of some logical aspects in the three languages is thus necessary in order to uncover the differences; this paper presents some findings concerning logic-linguistic aspects that are relevant in mathematical activities. Previous studies helped us to identify those aspects. Selden and Selden (1995) reported that undergraduates in the USA showed significant difficulties to translate informal mathematical statements (written in ordinary language) to formal language (using predicate logic): in particular, the logical connector 'or', 'if-then' statements and 'if-only-if' statements are generally misunderstood by students (Epp, 2003), Durand-Guerrier (2008) observed a semantic ambiguity with both French students and teachers, and Tunisian students about the statements using quantifiers and negation like: 'all As are not B'.

The next part, which is the core of this paper, is a comparison of the three languages (Arabic, dialect and French) regarding the aspects identified as being problematic for students in the literature; the last part of the paper is an outline of further research.

Differences concerning the logical structures of Arabic, dialect and French

The language used for writing mathematics and reading mathematics in textbooks is French; but teachers and students do not speak French in the class. Both Arabic and Dialect are used for explanation and discussion, due to the weak mastery of French language by students, but also by several teachers. By using Arabic or Dialect, teachers try to help students to understand better the concepts and encourage them to ask questions and share the discussions. But the three languages have different features concerning the way of expressing logical and conditional expressions and this results in relevant differences when mathematical reasoning needs to be expressed by words.

- The conditional structure 'if … then', which plays a crucial role in mathematics, is presented in different ways in the three languages.

In mathematics, the only Arabic form taught in schools (from upper secondary) is expressed 'idha…idhen' ('dh' is pronounced as 'th' in then), in the plain case of expression like this: *if* a function is derivable, *then* it is continuous. However, this structure (using idha...idhen...) is not regularly used in Arabic and does not sound good. In Arabic, like in other languages, the conditional sentence is composed by two parts: the protasis, i.e. the sentence of the condition (with the verb of the condition), as 'the function is derivable' in the previous example, and the apodosis, i.e. the consequence of the condition, as 'the function is continuous' in the previous example (with the verb of the consequence). The terms put in the place of 'if' and 'then' are many and are not used two by two necessarily (like 'if' is only used with 'then'), some term1 (as if) is used with some term2 (as then) but is also used with term 3 (as then). Focus in Arabic is on the conditional terms put at the beginning, there are many and fall into two groups: those that change the ending of the first verb and those that do not. Among those that belong to the first group, there are seven (inna, idhma, mahma, kayfama, haythouma, ayy). The second group is composed by (idha, law, lawla).

- ‘ithen’ (then in English) is rarely used, there is another word used more often, which is ‘fa’, or we use nothing, so the conditional structure is often expressed as: ‘*idha*…*fa*’…’, or ‘*idha*...nothing...’

- There are other words for if, which are in, law and lamma (synonym of indama:when), that would give: ‘in…fa…’, ‘law…fa…’.

The teaching is focused more on how to use these terms syntactically (with care for how ‘the ending’ of the verb of the condition and the verb of the response of the condition are pronounced). There is no correspondence with what happens in the same situations expressed by the conditional ways in French (and also in English) to express possible, probable or impossible situations, by changing the tense and the mode of the first verb and the second verb. Let us see some examples:

‘If you turn left, you will end up in the next street’ (expressing possibility) would be expressed in Arabic by using: *Law*+ verb+money *la* +verb+car, ‘in ...la...’, or ‘in ...sa ...’,

To express a condition that has never been realized, in Arabic we use ‘law…la…’, with the past tense for the verb of the first sentence and a particular mode for the second sentence. Even in mathematics, this situation would have been expressed so, and not by using the ordinary expression ‘idha...ithen’ for ‘if-then’.

Even though, in high school mathematics, we have introduced one structure to express a condition, which is ‘idha…ithen…’, it is not sure that students would consider other conditional sentences (formed with other terms, cited above) as being the same as this one. But more importantly, students might not be able to translate Arabic sentences with different conditional terms (other than ‘idha’ and ‘idhen’) to French sentences with ‘si… alors…’ (if…then…) and when they translate from French the conditional sentence (si...alors...) to Arabic, students might not consider the multiple Arabic terms to express the condition other than ‘idha...idhen...’, particularly in the oral form.

In the dialect, the words are different, the analogous of ‘if’ is ‘ki’ and there is no words for ‘then’ in the present; while for a condition that has not been satisfied, the analogous for if is ‘lukane’ and no words for ‘then’ or we use ‘lukane’. So we say: ‘lukane…(nothing)…’, or ‘lukane…lukane…’. In this structure, the verbs for the first and the second sentence are always in the past. Again here, we wonder if students, when thinking in dialect or listening their teacher who speaks in dialect would translate their conditional structure to the French one with ‘si…alors’ and when they translate from French (for instance when they read a textbook), whether they identify their corresponding structure in dialect. No need to mention how it is confusing and complicated when a student thinks in dialect speaks in Arabic and translates into French to write a text, or when a student listens in dialect, thinks in Arabic and writes in French.

Here, it is worth to consider also the meaning of the implication presented by the arrow at the university level, but presented by the verb ‘implies’ before in the different school levels. When we associate always the same conditional structure and the same markers (if…then…) to the form P Þ

Q, there will be no confusion with students, and this is what it is supposed to happen with the form ‘idha...idhen...’ when translated to or from the French form ‘si…alors’. The problem is how would students deal with other conditional structures in Arabic using other terms than ‘idha’ and ‘idhen’? How would they translate them to French, and are they aware that they would all fall under the form ‘PÞ Q’?

The mathematical negation structure in English is expressed like in natural language; it is formed by do (in the corresponding tense: present or past)+ not+verb (example: I do not eat meat). In French natural language, the negation is formed by ‘ne+verb+pas’ (example: ‘je ne mange pas de viande’ ‘I do not eat meat’). In classical Arabic, the situation is almost the same; we say in mathematics ‘nefye P’ for ‘negation of P or non P and in the natural language, we negate verbs by using only ‘la’ before the verb (la+verb), ‘la’ is also the opposite of yes in Arabic. I also negate by using ‘laysa’ which comes before a noun while ‘la’ comes before a verb. But in dialect, it is a little bit different: when we negate, we use a prefix ‘ma’ attached to the verb followed by a suffix ‘esh’ (ma+verb+esh: ma+eat+esh means ‘I do not eat’), while the word for ‘no’ is ‘lala’ or ‘la’ (according to the different regions in Algeria).

Moreover, when we begin a proof by contradiction, we say ‘if not’ (about the negation of the hypothesis), in French there is one word for ‘if not’ (‘sinon’), used also in natural language and very used in mathematical language (for instance when a function is equal to 1 for x=0, and 0 if not). In Arabic, there is one such word, which is ‘lawla’, but used at the beginning of the sentence and not at the end. For instance, if someone says in English: if I find it I will buy it, if not I will find someone who could lend it to me; in Arabic, the word ‘lawla’ cannot be used here as ‘if not’, the sentence would be expressed as follows: ‘idha (or any other similar conditional term) +verb (find) it, I will buy it, idha (or in case) I do not find it I will find someone to lend it to me.

Lawla is used as follows: verb1+lawla+verb2 means verb2 impeded the action of verb1, or because verb2, verb1 did not happen, for example: ‘I missed the flight ‘lawla’ it had been delayed’, means ‘because the flight had been delayed, I could take it’, or expressed fully in English as: if the flight had not been delayed, I would have missed it. We note here that this word (lawla) is used in very developed literature and is not used in current written Arabic language, thus this formulation might not be known by many students. There is another word used more often, in the oral language, which is ‘illa’ a contraction of ‘inla’ (if not) used with wa ‘and’. Example: ‘call me, if not I will leave’ would be translated as ‘call me+wa inlla+ I will leave’ which is ‘call me and if not I will leave’. This situation that is more similar to the French one and the English one, is not used in mathematics. In dialect, the similar word as ‘lawla’ does not exist; we rather use the word ‘wa illa’ contracted to the word ‘wella’. If we want to express a similar situation as (call me otherwise (or if not) I leave), we would say: call me ‘wella’ I will leave. We note here that wella is the same word as ‘or’, which is an exclusive choice.

For the logical connector ‘or’, in French we use only one word (‘ou’). In Arabic, we have ‘awe’ to make a choice. The choices might be exclusive or inclusive, like in French. But in dialect, we use ‘wella’ which is an exclusion of the precedent choice, which means that in dialect the choice is always expressed in an exclusive way. For example, if we want to say, ‘take the black or the blue’, in Arabic would be ‘take the white awe the blue’, in dialect would be ‘take the black wella the blue’ which is take the black and in case you did not take the black take the blue. The inclusive case is not expressed in dialect.

For the articles of generality and particularity ‘the’ and ‘a’, in French they are (‘le’ or ‘la’) and (‘un’ or ‘une’) (according to the gender); in Arabic the definite article is a prefix ‘al’ attached to the substantive, and without it, the word is considered indefinite, but also, if it is definite by belonging to anyone or anything, it is expressed without it (like saying: the car of my father, in Arabic we do not use ‘al’ because it is considered definite as it belongs to my father). In French, we can express the generality and the particularity with both ‘le’ and ‘un’; we might say in French:

- ‘la voiture est un moyen de transport’ (the car is a mean of transportation) which expresses generality. And ‘la voiture de mon voisin est rouge’ (the car of my neighbour is red) which expresses particularity. When using ‘un’ (or une), we might say in French:

- ‘un élève doit respecter son professeur’ (a student must respect his professor) which expresses the generality. And ‘un élève de John a réussi’ (a student of John has succeeded) which expresses the particularity.

In Arabic, with the first case, we would say ‘al+car...’ for the generality and ‘car of my...’ without any article, for the particularity. With the second case, we would say in Arabic ‘al+student....’ for the generality and ‘student of John ...’, without any article, for the particularity. It seems easier and simpler for Arabic, we use al+substantive to express the generality and use nothing (no ‘al’) to express the particularity. In dialect, it is the same as with the Arabic. The problem might happen with dialect when the pronunciation is contracted and the ‘al’ cannot be heard, so apparently no difference exists between the pronunciation of the word with and without ‘al’.

With respect to universal and existential quantifiers, students encounter, for the first time, in the university the symbols $\forall$ and $\exists$. They already write the full expressions (for all… and there exists ...) in the upper secondary school, but do not use the symbols. In this case there are equivalent expressions in Arabic and in dialect.

About mathematical terms: when students enter the university, they already master the mathematical terms used in the past three years, like: limit, function, continuous, derivative, set, line, equation, inequality, etc. When students start to learn mathematics in French at the university, they try to translate French into Arabic (Azrou, 2019) and would feel secure when they are given a mathematical term in French, for which they already know its translation into Arabic. The problem happens when they begin to learn new mathematical terms related to concepts and definitions of undergraduate mathematics not already met in high school (example: group, intersection, injection, morphism, restriction, inferior bound, etc). In fact, this is the case with students all over the world, but the difference is that with any other student (Italian, English, French, …), a new mathematical term is accepted as such. With our students, who base their learning on translation from French into Arabic, when they encounter a new mathematical term in French whose translation in Arabic does not correspond to a known term (or not known by the teacher), they panic, feel stuck and continue to look for a similar word in Arabic in order to be able to understand the meaning.

The mathematical terms formed by adding prefixes and suffixes inform about their meaning, for example: ‘iso’ is a suffix to express the similarity (‘isomorphism’), ‘dis’ (discontinuity) to express the opposite, ‘sub’ (subset, subgroup) to express a part. These processes (consisting in adding prefixes and suffixes that inform about the meanings of the words) are not usual in Arabic. As a consequence of the lack of familiarity with the prefix – suffix processes, when students read terms like isomorphism or mono-morphism, they might tend to memorize them without engaging in finding the meaning suggested by the prefix (or the suffix), and would often confuse them; which is another difficulty for them.

An outline of further research

The analytic comparison of the three languages (classical Arabic, dialect and French) with respect to some of their structural-logical aspects has put into evidence differences that might affect the learning of mathematics and mathematical activities. We have seen how the mastery of the three languages looks very different, related to how students meet and use the three languages before their entering the university: Arabic as the teaching and learning language for all disciplines in all the pre-university levels, French as a “second language” learnt as such since degree three, dialect as the commonly used language in everyday life. These considerations result in three research questions that should be dealt with in further research to be performed with first year university students in scientific areas:

1. How is students’ mastery of written French as an ordinary communication language?

This question is motivated by the fact that the teaching of French as a second language might not result in a sufficient mastery of that language for some students to interpret ordinary texts (thus preventing them from the access to verbal explanations provided by scientific textbooks, which are written in French), and it is also motivated by the requests of many students during the exercise sessions to explain what is written in French.

2. How is students’ mastery of Arabic at the academic level (Cummins, 2005), in particular in mathematics, by students who enter the university in scientific disciplines?

This question is directly related to the role that should be played by Arabic language in the teaching and learning of all the disciplines at all pre-university school levels, and to Cummins’ theory. Cummins distinguishes between different levels of mastery of a language: the Basic Interpersonal Communication Skills (BICS), which are the surface skills of listening and speaking and Cognitive Academic Language Proficiency (CALP) which are the basis for a child’s ability to cope with the academic demands in various subjects. Cummins states that while many children develop native speaker fluency (BICS) within two years of immersion in the target language, it takes between 5-7 years for a child to reach an academic level with a second language.

Moreover, according to Cummins’ interdependence hypothesis (2005), the more developed or proficient a language 1 is, the easier is to develop a language 2. As Cummins (2000) also states: ‘Conceptual knowledge developed in one language helps to make input in the other language comprehensible’, in other words if a student understands a mathematical concept (like continuity or function) in Arabic, all she (or he) has to do is acquire the new corresponding French terms. However, a student has a more difficult task if he (or she) has to acquire both the term and the concept in the second language.

3. How is students’ awareness of the semantic and syntactic aspects of logical structures of the three languages, and the related differences?

This question is motivated by the fact that (according to previous research) most students move to/from French – the official language of textbooks and of exam tests in mathematics – from/to the ordinary languages of communication and of thinking. In these translations/conversions students need, in particular, to deal with the differences concerning those logical structures of the three languages, which are relevant for mathematics.

In order to answer the three questions a suitable experimental investigation should be planned.

In the three cases, a crucial role should be played by written texts for the same and enough large population (e.g. all the students of the first year in Mathematics – about 150 each year) and subsequent interviews of a sample of them.

The first question might be answered through the choice of a plain text in French and related items aimed at ascertaining the students’ understanding of it; interviews might contribute not only to identify the reasons for possible misunderstandings or blockages, but also the mastery of ordinary French at the oral level.

The second question might be answered through the choice of short mathematical texts in Arabic, e.g. a simple definition, a simple statement of a property (or a theorem), a simple proof for a given statement – and related items to ascertain understanding. Also non-mathematical Arabic texts with logical structures should be used, in order to distinguish between difficulties depending on the level of mastery of Arabic needed to interpret them, and difficulties depending on the knowledge of mathematical notions.

The third question might be answered by asking students to translate a text containing some logical content (not necessarily in mathematics) from Arabic to French, and the same from French to Arabic, with items focused on making explicit the conversion from one structure in the first language to another in the second language. The interviews might provide information on the use of dialect as an oral communication means for logical content.

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The flows and scales of language when doing explanations in (second language) mathematics classrooms

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I develop the idea of sources of meaning by proposing the notions of scale and flow as a way to think about participation in mathematics classroom discourse. Flow emphasizes the dynamic nature of language use, while scale makes apparent its stratified and stratifying nature. Drawing on an ethnographic study of four second language mathematics classrooms in Canada, I examine the flow and scales arising in interaction in one 'language positive' classroom: a group of recently arrived immigrants to the province of Quebec. The analysis focuses on examples of the construction of mathematical explanations to illustrate the role of scale-jumping in the construction and stratification of mathematical meaning.

Keywords: Language diversity, second language learners, scale-jumping, social stratification, mathematics classrooms.

Introduction

The political role of language in mathematics classrooms has long been recognized in research in mathematics education. In particular, research has demonstrated the role of language in the reproduction of social structure. In the UK, Cooper and Dunne (2000), for example, showed how differences in children's interpretations of word problems were related to differential rates of success in solving such problems (where 'solving' refers to the production of responses that fit the conventional expectations of school mathematics). Cooper and Dunne showed that such differences were related to students' socio-economic backgrounds, such that students from working-class backgrounds were more likely to produce more 'realistic' responses, which were therefore often considered incorrect. Their work shows how language is not a neutral feature of mathematics classrooms, but contributes to the marginalization of some students (see also Zevenbergen, 2000). As a result, Morgan (2012) has argued for greater attention to social structure in the analysis of mathematics classroom discourse.

A political dimension has also arisen in research on language diversity in mathematics classrooms. This work has particularly focused on the perceived value of different languages as a way to explain how languages are positioned in mathematics classrooms in different contexts. Setati (2008), for example, showed how English was more valued as a language of instruction by parents and students in South Africa, since it was perceived as giving access to better jobs and education, despite making the learning of mathematics more challenging when students spoke other languages at home. In later work, Setati (now Phakeng) and others have argued for languages to be thought of as resources for learning, and that the use of language as a resource is mediated by the broader political landscape in which mathematics classrooms are embedded (Planas & Civil, 2013; Planas & Setati-Phakeng, 2014).

The above work points to the significance of the relationship between language, mathematics learning and marginalization. That is, we know that part of the explanation for differences in students' performance in school mathematics is their socio-economic background, including their racial or linguistic background, although the relationship is not simple (see, for example, Secada, 1991, but also Clarkson, 2007). This relationship seems, in part, to be mediated by language. The work summarized in the first paragraph draws on sociology to understand this relationship, using concepts like linguistic capital, arguing that mathematics classroom interaction is shaped by much broader social structures. The work referred to in the second paragraph draws on discursive perspectives, using the concept of language as a resource which students and teachers can use to access desired parts of the social structure. But how does the relationship between language use and social structure play out in the moment-by-moment exchanges of a mathematics classroom? In this paper, I draw on the notion of scale to explore this question, with a particular focus on students' explanations.

Theoretical perspective: flows and scales

To think about the role of language in mathematics classrooms, I use the idea of sources of meaning, in which language is seen as a kind of flow. In participating in interaction, speakers navigate this flow, making use of parts of the current to make meaning with each other. In so doing, they also alter the flow, so that downstream, language is slightly different from what it was before. The flow of language has several general features (Barwell, 2018):

- *Meaning-making is relational* – there are no absolutes in language; meaning arises from the relations between multiple features of language, from phonemes to discourses to languages; these relations are always situated and depend on where speakers are in the flow;
- *Language is agentive* – in navigating the flow of language, speakers are pushed and pulled by language, so that multiple, often unintended meanings are possible; this dimension of language contrasts with ideas like languaging, which emphasize the agency of speakers;
- *Language is diverse* – the flow contains many currents, including strong 'mainstreams', as well as eddies and backwaters, all of which are constantly changing; all parts of the flow interact and shift, with new currents forming and existing currents dispersing over time.

Thinking about language in terms of flow highlights the dynamic, agentive nature of language, into which humans insert themselves as they interact. One of the drawbacks of this image, however, is that, as shown in the literature, the role of language in society and in education is not neutral. Blommaert (2007) argues that metaphors like flow therefore fall short, and proposes the metaphor of scale. Integrating this idea into the sources of meaning framework gives a fourth feature of language:

- *Language is stratified and stratifying* – language is "a system full of inequalities, in which people and actions develop on or across different scale-levels, and in which moves across such scale-levels are moves within a power regime" (Blommaert, 2007, p. 15). Working with some streams of language connects across wide scales, while others are locally located; forms of language are indicative of higher or lower status (institutional, political, educational, etc.) within a given milieu; higher status generally corresponds to wider scales.

Overall, therefore, the sources of meaning framework emphasizes the dynamic, contingent nature of language in use, as well as the way language is ordered across different scales. The notions of flow and scale link the individual moments of language to broader social structure.

Background to the study

The data for this paper comes from an ethnography of four second-language mathematics classrooms in Canada, a country with two official languages, English and French. Data including observations, audio-recordings, interviews and students' work were collected in four different classrooms featuring second-language learners: a mainstream anglophone class in which some students were learners of English as a second language (class A1); an anglophone class specifically for learners of English as a second language, all of whom were of Indigenous Cree background for most of the year (class A2); a francophone 'welcome class' for new immigrants who were all learners of French as a second language (class B); and a French-immersion class in which most of the curriculum was taught in French to students who did not speak French at home, so that they would become proficient in that language (class C). Classes A1, A2 and B were of grades 5-6, while class C was of grade 3.

In recent analysis of the full dataset, I developed a distinction grouping these classes into two: I characterized classes A1 and A2 as *language neutral mathematics classrooms* and classes B and C as *language positive mathematics classrooms*. This distinction was on the basis of a detailed analysis of how students in each class appear to be socialized into the discourse of mathematics as well as the medium of instruction. Language diversity was present in both types of classroom. In language positive mathematics classrooms, this diversity was explicitly recognized and incorporated into discussion of mathematics in different ways. In language neutral mathematics classrooms, language diversity was more implicit and tended to be skirted around in classroom interaction. (For a detailed account of this work, see Barwell, 2020). This analysis, however, did not explicitly examine the stratified and stratifying nature of language in mathematics classrooms. In this paper, therefore, I present a preliminary analysis of language flow and language scales in a language positive classroom (class B), and offer a brief comparison with these aspects of interaction in the language neutral classrooms, with a specific focus on students' mathematical explanations.

The analysis is organized around the idea of scale-jumping, which designates moments in which participants indicate a scalar shift in language use or their interpretation of an utterance (Blommaert, 2007). Scale-jumping can involve participants marking shifts "from the individual to the collective, the temporally situated to the transtemporal, the unique to the common, […] the specific to the general" (p. 4). Here is an example of scale-jumping from class A1 (language neutral, observed 10 December 2008):

Teacher L:	number two Darryl, where does it [the decimal point] go?
Darryl:	(indicates where)
Teacher L:	how do you know Darryl?
Darryl:	I just know
Teacher L:	you know what if you write that on your exam, what do you get?

Whereas Darryl's response may be locally acceptable (e.g., to himself, to his classmates), the teacher introduces a wider scale, institutional reading of his response. Exams are widely used (in this case, the teacher is referring to provincial exams) and impose wide-scale mathematical discourses, including with respect to explanations. The teacher's framing is therefore bound up with the measurement and performance of students, in which 'success' involves particular kinds of discourses.

For this paper, I selected an example of students in class B (language positive) participating in mathematical explanations, which had been recognized in previous work as significant in terms of students' participation in mathematics (Barwell, 2020). I examined the transcript for examples of scale-jumping, which were then unpacked in terms of the other aspects of the sources of meaning framework. In the last part of the paper, I briefly compare the results of this analysis with those from a previously published analysis of interaction in an episode from class A2 (Barwell, 2016).

Scale-jumping in mathematical explanations in class B

This notion of scale is highly applicable to (second language) mathematics classrooms, since the goal is to introduce widely scaled mathematical discourses, as well as an educated variety of the medium of instruction. The following extract comes from a series of lessons on geometry, in which the students, all newcomers to Canada and learners of French, were introduced to various geometric forms and their properties. The teacher used a variety of activities and styles, including group work, student-generated ideas, worksheets with various tasks and whole class interaction. In this extract, the teacher is reviewing some different geometric forms and encouraging the students to think about similarities and differences and find ways to articulate them.[1]

Teacher N:	ok E5 est-ce que tu peux m'expliquer la **diff**érence entre un ovale et un cercle ? {*ok E5 can you explain to me the* ***diff****erence between an oval and a circle?*}
E5:	uhm ok
Teacher N:	explique moi la [difference {*explain the difference*}
E5:	[ah le cercle il fait la troue comme ça et le ovale il est comme ça (.) pas de lignes mais il est comme drette ici (.) pas tout ça comme ça {*ah the circle it does the hole like that and the oval is like that (.) no lines but it's like straight here (.) not all like that*}
Teacher N:	o:k: et si je fais ça comme ça ? qu'est que c'est ? {*o:k and if I do this like that? what's that?*}
E5:	c'est un ovale {*it's an oval*}
Teacher N:	pourquoi ? {*why*}
E5:	parce que il est comme ça ici (.) un cercle est tout comme ça ici {*because it's like that here (.) a circle is all like that here*}
Teacher N:	ok est-ce que quelqu'un est capable de t'as la bonne réponse E5 ? quelqu'un qui serait capable de l'expliquer plus précisément ? {*ok can someone you have the right answer E5 someone wuo could explain more precisely?*}
E6:	c'est comme que c'est un cercle mais c'est comme tu fais un cercle comme ça et fait comme ça {*it's like that it's a circle but it's like you do a circle like that and do like that*}
Teacher N:	c'est comme si tu pousses sur le cercle ? {*it's as if you push on the circle?*}
E7:	**ouais (.)** {***yes***}

[1] bold shows emphasis, timed pauses are in ( ), = shows latching, ? for rising intonation, : shows vowel extension, overlaps are shown with aligned [, my English translation given in { }.

Teacher N:	un cercle=hein quand tu regardes le milieu (.) le milieu (.) si on regard à n'importe [quel côté ça va être la **même** longueur bon alors moi mon cercle il n'est pas parfait (.) mais un ovale (.) il n'a pas la même longueur (.) c'est comme si on avait pesé sur le cercle {*a circle=eh when you look at the middle (.) the middle (.) if you look at any side it'll be the **same** length (.) it's like someone had pushed on the circle*}
E8:	[et là il va partout {*and there it goes everywhere*}

This extract concerns the difference between an oval and a circle and most of the exchange is devoted to constructing an acceptable explanation of this difference. The students' contributions are notable for their use of deixis: locally specific forms of reference, such as 'this' or 'there', as well as physical gestures. In the first half of the exchange, both students and teacher make repeated use of several deictic forms, including 'like that', 'here', and 'this'. These formulations are not somehow limited, imprecise, or deficient forms of mathematical discourse; it seems clear that the participants are able to follow each other's utterances. More generally, we see several formulations and reformulations that appear to concern the same general idea. There is, however, one example of scale-jumping: in the teacher's 5th turn, she acknowledges that E5's formulations have responded to the question, but then asks if anyone can explain 'more precisely'. E5's initial formulations are acknowledged as answering the question, but a revision is necessary. The problem with E5's formulations, framed in terms of precision, is that they are only locally meaningful. The heavy use of deixis all but guarantees that E5's explanation is locally constrained. His interaction with the teacher is in a part of the flow of language that involves French and some mathematics, as well as deixis, gestures and words like 'hole' that are not typically considered part of mathematical discourse. The teacher's request for precision, then, is really a request for a formulation with a wider scale of utility: she is calling for a shift to a more standard form of mathematical discourse as might be recognizable to mathematically knowledgable speakers elsewhere. E6 attempts to respond, largely drawing again on deixis and gesture. This response does not appear to meet the explicit request for precision, or the implicit request for a wider scale discourse, since the teacher provides her own more elaborated explanation.

The teacher's version does not simply involve a more sophisticated vocabulary or syntax – she uses 'middle', 'side' and 'length', and the construction 'as if' among other things. It also introduces a mathematical principle of a wider scale discourse: the idea of constant radius. Wider scale does not necessarily mean more frequently used; rather, it means used across a wider range of space and time. The idea of constant radius would be recognized by mathematically educated people in many places, in a way in which the exact words of the students can only easily be interpreted in the immediate location of their utterance. Interestingly, the teacher links this principle back to the students' versions by talking about 'pushing' again. The scale-jump was in this instance only temporarily effected, since the discussion returns to a more local scale immediately afterwards. The teacher's use of scale-jumping nevertheless serves a valuable pedagogical purpose: it introduces a distinction (labelled with the word 'precise') between local, informal, proximal mathematical discourses and wider scale, formal, distal mathematical discourses. The difference between them is neither absolute, nor inherent in the language; it emerges from the relation between the forms now marked as different.

A similar pattern can be observed in the next extract, which immediately follows the previous one:

Teacher N:	ok **E13** explique moi la différence entre un **rec**tangle et un **car**ré {*ok E13 explain to me the difference between a **rec**tangle and a **squa**re*}
E13:	il n'est pas il est comme (.) umm il a comme il a hhmm [(.) non ? il est carré c'est u peu haut petit petit {*it isn't it's like (.) umm it's got like it has hmmm (.) no? it's square it's a bit high little little*}
E14:	[égal {*equal*}
Teacher N:	donc le rectangle il est **long** (.) ok (.) alors si je fais ça (2.5) ça est-ce que c'est long ? (.) {*so the rectangle it's **long** (.) ok (.) so if I do that (2.5) is that long? (.)*}
E13:	le gros carré normal petite (.) c'est long (.) mais carré petit comme tu le fais c'est petit comme ça mais petit mais c'est la même le même {*the big square normal small (.) it's long (.) but square small like you do it it's small like that but small but it's the same the same*}
Teacher N:	t'es proche t'es proche {*you're close you're close*}
E13:	je pense aussi ça fait un ligne c'est pas la même mais là le carré c'est le même longueur {*I think also it does a line it's not the same but there the square it's the same length*}
Teacher N:	c'est ça si je prends un règle (.) le carré les **quatre** côtés vont mesurer la **même chose** (.) ça va être pareil (.) le rectangle il y en a deux qui peuvent être un peu plus petits que les deux autres qui sont plus longs tu comprends la différence ? {*that's right if I take a ruler (.) the square the **four** sides will measure the **same** (.) it'll be equal (.) the rectangle there are two that can be a bit smaller than the two others that are longer do you understand the difference?*}
Several:	**oui:** {***yes***}

Once again, the students contribute highly localized explanations for the difference between a square and a rectangle. Their utterances are also grammatically, syntactically and phonologically unconventional. The teacher nevertheless participates in mathematical meaning making with these utterances, drawing out particular details such as 'long'. As in the previous extract, she scale-jumps, in this case by providing an elaborated formulation of key differences between the two geometric forms. The wider scale of this formulation is not just due to the use of more formal mathematical vocabulary, but also in the reasoning that is shared i.e. measuring and comparing lengths of sides. As with the previous extract, the teacher introduces a wider scale mathematical discourse.

Discussion

These two extracts give a sense of how the notions of flow and scale allow for a dynamic understanding of meaning making in (second language) mathematics classrooms, as well as a sense of how these processes are stratified and stratifying. In both the above extracts, students participate in the flow of language. They bring larger repertoires than that which is audible in the classroom; they all speak one or more other languages at home. Within the institutional context of the school, French is privileged. The students are buoyed by this flow in which, nevertheless, multiple sources of meaning are available, including their non-standard French, deixis, gestures and the texts and diagrams found in worksheets and on the blackboard. I do not see the students as taking up resources to produce particular meanings, so much as hitching a ride on particular flows, forming them as best they can for their purposes, but having to accept what is already there in the stream.

There are, however, different scales of discourse available in this current. Deixis and gesture, and idiosyncratic pronunciation are local and proximal, while the teacher (in this case) offers formulations that draw on wider scale sources of meaning, both in the words and construction of her utterances, as well as in the forms of mathematical reasoning (as realized in her words) that she exemplifies. These different scales produce a clear stratification, with the teacher's 'precise' formulations drawing on higher status discourses that are connected to educational success, among other things. Students' utterances are positioned as of lower status; they are 'close' but not 'precise'. The broader institutional and political structures in which the class is embedded produce this ordering of discourses, from the official language policy, to the requirement for these children to attend a francophone school, to the mathematics curriculum and provincial assessment systems. To be successful in the school system, students must learn to produce the desired wide-scale mathematical discourses. As immigrants, however, they do not necessarily encounter the French language at home and may bring other discourses of educational success which do not map onto the mathematical and educational discourses of their new home. Students therefore begin in a relatively marginalized position.

In this language positive classroom, the teacher mediates students' engagement with mathematical and educational discourses. Much of the time, she focuses on students' mathematical thinking and meaning-making (rather than their pronunciation, for example), and makes connections between the different discourses. For example, she takes up students' suggestions of ovals being squashed circles, talking about 'pushing' circles into oval shape. She introduces wider-scale discourses but sets them in explicit dialogue with the students' local formulations. This kind of connecting is in contrast to the language neutral classrooms, where students encountered wider scale discourses as rather alien. For example, in earlier work, I showed how two Indigenous students worked on a word problem that referred to a situation and used a way of seeing the world that was alien to the students' experiences and worldview (Barwell, 2016). In this classroom, the students were expected to acquire a new discourse with less opportunity to connect it with their own. Moreover, these mathematical discourses were embedded in institutional and political structures that were linked to a history of oppression.

Conclusions

The political role of language in (second language) mathematics classrooms has been acknowledged for some time. In research on language diversity in mathematics classrooms, the focus has mostly been on students' choice of language. In the second language classrooms in my study, this choice was already made for students. Other aspects of language are, however, implicated in the stratification of mathematics classroom interaction. This stratification is related to broader social structures, but not in a simple way. The idea of language as flow emphasizes how language is dynamic and how students must navigate these flows to make mathematical meaning. The notion of scale emphasizes how classroom discourses are ordered, such that some forms of expression are considered less valuable, precisely because they are not widely recognized. From this perspective, succeeding in mathematics means learning to navigate the flow of language, to shape it to one's intentions, and to be able to shift levels within the prevailing order. My analysis of examples from a language positive classroom suggests that teachers can support the learning process by connecting discourses of different scales so that students can learn to make these shifts in their own mathematical meaning-making.

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The logical analysis of statements. A tool for dealing with ambiguities in multilingual context

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There is research-based evidence that logical analysis of language is a relevant tool for revealing ambiguities in a given language. We discuss in this paper in which respect logical analysis of mathematical statements might provide a common reference allowing identifying and dealing with unavoidable ambiguities that might occur in multilingual educational contexts.

Keywords: Didactic of mathematics, logical analysis of language, multilingual context.

Introduction

A main problem in multilingual educational context concerns the possible misunderstandings that can emerge as a consequence of differing grammatical structures between the language of instruction and the vernacular languages of students, as claimed by Edmonds-Wharten, Trinick, and Durand-Guerrier (2016):

> The impacts of features of grammatical structures on mathematical thinking are still underresearched. We have shown that languages express mathematical ideas in diverse ways. These different ways of exploring mathematical ideas provide an opportunity to enrich the mathematical experience of learners in multilingual contexts. They can also introduce ambiguities or misunderstandings between teachers and students and impede the process of mathematical learning." (p. 41)

In a first section, I will briefly provide arguments that logical analysis of mathematical statements in predicate calculus is a relevant tool for mathematics education (Barrier, Durand-Guerrier, & Mesnil, 2018; Durand-Guerrier, 2008; Selden & Selden, 1995), in particular for what concerns proof and proving (Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012; Epp, 2003). In a second section, I will focus on negation, showing first some features of negation in French likely to introduce ambiguities or misunderstandings in class (Durand-Guerrier, 2016), and giving then the main results of two studies, one in Tunisia, the other one in Cameroon, supporting the hypothesis of an impact of differing grammatical structures. In the third section, I will illustrate on the expression "deux à deux" (pairwise) that logical analysis provides a tool for overcoming possible grammatical misunderstandings in mathematics education.

Predicate Calculus - A logical reference for mathematics education

In mathematics classes, it is often assumed that every statement is either true of false. In a logical perspective, this refers to propositional calculus, in which the basic unit is the propositional variables, that are interpreted either by singular statement such as "119 is a prime number" (false statement) or "π is an irrational number" (true statement), or by close statements such as "All prime numbers except 2 are odd)" (true statement) or "There is a rational number whose square is 2" (false statement). However, we have evidenced in our research that this logical reference is not sufficient for the needs of mathematics education. Indeed, object properties and relationships are fundamental categories involved in mathematics activities, and issues of quantification play an essential role (Dawkins & Roh, 2019; Durand-Guerrier & Arsac, 2005; Durand-Guerrier, 2008). Classical first order logic (predicate calculus) allows analysing mathematical statement precisely, e.g. those proposed above that we recall here:

1. 119 is a prime number
2. π is an irrational number
3. All prime numbers except 2 are odd
4. There is a rational number whose squared is 2

In statements 1 and 3, the property ‘to be a prime number’ is involved. It is modelled by a one place predicate $P(x)$, where P is a letter for predicate, and x a letter for a free variable (a place holder). In the considered interpretation, this property applies to natural numbers. Assigning a natural number to the free variable provides a singular statement that is either true of false, depending on the assigned object such as in statement 1. In sentence 3, there is no assignation; a universal quantifier bounds the free variable; the quantification domain is mentioned (the set of prime natural numbers deprived of 2). Such sentences are closed sentences and are either true of false. For sentence 2 and 4, the analysis is rather similar, considering an existential quantifier.

In addition, in first order logic, relationships are modelled by two (or more) places predicate. For examples, the relation ‘is divisible by’ on the set of integers is modelled by a two places predicate $P(x, y)$; while the relationship ‘is the GCD of … and of …’ is modelled by a three places predicate. With a two places predicate, there are four possibilities to provide a close statement (a proposition) – 1. for all x, for all y $P(x, y)$ (AA statements); 2. For all x, there exists y $P(x, y)$ (AE statements); 3. There exists x such that for all x $P(x, y)$ (EA statements); 4. There exists x, there exists y such that $P(x, y)$ (EE statements). As evidenced by Dubinsky and Yiparaki (2000), in natural language, the interpretation of AE and EA statements are interpreted according to the context. For example, the statement “There exists a mother for each child” (apparently EA) will be interpreted as “For each child there exists a mother” (AE), due to the context. In mathematics, such flexibility in interpretation is not possible. For example, it is necessary to clearly distinguish between “There exists a number greater than all other numbers” (EA), which is false in the set of natural numbers, and “For every number, there exists a number that is greater” (AE) which is true in this set. Chellougui (2009) showed that Tunisian first year university’s students consider that EA and AE statements have the same interpretation, even in a mathematical context. The number of possibilities for providing a close statement with quantifiers is increasing with the number of places in the predicate: 2^3 for a three places predicate, 2^4 for a four place predicate and so on.

The complexity of the logical structure of quantified mathematical statements is still increasing as soon as logical connectors in particular negation and implication, are involved. In such cases, for a correct interpretation, it is necessary to identify the respective scopes of connectors and quantifiers.

Let us consider the following example (from Njomgang Ngansop & Durand-Guerrier, 2011):

For any function f from the set $\mathbb{R}$ of real number into itself, for any a in $\mathbb{R}$, if for any sequence u with values in $\mathbb{R}$ converging to a, $f \circ u$ converge to $f(a)$, then f is continuous at a. (1)

It is possible to formalize the statement as below, showing the complexity of its logical structure.

$$\forall f\ \forall a\ [(\forall u\ (F(u,a) \Rightarrow G(f,u,a))) \Rightarrow H(f,a)]\ (2)$$

where $F(u,a)$ formalizes the relation 'u converges to a', $G(f,u,a)$ formalizes 'the composition of f with u converges to $f(a)$', and H formalizes 'f is continuous in a.' The domain of quantification are: the set of functions from $\mathbb{R}$ to $\mathbb{R}$ for the variable f, the set $\mathbb{R}$ of real number for the variable a, the set of sequences with values in $\mathbb{R}$ for the variable u. It is to notice that there are three universal quantifiers, among two are in heading positions, the scope being indicated by the brackets, while the third one is in the antecedent of the external implication; the scope of this third quantifier being indicated by the parenthesis.

Being able to unpack the logic of mathematical statements is a core competence for proof and proving in mathematics education (Selden & Selden, 1995), in particular the logical structure of a statement is likely to orient the way to engage a proof. Coming back to example (1) above, as the antecedent of the external implication is a universal conditional statement, it is difficult to think of a direct proof consisting in considering a function and a real satisfying this antecedent. Opposite, the consequent of this external implication is an atomic formula. For this reason, the classical proof of this statement is a proof by contraposition. The proof by contraposition relies on the following logical theorem in first order logic (a statement true for all interpretation of its letters in any non empty universe, Quine, 1950)

$$\forall x\ \forall y\ (P(x,y) \Rightarrow Q(x,y)) \Leftrightarrow \forall x\ \forall y\ (\neg Q(x,y) \Rightarrow \neg P(x,y))\ (3)$$

A consequence of this logical theorem is that a quantified conditional statement has the same truth value as its contrapositive whatever the interpretation and the domain of quantification, so that a proof of the contrapositive counts as a proof of the statement.

The logical form of the contrapositive of the initial statement is

$$\forall f\ \forall a\ [\neg H(f,a) \Rightarrow \neg(\forall u\ (F(u,a) \Rightarrow G(f,u,a)))]\ (4)$$

Using the rules for negating a conditional statement in the formal system, we get

$$\forall f\ \forall a\ [(\neg H(f,a)) \Rightarrow (\exists u\ (F(u,a) \wedge \neg G(f,u,a)))]\ (5)$$

and finally, the contrapositive in the vernacular language:

For any function f from the set $\mathbb{R}$ of real number into itself, for any a in $\mathbb{R}$, if f is not continuous at a, then there exist a sequence u such that u converges to a and $f \circ u$ does not converge to a.

In this section we have provided examples supporting the claim that classical first order logic (predicate calculus) provides relevant tools to analyse mathematical statements, which is a core competence in proof and proving. In the last example, we have shown a methodology consisting in unpacking the logical structure of the statement in order to formalize it in the language of predicate calculus, then providing the contrapositive of this formalized statement and then give the contrapositive in the vernacular language. This last step is necessary because in order to engage in a mathematical proof, it is necessary to consider objects, properties and relations involved. Considering this, it is reasonable to conjecture that such work would be difficult for students learning mathematics in a non-native language. In the following section, I will present some features of negation in French and I will briefly report on two studies on negation in a bilingual context in the Francophone area.

Negation in French – impact in multilingual context

In French, the sentences in the form “For all *A*, *A* is not *B*” are ambiguous. In the linguistic norms, they must be interpreted as “some *A* are (is) *B*, and some are (is) not”, or in a more formal way “There exists some A that are (is) not B”. However, in everyday contexts it might be used with a different interpretation, namely, no *A* is *B*. For example, in a very cold winter in Lyon (France), the Public Transportation Company had widespread the following message: “Aujourd’hui, tous les bus ne circulent pas” (Today, all buses do not circulate). It seems that a number of people called to ask which buses circulated. In any case, three hours later, they changed their message to “Aujourd’hui, aucun bus ne circule” (Today, no bus is circulating). This non-standard interpretation is rather common in oral discourse, including radio and broadcast. In addition, it is noticeable that this linguistic norm does not respect the fundamental rule that “replacing a term by an equivalent expression should preserve the truth value of the statement”. Here “equivalent expression” means to be satisfied/not satisfied by exactly the same elements of a given domain. For example, in the domain of natural numbers ‘to be odd’ is equivalent to ‘not to be even.’

Let us consider now the following statement:

“Tous les diviseurs de 12 sont pairs” (All the divisors of 12 are even) - False

Its standard negation in French

“Tous les diviseurs de 12 ne sont pas pairs” (All the divisors of 12 are not even) - True

Then change “ne sont pas pairs” (are not even) in “sont impairs” (are odd)

“Tous les diviseurs de 12 sont impairs” (All divisors of 12 are odd) - False

In addition, the negation of the universal statement in French, according to the linguistic norm, is not congruent with the logical structure. Indeed, a word-for-word formalization would lead to the formalized statement “$\forall x \neg P(x)$” which is not the negation of “$\forall x \, P(x)$”

We might anticipate that such ambiguities inherent to the French grammar could be source of difficulties for non-francophone natives studying mathematics in French, and this especially as teachers are generally not aware of this. This has been confirmed by two studies, one In Tunisia, the other in Cameroon. I briefly summarize below the results.

In his PhD (Ben Kilani, 2005), Ben Kilani studied the differing grammatical structures between Arabic, French and predicate calculus. He showed that French and Arabic were not congruent for what concerns the negation of universal statements, while Arabic is congruent with predicate calculus. Indeed, in Arabic, when the negation is on the predicate, the scope of the negation is the predicate, not the sentence. The experimental results show that for most students, the French universal statements with negation on the predicate were not interpreted as the negation of the sentence, in coherence with the standard interpretation in Arabic and, as already said, in logic. He also showed that nobody took care of this: neither the language teachers (Arabic or French), nor the mathematics teachers, where in Tunisia, mathematics is first taught in Arabic from grade 1 to grade 9 (Ecole de base), and then in French at secondary school (Durand-Guerrier & Ben Kilani, 2004; Durand-Guerrier, Dias, & Ben Kilani, 2006).

Following the work by Ben Kilani, Njomgang Ngansop and Durand-Guerrier (2011) examined the impact of the grammatical structure of Ewondo in the teaching and learning of logical concepts. There are two educational systems in Cameroon, one is Anglophone, the other one is Francophone. The study concerns the latter, and among the great number of languages, the Ewondo. For what concerns negation, as for Arabic, the grammatical structure of Ewondo is differing from the French grammatical structure. The results of the exploratory study support the conjecture of an impact on the teaching and learning of mathematics.

Logical analysis as a tool for overcoming grammatical misunderstandings in mathematics education

The previous examples on negation highlight the impact of differing grammar on the interpretation of statements. It also emphasizes the fact that formalization in predicate calculus is a means to identify possible differing interpretations given by interlocutors. Summarizing briefly, we could claim that formalizing is choosing an interpretation.

Beyond negation, there are other mathematical expressions likely to introduce ambiguities in mathematical discourse. For example, in French, we often use in definitions involving a binary relation the expression “deux à deux” (in English: pairwise) as in the following examples.

> *Definition 1* (plane Euclidean geometry): A non-degenerated plan quadrilateral is a parallelogram *iff* its opposite sides are pairwise parallels.
>
> *Definition 2* (set theory): A partition of a given set E is a finite collection of non empty sub-sets of E that are pairwise disjoint and such that their union is exactly the set E.
>
> *Definition 3* (probability): Events are pairwise independent *iff* the occurrence of one event does not affect the probability of the other events.
>
> *Definition 4* (plane Euclidean geometry): Two triangles are similar *iff* the corresponding angles are pairwise congruent.
>
> *Definition 5* (space Euclidean geometry): A polyhedron is regular *iff* it is convex, its faces are regular polygons that are pairwise superimposable, and there is the same number of faces meeting at each vertex.
>
> *Theorem* (complex numbers): Given a polynomial with real coefficients, the non-real complex zeros of this polynomial, if any, are pairwise conjugate.

In a teacher training session, we had asked participants to determine all the regular polyhedral (definition 5), with the possibility of using materials to build such polyhedral, or of doing drawings or of making patterns. They were working in small groups. During the session, one of the groups of participants discussed the meaning of the expression “deux à deux” (pairwise). The discussion was: does it means *whenever you consider two faces, they are congruent (superimposable)* (interpretation 1) or does it mean *whenever you consider a face, you can find another one (different from the initial) that is congruent (superimposable)* (meaning 2)? The members of the group did not manage to reach an agreement, until the researcher came and gave the correct interpretation in this context (interpretation 1). It is noticeable that in this case, it was not possible to refer to the empirical possibility of the realization, because both interpretations allow it. At a first glance, and considering the strong willing of univocity in mathematical language, one could think that this was the standard interpretation in mathematics. However, the four other definitions and the theorem given above show that both interpretation are currently used in mathematics. Indeed, in definitions 2 and 3, pairwise refers to interpretation 1, while in definitions 1 and 4 and in the theorem, pairwise refers to interpretation 2.[2]

Once I presented and discussed this example in a seminar, a PhD student attending the seminar told me that, when he arrived in France for preparing a master degree, he encountered this expression in a topology course, and that it took him time before he understood that he misinterpreted it. His insufficient knowledge of the mathematical context did not permit him to choose immediately the right interpretation. It seems rather clear that such misinterpretation is likely to impact the understanding of the concept at stake, and consequently the learning. A path to overcome such ambiguities is to express the corresponding expressions in the formalized language of predicate calculus.

> Given a binary relation, it is modelled in predicate calculus by a two-places predicate. Let us name it $S(x, y)$. The two interpretations refer to the two following formalized expressions
>
> Interpretation 1 - $\forall x \; \forall y \; S(x, y)$
>
> Interpretation 2 - $\forall x \; \exists y \; (x \neq y \wedge S(x, y))$

Although the aim of mathematical language is to avoid ambiguities, such examples show that this is not always possible. Indeed, formalizing these statements has lead to considering two interpretations. In such cases, to formalize is to choose an interpretation.

It seems rather clear that in multilingual contexts, being able to deal explicitly with such ambiguities would open paths for remediation. As we have seen, in case of non-grammatical congruence, translating from one language to the other might change the interpretation, introducing misunderstandings likely to impede the learning process.

Conclusion and perspective

In this paper I have evidenced that logical analysis is likely to shed light on possible mis-understandings in mathematics education due to differing grammatical structures in multilingual educational context. There are two future lines of research. The first one consists in considering a greater variety of languages. The second one consists in testing empirically the following hypothesis: I hypothesize that, given a mathematical statement with possible problematic interpretation in multilingual contexts, asking students to first formalize the statement in predicate calculus, and then move to their own preferred language might help them to overcome misunderstandings resulting of

[2] For a wider presentation and discussion in English, see Durand-Guerrier (in press).

differing grammatical structures between the language of instruction and their preferred language. And, additionally, that this would contribute to a better understanding of the concept at stake.

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Multilingual preservice teachers evaluating mathematical argumentation: Realised and potential learning opportunities

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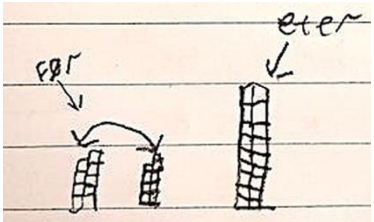
Preservice teachers' evaluations on school students' mathematical argumentation are rarely the focus of mathematics education research. Yet, understanding how they evaluate the quality of their future students' mathematical argumentation is important. In this paper, a discussion by a group of multilingual preservice teachers is analysed to determine the properties they considered to be connected to high-quality mathematical argumentation. The preservice teachers, who had two different dominant languages, discussed whether examples of Grade Four students' work, written in those two languages, displayed the qualities of being clearly written, mathematically correct and complete. From an analysis of this discussion, we identify how the preservice teachers' multilingual backgrounds provided potential and realised learning opportunities about students' mathematical argumentation.

Keywords: Multilingual preservice teachers, mathematical argumentation, multimodal representations

Introduction

Multilingual mathematics classrooms have been a research area for decades (see for example, Austin & Howson, 1979). However, it is only recently that mathematics teacher education for multilingual classrooms has been investigated (see Trinick, Meaney, & Fairhall, 2014). This lack of focus is also apparent in Norwegian teacher education, where preservice teachers [PTs] stated that they did not gain adequate knowledge on how to teach subjects, such as mathematics, in multilingual classrooms (Thomassen & Munthe, 2020). Nevertheless, the guidelines for the new teacher education programme clearly state, "teaching must be prepared from students' different needs, where different cultural, linguistic and social background are both taken into consideration, but also seen as a resource in the classroom" (UHR, 2016, p. 23, own translation). Viewing language as a resource has been promoted as a way of identifying a range of learning opportunities when different language backgrounds are included in mathematics lessons (Planas, 2014). In our project, *Learning About Teaching Argumentation for Critical Mathematics Education in multilingual classrooms* [LATACME], we aim to overcome both the lack of research and teacher knowledge by using PTs' discussions of school students' argumentation to gain understandings about using different language backgrounds as a resource.

In this paper, we explore how multilingual mathematics PTs, in a fifth-year Master level course, evaluated school students' argumentation about odd and even numbers. Our intention was to identify where their understandings, from different educational cultures, about mathematical argumentation provided both potential and realised learning opportunities about their future teaching. The course was part of an international programme and taught in English, with the PTs having either Norwegian or another European language (AEL) as their dominant languages. The examples, written either in Norwegian or AEL, were about adding two odd numbers together and came from Grade 4. We chose, these examples because we had incorporated them into a survey for our first-year PTs who we will survey again at the end of their second year. The survey responses suggested that we had a group of PTs who did not seem to make distinctions about the quality of children’s argumentation. In the survey, the PTs were asked the level of agreement (or disagreement) with statements about whether they could follow an explanation, whether the explanation was incomplete and whether the explanation was mathematically correct. Figure 1 shows one of the four examples in the survey.



(“før” means before and “etter” means after in Norwegian)

From Ure (2018)

Figure 1: A Year 4 student’s argumentation for why adding two odd numbers makes an even number

In relationship to the explanation in Figure 1, 63% of the first-year PTs completely agreed that they could follow and understand it. 13% completely disagreed and 29% somewhat disagreed that the explanation was incomplete and 32% completely agreed that it was mathematically correct. Although A. J. Stylianides and Stylianides (2009) might suggest that the reason for such responses was because PTs lacked mathematical knowledge about proofs and mathematical argumentation, it was important for LATACME to investigate PTs’ thinking about the students’ argumentation. As the original PTs are to be surveyed at the end of their compulsory courses, we had to explore this with another group. This Master course was chosen as it included a group of international PTs who were native speakers of AEL, which was used in one of the argumentation examples. Although the first-year PTs were unlikely to know this language, similar percentages considered the example to be understandable, not incomplete and mathematically correct. It was, therefore, necessary to investigate how PTs evaluated the argumentations and whether being in multilingual groups supported the development of evaluation criteria and if this was the case, in what ways. The multilingual nature of mathematics teacher education courses is often ignored, with the focus being on the language of instruction of the course (Chitera, 2011). It was, therefore, important to see how multiple languages could support the PTs’ possibilities for learning about school students’ argumentation.

Mathematical proof and argumentation

As Stylianides, Stylianides, and Weber (2017) noted, “proof is a mathematical argument” (p. 238), and in this paper, we use the terms interchangeably. G. J. Stylianides and Stylianides (2009) indicated that although proof in mathematics is a core activity, many students, including at university level, accepted empirical arguments as proof, suggesting this needs more focus in teacher education.

Balacheff (1988) distinguished between four types of proofs, based on school students’ reasoning. These were 1) naïve empiricism where the student asserts a result and verifies this with some examples. 2) The crucial experiment, where the student deals explicitly with the problem of generality and resolves it by using the outcome of a particular case. Stylianides (2009) described the crucial experiment as a strategic search for an example that corroborates the claim. 3) The generic example is when the student explicitly identifies some common characteristics, as representative of a class, and uses an example to demonstrate the properties of the class. However, unless the interpreters of the proof share an understanding of the importance of the characteristics, then the example can be interpreted as simply illustrating a particular case, that is naïve empiricism. 4) The thought experiment is when the student's proof is detached from a particular example of a class, such as when a general algebraic solution is provided. Any of these four types can be considered as proofs, when "they are recognised as such by their producers" (Balacheff, 1988, p. 218). However, there is also a need for them to be accepted by the community, not just the producer. Van Bendegem (1993) stated "*the proof* as a mathematically accepted proof, *exists only on a social level. Hence the basic unit to consider is not the individual mathematician but the mathematical community*" (p. 32, emphasis in the original).

The social acceptance is also related to expectations about appropriate representations. Stylianides (2009) illustrated how a proof of the sum of two odd numbers always equals an even number could be achieved through three different representations: by everyday language; by algebra; and by pictures. In the picture solution, there are drawings of pairs of squares, making a rectangle, representing even numbers or pairs of squares with one extra square that represented odd numbers. These kinds of representations are common when teaching odd and even numbers in Norway and can be considered cultural artefacts, providing meaning to those familiar with them.

A. J. Stylianides and Stylianides (2009) explored elementary PTs' constructions and evaluation of proofs. Some proofs included empirical evidence, in the style of Balacheff's (1988) naïve empiricism. However, when their PTs evaluated them, they considered them invalid (A. J. Stylianides & Stylianides, 2009). Over the semester, the PTs collectively identified criteria of a "good" proof. The criteria included that the proof had to: be correct; address the question or the posed problem; be focused, detailed and precise; and be clear, convincing and logical. The PTs elaborated on these criteria by stating that the language, representations, definitions had to be understood by the people to whom the proof was addressed, that the proof could be used to convince a sceptic and did not require the reader to make a leap of faith, key points had to be emphasized, if applicable supporting pictures or other representations had to be used appropriately, coherent, clear with complete sentences, and that the proof could be used by someone to solve similar problems.

In our investigation, we wanted to see if the fifth-year PTs identified the same or different criteria in relationship to the Grade 4 students' written argumentation and how they valued the different representations. We also wanted to see if differences in expectations about acceptable representations, from different cultural/language backgrounds, could provoke discussions about evaluation criteria.

Data collection and analysis

In this paper, we analyse a transcript of an audio-recording of a group of three PTs discussing the four student examples, in regard to which ones could be considered clear/understandable, complete and mathematically correct. The PTs were also asked to describe the future teaching, the students would need, based on their analysis. The whole class discussions, which followed both episodes of small group work, are also analysed. The PTs used English to communicate together but none of them were fluent users of English for discussing mathematics education. At the beginning of the workshop, the PTs were asked to construct a proof for why two odd numbers added together produced an even number and then produce a proof which could be used to convince Grade 4 students. Of the collected examples, all except one provided correct algebraic proofs, with $2n-1$ or $2n+1$ being used to define odd numbers. In the examples for Grade 4 students, all included pictures with no words, usually of groups of pairs of squares. In this paper, the Norwegian preservice teachers are PTN 1 and PTN 2, etc and the international student with AEL, PTI 1. Tamsin was the teacher educator, in charge of this class, and Toril attended as an observer who was known to the PTs from previous interactions.

We began our analysis by identifying discussion points about mathematical argumentation being clear, complete and mathematically correct. We then looked for what had prompted the clarification or elaboration of these points, to see if cultural differences, such as whether the appropriateness of the representations contributed to these discussions points being made.

The PTs were provided with mathematical argumentations from four Grade 4 students: Example 1 (E1) in Figure 1; Example 2 (E2) had a claim and $3 + 3 = 6$; with the two other ones having more elaborated arguments, one in Norwegian (E3) and the other in AEL (E4). Unlike Figure 1, E3 and E4 used both diagrams and written words to show that an odd number added to an odd number always resulted in an even number. There were also differences between E3 and E4. In the Norwegian student's argumentation (E3), an odd number was described as a number with a hole and there were pictures with three examples of odd numbers with arrows pointing at the hole and even numbers with arrows highlighting "no" holes. The drawing and explanation correspond to an algebraic definition of odd numbers as 2n-1. The child also wrote, "So if we have three and nine, we then can do like this so you fill the hole and then it will be an even number." E4 also included drawings of three different diagrams to illustrate even numbers, similar to those in E3. E4 stated that an odd number had an extra block, with two rows of squares forming a rectangle with one square on the top. This explanation corresponds to the algebraic definition of odd numbers as $2n + 1$. As well as a drawing, similar to Figure 1 (E1), this solution included the symbols $+$ and $=$ between the diagrams as a support to the argumentation that two odd numbers added together produced an even number.

Preservice teachers' evaluation criteria for mathematical argumentations

In the following sections, we describe the PTs' criteria for what made mathematical argumentations clear, mathematically correct and complete. Where possible, we then identify what seemed to prompt them to discuss these criteria. In the final section of the paper, we discuss how the different language backgrounds of the PTs provided both potential and realised mathematical learning opportunities and the implications of these opportunities for teacher educators.

Clear

The PTs in the small group questioned what it meant for a mathematical argumentation to be clear. One of the Norwegian preservice teacher, PTN 1, described E2 as, "Just giving an example, instead of doing a mathematically correct proof. So, the category would be clearly written or mathematically? It's mathematically correct but it's not clearly written." E2 could be classified as naïve empiricism (Balacheff, 1988), and although Balacheff (1988) did suggest that some producers would consider it a proof, these PTs did not see it as being sufficient. PTN 2 queried if the student actually understood what they had written, "It looks like they just read this somewhere and just wrote it on their own, but it doesn't necessarily mean they understood." So, it seemed that for the PTs clarity in the mathematical argumentation did not always indicate that a student had a deep understanding of a topic. In the whole class discussion, there was a similar response to a direct question from Tamsin about what clarity was, PTN 7 stated, “If they had examples and had clear reasoning for their claim, we said it was clear.” Clarity was, thus, connected to reasoning which had to be understandable by others, including teachers, with group members suggesting that there should not be a gap in the reasoning, which could be the case if a student just repeated someone else’s words. This description seems similar to those in A. J. Stylianides and Stylianides (2009) where the PTs considered that a proof had to be clear, convincing and logical, without requiring interpreters to make jumps of faith to understand what was happening.

Mathematically Correct

In the small group work, the PTs suggested that mathematically correct was the opposite of mathematically incorrect. Consequently, they did not discuss what would make a mathematically correct proof, but only about whether the symbolic, written or diagrammatic mathematics was correct. From this perspective, the PTs considered that all the examples were correct as included “incorrect” mathematics. This was exemplified in the plenary discussion when PTN 4 stated that to be mathematically correct, the example could not be wrong. Thus, there seemed to be a consensus at the social level of the PTs about the criteria to do with mathematical correctness.

However, when queried by Tamsin in the plenary session, PTN 5 stated, “that the equation has to be right at least or correct if there is an equation in the example and also the argumentation should be good.” This, then, switched the focus from the mathematics in the argumentation to what constituted a correct mathematical argumentation. However, for the PTs this seemed to be connected more to the criteria about completeness and so is discussed in the next section.

Complete

During the discussion, it became clear that for the argumentation to be considered complete, it had to be mathematically correct and clear. In discussing the elaborated Norwegian example (E3), the PTs distinguished between what other fourth grade students would understand and what was required of PTs providing a mathematical proof. This suggests they understood that the social group for whom the mathematical argumentation was produced would affect whether it was considered sufficient.

In discussing E2, PTI 1 stated, “It’s just an example, so is it mathematically correct but not complete.” This reinforces the earlier point from the whole class discussion about the difference between between being mathematically correct and being a mathematically-correct argumentation. PTN 2, in the small group, considered that complete argumentations often (if not always) included visualisations. The proofs with visualisation were acknowledged as “coming a bit further”, and PTI 1 reinforced this by stating that “they show the concept of even and odd number.” However, in discussing the argumentation in Figure 1 (E1), PTN 2 acknowledged that just having a visualisation was not sufficient, “It just shows that these two if you add them, they become an even number but it doesn’t say anything … It’s just two bits put together, it’s nothing.”

The importance of understanding how context affected whether a mathematical argumentation was complete came up both in the small group session, around whether just a visualisation was sufficient, and also in the plenary session. As a result of a direct question from Tamsin about how they knew that E1 showed a proof of how adding two odd numbers always resulted in an even number, the Norwegian PTs stated that they could use their background knowledge about these representations from textbooks, “I would say our background knowledge because we have seen this picture of the blocks many times before, but we have no idea what the students are thinking about this, so there is no way of knowing.” They used their background knowledge to understand that E1 (Figure 1) could be a generic example (Balacheff, 1988), where they could interpret essential features of the sets of odd and even numbers from the representations. Still, they also acknowledged that they could not be sure that the student intended it to be viewed in this way. It was their background and knowledge of common cultural artefacts that allowed them to interpret whether mathematical argumentations were complete or not.

For PTI 1, the example provided in their language was the clearest. Nonetheless, they insisted that it needed a definition of an odd number for it to be complete. In the example, even numbers were defined with a series of diagrams where boxes were paired to show 2, 4, and 6. PTI 1 seemed to value the use of mathematical symbols, $+$ and $=$, which provided more information about how to interpret the arrows in the diagrams. Neither of the other two PTs in this small group reacted to this point, by showing either agreement or disagreement. In contrast to the Norwegian PTs, PTI 1 struggled to consider E1 as being mathematically correct, let alone a complete mathematical argumentation. In the task about how to work with the students to improve their mathematical understanding, PTI 1 stated that as a teacher, she would have the student talk about what odd and even numbers were and what their diagram meant. She seemed to be indicating that she saw it only as a single case, naïve empiricism (Balacheff, 1988), rather than a generic example (Balacheff, 1988) which represented a whole group of numbers.

In the plenary, there was agreement that for a mathematical argumentation to be complete, it had to be both clear and mathematically correct. However, what this meant still seemed to be interpreted differently amongst the PTs, especially when they had different language backgrounds. In the task, where the PTs had to decide what they would do as teachers with a class that included the four students whose examples they had analysed, the Norwegian PTs described E3 written in Norwegian as being exemplary. By mentioning a “hole”, E3’s diagrams suggested a 2n-1 understanding of odd numbers. In contrast, PTI 1 suggested that the student, who had produced E2, could be shown E4, which included a diagram with one extra to represent odd numbers, to illustrate how to elaborate their mathematical argumentation of a statement and a symbolic example ($3 + 3 = 6$). PTN 2 agreed with this point because they stated that the student who had made E2 had a similar approach to the student who produced E4. This led to a change in thinking for PTN 1 and PTN 2 who had originally placed E4 with E1 and E2 as needing improvement. However, it was not until just before the final plenary, that the PTs realised that they had not discussed how to teach all students so that their contributions could be valued, including E4 written in AEL.

Potential and realised mathematical learning opportunities.

In the data, it is possible to see that the PTs were able to identify a range of criteria for evaluating mathematical argumentations, many of which had similarities with those identified by A. G. Stylianides and Stylianides’ (2009) PTs. This was perhaps not surprising considering that these PTs were in their fifth year of their teacher education. The different language backgrounds of the PTs also provided some opportunities for enriching these discussions. For example, in the discussion in the small group and the plenary, different interpretations of E1 (Figure 1) provided an opportunity to raise issues about how cultural artefacts provide meaning only to those who are from the same education systems. In the small group work, it was the PTI 1, who spoke AEL, who initially queried whether E1 was mathematically correct. Although the Norwegian PTs did not state that they disagreed with PTI 1, they seemed to be challenged by this suggestion and this led to a discussion about the importance of context. In the plenary, it was Norwegian PTs from another group, who had work in a group with another PT who spoke AEL, who voiced similar issues to do with the need for context to interpret the mathematical argumentation. Tamsin, who also did not have previous experiences of using drawings of blocks as representations of odd and even number, queried in the plenary whether E1 could be considered mathematically correct. When teacher educators and PTs share the same backgrounds, then expectations about mathematical ideas, including mathematical argumentation can go undiscussed. It was when expectations. connected to the different language backgrounds, were raised that there were increased possibilities for richer discussions and the development of broader understandings. Unlike most of the previous research which has shown that the multilingualism of PTs in mathematics teachers education is often ignored (see Chitera, 2011), multilingual PTs' knowledge about cultural artefacts for teaching mathematics made possibilities available to query assumptions connected to just one language/culture about what were acceptable mathematical argumentations.

Nevertheless, the analysis shows that there were many other possibilities where the PTs' backgrounds were likely to produce alternative interpretations that could have led to richer discussions about the evaluation criteria. This occurred, for instance, around the discussions of the two most elaborated examples where the differences between them, one connected to defining odd numbers as $2n - 1$ by highlighting the "hole" and the other defining odd numbers as $2n + 1$ by focusing on the extra box, were not described and discussed. The different definitions could have led to richer discussions about multiple representations–verbal, diagrammatic, symbolic– and thus may have contributed to clarifying what made mathematical argumentations clear, mathematically correct and complete.

There is likely to be several reasons why these potential mathematics education learning opportunities were never realised. One of them was that all the PTs were discussing these ideas, probably for the first time, in English, a language. This suggests that using an unfamiliar language can result in assumptions about mathematical argumentation not being raised. It also seemed that when the PTs did not know each other well, it was perhaps difficult for them to query each other's understanding. The teacher educator, therefore, has a role in moving the discussions beyond the hindrances caused by a lack of fluency and familiarity with each other. It is also important to explicitly bring the discussions with PTs back to what could be learnt about teaching mathematics in multilingual classrooms. As Thomassen and Munthe (2020) found, Norwegian PTs have noted a lack of input about how to teach subjects like mathematics in multilingual classrooms. It cannot be sure that just participating in this workshop would have made PTs aware of what they could have facilitated students to learn from seeing their language/cultural backgrounds as resources in mathematics lessons. As our project develops, we, as teacher educators, will explore how to support PTs to reflect on and query their own assumptions about mathematical argumentation, even when they speak the same language and have similar previous experiences and what this reflection could contribute to their teaching in multilingual mathematics classrooms.

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Bilingual undergraduate mathematics teaching examined through a commognitive lens

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In this paper we examine key principles of visual mediators and endorsed narratives as relating to the commognitive approach. We apply these concepts to the analysis of teaching in a bilingual undergraduate mathematics context. A central focus is on the lecturer's communicative practices and ways of facilitating discursive shifts (colloquial to literate), while examining which language(s) this occurs in. The findings suggest that the use of graphs as well as accompanying gestures, deictic expressions and symbols substantiate the claims about functions that the lecturer is making. The primary language of communication was Irish, establishing an expectation of utilising Irish when engaged in mathematical narratives relating to functions. English was utilised (with Irish) to clarify meaning, primarily when a discursive shift from an object to meta-level discourse was being established.

Keywords: Commognitive approach, bilingual university teaching, language use, functions.

Introduction

A primary mathematical objective in a first undergraduate year of study is to facilitate and develop advanced mathematical thinking. The focus of this paper is on a bilingual (Irish and English) lecturer's teaching of a variety of function constructs, explicitly graphing and limits to a group of bilingual learners. These students were participating in a bilingual first year undergraduate Calculus module at the National University of Ireland, Galway. The data presented in this paper forms part of a larger study examining mathematical meta-level discourses in the English and Irish languages in this bilingual undergraduate context. Previous findings from a student perspective (Ní Ríordáin & Flanagan, 2019), and adapting Sfard's Commognitive framework (2008) for data analysis, suggest that evidence of employing an advanced language trajectory (in either language) does not correlate with mathematical meta-level thinking for the students who participated in the study. Further, language preferences for communicating mathematics learning are situated and can impact the discursive processes of bilingual learners. It was found that when students considered their individual problem-solving routines as incompatible and/or dichotomous this impacted their mathematical discourse practices and contributed to a lack of language negotiation and meta-level discourse development.

Following on from this, we are interested in examining the teaching practices in this bilingual undergraduate mathematics context. A central focus of this paper is on the lecturer's communicative practices and ways of facilitating discursive shifts, while examining which language(s) this occurs in. We utilise Sfard's (2008) commognitive framework as an analytical lens to support our investigation of bilingual teaching practices. Nardi, Ryve, Stadler, and Viirman (2014) illustrate how the commognitive approach can be applied to university mathematics education, while emphasizing that 'this approach can be extended to topics hitherto untouched' (p.183). Examination of teaching practices in bilingual undergraduate mathematics education remains under-researched. Barwell, Wessall, and Parra (2019) acknowledge that much has been done to advance the field but that gaps still remain. They highlight the need for further research into how "multilingual students' language repertoires can be activated and developed in order to achieve deeper mathematics learning." (p.116). In particular, they highlight the need for discursive research in multilingual classrooms which examines mathematical thinking and interaction. Our primary aim in this paper is to illustrate the commognitive perspective in a naturally occurring bilingual undergraduate mathematics education context, and accordingly to contribute towards the use of the frameworks analytical tools to provide analyses of bilingual mathematics teaching and learning contexts.

Sfard's Commognitive Framework

Sfard's (2008) framework for analyzing mathematics discourse (both instruction and learning) was employed to explore a university lecturer's teaching of functions and in which language(s) (English and/or Irish) this occurred. The neologism *commognition* comprises communication and cognition and recognises the varied, purposive and unique nature of discursive approaches to examining mathematics teaching and learning. Sfard (2008) distinguishes between colloquial (everyday or spontaneous discourses) and literate mathematics discourse (requires deliberate teaching). The focus of the M^{2}EID study was how objectified talk signifies discursive shifts between colloquial to literate mathematical discourses and in which language this occurs. Therefore, we observed the meaning-making processes that the lecturer engaged in when mathematizing and in particular the relationships between object- and meta- level discourse and the language(s) of use. In particular, we aim to illustrate the potential of the commognitive approach (Nardi et al., 2014) in examining bilingual undergraduate mathematics teaching. The four tenets of Sfard's (2008) approach include 1) word use, 2) visual mediators, 3) routines, and 4) endorsed narratives. For the purpose of this paper we focus on visual mediators and endorsed narratives as these were the dominant themes emerging from the analysis.

Sfard (2008) describes visual mediators as visible objects that embody interlocutors' mathematical communication (e.g. concrete objects, symbols, tables, graphs, drawings, diagrams, charts, formulae). As a representational tool, such visual mediators can influence interlocutors' mathematical thinking and consequent actions (Tabach & Nachlieli, 2011). Sfard categorised visual mediators as: symbolic (expressions), iconic (graphs) and concrete (protractors). Symbolic mediators can be utilised to interpret the global properties of the mediator or they can be a basis for substitution process to, for example, simplify an equation, through a clearly defined set of rules or steps. Iconic mediators can be observed (graphs) or constructed (graphing). Concrete mediators are physical objects that can be manipulated, or can refer to mental images of objects. Visual mediators are a key symbiotic feature of mathematics discourse and consequently this research explored how this tenet was employed towards objectification, particularly with respect to the language(s) in which the lecturer discussed the mediator.

Narratives are descriptions (written or verbal) of objects, including relationships between objects and the processes involving objects and therefore, these narratives can be substantiated or rejected according to discourse-specific substantiation procedures (Sfard, 2008). These are the widely accepted rules within mathematical discourse that are sanctioned by the mathematical community (e.g. definitions, theorems, etc.). Narratives are comprised of construction, recall and substantiation procedures. Constructions result in new narratives, recall is the memorisation of endorsed narratives and substantiation involves the actions leading to the endorsing of a narrative. This paper examines how visual mediators support the development of the endorsed narrative, while being cognizant of which language(s) are utilised.

Context and Methods

At the National University of Ireland, Galway (NUI Galway), first year undergraduate mathematics students are afforded the opportunity to study honours mathematics through a bilingual approach. Modules offered bilingually are Calculus and Algebra. Four weekly lectures are provided through the medium of Irish, with all mathematics terminology offered bilingually (Irish and English). In addition, lecturers may opt to describe more complex concepts (such as the limit of a function) bilingually. The lectures are supplemented by the provision of a weekly tutorial in English in addition to an Irish-medium tutorial. It is important to note that the mathematics register through the medium of Irish has not been developed beyond this first year of undergraduate education and this is the only institution in Ireland that offers the opportunity to learn mathematics through this medium. A change in one's discourse practices is employment of a mathematics discourse that resembles the features of the discourse practiced by the mathematical community (Sfard, 2012). So as such, a mathematics community of which Irish bilingual learners can become a proficient member of, does not exist and is very much restricted to the context of these lectures/tutorials.

Given the scope of this research the authors required evidence of the lecturer's language use and the instructional methodologies employed within a natural educational context. Therefore, all lectures were video recorded. Video research is an effective method of examining teaching and learning experiences in naturalistic contexts and the affordances of modern technologies facilitate the documentation, sharing and intensive analysis of in-situ learning (Derry et al., 2010). In excess of 22 hours of video-recorded lectures were captured and analysed as part of a regular undergraduate mathematics programme.

Our analysis involved two key stages: (1) narration, preparation and immersion and (2) coding and categorization. The first stage involved providing a rich account of the research context (Huberman & Miles, 2002). Verbatim transcriptions of salient video vignettes were developed and subjected to detailed analysis. Immersion in the data facilitated understanding the diverse mathematical discourses generated by the lecturer in the bilingual context. Systematic selection of the vignettes identified key learning events that were characterized as illustrations of mathematical meta-level developments. The second stage is coding and categorization and involves first fragmenting the data into units for analysis (Denscombe, 2010). The codes employed in this study were Sfard's discourse tenets (word use, visual mediators, routines and endorsed narratives), type of discourse (colloquial/object-level or literate/meta-level) and language use (English and/or Irish). Therefore, data was analysed four times initially in accordance with Sfard's four discourse tenets while concurrently considering evidence of object- or meta- level developments and importantly, in which language this occurs (English and/or Irish). Erickson's (2006) deductive, part-to-whole approach to inquiry is a useful method for tracking patterns across codes and categories. Themes and relationships (such as between discourse tenets and language use, e.g. visual mediator and Irish language use) were identified and aligned with the overall analysis framework (Ní Ríordáin & Flanagan, 2019).

Findings

Analysis of the video recordings reveals the lecturer employs commognitive constructs such as visual mediators in an attempt to convey her meta-discursive expectations (as experienced interlocutor) to students towards the aim of producing an endorsed narrative about the topic of functions (Sfard & Kieran, 2001, p.49). Two examples of the use of such visual mediators employed by the lecturer are outlined below and include the explanation of mathematical symbols and the use of graphs. First, the lecturer explains the use of two symbols, one for infinity (∞) and one to signify 'there exists' (∃). The transcript below depicts how the lecturer explains the infinity (∞) symbol in relation to solving the mathematical problem: $f(x) = \frac{1}{x^2}$ (Video: W4L2 4.13mins):

L1: It looks like x is approaching 0 [writes $x \to 0$] from the right side; that

L2: this function is increasing. The values are always positive because this

L3: is squared [pointing to x^2]. We write this as such: [*writes* ∞ *symbol on board*].

[L1: Tá an cuma ar agus x ag druidim le 0 [writes $x \to 0$] ón taobh

L2: dheis; go bhfuil an fheidhm seo ag pleascadh suas. Bíonn na

L3: luachanna i gconaí deimhneach mar gheall ar go bhfuil seo cearnaithe

L4: [pointing to x^2] [...] Scríobhann muid é síos mar seo [writes ∞ symbol on board]

In the example above the lecturer used a combination of written and graphical communication to explain the concept of infinity as continuous or endless. The use of visual mediators supports the development of the endorsed narrative, but the lecturer also places emphasis on explaining the meaning of the symbols and their use. The explanation was provided through the medium of Irish which we expect relates to the fact that the students would have encountered these symbols at post-primary education and no major discursive shifts were evidenced (remained at object level).

Second, the lecturer explains the 'there exists' symbol (∃), which is outlined in transcript below as follows (Video W4L3 32.30mins):

L1: Do you know this symbol [pointing to ∃]?[*The students shake their*

L2: *heads to signify they do not know this symbol].* This means, in English,

L3: 'there exists' or 'there is'. So when I write that it means, 'there exists'.

*[**L1:** An bhfuil an siombal sin [pointing to ∃] ar eolas agaibh? [The students*

L2: *shake their heads to signify they do not know this symbol].*

L3: *Ciallaíonn seo i mBéarla 'there exists' or 'there is'. So when I write that it means, 'is ann do'.]*

The lecturer explains this symbol by switching to English to clarify its meaning as 'there exists'. This is translated as 'is ann do' in Irish, with a literal translation of 'it is in it, or 'there it is'. In comparison to the previous example, students would not be as familiar with the symbol ∃ and requires a discursive shift from object to meta-level. The lecturer employs both languages when engaged in such a discursive shift and to aid clarification of meaning.

Graphs are another recurrent visual mediator employed by the lecturer in the process of endorsing the narrative regarding functions. In the following example (Video W4L2 23.40) the lecturer is graphing the function $\frac{-2}{(x-3)^2}$, with $lim\ x \to 3$, to investigate and to illustrate if the function crosses the x-axis.

L1: So, find the limit in this case: $\lim_{x\to 3}\left(\frac{-2}{(x-3)^2}\right)$,

….

L53: [*Progressing the topic the lecturer now introduces the concepts of*

L54: *asymptotes*]. So I suppose, that you both have heard of things called

L55: asymptotes? [*Repeats the word in English*].

V7: Just like, when the line gets very close [*using hand gesture to signify*

V8: *vertical line*] but it does not hit the line [*referring to y-axis*].

L56: In this case [*referring to the graph from the previous example*

L57: *explained in class*] this line [*pointing to the y-axis*], is an asymptote, a

L58: vertical asymptote. Why? Well, as you [*referring to V*] said, the graph

L59: does not cross the vertical line, but the graph comes very close to the

L60: line [*pointing to the y-axis*] [*The lecturer then refers students to the*

L61: *explanation provided in the notes*]. So, when things like this happen

L62: [*pointing to the previous two examples that were utilised:*

L63: $lim\ x{-}{>}\ 3^{\mp}\ f(x) = -\infty$ *and* $domain\ x{-}{>}\ 3^{-} f(x) = \infty$*]* at a

L64: certain number, so in that case [*pointing to the example in the notes*]

L65: c is the equivalent of 3, we say that there is a vertical asymptote at

L66: $x = 3$. If you were to draw a graph, at 3, of course it is not defined,

L67: you need more information than that. [*L proceeds to graph the*

L68: *vertical asymptote*]…

….

L90: *running parallel with the x-axis*] [...] This demonstrates what is meant

L91: by vertical asymptotes. So, it is a vertical line and the function tends

L92: to infinity or minus infinity around it. [*L explains the various*

L93: *manifestations of the asymptotes through the use of graphs,*

L94: *and writes the explanation:*

L95: $lim\, f(x) = \infty\ nó - \infty\ agus\ x \rightarrow c^{+/-}$].

In this investigation of the problem: $\lim_{x \to 3} \left(\frac{-2}{(x-3)^2}\right)$, depicted in the above transcript, the lecturer commences with an investigation of whether the function crosses the x- and y-axes. In this regard, a substitution method is utilised in which the equation is written out, various values are substituted into the equation (L2-L18) and explained verbally. The lecturer also illustrates these explanations with graphs to highlight if and where the function crosses the axes (L58-61) and also to introduce the concept of asymptotes (L66-76). The primary language of communication utilised by the lecturer was Irish and was observed that mathematical language employed by the lecturer progressed from basic to more advanced. English was only utilised when providing key terminology – both the Irish and English versions were provided to students. First, the lecturer employed a process of calculating limits through substitution without necessarily referring to continuity. Then, this concept was developed further through the graphing of asymptotes to convey conceptual understanding of the notion of alternative limits that comprise variables tending to infinity or negative infinity. Therefore, the use of visual mediators facilitated a discursive shift from an object to a more meta-level (literate) discourse, with evidence of the use of both languages in this discursive shift.

Conclusion

A discourse is made clear by a community's word use, visual mediators, endorsed narratives and routines (Sfard, 2008). By employing Sfard's approach to research in the fields of mathematics learning and language use it provided us with an opportunity to examine a bilingual undergraduate lecturer's approach to teaching Functions in a naturally occurring bilingual context. In the examples provided above, the lecturer shifts between verbal and visual mediation of functions. Mathematical words and phrases such as *positive*, *squared* and *there exists* are employed throughout the developing discourse relating to functions to convey meaning such as the continuous nature of functions and tending towards infinity. Word use is then progressed by the utilisation of visual mediators such as gestures (pointing, upward and downward hand gestures signifying increasing/decreasing values), symbols (∞, $\exists$) and graphs. In the third example, a graph is employed by the lecturer to visually mediate the students' investigation into whether the function intersects the x-axis in the problem: $\lim_{x \to 3} \left(\frac{-2}{(x-3)^2}\right)$. Visual mediators are of immense importance in mathematics because "With the help of symbolic records, the inherently transient spoken discourse acquires permanence and the different discursive elements become simultaneously present" (Sfard, 2008, p. 159). The "symbolically encoded mathematical discourse" has the capacity to "become an object of metadiscursive activity" (ibid, p. 159). Of additional interest to us was which language(s) was utilised. In general, the lecturer switched to English to clarify meaning or to express the meaning of a particular visual mediator in its English form, particularly when progressing the discourse from object to meta-level (literate). This perhaps suggests that the endorsed narrative for meta-level mathematics requires initiation into the mathematics English-medium community, ensuring that students are in a position to participate in this community given that the Irish mathematics register has not been developed to support further study in mathematics.

The use of graphs as well as the aforementioned accompanying word use/symbols substantiate the claims about functions that the lecturer is making. Validation of a narrative is the means through which we “become convinced that a narrative can be endorsed” (Sfard, 2008, p. 231) and often forms a sequence “which is deductively inferred from previous ones and the last of which is the narrative that is being endorsed” (ibid, p. 232). Students do not offer any commognitive conflicts (challenges to or rejections of the narrative) in terms of these claims and their collective agreement (signified by head-nodding gestures, uttering ‘yes’ and subsequent performance (see Ní Ríordáin & Flanagan, 2019) endorses the particular narrative the lecturer is creating. This also highlights students’ awareness of the meta-discursive expectations of the lecturer, which they will be required to negotiate when engaged in problem-solving activities. The primary language of communication was Irish, establishing an expectation of utilising Irish when engaged in mathematical narratives relating to functions within this context.

The research and analysis presented here is an initial examination and we fully appreciate that further analysis and depth is required. The insights presented here suggest that the lecturer primarily employed the Irish language in their teaching and used English to clarify meaning, thus setting the expectation of utilising Irish when engaged in mathematical narratives relating to functions. However, English was utilised for clarification when a discursive shift from an object to meta-level discourse was being established. We propose that there is a need to examine in greater detail language use as relating to discursive shifts in the study of university mathematics and bi-/multi-lingual learners. Sfard’s Commognitive Framework (2008) has been utilised in many ways and contexts. However, it has not been adapted and utilised specifically to examine bilingual students and their language use when learning mathematics. Therefore, our research contributes to expanding this framework and its use, particularly within a university context.

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Language factors in teaching and learning mathematics: basic qualities of mathematical communication in L1 and L2

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Two opposite trends in evaluating the bilingualism effect on student's academic achievements in math are analyzed in the paper: bilingualism as a resource or as a problem. As the linguistic features and practices of bilinguals form a unitary linguistic system which interacts in dynamic ways, bilingualism should be considered a human's resource with a large potential. From this theoretical framework, the basic qualities of mathematical communication in L1 and L2 (correctness, accuracy, coherence, way of translanguaging) and their descriptors are described. These qualities are used to describe how bilingualism influences the mathematical communication skills of pre-service bilingual Tatar-Russian mathematics teachers.

Keywords: Learning and teaching mathematics, bilingualism, bilingual students, mathematical communication, qualities of mathematical communication.

Mathematical communication and bilingualism

The abstract character of mathematics highlights the important role of language in the teaching, and learning mathematics, because it is impossible to physically show the abstract nature of most mathematics concepts. They can be described only through language. Monaghan (2000) introduced the term “mathematical langscape” to denote the combination of mathematical meanings and the resources for communicating these meanings that make up the mathematical curriculum. Similarly, Sfard (2001) argued that thinking was communication and to consider learning mathematics as being equivalent to developing a mathematical discourse. According to Sfard (2001) discourse can be defined as an activity of communication with oneself and others.

The study of bilingual mathematical discourse is of particular interest since it is more complicated because of interacting linguistic features and practices of bilinguals. Many students are currently studying mathematics in their second or third language and this phenomenon is gradually becoming the norm in many countries around the world. One reason for this is the migration process into developed countries, as well as the legacy of colonialism and the diverse plurality of local languages in developing countries. The second reason is that the language of science, technology and the Internet is slowly narrowing down to several international languages, such as English, therefore, textbooks and other teaching materials are often provided only in these selected languages.

The effect of bilingualism on cognitive development and student's academic achievements in mathematics is often seen as being at one or other end of a continuum in which bilingualism is considered either as a resource or as a problem (Planas, 2014).Many researchers state that a lack of fluency in the language of instruction is one of the main reasons for the poor performance of many students in mathematics, especially those who are bilingual and multilingual (Secada, 1992; Salekhova & Danilov, 2016). From this perspective, fluency in the language of instruction in regard to in mathematical discourse requires fixing.

Other research has found that knowledge representations are often cultural artefacts, closely connected to the language of instruction (Spelke & Tsivkin, 2001; Campbell, Davis, & Adams, 2007). Negative effects on academic achievement have been observed when the languages of instruction for learning mathematics and languages of assessment differ because language is a necessary condition for understanding mathematics concepts. Bilingual students that are weak in the language of instruction tend to have poor comprehension and participation in classroom discourse (Setati, 2005). Consequently, they cannot gain the desired objectives of their studies due to a lack of communication skills.

The aspects of the language which are specific for mathematics have also been found to be another source of difficulty and confusion for bilingual students, who are learning both the language of instruction and mathematics in a new language. In particular, words used in mathematical terminology are often endowed with meanings that in most cases are completely different from their everyday meaning. For example, the words: root, similar, space, even or odd have a different meaning when they are used in mathematics. Sometimes it can be difficult, even for students who are not bilingual, to determine what the intended meaning of is "odd" is in a problem. The research of Durand-Guerrier and Ben Kilani (2004) in the Tunisian context showed the difficulties students experienced in understanding mathematical negation.

However, in recent years, more and more researchers consider bilingualism as an intellectual resource with cognitive benefits. This is considered to be because the experience of using more than one language can create unique opportunities in the bilingual brain. For example, the constant switching of bilinguals from one language to another leads to increase in executive function. Bilinguals sometimes have an advantage in inhibitory control, in selection, switching, working memory, representation and retrieval, which play an essential role in learning mathematics (Bialystok, Craik, & Luk, 2012). Planas (2014) also argued that bilingualism can create advantages for learners to deal more deeply with mathematics concepts. In alignment with this, Alòs i Font and Tovar-García (2018) analyzed a sample of 709 ethnic Tatar school students from Tatarstan and showed that those who spoke Tatar at home tended to outperform in mathematics their schoolmates who had Russian as the home language. The results of my previous research showed a significant difference emerged in favor of bilinguals in solving language-independent, symbolic mathematics tasks of high complexity (Salekhova, 2019). Similarly, Mielicki, Kacinik, and Wiley (2017) found that bilingual USA college students solved mathematical problems that required advanced abstract thinking better.

Research and experience of teaching mathematics to bilingual students in Tartarstan is that student’s bilingualism is a cognitive and linguistic resource. Nevertheless, there is a need to ensure that this potential is developed and used.

Context of mathematics learning and teaching in Tatarstan Republic

Russia is one of the countries with the highest language diversity in the world; representatives of more than 200 ethnic groups live here. Tatarstan is one of the ethnic republics of Russian Federation, and Tatars constitute 54,6 % of its population. Tatar is spoken by most of the people in Tatarstan either as the dominant (L1) or as the second language (L2). Russian–Tatar and Tatar–Russian bilingualism is widespread in Tatarstan. Both languages are used for teaching almost all school subjects and the choice of the language of instruction depends on the school location (rural or urban) or the model of bilingual education used in the schools.

There are currently two main trends related to using languages in mathematics teaching and learning contexts can be observed in the Tatarstan schools - submersion and immersion. They lead to supportive and unsupportive bilingualism. Unsupportive bilingualism occurs when a majority Russian language replaces a minority Tatar language, in this case students, whose dominant language is not Russian must adapt to mainstream education where the Russian language is used as the medium of instruction. In the case of supportive bilingualism, the Tatar mother tongue of the child is the majority language of instruction at school, and he or she is learning the Russian language as a second one and has some subjects taught in it. This situation is the case in rural areas of Tatarstan, where the majority of the population is Tatar.

The influence of bilingualism on the mathematical thinking of bilingual students can be both negative and positive depending on the conditions under which the interaction of the two languages in the educational context occurs.

One of the goals of studying mathematics according to the new «Federal state educational standard of basic education in Russia» (2016) is developing methods of thinking (analysis, synthesis, comparison, classification, generalization). The role of language and communication in teaching and learning mathematics is noted in the “Concept of development of mathematical education in the Russian Federation” (2013). The document states that it is necessary to facilitate communication in teaching and learning mathematics, to encourage students to speak, write, read and listen in a math class.

Similar requirements are formulated in “Principles to Actions: Ensuring Mathematical Success for All” of National Council of Teachers of Mathematics (2014). NCTM states that mathematics learning program should give opportunity to students to (1) arrange and link their mathematical thinking through communication, (2) communicating their logical and clear mathematical thinking to their friends, teachers and others; (3) analyze and assess mathematical thinking and strategies used by others; (4) using mathematical language to express mathematical ideas correctly. Such endorsements and recognition of the importance of language in teaching and learning mathematics can be found in education documents of many countries, especially in such multicultural countries as the United States, New Zealand, Australia and South Africa (Ellerton & Clarkson, 1996).

Despite the multiplicity of languages used in the Russian educational system and the issues they raise, the influence of bilingualism on mathematical thinking and mathematical communication is an under-researched field in Russia. The national educational system of the Russian Federation supports monolingualism and assimilation. This can be seen in the fact that state exams in mathematics must be conducted on the territory of the Russian Federation, including Tatarstan, only in the Russian language. Moreover, starting from grade 4 to 11 final tests in all subjects, including mathematics, are only in Russian. The purpose of the final testing exams is to ensure the unity of the educational space of the Russian Federation and support the implementation of Federal educational standards.

70% of mathematics tasks in the final testing are word problems. In order to solve word problems successfully, students need to understand the essence of the mathematics task, the lack of language proficiency may be a problem in comprehension.

Tatar parents want their children to study mathematics in Russian, as they understand that success in passing exams is associated with knowledge of the Russian language and mathematical terminology in Russian. Mathematics teachers also want their pupils to have high scores in mathematics exams; therefore they choose the bilingual approach using the Russian and Tatar languages in teaching mathematics. However, they implement bilingual instruction spontaneously, without experience or knowledge of scientifically developed teaching methods.

The Ministry of Education for the Republic of Tatarstan launched two projects to preserve national identity and education in the Tatar language in 2018. The goal of the first project “Adymnar - the path to knowledge and harmony” is to build multilingual schools in the Republic of Tatarstan in which teaching subjects will be in three languages (Tatar, Russian and English).

The second project, “Bilingual Teacher”, is aimed at pre-service teacher training in order to ensure that teachers have the competency to teach subjects in two languages. The preparation of bilingual teachers of mathematics, physics, music, foreign languages for these schools has begun at Kazan Federal University; at this stage, 150 students are studying in the first and second years of the courses.

Teacher educators working with pre-service bilingual teachers have many methodological issues because of little preparation in the development and use of bilingual materials and methodologies, in locating appropriate instructional materials etc. Since the teaching methods are not perfect, pre-service teachers have difficulties, in particular, associated with a limited vocabulary in one of the languages and, as a result, with academic communication in the subject-specific area, such as mathematics.

Theoretical framework and literature review

From a theoretical perspective (Vygotsky, 1986; Sfard, 2001), bilingualism should be seen as a resource for mathematical communication which is a window into mathematical thinking. The aim for our research is to identify how bilingualism influences the mathematical communication skills of Tatar-Russian bilingual pre-service teachers of mathematics. To do this, it was decided to highlight the basic qualities of mathematical communication, then to develop their descriptors and a scale for their assessment. In this paper, this assessment scheme is described.

Ben-Yehuda (2005) proposed four distinctive features of mathematical discourse: (1) the use of words that count as mathematical; (2) the use of uniquely mathematical visual mediators in the form of symbolic artefacts that have been created specifically for the purpose of communicating about quantities; (3) special discursive routines with which the participants implement well-defined types of tasks; (4) endorsed narratives, such as definitions, postulates, and theorems, produced throughout the discursive activity. (p. 182)

The analysis shows that the most frequently encountered qualities of mathematical communication are correctness, accuracy and coherence (Salekhova & Spiridonova, 2018). Therefore, they can be defined as the basic qualities of mathematical communication in the Russian language.

The bilingual mathematical discourse is more complicated than monolingual discourse as two languages are interacting in each utterance. One of the peculiarities of bilingual student's mathematical communication is that to express thoughts they can start a sentence in one language and end it in another. This phenomenon is known as translanguaging. Baker (2011) describes translanguaging as "the process of making meaning, shaping experiences, understandings and knowledge through the use of two languages" (p. 288).

García (2009) introduced the concept of dynamic bilingualism as enacted in translanguaging. Dynamic bilingualism does not simply refer to the addition of a separate set of language features, but acknowledges that the linguistic features and practices of bilinguals form a unitary linguistic system that interact in dynamic ways with each other. Despite the growing number of scholars using the term translanguaging, it is difficult for teachers, steeped in the monoglossic language ideologies that schools often promote, to accept it fully.

As García (2009) writes:

> Using translanguaging theory would mean that we would be able to separate the two types of performances. We would be able to assess if a bilingual student uses the lexicon and linguistic structures of a specific-named language in socially and academic appropriate ways—the named language-specific performance. And we would be able to assess if he or she is able to perform linguistically to engage in academic and social tasks regardless of the language features used—the general linguistic performance. (134).

According to Shohamy (2011), bilingual assessment can be placed in a continuum. At one end of the continuum, teachers can use multiple languages in the same assessment, but only responses in the target language are evaluated. On the other hand, all the student's languages are considered part of a single system-their linguistic repertoire, and students can use any language in the tests and even mix them.

The question is how can the basic qualities of mathematical communication (correctness, accuracy and coherence) be assessed in bilingual students' mathematical communication if two languages (L1, L2) are interacting in their speech? The answer is to introduce one more quality of mathematical communication - how translanguaging is conducted. By viewing translanguaging in mathematic classroom communication as positive, it is important to acknowledge that the linguistic features and practices of bilinguals form a unitary linguistic system that interact in dynamic ways with each other (García, 2009). Students can use their full language repertoire in explanations, but it is essential to pay attention to how translanguaging is conducted. Translanguaging needs to be conscious, in that there should be an awareness of the speech, characterized by the validity of the reasoning, and an ability to select language tools (Russian or Tatar), which meet the goals and conditions of communication.

Potential framework for evaluating the mathematical communication

In order to develop support mechanism for preservice bilingual teachers of mathematics, it was deemed important to provide descriptors of the basic qualities. Table 1 provides the descriptors for basic qualities of mathematical communication in L1 and L2.

Basic qualities	Descriptors
correctness	correct usage of mathematical terms, symbols and notations, correct spelling of mathematical terms and symbols, words and expressions of the natural language (Russian or Tatar), correct design of graphic images and drawings
accuracy	ability to choose a rational way to solve the problem (proof of the theorem), to present the mathematical material precisely, to document the process of solving the problem accurately and efficiently
coherence	knowledge of the basic verbal and logical constructions of the mathematical language, ability to present the material consistently, to build a text following its semantic structure (break into sentences, paragraphs, etc.).
way of translanguaging	translanguaging must be conscious which is understood as an awareness of the speech, characterized by the validity of the reasoning, ability to select language tools (Russian or Tatar) that meet the goals and conditions of communication

Table 1: Basic qualities of mathematical communication

In order to evaluate the preservice bilingual teachers' competencies so that support could be targeted to them, a four-point scale was developed to evaluate the level of each of the qualities: "high" (4 points), "average" (3 points), "low" (2 points) and "very low" (1 point). For example, the level of coherence of mathematical communication can be evaluated in the following way:

4 points: Text content is presented sequentially; splitting a text into meaningful units (sentences and paragraphs) is made clearly; verbal-logical structures of the natural language, namely, comparative quantifiers, conjunctive, disjunctive, implicative of the design and construction of negation are used correctly; there is a variety of used words and expressions denoting the verbal-logical design; one inaccuracy is allowed;

3 points: there are insignificant violations in the sequence of ideas presented, in breaking the text into meaningful parts; verbal-logical constructions are used correctly; there is a sufficient variety in the use of words and expressions denoting the verbal-logical structures; 2-3 errors may occur;

2 points: there are some violations in the sequence and the use of verbal and logical structures; the division of the text into semantic parts is not clear; the written text is characterized by the monotony of words and phrases denoting verbal and logical constructions; 4-5 errors are possible;

1 point: the sequence of presentation of thoughts is broken in all parts of the text, thoughts are not presented in the form of structural units of the text; there are numerous violations in the use of verbal and logical constructions.

Similarly, descriptors for other basic qualities of mathematical communication were developed.

Conclusion and perspective

Based on the view that mathematical communication is a window into mathematical thinking, that dynamic bilingualism is a human resource, that translanguaging in mathematic classroom communication is positive, the basic qualities of mathematical communication (correctness, accuracy, coherence and way of translanguaging) were highlighted and corresponding descriptors were developed. These descriptors and the accompanying rating scale will be used to evaluate pre-service teachers' solutions to mathematical problems presented in written form. From this evaluation, the basic qualities of mathematical communication in L1 and L2 will be used to explore whether bilingualism influences the mathematical communication skills of bilingual pre-service teachers', and if it does to what extent and why. Our plans are to use the results in order to design task and teaching-learning possibilities that are targeted at improving bilingual learners' language repertoires so that they have the best possibilities for deepening mathematics thinking and learning.

There are many studies on the communicative qualities of mathematical speech, but there are few that raise the question of how these qualities in bilingual pre-service teachers can be developed. Many researchers both within and outside Tartar and Russia along with our experience indicate that the mathematical achievement of bilingual students depends on the knowledge of the language of instruction. Therefore, it is important that future mathematics teachers are aware of specially designed teaching methods and techniques that will support school students' possibilities to learn.

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ST2
Language for meaning making

Discursive development of Eda on the concept of reflection

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The aim of this study is to explore one Turkish high school student's discursive development on reflection in relation to her teacher's discourse through the commognitive framework. The data sources for examining the teacher's and student's discourses included classroom observations and task-based interviews. The data was analyzed in terms of participants' word use, visual mediators, routines, and endorsed narratives from a commognitive perspective. The results indicated that the teacher's discourse was based on an algebraic-formal and objectified approach during his instruction, which differed from the student's geometric realization of reflection. Although instruction influenced the student's discourse in some ways, the teacher's discourse did not help the student move towards an algebraic realization of geometric reflection as a function.

Keywords: Mathematical discourse, commognitive perspective, reflection, high school.

Introduction

Geometric transformations help students identify mathematical patterns and support mathematical visualization and generalization (Clements, Battista, Sarama, & Swaminathan, 1997; Portnoy, Grundmeimer, & Graham, 2006). Geometric transformations also help students think about mathematical concepts like functions, similarity, and congruence, and improve their reasoning skills and use of different representations (Hollebrands, 2003).

Existing literature on geometric transformations focus on issues such as pre-service mathematics teachers' pedagogical approaches, knowledge, and understanding (Son & Sinclair, 2010) and elementary students' thinking (Xistouri, 2007). There are also studies on high school students' proof-processing (Miyakawa, 2004); problem-solving processes (Leikin, Berman, & Zaslavsky, 2000); interactions in technological environments (Hoyles & Healy, 1997); and use of different representations (Panaoura, Elia, Stamboulides, & Spyrou, 2009) on the concept of reflection. Studies have shown that high school students face challenges when working on reflection problems (Hoyles & Healy, 1997; Küchemann, 1981). Researchers indicated that when a line of reflection is oblique, students struggle while reflecting geometric shapes, whereas such struggles were not observed when the line of reflection was horizontal or vertical (Hollebrands, 2004; Hoyles & Healy, 1997; Panoura, et al., 2009; Xistouri, 2007). Researchers argue that such challenges may be resulting from the fact that students mostly work with vertical and horizontal lines of reflection while reflecting a geometric shape in the classrooms (Panaoura et al., 2009; Son & Sinclair, 2010; Xistouri, 2007). Studies show that when the reflection line is oblique, students have difficulties in realizing the shortest distance as the perpendicular distance with respect to the line of reflection (Hoyles & Healy, 1997; Xistouri, 2007). Edwards (2003) notes that students often consider reflection as a dynamic movement of geometric shapes and have difficulties in considering reflection as a static object in the form of a function. Edwards (2003) argues that it is critical for students to think about reflection both as a dynamic process and as a function to develop conceptual understanding of geometric transformations.

Most of the existing studies on reflection are based on cognitive perspectives and few of them explore students' development of learning in relation to instruction through socio-cultural perspectives. Further, there is a scarcity of research focusing on reflection in natural classroom settings at the high school level. In order to examine student learning within its context, utilization of social theories may offer additional insights. Our aim is to explore one Turkish high school student's discursive development on reflection in relation to her teacher's discourse. In our study, we use a commognitive perspective because it highlights the communicational nature of learning and provides us with the analytical tools with which to examine student development and juxtapose it with the teacher's discourse in the classroom. We address the following question: How does one student develop her discourse on reflection in relation to the teacher's discourse in a high school classroom?

Theoretical Framework

As a sociocultural approach, the commognitive perspective eliminates the dichotomy between thinking and communication by formulating thinking as self-communication (Sfard, 2008). Sfard (2008) defines discourse as a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions (p. 297)". Sfard (2008) notes that mathematical discourse can be identified through four elements: word use, visual mediators, routines, and endorsed narratives. Word use refers to the mathematical words used in the discourse. Visual mediators refer to visual objects created for mathematical communication (e.g., graphs, diagrams, algebraic notations, figures, and shapes). Routines are the "set of metarules defining a discursive pattern that repeats itself in certain types of situations" (Sfard, 2008, p. 301). Routines are defined how and when of a routine is performed (Sfard, 2008). The how of a routine "which is a set of metarules that determine, the course of patterned discursive performance; the when of a routine, which is a collection of metarules that determine, or just constrain, those situations in which the discursant would deem this performance as appropriate" (Sfard, 208, p.208). Closure of the routines comprises how routines ended (Sfard, 2008). Endorsed narratives are utterances about objects and their relations participants consider as true as substantiated by the other three elements of their discourses.

Sfard (2008) defines four developmental stages in participants' word use: passive, routine-driven, phrase-driven and object-driven use. Passive use is the stage where participants would not utter the mathematical word in their speech. For instance, students would not utter "reflection" in their word use; they could just say "it", "this" or "that" instead of "reflection". In the stage of routine-driven use, participants use mathematical words only in relation to specific and limited procedures and actions they perform. For example, when asked about reflection, they may describe all the actions and processes with which they reflect a geometric shape. The next stage is phrase-driven use, where "entire phrases rather than the word as such constitute the basic building blocks" of participants' utterances (Sfard, 2008, p. 181). At this stage, students may not refer to the actions they perform when reflecting a shape but instead use phrases such as "the size of a geometric shape is preserved in reflection" in their discourses. The last stage is object-driven use where participants utter mathematical words as if they refer to objects or end states by using a noun. For example, if students utter “reflection is a geometric transformation”, then they refer to reflection as a specific mathematical object.

Methodology

This is a case study investigating one 10th grade (16-year-old) high school student’s discursive development on reflection in comparison to the teacher’s discourse in a medium-size urban public high school in Turkey. We selected Eda (a pseudonym) as the student for our study, since she was a talkative student who communicated her experiences in an expressive and reflective way and was also willing to participate in the study—a form of purposeful sampling for rich and in-depth data collection (Patton, 2002).

The data sources for examining the teacher’s (Mr. Can, a pseudonym) discourse on reflection included the two video-taped classroom sessions during which he talked about reflection and a video-taped task-based interview that was administered eight days after the instruction. The reason of interviewing with the Mr. Can was to gain deep and detail information and strengthen the data about Mr. Can’s discourse on reflection in addition to his discourses that were obtained from classroom observations. The interview consisted of four tasks on reflection. In the first task, we asked Mr. Can to give an example for reflection. In the second task, which included geometric and algebraic visual mediators, we asked him to reflect a point across an oblique line, to reflect a triangle across an oblique line, and to reflect a line across a point. The third task involved reflecting a geometric shape across an oblique or vertical line. In the fourth task, we provided a geometric figure and asked what kind of geometric transformations (including reflection) can be seen in the figure.

The data sources for examining Eda’s discourse on reflection included three video-taped task-based interviews. Eda was interviewed individually and each interview lasted about 20 minutes. We conducted the first interview before Mr. Can introduced reflection in the classroom. The second interview was conducted right after instruction and the third interview was conducted eight days after instruction. Each interview consisted of four tasks on reflection, which were mathematically equivalent to those used in Mr. Can’s interview, but with different follow-up questions. Eda was silent during the classroom observations and the lessons were dominated by Mr. Can’s discourses, so we could not analyze students’ interactions during the classroom observations. To ensure the equivalence and validity of the tasks (Patton, 2002), we sought and incorporated feedback from five experts (one professor of mathematics, one professor of mathematics education, two Ph.D. candidates in mathematics education, and one high school mathematics teacher) regarding the parallelism and content of the tasks.

We conducted the interviews in participants’ native language and then translated them from Turkish into English. The transcripts of the interviews and classroom observations included participants’ utterances as well as their visual mediators and actions. The data was analyzed in terms of participants’ word use, visual mediators, routines, and endorsed narratives (Sfard, 2008).

Results

In this section, due to space constraints, we only provide a comparative analysis that outlines the differences and similarities between Mr. Can’s and Eda’s mathematical discourses according to the four components of their discourses (word uses, visual mediators, routines, narratives).

Mr. Can's word use on reflection was mostly object- and phrase-driven; he also used some routine-driven words in his discourse. Eda's discourse on reflection before the lesson (first interview), was mostly routine-driven but it also included some phrase-driven and object-driven word use. Eda's word use after the lesson (second interview) was mainly routine- and phrase-driven; there was also an occurrence of object-driven word use. Her word use in the final interview was mostly phrase- and object-driven with fewer occurrences of routine-driven word use. Table 1 demonstrates the frequencies of the word use categories for Mr. Can and Eda. Mr. Can introduced reflection in an objectified and algebraic manner, as a function, in the classroom and consistently referred to reflection as a function throughout his discourse. Before instruction, we observed Eda's word use to be dominantly routine-driven (through a geometric approach). However, during the last interview, her routine-driven discourse gave way to phrase- and object-driven word use. Although she used an algebraic approach in the second interview right after instruction, such use was rare and was not found during the third interview, where her word use was accompanied by geometric approaches to realizing reflection.

	Mr. Can	Eda 1st interview	Eda 2nd interview	Eda 3rd interview
Passive use	N/A	N/A	N/A	N/A
Routine-driven use	6	6	5	2
Phrase-driven use	15	3	6	5
Object driven-use	22	1	1	5

Table 1: Frequencies of word use for Mr. Can and Eda

Mr. Can's visual mediators consisted of algebraic notations, geometric shapes and lines. Although he used both algebraic and geometric visual mediators, he used the former more frequently than the latter. Eda's visual mediators consisted of geometric shapes and lines as well as algebraic notations. She consistently used a geometrical approach before and after instruction as well as during the last interview. Right after instruction, we identified three occasions in which Eda used the algebraic approach presented by the teacher in the classroom. During the last interview, she went back to her geometric approach, indicating that her use of algebraic mediators did not constitute a pattern in her discourse in that interview. Even in the interviews she used the algebraic approach, the frequencies of the algebraic visual mediators in her discourse were significantly lower than the frequencies of her geometric visual mediators. Eda's visual mediators during the first interview also indicated that she struggled reflecting a geometric shape across an oblique line of reflection, the details of which will be included in the following discussion regarding her routines.

Mr. Can used one algebraic and one geometric routine in his discourse, consistent with his visual mediators. His algebraic routine (AR) was to use an equation, where reflection is defined as a function, to determine where a vertex of a geometric shape will be transformed through the reflection. After determining the reflected vertices, he then drew the final shape using the mathematical properties of reflection (e.g., that it preserves size and overall shape of the original geometric shape). His geometric routine (GR) was to reflect each vertex of a given geometric shape visually by identifying its shortest distance from a given line/point of reflection and then reflecting the overall shape across the line/point by using mathematical properties of reflection.

Eda used two geometric routines during the first interview (GR1 and GR2). In this interview, Eda used the mathematically incorrect routine GR1 when reflecting a geometric shape across an oblique line of reflection. When using GR1, she reflected each vertex of a given shape in an equidistant manner with respect to the oblique line of reflection, without taking into consideration that the shortest such distance is the vertical distance to the line (see figure 1). In this interview, she also used GR2 as a routine, which was the same geometric routine Mr. Can used. During the second interview, when solving another problem involving an oblique line of reflection, Eda used the algebraic routine AR introduced by Mr. Can to correctly solve the problem. She also continued using her mathematically correct geometric routine GR2. In the final interview, Eda only used GR2 as a routine. It seems that Eda was influenced by instruction when she used AR to replace her mathematically incorrect routine GR1. On the other hand, she preferred using GR2 during the final interview, which was consistent with her overall geometric approach throughout the study. Although GR2 was identical to Mr. Can's routine GR, Eda seemed to know about this routine before instruction since she used it during the first interview.

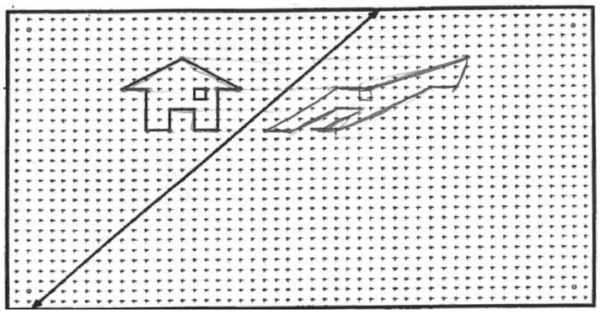

Figure 1: Eda's use of GR1 as a routine in the first interview

The most dominant narrative Mr. Can endorsed in the classroom was "Reflection is a function", which was consistent with the algebraic definition he provided when he introduced the concept. Although not as frequently, Mr. Can also endorsed some narratives through his geometric approach, for example, "The reflection of a point across a line means moving the point across the line so that it has the same shortest distance from that line". Eda's endorsed narratives during the first interview were as follows: " Reflection preserves the dimension of the shape, but not the shape itself" and "Reflection is the visual drawing of a shape in the coordinate system according to the unit distance from the x or y axis". In the second interview, Eda endorsed the narrative "Reflection preserves the dimension of the shape and shape itself, but changes its direction". The common narrative Eda endorsed in her last interview is "Reflection does not preserve the direction of the shape, yet preserves the dimension of the shape and the shape itself". During this interview, she also endorsed "Reflection is the visual drawing of a shape on the coordinate system according to the x or y axis in the same unit distance". This narrative was similar to the ideas involved in Mr Can's endorsed narratives based on his geometric approach. Although Eda's endorsed narratives resembled the teacher's endorsed narratives used in a geometrical context, Eda did not seem to adopt the realization of reflection as a function by the end of the study, instead consistently viewing reflection as the movement or visual drawings of geometric shapes.

Conclusion and Discussion

In this study, we investigated one Turkish high school student's discursive development on reflection in relation to her teacher's discourse. The results indicated that Mr. Can's discourse was mainly based on an algebraic-formal approach during his instruction. Mr. Can taught the concept of reflection in an objectified manner as a function. One explanation for his choice may be due to his assumption that his students had previously learned about reflection. The Turkish mathematics curriculum is a spiral curriculum; students learn about geometric transformations (including reflection) as dynamic motion through grades 6 – 9. It is likely that Mr. Can was unaware of the transition his students had to go through from a dynamic motion view to an object view of reflection in his classroom. It is also possible that Mr. Can was aware of such transitions but the opportunities he provided for his students in the classroom were not explicit enough to be picked up by his students. For example, despite adopting AR as a means to correct her incorrect approach to reflection problems involving oblique lines of symmetry after instruction, Eda consistently used her geometric routine GR2 in the third interview. Similarly, despite her realization of AR as a useful routine, Eda did not adopt the objectified realization of reflection as a function as a consistent aspect of using AR. On the other hand, throughout the study, her word use transitioned from being mainly routine-driven to being mainly phrase- and object-driven. Therefore, for Eda, instruction seemed to be a means for solving her mathematical difficulties while preserving her previous realizations that remain to be useful in the context of working on reflection problems, indicating a complex relationship between instruction and student learning. Edwards (2003) argues that it is important for students to realize geometric transformations both as motion and function since the concept of function has vital importance in learning of transformations (Flagan, 2001). In the context of our study, Mr. Can could not help students transition from a motion-based realization to a function-based realization of the concept of reflection—a result similar to our previous study on the concept of translation (Emre-Akdoğan, Güçler, & Argün, 2018).

Studies have shown that if teachers do not have consistent, clear, and transparent mathematical discourses in the classroom, students may have difficulties understanding the teachers' discourses and may imitate the teacher inaccurately or without thinking about the reasoning behind the teachers' discourses (Güçler, 2013, 2014). It is important for teachers to make their discourses clear and relevant for the students to prevent communicational failures in the classroom. Further studies using sociocultural approaches to explore student learning in relation to classroom discourse are needed to shed additional light on how to improve classroom communication and support student learning of geometric transformations.

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Developing new discourses to deepen students' conceptual understanding in mathematics

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Previous studies have led us to identify some objects or aspects of mathematical activity as being "transparent" for teachers and hence as causing some learning difficulties. This paper examines perspectives and challenges in explicating these transparent mathematical objects and developing new mathematical discourses to increase students' conceptual understanding of geometry and algebra. We present here the first results of an exploratory study conducted in collaboration with a group of teachers and which show similar phenomena in geometry and algebra.

Keywords: Language role, mathematics activities, discourse, measurement, algebra.

Introduction

The work of Lev Vygotsky highlighted the fundamental role of language in mediating individuals' access to concepts, thereby pointing at its epistemic role together with its social, historical, and cultural dimensions. Language constitutes one of the main supports of mathematics teaching and learning activities in classrooms, which has made it an important object of research in mathematical education for some time now (Austin & Howson, 1979). Mathematical and mathematical education discourses (Pimm, 2004) thus constitute a resource for mathematical thinking in classrooms, but also an object of learning: learning mathematics includes learning how to talk like a mathematician. This crucial issue has been expressed and explored from various theoretical perspectives, which vary by the way they link mathematics and language: for example, Vergnaud (1998), who states that "mathematics is not language but knowledge", or Pimm (2004) or Sfard (2008), who base their approach on the idea of "mathematics as a language".

This has led many researchers in mathematics education to focus on the development of discourse (e.g. Moschkovitch, 2010; Planas, Morgan, & Schütte, 2018; Radford & Barwell, 2016), considering both the epistemic role of language, based on Vygotsky's theory, and the dialogical form of its development from a more Bakhtinian perspective (Barwell, 2016; Barwell & Pimm, 2016). The teacher's role in the learning process thus appears to be crucial, and it includes the fact of offering opportunities for discussion by the tasks s/he chooses, by the way s/he implements these tasks, and especially by the way s/he organizes discussions; finally, what also appears to be important is discourses that the teacher her/himself produces (reformulations, new words or grammatical forms s/he introduces, etc.). The resulting difficulties for the teacher can be formulated in terms of tensions and dilemmas (Schleppegrell, 2010). The "dilemma of mediation" (Adler, 1997) concerns the fact of letting students develop their own language within the conceptualizing process on the one hand and driving them towards the use of conventional expert language on the other, this latter language being socially and historically constructed, meaning that it is unlikely to appear spontaneously. The crucial "dilemma of transparency" (Adler, 1999) concerns the difficulty teachers experience in making things clear for students in their discourse and, at the same time, the importance of not losing focus on mathematical activity by paying too much attention to language.

Our contribution to the investigation of these questions has previously consisted in linking some learning difficulties to elements that remain implicit in mathematics classroom discourses. We hypothesized that these elements remain implicit because they correspond to 'naturalized' practices for teachers: Chesnais (2018, 2019), Munier and Chesnais (2016), and Chesnais and Munier (2016) highlighted the implicitness in the language used by teachers in geometry lessons, while Constantin (2014) identified some implicitness relating to some aspects of the manipulation of expressions in algebra. Our postulate (supported by classroom observations) is that the implicitness of some objects or aspects of mathematical activity might prevent some students – especially the ones whose sociocultural background is distant from school – from achieving a conceptual understanding of mathematics. Our intention in this paper is to present some first attempts to investigate ways of enriching mathematics classroom discourse in order to remedy some students' difficulties concerning these two subjects. After revealing the context of our research and the research question we are tackling in this paper, we will present our theoretical framework and methodological tools. The next part will be devoted to the exposition of the first results. The final discussion will lead us to expose the questions raised by these first attempts and the need for further research.

Context of the research

Our previous work has led us to identify mathematical objects and procedures which are "concealed" in most classroom discourses where implicitness seems to be causing difficulties for some students. In particular, we identified elements related to the role of measurement in geometry at the beginning of secondary school (6^{th} grade) and the distributive law in algebra in 8^{th} grade.

Concerning geometry, we identified that the word *measure* is sometimes used by teachers to label either values which are obtained using an instrument (which are decimals and subject to uncertainty) or values which are obtained by calculations (which are real numbers and exact values) (Chesnais & Munier, 2016). This polysemy appeared to cause some difficulties for students when, for example, the two values are not equal or when they do not understand why teachers will not accept measuring with an instrument in some situations (situations where a deductive proof is expected). Most teachers' discourse[3] on this point is often limited to some "rules" such as "you cannot prove a property by measuring a drawing". This led us to suppose that explicitly tackling the distinction and the relationship between the empirical and theoretical aspects of measuring is necessary to support a conceptual understanding of the role of measurement and proof in geometry. It then necessitates new discourses, including words to label these two "kinds" of measures differently. As researchers, we labeled them "empirical measure" and "theoretical measures". The distinction hence appeared to be a promising tool for didactic analyses (Chesnais & Munier, 2016), but the matter of its operationalization in mathematics classrooms remained an open question.

[3] Such an allegation is supported by multiple observations in classrooms but also by textbook surveys, where discourses in textbooks are considered to approximate teachers' discourses in classrooms.

Concerning algebra, we identified implicit substitutions in teachers' discourse during transformational activities (Constantin, 2014). When stressing the use of formulas such as $k(a + b) = ka + kb$ for factoring or expanding expressions such as $3x(4x + 2)$ or $3(a - 6b + 9)$, teachers do not consider substitution as a potential tool for explaining what is at stake in these computations. Some difficulties may appear for students when k and a denote an expression instead of a number, such as $3x$ or $a - 6b$. These expressions must no longer only be considered to represent a computational operation; now, they also represent the result of such a procedure. Sfard and Linchevski (1994) use the term *reification* "to denote the switch in the pupil's conception which is necessary to turn a process into an object" (p. 8). Their study yields evidence that this is a long and difficult process and that many pupils see algebraic manipulations as being governed by arbitrary rules. Several studies in France or Lebanon (Abou Raad & Mercier, 2009; Coulange et al., 2012) have also shown how teachers' discourses rely on syntactic descriptions, talking about "taking what is before the brackets" or "after the plus sign", without linking them to the properties of the operations. Mok (2010) also points out that when they achieve conceptual understanding of the distributive law, pupils are able to justify their claims with different strategies, using substitutions to examine whether a distributive pattern is correct or not and explaining their ideas using functional meanings of symbols and explicit properties. Following this research, our claim is that substitution may contribute to enriching mathematical discourse about the distributive law. These new discourses might necessitate new names such as "substituent" and "substituted" to specify sub-expressions.

Our research question is based on the hypothesis that the options we have identified may potentially constitute good "tools" for fostering mathematical activity and discourse in classrooms that would be more likely to support conceptual understanding than the teacher's usual methods. The experiments we present constitute a first attempt to investigate how talk and argument progress when substitution and a distinction between theoretical and empirical measures are explicitly used.

Theoretical framework

Following Vergnaud and including a Vygotskian perspective, we consider that managing specific linguistic signs – particularly verbal language (oral or written, using words) – is an integral part of the conceptualizing process. From both a Vygotskian and a Bakhtinian perspective, we postulate that new discourses emerge in dialogical forms of activity (Barwell, 2016; Barwell & Pimm, 2016) before they are interiorized by individuals to support their own mathematical activity. Conceptualization and discourse development thus constitute dialectical processes that are highly complex and subject to dilemmas and tensions (Adler, 1997, 1999; Schleppegrell, 2010).

We consider the development of discourse supporting conceptualizing as an evolution from informal genres towards "more formal" genres of discourse (Bakhtin, 1981; Barwell, 2016). This secondarizing process (Bernié, 2002; Jaubert & Rebière, 2012) refers to a "densification" of discourse. In particular, new common nouns appear together with new objects in mathematical activity as a necessity: this is what Vergnaud (1998) calls "substantification". New discourses may also include new expressions or evolution in the usage of some words in order to support the conceptual complexity in intermediate forms, thus explicating what is transparent. Those intermediate forms might suppose a high syntactical complexity (Prediger & Sahin-Gür, 2019).

Experiments and methodology

Our collaborative research project involves researchers and teachers in iterating processes of elaboration-implementation-analysis. The project is intended to last for at least three years, but we will present only the first step in this paper. The project in geometry included five teachers teaching in the sixth grade. The project in algebra concerns only one class in the eighth grade. In geometry, the experiments concern the subject of angles and last for about 10 sessions. This choice was dictated by the fact that it is an emblematic subject for the distinction and articulation of empirical and theoretical aspects of measure: the French curriculum for the 6^{th} grade includes the use of the protractor, but also work on short proofs concerning theoretical measures (like determining whether angle ABC is a right angle if one knows the size of two adjacent angles ABD and DBC).

Our discussions with teachers about the conceptual challenges that we detailed in the first part of this paper revealed that they correspond to actual difficulties that they had identified in their classes which they did not know how to resolve. Researchers and teachers then developed ideas in order to remedy these difficulties based on what teachers usually do with their classes while attempting to enrich it with elements brought into the discussion by researchers: the explicit distinction between empirical and theoretical measures in the 6^{th} grade, and the idea of substitution in the 8^{th} grade.

Lessons were videotaped and students' written productions were collected. Videos were transcribed and analyzed in order to identify the emergence of new discourses on the two critical points (measure and substitution). This included detecting opportunities for discussions on these points and collecting all the associated discourses, then identifying changes. Our indicators of an evolution were related to lexical and syntactical aspects of the discourse.

Two common findings

Although experiments were conducted on different subjects and levels, some critical phenomena appeared in both contexts. Because of the lack of space, we have decided to illustrate our findings on one particular task for each subject. In geometry, one teacher decided to propose the following task: "Here is an exercise proposed to sixth-grade students and one student's answer: In the figure below [showing an angle DBE which appears close to a right angle], is the angle DBE a right angle? Student's answer: 'I have measured EBD = 90°, and therefore EBD is a right angle'. Is the student's answer correct?" In algebra, two sessions were devoted to explicitly making substitutions in k(a + b) = ka + kb in order to expand the expressions. In the first exercise, pupils were asked to substitute k for $2x$, a for 7, and b for $3x$, while in the second exercise, they had to find possible substitutions in the formula in order to expand $-5(3n + 2)$ or $(4n + 3)(7n + 1)$.

Our first main finding is that while new discourses appear within students' and teachers' discussions, learners invent new words or expressions which do not necessarily correspond to those anticipated by researchers and teachers. While "theoretical measure" and "measured value" were the expressions that teachers had decided to use, and which were more or less used by students at some points, some other expressions appeared within the lessons. Here is an excerpt from a discussion about the task we mentioned above:

Student 1: It could be 91° […] it is not a precise measure.

Teacher: So, what kind of measure is it, according to you?

Student 1: It is a *measured measure*.

Teacher: A measured measure. Ok. So it is not a theoretical measure, at any rate.

We interpret this excerpt as revealing that students are able to invent their own – ingenious and mindful – words ("measured measure") to make sense of the situation, even if the expression does not seem completely acceptable to the teacher. We consider this nominalization to be a significant step towards substantification and conceptualization. Working on some other tasks also led one of the teachers to label the geometrical figures in order to distinguish them from their representations. She decided to talk about the "imaginary, ideal, and perfect figure" which is "schematized by the drawing" and she had the students rewrite the questions in exercises more explicitly in order to have them understand that measuring using a protractor was not the correct procedure because the question was about the theoretical value.

In the substitution experiment, pupils were asked to find possible substituents in order to use substitution in $k\text{x}(a + b) = ka + kb$ to expand -5(3*n* + 2). Pupils suggested replacing *k* with -5, *a* with 3*n*, and *b* with 2. When the teacher asked them how they had found the substituents, a student explained: "We look at the given computation [and] because *k* is the first, we take the first number, *a* is the second so we take the second number, and [for] *b* we take the third number". Students also discussed about how to produce an equality using substitution:

S 1: As *k* is hmm, what do you call it? Well the 2, the 2*n* + 3, I don't remember what it's called.

S 2: *k*.

S 3: The factor […].

S 1: The factor, well, as *k* is the factor, well, hmm, we will multiply it with the terms.

These episodes show the necessity of labeling the different sub-expressions in relation to the identity to be used in the transformational activity. In doing so, two different methods appear: talking about a number (including 3*n*), or a factor referring to either what the sub-expression denotes or its syntactic function. This leads us to the idea that substitution may play a role in the process of reification considering both systemic and syntactic structure in algebraic expressions (Kieran, 1989). It also contributes to a certain densification of the discourse, all the more so as several students show how difficult it can be to identify a factor in an expression such as (4*n* + 3)(7*n* + 1), saying at first that there is no *k* or no factor. When expanding (4*n* + 3)(7*n* + 1), the question of the choice of the substituent for *k* was then raised by the students when they were asked about how to find the factor, or "the *k*". One of the pupils argued that it is always at the beginning of the expression. Another challenged this statement by saying that (7*n* + 1) could as well have been chosen. Two students then asked why the other was chosen at first and put forward the idea that it might not have given the same result, leading them to try another substitution, comparing and discussing the validity of both transformations in relation to the distributive and commutative law.

Our second finding is also illustrated by the two episodes presented above: introducing new tasks generated discussions which do not usually appear in classrooms since, according to teachers who participated in the experiments, "things are usually swept under the carpet". Indeed, discussions like the ones we have just evoked seem far from the idea of "applying rules".

Discussion and conclusion

Though concerning different domains, our findings suggest that fostering the development of new discourses based on new language artifacts may help deepen students' conceptual understanding.

However, developing new discourses is of not easy for teachers. Unsurprisingly, introducing new objects destabilizes teachers' practices and some confusion still occasionally appears in their discourse; for example, between theoretical and empirical measures or between substituents and substitutions. Our first experiments also show how difficult it can be to seize pupils' ideas in intermediate forms and link them to concepts or identify potentials for new lexical forms. In the case of substitution, pupils use *number* to refer indiscriminately to either 3 or 3*n*. It appears to be both a lever for generalization and a hurdle for others, who will only consider 3 or *n* as substituents. The teacher also misses an opportunity to talk about a property of substitution: students use reverse substitutions to recognize the identity and to control their choice of substituents, but neither teacher nor students ever put into words what is happening here. Substantification takes some time: substitution is at first considered as a technique (or gesture) and not an object. These observations point out the necessity of long-term processes to provoke evolutions.

However, new names or labels are not sufficient conditions for developing conceptual understanding. Focusing pupils' attention on "the right word", without relating it to the mathematical activity at stake, probably prevents the students from developing their own discourse.

Our case study also raises methodological questions about the intermediate forms: Since they do not appear in expert practices, where could they have come from? The idea we would like to explore more deeply is to investigate the way in which these mathematical objects emerged historically. Two other limitations of our research still need deeper investigation: first, we need to find ways to appreciate the relationship between discourse development and learning; second, we need to identify some indicators of the secondarizing process which would correspond not only to lexical and syntactical elements, but also to each of the three items identified by Pimm (2004), and especially the question of the voice. What appears certain, however, is that this kind of research necessitates long-term collaborations between researchers and teachers.

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The School Mathematical Discursive Community: Diversity and the role of language in meaning-making

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Our theoretical framework allows us to consider the mathematics class as a School Mathematical Discursive Community, relying on Bakhtin's theory of genres of discourses and heteroglossia. In our paper, we question how the teacher can orchestrate heteroglossia in order to establish a Relevant School Mathematical Discursive Community. We show that such an orchestration, which relies on a secondarising process, is productive for teaching and learning mathematical concepts (in this case, decimals and fractions) at primary school.

Keywords: (Relevant) School Mathematical Discursive Community, heteroglossia, secondarising process.

Theoretical framework: The School Mathematical Discursive Community

Our theoretical framework is based on both Vygotskian and Bakhtinian perspectives. In particular, we use Bakhtin's distinction between first and seconde of discourse in order to analyse the language practices of teachers and students. First genres of discourse relate to everyday or informal contexts and refer to spontaneous or commonplace activities. Second genres of discourse are more elaborate and formal and relate to specific knowledge. Like other researchers (Pimm, 1987; Barwell, 2012; Barwell & Pimm, 2016; Planas & Setati, 2009; Prediger & Wessel, 2011), we are interested in the evolution from first genres of discourse to second genres of discourse in the mathematics classroom. From our theoretical perspective, we tackle this evolution through a specific dynamic called the *secondarising process* of language practices (Bernié, 2002, 2004; Jaubert & Rebière, 2012). The secondarising process also relies on the idea that the students' and teacher's activities always reference diverse contexts. This plurality of contexts potentially provides diverse meanings to their activities. The resulting heteroglossia (Barwell & Pimm, 2016) includes different voices carried by the students' and teacher's discourses, seen as a language diversity that is inherent in teaching and learning. Thus, we consider the mathematics class to be a *School Mathematical Discursive Community*. Within a teaching and learning perspective, it is necessary orchestrate heteroglossia through interactions between students and teacher. In order to establish a *Relevant School Mathematical Discursive Community*, this orchestration also needs to come about via a *secondarising process*, gradually leading the students' language from first genres of discourse to second genres of discourse in order to mediate their access to mathematical concepts.

Methodology

Our research study took place over four years (2015–2018) in a French primary school with students (grades 4 and 5) from socially disadvantaged backgrounds. The table below describes the data collected and presented in this paper and examples of the phenomena we will discuss thereafter.

Grade – Mathematical content		Data	Phenomena
Grade 4	Decimal numbers Positional notation	3 sessions, videotaped (October 2015) and transcribed	Heteroglossic linking (local event)
Grade 4 Grade 5	Fractions Diagrams and problems	Questionnaire (January 2018 – grade 4) – students' written productions 6 sessions, videotaped and transcribed (February–March 2018 – grade 4) Questionnaire (September 2018 – grade 5) – students' written productions	Establishing a Relevant School Mathematical Discursive Community (several steps)

Table 1: Data collected and analysed – Phenomena

The analyses of the collected data aim to highlight two kinds of phenomena. In the first section, we will show how we tackle *heteroglossia*. We will illustrate the role played by *heteroglossic linking* in (re)constructing the mathematical meaning of decimals (positional notation) through an analysis of an episode from the end of the first videotaped session. In the second section, we will study dynamics related to the orchestration of *heteroglossia* in order to establish a *Relevant School Mathematical Discursive Community*. Our analyses of several sessions and of students' written work will allow us to illustrate a *secondarising process* in the teaching and learning of fractions (diagrams, equitable sharing problems).

Heteroglossic linking and constructing the meaning of decimal positional notation

During the session, the teacher asked grade 4 students the following question: "If you know that the length of a given line is 1 cm, how can you give its length in dm?" Students had several rulers: each ruler was graduated in a given unit (cm, dm, mm) and they had previously used them to measure the length of several lines. They had also studied simple fractions ($\frac{1}{2}$;$\frac{1}{4}$;$\frac{1}{8}$;$\frac{3}{4}$) as "unit-splitting", but they had not yet encountered decimal fractions ($\frac{1}{10}$;$\frac{1}{100}$) or decimal positional notation (0,1; 0,01). The teacher aimed to discover whether students would manage to extend the "unit-splitting" in such a problem: by "reversing" the idea of dividing a group of units by 10 (10 cm = 1 dm) in order to reach the idea of dividing a unit into 10 (1 cm = $\frac{1}{10}$ dm). This is quite a new perspective towards measurement units (grouping/splitting units) which seems to be difficult for students to conceptualise (Coulange & Train, 2019).

One of the students, JOS, managed to envisage that as "one dm was equal to ten cm", so "one cm was ten times smaller than one dm", and then "one dm shared by ten or one-tenth of a dm". Another student, TAR, wrote a decimal notation, "0,1 dm", which relied on a kind of written literacy. TAR used units on a measurement place value chart and entered "1" in the cm column, then "0" in the dm column. TAR added a comma between the two digits. At the end of the session, the teacher spoke to both JOS and TAR. Our analysis focuses on this final episode:

JOS:	You say that one centimetre is ten millimetres. So, it is ten times larger. You have to make it ten times larger, so, if you want to do it in decimetres, it should be ten times smaller […].
Teacher:	So how many many centimeters we can put how many in one decimetre?
Students:	Ten.
Teacher:	Then, a centimetre is how many times smaller? So I will wait for your answers […].
JOS:	One-tenth.
Teacher:	One-tenth of a decimetre, does everybody agree? [Students: Yes.] So you had another suggestion, what did you suggest? Everybody, you have to look. […] TAR suggested another notation.
TAR:	As there are zero decimetres, I put a zero, but comma one. I think this is the same as this [pointing at JOS's 1/10 – at the same time, JOS points to the "1" of the "0,1" written by TAR].

Figure 2: Linking decimal notation "0,1 dm" and fractional notation "$\frac{1}{10}$ dm"

Teacher:	You think they are the same?
JOS:	Yes, because here it is one centimetre and ten millimetres, and here we say it is one-tenth. So, if he puts 'zero comma one', it means it is one-tenth. As you add the tenth, it makes one-tenth.
Teacher:	This one, it could be the one-tenth. I don't know, we will have to speak more about this

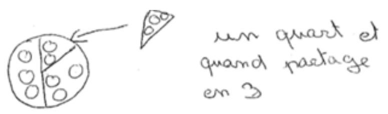
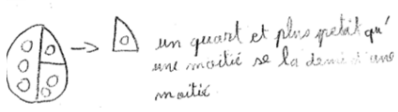
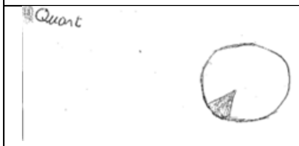
During this collective episode, the two students managed to simultaneously construct coherence between two voices, one relating to decimal notation, "0,1 dm" (relying on written literacy), and one relating to "one-tenth of a decimetre" (relying on the mathematical concept of "splitting units"). From our point of view, this is a heteroglossic linking (Lhoste et al., 2011) which was initiated by TAR ("I think that this is the same as this") and completed by JOS ("zero comma one, it means it is one-tenth"). This heteroglossic linking seems to allow meaning-making of decimal positional notation through a secondarising process. The teacher adressed this issue ("we will have to speak more about this") and pursued this question further during the following sessions.

Moreover, during the following years, this strategy of making meaning of decimal positional notation through heteroglossic linking was repeated. Students investigated the relationships between two systems of symbolic notation used for expressing measurements of length: decimal positional notation (18,2 cm) and complex numbers and metric units (18 cm 2 mm). Then, they had to (re)construct the meaning of the decimal positional notation “18,2 cm” in relation to decimal fractions (the “2” of 18,2 cm interpreted as 2 tenths of a unit) (Coulange & Train, 2019). This teaching strategy relies on the orchestration of diverse voices regarding mathematical notation within the School Mathematical Discursive Community. These voices come from diverse contexts, such as social use, written literacy, and mathematical concepts. Even though it is not easy to orchestrate such a mathematical discussion in a productive way, it nevertheless allows the secondarising process of the students’ discourses and the conceptualisation of decimal (positional) notation to take place.

Establishing a Relevant School Mathematical Discursive Community regarding fractions and diagrams

First step: A strong *heteroglossia* related to words and diagrams

We had observed that students seemed to use familiar words, such as “quarter” or “half”, before studying fractions at school. In order to learn more about the meaning that students attached to these words, we formulated a questionnaire and collected written productions from students (grade 4, before being taught fractions): students were how they understood terms such as “a half”, “a quarter”, “two quarters”, “a half of a quarter”, and “a quarter of a half”. The results of this study revealed diverse understandings of such terms: for example, some students considered a quarter as “a half of a half” or “sharing into four equal parts”, while others described it as “a piece smaller than a half” or “sharing into three parts” (with some different arguments related to sharing actions or the results of these actions). Moreover, a strong heterogeneity of writing practices regarding diagrams was revealed. Some students made drawings (of a pizza or a slice of pizza), sometimes without information about “a unit”, while others produced more elaborated models of “unit-splitting” and used them to make a mathematical argument (for example, in order to form relationships between “a half of a half” and “a quarter”).

	“Here it is as a pizza / A fourth is when you share into three / I said it was like a half / Actually I think it is not that” Drawing of a pizza (unit) to share – Quarter as “sharing into three”
	“A fourth is maybe smaller than a half? It is maybe a part of a half / I think it is a half / a half of a half […] it is the same as a clock / Actually a quarter of an hour” Drawing of a pizza (unit) to share – Quarter as “the half of a half”
	“A quarter is a small piece” Circle diagram with a unit. Quarter as a “small” part

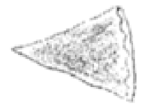
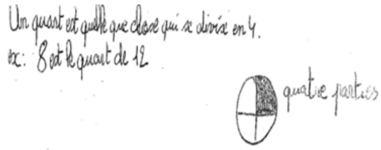
	“A slice of pizza / As it is cut like this to eat it” Drawing without the pizza (unit) – Quarter as a “small” part
	“A quarter is something you divide into four […] An example / I am not sure because eight, eight plus four, it makes twelve / I think eight is a quarter of twelve” Circle diagram with a unit. Quarter as splitting a unit into four (diagram) and as adding four (numerical example)

Table 3: “What is a quarter?” Students’ answers and diagrams or drawings

We used these diverse language practices relating to both the meaning of “a quarter” (and also to “two quarters” or “a half of a quarter”) and the diagram of “equal sharing” or “splitting a unit” in order to design activities about fractions. Our aim was to orchestrate this kind of *heteroglossia* and to establish a *Relevant School Mathematical Discursive Community* through a *secondarising process*. In particular, as previous results (Champagne & Coulange, 2019) led us to find that drawing or interpreting diagrams (circles or rectangular models) related to fractions may help to resolve persistent difficulties for some students, we especially focused on this point. We designed activities with the teacher in order to help students to conceptualize graphical models of fractions.

Second step: Question about “equitable sharing – splitting a unit” drawing

In a previous session, students had to split a semi-circle into six equal parts. They had produced diverse drawings and diagrams, but they did not manage to conclude whether their propositions were valid or not during the discussion. The teacher came back to this question.

Students: Splitting into six parts.

Teacher: Splitting into six // what was your problem / you had made drawings, but what was the question [*pointing on drawings on the whiteboard*]?

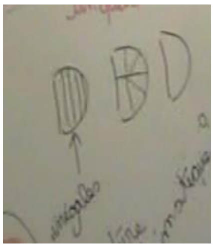


Figure 3: Splitting a semi-circle into six equal parts – First diagrams proposed by students

Student: The problem was / at first we had big parts and then smaller.

Teacher: Actually, it seems we didn’t have equal parts / if I colour this part in green and this part in red / we agree / it is not equal.

Student: The other one / there are equal parts.

Teacher: What do you think? These parts could be equal [*colouring parts on the second drawing*] / You think that / Everybody look / it is important / Do you think this part could be the same as this one? […]

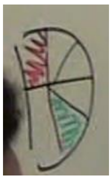


Figure 4: Splitting a semi-circle into six equal parts – Discussion of a diagram

Students: Yes / No / Not necessary because what you coloured in red is small […].

Teacher: So I will propose something to you / We will continue with this / Today, our question is how we can split this semi-circle [*showing some semi-circular surfaces made of paper – one for each student*] into /// At first, we will try to split it into three equal parts.

Students produced several drawings in order to answer to this question (splitting a semi-circle into three).

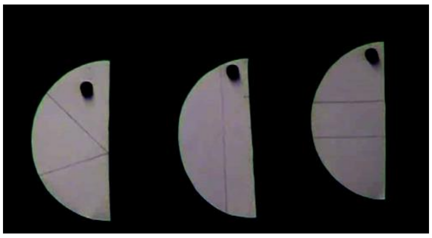


Figure 5: Splitting a semi-circle into three equal parts – New diagrams proposed by students

In a new discussion, they seemed more or less in agreement that only the drawing on the left could be the right answer (a student argued "the point should be in the middle [of the diameter]") In order to check this drawing, a student suggested folding the paper into three. His proposal allowed him to establish relationships between graphical representations and material gestures (folding and superposing). This session played a crucial role in the students' comprehension of the constraints on "equitable-sharing/splitting units" drawings. Then, the teacher encouraged them to use diagrams for reasoning, for showing their argument, and so on. The students gradually seemed to adopt such written practices for themselves and to become particularly proficient in this kind of use.

Third step: Establishing a Relevant School Mathematical Discursive Community – Some consequences

The following written productions come from the same students (at the beginning of grade 5). They had to solve the following problem: "The teacher has 5 biscuits. Could you help her share all the biscuits among 5 students?"

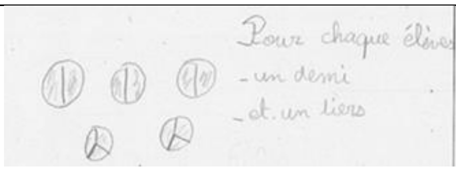
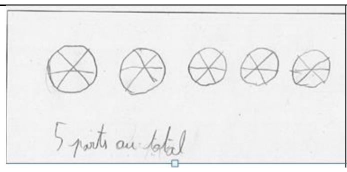
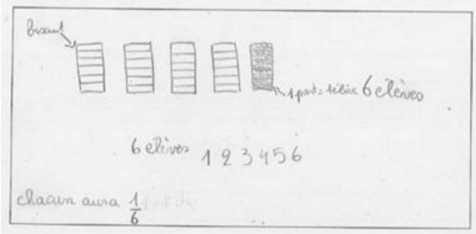
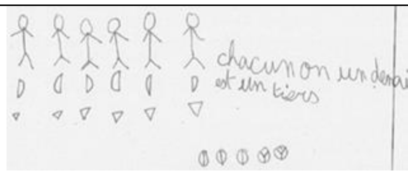
 Correct answer – valid diagram, “6 halves” and “6 thirds”	 Correct answer – valid diagram, “5 sixths”
 Wrong answer – valid diagram, “5 sixths”	 Correct answer – valid diagram, “6 halves” and “6 thirds”

Figure 6: Diagrams proposed by students in order to solve a sharing problem

The analysis of these written productions shows that the students were able to use models in order to represent fractions and to solve “equitable sharing” problems. It also highlights that this secondarising process did not completely reduce the diversity of the students’ language practices: they produced diverse diagrams (circular or rectangular models) and reasoning. However, the students’ written productions also have common features that are relevant to mathematical concepts of “equitable-sharing/splitting units”. Therefore, we can consider that a Relevant School Mathematical Discursive Community was established through a secondarising process: it allowed all the students to evolve their language practices about “equitable-sharing/splitting units” and to access new concepts related to fractions.

Conclusion and discussion

In this paper, we have highlighted phenomena relating to establishing a School Mathematical Discursive Community. Our first example gives a local illustration of how the orchestration of heteroglossia relying on heteroglossic linking is likely to make meaning of decimal positional notation. It also shows that taking heteroglossia into account may be useful for designing efficient teaching and learning trajectories. The second example investigates the steps involved in such a perspective. It shows how establishing a Relevant School Mathematical Discursive Community relies on consideration of the students’ prior language practices in the mathematics classroom. This initial language diversity needs to be gradually orchestrated through a secondarising process guided by relevant mathematical questions acknowledging initial heteroglossia. Such a dynamic process encourages the evolution of students’ language practices in order to mediate mathematical concepts (related to fractions in this case). Therefore, we consider that establishing a Relevant School Mathematical Discursive Community may provide new directions for mathematics teaching and learning. We hope that our contribution highlights some of these perspectives. From a more theoretical point of view, it may be interesting to how Bakhtin’s theory may contribute to studies of language and mathematics learning and teaching (Barwell & Pimm, 2016).

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Disentangling discourse practices and language means for developing conceptual understanding: The case of pre-algebraic equivalence

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Explaining meanings has been shown to be the crucial discourse practice for students' conceptual development. However, more research is required to disentangle what kind of language means are involved in different mathematical topics. The paper reports on a Design Research project aiming at developing language learners' conceptual understanding for pre-algebraic equivalence. The qualitative analysis of the first design experiments with fifth graders shows that activities of connecting representations can foster students' learning, but language support is required for deepening a meaning-related transformational approach to what we call restructuring equivalence.

Keywords: discourse practice, explaining meanings, pre-algebra, equivalence of expressions

Background: Explaining meanings of equivalence of expressions

Discourse practices as key language demands for students' processes of meaning making

For supporting (monolingual and multilingual) language learners in developing conceptual understanding for mathematical concepts, engaging all students in rich discourse practices has been shown to be crucial (Moschkovich, 2015; Setati, 2005). Thereby, the attribute "rich" is not interpreted by "the more students speak, the better it is", but with respect to the quality of the discourse practices: The learning opportunity gap for language learners occurs when they mainly participate in procedural rather than conceptual talk. Various empirical studies have started to identify typical discourse practices and their relevance for students' processes of meaning making (e.g., Erath, Prediger, Quasthoff, & Heller, 2018; Moschkovich, 2015; Prediger & Zindel, 2017). The main distinction is illustrated by the task in Figure 1: *reporting the procedure* of transforming one expression into an equivalent expression is much less demanding with respect to language than *explaining the meaning* of equivalence or *justifying why* the transformational rules guarantee the transformation into equivalent expressions. Although empirical studies have generally shown the relevance of the three discourse practices (e.g. Erath et al., 2018; Moschkovich, 2015; Setati, 2005), little is known about what students need to learn to engage in each of them for a specific topic such as equivalence of algebraic expressions. In order to close this research gap, this paper reports on a study in topic-specific Design Research methodology that aims at developing language-responsive teaching-learning arrangements and at analyzing the students' learning pathways, based upon the existing topic-specific state of research as presented in the next section.

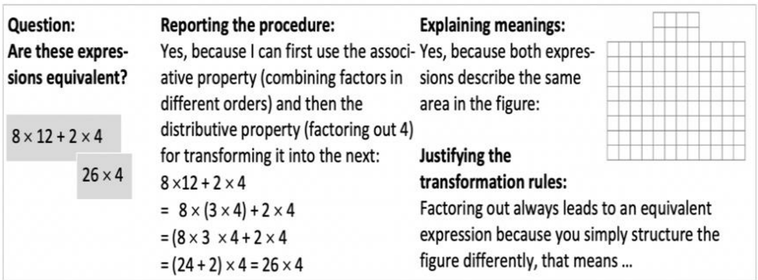


Figure 1. Three discourse practices with equivalent expressions in a pre-algebraic context

Findings of algebra education research on procedures and concepts for equivalence

The research overview by Bush and Karp (2013) reveals that students have multiple difficulties and misconceptions in algebra, among them procedural difficulties with transformations between algebraic expressions, and the understanding of equality as equivalence of expressions instead of "results". For understanding equality, three different characterizations of equivalence of expressions have been identified as relevant (Kieran, 2004; Knuth, Stephens, McNeil, & Alibali, 2006; Zwetzschler & Prediger, 2013):

(1) In the *operational approach*, two expressions are said to be equivalent if they have the same value; when containing with variables, then same value for each evaluated number (*result equivalence*).

(2) In the *relational approach*, two expressions are said to be equivalent if they describe the same constellation, i.e. when they can be related to the same everyday situation or the same geometric figure (*description equivalence*).

(3) In the *formal transformational approach*, two expressions are said to be equivalent if they can be transformed according to formal transformation rules (*transformation equivalence*).

Many instructional approaches in existing textbooks have been criticized to promote only the formal transformational approach without providing learning opportunities for meaning making (Kieran, 2004; Knuth et al., 2006). When meaning making is promoted, students' strong operational approach (which is often focused in arithmetic: an equal sign only signifies "result") often hinders the development of a relational approach with a description equivalence (Kieran, 2004; Zwetzschler & Prediger, 2013). For developing a conceptual understanding, however, the relational approach is crucial and must be tightly connected to the transformational approach (Knuth et al., 2006).

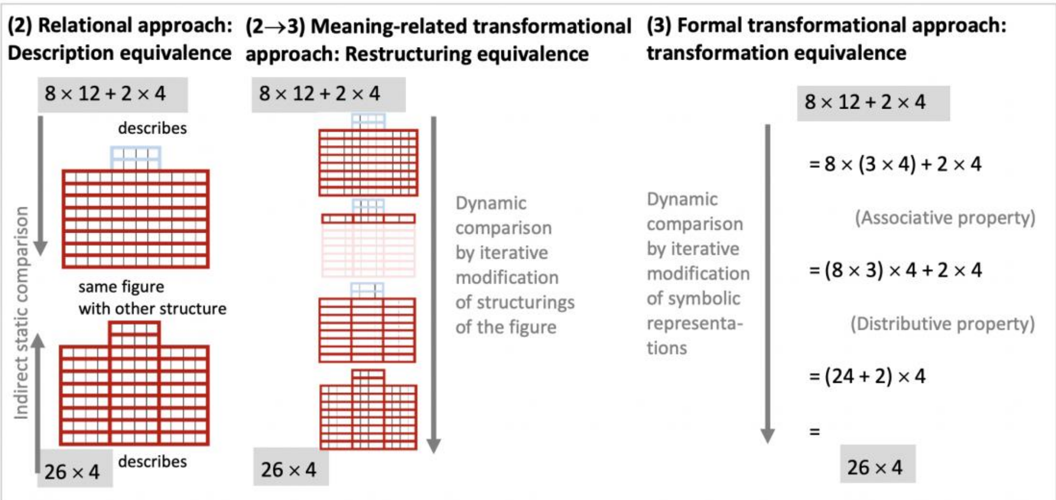


Figure 2. Restructuring equivalence: Bridging between static meaning related relational approach and dynamic formal transformational approach for comparison of expressions

This connection is not trivial: The relational approach in the description equivalence requires an indirect but *static* comparison whereas the transformational approach refers to a *dynamic* comparison. Most students can learn to explain the meaning of description equivalence, but many of them (especially, but not only language learners) struggle with connecting the two approaches in the discourse practice of justifying the rules. Figure 2 illustrates the suggestion we make in this article to overcome this gap between the meaning-related static relational approach and the dynamic transformational approach: We introduce a fourth approach between (2) and (3) and explore empirically how to support students to overcome the conceptual and language-related challenges:

(2→3) In a *meaning-related transformational approach*, two expressions are characterized to be equivalent when we can explain how the structuring related to one expression can be modified into the structuring of the second expression (*restructuring equivalence*).

Design principles for the teaching-learning arrangement on restructuring equivalence

This new bridging approach was implemented in a language-responsive teaching-learning arrangement that follows three language-responsive design principles (cf. Prediger & Wessel, 2013):

(DP1) engaging students in rich discourse practices;
(DP2) connecting multiple representations and language registers;
(DP3) macro scaffolding, i.e. providing discursive and lexical language learning opportunities for each step of the conceptual learning trajectory.

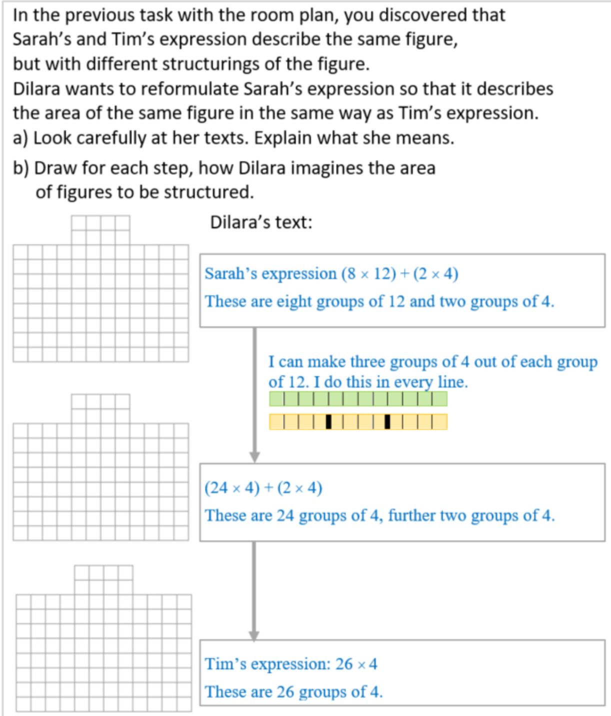


Figure 3. Core task for establishing restructuring equivalence

Figure 3 shows the core task for establishing the meaning-related transformational approach and promoting students' mental and discursive construction of the conception of restructuring equivalence. The example in Figure 3 indicates an important language means required for explaining how the expressions fit the structuring of the picture: Multiplicative language means for expressing unitizing, e.g. two groups of 4 or two fours.

Research question

The Design Research project pursues the following research questions: *How can the bridging approach of restructuring equivalence support students in constructing meanings for transformations, and which language demands occur for students when involved in discourse practices of explaining meanings and describing meaning-related transformations?*

Methodology of the case study

Research context. The qualitative case study presented in this paper is embedded in the project *MuM pre-algebra*. It follows a Design Research methodology (Gravemeijer & Cobb, 2006) and aims at developing a language-responsive teaching-learning arrangement for fostering language learners' conceptual understanding of pre-algebraic equivalence in Grade 5 and at generating theoretical contributions to unpack the conceptual and language-related learning pathways.

Methods of data gathering. Conducting design experiments in laboratory settings ise the central method for data gathering. So far, two design experiment cycles have been conducted with seven pairs of fifth graders (10-11 years old) with varying language proficiency. In total, 27.5 hours of video were recorded and partly transcribed. The case studies analyzed for this paper focus on two pairs chosen for illustrating contrasting phenomena for the research question in view.

Methods of data analysis. The transcripts are analyzed qualitatively in three steps: In Step 1, the students' utterances are coded according to their conceptions-in-action on equivalence of expressions, structurings of figures and the match between them. In Step 2, students' utterances are disentangled with respect to the activated language means in static and dynamic views. In Step 3, all inventoried language means and coded conceptions are systematized in order to generate hypotheses about typical learning pathways.

Empirical insights: Students' language in handling restructuring equivalence

Episode 1: Mira and Victoria's dynamic language

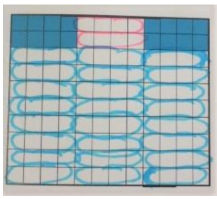
Episode 1 illustrates nicely the claim that restructuring equivalence (2→3) can provide a bridging approach between the static description equivalence (2) and the dynamic symbolic transformation equivalence (3).

Mira and Victoria (11 years old) work on the first step of the task in Figure 3, trying to construct meaning for substituting 12 by 3×4 (and later 8×3 by 24). They do not talk about factorizing or substituting in a symbolic representation, but use different language means for expressing Dilara's restructurings in a meaning-related way (Dilara is erroneously taken as a male by the girls):

15 Mira He has divided these rows here [*hints to the rows of 12 in the first figure*] into three.

…

17 Victoria He has broken this apart [*hints forward and backward between the expression 26 × 4 and the groups of 4 in their figure*]



18 Mira He has always divided into three.

19 Victoria Yes, thus, and these are three rows, then [*7 seconds break*].
Actually, for example, he has cut it here and then together [*hints to the area with eight groups of four*] and here, he has also cut it.

Mira and Victoria investigate the existing figures and seem to understand the undertaken restructurings. In Turns #17 and #19, Victoria refers to the concrete activities, but without making these references very explicit, a typical phenomenon in students' everyday language. Mira uses a more explicit and concise language ("divided into 3", #15, similarly in #18). Both students adopt a dynamic language of describing the changes as concrete activities, rather than a static language of expressing description equivalence.

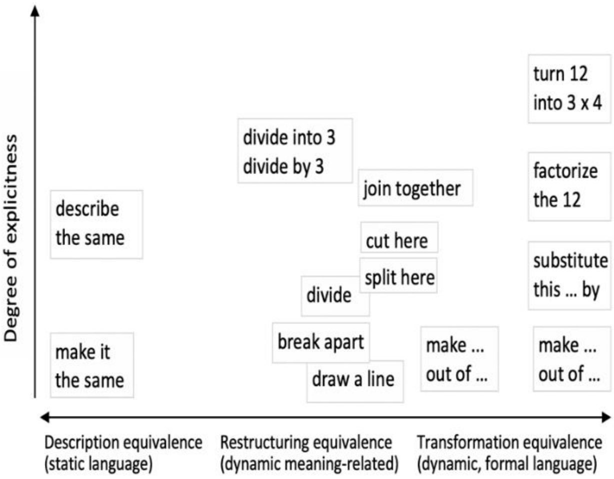


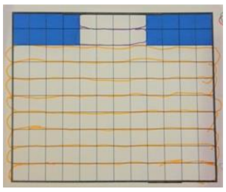
Figure 4: Ordering students' language

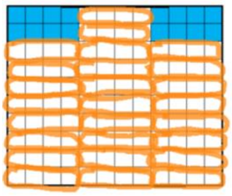
Beyond the case of Mira and Victoria, Figure 4 provides an overview of language means activated by several students to explain the equivalence of both expressions. Students' utterances substantially vary with respect to their degree of explicitness. Whereas some expressions could also be used for the formal transformation in the symbolic representations, nearly none of their words refer to a static view. Hence, the restructuring equivalence seems to have indeed the potential to bridge the gap between the meaning-related but static approach and the formal but dynamic approach. According to the current state of analysis for seven pairs of students, the task in Figure 3 seems to support students in developing a language for explaining the meanings of transformations and justify their match to the description equivalence.

Episode 2: Jessica's and Annica's struggle with seeing and expressing structures

Even if the general design decision of enriching the learning trajectory (by asking students to explain the restructurings in a meaning-related approach to equivalence) seems to have a great didactical potential for fostering students' learning pathways, many students still struggle with seeing and expressing structures. Episode 2 with Jessica and Annica can provide an insight into typical challenges which still occur on this pathway.

Jessica and Annica (11 years old) also work on the first step of the task in Figure 3, trying to construct meaning for substituting 8×12 by 24×4. They have mastered the first substep of factorizing 12 into 3×4 and now try to construct meanings of transforming $8 \times (3 \times 4)$ into $(8 \times 3) \times 4$. The girls struggle to articulate that they wonder why there are exactly 24 groups of four.





75 Teacher — What – What has Dilara changed, now? [*hints forward and backward between their first and second figure*]

76 Jessica — She has – ehh – divided the eight groups of 12 into 24 four-groups

77 Annica — groups of 4, […] nothing said

78 Teacher — Why are these **24** groups of 4?

79 Annica — Because, she has written this

…

82 Annica — One, two – wait – [*starts counting the groups of 4 in the second figure*] 1, 2, 3 ,4, 5, 6, 7, 8, [*tips on the figure without counting 9 aloud*] 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22. Oh, that does not fit!

Annica does not seem to recognize the underlying structure in the figures. She counts the groups of four without using any structure and does not come to the right result (#82) and does not notice her counting mistake. This activity is an indicator for her lack of seeing structure: she treats the figures in a quasi-empirical way. Jessica can go beyond Annica's quasi-empirical ideas:

97 Jessica — Let us see, in each row, there are three, aren't they? Then, you only need to count down here [*gestures down a column of fours*]

98 Annica — And then, times two?

99 Jessica — Eh, yes. That means, it goes eight times here [*hints to the first column of fours*]. This way down, then you must calculate 3×8.

…

102 Annica — Three times four. [*three seconds break*] Why three times eight?

103 Teacher — [*To Jessica*] Explain it. Annica has not yet understood. Explain again, longer and more precisely. Try to use the language of groups. Perhaps, this works.

104 Jessica — Ok. [*two seconds break*] How shall I explain that?

In contrast to Annica, Jessica discovers the underlying structure of 8×3 (#97 & # 99), but she struggles with finding a meaning-related language to explain her ideas (#104). The analysis of her struggle reveals empirical insights into the particular challenges of the transformation from $8 \times (3 \times$

4) to $(8 \times 3) \times 4$: When thinking in the graphical figure, students have to reinterpret the number of groups and the size of each group. In the first structuring, there are eight groups, each group consists of three groups of four. Therefore, the nested structure is in the *size* of the eight groups. Using the associative property means reinterpreting this structure: In $(8 \times 3) \times 4$, the *number* of groups is described as a structure of eight times three groups, while the size of each group is four.

Annica's challenge in explaining this step can be located in recognizing the structure. For Jessica, the challenge is to verbalize the restructuring. The language of grouping (which we use successfully to speak in meaning-related ways about multiplication) reaches limitations as the nested structure of three factors must be further explicated:

Symbolic expression	**English verbalization**	**German verbalization**
$8 \times (3 \times 4)$	eight groups of (three groups of four)	acht Gruppen mit drei 4ern
$(8 \times 3) \times 4$	(eight groups of three), consisting of four elements each	acht Dreier mit je vier Elementen

The more complicated nested mathematical structure results in a more nested language with unclear references. Therefore, the grammatical structure gets more complicated, too.

However, with the support of the teacher, Jessica overcomes the challenges:

108 Jessica	Look, here are always three, aren't they? In – three groups in each row [*hints to a row in 2nd figure*]
109 Annica	Yes.
110 Jessica	Though, then you simply have to – you
111 Annica	Oh, I am clever. Because there are three [*hints to a row with three fours*]. Now three, and if – well, you must calculate three times eight, because there are three [*hints to a row with three fours*] and there, eight downwards [*hints to a complete column*]
112 Jessica	Yes

Jessica overcomes the obstacle of explaining the structure of nested groups by excluding the *size* of four of each group in her utterance, she merely focuses on the *number* of groups. Thus, she can explain this by classifying it as number and size of the groups again. She marks the reinterpretation in her language when she signifies "groups" as new elements and highlights that by stressing "three groups" (#108). She also co-uses "groups" and "rows". By this she can distinguish the external structure of 8×3 (eight rows, three in each row) and the internal structure: each element is a group (of four) itself. Although Annica does not articulate the nested structure explicitly, her explanation with deictic means shows that she has discovered the new structure. Therefore, she finally understood the restructuring, even if the utterance in #111 might also hint to a further challenge in her multiplicative understanding, talking only about rows and columns but not groups of rows.

Whereas the associative property is often treated as trivial in the symbolic representation, its verbalization has shown to be a major challenge for students' conceptual understanding and for their participation in meaning-related discourse practices, particularly in justifying why a symbolic transformation is valid.

Discussion and Outlook

Main results of the case study

Summarizing the observations, we conclude that the empirical insights add plausibility to the assumption that the meaning-related transformational approach of restructuring equivalence can indeed reveal a bridging approach between the well-established approaches of static description equivalence and dynamic transformation equivalence. However, restructuring turns out to be much more challenging than anticipated, especially the meaning-related explanation of the associative property. Figure 5 summarizes the substeps and shows how they are expressed in the different representations.

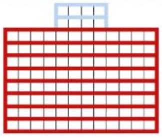
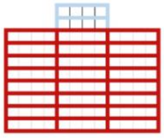
Trans-formation	Formal steps	Meaning-related verbalization of the steps	Graphical represen-tation of the steps	Meaning-related verbalization of graphical transformation
	$8 \times \mathbf{12}$ $+ 2 \times 4$	Eight rows of groups of 12 and two rows of groups of 4		
12 is splitted into 3×4				I cut every group of 12 into three groups of 4 (splitting)
	$8 \times (\mathbf{3 \times 4})$ $+ 2 \times 4$	Eight rows of (three groups of 4) and two rows of groups of 4 **Unfolded into:** Eight rows. And each row consists of three groups of 4		
Asso-ciative property				I see the structure differently: In the step above, I see each row and then count eight of them. In the step below, I first see the number of rows and the number of groups in a row. Then I see the group size.
	$(\mathbf{8 \times 3}) \times 4$ $+ 2 \times 4$	Eight rows of three, each of the three is a group of 4 and two rows of groups of 4 **Unfolded into:** Eight rows of three groups each. And each group is a group of 4.		

Figure 5: Summary of representations for the transformation steps

The first substep can easily be expressed as concrete actions in the graphical representation, the analysis shows that students have multiple language means for articulating it (cut, split, make … out of …, etc.), and some of these means can also be referred to the symbolic representation. The second substep, applying the associative property, does not correspond to restructuring in the graphical representation, but to another look at the same structure by flexibly re-unitizing. Here, the meaning-related language for the units require a further unfolding into a stepwise unitizing. Unfolding the unitizing is challenging, but (at least in the presented case of Annica and Jessica) it is successful in order to enable them to express the restructuring step (even if they do it in a less condensed way than presented in Figure 5).

Limitations of the study and future steps of Design Research

Even if the presented results of our Design Research study provide an interesting insight into the didactical potential and affordances of restructuring equivalence, future research is required to overcome the current methodological limitations of the study. The major limitation is its contextualization within the specific tasks in the teaching-learning arrangements. We can assume that other tasks for restructuring expressions graphically can extend the list of language demands. Additionally, we have not yet investigated whether the bridging function from description to transformation equivalence can be transferred to other contexts and situations.

So far, the presented potentials and affordances indicate specific challenges in the verbalization and the requirement of further unfolding the language for which not all students might be prepared. However, supporting the students' language unfolding processes might be the key to providing access to algebra for more students. In further design experiment cycles, we will try to overcome some of these limitations and thereby extend the scope of the results. We will specifically investigate in which way the bridging construct can really strengthen the students' abilities to see structures also in the formal representation of the symbolic expression.

Acknowledgment.

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Toys as dual pivots for imaginative play and mathematics

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Presently in Malta there is a move towards an informal pedagogy in kindergarten (ages 3 – 5), through which the teacher is expected to support the learning of mathematics by interacting with children as they play. Taking the role of teacher-researcher, I carried out a small scale study wherein I first observed children play on their own, then later interacted with them as they played. Drawing on Vygotsky (1967), Walkerdine (1988) and Sfard (2008), I considered 'learning mathematics' as a shift from a play discourse to a mathematical discourse. In particular, play items were considered to serve as 'dual pivots' for new meanings. In this paper I focus on one child, using the empirical data to articulate a theory with which one can discuss children's learning of mathematics in play settings.

Keywords: Early Years education; mathematics discourse; pivots; chains of signification

Introduction

The present direction in early childhood education in Malta is to move away from structured pedagogy. Teachers are now expected to act as 'responsive' adults, engaging in "authentic conversations" with children (Stacey, 2009, p.14), to support learning through play. In this regard, I wished to theorize about what it means to 'learn mathematics through play'. I wished to present a theoretical approach through which one might consider a child's play as 'mathematical' or otherwise. I also wished to investigate if children attended spontaneously to mathematical ideas when playing on their own, and then see how an adult's interaction (mine) might influence their talk/actions. I undertook a small scale study in which I took the role of teacher-researcher with four 4-year olds, and explored their talk and action in a context of play initiated by the children themselves. The play items presented were 'loose parts' e.g. pebbles, blocks, connecting camels, and so on. The main subject of the present paper is one of the children, Mario. The mathematics in focus is counting and foundational aspects of measurement (length); while I had taken *a priori* decision to focus on counting, due to the countable nature of the loose parts, the element of measurement emerged from Mario's play itself.

Although young children are sensitive to quantity, counting is socially constructed and can only be learnt through interaction with others (Montague-Smith, Cotton, Hansen, & Price, 2018). According to Gelman and Gallistel (1986) learning to count involves an appreciation of five principles of how to count, namely one-to-one, stable order and cardinality, order irrelevance and abstraction. Measurement involves assigning numbers to physical quantities such as length (Smith, 2013). As discussed by the Early Math Collaborative (2014), the idea that counting can be used to compare not only sets but also the length of objects is an important revelation for young children. Indeed, it is this non-obvious application of counting to the comparison of attributes of size that creates the concept of the repeated or iterated 'unit, which is a key idea of measuring length (Clements & Stephan, 2004).

Theoretical framework

Vygotsky (1967) believes that imaginary play had an enormous role in a child's development, in that it is a transitional stage through which a child can 'sever thought from an object' (p.12). The liberating of thought and meaning from their origin in the perceptual field provides the foundation for the further development of speech and its role in advanced forms of thinking. This liberation of meaning from object is facilitated by means of what Vygotsky referred to as 'pivots'. For example, if a child uses a stick as a horse, the stick acts as a pivot to transition of the child from a real situation to an imaginary one. Vygotsky (1978) further explains that when using a stick as a horse, the child retains the property of the thing, but changes its meaning. It is the meaning, in play, that now becomes the central point and objects are moved from a dominant to a subordinate position.

Hodge and Kress (1988) define discourses as sets of practices within which spoken or written language are embedded. Play situations provide contexts for the intersection of multiple discourses (Wagner and Andersson, 2018). For example, in the kindergarten classroom, one discourse may relate to how children are expected to play around a table (e.g. by taking turns), another discourse relates to how the student-adult relationship is articulated. The imaginative narrative of the play may be an enactment of discourses learnt outside school. For example, when children enact a birthday party, the discourse is that of the social practice of preparing cakes, decorating the environment and inviting guests. In this paper, I use the expression 'play discourse' in an encompassing sense in order to distinguish 'non-mathematical' discourse from mathematical discourse. Sfard (2008) identifies four characteristics that render a discourse 'mathematical': (1) Word use that is responsible for what the user is able to say about (and thus to see in) the world; (2) visual mediators, especially symbolic artefacts that are operated upon as part of the communication; (3) endorsed narratives, or established constructs such as definitions, proofs and theorems; (4) Routines, or repetitive patterns, e.g. regularities in forms of categorizing. Applying these characteristics to the kindergarten play context, I conceptualize the characteristics as follows.

Word use. The adult can use introduce terminology into talk with young children, while s/he notes whether the child uses words that might be considered 'mathematical'. While some words may easily be identified as 'mathematical' (e.g. *seven*, *subtract*), Walkerdine (1988) draws our attention to instances when the use of a word needs further reflection in order to draw conclusions about whether it is used in a mathematical sense or not. In her observations of parent-child conversations, Walkerdine noted how the 'everyday' word "*more*" was used in relation to the consumption of food and was contrasted with "*no more [food*]" or "*not as much [food]*". Thus, the relational pair in use was *more/no more*. On the other hand, in mathematical discourse, "*more*" is contrasted with "*less*", and this particular contrast sets up a different relation.

Endorsed narratives. Young children begin to gain experience of simple definitions and conventions, such as names of shapes and the appropriate way of writing numbers and so on. They might bring the ideas into their play, or an adult might make reference to established conventions.

Routines. For the young child, this might include the activity of counting to establish 'how many', measuring length using units, using instruments to establish weight, using simple graphical representations to show information, thinking of shapes in terms of their properties and so on.

Visual mediators. In a play situation, the adult may prompt the child to attend to, and to use, certain words or to engage in a narrative or routine that the adult knows to be established norms for the discipline. For example, suppose a child is building a dragon's tower with pebbles and is focusing on the smoothness and colour of the pebbles, creating a doorway for the dragon. At this stage, the pebbles - originally river related objects – serve as what Vygotsky (1967) calls ‘pivots’ for imaginative thought. If an adult draws the child’s attention to the *number* of pebbles being used, or their weight, or relative size, then the pebbles come to serve as a pivot for a mathematical meaning. Hence I consider that the pebbles in this play situation can serve as a *dual pivot.*

As explained by Walkerdine (1988), one way of achieving a shift is through chains of signification. During the parent-child conversations that she observed, Walkerdine noted the mother and child talking about inviting friends to a party. In the ensuing interaction, the friends were first represented by their names (*Mark, Michelle, Kirstie*); these names were then represented by fingers, and finally the fingers were referred to by numerals (“*one, two, three*”). In my study I aimed to investigate the process of shifting discourse in a particular play context, that of playing with ‘loose parts’. In my own study, I was interested in the children’s verbal interaction between themselves and /or with an adult (myself) in relation to play items - it was the combination of talk and action that realized the discourse.

Research Design and overview of children’s play

Wagner and Andersson (2018) note that children can develop proficiency in mathematical language practices by drawing on their repertoires of language practices in other discourses. In this study, my intention was to use the play context and conversation (‘play discourse’) as the basis for a shift to a mathematical discourse. The four children (pseudonyms) who participated in my study were Mario and Sarah who were Maltese, Dorina who was Hungarian and Ling who was Chinese. They were suggested by their teacher as being comfortable to play together; parental and child consent was also sought. The children were presented with various sets of loose parts to play as they wished. For Sessions 1 - 5, I took the children together and allowed them to play with minimal interaction from my part. For Sessions 6 – 8, I intended to pair the children since I planned to interact much more with them. However, Dorina stopped attending school and hence Mario, the subject of this paper, was taken alone, since I had already paired up Sarah and Ling. The language used during the study was English, which was the *lingua franca* of the class due to the large number of language groups represented in the class. All four children could understand English and they appeared comfortable using the language to communicate with me and with each other as they played. Throughout these sessions, a recurrent theme was birthday parties and cakes. The children also made crowns and necklaces with small coloured camels (of three sizes) that connect as shown in Figure 1. Mario was particularly drawn to the camels, and it is his play with these items that are the subject of this paper. (See Farrugia, forthcoming, for a discussion of Sarah and Ling’s discourses).

Figure 1. Connecting camels

The 25-minute sessions were video-recorded to allow for later transcription. Analysis was done by first viewing the videos and noting the children’s conversations as they played without my intervention, and then noting the development of the conversation and actions when I *did* interact with them. I kept Sfard’s (2008) discourse characteristics in mind in order to select excerpts for discussion. My main *a priori* interest was counting but, as a result of the direction of the play, I was also able to link counting with size, thus touching on the foundations of measurement.

Mario's play with connecting camels - from play to mathematical discourse

When the children played without my intervention, it was rare that an instance could be classified as 'mathematical discourse' in terms of the characteristics identified by Sfard (2008). Examples of Mario's talk/actions during these sessions are the following (the reader is reminded that English was not Mario's home language). During the first session, Mario was taken up playing with blocks and shells and an excerpt of his talk is: "Miss, I'm going to play with the shells. I need the 'carrot' one. It looks like a carrot. I don't know where it is." Mario's first use of the camels came in the second session, when he made a four-camel circle declaring to me "Miss, look! This is a crown". On the third day, I did not take the camels along with me, in order to vary the items presented, so there was no reference to them by Mario. On the fourth day, Mario scoffed the girls for making 'cakes' yet again and he preferred to play with the camels on the carpet; he used language to express what his creations 'looked' like, naming them as traffic, crowns and so on. For example:

> Miss, I'm going to make a crown (*he makes two rings of camels, then wears one on his head*). *He then takes it off, opens out both crowns).* Sarah, now I'll make a big one, so we'll have a big crown for us. (*He makes a circuit of 18 camels – see Figure 3*). Wow, a big crown! It's a bit [like] tracks … like, like a[n] escalator. Escalator! Escalator!

Figure 2. Mario's big crown / tracks / escalator

On the fifth day, I added various sets of numerals to the play items. Mario initially showed a brief interest in the set of large plastic numerals. However, he quickly turned his attention to the camels and made a crown, which he wore, and then a necklace. I concluded that up to this point in Mario's play, his talk appeared to focus mainly on describing his creations. In terms of Sfard's characteristics, there was only a brief instance of naming of some of the plastic numerals ("two, three …"), which can be considered as an endorsed narrative. No routines normally associated with mathematics were observed. There were a few words that I examined more closely in order to evaluate whether they formed part of a *mathematical* discourse but concluded that they did not, as explained below.

- *Big*. Mario attended to size three times, for example "Sarah, now I'll make a big one, so we'll have a big crown for us … wow, a big crown. Mario's use of the word *big* confirmed the suggestion by Montague-Smith et al. (2018) that children aged 3 to 4 years use one of three standards to judge size: perceptual (what the object looks or feels like), normative (comparing it with a mental image of what is 'normal') and functional (comparing it with what it is used for). The crowns were too large to sit on a head and would fall onto one's chest like a necklace; the party was 'big' in the sense that several cakes were to be made and placed on the couch. Hence, bigness was not quantified, nor was it compared - both key aspects of measurement.
- *More*. The word *more* was used once when the focus of play was on creating decorations ("I'm bringing more"). However, the play context allowed me to conclude that *more* was not contrasted with *less* , but appeared to be used in the sense of '*further items to add to the play'*.
- *Some*. On two occasions, Mario used the word '*some'* as an expression of the need for play items; one instance was when the girls had monopolized the shells ("Give me some!") and another was when he couldn't find the camels ("I'm going to need some camels. Where ARE

these camels?!"). Hence, *some* was used in the sense not of a specific quantity but in the sense of 'something to play with' implying that *some* was contrasted with *none* or *no play items*.

Given this picture of Mario's play, I concluded that the play items, in particular the toy camels, served only as pivots for imaginative play.

At the end of the fifth session, Mario joined up four camels and tried to wear the circuit as a bracelet. It was too small to pass over his hand and as he tried to force it, two camels disconnected. He tried to close the bracelet again. The end camels touched, but could not connect. He addressed me for help ("Miss, can you close this?"). I did manage – just – but the camels disconnected again.

MTF	I think you need an extra one. I think you need another one.
Mario	(*The string of camels falls into a box. Mario picks up another camel and adds it to the string, thus closing the bracelet. He wears it around his wrist. Then he makes another similar bracelet, by using camels of similar colours. He wears the second bracelet on the other wrist*). Miss! Have a look at my bracelets!
MTF	Now nice! They're the same.

My contribution to the discourse here was to introduce the words *extra/another one* thus introducing the idea of *more* [implied] in the sense of an exact quantity: *one* more. By following my suggestion, Mario managed to create a bracelet that fitted him. Following this, he made another bracelet of similar length. Although at this point it appeared that Mario's attention was on creating a similar-coloured bracelet, this episode formed a link to further interaction on the following days as will be explained.

During Sessions 6-8, Mario was taken alone and I interacted with him with the intention of directing his attention to aspects of quantity. I started off by asking Mario to make a bracelet again, with the excuse that the photo taken the previous day had not turned out well. Thus, I went along with the child's interests in order to address cardinality and introduce the notion of unit iteration.

<u>Establishing cardinality</u>. At several points in the three sessions, I asked Mario "*How many*"-type questions, for example: "*How many camels did you use to make it*?" Each time, Mario counted willingly, albeit not always applying accurately the one-to-one and stable-order principles (Gelman & Gallistel, 1986). Furthermore, he did not seem to know all numbers. A typical example of his counting beyond ten was: "… eleven, twelve, fourteen, sixteen, nineteen, twenty-two, twenty-three, twenty-four, twenty-one, twenty-two, twenty-zero, twenty!" *Twenty* was often the last number Mario stated, and a number he used when he wished to indicate a large quantity. Despite inaccuracies, it was clear that Mario was aware of the social purpose of counting to establish quantity ("I can count them very easy!" [sic]). Through my use of the expression *how many*, and Mario's socially appropriate response to it, the play discourse shifted to a mathematical one as indicated below:

- *Words*: How many? One, two … twenty etc., count.
- *Endorsed narrative*: the social activity to count to establish quantity
- *Routine*: the actual recitation of numbers by tagging items by pointing/touching (a routine still being developed by Mario)

As a result of these aspects now included in the interaction, the camels' meaning changed. Originally they had been interpreted as coloured components of a crown, necklace or decoration, and the action carried out was connecting them mouth to tail. Hence, the items served as a pivot for the imaginative narrative. Then the camels came to serve as a visual mediator for quantification; the action carried out was tagging with the pointer finger and referring to quantity: "one, two, three…twenty". Here the items served as a pivot for mathematical abstraction. Hence, the play items served as 'dual pivots'.

Linking number with size (unit iteration). Following the incident when Mario asked me to help him connect his bracelet, the ensuing play in the following days offered an opportunity to link quantity with size. For example, the following excerpt is taken from Session 6.

(Mario has just made a small circuit of camels).

MTF So how many camels did you have to use to make it?

Mario (*Mario counts the camels touching one camel at a time*). "One, two, three, four, five, SIX".

MTF Six camels! I wonder if you can make it a bit bigger, to fit ME.

Mario Yes. (*Starts connecting camels*)

MTF What shall we do to make it a bit bigger?

Mario MORE camels. We can make different colours (*continues connecting*)…

MTF That's fine. Do you think that will fit me? How many camels did you have to use to make one for me?

Mario (*Counts, touching one camel at a time*). One, two, three, four, five, six, seven, EIGHT.

MTF Eight! Let's see! Does it fit me? (*Puts out her wrist*).

Mario *(He slides the bracelet over MTF's hand. The bracelet hangs loose over MTF's wrist).*

MTF Oh! (*They both laugh, then Mario takes off the bracelet*).

van Oers' (2010) states that if a spontaneous, action/utterance by a child is be taken as a cultural form [e.g. mathematics] by an adult, who reacts accordingly, then in time, through participation in such interactions, the child him/herself may acknowledge the [mathematical] meaning of the adult reaction, and finally, of his/her own actions as well. Indeed, over the three sessions in which I interacted with him, Mario himself began to relate quantity with size. For example, towards the end of the 7th session Mario was making a long circuit of camels on the carpet under the table and I was preparing short strings of camels to save Mario some time.

MTF I'll join them up and you can put them around the table...Oh! I don't know how many. How many shall I join up?

Mario Many.

MTF D' you know how many?

Mario Yes.

MTF Hmm… maybe five?

Mario No, twenty! (*By now Mario has a line of camels going around two legs of the play table*).

MTF Ah, twenty.

Mario	Twenty’s a LOT.
MTF	Here. (*Hands Mario a string of camels – seven or eight, not clear from video*). Can you put those? Do you think we’ve got twenty up to now?
Mario	No. It needs longer.
MTF	OK.
Mario	(*He has taken the new string of camels and goes to attach it to the developing line on the carpet*). Look, we have to make it a bit longer.
MTF	I wonder if I need a hundred.
Mario	Yes, we need big of hundred…up to orange.
MTF	Oh how nice. (*Passes another string of camels*). Here you are, look, what a long one!
Mario	(*Takes the string*). Now I’ll make the red one up to… up to the orange. Almost there, Miss.

In terms of characteristics for mathematical discourse

- *Words*: make bigger, fit, more, twenty’s a lot, needs longer, bit longer, big of hundred, up to, one, two …
- *Endorsed narrative*: quantifying length (unit iteration for measurement)
- *Routine*: recitation of number sequence

Once again the camels served as pivots twice over – firstly the line of camels was a decoration for the party, then they were counted and hence given a new, mathematical meaning. Furthermore, the similar toys attached in a line now served as visual mediators for the notion of unit iteration which, as stated by Clements and Stephan (2004) is a key idea of measuring length.

Conclusion

An informal pedagogy in Early Childhood settings presents a challenge with regard to how mathematics is actually learnt through play. In my study, I observed that children did not engage spontaneously in talk and action that might be described as ‘mathematical’, but it was evident that the ‘responsive’ adult can have a significant impact on shifting the discourse from a ‘play’ to a ‘mathematical’ discourse. The chain of signification occurring is shown in Figure 3.

Camels are components of a necklace or party decoration → *More* camels (contrasted with no camels) → *Five* camels (cardinality) → *More to make bigger* / *Hundred up to* [the] orange / it needs *longer* / *eight* camels *to fit* me (linking number to length)

Figure 3. Chains of signification

Affording the play items a new interpretation implies that the items serve as dual pivots – first for imaginary play and then for mathematics. My key observation is that through the introduction of words, narrative and endorsed routines, the play items are rendered visual mediators for mathematics. Thus, I suggest that the shift in discourse to a mathematical one centres on establishing the play items as visual mediators, that is, on introducing children to the key mathematical aspect of representation.

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Decoding and discussing part-whole relationships in probability area models: The role of meaning-related language

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Although the epistemic role of academic language for developing conceptual understanding has been shown for some mathematical concepts, more topic-specific research is required for specifying the academic language demands for students' meaning making processes in different mathematical topics. The design research study presented in this paper contributes to this research agenda by investigating tenth graders' learning pathways towards conceptual understanding of conditional probabilities in area models. The qualitative analysis of a design experiment shows that students' processes of decoding and discussing meanings of different probability statements require a systematic scaffold for explicating the underlying part-whole relationships: the area model is helpful, but not sufficient.

Keywords: topic-specific language demands, probability area model, connecting multiple representations, explicating relationships

Background: Epistemic role of language for the part-whole relationship in the mathematical topic of conditional probabilities

Epistemic role of language in learning to discuss structural challenges

Since Vygotsky's early writings about thinking and speaking (1934/1962), the epistemic role of language as a thinking tool has been emphasized by many researchers (e.g., Pimm, 1987): Language is not only relevant in mathematics classrooms as a means for communication but also for the mental processes of knowledge construction. Whereas procedural knowledge can be developed also with restricted language resources, the epistemic role of a school academic language has been shown to be particularly crucial for constructing conceptual understanding, e.g. for functional relationships (Prediger & Zindel, 2017). This and other studies indicate that a major challenge seems to be the articulation of abstract relationships.

These general insights into the epistemic role of academic language for constructing conceptual understanding have immediate impacts for instructional designs: If we want to support language learners' development of conceptual understanding, we should identify those language demands which are crucial in this conceptual learning process and organize systematic learning opportunities so that all students can participate in discourse practices of explaining meanings (Prediger & Zindel, 2017). Although some aspects of this identification are generic for all mathematical topics, the detailed specification for a specific mathematical topic also requires empirical research (ibid.). Thus, the topic-specific identification of relevant meaning-related language demands has become a substantial step in the topic-specific design research agenda of the MuM research group in Dortmund. In this paper, we contribute to this research agenda for the mathematical concept of conditional probabilities and the underlying part-whole relationship in Grade 10/11.

Topic in view: Conditional probability and the probability area model

Conditional probability and Bayes' rule have been shown to raise difficulties for many students. Already Shaughnessy's (1992) research overview summarizes typical misconceptions and challenges, e.g., determining the conditioning event or the confusion between a conditional and its inverse or the challenge to distinguish conjunctive and conditional probabilities (as in the example in Figure 1). Within the succeeding decades, researchers in stochastics education developed and investigated approaches for fostering students' reasoning, especially by making the structure of nested sets more explicit for students (Sloman, Over, Slovak, & Stibel, 2003), e.g., in an area model as in Figure 1. Based on this research tradition, Böcherer-Linder and Eichler (2017) investigated which representations best support students' access to complex part-whole relationships. By comparing students' success rates of task completion for different representations, the authors showed that the area model and the tree diagram provide similar support for students in simple cases, but the area model is much more helpful for when horizontal sub set relations (i.e. Bayes' rule) are addressed. Other researchers have investigated different complexities on the language side; for instance, Watson and Moritz (2002) found that students master statements about natural frequencies better than statements about probabilities and track this back to the complexity of the grammatical structure of the statements, particularly the phrase "out of" proved to be more accessible.

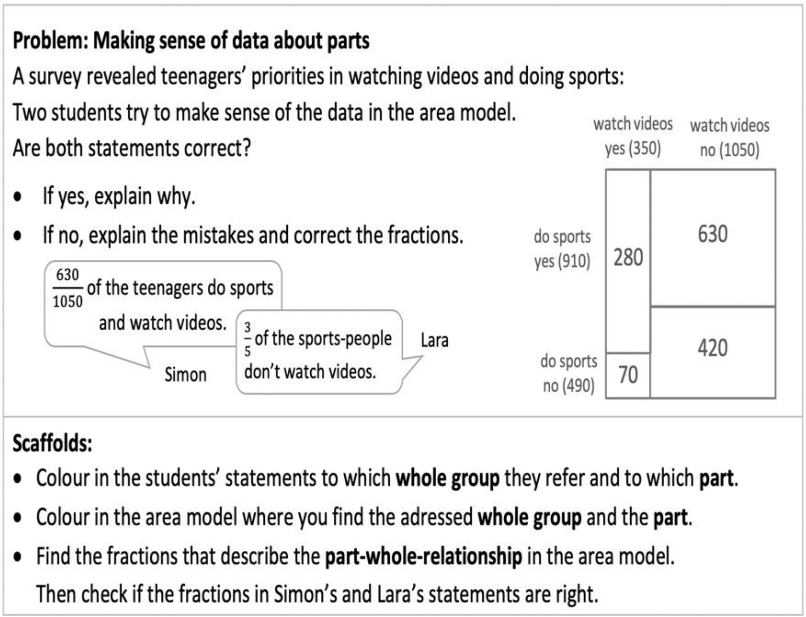

Figure 1: Exemplary problem on decoding conditional probabilities (or fraction statements) in area models

From these lines of research, we conclude that decoding the part-whole information can raise different complexities for different formulations and grammatical structures. Decoding the part-whole-information also seems to be challenging in written texts that contain no probability but only fraction statements such as in Figure 1. Whereas the existing literature suggests that decoding the nested structure of sets, the parts and the wholes, seem to be the crucial step in mastering conditional probabilities, this paper shows that the *part-whole relationship itself* is equally important for the process, not only the part and the whole and their nested structure. From the existing literature, we draw the need to visualize the parts and wholes, and will show that language is required in its epistemic role to focus the part-whole relationship besides the sets themselves. While existing studies have mainly focused on the comparison of students' performance in different tasks or items and indicated what kind of task presentation could increase students' access, we focus on developing students' conceptual understanding and problem-solving strategies to master also complicated texts. In this design research perspective, the scaffolds printed in Figure 1 do not serve as the permanent support in each problem but as a strategy to be internalized for students' independent access.

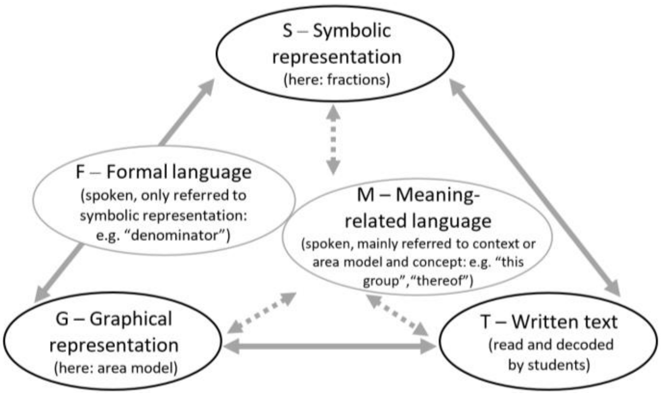

Figure 2: Connecting multiple representations, with formal and meaning-related language

This additional design ambition can build upon the previous research in stochastic education which resonates with the *general design principle of connecting multiple representations and language registers*. The principle has proven successful for developing language learners' conceptual understanding (Prediger & Zindel, 2017) and is applied in our design. Figure 2 shows the representations in view for this article for the specific topic of conditional probability. We choose the area model with natural frequencies as the graphical representation that students are supposed to connect to the written text of the problem and the symbolic-numerical representation of fractions. Formal language means are those which refer to the symbolic representation. Meaning-related language means are all utterances that refer to the meaning as part-whole relationship (e.g., "group of sports-people", "part", "thereof"). The empirical section of this paper will show that the distinction between formal and meaning-related language is more crucial here than the registers because students require the meaning-related language as the epistemic tool for promoting students' conceptual understanding for the distinction of conditional and conjunctive probabilities, prepared by fraction statements such as in Figure 1.

Research gap: Language demands in students' learning pathways towards conceptual understanding of the part-whole relationship

Although the area model with natural frequencies has already been identified as the most accessible graphical representation for simplifying students' access to complex fraction statements in conditional probabilities, little is known on how to foster students' ability to decode the structures independently, and on language demands that occur on this pathway. To explore the role of language as an epistemic tool in this process, the design research pursues the following research question: *Which academic language demands do students meet while developing conceptual understanding for conditional probability in the area model?* Once these language demands are identified, language support can be designed and provided throughout the conceptual learning trajectory.

Methodological framework: Data gathering and qualitative data analysis

The methodological framework chosen for this project is design research because it combines two aims: designing a teaching-learning arrangement and developing an empirically grounded local theory of students' learning pathways and the demands they meet (Gravemeijer & Cobb, 2006). For data gathering, four design experiment series were conducted in four classes in Grades 10 or 11 with 94 students between 15 and 18 years old. All sessions were video-recorded, a selection was transcribed. The episode presented in the empirical part of this paper stems from Cycle 1 in which a class of 24 students worked on the teaching unit for 12 sessions of 60 to 75 minutes each. This episode from Cycle 1 was chosen to show best the epistemic role of meaning-related language.

In order to identify students' language demands while learning to decode written texts and while discussing part-whole relationships, the transcripts were qualitatively analyzed in three steps of deductive coding: In Step 1, each of students' utterance was coded when addressing the part, the whole or the part-whole relationship. Step 2 identified in which representations the components were addressed (with the letter codes S, F, M, G, T according to Figure 2). Correctly drawn connections are symbolized with continuous black lines, wrong connections with crossed lines, and not addressed connections are marked in grey. For distinguishing students' formal and meaning-related language, the interpreter analyzed how the students articulated the elements and their relation. All utterances referring only to the symbolic representation ("numerator", "the number above", articulation of numerator by rephrasing labels "350 are those who watch videos") were subsumed as formal language F. All utterances expressing references to the context of the data ("sports-people", "this group here") or the area model ("this rectangle") and at the same time referring to the underlying meaning-related concept/idea of fraction as part-of-the-whole ("thereof") were coded as meaning-related language M. Many utterances were double coded as F and M when referring to both. Interrater coherence was assured by consensus between two researchers. In Step 3, the codings of each sequence were graphically summarized (e.g., in Figure 3 and 4). The graphical summaries reveal what is made explicit or left unattended in a sequence. Hereby, we identify the function of the area model and the two languages and identify language obstacles in developing conceptual understanding.

Empirical insights into decoding and discussion processes

The presented episode shows the whole class discussion of a Grade 10 classroom after the students' group work on the problem in Figure 1.

Sequence 1: Celina's separate look on part and whole. Sequence 1 starts when Celina presents her group's ideas about the wrong fraction statement of the fictitious textbook student Simon.

10 a	Celina	The first statement was, um, that $\frac{630}{1050}$ of the teenagers do sports and – at that – watch videos	
10b		And, starting with explaining the fraction, this statement is wrong, actually, because ... Um, first, um, the 630 you can see here [*hints to the number 630 in the area model*]. They do not watch videos and do sports [*hints to the labels of the area*]. And, sort of, that does not fit, first, because, um, sort of, because it is about, if teenagers watch videos [*hints to Lisa's statement*]	**Part:** S-G-F-T (numerator 630)
10c		And, um, to the second part of the fraction. These are, though, the teenagers who – sort of – do not watch videos, in total.	**Whole:** S-F-T (denominator)
10d		And the corrected fraction would be, um, I noted $\frac{280}{350}$. Because, um, sort of, you should colour the important things in the text, and I have coloured "do sports" and "watch videos" [*hints to both text fragments*]. And, I wrote that 280 people watch videos and – at that – do sports [*hints to the upper left rectangle 280 and their labels*] And – sort of – this 350, these are all who watch videos then [*hints to both left rectangles*]	**PWR**: S-F **Part**: T-S-F-G (correctly identified) **Whole**: T-S-F-G (false-ly identified in T)

10e		That means, 350 are those who watch videos and the 280 those, those who – at that, even also - do sports [*hints the 280 in upper left rectangle*]. So I got $\frac{280}{350}$.	**Whole**: S-G **Part**: S-G **PWR:** S

Celina starts by explaining the fraction (focus on symbolic representation S) and by locating the numerator in the graphical representation (S-G-F-T for numerator in Turn #10a/b, see Figure 3). She correctly identifies that Simon connects the numerator to the wrong group in the text and the graphical model. From #10c on, she does not explicitly refer to the graphical model but only to its labels which also appear in the text (S-F-T for denominator in #10c). She corrects the fraction (addressing the part-whole relationship solely in the symbolic representation with the formal word fraction S-F in #10d), but does not verbalize the part-whole relationship (abbreviated PWR) beyond the word "at that" ("dazu" in German original). For explaining her choice of the fraction, she only focuses the part and the whole separately, but not the PWR. Her main concern is to express the nested structure of the conditions "watch videos" and "do sports and watch video", but she neither expresses the part-whole relationship between the two, nor she realizes that the whole in view should be all teenagers. Thereby, she confuses the conjunctive with the conditional fraction statement.

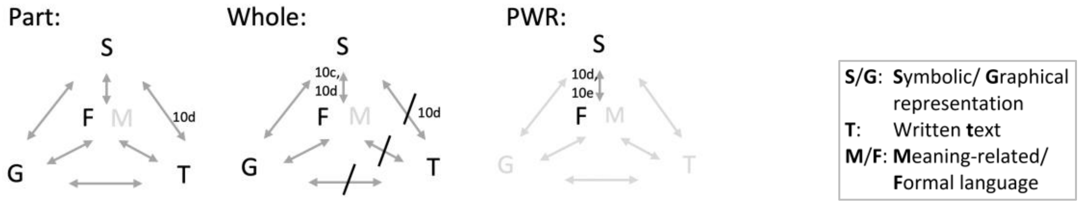

Figure 3: Graphical summary of the analysis of Sequence 1: Celina's separate focus on part and whole

Figure 3 gives the graphical summary of her focus and expressions. The PWR is hardly addressed and this is characteristic of the sequences. Her own meaning-related language beyond rephrasing parts of the texts or labels is not activated, for none of the part, whole, or the PWR.

Sequence 2: Meaning-related language focusing on part and whole. Sequence 2 immediately follows Sequence 1 when Piotr and Efkan react to Celina's statements.

12	Efkan	Well, I first had written $\frac{280}{1400}$. Because, there are 1400 students, altogether, and 280 – as Celine has already said – do sports and watch videos. But, um, I still go hold on with my statement.	**PWR**: S **Whole**: S-M ("altogether") **Part:** S-T
19	Piotr	I have had another idea, that is: 630 of 1050 of the teenagers do sports, but do not watch videos. Would do this there. There are 630 who, um, do sports but don't watch videos. And there, we have "do watch videos". I would only correct the statement, "but do NOT watch videos".	**Part:** S-F-T

Piotr (in #19) seems to continue Celina's pathway by starting with the fraction and only correcting the reference in the written text, but still explains a conditional fraction statement rather than a conjunctive one. In contrast, Efkan focuses the correction of the whole and succeeds in rephrasing "of the teenagers" in his own, meaning-related words ("students, altogether", #12). He identifies the relevant groups, but still does not verbalize the part-whole relationship (Figure 4).

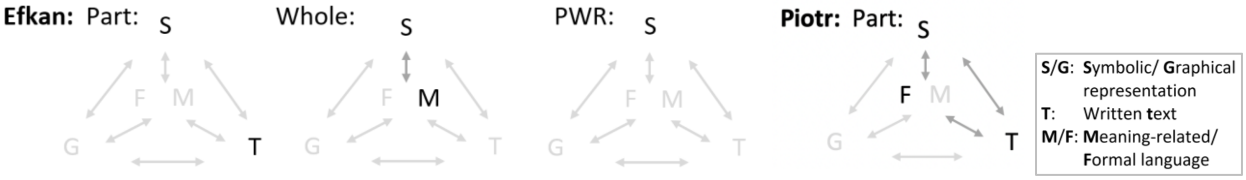

Figure 4: Graphical summary of the analysis of Sequence 2: Efkan and Piotr focus differently

Sequence 3: Identification of whole by focusing on syntax and explicating the relation. A little later, Tom presents his group work results for the second statement (of Lara).

55a	Tom	Yes, well, 3/5 is just this here, simplified. This would be 3/5 and this 2/5 [*hints to the lower right rectangle*]. And this would be the denominator, in this case [*hints to the label "watch videos no"*] and these both the numerators [*hints to the two right rectangles*].	**PWR**: S-vaguely G **Part**: G-S-F **Whole**: G-S-F
55b		And, um, that, well, you just have only this group here, considered [*hints to the label "watch videos no"*], so those who do watch no videos, but not those who watch videos. And sport, so all the persons [*hints to the left rectangles*], these are not considered in the denominator.	**Whole**: G-M-S-F
55c		And, um, it says "3/5 of the sports-people" [*hints to the text*], that means, you would need to consider them here [*hints to two upper rectangles*] and not these here [*hints to two right rectangles*]. That means, the denominator should actually be 910 of the sports-people and THEREOF, then 630, the numerator [*hints to the upper right rectangle*].	**Whole:** T-G-S-F-M **Part**: S-F-G **PWR:** M-G

Tom starts with connecting the symbolic and graphical representation for the part and the whole. His connection to the graphical area model is realized only by deictic means, e.g., "this would be" in #55a. In contrast to Celina, Tom does not only verbalize the fraction, instead, he articulates the whole in own meaning-related words (in #55b) and connects them to the symbols. This allows him to connect formal and meaning-related language. With "thereof", he also offers an explicit articulation of the part-whole relationship, supported by gestures on first the whole and then the part in a connection.

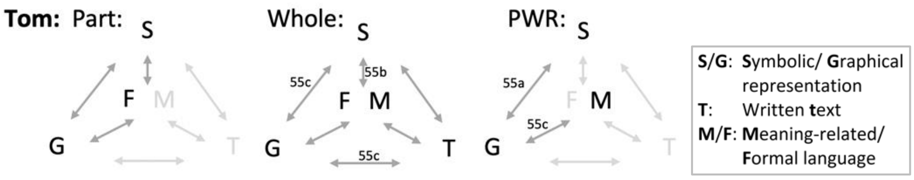

Figure 5: Graphical summary of the analysis of Sequence 3: Tom's full reconstruction of the whole

Discussion and outlook

The comparison of the three graphical summaries reveals typical profiles which we have also found in the analysis of many other episodes: Celina picks two conditions out of the text and interprets them as standing for numerator and denominator of the fraction. Thus she connects symbolic, graphical and textual representations for part and whole without searching for textual indicators of the part-whole relationship or for signifying the whole (part, whole: S-T-G-F, PWR: S-F). Hence, she adopts a direct translation strategy which often occurs in students' comprehension processes and frequently results in misconceptions. Tom does not only pick out conditions, he also pays attention to the grammatical structure and the relationship between part and whole. Accordingly, he identifies the correct whole and justifies how to identity the whole in the text by the grammatical features of genitive construction (translated to English by “of”). These mental processes are reflected by students’ use: Celina uses no meaning-related language besides the phrases given in the text and the labels in the area model (which are therefore not counted as own meaning-related language). This hinders her building an adequate model of the situation. Tom, in contrast, can articulate part (S-G-F), whole (S-G-FM-T) and the part-whole relationship (S-M-G) which gives him access to decoding and negotiating the adequate part-whole relationship.

Like Celina, Piotr also only concentrates on correcting the part without recognizing the wrongly decoded whole and without meaning-related verbalization of the part-whole relationship. In contrast, Efkan can recognize the wrongly decoded whole (S-M for whole; S-T for part) and expresses the whole using meaning-related language, but does not explicate the part-whole relationship. In his articulations, it becomes evident that the fact that the PWR has no direct counterpart in the graphical representation (being the relation between two rectangle areas) might be part of his challenge.

Especially the comparison between Tom and Celina shows: For distinguishing fraction statements with different structures and different referent wholes, it is crucial to articulate not only the part and the whole, but also the part-whole relationship in order to be able to express it, reason and negotiate about it. As long as the students do not find a language for expressing the part-whole relationship, the direct translation strategy for the separate conditions has a certain instrumental rationality. However, the empirical insights into these and also other episodes repeatedly show the high relevance of language, here in its epistemic function of allowing students to internally construct adequate models of the situations and models for distinguishing conditional and conjunctive fraction statements.

The observation that some students (as Piotr in #19) cannot follow other students’ reasoning (such as Efkan’s in #12) strengthens the need to support students in developing their meaning-related language for identifying parts, wholes, and particularly part-whole relationships. The area model alone provides a helpful scaffold for the part and the whole, but not really for the part-whole relationship. Hence, language – in these cases – appears as crucial not only for the textual representation of the situations but mainly for students’ thinking processes during decoding and discussing. We therefore deem it necessary to compile an inventory of all phrases that students used to express the rarely articulated part-whole relationship. Including observations from other episodes of the data material, Figure 6 summarizes options of increasing density for articulating the part-whole relationships in meaning-related ways. These kinds of phrases were identified as those language means for which most students require focused language-learning opportunities.

Articulation of PWR with increasing density	Two-step phrase, with “thereof”	Two-step phrase with “out of them”	Connective “(out) of”	Genitive construction *(not translatable)*	Condensed term “share”
Examples	There are 91 sports-people altogether and thereof 63 don’t watch videos.	There are 54 female persons and 45 out of them do sports.	Out of the girls 45 do sports. $\frac{52}{75}$ of the girls do sports.	(*German original:* 52 der 75 Mädchen …)	Share of sportsmen among the boys is $\frac{20}{90}$.

Figure 6: Meaning-related phrases for articulating the part-whole relationship in the probability area model: Inventories from several design experiments (literally translated from German)

Of course, the empirical findings of the case study must be interpreted with respect to their methodological limitations, most importantly (a) being bound to the specific teaching-learning arrangement in view, (b) the small sample size, and (c) the missing language support in the currently investigated Design Experiment Cycle 1. Future research should extend these boundaries.

However, the empirical findings have substantially informed the re-design of the teaching-learning arrangement which now includes the structural scaffolds as already printed in Figure 1 (which explicitly ask for the part-whole relationship and not only for part and whole separately) and a lexical work on meaning-related phrases for expressing it in the different contexts has been included by providing meaning-related language frames to verbalize the graphically given part-whole relationships. Once again, the study could replicate the high relevance of the epistemic role of language, especially for expressing and thinking about abstract relationships. Whereas concrete objects and sets can be visualized easily, understanding the abstract relationships requires their verbalization, and this phenomenon has now been reconstructed for the part-whole relationship in probability area models as well. Rather than simplifying all texts until the part, the whole and the PWR are easily decodable (as many of the existing studies in the field of stochastics education might suggest or be misinterpreted to suggest), the ultimate aim of language-responsive classrooms must be to equip all students with the language means to master also difficult demands. Before reaching this aim, further research is required.

Acknowledgement

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Language elements helping to see enlargement as a multiplicative situation

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This paper presents an analysis of Grade 5 pupils' work with similarity, based on an adaptation of Brousseau's classical puzzle. My main interest is in investigating the pupils' language repertoire and mathematical strategies for expressing enlargement in a situation involving similar shapes. A crucial point is to identify elements of the language use that contribute to seeing enlargement to preserve similarity as a multiplicative situation.

Keywords: Enlargement, similarity, multiplicative structures, Theory of Didactical Situations

Introduction

A large part of the mathematics taught in compulsory school in Norway and elsewhere can be related to multiplicative structures (Vergnaud, 1983). The span reaches from the early work with multiplication, usually seen as repeated addition, in the lower grades to more advanced topics such as proportionality, similarity, combinatorics and growth rate in the higher grades. In the project Language Use and Development in the Mathematics Classroom (LaUDiM) — an intervention study carried out in collaboration between researchers at the Norwegian University of Science and Technology and two local primary schools, we have acknowledged the central role of multiplicative structures by making this a recurring theme over several years, from basic models for multiplication and division, to combinatorial problems, and, as in this paper, where the pupils in Grade 5 work with proportional growth in a task which can be seen as an introduction to similarity.

It is widely acknowledged that an important achievement in the development of pupils' numerical thinking is the transition from additive to multiplicative (proportional) thinking and that humans acquire additive reasoning (AR) before proportional reasoning (PR) (Gläser & Riegler, 2015). This may lead to an application of AR in situations where PR is appropriate but there is also evidence to show that when pupils have become familiar with PR, they tend to apply PR in situations where this is *not* appropriate (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2012; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). These studies seem to indicate that language use may play an important role for pupils' success when choosing the correct type of reasoning.

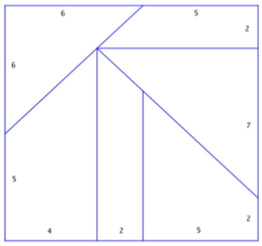
By zooming in on discussions between pupils, and pupils and teacher, I will identify language elements that play a role when pupils development their reasoning from additive to multiplicative (proportional).

The task given to the pupils

The task, presented in Figure 1, is an adaptation of the puzzle described in (Brousseau, 1997, p. 177).

The pieces in the puzzle to the right shall be **enlarged** to a puzzle having the same shape as the one shown to the right. This should be done in such a way that the length which is **4 cm** in the original puzzle becomes **6 cm** in the enlarged puzzle.

1) Each member of the group shall enlarge two pieces.
2) Put the pieces together to an enlarged puzzle.



In Brousseau's example, the given enlargement is from 4 cm to 7 cm. We chose the measures 4 cm and 6 cm in order to make calculations simpler. A similar situation is reported in Erath (2019), with the enlargement factor 7/4 but with slightly different, and fewer pieces.

Multiplicative structures

Greer (1992) presented a classification of situations modelled by multiplication and division. A shortened version of his table is given below (Greer, 1992, p. 280):

- Equal groups and equal measures
- Rate
- Measure conversion
- Multiplicative comparison and multiplicative change
- Part/whole
- Cartesian products
- Rectangular area

The situation in the task in Figure 1, fits with Greer's class multiplicative change, as in his example: "A piece of elastic 4.2 meters long can be stretched to 13.9 meters. By what factor is it lengthened?" (Greer, 1992, p. 280). The crucial point in the task in Figure 1 is to identify the enlargement factor, 1.5, although it is not explicitly said that one is looking for an enlargement *factor*.

Vergnaud sees multiplication, division, fraction, ratio, proportion, similarity, linear functions, and other concepts as belonging to one *conceptual field* and presents three classes of multiplicative structures (Vergnaud, 1983, 2009): *Isomorphy of measures*, *Product of measures*, and *Multiple proportions*. Isomorphy of measures is described as a direct proportion between two measure spaces, M_1 and M_2, as illustrated in Table 1.

M_1	M_2
1	a
b	x

Table 1: Isomorphy of measures

The terms *scalar operator* and *function operator* are used to denote the numbers b and a, respectively. The scalar operator operates *within* one measure space, whereas the function operator operates *between* two measure spaces. This model contains both equal groups, multiplicative comparison and change, as well as rate in Greer's list. Problems fitting into Vergnaud's model *Isomorphy of measures* are of a type referred to by Behr, Harel, Post and Lesh as "the rule of three problems" (1992, p. 297). One can construct problems that on the surface look like rule-of-three-problems, but that not are multiplicative. Several authors have studied and compared pupils' reasoning in such problems

(Fernández et al., 2012; Van Dooren et al., 2005, 2010). Van Dooren et al. (2010) report that there is an increase in the use of PR when AR is appropriate as pupils get older but for young pupils the inappropriate use of AR is dominating. This seems to indicate that when the ability for PR is established it tends to be used also in situations where it is not appropriate. An example of a problem requiring AR, which has the structure of a rule-of-three-problem, is the following:

> Ellen and Kim are running around a track. **They run equally fast but Ellen started later.** When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run? (Van Dooren et al., 2010, p. 368, my emphasis)

In the study by Van Dooren et al. (2010) the percentage of pupils that solved additive problems of this type using PR increased from 17 % in Grade 3 to 67.9 % in Grade 6. The language cue indicating that this is an AR task is "[t]hey run equally fast but Ellen started later". In the task in Figure 1, the cue indicating PR is that the new puzzle should have the same shape as the original one. In addition, the physical fitting together of the pieces was intended to support the choice of PR. As will be seen, this did not suffice, and only after additional language support, elements of PR started to develop. My interest in this paper is to investigate the nature of these elements of language support.

Method

Throughout the whole LaUDiM project, classroom sessions have been designed based on principles from the *Theory of Didactical Situations (TDS)* (Brousseau, 1997). A central feature of TDS is to create an *adidactical situation,* a situation in which the pupils take a mathematical problem as their own and try to solve it without the teacher's guidance and without trying to interpret the teacher's intention with it. For an adidactical situation to be successful, it is important that the pupils get relevant feedback from the *milieu* (Brousseau, 1997) so that they have a chance to know when the task is solved correctly without being told so by the teacher. For this to happen, the task must contain an inner logic, guiding the pupils to the correct solution without the teacher's intervention. To create tasks with such an inner logic has proved to be very challenging (Rønning & Strømskag, 2017). In the task discussed here, the feedback from the milieu is that if the pieces are not enlarged in the correct way, they will not fit together as a puzzle. It is then expected that this feedback will strengthen the connection between the language cue "same shape" and proportional reasoning.

The task was provided for the pupils to work with in groups of three-four, after a brief introduction by the teacher. After some time, the teacher interrupted and gave some additional information, and the pupils continued their work. In the next mathematics lesson, three days later, the lesson started with the teacher giving an introduction where she drew on what she had observed on the first day. During this whole class session, the correct solution of the task was established, with active participation from the pupils. All whole class situations are videotaped, as well as the group work of three groups on day 1 and five groups on day 2. For this paper I follow one group of three pupils, Frances, George and Mary, throughout their work on day 1. In addition, I base my analysis on the teacher's intervention in whole class situations as well as on written work collected from the pupils. In the whole class session on day 2 it became evident that she had observed something interesting in the group with Frank, Roger, Nora and Brenda, and to look closely at this, I also use part of their discussion as data for the analysis.

Analysis of the task

The task given in Figure 1 can be seen as a rule-of-three-problem where the measure spaces M_1 and M_2 are the original and the enlarged puzzle, respectively. Since it is not given what length in the enlarged puzzle corresponds to the length 1 in the original puzzle, a slightly modified version of the Vergnaud table is therefore appropriate for this task (Table 2).

M_1	M_2
4	6
b	x

Table 2: The Vergnaud table for the task

Here b can be any of the other numbers in the original puzzle and the task is to find the corresponding x. The givens in Table 2 will make it possible to compute the enlargement factor, 6/4 = 1.5, which in Vergnaud's (1983) terms is the *function operator*.

The task requires that the pieces should be *enlarged*, and to secure a multiplicative and not an additive enlargement the cue is given that the enlarged puzzle should have *the same shape* as the original puzzle. The task is therefore to produce a new puzzle where each piece, and therefore the whole puzzle, is *similar* to the original one. This can be seen as what Hölzl (2018) describes as a *dynamic* approach to similarity, where enlargement is seen as a dynamic process, something should be done to the original figure to get the new one. This can be contrasted with at *static* view, where only the ratio between corresponding distances is considered.

The Norwegian word used in the task is 'å forstørre', which has the same meaning as 'to enlarge', namely 'to make something larger'. To make something larger can be done by adding a positive number, or by multiplying with a number greater than 1. In the task, the enlargement should lead to new shapes *similar* to the original shapes, meaning that the ratio between corresponding lengths should be the same. Hence, the dynamic enlargement process should lead to a static situation with a constant ratio. Both Brousseau (1997) and Erath (2019), experienced that the first attempt of the pupils was to add a constant number to each length, and indeed, this also happened in our case. Van Dooren et al. (2010) investigated pupils from Grades 3-6 and found that the tendency to use AR for PR was decreasing as pupils got older, whereas the tendency to use PR when AR was appropriate was increasing. One would therefore expect that the pupils in my study (Grade 5) would have developed PR some extent.

The research question that is addressed in this paper is: *What language cues can be identified in the classroom sessions that support a transition from additive reasoning to multiplicative reasoning?*

Analysis of the teaching sequence

Group work, day 1, Frances, George and Mary

The pupils are given the pieces of the original puzzle cut out in blue cardboard with the correct measures. The task is printed on a piece of paper and the pupils are given a large piece of yellow cardboard on which they are supposed to draw the enlarged pieces. Each pupil makes two pieces. When they have drawn their pieces, they are supposed to cut them out and place them together. Almost immediately, when looking at the task, Mary says "we can take plus two. Since this was plus two, we can take all the others plus two". There seems to be agreement in the group that this is the thing to do and based on this, they make their pieces. However, already when cutting out the new shapes they start to suspect that something is wrong. Mary says that "the shapes are not the same". It seems that when enlarging the right-angled triangles, they have added 2 cm to all three sides, and then the new triangles are no longer right-angled. Obviously, the pieces then don't fit together. They think this is because they have not been accurate enough and they try to make new versions of some of the shapes. Although they see that the pieces don't fit, they don't question the "plus 2 strategy". Discussing with the teacher, they realise that when adding 2 to all pieces, the new total shape will no longer be a square. George observes that by adding 2 to each length, the sides composed of three parts will be longer than the sides composed of two parts. Therefore, they decide to concentrate on the frame and make this into a square of 13 x 13 cm (instead of 11 x 11 cm) and adjust the interior parts to fit into the frame. They don't finish this strategy before the teacher intervenes.

Teacher intervention in whole class

The teacher has observed that all groups have used the strategy of adding 2. She decides to help them further by giving the additional information is that the sides which are 5 should become 7.5.

Group work, day 1, Frances, George and Mary, continued

The teacher's new information leads to a new hypothesis in the group, that the even numbers get +2 and the odd numbers get +2.5. This is consistent with the information that 4 → 6 and 5 → 7.5. They use this procedure to compute the new lengths, in combination with trying to get a square. They realise that this procedure will give 15.5 cm for two of the edges and 17.5 cm for the other two, hence not a square. The discussion gives rise to two different hypotheses to solve this problem. One comes from George who suggests that if the length that is 6 becomes 10 and not 8, they will get a 17.5 x 17.5 square. The second hypothesis is to let the lengths that are 2 remain as 2. Then they think that they get a 15.5 x 15.5 square. Here they overlook one of the lengths of 2. They don't get time to finish testing their hypotheses but they seem to be prepared to construct rather elaborate additive structures to solve the problem, where the operator takes on a different form depending on the values of the side lengths.

Teacher in whole class, day 2

The teacher starts by presenting the methods she has observed in the groups on day 1, first the "+2 strategy", and then variations of this, mainly based on distinguishing between even and odd numbers. In the group with Frank, Roger, Nora and Brenda, the teacher has observed something different, and she asks Nora to present the strategy from her group.

Nora:	We thought first that four should be six. So then we added two. Then we found that on the other we should add two point five. So we added one half more on the next. Then we tried to do that upwards on the other numbers.
Teacher:	What we add, should be larger and larger the longer the edges are.

Nora's utterance "added one half more on the other" gives an indication that they are moving away from adding a constant to adding something which varies, which could be a first step towards PR.

Looking into the video from this particular group on day 1, I can see that these pupils also started with adding 2 for even numbers and 2.5 for odd numbers and when they realised that this did not work they were stuck. They sit for some time, not knowing what to do. Then the teacher comes and asks how they are doing. Frank says “not good”. He has written 4 = 6 and 5 = 7.5. The teacher points to this and the following conversation takes place.

Teacher: Is there something else with these apart from that here we add two and here we add two point five? Is there another pattern we can think of in addition to this?

Frank: Perhaps that when there is a larger number, then we add one half more.

Teacher: OK, so what do you think if it had been six here [points to the number 5]?

Frank: [writes 6 = 9 below 5 = 7.5]

Teacher: You think it will be nine. What about seven?

Frank continues writing and finally he produces the table shown in Figure 2. The teacher turns to the other pupils and encourages them to try out this method.

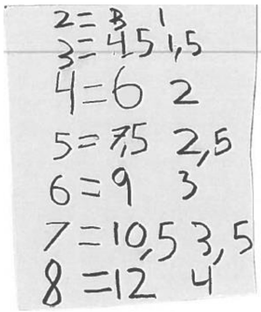


Figure 2: From the notes by Frank

In the whole class session on day 2, the teacher presents the table that Frank had made and says:

Teacher: Do you see any connection between the lengths in the original puzzle and how much we should add?

Frances: I think we have to add a half each time.

Teacher: What half then, do you think? What is it half of?

Mary: Perhaps when it is eight, we have to add half of eight.

When Frances says that “we have to add a half each time” it is reasonable to think that she means that she thinks about the increase in the increment and not the increase itself, since her utterance comes after the teacher has pointed to the table in Figure 2. A language cue to bring the reasoning forwards may be when the teacher then asks “what is it half of?” Then the word ‘half’ changes from being an additive constant to being a multiplicative operator. It is no longer the number ‘one half’ but it is ‘one half of something’. There are also traces of the operator aspect in what Frank said in the group, “when there is a larger number, then we add one half more”. When the teacher has emphasized “what is it half *of*”, Mary is able to express the general function $x \rightarrow x + ½ x$, using ‘eight’ as a generic example, “when it is eight, we have to add half of eight”. Finally, the teacher produces the table shown in Figure 3 on the board.

1	2	3	4	5	6	7	8	9	11
1.5	3	4.5	6	7.5	9	10.5	12	13.5	16.5
+0.5	+1	+1.5	+2	+2.5	+3	+3.5	+4	+4.5	+5.5

Figure 3: The table produced by the teacher

Here the teacher has indicated that what is to be added is half of the original value of the lengths, so in effect the teacher has made a representation of the function $x \rightarrow f(x)$ shown in Figure 4.

$$f(x) = x + \frac{1}{2}x$$

Figure 4: The enlargement function

Summary

All the groups started by adding 2 cm to each length in the puzzle, an indication that the word *enlarged* made them think of adding a constant. During their work, they got feedback from the milieu as they observed that the new pieces did not have the same shape as the original pieces and also that they did not fit together without gaps. However, they explained this by inaccurate measuring and inaccurate cutting. They did not question the procedure. This is completely in line with the experiences made by Brousseau (1997, p. 177): "It is not the model, it is the realization that is put into question." When the teacher gave the information that 5 → 7.5, in addition to 4 → 6 they realised that they should not add the same number each time. This information led to new procedures, which, in the effort to preserve the shape, turned out to be rather complicated.

Already in grade 3, the pupils worked with situations of multiplicative comparison, which is closely related to multiplicative change. These situations were of the type "the short walls of a room are 3 meters long, and the long walls are five times longer. How long are the long walls?" In Vergnaud's model, this means that the measure spaces M_1 and M_2 are the short and the long walls, respectively, and "five times longer" is the function operator. Hence, in $f: M_1 \rightarrow M_2: a \xrightarrow{k\cdot} b$, a and k are given and b is unknown. In the current situation, a and b are given and k is unknown.

In line with previous research (Ahl, 2019; Erath, 2019; Fernández et al., 2012; Gläser & Riegler, 2015; Van Dooren et al., 2005, 2010) done with students in different age groups, the present study shows that using AR instead of PR is firmly established in problems of the kind studied here. Seen from a language perspective, an interesting question is, what elements of language use may support a transition from additive to multiplicative thinking. A turning point towards multiplicative reasoning takes place when Frank suggests that "when there is a larger number, then we add one half more". Since only integers are considered, I interpret the statement "when there is a larger number, then we add one half more" to mean that when a length in the original puzzle is increased by 1, the increment increases by a constant, ½. This is extended by Mary to say that "when it is eight, we have to add half of eight". Although the multiplicative factor is not clearly identified and the function in Figure 4 is a mixed additive/multiplicative model, it seems that the pupils are beginning to develop a language involving a multiplicative structure.

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ST3
Language in interaction

Online discursive interactions concerning mathematical issues within digital interactive storytelling

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This paper focuses on analysis of online discursive interactions among students from the first years of high school in problem-solving situations requiring the production of conjectures, arguments, and proofs. The study is framed in Vygotskian perspective, focusing on the language used to communicate and to manipulate and solve the tasks. We refer to the one-line proof model in order to analyze the students' path towards argumentation and proof and to discuss the outcomes of an experiment.

Keywords: Language, online interaction, proof, argumentation.

Introduction and theoretical background

This paper aims to analyze the impact of online discursive interactions among students in problem-solving situations, requiring the production of a conjecture as well as arguments to support and prove it. The study is part of a larger project framed in a Vygotskian historic-cultural perspective (Vygotsky, 1934), assuming that cognitive development is a social process and reasoning capabilities increase through interactions among peers and with an expert (e.g., a teacher). The children's participation in social interactions and communication is pivotal to their evolution and appropriation of cultural tools. Language is one of the most important cultural artifact and it has a fundamental role in mathematics learning processes (Planas, Morgan, & Schütte, 2018; Ferrari, 2004). On the one hand, it has a basic role as a communication and thought-organization tool in the social practices of the classroom. Language has a prominent communicative role in the development of mathematical thinking (Sfard, 2001): "Thinking may be conceptualized as a case of communication, that is communication with oneself" and it "arises as a modified private version of interpersonal communication" (p. 26). On the other hand, language assumes a key role in the process of objectifying and manipulating linguistic objects and symbols that is typical of mathematical activity (Coppola, Mollo, & Pacelli, 2019). This is still more true in our case, as the students predominantly interact through written communication via an online chat room. What may seem a constraint can instead be assumed to be a strength. In fact, writing is considered a semiotic tool of objectification, used by individuals in social processes of producing meaning in order to achieve a stable form of awareness, make their thoughts explicit and visible, and perform actions (Radford, 2002).

The problem-solving activities in our study engage students in producing conjectures, arguments, and proofs, which characterize mathematical activity. It is widely recognized that they are worth considering from an educational viewpoint. In this respect, we refer to *one-line proof* model, by Gholamazad, Liljedahl, & Zazkis (2003), for the fine-grained analysis of students' short proofs and the diagnosis and remediation of their productions, involving the following steps:

- *Recognizing the need for proof*: It is essential that students understand that the production of examples may not be sufficient to establish the validity of a statement and that they recognize that proof is essential.

- *Recognizing the need for representation*: The need for proof inevitably leads to the need for generalization, which requires students to select some form of representation.

- *Choosing correct and useful representation*: It is not enough for students to recognize the need for representation, but they also should select one that is correct and useful in order to arrive at a proof.

- *Manipulating the representation correctly*: Having chosen a representation, students should be able to work with it, thus carrying out the necessary manipulations in order to reach the proof.

- *Interpreting the manipulation correctly:* At the end of the manipulative process, students should be able to interpret the result of their manipulation and understand whether the proof is complete or not.

In the above framework, we want to investigate:

RQ1: How and to what extent can online discursive interactions foster the processes of explanation and argumentation in order to arrive at a proof?

RQ2: Furthermore, is it possible – and if so, to what extent – to transfer the teacher's expert role to a suitably designed digital environment?

Methodology

The design of the activity

The activity was designed within a digital storytelling platform (Albano, Dello Iacono, Fiorentino, & Polo, 2018). The tasks are embedded in a narrative in which there are characters who will act as the avatars of the various participating students. Each student takes a role and has actions to perform, sometimes individually, sometimes together with others. There is also an expert who interacts with the students by orchestrating discussions on general topics.

Each task will develop according to a scheme that provides the following actions:

Inquiry & Conjecture, referred to as Episode 1, is related to the exploration of a mathematical situation. Starting from an event that relates to the mathematical problem, through the investigation, the student arrives at the formulation of a personal conjecture that she is called to share with her peers. Subsequently, starting from the shared findings, a comparison between peers opens up, with the explicit aim of formulating a common conjecture and then communicating it to the expert. While the comparison also takes place in the Chat tool, the communication with the expert takes place through the Forum tool, which encourages the students to produce a response that is expressed in a more evolved, literate register.

Arguing & Proof, referred to as Episode 2, leads, through discussion with the expert, to the comparison of the conjecture that emerged in the previous action with a formal proof. The expert manages this discussion with the aim of guiding the students towards the development of a formal proof, if necessary. At the end of the activity, the expert is responsible for institutionalizing what they have found.

Summing Up & Refining, referred to as Episode 3, draws the activities to a close. After this, the students, either alone or assisted by the expert, will have achieved a formal proof, providing the solution to the mathematical problem.

The problem

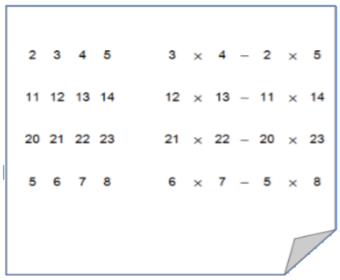
The following mathematical problem is posed to the students (Mellone & Tortora, 2015, p.1436): Choose four consecutive natural numbers, multiply the two intermediate numbers, multiply the two extremes, and subtract the results. What do you get?

The aim of the problem is to introduce students to algebraic modeling and to develop their argumentative and proving skills. Students should conjecture that the result is always 2 and prove it using suitable algebraic representation, e.g., n, n + 1, n + 2, n + 3, obtaining the expression (n + 1) (n + 2) - n (n + 3) and manipulating it in order to obtain the constant 2. It could also promote linguistic and logical discussions about terms such as “all” and “always”, as well as stimulating students’ thinking about key mathematical concepts such as the meaning of “consecutive” numbers.

This problem has been reframed in a narrative framework (Zan, 2012): a group of four friends, Marco, Clara, Federico, and Sofia, receive mysterious messages from aliens and collaborate with each other to understand it. They ask for help from Federico’s uncle, Gianmaria (Figure 1).

Figure 1: The four friends and Gianmaria

In the story, the original problem has been modified as follows. The students see a sheet showing some quadruplets and operations corresponding to the subtraction of the product of the second and third terms from the product of the first and fourth terms (Figure 2):



2 3 4 5	3 × 4 – 2 × 5
11 12 13 14	12 × 13 – 11 × 14
20 21 22 23	21 × 22 – 20 × 23
5 6 7 8	6 × 7 – 5 × 8

Figure 2: The sheet

In Episode 1, students receive the sheet in Figure 2, with the request to explore in order to try to understand the meaning of these numbers. They should note that these are groups of 4 consecutive numbers and should conjecture that when the operation to the right of each group of 4 numbers is carried out, the result is always 2. In Episode 2, the conjecture is institutionalized and students are asked to come up with a general proof. Various possibilities to be examined are presented through the story and its characters: “large numbers”, “multiple symbols”, “one symbol”. The comic, therefore, acts as scaffolding for recognizing the need for a proof and a representation. In this episode, the students should come up with an adequate formalization of the mathematical problem and prove the conjecture, with Gianmaria’s help. Finally, in Episode 3, students should send a response to the aliens that proves that they have understood their message; namely, the mathematical statement underlying the numbers received and its proof. The start of the task resolution process is scaffolded by a comic strip that lists several kinds of messages to be sent, such as: "let's send 2", "let's send a formula", "let's send a 4-upla", "let's send a proof".

The experiment

The experiment carried out and analyzed in this paper involved 24 students from the first year of high school. Students worked at a distance on Episode 1 and Episode 2, while they worked on Episode 3 in class, a month after Episode 2. They worked in groups of 4 students. In each group, each student assumed a specific role: Boss, Pest, Nerd, and Blogger. The Boss is task- and group-oriented; the Pest insinuates doubts and asks questions; the Nerd supports the friends in their use of computer tools, and the Blogger takes notes and summarizes the group's answers. The role of the expert, Gianmaria, was played by a researcher in Episodes 1 and 2, while it was played by the class teacher, assisted by a researcher, in Episode 3. The story was implemented using the Moodle platform; specifically, its collaborative Chat and Forum tools. All data – that is, student interactions – were collected on the platform. The story was implemented through comics, integrated into the Moodle platform.

Data analysis and discussion

In this section, we analyze the interaction of two groups of students, using the one-line proof model. The first excerpt is taken from group G1's interactions in the Chat during Episode 2 (section 2.2):

1	Federico G1:	I thought we could send either the quadruplets we came up with (even if we don't know if they are correct or not) or the formula which we found, that is = 2
2	Sofia G1:	I thought we could send "middle items-external items = 2"
3	Federico G1:	I thought we could send one of these things because they are the things we have developed and reflected about the most
4	Sofia G1:	Yes, I agree, maybe the second one is better
5	Federico G1:	If we send that one, we should also add a proof and maybe a reason to justify this choice
6	Sofia G1:	We should say that we have decided to send this proof because when we calculate the quadruplets that the aliens sent us, the result is always 2.
		That is, a, b, c, d = bxc - axd
8	Federico G1:	Sending number 2 alone is useless without explaining how we found it. The proof without the formula is similarly useless
9	Sofia G1:	Summing up, we thought that we could send the aliens the proof of why it is always 2, that is: a, b, c, d are generic letters, so they assume the value that we want to give them, that is consecutive numbers. So if we find the product of the middle items minus the product of the external items always is 2. So we should send: a, b, c, d: bxc - axd

At the beginning, the students discuss whether they should choose an example or a formula (lines 1–4). It is noteworthy that they agree that the formula is better (line 4), which seems to highlight the awareness that an example is not enough to prove what they have understood. Moreover, the need to justify their choice emerges (line 5). In fact, Federico G1 seems to distinguish between giving a proof and giving a reason for their decision, and he later underlines the need for completeness (line 8). Then, they try to formalize what they have observed (lines 6) and Sofia (as the Blogger) sums up what they have found (line 9).

We note that the students have grasped the fact that the numbers in the given quadruplets are consecutive, but that they use the representation “a, b, c, d”. They make use of an isomorphism that matches the numbers’ consecutiveness with the order of the letters in the alphabet. The analysis of further Chats from this group shows that they were not able to choose a correct and useful representation and thus they did not produce any proof of their conjecture.

The following two excerpts are about another group’s interactions. The first concerns the Forum in Episode 2, after they verify their conjecture in various examples using both small and large numbers:

10	Clara G2:	It is generally true because if we do (9 × 8) - (10 × 7) = 2 (3 × 4) - (2 × 5) = 2 [*some more examples are given*] we can note that in each line the result is 2, then the answer is yes, it is always true. We can note one more thing: the operations always occur according to the pattern “even × odd – even × odd” or vice versa Let’s try using symbols! a = even number b = odd number c = even number d = odd number (b × c) - (a × d) = 2
11	Marco G2:	We can prove it using letters a = 12, b = 13, c = 14, d = 15, e = 16, f = 17, g = 18, for instance, (e*f) - (d*g) = 2
12	Sofia G2:	For instance, let us prove it with large numbers 200, 201, 202, 203 (201 × 202) - (200 × 203) gives exactly 2 as a result! So I would say that with any sequence of numbers, the result is always the same

Note that the students focus their attention on the pattern relating to the order of the operations; that is, the difference between products such as an even number times an odd one and subsequently the alternation of even and odd numbers in the sequence (line 10). There is no recognition of the fact that the numbers are consecutive, except in the examples (line 11). It is worth noting that the students seem to recognize the need to go beyond the example by using letters, but they are unable to do so: they merely substitute the letters for some specific numbers. Indeed, in the following line, they again “prove” their conjecture using numbers, although they use large numbers. From the fact that it works with large numbers, they conclude that it always works: note “any sequence of numbers” (line 12); that is, their conjecture is proved. In their discourse, the students do not refer to “consecutive” numbers in the quadruplets. Their awareness of this aspect is revealed by the use of the above isomorphism by means of consecutive letters, as well as in the colloquial use of the words “sequence of numbers”, “sequence” meaning “consecutive”.

Let us see how they continue the discussion in the Chat during Episode 3.

13	Federico G2:	Let's send a formula
14	Clara G2:	A proof would not be a bad idea
15	Marco G2:	We should choose from formula, proof, number
16	Clara G2:	Let's send them a formula using some numbers
17	Federico G2:	I prefer a formula made of symbols
18	Sofia G2:	Mhm…a formula about what?
19	Marco G2:	A formula made of natural numbers, such as n
20	Clara G2:	We need to use the generic ones because the numbers are infinite
21	Sofia G2:	What kind of formula do we want to use?
22	Clara G2:	$n + 3, n + 4, n + 5, n + 6$
23	Marco G2:	$(n) (3 + n) - (2 + n) (1 + n)$
24	Sofia G2:	A proportion that corresponds to the given rule; in a proportion where the product of the middle items is equal to the product of the external items
25	Federico G2:	$n : n1 = n2 : n3$
26	Sofia G2:	Actually, they use a sequence of numbers that we should denote as A B C D $(A \times D) - (C \times B) = 2$
27	Marco G2:	We should justify the reason for our choice
28	Clara G2:	But you have used 4 random letters not in a sequence In my opinion A could be 300, B 3, C 6, and D 90000 They should be a sequence, they must be consecutive They must have a sequence, they must be consecutive numbers
29	Sofia G2:	Using the exact numbers is wrong
30	Marco G2:	Then we should use either the letters or n
31	Federico G2:	It is an equation
32	Sofia G2:	$(n + 1) (n + 2) - (n) (n + 3) = 2$ Let's try to replace n with other numbers, it should always be correct
36	Federico G2:	But what is n?
37	Marco G2:	Any natural number
38	Clara G2:	n is all the numbers

Looking at the beginning of the discussion, we see that the students are looking for a representation (lines 13–17). This need is born within the story's context, so the comic strip acts as scaffolding. Then someone makes explicit the link between the need for a representation and the need for proof, without merely relying on the examples (lines 18–20). Later, the students recognize that there can be various representations, so they discuss looking for the most useful one (lines 21–24) and find they need to justify their choice of representation (lines 25–27). It is worth noting that Sofia G2 uses the word "sequence" linked to "A B C D" (line 26), probably meaning consecutive numbers, but Clara G2's reply highlights that A, B, C, and D are merely variables for identifying numbers and not consecutive numbers (line 28). It is worth noting that this remark is consistent with Clara's role (that is, the Pest). Going on, the students recognize that they cannot use examples, but need a general representation (lines 29–30). Finally, they reach a correct and useful representation and give an equation (line 32). They stop here, without manipulating the equation in order to show why it works. Sofia G2 merely suggests verifying the equation by replacing n with numbers. This invitation stimulates a very interesting interaction concerning the values with which n can be replaced and the universal quantifier naturally comes into play (line 37–38), which is one of the issues we expected the problem to pose.

Results and conclusions

Concerning RQ1, from the analysis of the students' interactions in the Chat and Forum tools, supported by comic strips acting as scaffolding at the beginning of the solving process, a co-construction of explanations and arguments seems to emerge. Students work together, each one in turn, starting from what other students in the group have said, answering and rebutting, and together they attempt to construct the solution to the proposed task. This process seems to support the path through the steps of the *one-line proof* model (Gholamazad et al., 2003), which also allows them to individuate the failure points. Indeed, the students move along the first step on the left-hand side of the model, recognizing the need for proof: line 5 shows the emergence of the need to justify their choice, and thus the need for representation (the second step in the model). Unfortunately, they fail this step by choosing an incorrect – or at least unhelpful – representation (line 9 – the consecutive numbers are indicated by the letters a, b, c, d) and consequently manipulating them incorrectly (line 11 – the letters are substituted with numbers). The conducted analysis thus seems to suggest, with respect to RQ2, that neither the group of peers nor the technological environment is sufficient to guide students to manipulate the chosen representations in order to correctly interpret them and arrive at a proof of their conjectures. Moreover, we observe that when students talk to each other in the Chat function, they often use a colloquial register (Ferrari, 2004), even though they have to write their answers. This is probably because the chat is perceived as a shared context, almost as if they were in each other's presence. A change can be observed when they have to communicate their answers to Gianmaria (or to the aliens); that is, asymmetric communication. In this case, the environment also changes, from Chat to Forum: they no longer have the same shared context, so they need to be more "formal" and to use a literate register (Ferrari, 2004). The interaction with the expert and the consequent change of register could spur the process towards a proof.

These remarks suggested a change in the design, involving the possibility of an expert intervening in the Chat. The work on this new design is in progress. Another suggestion that emerged from this analysis is that the different characters played by the students seem to work well, in accordance with the characteristics of the roles (Albano, Pierri, & Polo, 2019). Further analysis is in progress.

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About real-world contexts in mathematics education and their impact on language and learning

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In mathematics education real-world contexts are frequently used to catch the students` attention and to foster their understanding of the learning content. Presumably, especially in primary school they are used with the aim of building bridges between the familiar 'everyday world' of the students and the more formal and abstract 'world of mathematics'. The presented study questions the meaningfulness of real-world contexts in mathematics lessons in primary school and reveals some limitations of them. Interactional analyses of selected passages of mathematics lessons will illustrate some scenes with real-world contexts and their impact on language use and the opportunities to learn.

Keywords: Context effect, interaction analysis, language styles, primary school, teaching methods.

How this study came to its focus – Research goals, questions and methods

One fundamental theoretical assumption of this research project is, that the use of language differs concerning the situation. Taking into account the sum of linguistic events taking part during (mathematics) lessons could help considering language as "a complex meaning-making system" (Moschkovich, 2018, p. 39). Recognizing that people participate in different manners during varying situations the central purpose of this study is to examine language use during different situations of mathematics lessons. Findings about the situatedness of language within the mathematics classroom will be significant for teachers and designers of text books as they can be sensitized for specific language-based particularities of establishing a real-world context and, as a result, use different didactic approaches during different situations. Of the multitude of language-influencing factors – like the social formation, the (non-)existence of the teacher, happenings before the situation, the participants' aims and experiences, and many other –, during the observations I recognized scenes in which language use and its impact on the participants' supposed opportunities to learn were of a special character: Especially in the observed primary school mathematics lessons, the mathematical content was frequently and in different manners connected to a real-world context. Derived from the main question about how language use in mathematics classrooms changes depending on the situation – what could be hardly answered by a single dissertation project –, the focus is on how the "'pure mathematics' task [is] dressed up in a real-life context" (Palm, 2006, p. 46), what led to the question:

Q1: How does the use of language look like when the mathematical content is 'dressed up' or connected to a real-world context?

The first research question aims to give answers about how language use might differ from other classroom situations when teachers (and mathematics textbooks) connect mathematics to the students' supposed real-world experiences. For example, it might happen that language is less explicit and formal and, instead, tends to be imprecise, context-bound and less mathematical. Another important aspect is the impact on learning mathematics. A central assumption of the presented study is that different situations create different possibilities of using language and, as a result, also create different demands to the students which form of language is seen as the 'more appropriate' one in the current situation. As the presented research project is situated in *interpretative classroom research* and *symbolic interactionism*, we take up a social-constructivist view on learning: Here, learning is seen as a collective and interactional process (e.g. Blumer, 1969; Cobb & Bauersfeld, 1995). In this regard, differences in language use result in different opportunities to learn and to participate within the mathematics classroom. From this the following question can be derived:

Q2: Which language-based opportunities and requirements and which opportunities to learn mathematics go along with the use of real-world contexts and learning material?

As one is not able to look in the students' heads and to make definite statements about their progress in learning, the second research question does not aim at giving absolute and complete answers about arising learning opportunities. It is more about giving tentative assumptions about how the connection of the mathematical content and a real-world context might affect the opportunities to learn.

To gain a broad impression of the language use during different situations, mathematics lessons of several classes (different school types and different class levels as well) have been video-recorded during the period from March 2017 to May 2018. The duration of the recordings in each class varies from two to four weeks (just mathematics lessons). For each observed class there are at least six video-recorded mathematics lessons which follows each other. Selected passages were transcribed and analyzed with the use of interactional analysis. One fundamental assumption of this approach is

> that meaning is negotiated in interactions between several individuals and that social interaction is thus be understood as constitutive of learning processes, speaking about mathematics with others is in itself to be seen as the 'doing' of mathematics and the development of meaning (Schütte, Friesen, & Jung, 2019, p. 104).

Following this, social interactions are at the center of attention and the 'place', where (mathematical) meaning is negotiated. After setting and structuring an interactional unit that should be analysed in detail (here, these are scenes where the mathematical content is connected to a real-world context) a further step is to give a general description of the scene and to make a detailed sequential interpretation of the individual utterances. This detailed step-by-step analysis helps to look at the transcript scene in detail and to reconstruct the development of the interaction and is often interwoven with the turn-by-turn analysis, where some interpretations of the sequential analysis are regarded as more or less applicable interpretations of an utterance. Finally, the summary of the interpretation helps to reduce the diversity of the interpretations that were made before and illustrates only these interpretations, which can best be justified (for a detailed English description of the approach, its basic concepts and analysis steps see Schütte et al., 2019). For reasons of space, the analysis below is limited to the first and last step of the interactional analysis: A description of the scene and summary of the interpretations. Beforehand, it seems important to take a closer look at real-world contexts.

Real-world contexts as bridges and obstacles for learning

Although this paper makes use of the term *real-world context* there exist a number of similarly used terms (like narrative, real-world connection, storytelling, word problem, or story problem) and there is a lack of an established and universal specification of the term (Gainsburg, 2008; Larina, 2016). Additionally, a range of different activities can be called a real-world context in mathematics education: From classic word problems and analogies to the discussion of mathematics in the community, the analysis of real data, or processes of mathematical modeling (Gainsburg, 2008). In this context, Larina (2016) also underlines that because of the missing universal definition but simultaneously the continuous claim to use real-world problems, teachers are guided by their own criteria for the selection of them. Within the frame of this paper, the term real-world context is used to refer to a – more or less explicit – connection between the formal symbols and procedures of mathematics on the one hand, and activities, objects and actions from the world we are living in, on the other hand.

There are beneficial and limiting aspects of conducting a real-world context in the mathematics classroom. Previous studies document, that teachers connect mathematics with real-world with the aim of enhancing the students' understanding, motivate them to learn mathematics and to participate in the mathematics lesson, and to help them applying mathematics to real problems (e.g. Boaler, 1993; Gainsburg, 2008; Sullivan, Zevenbergen, & Mousley, 2003). In this regard, real-world contexts have the potential to be viewed as a *bridge* between mathematics and real world. For many – children as well as adults – the abstractness of mathematics is a synonym for a detached, inflexible and cold body of knowledge. Through the use of real-world contexts, the mathematical task becomes more subjective and personal in order to get involved with mathematics more easily (Boaler, 1993).

Despite the high variety of approaches in establishing a real-world context in the mathematics classroom and the potentials named above, research suggests that they are used infrequently and cursory (Gainsburg, 2008). Often, especially the presumably most common form of real-world contexts – word problems – come from textbooks and less from the teacher him- or herself, what might rather cause fuzziness than enlightenment and does not open the usage of the underlying mathematical content for real-world situations (Crespo & Sinclair, 2008). Greer (1997) and Nesher (1980) point out the stereotyped nature of their presentation: an unrealistic numerically clean nature including all necessary data; single-step and narrow story problems; superficial features like keywords; the fact that they seem to be always-solvable in only a single way; or the direct match with exactly one operation or mathematical concept (see also in Crespo & Sinclair, 2008; Gainsburg, 2008; Zan, 2017). These shortcomings often result in an unreflected treatment with the mathematical content within the presented real-world context: Students become trained in solving a context task exactly in the expected way. They often believe what they are told without questioning the distance of the mathematical task from reality (Boaler, 1993). In this regard, many students could not sufficiently identify with the presented real-world context, as they had no contact with them before (e.g. paying bills and wage slips); or they fail to recognize the purpose of the task and the mathematical content within (e.g. Boaler, 1993; Sullivan et al., 2003; Zan, 2017). Taking also into account these aspects, real-world contexts could be also seen as *obstacles* for understanding mathematics and recognizing the usefulness of mathematics for life. Following Sullivan et al. (2003), teachers need to develop a certain sensitiveness and should be able to make judgements about the use of contexts. This concern, for example, the (mathematical) suitability, the relevance to the students, the potential motivational and emotional impact, and even the possibility of negative effects for the students' opportunities to learn.

Considering the related literature, many studies and literature reviews focus mainly on the usage of real-world connections in higher grades (e.g. Gainsburg, 2008) – but especially in *primary school* we expect a more frequent (and different) usage of real-world contexts and that the relation between the everyday world and mathematics could be problematic especially for young students: While the everyday context is mostly unquestionable and plausible for them, the world of mathematics seems to reveal a 'new way' of thinking about this everyday world, as nearly everything could be the topic of a mathematical task (Neth & Voigt, 1991). Presumably, especially primary school children take the presented real-world context too much into account and fail to identify the mathematical content of it. While the teacher who presents the real-world context puts his or her focus on the mathematical content, the students might think more directly and emotional with connection to their world experiences (Neth & Voigt, 1991). In this regard, the use of real-world contexts for illustrating a mathematical content could have consequences for the use of language and the opportunities to learn mathematics as well, as the presented scenes below will illustrate.

Research example "A real circle in mathematics" versus "Our circle in here"

Within the frame of this paper, the focus is on one scene from a geometrical topic in which the teacher explicitly asks the students about differences between mathematics and an example from the real-world. In this second-class mathematics lesson the teacher starts the topic (circles and circle patterns) by forming a circle of people with the whole class around some desks in the classroom (see Figure 1). Additionally, she also prepared a circle with diameter, radius and center on the board before the lesson begun (see Figure 2). She opens up the comparison of a 'real circle in mathematics' and the circle of humans in the classroom ("...our circle in here...") and starts the discussion as follows:

Figure 1

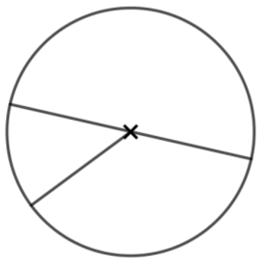

Figure 2

Teacher: now here we have a circle and we say this is a circle\ and certainly for a real circle in mathematics there are some requirements\ these we have already learned right/ and/ . maybe we we could collect them together\ . why is this not a real circle in mathematics/ I am curious if we already know this and have memorized it\ Matty is allowed to tell what is missing here\

Matty: the round edges there are missing a bit here\ like this the round ones [shows along the circle of humans]

Teacher: the . yes\ how is this called in a circle/ . I have brought up this question once already\ what do we call this all the way round a circle/ a /

Eddy: erm at the moment this [pointing along the circle of students] is rather a rectangle

Teacher: yes\ this is not a circle\ we only say this is a circle for us ok/ but in mathematics this is not a real circle\ . . what is a real circle/ what is missing here/ how does this looks

	in mathematics what we are forming now in here with the children/ [goes to the board and points at the circle line]
3 Students:	an oval
Teacher:	. . actually we are forming the c i r c l e/ line\ because we closed the circle with our hands\ so . and why is this not a real circle yet/ what is still missing here/ Polly\
Polly:	well but the the desks are also standing around that way .. so all in all this is a quadrangle\ [pointing on the desks in the middle of the human circle and then painting a circle in the air with her hand] one cannot form a circle here\
Teacher:	why not\ . we now could stand in a way that it will be exactly a circle\ ..why can this never be a real circle as in mathematics/
Eddy:	erm every time when we are touching each other like here for example there is an edge backwards when one is doing this [pats slightly on the wall behind him and his left partner with both their hands while they hold each other]
Teacher:	yes this is ok\ now let us pretend we are the circle line\ but what is still missing/ we do not know if this is a real circle line . Nock
Nock:	there is too less space .. one rather could make a zig-zag line. it could never become straight\ . or it should be equal on all sides\ so not that there stand less and on the other side many more
Teacher:	then we do it like this/ we imagine that we are all points on the circle line\ .. maybe you will get it\ . is the distance the same everywhere/

In this example the real-world context was established at the beginning of a new (sub-)topic in the first minutes of the mathematics lesson. For a better structure, the analysis is organized in two areas: (1) Characteristics of the use of language, (2) and the assumed opportunities to learn. As it is hardly possible to make certain statements about what the students have learned, this part of the analysis is limited to *assumptions* about learning opportunities that could arise (Schütte et al., 2019).

(1) Concerning the use of language, we could identify some meaningful points about establishing the comparison between different types of circles: In the presented situation this leads to an indefinite and ambiguous use of language. By opening up the contrast between ‘a real circle in mathematics’ and ‘our circle here ... this is not a real circle’ the teacher implicitly points out, that there exist two (or even more) different kinds of circles which differ depending on the context. She also indicates, that only the mathematical circle is ‘a real circle’ while the human one does not fulfill the same (mathematical) requirements. Presumably, the teacher intends to repeat some important characteristics and terms of a circle like ‘radius’ or ‘middle point’. But the students’ answers seem more to refer to the question, why this (human) circle is not a ‘real’ circle and less to the necessary requirements a mathematical circle has to fulfill. The visual and haptic existent circle of humans could help them to participate in the class discussion (for instance, by referring to it through deictic expressions or a contextualized use of language). Maybe because of this, their utterances seem to be oriented to the visual and haptic human circle in the classroom. They use deictic expressions instead

of a decontextualized language that includes mathematical technical terms like ‘circle line’ and ‘radius’ Accordingly, the ‘place’ of the ongoing classroom discussion is somewhere ‘between’ mathematics and everyday world, as neither the connection of both types of circles nor the teachers’ goals were made explicit. The ongoing classroom discussion seems to be less a mathematical oriented discourse about a circle than on a quiz about differences between the ‘real’ and the human circle. The students seem to play two different roles in this scene. First, they could be regarded as part of the real-world connection, as their bodies shape the human circle. Through their individual position within the circle, everyone has a different view on the scene. Second, they are actors in the teachers established “playful” introducing performance of collecting characteristics of a mathematical circle, as they are asked for their ideas and already existing knowledge. This might also affect their language use.

(2) Through visualizing a circle the learners’ knowledge is recalled from their memory. But we could question if the circle of humans or the dichotomous distinction between two different kinds of circles is the source of misconceptions and misleading statements of the students. As through the learners’ utterances we cannot make a definite and confirmed statement about their orientation (either to the formal mathematical circle or to the human circle) we can only assume, that they are aware of the characteristics of a mathematical circle. That is the regularity and ‘smoothness’ of the circle line, and the non-existence of edges and irregularities. The reasons for the students’ ‘less mathematical’ answers could be diverse: First, the teacher simultaneously asked for two aspects (characteristics of a ‘real circle’ versus why is this not a ‘real circle’) and they are referring to the second and maybe more easily one. Second, the visual and haptic existence of the human circle ‘grabs’ the students’ attention too much. Third, the teacher did not make explicit, which question is the more important one and that the students have to tell technical terms of a mathematical circle. Unfortunately, the teacher only seems to realize her own script of the lesson. And this is about the words “circle line, middle point and radius”, as her closing words for this scene illustrate: “The radius is, so to speak, the bar between the middle point and the circle line and has to be equal everywhere. And if we do not have a middle point we do not know if we all stand in the same distance to the middle point”. It is questionable, where the discussion might lead, if the teacher does ask the students about the similarities of the human and a mathematical circle. Maybe then the teacher might rather identify the mathematical important and correct aspects in the students’ utterances. Additionally, a circle can also be defined through optical aspects like the circle lines’ regularity but the teacher seems to assume only the technical terms as necessary characters for a (mathematical) correct definition of a circle.

Putting the mathematical content of the formal characterizing aspects of a mathematical circle in relationship to the circle of humans lead – this is our assumption – to differences in *definitions of the situation* (Schütte et al., 2019) between the teacher and the students. And in result, also to irregularities and ambiguities in language use. With better competences in interpreting and more sensitivity to the mathematical in the students’ answers, the teacher would be able to better comprehend and connect their utterances with her own expectations and with each other.

So no real-world connections?

Considering the impact of real-world contexts in mathematics primary school classrooms for language use the presented scenes illuminate the following: At first glance, the language seems to be less mathematical and academic. Rather, one would say it is like informal everyday language and without a close connection to school mathematics. This seems to be mainly affected by the established real-world contexts, as the participants are able to directly refer to it and their own experiences. Additionally, the scenes illuminated a high frequency of deictic expressions and gestures that were used instead of 'concrete' and technical verbal expressions. The teachers' language is a mix between the real-world context (mainly expressed through less technical and less academic words) and the underlying mathematical content (mainly expressed through technical terms of mathematics) without explicitly expressing the connection between both. Taking a closer look (e.g. with the help of interactional analysis), we could also identify the mathematical meaning in the teachers' and the students' utterances. Especially the first scene shows clearly, that teachers need competences in interpreting their students' answers, as all of them contain the mathematically relevant idea of a circles' regularity. Teachers should be aware, that they should be a linguistical example for the students on the one hand (and therefore use technical terms and a more formal and academic language register), and that the mathematical content should be specific and relevant for the students on the other hand (e.g. through real-world contexts that are mainly presented in a more informal and everyday language register).

With the help of interactional analyses of the presented scenes we can also assume the following effects of the real-world contexts on the students' opportunities to learn mathematics: As the connection between the established real-world context and the mathematical topic was not explicitly expressed by the teacher, this could end up in students' misconceptions and misunderstandings of the mathematical content. In some way, the presented real-world contexts rather seem to be a 'decorative mask' of the mathematical 'core', than a necessary tool to explain it. Additionally, the construction of an adequate representation of the real-world context and the mathematical problem could be more difficult for the students, than its solution (Zan, 2017). In this regard, teachers, teacher educators and the designers of mathematical textbooks should question the usefulness and necessity of real-world contexts critically. In both presented scenes, we see inhibitory aspects for the students' opportunities to learn mathematics and implicit norms (see also Zan, 2017).

Shall we end up in the resulting conclusion about not establishing real-world contexts in mathematics lessons? In view of the fact, that we could identify positive as well as negative aspects of their application, this answer has unambiguously to be *no*. Especially in primary school, connecting the learning content and the students' experiences from their real-world is considered as a central method. Regarding the use of language and the students' chances to participate in the classroom discourse, real-world contexts are viewed as creating familiarity and reducing inhibitions. Nevertheless, teachers need to be sensitive about the usefulness of real-world connections, and, in result, also to aspects of language use and opportunities to learn.

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Rational communication in the classroom: Students' participation and rational mathematical templates

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This paper presents the ongoing refinement of a construct (a Rational Mathematical Template, RMT) emerging from observations of effective classroom discussions that are aimed at producing a definition, statement of a theorem, proof for a given statement, mathematical model for a physical situation, etc., as key elements of mathematical rationality. An RMT concerning one of these elements provides students with guidelines for how to manage the related process of production in a conscious way and how to check the validity of the product. In order to introduce the RMT construct, the paper refers to two episodes of definition production. A brief discussion of the analogies and differences with other constructs follows. The paper ends with an outline of possible developments of the refinement of the RMT construct and related educational issues.

Keywords: Habermas' rationality, routines, rituals, schemata, *rational mathematical templates.*

Introduction

The motivation for the study reported in this paper came from recent developments in our work on the implementation of Habermas' construct of rationality in mathematics education and from recent research advances regarding routines and rituals which resulted in a special issue of *Educational Studies in Mathematics* (2019, vol. 101[2]). The advances that are of interest for us concern the relationships between "ritual-enabling and exploration-requiring opportunities to learn" (Nachlieli & Tabach, 2019, p. 253) and the elaboration by Lavie, Steiner, and Sfard (2019) regarding the notion of routine as "repetition-generated patterns of our actions" (p. 153); see more in the next section.

In our recent research work, we have considered what happens in classroom discussions when the teacher attempts to promote students' rational behavior in their approach to theoretical mathematics, particularly as concerns defining, conjecturing, proving (in synthetic geometry and elementary arithmetic), and mathematical modeling. In order to participate productively and develop their rational behavior in those activities, students need to interiorize specific ways of organizing their thought processes related to the content at stake, which share some aspects with routines that combine ritual and exploratory aspects (hence our interest in Nachlieli and Tabach's and Lavie, Steiner, and Sfard's development of routines, rituals, and explorations). However, in spite of these shared aspects, our objective (the promotion of students' effective approach to rational behavior in theoretical mathematics) produces important differences, with implications for the construct needed to plan and analyze activities aimed at achieving our goal. The (provisional) name for this construct is the Rational Mathematical Template (RMT), and the choice of name alludes to what we mean by it. For instance, in the case (dealt with in this paper) of producing a definition and, at the same time, gradually becoming aware of what "definition" means in mathematics (a crucial condition for mathematical rationality), students gradually interiorize a specific "model" of rational behavior for this activity. This "model" provides the students with guidelines for the exploration and production of a definition (the *teleological* component of rational behavior) and with criteria for validating and communicating the produced definition (its *epistemic* and *communicative* components). In a classroom interaction situation aimed at producing a definition, an RMT offers students an opportunity for purposeful interaction towards that common goal, according to Habermas' development of communicative rationality:

> Communicative rationality is expressed in the unifying force of speech oriented towards understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (Habermas, 1988, p. 315)

Moreover, in an RMT-led process, an affective shared dimension plays a relevant role (see Discussion). In a long-term perspective, it is the driving force that ensures a (relative) autonomy from the teacher and the purposeful development of the process.

In this paper, we will move from a brief account of theories of interest for our work to two episodes concerning two steps in the students' interiorization of what we will call the RMT of mathematical definition. Then, we will discuss the RMT's relationship to other constructs from mathematics education. Finally, in the Discussion, we will present the current status of our refinement of the RMT idea based on classroom observations and reflections by the second co-author of this paper.

Habermas' rationality, routines, and concepts

Habermas (1998) proposed the construct of rationality (and that of rational behavior) in order to account for discursive practices that are characterized by *awareness* when checking the truth of statements and the validity of reasoning according to shared criteria (*epistemic rationality*), evaluating strategies to attain the aim of the activity (*teleological rationality*), and choosing suitable communication tools to reach others in a given social context (*communicative rationality*), the three components being strictly interconnected. In the past years, researchers both in our group and outside it have attempted to adapt Habermas' construct to mathematics teacher education (one of these studies is reported in Guala & Boero, 2018) and to plan teaching aimed at developing and analyzing students' rational behaviors (see Boero & Planas, 2014 for a general account and a presentation of five studies).

Lavie, Steiner, and Sfard (2019) considered routines to be "repetition-generated patterns of our actions" (p. 153); they further elaborated this notion by distinguishing between "process-oriented routines" (called rituals) and "product-oriented routines" (called "deeds or explorations, depending on whether the routine is practical or discursive"), in order to support their claim that

> in the process of learning mathematics, the germinal routines, from which a discourse new to the learner is to emerge, are initially implemented as rituals. In the longer run, these routines are expected to undergo gradual de-ritualization until they become fully fledged explorations. (p. 153)

Exploration and ritual are "different genres of routines, corresponding to different mechanisms of utilizing our past experiences" (p. 154). The following quote highlights the difference between them in terms of awareness and autonomy:

> Deeds and explorations may be seen as acts of production: their execution is judged exclusively by the artifacts they generate. In the case of a ritual, […] the performers do not ask themselves "what is that I want to get?" as do those involved in deeds and explorations. This time, "how do I proceed?" is the only question that guides their actions. Since the resulting performance does not count as an act of production, one can only have social reasons for a ritual activity. Indeed, we perform rituals when we feel expected to do so, and in particular, when the expectation comes from those whom we see as in any way superior to ourselves. (p. 166)

Vergnaud (1990) defines mathematical concepts as triads consisting of reference situations, operational invariants, and linguistic representations. In Durand-Guerrier et al. (2012), proof is considered to be a mathematical concept according to Vergnaud's definition, and in the same perspective, definition and mathematical model can also be considered concepts.

Two episodes from the same classroom

A teacher (Marina Molinari) belonging to our research team wanted to introduce a class of 22 seventh-grade students to the process of defining and the concept of definition in mathematics. In the previous school year with the same class, the teacher had gradually developed the students' methods of productive interaction both among themselves and with the teacher in the perspective of Habermas' rational behavior (i.e., with attention to their conscious consideration of the truth of statements according to shared epistemic criteria, the effectiveness of strategies, and the quality of communication in the classroom).

The reported excerpts come from audio recordings, with supplementary information taken from a student teacher's field notes.

The first episode: How to define the diameter of a circle

Students had already encountered the term "diameter" in elementary school, where it was associated with a segment passing through the center of a circumference with extremities on that circumference. According to a classroom discussion after the second episode, they had no experience of constructing a definition or of reflecting on what "definition" means in mathematics (even if they knew, for instance, that "a right-angled triangle is a triangle with an angle of 90°"). The teacher's aim in the reported discussion was to allow the students to experience a guided defining activity and become aware of its goal. She drew a circumference (with its center) on the blackboard.

(T is the teacher; students [Sn] are numbered according to the order in which they entered the discussion)

1. T: Are you able to draw a diameter of this circumference?
2. S1: Yes, I will do this. [*S1 goes to the blackboard and draws a diameter for the drawn circumference.*]
3. T: Could you explain to a friend of yours what a diameter is?
4. S1: Yes, it is when I draw a line through the center of the circle. [*He turns to look at his classmates.*]
5. S2: A straight line with a ruler.
6. S3: A straight line through the center of the circumference.

7. T: A straight line like this? [*She draws another segment passing through the center of the circumference that is shorter than a diameter.*]

[*The discussion develops as to how to describe the construction of a diameter*.]

13. S1: Thus I should say: A diameter is a segment that I draw by choosing one point on the circumference and then by drawing a segment from that point through the center until it reaches the circumference again.

14. T: OK, this is a fine description of how to draw a diameter. [*She writes it down on the blackboard*.] But what is a diameter? How do you distinguish this segment which is not a diameter [*she draws a chord that does not pass through the center*] from this segment [*she draws a diameter*]?

15. S2: The segment must pass through the center, as S1 said.

16. T: Thus, what is a diameter?

17. S2: It is a segment that passes through the center.

18. S8: And which moves from one side of the circumference to the other side.

19. S2: A segment through the center joining two points of the circumference.

20. T: [*She writes: A diameter is, etc., as S2 said*.] What is the difference between this sentence and the previous sentence?

21. S9: The first one describes how to draw a diameter.

22. S2: While the other says what a diameter is.

23. T: [*She draws a chord that does not pass through the center*.] Is this a diameter?

24. S2: No, it does not pass through the center.

[*The teacher underlines the corresponding part of the definition with red chalk, then she draws a very short segment through the center, she asks if it is a diameter, etc.*]

The second episode: How to define prime numbers

In the same classroom, one month later, students had refined their understanding of the definitions of even numbers and adjacent angles; gradually, the teacher's support and guidance had become less concerned with the process of orienting the students towards their goal (the students' labels correspond to the students from the first episode).

1. T: Do you know what a prime number is?
2. S2: A number without divisors, like three, seven, eleven.
3. S7: I think that it is not necessary to give examples.
4. S8: The words are sufficient; like for even numbers, it was not necessary to say two, eight, ten.
5. S7: Yes, it is sufficient to say that it is a number which has no divisors.

[*The teacher writes "a prime number is a number which has no divisors" on the blackboard.*]

6. S1: But [*pause*] every number is divisible by one [*emphasis*]
7. S3: Yes, we should say [*pause*] a prime number is a number which has no divisors, except one.

[*The teacher completes the sentence this way*.]

8. T: Do you agree?

9. S2: A number [*pause*]. What happens with decimal numbers?
10. S12: One point two divided by two makes zero point six.
[*A discussion follows about how to exclude cases like that or like 3:1,5 from the definition*.]
16. S8: We need to define [*pause*] exactly the numbers that we know as prime numbers from elementary school [*pause*].
[*students produce and discuss some examples of prime numbers*.]
21. S7: We must add that we take only integer numbers that are divisible by one.
22. S4: With an integer result [*emphasis*].
23. S1: Yes, we need to be precise: divisibility means [*pause*].
24. S2: We need another definition, that of the divisibility of an integer.
25. S3: Divisibility [*pause*]. Divisibility by another integer.
26. S12: I think that we may say that a prime number is when we have a number and we find whether it has any divisors by dividing it by the numbers that are smaller than it.
27. S9: Yes, if we find that it is only divisible by one, it is a prime number.
28. T: Thus, what is a prime number for you?
29. S9: It is a number that if I divide it by all the numbers that are smaller than it, I find that its only divisor is one.
30. T: [*Writes what S9 said on the blackboard.*] What do you think about what S9 is proposing as a definition of a prime number?
31. S8: We need to explain what a prime number is, not how to ascertain if a number is a prime number. It is like in the case of a diameter. [*Pause.*]
32. S6: It does not work, I remember when we worked out the definition of a diameter that how to draw it was not a good definition
33. S15: I do not understand. Explain better what you mean
34. S2: Yes, I understand: in that case, how to draw a diameter was not a definition, while saying that a diameter is a segment through the center, etc., is a definition.
35. S8: In this case, we must say that a number is a prime number if its only divisor is one.
36. S14: Or, to be more precise: A number is a prime number if it is an integer number and its only integer divisor is one.
37. T: [*Points at the definition read by S8.*] Thus, what do we need to add to what is written on the blackboard? And why?

[*The discussion continues regarding the need to consider "integer numbers" and also accept the prime number itself as a divisor*.]

Two weeks later, after a similar activity on the definition of a circumference, this individual test was proposed (the definition of a square had not been previously discussed in class):

Are the following sentences definitions of a square? If not, explain why, for each sentence that is not a definition of a square.

- A square is a quadrilateral with four right angles

- A square is a quadrilateral with four sides of the same length
- A square is a quadrilateral with opposite parallel sides

Nineteen students out of 22 answered that none of these sentences was a definition of a square; among them, 16 students found valid justifications, based on salient features of a definition, for rejecting the three sentences: e.g., "the first sentence includes also those rectangles that are not squares", "the third definition also includes all parallelograms".

Analyzing and reflecting on the episodes: Rational Mathematical Templates

In the first discussion, the teacher leads an interaction that involves students in the guided construction of a definition; during the discussion, the teacher mediates salient features of defining as a rational behavior: epistemic criteria for judging the validity of a statement as a definition (16, 22); the need to ascertain whether the statement includes diameters, and only diameters, and the ways of doing so (production of various kinds of segments, 23); the precision of wording and communication (7).

By comparing what students do in the second discussion (and how the teacher orchestrated the discussion in the first discussion) with their behavior in the first discussion, we may observe how the process of defining developed in strict relation to the development of mathematical rationality in the specific domain of theoretical mathematics: in the second episode, the students take care of criteria that characterize a mathematical definition (3, 4, 5; 31, 34) and that allow them to decide whether a formulation satisfies them (9, 21) (*epistemic rationality*); they engage in a goal-oriented process (*teleological rationality:* 16, 33); during the discussion, the students are concerned with the formulation of the statement that is to represent the set of objects (which constitutes the semantic field of the definition and needs to include all the objects associated with the expression "prime numbers") and engage in mutual corrections about it (*communicative rationality*: 23, 36). This is the reason why we have chosen the provisional expression "Rational Mathematical Template" to name *the couplet consisting of a mathematical entity (in our case, the definition) and a rational process that is purposefully oriented to produce an instance of that mathematical entity* – for the moment, a simple provisional definition of a more complex object, as we will see in the Discussion.

Comparison with other constructs

What students have acquired is more than a concept, according to Vergnaud's definition of concepts as adapted by Durand-Guerrier et al. (2012) to the case of proof. In Episode 2, we find *reference situations* (e.g., the reference to previous activities about the definition of even numbers and diameter, interventions 4 and 31); *operational invariants* (implicit in most cases – like "theorems in action" in Vergnaud's definition, but also explicit, e.g., 3, 5, 33); and *linguistic representations* (e.g., 34). However, students develop the second discussion according to interiorized "how-to" guidelines and move towards a goal to be achieved. This is not included in a concept. Note that in the case of the RMT of definition (or theorem, or mathematical model), the "mathematical entity" component of the RMT is a concept, according to Durand-Guerrier et al. (2012).

In spite of some surface analogies (the RMT process of definition looks like an exploration routine, in situations of definition production), there are profound differences between routines and RMTs concerning the content-specificity of the definition of RMTs as couplets (mathematical entity, process); the crucial role of awareness (related to the rationality perspective) in RMT-led activities, be they guided by the teacher or autonomously developed by students; the characterization of the RMT process as “rational”, which implies a reference to the components of rationality in the analysis of RMT-led activities; the narrow interconnection between epistemic and teleological dimensions of the RMT-led activities; and the attention to communication issues during them.

Discussion: Hints for further research

In the last year, we have realized that there is more to say about the object emerging from the analysis of classroom discussions aimed at the development of rational behavior than what is represented in the provisional definition above. In particular, the second author’s observations in her classes and the related reflections have led us to focus on both the affective and the cultural dimensions of how RMTs work in a “satisfactory” classroom interaction. At present, we see the RMT as a medium in the 1:1 teacher-student relationship, in the 1:n teacher-classroom relationship, and in the network relationship among the members of the classroom. It seems to us that the RMT as a medium plays a double role: both towards the epistemic dimension and towards the affective relationship. The medium role of the RMT strengthens and inter-exchanges with the medium role of the teacher in an interaction space, which looks like a game space; the participants gradually enter it with different roles. This game space, constituted/made possible by the RMT, is the space of cognition and affect, which develops students’ subjectivity, identity, autonomy. In the classroom, we may interpret Habermas’ communicative rationality thus: she who behaves according to communicative rationality is someone who wishes to get in touch with another person and with the Other (the Culture): thus, her driving interest is to be understood by, and to understand, the other (and the Other) in order to create a discourse – i.e., something which dynamically flows among the participants in the communication; communication is conditioned by their communicative productions, and at the same time, they are modified by it. Not only do RMTs allow communicative rationality to be exercised and connect communicative rationality with epistemic rationality (a connection already hypothesized by Habermas), but they also contribute to creating (thanks to the teacher’s guidance) and maintaining a space of wellbeing for all participants under the driving force of reaching the other (and the Other).

In the last few months, teachers who collaborate with us have been engaged in designing and reporting classroom situations of the initiation and development of RMTs and of communicative rationality “in action” based on those RMTs. Those teaching experiments should contribute to answering the following questions: How should one characterize and (possibly) frame the dialectic relationship between the evolution of the RMTs and the development of the communicative rationality whose well-functioning depends on one (or more) RMTs being available to students? What about the affective dimension (motivation, interest in reaching others and being reached by others) of classroom RMT-led interactions? How does it depend on students’ familiarity with RMTs, and how does the growth of students’ familiarity with RMTs depend on the affective quality of interactions in the classroom? And what about the possibility that students who have experienced RMTs after the teachers’ initial guidance are enabled to develop other RMTs by themselves in new situations? The last two questions are connected to another important question: How should students be initiated to RMTs? The mediating role of the teacher should play a crucial role in engaging the students in the “game” and in making the aim of the activity explicit. It is important to avoid the danger of a one-sense transmission with the related negative contractual effects on students, because the goal is to develop their rational behavior. A participated, gradual appropriation of the RMTs (see the first episode) might avoid this; the "voices and echoes game" (Garuti & Boero, 2002), a participated cultural transmission method relying on the dialogue with the Other, might be another valid method. In the classroom context, the above perspective of the dialectic-dynamic relationship between the development of the communicative rationality space and the evolution of the RMT is aimed at establishing a constructive relationship with the Other (the Culture), intended not as an object to be received and conserved "as such" by the students, but as a means for their cultural growth, as they use and simultaneously transform it in the "intersubjectively shared lifeworld" (Habermas, 1998, p. 315). This consideration establishes a close connection with Engeström's construct of "expansive learning" (Engeström & Sannino, 2010), to be further explored.

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Interactional obligations within collaborative learning situations bringing forth deeper collective argumentation

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Following an interactionist approach to learning, social interaction is to be understood as constitutive of learning processes. Based on this idea, learning of mathematics can be described as the increasingly autonomous participation in processes of collective argumentation (Krummheuer & Brandt, 2001). This study aims at describing, on an empirical level, how children receive optimised conditions for learning opportunities when, within the interaction with other learners, the children are interactively "obligated" to participate in bringing forth warrants or backings within collective argumentations. Specifically for this research, children between the ages of 6 and 12 were filmed when learning collaboratively in multi-age groups. The three interactional obligations which have been reconstructed so far – a strong contradiction, a mistake and certain types of questions – are described in this paper with the help of exemplary interactional sequences.

Keywords: Collective argumentation, collaborative learning, heterogeneous grouping, interactional analysis.

Learning in pair and group work

Within mathematics education research, mathematics is increasingly seen as being mediated and constructed by language. Social interaction is thus to be understood as constitutive of learning processes as social-constructivist theories of learning, including interactionist approaches to learning, view individual processes of learning as being constituted in the social processes of the negotiation of meaning (e.g. Vygotski, 1978; Voigt, 1994). This also results in a greater demand within mathematics education to incorporate group and pair work into mathematics classrooms, instead of using predominantly whole class discussions and individual work, as such collaborative activities are seen as holding a unique potential for providing opportunities for students' learning (e.g. Brandt & Höck, 2012; Jung & Schütte, 2018). From an interactionist perspective, learning can be seen as the development, expansion or modulation of framings which becomes visible in an increasingly autonomous participation in collective argumentation within classroom interaction (Krummheuer & Brandt, 2001; Jung & Schütte, 2018). As this research follows this interactionist perspective, two main concepts of this perspective on learning - 'framing' and 'collective argumentation' - will be explained in the following chapter, as well as the relevance of 'interactional obligations' for bringing forth collective argumentation. Then the guiding research question for this study "Which interactional obligations for bringing forth warrants or even backings for conclusions can be found within collective argumentations in group and pair work?" is derived from the underlying theory and preliminary results are described.

Central concepts of the interactionist theory of learning

The interactionist theory of learning is based on the theory of Symbolic Interactionism which views meaning not as something inherent to an object but rather something that develops in the interaction between people about the object and is therefore a social product (Blumer, 1975). Different participants of an interaction have their own interpretations of a situation on the basis of their individual experiences and knowledge – called definitions of the situation (Krummheuer, 1992). While the participants negotiate these different definitions of the situation within collective argumentations, ideally these definitions will become more aligned with each other and a working consensus (Goffmann, 1959) is brought forth enabling the participants to further work together. The working consensus has the potential to stimulate cognitive restructuring processes in the individual as it possibly goes beyond their previous experiences.

Framing

Especially, when the working consensus is brought forth repeatedly in the interaction, the participants' individual definitions of situations can become standardised and routinised and can therefore later be recalled in similar situations. Bauersfeld et al. (1988) call these individual definitions which outlast the situation *framings*. The process of developing, expanding or modulating individual mathematical framings is when learning takes place. Schütte and Krummheuer (2017) further differentiate between development of framings related to mathematical terms and procedures, which equals the acquisition of mathematical content, and the development of framings related to methods for reasoning and explaining, which they describe as the development of mathematical thinking. However, it often happens that the framings are not in alignment with each other (Krummheuer, 1992). In order to continue the process of negotiating meaning, the *differences in framing* between the participants need to be coordinated. While these differences in framing can make it more difficult for the participating individuals to adjust their definitions of the situations to fit each other, they also provide the "'motor' of learning" (Schütte, 2014) since, on the interactional level, they generate a necessity for negotiation. This is why, hereafter the focus will be on students participating in collective negotiations of meaning or argumentation, as this seems to hold the potential for learning to take place.

Collective argumentation

Krummheuer takes the idea of *collective argumentation* from a sociological learning theory by Miller (1986), who sees participating in collective argumentation as essential for fundamental learning – meaning learning something 'new' – for young learners. Contrary to Miller, for whom a collective argumentation is a communicative type of action which serves to solve a socio-cognitive conflict by bringing forth different arguments collectively and negotiating them, this study agrees with Krummheuer (1995) for whom there is no need for an explicit socio-cognitive conflict. He involves an ethnomethodological perspective which leads him to the conclusion that participants always indicate the rationality of their behaviour in the interaction (see also Jung & Schütte, 2018). Out of this concept Krummheuer and Brandt (2001) develop an interactional theory of learning where participating in collective argumentation within classroom interaction becomes both an opportunity for learning and the increase in autonomous participation an indicator that something has been learned. If collective argumentation is so central to the learning process, the question arises, when are collective argumentations brought forth? In whole class discussion, the teacher is predominantly the one fostering arguments to be brought forth, e.g. by asking for clarification or reasoning, or by giving counterexamples to claims (O'Connor, 2001; Schwarzkopf 2001). But often times only few students have the opportunity to actively engage in these interactions. On the other hand, in group and pair work more students have the opportunity to engage more actively within the interaction and therefore possibly within an emerging collective argumentation. So the question arises: When do students bring forth collective argumentations?

Interactional Obligations

According to Voigt (1994), participants within an interaction do not act independently of other participants' utterances and actions. Rather, individual utterances and actions create *interactional obligations* connecting them with each other. Because of these obligations, individual participants develop expectations of how others will react to an utterance or action. However, these expectations typically remain implicit unless conflict arises between the participants. Therefore, whether or not a person interprets an utterance or action as creating an interactional obligations for him- or herself to respond to and in what way he or she should respond, is very much based on his or her framings. Thus one can only reconstruct an interactional obligation by the response of the other participants and by how the interaction unfolds. For Schwarzkopf (2001), interactional obligations are crucial for initiating argumentations. However, like Voigt he only analyses the obligations between teacher and students and not between students within collaborative learning situations. So the question arises whether there are interactional obligations for collective argumentations in students' interaction during group and pair work? This is the guiding research question for this paper.

Interactionist approach for analysing interactional obligations

Methodologically, this study is located within interactionist approaches of mathematics educational classroom research (e.g. Krummheuer & Brandt, 2001). Videos were taken in several multi-age classrooms with students learning together in different combinations of grade levels one through six (ages 6 to 12) depending on the school. Only schools which incorporated multi-age education in their school's pedagogic concept were chosen, however how multi-age learning is part of their everyday mathematics lessons is very diverse. In general, multi-age learning groups were chosen, as the overall goal of the study is to show how students with great diversity in their learning preconditions work together in collaborative settings. Furthermore, in multi-age learning groups it seems more likely that collective argumentations are brought forth because of the possibly more diverse framings the students may contribute than in single-age learning groups.

Only interactions of pupils from different grade levels working together in group or pair work on the same mathematical task are then transcribed and analysed using the interactional analysis first developed by Bauersfeld et al. (1988). The core of this analysis is a sequential analysis of the individual utterances combined with a turn-by-turn analysis in order to find possible interpretations of utterances taking into account the sequential organization of the conversation. The interactional analysis allows research to reconstruct the ways in which negotiations of mathematical meaning are interactively constituted by individuals. Furthermore, it can help to reconstruct patterns and structures of verbal actions of the teacher and the students (see also Schütte, Friesen & Jung, 2019) – as e.g. interactional obligations. Subsequently, the analysis of argumentation was used which is based on Toulmin (1969) and identifies which utterances or actions contribute to which of the four functional categories of an argumentation: data (undoubted statements), conclusion (inference together with the data), warrant (contribution to the legitimation of the inference) or backing (undoubtable basic convictions which refer to the permissibility of the warrant).

According to Krummheuer and Brandt (2001), one of the aspects which increases the possibility of mathematical learning is seen when processes of argumentation with a complete ‘core’ of an argumentation - meaning data, conclusion and warrant - are produced and not only the data and/or the conclusion which is often the case. The research question therefore more specifically is “Which interactional obligations for bringing forth warrants or even backings for conclusions can be found within collective argumentations in group and pair work?” In the next chapter, interactional obligations which have been identified so far will be presented each by describing it and by giving an exemplary interactional sequence.[4] For each presented transcript, the grade levels of the children participating in the interaction will be given.

Preliminary results

Interactional obligation nb. 1: a strong contradiction

A reoccurring interactional obligation seems to be a strong contradiction to a previous utterance or action especially when the working consensus emerges among the students that they have to agree on the solutions before continuing their work. An example can be seen in the following transcript between Isabella (grade 1), Hans (grade 2) and Elias (grade 3) working on a combination problem about creating Christmas ornaments using the colours blue, green and red and either stripes or dots in those same colours as the pattern. They are allowed to use only two colours for one ornament and have paper circles, dots and stripes in the different colours as a support.

180	Elias	and now we will do it the other way around [*puts a red circle on the table*]
181		< [*takes a blue dot and puts it on the red circle*]
182	Isabella	< I wouldn’t do that
183	Hans	with two dots\ [*hands Elias another blue dot*]
184	Elias	no with one\ [*has his finger on the one blue dot*]
185	Hans	< nooo
186	Elias	< because because we only have so little of blue left and we still have some
187		*more left here right/ [points to the remaining blue dots and then points to the*
188		*other circles which are left*] there we maybe still want to (also) #
189	Hans	#oh come on\ look we still have so many of the other colours left\ . [*first*
190		*points to the green dots and then puts a second blue dot on the red circle*]

Table 3: Transcript Elias, Hans & Isabella

What can be seen is that even though Isabella contradicts Elias’ suggestion (182), he does not respond to her. He does not seem to feel obligated to give a reason for his action by her “simple” contradiction. In the same way, when Elias in (184) contradicts Hans’ suggestion to add two dots in (183), Hans only contradicts Elias stronger by emphasizing the “no” (185) but also at first without giving a reason. However, Elias starts giving a reason in (186-188) – here the warrant that they should only use one dot because otherwise they do not have enough dots left for the other circles – defending his own suggestion. This happens at the same time as Hans emphasizes his contradiction. One could argue that Elias feels obligated to explain his action either because two different children have contradicted

[4] Transcripts are translated by the authors from German.

his solution (Isabella in 182 and Hans in 183) or because Hans contradicted Elias more strongly by emphasising the "no" in (185). Because Elias gives a warrant for his conclusion (186-188), his contradiction of Hans' suggestion is now stronger than in (184), and this seems to in return put Hans under obligation to now give a warrant for his conclusion as well, which he does in (190, 191).

So this analysis suggests that simply contradicting a fellow student does not put them under obligation to further explain their action or conclusion. Yet, when the contradiction is strong it does, which seems to be either when a "no" is emphasised by saying it louder, longer or repeating it, or when a second contradiction of another child is added, or else when a warrant is added to a contradiction.

Interactional obligation nb. 2: a mistake

Another utterance or action which seems to be an interactional obligation to give a warrant or backing for an argument is when a person interprets this utterance or action as being a mistake. This holds true both for own mistakes and mistakes of others. However, as mentioned above, since it is all about one's definition of the situation within the interaction, it is irrelevant whether it is an actual mistake or not. An example is when Kim (grade 1), Erich (grade 2) and Hannes (grade 3) are working on the same task as the three kids above:

58	Erich	< [*takes two red dots and puts them on the blue circle in front of Kim*]
59	Hannes	< and green strokes/ [*touches the green stripes*]
60	Erich	yes/ I have an idea\ so > here there there a green stroke has to go and there
61		also (unintelligible) stroke has to go\ so that
62	Hannes	> [*holds a green stripe towards the blue circle with the red dots*]
63	Kim	> we could also do (this)
64	Hannes	no/ there is another red dot missing put the stroke there . < no I will do the stroke/
65	Erich	< [*takes another red dot and holds it next to the blue circle*]
66	Kim	wait
67	Hannes	[*puts a green stripe underneath the two red dots on a blue circle*]
68	Kim	yes like that
69	Erich	smiley\ with nose\ [*puts red dot on the blue circle*]
70	Hannes	no/ we are only allowed to use two coolloouurs/ . shoot . green gone [*takes*
71		*green stripes off the circle*] . so

Table 2: Transcript Hannes, Erich and Kim

The three kids have been collaboratively working on this solution, all contributing at different times. Hannes is the one who suggests to add green stripes as well on the blue circle that already had red dots in (59) and in (67) places them there himself. As long as the children agree with each other's suggestions, no one feels obligated to go into further details and give reasons for their actions or suggestions. However, in (70, 71) he clearly interprets his own (or their joint) solution of a blue circle with red dots and green stripes as being a wrong solution – and therefore as having done a mistake – which seems to make him feel obligated to give a warrant for his conclusion of taking off the green stripes again (70, 71). His warrant here is that only two colours are allowed to be used and so the third colour has to be taken off.

Interactional obligation nb. 3: certain types of questions

The third interactional obligation is more difficult to clearly define as it seems to not be the question which obligates students to argue more deeply but what is expressed implicitly through the question.

One possible type of questions seems to be when the question is the suggestion of a conclusion but phrased as a question suggesting uncertainty. An example can be seen in the following transcript of Isabella (grade 1) and Henriette (grade 3) trying to find as many pentominoes[5] as possible by using five small paper squares and a worksheet (WS) in chequer pattern.

222	Isabella	[*moves one square of the pentomino lying on the table*] . voila/
223	Henriette	we had thiiis/ (unintelligible) [*looks at the WS and turns it 90°*]
224	Isabella	mhm
225	Henriette	well here < [*turns the WS further around*]
226	Isabella	< or is it this/ . this this this this this [*points to the blue pentomino on the WS*]
227	Henriette	this [*points to the blue pentomino on the WS*] because > el [*moves her finger*
228		*along the pentomino from the top down two squares and then one to the right*]
229	Isabella	> no\ no no no#
230	Henriette	yes\ el el [*moves her finger like the letter ‘L’ twice along the pentomino on*
231		*the table made of little paper squares*] el el [*then moves her finger like the*
232		*letter ‘L’ twice on the blue pentomino on the WS*]

Table 3: Transcript Henriette and Isabella

In (226), Isabella asks “is it this?” and point to the worksheet which seems to be on the one hand suggesting the conclusion that the pentomino she created with squares is the same as one on the work sheet they have already created earlier, and at the same time expressing uncertainty through formulating it as a question. Henriette then starts giving an explanation with a warrant and even a backing of why the pentomino they created on the table is the same one as the light blue one on their worksheet. She could feel obligated to give this deeper argument because of Isabella’s question suggested uncertainty or because she interprets Isabella’s suggestion as a mistake, making (226) an interactional ‘obligation’ in the category of a mistake. In the utterances and actions (229-232), one can also see another example of the interactional obligation number one “a strong contradiction” by Isabella which then obligates Henriette to further deepen her argument.

Another type of question acting as interactional obligation can be seen in the following interaction between Lia (grade 2) and Lara (grade 1) while comparing pentominoes they have found with two 3rd graders. Here the question seems to explicitly request an explanation or justification of a contradiction. Lia makes a suggestion which Lara seems to interpret as a mistake but does not follow the interactional ‘obligation’ to give a warrant for her conclusion when contradicting Lia. By specifically asking Lara to justify her contradiction, it becomes visible that Lia seems to interpret the situation in a way that she thinks Lara should give a warrant for her conclusion. She therefore makes the interactional ‘obligation’ more explicit for Lara to respond, or only now Lara interprets the situation in a way where she feels obligated to give a warrant which she does in.

Conclusion

So far three interactional obligations have been identified after which students seem to feel obligated to give a warrant or backing for an argument: a strong contradiction, a mistake and certain types of questions. Therefore, within the interaction these obligations open up opportunities for students to participate more autonomously within the collective argumentation and bring forth a complete ‘core’

[5] A shape consisting of five squares joined together at their sides.

of an argument. As a result, this gives all students participating in the interaction optimised condition for learning as they can possibly either expand their framings by comparing their own definitions of the situation with their interpretations of the arguments brought forth in the interaction and therefore develop new or expand existing framings, or modulate their framings by bringing forth arguments because of interactional obligations in the interaction and therefore change their own framings related to methods for reasoning and explaining (see also Friesen, 2019; Jung & Schütte, 2018).

In the future, these findings will be compared to other interactions in order to describe these obligations in more detail and possibly describe further obligations as well. As all the analysed interactions in this study are from heterogenous groupings in collaborative settings, analysing interactions in other types of group and pair work, could reveal different interactional obligations. However, the results of this study can still help mathematics educators understand more deeply which interactional patterns can be found within group and pair work and how learning takes place within collaborative situations. Furthermore, it can help teachers when listening to students giving arguments to understand the interactional patterns at work in the students' interaction, and possibly help the teacher foster collective argumentations within pair and group work by "sensitising" students to these interactional obligations or even "modelling" them in whole class discussions.

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Trying to improve communication skills: the challenge of joint sense making in classroom interactions

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In this paper we examine the efforts of one teacher working to improve her students' communication skills as part of a collaborative project with teachers and teacher educators/researchers. The paper reports on a project meeting where the teacher presents a short video clip featuring two student explanations. Yet only one explanation is treated in the lesson as an example of good communication. Following discussion and multiple re-viewings of the video clip in the meeting, what counts as good communication is critiqued by the teachers. Driven by an emphasis on the two-way nature of communication, the need for joint sense-making between teacher and students, and privileging explanations that communicate mathematical understanding, alternative teacher actions are suggested during the meeting that are related to how different teachers interpreted the students' explanation.

Keywords: classroom interaction; sense-making; video-based professional development; mathematical communication; explanations.

Introduction

Students explaining mathematical ideas, relationships and reasoning is something that students often find difficult to do, and teachers often find difficult to support students to do (Michaels & O'Connor, 2015). There is now a wealth of literature and research detailing the different ways in which teachers can support students in their communications skills in mathematics including through their questioning (e.g. Boaler & Brodie, 2004), how they follow up on students' answers (e.g. Lim, Lee, Tyson, Kim, & Kim, 2019) and by giving students time to think (Ingram & Elliott, 2016; Sohmer, Michaels, O'Connor, & Resnick, 2009). One of the challenges is the need to provide students with the opportunities and support to give explanations whilst at the same time ensuring that the content of what students say supports the learning of all the students within the classroom, the difference between the interactive and dialogic aspects, and the epistemic aspects (Erath, Prediger, Quasthoff, & Heller, 2018). Whilst explanations can support students in developing new understandings of mathematical ideas at the same time as accomplishing linguistic goals (Moschkovich, 2015) by offering students opportunities to use mathematical terminology in meaningful ways.

Taking mathematical explanations to be "giving mathematical meaning to ideas, procedures, steps , or solution methods" (Hill, Charalambous, & Kraft, 2012, p. 63) an issue arises where ambiguities in the mathematical meaning of what a student is saying arises. In this paper we focus on a particular challenge that teachers can face when offering students the opportunities to explain their thinking – where the teacher has difficulty making sense in the moment of what the student is saying. These contingent moments place significant demands on teachers' subject and pedagogic content knowledge (Rowland, Thwaites, & Jared, 2015) both in terms of interpreting what students are saying as well as making the decision of whether pursuing the line of thinking will be beneficial to other students. At these points in time teachers have a range of talk moves or strategies (Howe, Hennessy, Mercer, Vrikki, & Wheatley, 2019; Michaels & O’Connor, 2015) which they could use such as revoicing (O’Connor & Michaels, 1993), asking the student to elaborate on what they’ve said, asking other students to elaborate or build on what has been said, or asking other students to agree or disagree or contrast with what has been previously said, though often the most common reaction is to move on to a new discussion. Where a teacher has difficulty making sense of what students are saying, it can also be difficult for teachers to make connections with other student contributions or the original task, or to support the student in improving the clarity of what is being said.

The interactional perspective taken in this paper (Blumer, 1969; Ingram, 2018) emphasises the situatedness of explanations within the context in which they occur. What counts as an explanation, and what counts as a mathematical explanation depends upon how they are treated by teachers and students as they interact (Ingram, Andrews, & Pitt, 2019) as well as the sociomathematical norms established in the classroom (Yackel & Cobb, 1996). This paper considers an example where an explanation is offered, and therefore is treated as an explanation by the student, that is not treated by the teacher in the moment as having mathematical meaning to the task being considered. This type of situation is significant in that it highlights the distinction made by Erath (2016) between teachers offering opportunities for students to give mathematical explanations, and teachers giving support for students to give mathematical explanations.

Discipline of Noticing

The way of working with teachers described in this paper is based on Mason’s Discipline of Noticing (2002) which combines reflective practice and action research. From this perspective, teachers shape their own professional development by reflecting and acting upon their own practice with a supportive group. The principle underlying the professional development is that by teachers noticing aspects of their own practice, they become sensitised to noticing this aspect in the future which opens up opportunities for acting differently (Mason, 2012). When working with videos of teaching, the practice of giving *accounts of* before *accounts for* is key to reflecting on practice using what is actually happening in classrooms, rather than our interpretations and impressions of what is happening. *Accounts of* describe the video clip in a way that others can recognise, whilst *accounting for* includes interpretations, explanations, justifications or criticisms (Mason, 2002, p. 41).

Method

Anna shared the clip of her teaching during a meeting involving 5 mathematics teachers and 2 mathematics teacher educators/researchers. The meeting used the CoNCAV (Collaborative noticing through close analysis of video) approach based on Mason’s Discipline of Noticing (Mason, 2002, 2012). This involves teachers sharing and discussing a 2-3 minute video clip, chosen by them, of their own practice. The meetings involved the analysis of the clip from the teachers’ own practice, focusing first on *accounts of* before *accounting for* (Coles, 2013) what is seen in the clip, and then considering alternatives for acting differently in the future. The meeting reported on is this paper occurred half-way through the second year of a two-year project in which a group of mathematics teachers from the same school met 6 times a year.

The data for our analysis in this paper are the verbatim transcripts of the meeting and Anna’s lesson, with both teacher and student voices being recorded in the transcript as well as there being a record of what was displayed on the board in the classroom. Anna's clip came from a Year 7 class (students typically aged 11 to 12 years) in England. In order to reflect the close collaborative nature of the project between the teachers and the researchers we have used the teachers' and researchers' own words as often as possible in the analysis, and furthermore we make no distinction between the contributions of teachers and researchers. We mark out instances of talk from the lesson transcript by including these as extracts, whereas talk from the meetings is generally included within the body of the text unless we are offering an example of an extended conversation from the meeting. Across the project as a whole, all of the meetings were audio recorded. In addition, the teachers shared with the researchers the video from all lessons they recorded, and not just those from which they chose clips to share with the group.

Results and analysis

The Clip.

The CoNCAV approach starts with the teacher presenting the 2-3 minute clip they have chosen for discussion. Anna introduced her clip by commenting: "*This is Year 7 and they've done some work, previous lesson, on sequence, just term to term rule of sequences and I've just put a picture up on the board of pentagons linked together like a matchstick pattern. They had to find the next couple of patterns and then the 12th pattern*." (See Figures 1 and 2).

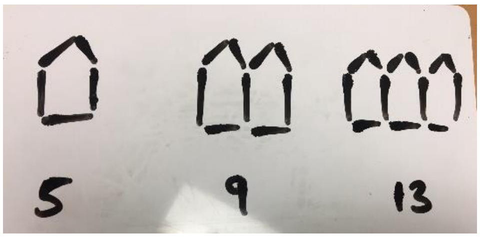

Figure 1: Matchstick pattern

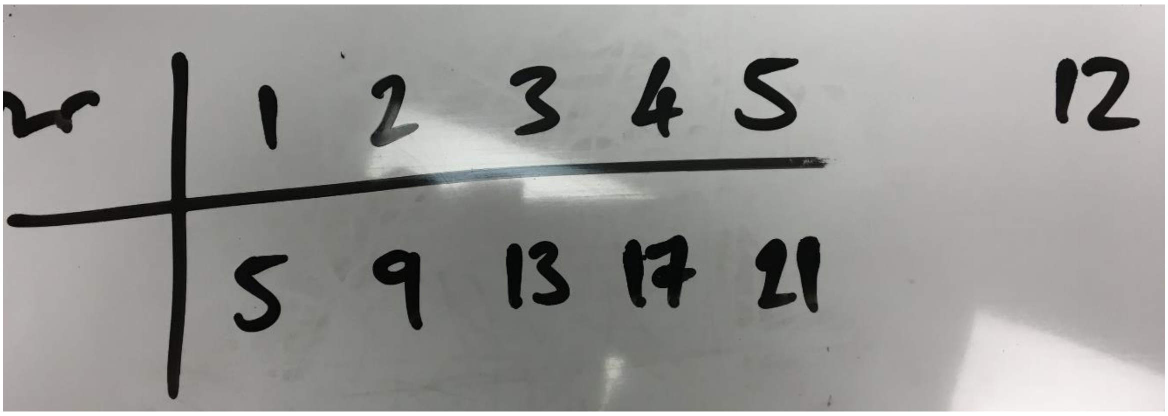

Figure 2: Number pattern

Typically the next step in the CoNCAV approach is for the teacher to give an *account of* the clip. Anna had been "*trying to improve their communication skills*" and one way she reported doing this was by leaving students time, or pausing, and her attention during the discussion following the first viewing of the clip was how she had been "*doing displacement activities. You know what people do when you don't like pauses? And, so I wrote on the board when the boy was thinking, to give people thinking time. So, rather than them feeling so much on the spot, I busied myself doing something on the board, so they could buy a bit more time*." She described the clip as showing improvement in the students' communication, but reported that Sam "*knew exactly what he was doing, he just wasn't quite clear enough*".

A critical feature of the CoNCAV approach is the re-viewing of the video clip. After the second viewing of the clip the discussion focused on Sam's explanation, given in Extract 1, of why there will be 49 matchsticks in the 12th picture.

25	Anna:	so, um, why is it 49 then, Sam? Can you explain in your own words, please?
…		
39	Anna:	how do you get from five to nine? What are you adding?
40	Sarah:	four, four, four, four
41	Anna:	what are you adding?
42	Sam:	Four
43	Anna:	yeah. Are you adding four there as well?
44	Sam:	yeah
45	Anna:	yeah. Are you adding four there as well?
46	Sam:	yeah
…		
58	Sam:	you times four by seven
59		(0.8)
60	Anna:	why?
61	Sam:	you add twenty eight to it and then that
62	Anna:	oh!
63	Sam:	no, 'cause if you times four by seven, we have seven until we reach like the number at the top, seven till we reach 12, and then ;cause you're adding four every time, it's four times even, like that, so that's 28.
64	Anna:	so four times seven is
65	Sam:	twenty eight
66	Anna:	so you think the answer is 28 sticks do you?
67	Sam:	um
68	Anna:	no, we agreed there was actually 49 sticks
69	Students:	yeah
70	Anna:	yeah
71	Sonia:	um, you do, um and so it's plus four each time
72	Anna:	mm
73	Sonia:	so you do twelve times four plus one, because twelve times four, no four times twelve is forty eight, so you have to plus one. Or if you do that, because it's just one more than the actual four times twelve, so you go four, eight, twelve, so it's four, nine, seventeen
74	Anna:	okay
75	Sonia:	so I think Sam means (0.3) you have to have twenty eight more sticks.

Extract 1: Sam explaining why there are 49 matchsticks in pattern number 12

Anna stated at the beginning of the meeting that she chose the clip because although Sam had got the right answer he had not communicated his ideas clearly. This meant that during the interaction with

Sam in the lesson, she had “*persevered with him, because I knew that, really, he knows what he was doing and after a bit of thought he could do it*”, but she also reached a point where she “*lost the will to live and moved to Sonia*”. Following this second viewing of the clip Anna acknowledged that in that moment, during the lesson she did not know “*exactly what he meant*”, and was not aware that his reasoning was valid, that “*he had got the right answer*”. Thus Sam did not communicate what he was referring to when he said that there were 28 matchsticks and he “*wasn’t able to just argue that point*” in turn 67.

The discussion then turned to possible actions Anna could have taken instead of moving to Sonia, and this represents the shift in the CoNCAV approach from focusing on the specific case of the clip to considering alternative ways of acting. Anna began by suggesting that she “*should have stalled for a bit longer and gone and watered the plants or something and let him think about it and then he could have explained better.”* Laura then suggested that another possibility would be to use Sonia’s explanation of what Sam means given in turn 75. This strategy involves treating Sonia’s explanation as a model of “*a clear explanation”* and then asking Sam if “*that is what you meant*”. Yet this raised another issue for Laura and Freya as illustrated in Extract 2.

Laura:	But, giving him that ... yes, getting him to then be able to do that next time is probably going to be hard, because he's just gone, "Yes, yes. What she said”.
Freya:	Yes, that thing of making him say it, exactly what he meant, then or even talking a bit about, "Why didn't I understand when you were saying it, but I've understood when Sonia has? What's different about the way she explained what you meant?
Anna:	I know. That's why we keep plugging away, don't we?
Freya:	Yes, but if they're let off the hook by just letting someone articulate explain it and they get to go, "Yes, I meant that," does that actually make him any more able to do it next time?
Anna:	No, it doesn't. So, tomorrow I'd put him on the spot again. We'll do another starter similar to that and I'll ask him again.
Freya:	I think some explicitly need teaching how to articulate something like that, don't they?
Anna:	Yes
Freya:	Like you were saying you could have gone and watered the plants instead, but I don't think he would have got there, because he doesn't actually know how to do it any differently. He was saying it the best he could. It was only when he heard her that he could see how it could be said.

Extract 2: Developing students’ explanations

The challenge of trying to improve students’ communication skills has become multifaceted for these teachers, involving giving students time to think and to articulate their explanations, using students’ explanations as models, but also explicitly teaching students what it means to articulate an explanation. Yet so far, the discussion has not considered the mathematical nature of the explanations being given. This shifts when Dave contrasts Sam's and Sonia's explanations in Extract 3.

Dave: although, I would say, out of the responses, he's got the clear insight of the nth term of arithmetic progression, so the a plus n minus 1 d. ((a + (n-1)d))

Anna: well, yes actually

Dave: whereas, some of the others are saying 'right, I can see the formula that I've got to multiply by four and add 1' without any idea of actually where that's coming from.

Extract 3: Contrasting the mathematical nature of students' explanations

The contrast becomes between an explanation that reports "*the calculation that I did*" and explanations that focus on the mathematical meaning behind finding the nth term. This requires the teachers to re-view the video once again in order to identify whether the students are making connections to the image of the matchstick patterns in Figure 1, or connections to the numeric sequences in Figure 2 or are solely reporting the calculation they did. This includes considering the differences between (a) Sonia saying 12×4+1 and 4×12+1, (b) the formula 4n+1, (c) the first term plus multiple common difference formulation 5 +11×4, and (d) Sam's 21 + 7×4. This leads to another possibility to support Sam in explaining his thinking by offering "*sufficient variation in the examples to actually highlight what he's saying".* Thus offering another example, for instance another term later in the sequence to identify or the twelfth term of the linear sequence 5, 11, 17, 23, 29 … to allow him to articulate his thinking on a similar problem, rather than just pausing.

The challenge these teachers are facing is that "*you're not just trying to teach them how to find the `nth' term rule of a linear sequence, you're trying to teach them how to explain how they find it, which is a whole different thing and means they've got to know what is a good explanation and what isn't, which goes back to what do we accept as a good explanation? So, are we accepting, because 4 x 12 + 1 is 49? Is that an explanation of wh*y?". In order to improve students mathematical explanations we need to consider what counts as a mathematical explanation as well as to consider the discursive moves to support students in articulating their explanations. We offer the example of the case of Sam's explanation and the CoNCAV approach as a contribution to the development of this aspect of mathematical work of teaching.

Conclusion

In this paper we examined the efforts of one teacher working to improve her students' communication skills. Following discussion and multiple re-viewings of the video clip in the meeting, what counts as good communication is critiqued by the teachers. Different future teacher actions were suggested which related to the different interpretations of Sam's explanation, in contrast to Sonia's. Anna focused initially on whether Sam's explanation was correct and the issue being the clarity of his explanation. She then suggested giving Sam more time to articulate his explanation until it became clear. Laura focused solely on the clarity of Sam's explanation and suggesting using Sonia's explanation as a model of clarity whilst also raising the issue of whether this would be sufficient to enable Sam to give a clear explanation for himself. Finally, Dave considered the mathematical content of the two explanations given and suggested the inclusion of an extension to the task being worked on or a similar task to enable Sam to explain across more than one example.

The CoNCAV approach thus enabled the teachers to view the students' explanations from different perspectives in a collaborative setting, whilst focusing on the key issue of working on students' ability to communicate their thinking. This led to an emphasis on the two-way nature of communication, the need for joint sense making between teacher and students, and consequently privileging explanations that communicate mathematical understanding both in terms of clarity and in terms of their mathematical content. The meeting also resulted in the identification of a range of potential actions for responding to students' explanations that are not clear, or where it is difficult to make sense of the student's explanation, in the future.

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Dissent and consensus situational structures in interaction based on mathematics and computer science learning

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How exactly and under which circumstances does the learning of the fundamentally new occur? To focus on this question, learning environments with core topics of mathematics and computer science are examined. In the focus of digitalization, computer science and its connection to mathematics will play an important role in future curricula, making it an interesting object of investigation. Furthermore, at primary level, computer science topics are about to be included in every subject, including mathematics. The paper presents an ongoing study that examines, how the topic of computer science connected to mathematics learning can be approached in primary schools and what and how meanings are negotiated. The focus will be on the question, what roles consensus and dissent play in interactional processes of negotiation and how the learning of the fundamentally (Miller, 1986) new occurs in collective argumentation between pupils.

Keywords: Interaction, Dissent, Consensus, mathematics, computer science

View on learning mathematics and the learning of the fundamentally new

When talking about learning, we see learning from an interactionist point of view. Miller (1986) developed a sociological theory, where he focuses on collective learning processes of at least two individuals to distinguish himself from psychological approaches to the term learning. Fundamental mathematical learning occurs in collective negotiation processes (Schütte, 2009). On the basis of a theory developed by Krummheuer and Brandt (Krummheuer, 2007) learning in mathematics is seen as an increasingly autonomous participation in collective argumentation processes. Hitherto, these theories focus on mathematics learning. To extend the theories a topic with a close relationship to mathematics is chosen but which is fundamentally new to primary school children: computer science. Generally, primary school pupils in Germany do not have any structured knowledge about computer science concepts or the connected underlying mathematics. To understand how pupils can connect mathematics and computer science, one has to realize, how and where mathematics is related to this specific field. Mathematics can be connected to computer science on two levels: the content level, which includes the specific topics, such as algorithms, calculation, logic, as almost all mathematics content is relevant for computer science, and the competence level, which includes e.g. argumentation, modeling, communication. Especially the content-related mathematical competences can be linked to competences for computer science education in many ways (Ludes-Adamy & Schütte, 2019). This is of course true about many other domains as well, but the relation between these two topics seems to be especially interesting. The general idea is to transfer the findings in mathematics education to the learning of the - on primary level - fundamentally new but closely related topic of computer science and later on reconnect these new ideas to the learning of mathematics to possibly pave new ways of learning mathematics content and develop additional approaches to mathematics education.

Working cooperatively

The approach to cooperative learning is based on the ideas of Johnson and Johnson (1999). They named three organizational types of lessons called individualistic, competitive and cooperative. The first two types place the individual in the center of action, where no or negative interdependence occurs. The last structure, the cooperative one, follows the idea that pupils cooperate in a way that the group, as well as the individual are responsible for the successful handling of the task, which is called positive interdependence, as the different members have to rely on each other's individual work, although it seems difficult to provide students with such a task and keep them inside this working scheme. The study tries to connect interdependencies with dissent and consensus, which possibly arise from negotiation processes where socio-cognitive conflicts appear (Miller, 1986; Nührenbörger & Schwarzkopf, 2010). Thus, different interactional structures might emerge from these specific situations that can be identified and structured. When working competitively or cooperatively different consensus and dissent situations arise on the basis of interpretational differences. On the basis of these, new mathematical meaning can be negotiated at the least, in a situational frame. The research question that will now be focused in this paper is how situations in which dissents and consensuses emerge are structured when primary school pupils learn a subjectively new topic (Schütte, 2014) and negotiate and construct computer science and mathematics meaning collectively.

Methodology

Learning environments in the intersection of mathematics and computer science with the topics logic, algorithms, cryptography, programming, binary code have been designed over several semesters, working closely together with students and teachers. The tasks themselves are designed closely to the concept of natural differentiation (Krauthausen & Scherer, 2014), although it became apparent that this premise is rather difficult when trying to convey fundamentally new content and there is little previous knowledge that can be activated. The children work on the tasks in groups of two to four. The data for the analyses originates from two pilot studies that have been run in small groups (11 and 13 pupils) in grade 3 and 4 and two main studies that took place with a larger number of pupils (48 and 47) in grade 4. The situations are recorded outside the usual lessons within the scope of a project week where the focus could be put solely on our learning environments. The recordings are transcribed and analyzed using methods of interpretive classroom research, primarily, interactional analysis (Schütte, Friesen, & Jung, 2019). The students that supervise the situations can provide help to the pupils if needed but are advised to maintain a low profile and merely observe if possible.

Tasks and analysis

The data shows different interactional structures, that will be described as dissent/consensus situational structures in the following. These can be differentiated according to the initial interactional relationship between the individuals and the emergence or transformation of the final interactional structure. The individual cases will be defined when describing the corresponding situation. One type has been described previously by Ludes-Adamy and Schütte (2018), where a *dissent-consensus-*

situational-structure emerges in which the collectively originated dissent transforms into a collectively negotiated consensus to construct situational-dependent new mathematical meaning (Krummheuer, 2007). The following examples will show other situations that indicate other interactional structures as well. The first task is a logical structure, with different statements that lead to logical conclusions. The 11 statements are as follows: 1. All numbers of this puzzle are integers. 2. The number D is 9 times A. 3. The sum of G and J is 100. 4. Three times D is G. 5. The fifth part of the number E is J. 6. H is half of D. 7. B is double of D. 8. If I subtract 2 from the quotient of E and J, I receive C. 9. The digit sum of all numbers is F. 10. I is double of C. 11. D is a number of the nine times table and less than 30. (Solution: A=2; B=36; C=3; D=18; E=230; F=62; G=54; H=9; I=6; J=46).

Kai	well, it is best, it is best . actually it is best/ well it can be nine . actually it is best if we get D
Jonas	is it now twenty-seven or eighteen . D/..
Kai	because with D there are a lot of tasks
Jonas	yes but then we have to find out B first
Kai	do we want to do the A eh do we want to (5) do we want oh man that is difficult(12)
Teacher	what did you want to say/ do we want/ ..
Kai	here . well (incomprehensible) . write down the eighteen for D
Jonas	(a moment) D/ . but it could be twenty-seven or nine as well
Kai	I mean to try out
Jonas	moment (9)
Kai	so do we want to write down eighteen for D/ .
Jonas	mhh let's try ..
Kai	so eighteen .. so and A ehh . no that doesn't work
Jonas	wait . a moment/ .
Kai	because when this is eighteen A must be two
Jonas	B is that . a moment
Kai	no look, when you look at the second
Jonas	H is half of .. D so it must be fourteen
Kai	no .
Jonas	if it is eighteen . .
Kai	nine /
Jonas	so .
Kai	H is nine if it is half of eighteen
Jonas	moment yes the eighteen is right . otherwise this would be . ehm . the sixth task one would not be able to solve it
Kai	so eighteen is right/
Jonas	yes

Figure 1 Transcript Kai & Jonas

Throughout the task, Kai and Jonas cooperate and have a relatively good fitting to each other's ideas. A working consensus (Schütte et al., 2019) seems to emerge, in which the two boys freely interact with each other, without ever leaving the working consensus. Kai's first utterance shows a logical deductive conclusion as he proposes to find the number D, because it is used in many other statements. He seems to understand that the possibility to find a successor increases, if a number that is often used in other statements is known. Jonas does not explicitly agree, but he seems to accept it as a good suggestion as he immediately proposes two possible suggestions for D. The two pupils show an almost perfect cooperation. They work together without specifically dividing tasks among themselves or discuss their collaboration. It seems a little bit confusing to follow their discussion, but they never seem to lose their goal out of sight. Kai and Jonas cooperate and deduct logically that eighteen can be the only possible candidate for D. In this first example, a general *consensus-situational-structure* seems to prevail throughout the cooperation. A *consensus-situational-structure* would be characterized through an interactional process, wherein the contradictions and clashes are so minimal that one would not recognize a general dissent. We call these small clashes *micro-dissents*. A *micro-dissent* is seen as a turn in a conversation, where one individual question the utterance of the other implicitly through short questions or injections but does not change the flow of the idea. The emerging idea is changed very subtle through these interjections and the outcome can be seen as a more solidified solution.

Another example of the same task will show a different situational structure. It is taken from a group of four girls who seem to be at different mathematical skill levels but try to work together. As they are discussing but do not pick up each other's inputs immediately, the conversations often stall, and the supervising student has the feeling that she needs to support through statements or help. This rather destroys the aspect of cooperative learning as the teacher takes all hurdles out of the way.

Lisa	I start. I would say we start with that G and J are simply fifty
Eva	so
Anna	no, because it also could be twenty and eighty
Lisa	Yeah sure, but how do you want to start . other than with that number . nine
Anna	Yes, then let's start with twenty and eighty or with seventy and thirty
Lisa	I don't care but we have to agree on one #
Eva	D is a number from the 9-series
Anna	or with forty and sixty or with thirty-five
Lisa	ten
Lisa	I would say we start with ninety and ten and then twenty hm no twenty and eighty
Anna	She just said something
Eva	D can be nine or eighteen or twenty-seven
Teacher	Well that is a nice overview of numbers . with those, you can #
Anna	Yes right, the nine, the eighteen and the twenty-seven
Teacher	mhm (affirmative) (5) you could fill it in for a start what you think what it could be
Lisa	so well/ . fill in what/ whatever
Eva	D
Anna	So are there any more information about D
Anna	yes three times D is G . mh that means one of these numbers has to be divided by three
Lisa	Wait, what should I put in?
Eva	For D you can write nine, eighteen and twenty-seven
Anna	yes both all of them (7) mh . here three times D is G that m e a n s . mh no idea
Eva	B is the double of D . H is half of D
Anna	H is half of D B is double of D but we don't know what these are
Teacher	Then let us think . think . if H is half of D then have a look at the numbers what the half was
Lisa	look . nine and half of that is three
Anna	point five but they have to be integers so nine
Eva	what/ eighteen only eighteen works with seven only eighteen works so it is eighteen because only eighteen works [all of them start to write]
Teacher	very well

Figure 2 Transcript Lisa, Eva & Anna

The girls all offer different ideas and they try to act upon those inputs. As the conversation seems to stall, the teacher helps. She suggests that the girls write down their possible answers. Lisa and Anna do not quite seem to understand what they are supposed to write down but Eva sums everything up after the teacher interjects a second time. It seems that Eva has more of an overview on the situation than the other two girls. Unfortunately, the teacher intervenes in a way that makes cooperative learning hardly possible as it destroys the exact situation where Anna and Lisa would eventually require help from Eva to understand what she already might have grasped. Eva seems to have a general idea and she could probably explain this idea to the others, but the teacher moves all hurdles out of the way. Whether the two other girls now have a better understanding of the situation or the solution does not become clear. Through the teacher's interference, the framing differences cannot be coordinated (Schütte et al., 2019). This is a difficult situation to observe real interactional processes between the children as this crucial point is covered up. As the pupils interact with each other and seem to try to at least pick up the sentences of each other but do not interact on a level that resolves their differences, we would describe this situation as a *consent-dissent-situational structure*. Although this denomination seems odd, it shall describe a situation, where the individuals are working collectively and interact but cannot align their framing differences to reach a working consensus.

A further example that illustrates a third type of situational structure is taken from the Algorithms learning environment. After learning what an algorithm is and how it is connected to computer science the pupils have the task to find algorithms in mathematics. Most groups decide to use long forms of calculation that always follow a specific pattern. Here the connection of computer science and mathematics becomes very obvious as the concept of an algorithm exists in both fields equally but at least in German primary education the term algorithm and its meaning are not taught to the children explicitly. Theo and Karl try to find an example to illustrate the long form of a calculation before writing down a general algorithm.

Theo	Let's say . . 500 minus 55
Karl	This is not the long ways Theo, I can do it in my head . . . 445
L.	What is the long way/
Karl	The long way is when you have 345 divided by 7 and then you write down this numbers and underline them and then write the equal sign
Karl	Now, I say 345 divided by 7 then it works.

Figure 3 Transcript Theo & Karl

Theo proposes an example that is perfectly suitable to write down the algorithm for a long form of calculation. Karl on the other hand thinks that only an example, where the long form of calculation is compulsively necessary (mentally), qualifies. Therefore, he does not consider Theo's example. An implicit dissent emerges that is not resolved in any manner but rather kept on a meta level. The basis for moving on is just Karl imprinting his opinion as correct onto the situation. This would be what we would call a *dissent-situational-structure*. This structure would be characterized through a dissent, that is not resolved in any way and exists throughout the situation, either implicitly or explicitly.

The tasks on logic and algorithms had a closer relation to conventional mathematics lessons as they deal directly with numbers, although being situated in a computer science context. To see how pupils interact in a new context and to elevate the theory from purely mathematics education we will look at a situation where the pupils talk about programming, a for primary school children fundamentally new topic, where specific framings were not yet differentiated. Programming melts computer science and mathematics together, as almost always calculations and mathematical thinking is required to write code and on the other hand, computational thinking, which is a core element of programming, is necessary and helpful in mathematics. Theo and Maria, who have never been taught about programming other than perhaps in their everyday language, talk about possibilities to program a microcontroller (Calliope Mini), that has been specifically designed for primary school children. They utter their ideas what programming is and what to do with the calliope. They have a laptop with the programming environment where visual code blocks can be linked together to create a working program. Elements that cannot be connected will not connect to each other. The pupils can therefore find out on their own what elements fit and how they interact. Theo and Maria test the programming surface with a trial and error method. The task is to try to create the code for a self-chosen program. They decide to write a program that makes the calliope play different tunes when touching the corners.

Maria	Let's use the element that makes sound. The calliope can make a sound when pushing a button so you can play music
Theo	But we can use the corners then we can make more sounds
Maria	Yes but we can use the buttons and make music
Theo	Yes but there are only two and then we only have two sounds
Maria	I use the buttons . . so 'if button A is pressed' play note c . . okay . . then 'if button A is pressed' play note d . . we can play Alle meine Entchen
Theo	No that is all wrong . . how is the thing supposed to know what to play . . you always say button A
Maria	yes first c and then d that is how the song starts
Theo	but it doesn't know when . . we cannot use the button more than once
Maria	why/ yes we can
Theo	no we have to use different buttons but we have two only . so let us use the corners there are four corners then we can play four notes
Maria	I don't know what you mean . you do it then
Theo	Okay . If corner 1 is touched play note c (7) if corner 2 is touched play note d (5) and so on (finishes for the other two corners) but we only can play now four different notes that is not enough for the song but we can play another song
Maria	But it does not work again only once . you made a mistake
Theo	no . what no that is awkward . no i don't know
Maria	let me try
Theo	alright have a look

Figure 4 Transcript Theo & Maria

Maria and Theo have a dissent on what method would be best. Maria tries her solution with the buttons but according to Theo makes a bad engineering decision as she assigns different tasks to the button. Theo tries to explain his thinking which he does very well. In this situation Theo seems to be willing to work cooperatively. He does not only present his idea, he also explains why Maria's idea is, in his eyes, less suitable. Maria on the other hand is not convinced of Theo's criticism and tries to stick with here solution until she finally suggests that Theo does what he wants. Then, when Theo hits a hurdle himself, Maria also suggests he might have made a mistake and wants to try and help which Theo accepts. The classification of this situation is rather difficult. The first impression is that this is a *consensus-situational-structure*, as Theo is willing to take Maria's ideas and tries to cooperatively tackle the mistakes she makes. Maria’s unwillingness to accept her mistakes and work together with Theo to come to a joint solution on the other hand would be more than a *micro-dissent*, as the general idea of the solution does not move forward. This situation seems to undergo a certain transformative process as it inherits marks of both the *consensus-situational-structure* and the *dissent-situational-structure*. This could imply, that dissent and consensus situations that are negotiated working a fundamentally new topic are rather harder to be resolved into a permanent working consensus. In comparison to the other situations, where mathematics itself was more dominant and obvious, the pupils here have no previous knowledge on no general idea so it could be harder for them to reconstruct each other’s ideas and articulate possibly helpful utterances. as the pupils have not yet developed framings of computer science meaning for themselves and therefore cannot follow the other’s ideas. We would therefore classify this situation as an *alternating-consensus-dissent-situational structure*.

Relating back to Johnson and Johnson (1999) different basic forms of cooperation can be identified that describe different positive interdependencies between pupils when working on the fundamentally new. Thus, these different interdependencies have varying influences on the learning opportunities of the new. Currently, the situations are examined to see, how these basic structures form patterns that can be found throughout the situations can be categorized into a descriptive system, to find out in what way dissent and consensus and their resolution can build the basis for the negotiation of computer science/mathematics meaning and through this provide a basis for learning opportunities of the fundamentally new within this domain. Especially the more computer science related task of programming suggests, that the cooperational aspects between the group members seem to faint, when the individual itself has to cope with sharpening framings for themselves about the new content and therefore struggles to reconstruct the other one’s ideas.

To this point, the situations that have been presented can be brought into a schematic. We could reconstruct four different structures:

Consensus-situational Kai & Jonas (Logic-Puzzle)	**Dissent-situational** Theo & Karl (Algorithms) (not shown in this paper)
Alternating-consensus-dissent-situational Maria & Theo (Programming)	
Consensus-dissent-situational Anna, Lisa & Eva (Logic-Puzzle)	**Dissent-consensus-situational** Simon & Arne (2018, Logic-True/False)

Table 1 Dissent/Consensus-situational structures

Further analysis will show, whether the situational structures that could be identified here, are also prevalent in other situations what their specific characteristics are and whether the finding of the fluent alternating structure is restricted to fundamentally new content.

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Senses of time and maturity structuring participation in advanced mathematics

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This paper examines how young people account for choosing advanced mathematical pathways, specifically how their language practices around time and maturity, inscribed in classroom and educational discourse, sustain, or not, their continued participation. It draws on a 2-year qualitative study of 24 young peoples' accounts of following advanced mathematical pathways within a widening participation programme. Working from a post-structural perspective, I identify distinct discourses –moving/improving and getting ahead - that structure the intelligibility of participation in the two pathways. I argue that tracing the alignments and tensions between these discourse offers potential to understand emergent practices in mathematics participation in terms of ongoing inclusions and exclusions that render individual student choices secure or fragile.

Keywords: Time; mathematics; aspiration; inclusion; social class.

Introduction

This paper starts from the perspective that language tools and rhythms contribute to the social structuring of time in educational settings, which in turn frame the educational choices of young people (Lingard & Thompson, 2017). Mathematics has been understood as culture-free and timeless, but is also pressed into service as the 'bright future' for individuals and nations. It is in this context that, in England, between 2005 and 2018, the government funded a national programme specifically to widen university-track 16-year-olds' participation in an enhanced mathematics pathway involving longer teaching hours, broader and deeper content (such as linear programming and complex numbers) and extra examinations. This pathway had entered a spiral of decline, available only in large, well-funded or fee-paying schools but still necessary for entry to elite university courses. It remains small: completed by 4.8% of academic-track students in 2018, while 27.4% complete standard mathematics. In England, 16-year-olds who have reached a threshold examination level can choose to study no more mathematics; thus the meanings that circulate in secondary classrooms about what it means to choose, or 'do', mathematics have material consequences. This study conceptualises the national programme as a serious attempt to reimagine the discursive structures of participation that reconstruct "knowing and thinking in/about mathematics" (Morgan, 2013) and patterns of exclusion. It draws on the accounts of 24 young people whose schools newly offered the enhanced pathway. These are analysed to trace the discourses that describe and construct these young people's choices and lived experiences, focusing on the senses of time and maturity that shape identities in relation to future and past selves. A full report of the research can be found in Smith (2020).

The research question driving this longitudinal, interview-based study was: What senses of time are circulated in students' accounts of choosing to study advanced mathematics pathways in a widening participation programme, and what discursive positions do they make available?

Theorising subject choice and time

I approach this argument and analysis using the Foucauldian concepts of discourse and positioning. Choosing mathematics is a discursive practice of the self, a way that power is circulated by producing knowing beings who judge their own and others' behaviour with respect to social norms (see, for example Smith, 2010). Discourses can be understood as webs of practices "that systematically form the objects of which they speak" (Foucault, 1972, p. 49). They are historically, linguistically and culturally contingent, but they present as unarguable regimes of meaning that gain currency through use. People are positioned relationally by discourse, by how they fit within webs of meaning, and they also understand themselves and position themselves through discourse. A sense of time frames people's explanations, purposes and imaginations, collectively and individually (Lingard & Thompson, 2017) and thus time was identified as a fruitful, and previously unexplored, focus through which to examine how discourses align locally to enable or prevent young people's participation in mathematics.

A post-structural method focuses on regularities in discursive practices, traces how these arise, how they align or interact, and the positions they make available. It thus shares a concern with Bourdieuan and cultural theory in how social patterns of aspiration are reproduced through individual behaviours. However, it focuses on the diffuse and productive functioning of power within particular situations, such as this national programme, and through specific social constructs, such as time, paying attention to alignments or tension between meanings that may include or exclude individuals. To ground this argument, I briefly outline macro- and micro-level discourses proposed by previous research in the three areas of: choosing mathematics, contemporary adolescence, and negotiating age imaginaries.

Discourse at a macro-level: choosing mathematics as a practice of the self

Within modern Western societies, choosing is constituted as a means by which individuals are enjoined to express themselves as agents with autonomy and subjectivity, and are held responsible for the processes and outcomes of those choices (Rose, 1999). Analyses of students' accounts have identified choosing mathematics as a practice of proving something *about* oneself in ways that are contingent on dominant discourses about the nature of mathematics as hard, competitive, rational, requiring natural aptitude and real understanding (Mendick, 2008; Stinson, 2013). These researchers show how students speak about their successes in "White male" mathematics as a counter to narratives of deficit or rejection that position them through race or gender. Such accounts of resistance are welcome, but their perceived rarity is part of what lends them discursive impact.

Discourse at a macro-level: adolescence as an emblem of modernity

Lesko (2001) argues that modernity privileges a *development-in-time* view of adolescents as always 'becoming', held between childhood and adulthood, at risk to themselves and others, and comprehensible in time without knowledge of their contexts. Lesko's argument rests on demonstrating how two main constructions of time are woven into the language that describes 20th century adolescence. Through *panoptical* time, adolescents are continually watched and measured normatively by age (for example, by schools' progress trackers); through *expectant* time they are positioned as not-yet-adults, unable to act until given social permission (e.g. everyone studies calculus aged 16). This means that for adolescents, time is necessarily involved with age and becoming mature.

Together, these temporalities construct knowledge about progressing towards a desirable end, a discourse that is valued in modernity but is not neutral since it recapitulates cultural stories about ‘progress towards’ Western civilisation.

Discourse at a micro-level: negotiating age imaginaries

Schools teach young people to express acceptable ambitions for employment even as they simultaneously consider contradictory dream jobs or realistic outcomes. The notion of “age imaginaries” (Alexander, 2014) acknowledges these concurrent constructions of young people’s future identities. Similarly to Lesko, Alexander suggests that time (and age) are central to modern youth, whose present conceptions of themselves are given meaning by the anticipation of consequence. Alexander’s research attends to what is mobilised in conversations as a marker of age differences – for example music choice or debt levels – and how transitions are organised. Whereas Lesko’s work emphasised the dominance of one temporal discourse, of adolescence as ‘becoming’, Alexander (2014) suggests that contemporary youth have a mercurial sense of temporality: that is, they imagine themselves in relation to multiple concurrent age-imaginaries that both follow and resist dominant discourses.

Study

The data comes from 31 interviews and 51 e-mail questionnaires with 24 students in three sites in England newly offering the enhanced mathematics pathway. The sites exemplified differing socio-geographic settings and teaching contexts: with the extra lessons scheduled in- or after-school, and either a full- or half-course offered. Almost all enhanced-course students at each site agreed to participate, a total of 10 girls and 14 boys. 13 participants completed an enhanced course, 9 stopped during their study and two had chosen not to take it. Using parents’ occupational classifications, 5 were working-class, 13 administrative/professional middle class and 3 unattributable.

The research data consisted of accounts of study experiences and choices, collected over two years using semi-structured interviews, half-termly emails, and observations of 2-6 mathematics lessons in each site. The students were interviewed by me either singly or in groups of 2-3, 18 during their first year after choosing their courses and 21 in the second year; with 15 interviewed twice. Interviews included direct questions about choosing subjects, how their class interacted in lessons, how they worked at home, and memories of learning mathematics. I also asked questions that involved talking about school and mathematics using unfamiliar adjectives. For example, students chose from a list of 12 adjectives to describe school subjects (such as *warm, talkative, straight*; derived by me from research into perceptions of mathematics, e.g. Gerofsky, 1997)), and explained their selections. I asked questions by email at significant transitional times, e.g. after receiving first module results, applying for and joining university, and to follow up any interesting responses in a reflective conversation The longitudinal aspect allowed me to follow discursive patterns in the students’ accounts as they made study choices. Observations documented lesson practices in mathematics so that I could trace their interactions with student accounts. Language use around time was not systematically recorded during the observation, but language used to describe the two pathways was recorded, providing a check for the later claim of different discourses.

Analysis involved coding the student accounts to locate language related to time or age (underlined in the following extracts), and examining whether/ how this language was associated with narratives of choosing and participation (Robson & Bailey, 2009). The codes and subcodes relevant to this paper were Time (as a resource/ age-related/ memories/ futurity), Maths and Further Maths (i.e., the names of the two pathways). Transcripts were initially coded for Time by searching for utterances (in interviews) or written answers (in emails) that included: time-specific words (e.g. before, always, future) or markers (at school, at university); verbs in association with time (e.g. spend, waste, have); age-specific words (e.g. child, mature); juxtaposing different times in reported actions (remembering, planning) or in explanations (e.g. I used to …, now I …; It will carry on ….). Coding for the pathways sought descriptions or explanations that were clearly ascribed to one pathway or the other, with subcodes for explicit comparisons (similarities; differences). Coded text was reviewed to ask, first, how, and in what contexts, students used these senses of time and age to position themselves and others as choosing and doing mathematics; and, secondly, whether there were alignments or tensions between these uses. This resulted in identifying concurrent discursive patterns that structure relationships between 'what can be said' and the power effects of saying it.

Results

My argument in this paper is that the senses of time and maturity mobilised when mathematics students account for their participation result in different discourses for the standard and enhanced mathematics pathways. The following sections show how these two discourses– that I named *moving/improving* and *getting ahead* – are each constructed through language patterns that combine into understandings of mathematics and selfhood. These were patterns that occurred across the students' talk; the quotations below are chosen to illustrate the language use while providing context.

Standard mathematics as moving/improving

This discourse was constructed through three patterns that align the futures for individuals with the temporalities of mathematics. The first pattern, *Securing progress in modernity*, occurs in students' explanations that their present choices are shaped as responses to their state of expectancy and of how mathematics projects itself as foreseeable security. In this pattern, students responded to the question 'What do you think is most important to you in choosing your subjects?' by invoking the future, for example:

> Well, would they be helpful to me <u>in the future</u>? Would they look good on my application forms? Cos I don't want to do subjects well like - not being harsh - but subjects that aren't as well thought of, like easier ones. (Clive)

Clive foresees a continuing process of being scrutinised, part of the *panoptical time* associated with adolescence. Within this process, he positions himself as knowledgeable about the relevant technologies of "my application forms" and the exchange value resulting from future employers' respect for mathematics. In a later interview he described family conversations about school and university choices as "finding the right course where <u>at the end of it</u> you have got a job set in stone, ready for you". Security was an aspiration and, for him, an expectation. Language such as his "set in stone", and others' "build and progress" illustrates how participants reproduced mathematics as central to a predictable modernity, and themselves as sharing in that security through their choice.

The second discursive pattern, *Doing mathematics as moving on*, aligns the futurity of mathematics with a sense that individuals learning mathematics are in continuous movement. It appears, for example, in describing being persistent through mathematics “Once I get started, if I can’t actually work it out then I’ll keep on going till I’ve worked it out” (Steve). It also appeared in responses to the adjectives task., where the two most commonly chosen for mathematics were ‘safe’ and ‘straight’, and students’ explanations positioned mathematics as of guaranteeing movement in time:

> When you learn one thing it goes on to another all the time. You are always progressing slightly. It gets harder as you go on through. (Joe)

This pattern is marked by a combination of linearity and repetition that conveys an *enduring* sense of time. There is ambiguity over who/what is progressing – ‘you’/ ‘I’ and/or mathematics/‘it’ - and repetitions that bring the person close to the subject, associating the future power of mathematics with the individual. The continuity of past going to present and future is emphasised, with echoes of Lesko’s (2001) discourse of expectant development in “you are always progressing slightly”, and in looking ahead to the endpoint of “hard” adulthood.

The third pattern, *Inheriting mathematics,* combines enduring and cyclical temporalities to position mathematics as inherited. This aligns with a dominant discourse that mathematical ability is ‘natural’ and timeless rather than achieved (Foyn, Solomon, & Braathe, 2018; Mendick, 2008). Students themselves summarised this concisely in one of the reasons most gave for choosing advanced mathematics: that they had always been good at it. When asked for memories or images of themselves doing mathematics, many gave examples of events involving parents. This emphasised mathematics as a natural and enduring inheritance heightened through family stories:

> My dad always has this story [laughs] when I was about 5 [...] And Dad finds this story so funny. He just sort of “oh we knew back then she was going to do maths.” (Charly)

> I remember we used to ask for more as well. I used to go up to him and say ‘Dad can you give me some more questions?’” (Joe)

In such stories, a child-like experience of ‘doing mathematics’ is understood as relevant to a present sense of belonging and a promised future, connecting the age imaginaries. These memories position participants within the family but also as agentic. Mathematics is inherited *and* it prepares them for the challenges of adulthood. Thus this discursive pattern positions mathematics within the dominant *expectant* time of adolescence – progressing without arriving - and layers this with concurrent, connected imaginaries (Alexander, 2014) of childhood and future.

Enhanced mathematics as getting ahead

The *Getting ahead* discourse, arose from the combination of three different patterns that make use of a sense of time associated with speculation and risk, where the present is used to compete for the rewards of the future. In the adjective task, students described the enhanced pathway as *not* safe, but hopeful. The first pattern, *Doing extra,* indicates the value of activities that run alongside what is seen as normative development and positions students as consuming time in a way future employers will like. Doing extra is not a guarantee in the way that straight mathematics is; rather, it concerns appearance and impressions:

> To start with I did it because it was an extra A-level and I thought it would look good, to be honest. (Charlotte)

‘Doing extra’ thus presents as an age imaginary of an adolescence supplemented with some aspects of adulthood – hard work, awareness of the adult gaze - and expecting rewards from this alignment. The second pattern, *A head start,* treats time subtly differently: it accelerates the normal linear progress of mathematics towards adulthood. Many students reported hearing from family, friends or teachers that the enhanced mathematics pathway resembled university work. In choosing it they expressed themselves as willing to secure a “head start” and project themselves into the future.

> At university they go straight into stuff... They go straight into the university stuff, they don't give you... They don't teach you the in-between stuff. I am glad I do Further Maths because that way I've kind of got a head start to students who aren't doing Further Maths. (Sukina)

There is rationality and pleasure in this reasoning and in emphasising the fast pace of university mathematics that she has adopted early. Enhanced mathematics is thus constructed as accelerating the dependable, staged progress of standard mathematics, and moving students more quickly towards an imagined next stage. This can function as an enabling discourse that provides doxic ways for students in these state-school sites to claim a privileged position. Nevertheless, it can very easily be threatened, as Sukina found when she visited a prestigious university admissions event and found that the half-course offered by the school was not accepted. Although a “gap year” is part of the adolescent story for middle-class students, and for this tutor, it was unthinkable for Sukina. The age-imaginaries offered by the admissions tutor were incompatible with the ways that Sukina and her British-Bangladeshi family were negotiating new imaginaries of young adulthood that combined familiar narratives, material resources and degree-level study.

In the third pattern, *Bright lights and im/maturity,* students on the enhanced course were commonly described by others, and described themselves, as missing the adolescent “play” appropriate to their age. Seen positively, this positioned them as accelerating to adulthood. However, this dominant discourse was also used, in reverse, to reason against enhanced study. Students critiqued the demands of the national programme for after-school learning: “really we shouldn't have been made to do that anyway, should we, at this age? We're still in A-levels”. In a late interview, Tom and AgentX looked back and contested the discourses that led to their original choice. Although Tom described still feeling “the lure” of enhanced mathematics, he framed his decision to stop as developing maturity, a matter of understanding his own limitations, managing school planning technologies, and “sacrific[ing] one thing to be better at other things”. They were humorously vocal in questioning the maturity of others on the enhanced pathway, condoning aspirations only for young or clever students.

> There's a lad I worked with who's in Year 12, and he's doing Further Maths, exactly like I was when I started it. I think he's cleverer than me, or than I was in Year 12. But he's not … And he tells me. He's got that look in his face, he says ‘Oh I'm doing really good; I'm doing Further Maths’. So I think he's kind of got hit by bright lights as well if you like. But I think he'll be alright at it because he's quite clever.

Despite allowing these exceptions, their use of the theatrical “bright lights” metaphor suggests that for most students claiming such cleverness may in fact be a self-deceiving and naïve performance, distanced from authentic adolescent development.

Discussion

This analysis suggests how students incorporate senses of time and age in forming discourses of doing mathematics. One mathematics pathway is positioned via senses of enduring, cyclical and inherited time, offering security as a relevant force in a technological but uncertain world. Students associate these qualities with mathematics; but the moving/improving discourse does more: it enables them to claim them for themselves through the practices of choosing. The sense of enduring time positions students within the developmental view of adolescence, subject to panoptical and expectant temporalities (Lesko, 2001). Childhood and family were also evoked in this discourse of choosing mathematics, giving a sense of time as inheritance that adds continuity to students’ projects of the self in mathematics. Mathematics is a doxic choice for students with prior achievement, but this discursive pattern locates it as arising also from family practices and sustains a sense of selfhood as authentic and persisting through time. The temporality of inheritance layers and threads age imaginaries together rather than keeping them apart, mirroring Alexander’s (2014) finding that young people conjure life trajectories where imminent and distant futures are mixed.

The discourse associate with the enhanced mathematics pathway was produced via a different sense of time, that of getting ahead. Students construct participation as not merely oriented towards their future study and career intentions but starting to access them now. This discourse provides a way of proving themselves within the school environment and justifies their thinking in terms of neoliberal dreamscapes in which they achieve more than others expected of them. They are positioned as already accessing an age imaginary that secures unexpected privilege – and the novelty of the pathway in these sites reinforced this sense. Nevertheless the discourse of getting ahead is precarious. It is readily intelligible by students but so are ways of resisting it, and of positioning oneself and others as not authentic participants. Notably, maturity was constructed as disciplining oneself to educational technologies of the present and resisting aspiration. In this positioning, participation in the programme excludes the normative understanding of schooling as developing adolescent maturity; students teeter instead between being a child and an adult. Lesko (2001) argues that precocious individuals (young drivers, parents) are understood as dangerous in the dominant discourse of development because they raise concerns that subordinate positions will become entwined with dominant ones. This wider cultural concern about precocity renders student choices fragile. Some students use the tension productively to perform new forms of aspiration. They challenge stereotypes about who can do mathematics by giving accounts of exceptional progress. However, these are easily threatened by calls to become appropriately mature and by encounters with inflexible institutional temporalities.

Finally, I consider what this means for advising students about studying mathematics. Overwhelmingly, these students relied on mathematics for security in the performative world of school classrooms and examinations; they felt hopeful but less safe accepting that mathematics requires risks. Knowing that you will do well in mathematics tests, and then actually doing well, was a strategy for simultaneously and publicly controlling time and success. Despite their attainment, losing the opportunity to produce this narrative of self-governance evoked the same reactions as with the majority of students who had already rejected the subject aged 16. This suggests the importance of creating a discourse that supports cognitive risks in the mathematics classroom and provides ways of valuing learning for what it makes possible in the present as well as the examination–mediated promise of future selves.

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Support systems for participation in collective argumentation in inclusive primary mathematics education

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This paper deals with participation in collective argumentation in inclusive mathematics education with a special focus on support systems. Since the ratification of the UN Convention on the Rights of Persons with Disabilities, the diversity of students in German schools is increasing. Different students need different support to participate autonomously in class or in collective argumentations, which can be considered as a prerequisite for mathematical learning. On the basis of an interactionist perspective on learning, the aim is to reconstruct different support systems between students, primary school teachers and special needs education teachers in order to work out the potential effects of these support systems on participation in collective argumentation in everyday inclusive mathematic lessons.

Keywords: inclusion, support systems (MLSS), collective argumentation, participation

Introduction

Inclusion is a central topic in current debates regarding education and school system in Germany. The discussions on inclusion in school have been stimulated by educational policies like the ratification of the UN Convention on the Rights of Persons with Disabilities (in the following CRPD) (UN General Assambly, 2004) and social and demographic changes, e.g. increasing linguistic-cultural differences because of a higher number of children with an immigrant background (Decristan et al., 2017). In addition to the discussions of an inclusive school system the current pedagogical discourse indicates multiple unresolved problems, and there is some criticism of previous approaches dealing with diversity in school, like the homogenisation of learning groups through selection and forms of external differentiation (Trautmann & Wischer, 2011).

In Germany, the school system is characterized by homogenisation. All children from five or six to nine or ten attend primary school after which they are subdivided according to their abilities into an academic high school or middle school. From the middle of the 18th century, a separate school system for children with special educational needs with special needs education teachers has been developed in addition to the regular school system. However, since the ratification of the CRPD, there is a legal right for persons with disabilities of a free choice of school and equal access to an inclusive education system. (CRPD, 2007, article 24 (2)) Countries must ensure that

> “Persons with disabilities can access an inclusive, quality and free primary education and secondary education on an equal basis with others in the communities in which they live” (CRPD, 2007, article 24.2).

To realize this right the countries have committed themselves to do everything necessary to ensure that persons with disabilities have equal participation within the regular school system (CRPD, 2007, article 24,2). Specifically, this means that more and more children and young people, who have been taught at special schools so far are enrolled in regular schools (Vock & Gronostaj, 2017). The fundamental idea of inclusion is that all students can participate equally in school. The diversity of the pupils should be perceived as something positive (Hinz, 2015). The implementation of inclusion requires schools to offer lessons that support and dare all students, regardless of their level of competence (Vock & Gronostaj, 2017) For this to be achieved adequate and new arrangements have to be made within regular schools in order to ensure the educational success of each individual. To comply with the equal participation of each student there is an increasing appointment of special needs education teachers in regular schools.

New discourse on the role of special needs education teachers in Germany

The new field of application of special needs education teachers in regular schools leads to a controversial discourse on the role of special needs education teachers in an inclusive school system in Germany (Lütje-Klose & Neumann, 2015). Are special needs education teachers as well as primary school teachers responsible for all students of the learning group or are they specialists for individual pupils? Through different academic studies special needs education teachers have a different perspective on learning and mathematics education. The training focus of special needs education teachers is more on the individual children, whereas primary school teachers must rather have the entire class in mind. Bock, Siegemund, Nolte, and Ricken (2019) have designed a university course in this context for students of mathematics education and students of special needs education.

> "The aim is to sharpen the prospective teacher's own professional perspective and to learn about the content and perspectives on other disciplines by strengthening and entangling perspectives (Bock et al., 2019)."

The focus of the study is on the differences of *perceptions*, *interpretations*, and *decision-making* of inclusive mathematics learning between the students and how they change during the seminar. The authors found that the students of special needs education focus on the individual child's behavior and needs. The decisions they make relate to the individual. For the students of mathematics education, mathematical content and mathematical thinking are more important, and their decisions are more likely to be made with regard to the entire class. (Bock et al., 2019).

As described above there is an increasing diversity among the students and this leads to a higher variety of professions in the classroom. Due to their different training, the teachers have a different view of learning mathematics, which may also establish different support systems between the pupils, the special education teachers and primary school teachers. The research project presented in this paper aims to provide theoretical considerations for learning mathematics from an interactionist perspective in conjunction with the theory of support systems. The research is guided by the question: *Which support systems develop in the inclusive mathematics classroom between the different teachers and how do they affect the participation possibilities of the students?*

Theoretical background

The theory development of the presented research project is carried out with the help of qualitative research methods. Therefore basic theoretical assumptions about learning in mathematics lessons, which at the same time represent the theoretical framework (Kelle & Kluge, 1999), are presented in the following section. The interactionist view on mathematical learning is based on the idea that the content to be learned or the topic of a lesson is negotiated between the participants of the interaction:

> The general assumption of the interaction theory of learning is that learning as a construction of meaning which outlast the situation has its origin within social interaction (Krummheuer, 1992; translated by the author)."

The meaning of things[6] is thus not derived from the things themselves, but arises from the interaction between persons and is thus a social product. These social and interactive meanings are reinterpreted by each individual and guide the person's actions, but they are also subjects to change in the process itself. From an interactional perspective, learning is thus seen as a social interpretive act in which meanings are constructed through mutual negotiation processes (Blumer, 1969). Miller (1986) also emphasizes the importance of the collective in the learning process. He describes that only in the social group and due to social interaction processes between the individuals of this group, the individual can made fundamental learning steps (Miller, 1986).

One basic idea of the interaction theory of mathematics learning states that the development of topics in the classroom is not predetermined by the teacher, but is negotiated together in an interactive exchange with the children. Based on their experiences and knowledge each individual has its own interpretations of a situation. This leads to a development of preliminary interpretations of the situation which, however, can be rejected or transformed in the process of interaction (Blumer, 1969). The participants attempt to attune these to each other which can lead to a taken-as-shared meaning or working consensus (Goffmann, 1959). The working consensus is a condition, which is created by the members of the interaction and also a basis on which the interaction can be continued. If the taken-as-shared interpretation is repeatedly produced in the interaction, the definitions of the situation can become standardized and routinized, which are then called framings (Krummheuer, 1992; Schütte, Friesen, & Jung, 2019; Jung, 2019). During the interaction the framings between the individuals sometimes do not coincide and thus can often lead to so-called framing differences. Schütte (2014) describes this as the "motor of learning". Through negotiation processes, the individual gains the opportunity to build up new framings and thus gains a new perspective on reality (Schütte, 2014).

Argumentation and participation as substantial concepts for mathematical learning

Collective argumentations are interactive negotiation processes in which, according to Miller (1986), collective solutions for interindividual coordination problems (dissent) are negotiated. However, Krummheuer (1992) describes in this context that for a collective argumentation there has to be no dissent only the common production of a working consensus is important. Collective arguments in the classroom are seen as learning-enabling and learning-conducive interaction processes (Krummheuer, 1992; Miller, 1986; Jung, 2019; Jung & Schütte, 2018). Following this idea, successful mathematical learning processes are manifested in the increasingly autonomous participation in collective argumentations. (Krummheuer, 1992, 2011a).

[6] "Such things include everything that the human being may note in his world – physical objects, such as trees or chairs; other human beings, such as a mother or a store clerk; categories of human beings, such as friends or enemies; institutions, as a school or a government; guiding ideals, such as individual independence or honesty; activities of others, such as their commands or requests; and such situations as an individual encounters in his daily life (Blumer, 1969)."

In this regard students have different opportunities for participation in collective argumentations in the mathematics classroom. An individual can assume different status (Fig. 1), like an “author” or a “spokesman”, regarding participation in the interaction which can change frequently. A speaker is called an “author” if he or she is responsible for the content and the formulation of an utterance. If he or she is neither responsible for the content nor for the formulation he or she is called a “relayer”. A Speaker is called a “ghostee” if he or she is responsible for the content of an utterance with using identical formulations of somebody else. If a speaker takes over an idea of somebody else with using his own words, he or she is called a spokesman (Brandt & Höck, 2012; Krummheuer 2011b, 2015). Brandt (2004) describes this as the *swarm of participation* (Brandt, 2004; translated by the author) of an individual. In this context, Brandt (2004) introduces the concept of the *scope for participation*. This term describes under which emergent conditions in the interaction a person can shape their participation in the sense of a participation swarm. Such conditions can be restrictive, so that, for example, a student is offered only certain participation opportunities like a “ghostee”. (Brandt, 2004). Support systems should be seen in this context as a possible condition for improving participation in collective argumentations in inclusive mathematics education. The present study aims to investigate possible effects of support systems on student participation in collective argumentations and therefore the theory of the "Mathematics Learning Support System (MLSS)" (see Krummheuer, 2011a) is to be transferred to inclusive primary school mathematics lessons.

	Responsibility for the **content** of an utterance	Responsibility for the **formulation** of an utterance
author	+	+
relayer	-	-
ghostee	+	-
spokesman	-	+

Figure 1- The production design (Krummheuer, 2011b)

Support systems for participation in collective argumentation

To ensure that each individual can participate and becomes more and more autonomous at collective argumentations there are potentially different and variable support systems that emerge within the interaction for each student. According to Bruner (1983) a support system “ […] frames the interaction of human beings in such a way as to aid the aspirant speaker in mastering the uses of language". Bruner's research focused on the early language acquisition of children. In this context, he has established the existence of a “Language Acquisition Support System (LASS)” which allows the child to learn how to use the language. The LASS emerges as a format[7] between a child and his mother. These routine procedures represent the support system through which the child can increasingly participate autonomously in the interaction and become a part of the mother’s culture. It can thus be said that a support system structures how the language and interaction affect the child (Bruner, 1983). From an interactionist perspective, support can be located between the participants

[7] “A format is a standardized, initially microcosmic interaction pattern between an adult and an infant that contains demarcated roles that eventually become reversible.” (Bruner, 1983)

of the interaction. Support is negotiated in the interaction and is not to be seen as an activity of an individual person e.g. the teacher. Utterances and actions are established as a support system, if the child orientate their interpretations on it (Tiedemann, 2012). I follow Bruner and Tiedemann that an increasing autonomy in the interaction within or whereby a support system can be seen as learning. Krummheuer (2011a) dealt with support systems and introduced the term MLSS (Mathematics Learning Support System) into the discussion. The MLSS describes support systems for processes of appropriation of mathematical terms and methods as well as their logical-argumentative anchoring within a mathematical content system. Tiedemann (2012) has transferred the theory of the support systems to early mathematical education, especially to family situations. Subject of the study are mother-child discourses in reading and playing situations in which support systems she called MASS (Mathematics Acquisition Support System) are examined. She reconstructed support systems between mother and child and identified three different types of support: participation, improvement and exploration (Tiedemann, 2013).

Since the establishing and adapting of different support systems could be a high cognitive requirement for students especially for students with special educational needs, the research project would like to examine the support systems in inclusive primary mathematics education and focus on possible differences between primary school- and special needs education teachers. In the following, the planned study design will be presented in order to reconstruct and describe its support systems in inclusive mathematics primary school education.

Methodology and analytical methods

The planned research project can be located in qualitative social research following a reconstructive-interpretative methodology (Bohnsack, 2007). The focus is on everyday teaching situations in inclusive primary school mathematics classroom which will be videotaped and transcribed. The research project follows a broad concept of inclusion: inclusive mathematics education means that all students, with their differences, are recognized and appreciated in the classroom, regardless of whether they need special support or not. To analyze the transcripts concerning the support systems which are developed in the interaction, I follow Tiedemann (2012) in using the (2) support analysis based on the (1) interaction analysis (Schütte, Friesen, & Jung 2019). The interaction analysis was developed in the working group around Bauersfeld and serves the analysis of negotiation processes in mathematics lessons (Bauersfeld, 1995; Krummheuer, 1992; Schütte, 2009; Tiedemann, 2012). After analyzing the data with the interaction analysis, the scenes that prove to be supportive are selected. Then the support systems will be reconstructed which establishes itself between the participants of the interaction. In order to make statements about the increasingly autonomous participation in collective argumentation, the scenes are subjected in a third step to a (3) participation analysis. (Krummheuer & Brandt, 2001). The participation analysis involves two different concepts, the concept of the *recipient design* and the concept of the *production design* (Brandt, 2004 & Krummheuer, 2011b). With the concept of the recipient design the audience and the relationship of different interaction strands in the classroom are worked out. However, the main focus of the study is on the concept of the production design. This concept is related to "[…]any person who is involved in the production of an utterance and that person's role as he/she participates in this production (Krummheuer, 2011b)". With the analysis of the production design it is possible to reconstruct the swarm of participation of a student. The study is applied longitudinally - there are three phases of data collection in which the same students are videotaped over several weeks - so that possible changes can be made visible. By comparing scenes that follow each other in time, a possible increase in autonomy of learners in the participation in collective argumentations through support systems should be reconstructed and described.

With the help of the presented analysis methods the potential effects of the support systems on participation in collective argumentation in everyday inclusive mathematic lessons should be investigated. The project is currently in the process of data collection. At the time of the conference first analysis results can be presented. Based on this empirical data, concepts for primary teacher education and special needs education should be developed as a result of the research project.

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Teacher moves for promoting student participation when teaching functional relationships to language learners

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Students' participation in rich discourse practices is important for their understanding of the meaning of mathematical concepts. It is the job of the teacher to enable and foster this participation. But how exactly do teachers' practices for promoting beneficial student participation in classrooms look like for a specific mathematical concept? This paper investigates two contrasting cases to explore the range of possible topic-related moves teachers use when teaching functional relationships to monolingual and multilingual academic language learners. The two cases illustrate topic-specific and topic-independent teacher moves by which teachers try to promote student participation in a discussion on comparing functional relationships with regard to their meaning in the given contexts.

Keywords: Teacher moves, student participation, functional relationships, comprehension elements.

Introduction

The quality of communication and discourse in classrooms is crucial for students' learning of mathematics (e.g. Barwell, 2012). The teachers' role is to provide appropriate learning opportunities through language-responsive teaching. However, Ing et al. (2015) indicate that teacher practices do not have a direct impact on student achievement. The missing link is student participation (ibid.). By participating in rich discourse practices, students can gain a deeper understanding of mathematical concepts (ibid.). Teachers should encourage and support beneficial student participation in rich discourse practices in their classrooms in order to foster student achievement. However, O'Connor and Michaels (2019) point out that solely using tools is not sufficient, as they need to be embedded within the mathematical content:

> "[Talk moves] are not themselves the substance – the food – of instruction. [They] are the forks, spoons, and spatulas. Like the tools that skillful cooks must use, they are not the meal itself. The meal is the intellectual content, and the talk tools must be used in relation to ever-changing content." (ibid., p. 185).

This paper investigates which moves teachers use for teaching functional relationships in secondary school, and to what extent these moves promote beneficial student participation.

Theoretical background

Teacher moves for facilitating participation

Teachers have to fulfill various jobs in mathematics classrooms. Jobs are "the typical, often complex situational demands of subject-matter teaching" (Prediger, 2019, p. 370). Bass and Ball (2004) name several examples of core tasks and problems of teaching. For example, relevant in whole classroom discussions are: managing productive discussions, analyzing and evaluating student responses, and analyzing and responding to student error (ibid., p. 296). For coping with the jobs, teachers use pedagogical tools like tasks, activity structures and (facilitation) moves (Prediger, 2019, p. 370).

Such facilitation moves can be differently successful for promoting beneficial student participation. Bass and Ball (2004, p. 309) identify supportive teacher moves in this sense, e.g. "*moving of individual ideas into the public space*", "*helping those ideas be articulated in ways that others can work on them*", "*revoicing, with clarification, of student offerings*", or "*inviting peer-evaluation*". They emphasize that a prerequisite for these moves is that teachers are able to interpret student thinking and to link it to the current mathematical issue (ibid., p. 309). Their results apply especially for the topic of mathematical reasoning.

Although the study points out that the identified moves apply for the specific topic, many of the moves themselves seem to be topic-independent. They first have to be made concrete for the specific mathematical topic in order to be suitable for the current situation (for further examples of possible talk moves see O'Connor & Michaels, 2019). O'Connor and Michaels (2019, p. 185) describe their observation that some teachers picked up the suggested talk moves whereas other (in particular less experienced) teachers had problems. A possible explanation could be that less experienced teachers use the moves "robotically", which means that a revoicing move could be used when there is no reason to revoice (ibid.). In this case, the revoicing move is not suitable for the situation.

As a consequence, teachers need to command several moves, they need to be able to use a suitable tool that fits to the situation, and they need to be able to embed it in the specific mathematical content. This paper investigates which moves teachers use for moderating a whole classroom discussion on the mathematical content of 'meaning of functional relationships'. Additionally, the focus lies on how teachers link students' utterances to the mathematical learning goal of the lesson. Therefore, the paper first presents a conceptualization of understanding functional relationships and the intended learning goal of the teaching unit in focus.

Learning content: understanding functional relationships in contexts

The mathematical learning goal of the teaching unit in focus is the deepening of students' understanding of functional relationships in contexts. This paper refers to the conceptualization of the 'core' of the function concept (Zindel, 2017; Prediger & Zindel, 2017). Following Drollinger-Vetter (2011), understanding a concept appears in flexibly unfolding and compacting of comprehension elements of a concept. Zindel (2017) introduces the core of the function concept (Figure 1) that contains those comprehension elements that are important for every representation and every type of function in the middle grades (Zindel, 2017; Prediger & Zindel, 2017). There are three important insights concerning functional relationships (Fig. 1): (1) there are two involved quantities, (2) these quantities vary, and (3) there is a direction of dependency (one variable depends on the other variable). In situations where students have to identify the meaning of a functional relationship in a context, it is necessary to be flexible in unfolding and compacting the functional relationship into its smaller comprehension elements (Prediger & Zindel, 2017, p. 4165 f.).

As a consequence, it is important that teachers initiate and support the addressing of comprehension elements of the function concept as well as processes of unfolding and compacting. This requires moves like demanding explanations of the meaning of functional relationships (with regard to the involved quantities and their relationship). Thus, the main goal of the focused teaching unit is sensitizing for the comprehension elements from Figure 1 by contrasting and explaining the meaning of different functional relationships (Figure 2).Student participation is investigated by regarding the students’ addressed comprehension elements within the classroom discussion.

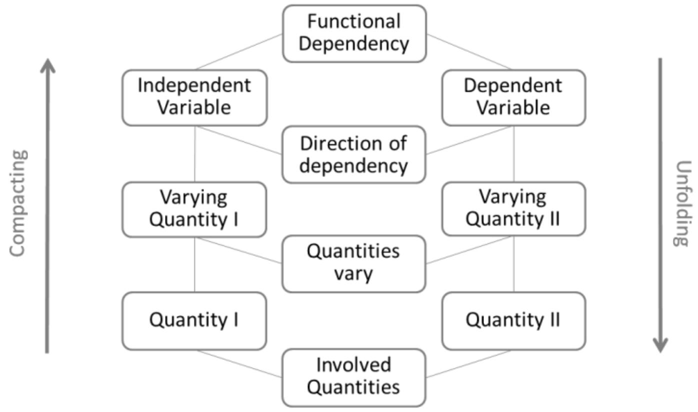

Figure 1: Comprehension elements of the core of the function concept (Prediger & Zindel, 2017, p. 4165)

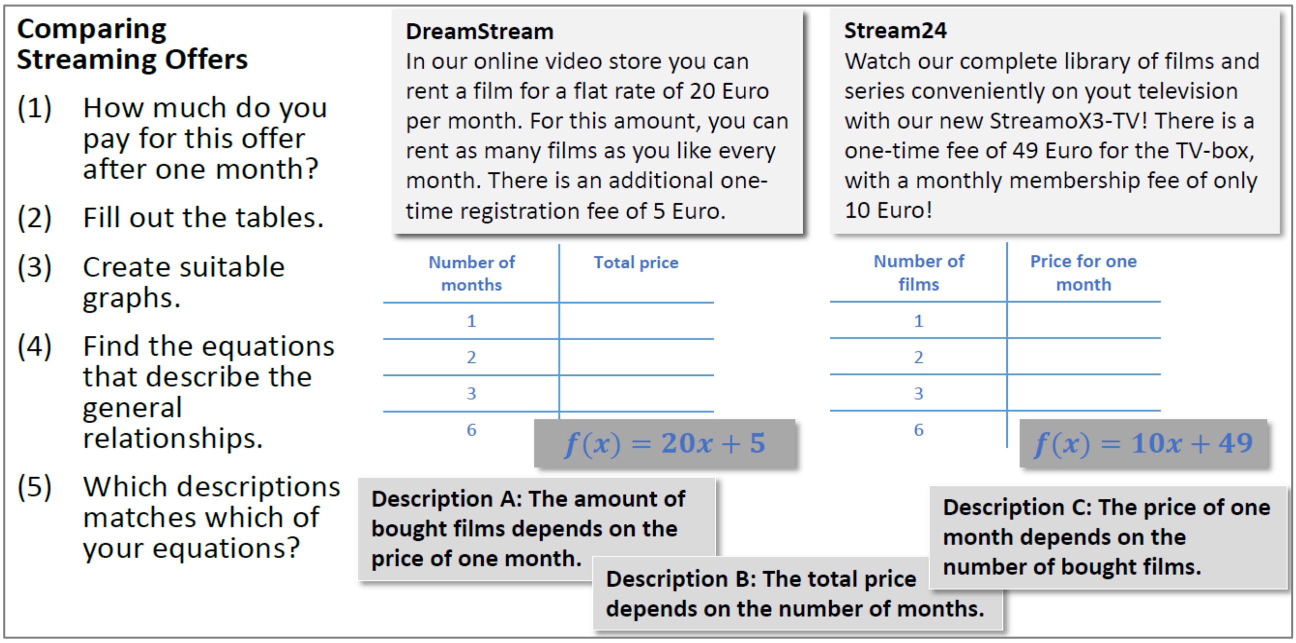

Figure 2: Identifying different functional relationships in streaming offers

Research questions

So far, the paper presented some examples for possibly supportive moves, as well as important comprehension elements of the function concept that form the focus of the teaching unit. The following research question can be derived:

RQ: Which moves do teachers use when moderating a discussion on the meaning of functional relationships? To what extent do their moves promote beneficial student participation?

Methods

Sampling. Overall, the data set consists of five teachers who were videotaped when teaching functional relationships in their grades 9. As this paper focuses on the whole classroom discussions after the students worked in groups, the videos were partly transcribed with regard to the whole-classroom discussions. In order to identify a broad range of possible moves, this paper adopts the approach of an exploratory case study (Yin, 2002). Two meaningful cases were selected for comparison that are distinct enough from each other to allow insights into their differences, yet not so far apart as to be incomparable.

Data analysis. The data analysis focused on the whole classroom discussion at the end of a group work. First, the teacher moves were identified by inductive category formation (Mayring, 2015): Excerpts from transcripts of whole-classroom discussions were collected and analyzed with regard to the teachers' demands and their way of leading the discussion. After that, the addressed comprehension elements were identified by a deductive-inductive analysis based on Figure 1.

Teaching material. Both teachers worked with the same teaching material (Figure 2) provided by the author. The students were divided into groups working on one streaming offer each before they compared their results and the different streaming offers in the whole classroom. The descriptions varied with regard to the involved quantities and the direction of dependency. This task aimed at increasing the students' awareness for these comprehension elements.

Empirical insights in two case studies

Case 1: Erin

Erin, the teacher, starts the discussion (#1) by the move *asking for the calculated result* of the first table row (Task 1a). She makes the communicative demand explicit by *demanding a full response*.

1	Erin	What is your result for the first task? How much do you pay after one month? Whole sentence!
2	Student 1	Well, for task 1a we have, in the first month you pay only 25 Euro.
3	Erin	25 Euro. *[writes 25 at the board]* Okay. *[...]* And what do you pay after two, three and six months? Student 2.
4	Student 2	Always 25 plus 7.
5	Erin	No.

Although the student answers in a whole sentence (#2), Erin refers to the calculated number only and records it at the blackboard (#3). Afterwards, she continues by *asking for calculated results* of the other table rows (e.g. #3) and *evaluates the students' answers only with "correct" or "no"*. She does not address any comprehension elements because she refers to the calculated numbers only. Later, a student gives a wrong result for the last table row. Erin reacts by *demanding the discourse practice of reporting procedures* (#13).

18	Erin	How would you calculate?
19	Student 4	130
20	Erin	130. What did you calculate?
21	Student 4	Simply calculated plus 65.
22	Erin	Is it correct?
23	Student 4	Yes. Because there are three, and two times three are six, and then simply two times 65.
24	Erin	Then look at the text again. *[...]*

In this excerpt, Erin *repeatedly asks for the discourse practice of reporting procedures* (#13, #18, #20). The student struggles because she seems to calculate two times the price after three months based on the assumption that it is a proportional relationship. Erin reacts by *asking whether this is correct* (#22) and the student reports her procedure in a more detailed way (#23). Nevertheless, Erin does not refer to the conceptual problem but gives the advice to look at the text again (#24). After another student has given the correct answer, Erin continues by collecting the results of the other groups in a similar way. Another problem when assigning descriptions to the function equations:

111	Erin	Does the amount of bought films depend on the price of one month? Student 12.
112	Student 12	I say no. We all have the same because the sentences do not make sense for us.
113	Erin	hmm *[doubting]*. Let's keep that as it is. *[...]*

One student explains the group's difficulties by saying that the descriptions do not make sense for them, showing that the students' difficulties can be traced back to unavailable conceptual understanding of a functional relationship. Nevertheless, Erin does not demand or promote any explanation of the functional relationships. Instead, she closes the discussion. Summing up, Erin goes through every task and only demands the discourse practice of reporting procedures.

Case 2: Natalie

Natalie starts the discussion (#1) by showing a graph (of the Stream24 offer) one group has prepared in the group work and *asking the other students connect it back to the alternative texts*.

1	Natalie	*[...]* Well, which streaming offer fits to this graph, what do you think? *[...]*
2	Student 2	I think that is the graph of the Stream24 offer, because you don't have to pay a registration fee or TV box at the beginning and it gets even 10 Euro more.
3	Natalie	Do you all agree?
4	Students	Yes.
5	Natalie	Makes sense, doesn't it? Good. The others who had this offer, too, did you draw the same graph? Who had this? Student 3, did you draw the same?

One student answers by referring to the comprehension elements of the meaning of the constants that he sees in the offer as well as in the graph (#2). Natalie does not evaluate this answer directly but calls on several students' opinions (#3, #5). Here, she does not explicitly address any comprehension elements. But in the next turn she *asks for the underlying assumptions* the drawer of the graph has made (#7).

7	Natalie	Yes, good okay. Well, what did the drawer took as a basis here? We all know how the Stream24 offer looks like. What did he consider with regard to the current films?
8	Student 2	That one doesn't watch any current films in a month but only old films.
9	Natalie	*[...]* Exactly. *[...]* They thought current films are too expensive, we just watch the classics. Good. So then there was a second task *[...]* You should set up an equation for Stream24. Which equation did you set up? *[...]*

After a student has given the correct answer (#8), Natalie *builds on the student's utterance to extend the main point* (#9). Then she refers to the task of setting up the function equations (#9).

10	Student 5	Erm. f then an equal sign. *[...]* in brackets x times four, right bracket plus 10.
11	Natalie	Some are putting up their hands already. Student 6.
12	Student 6	I would *[...]* exchange the four and the ten. So, f left bracket, x times 10#
13	Natalie	#go on first.
14	Student 6	Right bracket plus four. But the plus four ought to be written down only if you buy only one film per month.
15	Natalie	Hmm. That is not completely true, too. But erm, first it's Student seven's turn and then we come back to this.
16	Student 7	Left bracket, a times four, right bracket, plus c, because there are a films that have to be multiplied by four.
17	Natalie	Yes.
18	Student 7	So four Euro for each film and then plus 10 you have to pay for one month.

The students suggest two different variations for a function equation for Stream24 (#10, #12, #16, and #18). Here, Natalie *calls for different ideas and suggestions* again and thereby for students' participation (#11, #13, #15). Afterwards, she takes up one of the suggested equations and a*sks the others for the meaning of the equation* (#19). By that, she addresses the comprehension elements of the involved quantities (#19).

19	Natalie	What did Student 7 calculate here? For how many months does he calculate and for how many films? Student 8.
20	Student 8	Erm so for one month.
21	Natalie	And how many current films?
22	Student 8	Erm one.
23	Natalie	No.
24	Student 8	Well that is not given.
25	Natalie	Exactly. *[...]* That means you *[points to Student 7]* have set up a very nice equation for the case that I watch exact one month and I like to know how this depends on the number of films. But this is not what Student 4 has drawn, Student 4 assumed that he doesn't watch any current films. *[...]* And that is why we closely look at the first equation again.

A student suggests that the equation refers to one month and one film (#20, #22). Natalie *builds upon the student's statement* and *extends it by explaining the meaning of the two function equations herself* (#25). Summing up, Natalie initiates the discussion by using the not intended graph for making explicit the two possible functional relationships that can be seen in the Stream24 offer with respect to the different meaning of the variables. Thereby, she addresses the mathematical core of the function concept.

Comparing the cases

The identified moves and addressed comprehension elements are summarized in Table 1. Erin's move to go through every task and to ask for the calculated results addresses less comprehension elements concerning the meaning of the functional relationships. The overarching mathematical core the teaching material aimed at remains implicit. In contrast, Natalie commands moves that take into account topic-specific characteristics (e.g. clarifying the meaning of the variables). Besides, some of her moves seem to initiate students' addressing of comprehension elements. These beneficial moves are labeled as topic-specific moves in Table 1.

	Erin	Natalie
Topic-independent moves	• Going through every task and asking for calculated (numeric) results (#1, #3, #111) • Demanding the report of procedures (#18, #20) • Asking for and giving feedback on correctness (by yes/ no) (#5, #22) • Making communicative demands explicit (whole sentences) (#1)	• Calling on several students' results (#3, #5, #11, #15) • Giving face-saving evaluations (#15) • Extending student utterances in a general way (#13)
Topic-specific moves	• *none*	• Building upon a student product in order to initiate a discussion on the different possible functional relationships that can be seen in the Stream24 offer (#1) • Building upon students' utterances and clarifying the different meanings of the two suggested function equations for Stream24 (#19, #25) • Extending student utterances by focused and topic-related inquiries (#9)
Addressed comprehension elements by the teacher	• *none*	• Involved quantities (#9) • Involved variables (#19) • Functional dependency (#25)
Addressed comprehension elements by the students	• *none*	• meaning of the constants (#2, #14, #18) • involved quantities (#8) • involved variables (#16)

Table 1: Identified moves and addressed comprehension elements in both scenes

Conclusion and Discussion

Although they work with the same teaching materials, the two teachers moderate the classroom discussions in very different ways. Both teachers use moves that enable participation in a general way, but not every move is beneficial for student participation. Only one of the two teachers commands moves that initiate addressing relevant comprehension elements of functional relationships (building upon students' utterances and clarifying the different meanings in the function equations). These are moves that promote beneficial student participation whereas the other teacher uses moves that do not initiate beneficial student participation here (naming numeric results, reporting procedures). In this case, the mathematical core remains unclear as can be seen in the lack of understanding explicitly addressed by the students. Of course, the two cases can only give first insights in the field of teacher moves for teaching functional relationships, so that further research is needed. If the results can be proved for other cases, a possible consequence for PD-courses could be that teachers need more support in implementing general moves in a topic-specific way.

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Developing a collaboration tool to give every student a voice in a classroom discussion

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In this paper we outline suggestions for a new prototype tool that is currently under development. This new tool will combine a powerful mathematic software and a student response system, which allows teachers to create a classroom culture where students can give explanations. It will shorten the time for collecting students' responses and offers possibilities to work with these responses afterwards. By conducting semi-structured interviews with teachers highly knowledgeable in using technology, we found out that they need new features for formative assessment to foster classroom discussions. These include monitoring and collecting students' work in a convenient way and using students' responses in varying follow-up activities in a classroom. With this tool, teachers could have the advantage of encouraging students to explain their thinking by giving each student a voice during classroom discussions by letting them write down their thoughts and sharing them easily in class.

Keywords: classroom communication, formative assessment, student response systems, student participation.

Introduction

Our recent project is about the process of developing a new online collaboration tool, in which we want to use a powerful mathematics software and to further develop an existing online tool into a connected classroom technology application by adding new features and functionalities. "Connected classroom technology refers to a networked system of personal computers or handheld devices specifically designed to be used in a classroom for interactive teaching and learning" (Irving, 2006, p. 16). In literature, similar names appear for connected classroom technology (CCT), such as student response systems (SRS) or classroom response systems (CRS). Their common features are that they facilitate the communication between teachers and students, they display student responses in real time, and allow rapid aggregation of student work by teachers (Fies & Marshall, 2006; Irving, 2006; McLoone, Kelly, Brennan, & NiShe, 2017; Shirley, Irving, Sanalan, Pape, & Owens, 2011). According to Wright, Clark, and Tiplady (2018) connected classroom technologies offer a broad range of innovative features, which are summarized in various researches and also explored in a variety of approaches within a project. One great potential of connected classroom technologies is that they offer immediate information about students' progress to teachers as they could monitor their students' work real-time (Irving, 2006; Shirley et al., 2011; Wright et al., 2018). Moreover, connected classroom technologies enable most of the students to contribute to activities and taking a more active part in a classroom discussion (Roschelle & Pea, 2002; Shirley et al., 2011; Wright et al., 2018).

The used technology in our case is GeoGebra, a tool on teachers' and students' devices that are connected online and can be used for teacher and student interaction. We want to develop a new tool, called GeoGebra Classroom, that offers new opportunities for mathematics teaching to support teachers as well as assists students' learning. Furthermore, we want to explore how this tool may contribute to mathematics education and how teachers are using it for formative assessment. “Formative assessment refers to frequent, interactive assessments of student progress and understanding. Teachers are then able to adjust teaching approaches to better meet identified learning needs” (OECD, 2005, p. 13).

At the beginning of the project we wanted to find out whether or not teachers are satisfied with existing collaboration tools, why they are using a specific tool or feature, and if they need any additional features or functionalities for their classroom teaching with an online tool. Therefore, we conducted interviews with teachers where we focused on the research question “What do teachers need from an online collaboration tool to support their mathematics classroom teaching?”.

Methods

To approach this research question, we conducted semi-structured interviews with six experts, three men and three women, who are highly knowledgeable in using technology. We chose four experts from Europe and two from North America as they could give us broad international insights for our research. They are experienced with using online tools in their teaching because they work with several online tools in their mathematics teaching regularly. One expert works as a mathematics online teacher, where the use of online tools is indispensable. The other experts work in secondary schools or in teacher education. The interview guide focused on experts’ opinion and experiences on the following topics: uses of technology and different tools in mathematics teaching; features and functionalities of online (collaboration) tools; expectations, needs, and requests for new collaboration tools or features.

The interviews with two experts were conducted face to face and with four experts via an online video conferencing platform where audio and screen recording was used to collect the data. The duration of the interviews was between 30 and 60 minutes. During the interviews experts also presented examples how they are using different online tools in their teaching or in their research and where and for which purpose they need additional features. One interview was conducted in German and all the other interviews were conducted in English. All interviews were transcribed and analyzed by using qualitative data analysis based on qualitative content analysis (Mayring, 2014, 2015, 2020). According to Mayring (2020) the qualitative content analysis' approach is category-based, where the categories are referring to aspects within the text and the process is research question oriented. Characteristically, this analysis is systematic and follows a strict rule management, where the process is described step-by-step.

For our analysis, we chose the “Inductive Category Formation”, which is one specific technique of the qualitative content analysis, described by Mayring (2014, 2015). By using this technique, our categories derived directly from the material in a generalization process. First, we defined the selection criterion, where we focused only on those phrases or sentences of the interviews that answer the research question. When working through the transcripts line by line, the categories were formed, either by adding the phrases or sentences to an existing category if it fits, or by forming new categories. After the analysis of the first interview, a revision of the category system and the level of abstraction was done and then the other interviews were analyzed the same way. The following paragraph highlights results and main categories that extracted from the data.

Interview Findings and Categories

Based on the interviews, we found out that the experts were using several online tools in their mathematics teaching and each of those tools for a specific purpose. Nevertheless, they wanted new features or functionalities for existing tools or newly designed tools for formative assessment that offer them additional ways of teaching. They came up with various ideas how they might want to use specific new features aligned with their current teaching methods. The findings show that teachers request features that allow and support more interaction and communication between teachers and students and among students. They wanted connected technologies to monitor students in real time and to be able to use students' responses in varying follow-up activities. Such a tool should collect the responses in a convenient way on a dashboard where teachers can sort and organize responses so that they can be presented to the whole class during the same lesson. In a follow-up classroom discussion, they may want to compare different responses or highlight individual exemplary responses. Suggestions of the experts were grouped into the three main categories "Collaboration", "Real-Time Monitoring", and "Whole-Class Discussions". The following subsections highlight some of the experts' opinion regarding how they think a new tool can assist students learning and support teachers in creating a classroom culture where students give explanations, and which can be used for an upcoming whole-class discussion.

Collaboration

Teachers wanted students to collaborate and to interact with other students. As one example, one expert told us about her experience in her teaching.

> Expert 1: "I think that learning, probably anything, but specifically math works better socially. So, when you get your kids talking about your own learning, either to me or to each other. I think that is a more natural way of learning. […] So, any kind of tool allows that to happen, I am gonna find the best ways to use it."

Real-Time Monitoring

This expert also told about her experiences with an existing online tool that allowed her to monitor the students' work on her screen. The teacher experienced that every student was participating and working through the activities, when they knew that they were observed. As a result, students actively participated in the conversation afterwards, either by giving their own input or by asking questions.

> Expert 1: "I see them all doing the task. Every single student is participating and mowing through it, and as a result, when it comes have a talk, everybody is participating in the conversation."

The possibility to monitor students' work in real time had also an additional benefit according to two experts. They used this feature in an existing online tool for formative assessment and appreciated the possibility to make students' thinking visible.

> Expert 2: "… make students' thinking visible and use it as a basis for classroom discussion."

Whole-Class Discussions

Some experts also made suggestions on how to use student responses to foster classroom discussions. They wanted to present students' responses, either in written form or submitted in multiple ways -

via pictures, words, symbols, or in some other format such as geometrical constructions or graphing functions and to use them to discuss mathematical issues.

> Expert 3: "I would like to see only three different answers about three different aspects, and I would say, ok there were more different answers, but now we are going to discuss these."

> Expert 4: "If I wanted to show them all I could do that with the screen. So, in some cases if there was a big discussion, I would actually do that. Throw up every answer and we would discuss. But maybe not from the content of the answers, but from the type. Like we would discuss: Ok this answer is not complete, because they only say what they mean, and they never explain why, or something like this. So, I was pointing out to them more how do we work in math class than this is a correct answer."

> Expert 1: "… it facilitates such high participation in such high-level conversation about maths …"

As it can be concluded from the experts' quotations, students and their work should be in the center of the lessons. Experts want to put student explanations in the focus of classroom discussions and to use a tool to facilitate formative classroom assessment. They need a tool to encourage students to participate in classroom activities, to assist students in explaining their thinking, and to support classroom discussions.

Features for GeoGebra Classroom Tool

We took experts opinions into consideration when designing the new features for the GeoGebra Classroom tool that can be used with the GeoGebra resources. Even though we interviewed only a few experts and cannot draw any general conclusions, the results are good indications for the start of the development of the prototype. The GeoGebra resources can be used for all levels of education and thereby the GeoGebra platform offers a broad range of applications in a classroom setting. There are already more than one million free activities available on the GeoGebra website (GeoGebra, 2020).

Figure 1 and Figure 2 show two examples of existing characteristic GeoGebra resources. The first worksheet includes a prepared applet, where students can move the slider and drag the points to explore a mathematical concept. Below the students are advised to answer the two open questions. The second worksheet includes an empty applet, where students are advised to construct an equilateral triangle and then answer the multiple-choice question. Every worksheet can be created or edited to provide the powerful applet along with other question elements. This way students have mathematical tasks, explorations or constructions and response elements at one page close to one another.

The following subsections describe how a new and improved system on the GeoGebra platform, the GeoGebra Classroom, put students' responses in the focus, give every student a voice, and how it could support interaction in mathematics classrooms. Furthermore, we present possible scenarios how teachers could apply the GeoGebra Classroom tool for their classroom teaching.

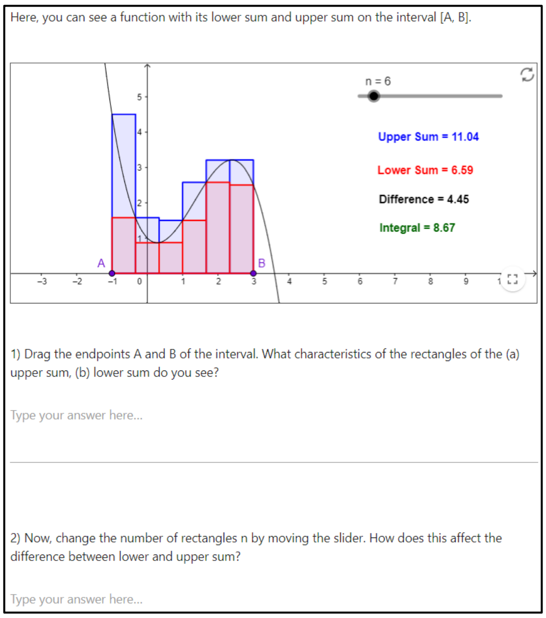


Figure 1: Worksheet 1[8]

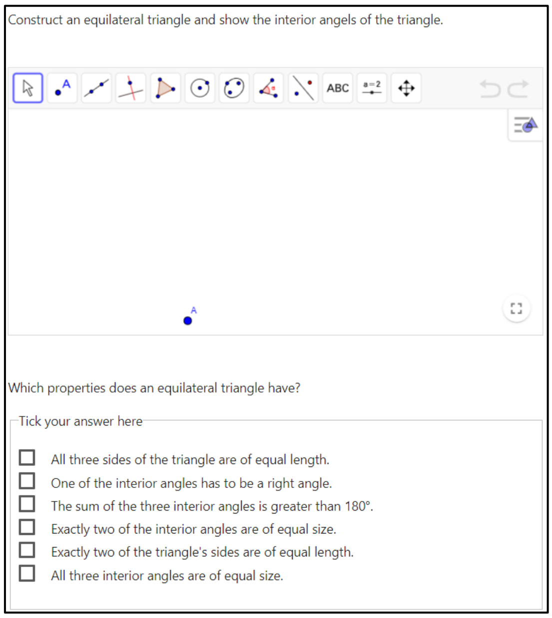
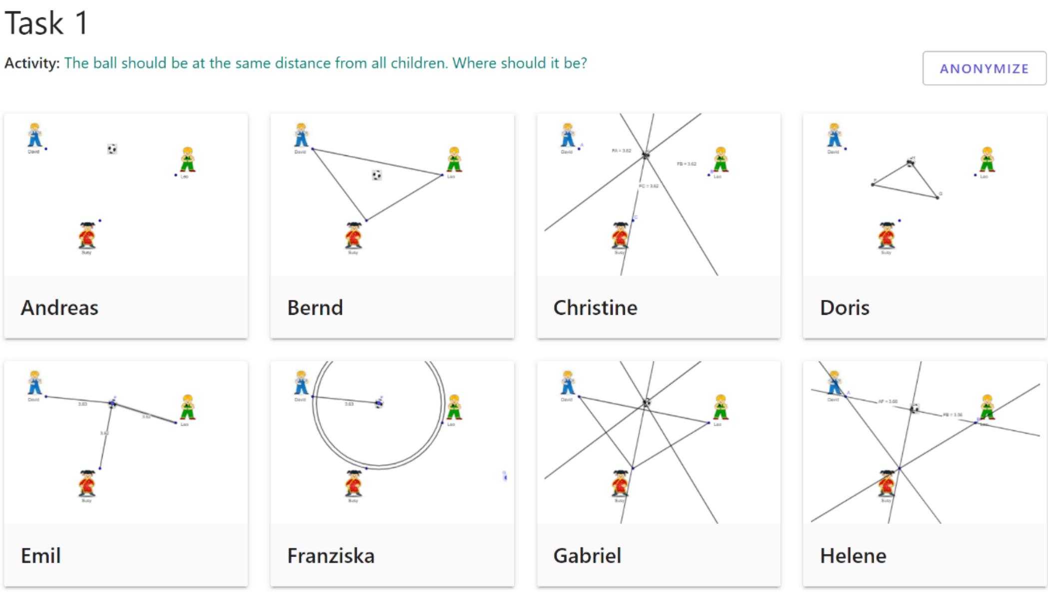


Figure 2: Worksheet 2[9]

Encouraging Students to Think (and Talk)

First, teachers hand out a sharing code for a GeoGebra resource that allows each student to enter a joint workspace online. Such an activity could consist of an interactive GeoGebra applet, where students can drag points, move sliders, change data, or construct objects in one of the available views, accompanied with question elements. Students could work on their own or in pairs sharing one device, which could be their phone, tablet, or laptop. No matter whether they work on their own or in pairs, each student is advised to think about the task and do the manipulations or constructions in the applet.



8 GeoGebra. https://www.geogebra.org/m/wcbcr49k, last visited Apr. 12, 2020.

9 GeoGebra. https://www.geogebra.org/m/ec9ysby5, last visited Apr. 12, 2020.

Figure 3: Teacher dashboard with student applets

The new features allow teachers to see at a glance and in real time all work done by their students. Teachers could observe all the manipulations that students are performing with an interactive applet (e.g. geometric constructions or function graphing, see Figure 3) while the students are working with it. When students are working together, they should talk with each other about their observations, manipulations or constructions.

Encouraging Students to Explain or Reason

Equally important is the fact that teachers can see all explanations or reasonings of students as written responses on a teacher dashboard. On the worksheet the students are advised to write down a response or explanation in the provided input box. This way, teachers have an overview of all the responses and can read through them without losing time by collecting them and can easily compare all students' responses in a quick and convenient way. Each student's response appears one after the other, and students know that their response is noticed by the teacher (see Figure 4).

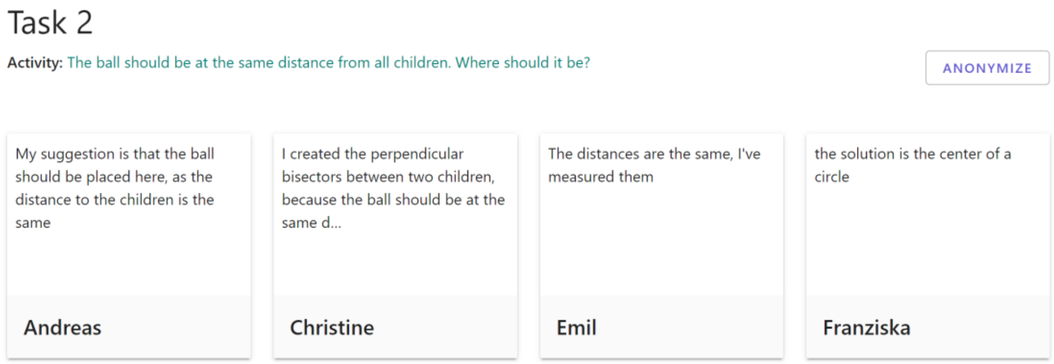


Figure 4: Teacher dashboard showing student explanations

According to one experts' experience, every student in the class is participating, when they know that they are observed. This experience coincides with a study result, where different student response systems were used in a classroom. "The students' observed behaviors indicated overall engagement during the learning process as they used the SRS" (Fuller & Dawson, 2017, p. 383). The participation included answering the teachers' question by using technology, responding to the collected data, and discussing questions with other students (Fuller & Dawson, 2017).

One big advantage and improvement of the question element will be the possibility to write mathematical formulas or expressions by using an equation editor. Students will then be able to submit a written text together with different mathematical symbols, write fractions, or add Greek letters in one input field. In this way students will be able to explain their mathematical observations not only in words, but also by using the correct mathematical language.

Encouraging Students to Discuss Mathematical Explanations

Teachers can drag and drop responses to organize and sort them to address a specific issue. When it comes to a classroom discussion the teacher can pick any response, either one or more, and present them to the other students in the classroom. If needed, the teacher can also show all of them and let the students read through them quickly. This way, each single response will be read, and each student could give an input for the discussion by submitting the response. By doing that, the teacher and the students can analyze the type of responses or mathematical meaning in a classroom discussion. Students will then be advised to take part of the classroom discussion as well. As a result, by using this new tool, the teachers can assist students in developing their skills in writing an explanation or in explaining their reasoning.

Another benefit of using such a system is that teachers can anonymize the responses. Therefore, each student can participate actively, and no one needs to worry that their name will be shown during the discussion. In addition, teachers can select responses randomly without knowing which student wrote it. The anonymous feature creates a safe environment in a classroom where all students' responses are taken seriously without fear of embarrassment.

Further Research

The continuation of this project will be based on a prototype system of the GeoGebra Classroom tool, including the features described above, where we will conduct further research focusing on the use of the new features in a classroom setting and analyzing how this tool affects mathematics teaching. We are particularly interested in how teachers will use this new tool and how they and subsequently their students will benefit from using it in the classroom. Furthermore, we will test it for several reasons as it should be easy and pleasant to use for teachers and as well for students. Moreover, we will create or adapt tasks, that have an additional benefit for students when they are working with this prototype. Finally, we want to find out how and to what extent the prototype, and subsequently the tool, can support teachers and how it can offer them new opportunities in teaching.

The contribution of this paper is to point out to teachers and other researchers that this new designed tool could have advantages in various fields. As it will allow a variety of possible applications in teaching methods, teachers should adapt it for their teaching needs in classroom. It could help researchers who are investigating students' use of language in mathematics classroom as they could easily collect students' responses and focus on those for an upcoming discussion. It could be useful for researchers who are working on orchestrating whole-class discussions by using online resources (e.g. Fahlgren & Brunström, in press). Moreover, it could be useful for new types of activities, such as the MERLO items, which are including interactive applets and explanation tasks for students (e.g. Arzarello et al., 2015). Additionally, it could also assist teachers, who are teaching online regularly or assist teachers in remote or distance learning, as those teachers could then see the work of their students in real time, although they cannot really see the students.

In addition, several follow-up questions arise regarding the content of teacher trainings, where teachers can discuss how to use such a new tool in order to focus on students' responses and to foster interaction in the classroom. This includes strategies on how to choose or design appropriate tasks, how to efficiently observe student responses, how to decide which responses to pick and how to orchestrate classroom discussions.

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