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First order periodic Mean Field Games in Heisenberg group

PAOLA MANNUCCI,* CLAUDIO MARCHI †, NICOLETTA TCHOU ‡

Abstract

In this paper we study evolutive first order Mean Field Games in the Heisenberg group \mathbb{H}^1 ; in particular, the state of each agent can move only along “horizontal” trajectories which are given in terms of the vector fields generating \mathbb{H}^1 while the kinetic part of the cost depends only on the horizontal velocity. In this case, the Hamiltonian is not coercive in the gradient term and the coefficient of the first order term in the continuity equation has a quadratic growth at infinity. The main results of this paper are two: the former is to establish the existence of a weak solution to the Mean Field Game system while the latter is to prove that this solution is a *mild* solution in the sense introduced by Cannarsa and Capuani [19] for state-constrained Mean Field Games. Roughly speaking, this property means that, for a.e. starting state, the agents follow optimal trajectories of the optimal control problem associated to the Hamilton-Jacobi equation given by the evolution of the population. In order to prove these results, we need some properties which have their own interest: some uniqueness results for a Fokker-Planck equation of the second order with respect to the vector fields and a probabilistic representation of the solution to the continuity equation.

Keywords: Mean Field Games, first order Hamilton-Jacobi equations, continuity equation, Fokker-Planck equation, noncoercive Hamiltonian, Heisenberg group, degenerate optimal control problem.

2010 AMS Subject classification: 35F50, 35Q91, 49K20, 49L25.

1 Introduction

In this paper we study evolutive first order Mean Field Game (briefly, MFG) in the Heisenberg group \mathbb{H}^1 . Let us recall that the MFG theory started with the works by Lasry and Lions [34, 35, 36] and by Huang, Malhamé and Caines [31] (see the monographs [1, 22, 9, 29, 30] for the many developments in recent years) and studies Nash equilibria when the number of agents tends to infinity and each agent’s aim is to control its dynamics so to minimize a given cost which depends on the distribution of the whole population. On the other hand, the Heisenberg group can be seen as the first non-Euclidean space which is still endowed with nice properties as a (noncommutative) group operation, a family of

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dilations and a manifold structure (see the monographs [16, 39] for an overview). From the viewpoint of a single agent, the Heisenberg's framework entails that its state cannot change isotropically in all the directions but it can move only along *admissible* trajectories.

We shall consider systems of the form

$$(1.1) \quad \begin{cases} (i) & -\partial_t u + \frac{|D_{\mathcal{H}}u|^2}{2} = F[m(t)](x) & \text{in } \mathbb{H}^1 \times (0, T) \\ (ii) & \partial_t m - \operatorname{div}_{\mathcal{H}}(m D_{\mathcal{H}}u) = 0 & \text{in } \mathbb{H}^1 \times (0, T) \\ (iii) & m(x, 0) = m_0(x), u(x, T) = G[m(T)](x) & \text{on } \mathbb{H}^1, \end{cases}$$

where $D_{\mathcal{H}}$ and $\operatorname{div}_{\mathcal{H}}$ are respectively the *horizontal gradient* and the *horizontal divergence* while F and G are strongly regularizing coupling operators and are $Q_{\mathcal{H}}$ -periodic w.r.t. x with $Q_{\mathcal{H}} := [0, 1]^3$: see assumptions (H1)-(H4) and section 2 below for the precise definitions. Hence, we shall focus our attention on $Q_{\mathcal{H}}$ -periodic solution to (1.1). It is worth to stress that the $Q_{\mathcal{H}}$ -periodicity is the periodicity in the group operation of the Heisenberg group: see Section 2.1 below for the precise definition.

For readers which are not familiar with intrinsic calculus, in Euclidean coordinates, system (1.1) becomes

$$(1.2) \quad \begin{cases} (i) & -\partial_t u + H(x, Du) = F[m(t)](x) & \text{in } \mathbb{R}^3 \times (0, T) \\ (ii) & \partial_t m - \operatorname{div}(m \partial_p H(x, Du)) = 0 & \text{in } \mathbb{R}^3 \times (0, T) \\ (iii) & m(x, 0) = m_0(x), u(x, T) = G[m(T)](x) & \text{on } \mathbb{R}^3, \end{cases}$$

where, for $p = (p_1, p_2, p_3)$ and $x = (x_1, x_2, x_3)$, the Hamiltonian $H(x, p)$ is

$$(1.3) \quad H(x, p) := \frac{1}{2}((p_1 - x_2 p_3)^2 + (p_2 + x_1 p_3)^2) = \frac{|pB(x)|^2}{2} \text{ with } B(x) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x_2 & x_1 \end{pmatrix} \in M^{3 \times 2}$$

while the drift $\partial_p H(x, p)$ is

$$(1.4) \quad \partial_p H(x, p) = pB(x)B(x)^T = (p_1 - x_2 p_3, p_2 + x_1 p_3, -p_1 x_2 + p_2 x_1 + p_3(x_1^2 + x_2^2)).$$

These MFG systems arise when the generic player with state x at time t must follow *horizontal curves* with respect to the two vector fields X_1 and X_2 generating the Heisenberg group (see (2.3) below):

$$(1.5) \quad x'(s) = \alpha_1(s)X_1(x(s)) + \alpha_2(s)X_2(x(s))$$

namely

$$x'_1(s) = \alpha_1(s), \quad x'_2(s) = \alpha_2(s), \quad x'_3(s) = -x_2(s)\alpha_1(s) + x_1(s)\alpha_2(s).$$

Each agent wants to choose the control $\alpha = (\alpha_1, \alpha_2)$ in $L^2([t, T]; \mathbb{R}^2)$ in order to minimize the cost

$$(1.6) \quad J_t^m(x(\cdot), \alpha(\cdot)) := \int_t^T \left[\frac{1}{2}|\alpha(\tau)|^2 + F[m(\tau)](x(\tau)) \right] d\tau + G[m(T)](x(T))$$

where $m(\cdot)$ is the evolution of the whole population's distribution while $(x(\cdot), \alpha(\cdot))$ is a trajectory obeying to (1.5).

Let us observe two important issues of these MFG systems: (i) the Hamiltonian H is not coercive in p uniformly in x , (ii) in equation (1.2)-(ii) the coefficient of the first

order term has a quadratic growth in x .

Point (i) prevents the application of standard approaches for first order MFG (for instance, see [9, 22, 24, 30]). Moreover, we recall that the papers [2, 20, 37] already tackled MFG systems with noncoercive Hamiltonians for first order MFG while paper [28] dealt with second order subelliptic MFG. However, the results in [20, 2] do not apply to the present setting because these papers consider a different kind of admissible trajectories. Note that the present case is not encompassed neither in our previous work [37] because the matrix B in (1.3) does not fulfil the assumptions of [37, Section 5], in particular $\det B(x)B^T(x) = 0$ for any $x \in \mathbb{R}^3$ hence the set where the matrix BB^T degenerates does not have null measure. On the other hand, point (ii) gives rise to some difficulties for applying the vanishing viscosity method, especially for the well-posedness of the Cauchy problem for equation (1.2)-(ii) with the viscosity term and for its stochastic interpretation.

The aims of this paper are two; the former one is to prove the existence of a weak solution to system (1.1) in the sense of Definition 3.1 below. To this end, we establish several properties of the solution to the Hamilton-Jacobi equation (1.1)-(i) (as semiconcavity, Lipschitz continuity, regularity of the optimal trajectories for the associated optimal control problem) and we perform a vanishing viscosity procedure with the *horizontal* Laplacian adapting the techniques introduced by P.L. Lions in his lectures at Collège de France [22, 36] (see also [2, 37] for similar approaches for some noncoercive Hamiltonians). The latter, and main, aim of this paper is to prove that this weak solution is also a *mild* solution in the sense of the definition introduced by Cannarsa and Capuani [19] when the control is on the velocity of the trajectory. Roughly speaking, as in the Lagrangian approach for MFG (see [8, 19]), this property means that, for a.e. starting state, the agents follow optimal trajectories for the optimal control problem associated to the Hamilton-Jacobi equation. As in the state-constrained case treated in [19], here the uniqueness of the optimal trajectories a.e. with respect to the initial points is not granted; this implies that the measure representing by the superposition principle the behaviour of the mass of agents is not unique. However, it is worth to observe that, in contrast with the cases tackled in [2, 22, 37], in the present setting it is not clear if for a.e. initial starting point the optimal trajectory is uniquely determined. This is a consequence of the degeneracy of the matrix BB^T at *every* point of the space. The issue of finding necessary or sufficient conditions ensuring the uniqueness of optimal trajectories for a.e. starting points with respect to the initial distribution of players is challenging and open even in the Heisenberg group; we hope to study it in a future work. In order to prove that our solution is in fact a mild solution, we shall adapt to the periodic Heisenberg setting the superposition principle [3, Theorem 8.1.2]. In particular we shall prove in the Heisenberg framework: (1) an optimal synthesis result, (2) a superposition principle, (3) a uniqueness result in a viscous setting for the Fokker-Planck equation with unbounded coefficients. In our opinion, these results can have an interest independent of the MFG context.

As in the papers [34, 35, 36], we tackle the periodic setting taking advantage of the invariance of the structure of the operators in the Heisenberg group and of the definition of a periodicity cell introduced in [12, 13] (see also [11]). Let us stress that our results could be generalized with suitable adaptations to the case of invariant operators in other Lie groups [38, 41]. The techniques for the nonperiodic case are different and the results will appear in a future paper.

This paper is organized as follows. Section 2 is devoted to define the Heisenberg group, the periodicity and the convolution in the group. Section 3 contains the assumptions and the statement of the main theorem. In Section 4 we study the main properties of

the solution of the optimal control problem associated to the Hamilton-Jacobi equation (1.1)-(i). The main contribution of Section 5 is the proof of Theorem 5.1 which gives the needed regularity of the solution m to prove the main Theorem 3.1 in Section 6. The three appendices deal with A) the definition of the differential operators in the Heisenberg group, B) uniqueness results for a general degenerate Fokker-Plank equation with unbounded coefficients, C) an adaptation of the probabilistic representation result of [3] to a generic “intrinsic” continuity equation in a periodic setting for the Heisenberg group.

Notations. For any function $u : \mathbb{R}^3 \times \mathbb{R} \ni (x, t) \rightarrow u(x, t) \in \mathbb{R}$, Du and D^2u stand for the Euclidean gradient and respectively Hessian matrix with respect to x . For any compact set A of \mathbb{R}^3 , we denote $C^2(A)$ the space of functions with continuous second order derivatives endowed with the norm $\|f\|_{C^2(A)} := \sup_{x \in A} [|f(x)| + |Df(x)| + |D^2f(x)|]$.

For any complete separable metric space X , $\mathcal{M}(X)$ and $\mathcal{P}(X)$ denotes the set of non-negative Radon measures on X , and respectively of probability measures on X . For any complete separable metric spaces X_1 and X_2 , any measure $\eta \in \mathcal{P}(X_1)$ and any function $\phi : X_1 \rightarrow X_2$, we denote $\phi\#\eta \in \mathcal{P}(X_2)$ the *push-forward* of η through ϕ , i.e. $\phi\#\eta(B) := \eta(\phi^{-1}(B))$, for any B measurable set, $B \subset X_2$. (see [3, section 5.2] for the precise definition and main properties). For a function $m \in C([0, T], \mathcal{P}(X))$, m_t stands for the probability $m(t, \cdot)$ on X .

2 Preliminaries: The Heisenberg group

We introduce the following noncommutative group structure on \mathbb{R}^3 . We refer to [16] for a complete overview on the Heisenberg group.

Definition 2.1 *The 3-dimensional Heisenberg group \mathbb{H}^1 is the vector space \mathbb{R}^3 , endowed with the following noncommutative group operation, denoted by \oplus :*

$$(2.1) \quad x \oplus y = (x_1, x_2, x_3) \oplus (y_1, y_2, y_3) := (x_1 + y_1, x_2 + y_2, x_3 + y_3 - x_2y_1 + x_1y_2).$$

for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

The law $x \oplus y$ is called the x *left translation* of y . We call x^{-1} the point such that $x^{-1} \oplus x = x \oplus x^{-1} = 0$. Note that $x^{-1} = (-x_1, -x_2, -x_3)$. Hence we define

$$x \ominus y := x \oplus y^{-1}$$

In \mathbb{H}^1 we define a dilations’ family as follows.

Definition 2.2 *The dilations in the Heisenberg group are the family of group homeomorphisms defined as, for all $\lambda > 0$, $\delta_\lambda : \mathbb{H}^1 \rightarrow \mathbb{H}^1$ with*

$$(2.2) \quad \delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3), \quad \forall x = (x_1, x_2, x_3) \in \mathbb{H}^1.$$

We define the two vector fields

$$(2.3) \quad X_1(x) = \begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix} \quad \text{and} \quad X_2(x) = \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}, \quad \forall x = (x_1, x_2, x_3) \in \mathbb{H}^1.$$

By these vectors we define the linear differential operators, still called X_1 and X_2

$$(2.4) \quad X_1 = \partial_{x_1} - x_2 \partial_{x_3}, \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3}.$$

Note that their commutator $[X_1, X_2] := X_1 X_2 - X_2 X_1$ verifies: $[X_1, X_2] = -2\partial_{x_3}$. Hence we say that the vectors $X_1(x)$ and $X_2(x)$ are the generators of \mathbb{R}^3 because, together with their commutator $[X_1, X_2]$, they span all \mathbb{R}^3 . The fields X_1 and X_2 are left-invariant vector fields, i.e. for all $u \in C^\infty(\mathbb{H}^1)$ and for all fixed $y \in \mathbb{H}^1$

$$(2.5) \quad X_i(u(y \oplus x)) = (X_i u)(y \oplus x), \quad i = 1, 2.$$

Note that the matrix $B(x)$ defined in (1.3) is the matrix associated to the vectors X_1 and X_2 . For any regular real-valued function u , we shall denote its horizontal gradient and its horizontal Laplacian by $D_{\mathcal{H}}u := (X_1 u, X_2 u)$ and respectively $\Delta_{\mathcal{H}} := X_1^2 u + X_2^2 u$ and we observe $D_{\mathcal{H}}u = Du B(x)$ and $\Delta_{\mathcal{H}}u = \text{tr}(D^2 u B B^T)$ where Du and $D^2 u$ denote the Euclidean gradient and respectively the Hessian matrix of u . For any regular $u = (u_1, u_2) : \mathbb{H}^1 \rightarrow \mathbb{R}^2$, we denote its horizontal divergence by $\text{div}_{\mathcal{H}} u := X_1 u_1 + X_2 u_2$ and we note that the left-invariance of X_i ($i = 1, 2$) entails the left-invariance of $\text{div}_{\mathcal{H}}$. We have: $\text{div}_{\mathcal{H}}(D_{\mathcal{H}}u) = \Delta_{\mathcal{H}}u$.

Let us define

$$(2.6) \quad \|x\|_{\mathcal{H}} := ((x_1^2 + x_2^2)^2 + x_3^2)^{1/4}$$

and the distance associated by the group law

$$(2.7) \quad d_{\mathcal{H}}(x, y) := \|x \oplus y^{-1}\|_{\mathcal{H}}.$$

Remark 2.1 *Let us recall that there holds*

$$d_{\mathcal{H}}(x, y) \leq |x - y| + (1 + |x_1|^{1/2} + |x_2|^{1/2})|x - y|^{1/2} \quad \forall x, y \in \mathbb{H}^1.$$

For the sake of completeness, let us briefly recall the proof. We have

$$\begin{aligned} d_{\mathcal{H}}(x, y) &\leq [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} + |x_3 - y_3 + x_2 y_1 - x_1 y_2|^{1/2} \\ &\leq |x - y| + |x_3 - y_3 + x_2 y_1 - x_1 y_2|^{1/2}. \end{aligned}$$

On the other hand, there holds

$$\begin{aligned} |x_3 - y_3 + x_2 y_1 - x_1 y_2|^{1/2} &= |x_3 - y_3 + x_2(y_1 - x_1) + x_1(x_2 - y_2)|^{1/2} \\ &\leq |x_3 - y_3|^{1/2} + |x_2|^{1/2}|y_1 - x_1|^{1/2} + |x_1|^{1/2}|x_2 - y_2|^{1/2} \\ &\leq (1 + |x_1|^{1/2} + |x_2|^{1/2})|x - y|^{1/2}. \end{aligned}$$

Replacing the last inequality in the previous one, we accomplish the proof.

Using the definitions (2.4) and (2.6) we easily prove the following equalities:

Lemma 2.1 *The following equalities hold*

$$\begin{aligned} (i) \quad X_1(\|x\|_{\mathcal{H}}^2) &= \frac{2x_1(x_1^2 + x_2^2) - x_2 x_3}{\|x\|_{\mathcal{H}}^2}, \quad X_2(\|x\|_{\mathcal{H}}^2) = \frac{2x_2(x_1^2 + x_2^2) + x_1 x_3}{\|x\|_{\mathcal{H}}^2}, \\ (ii) \quad |D_{\mathcal{H}}(\|x\|_{\mathcal{H}}^2)|^2 &= \frac{4(x_1^2 + x_2^2)^3 + (x_1^2 + x_2^2)x_3^2}{\|x\|_{\mathcal{H}}^4}, \\ (iii) \quad X_1^2(\|x\|_{\mathcal{H}}^2) &= \frac{6x_1^2 + 3x_2^2}{\|x\|_{\mathcal{H}}^2} - \frac{|X_1(\|x\|_{\mathcal{H}}^2)|^2}{\|x\|_{\mathcal{H}}^2}, \quad X_2^2(\|x\|_{\mathcal{H}}^2) = \frac{6x_2^2 + 3x_1^2}{\|x\|_{\mathcal{H}}^2} - \frac{|X_2(\|x\|_{\mathcal{H}}^2)|^2}{\|x\|_{\mathcal{H}}^2}, \\ (iv) \quad \Delta_{\mathcal{H}}(\|x\|_{\mathcal{H}}^2) &= \frac{9(x_1^2 + x_2^2)}{\|x\|_{\mathcal{H}}^2} - \frac{|D_{\mathcal{H}}(\|x\|_{\mathcal{H}}^2)|^2}{\|x\|_{\mathcal{H}}^2}. \end{aligned}$$

2.1 Periodicity in the Heisenberg group

The notion of periodicity is introduced by the group law \oplus . We follow the definition and the results given in [12, 13].

Let $Q_{\mathcal{H}} = [0, 1]^3$. We can construct a tiling of \mathbb{H}^1 by the *property of pavage*: for every $x \in \mathbb{H}^1$ there exists a unique $z \in \mathbb{Z}^3$ such that there exists a unique $q_{\mathcal{H}} = q_{\mathcal{H}}(x) \in Q_{\mathcal{H}}$ such that $z \oplus q_{\mathcal{H}} = x$.

We can now define the $Q_{\mathcal{H}}$ -periodicity on \mathbb{H}^1 with respect to this reference pavage.

Definition 2.3 A function f defined on \mathbb{H}^1 is said $Q_{\mathcal{H}}$ -periodic if for any $x \in \mathbb{H}^1$,

$$f(x) = f(q_{\mathcal{H}}(x)),$$

where $q_{\mathcal{H}}(x)$ is the unique element of $Q_{\mathcal{H}}$ such that $x = z \oplus q_{\mathcal{H}}(x)$ for the unique $z \in \mathbb{Z}^3$.

We will denote by $C_{Q_{\mathcal{H}}, per}^{\infty}$ the set of the functions $f \in C^{\infty}(\mathbb{H}^1)$ that are $Q_{\mathcal{H}}$ -periodic. The definition of $Q_{\mathcal{H}}$ -periodicity is equivalent to the following definition of $1_{\mathcal{H}}$ -periodicity:

Definition 2.4 A function f defined on \mathbb{H}^1 is said $1_{\mathcal{H}}$ -periodic if for any $x \in \mathbb{H}^1$ and any $z \in \mathbb{Z}^3$ there holds

$$f(z \oplus x) = f(x).$$

Lemma 2.2 A function f is $Q_{\mathcal{H}}$ -periodic if and only if is $1_{\mathcal{H}}$ -periodic.

Proof. Note that by the pavage property if f is $1_{\mathcal{H}}$ -periodic then is $Q_{\mathcal{H}}$ -periodic. Conversely, for any $x \in \mathbb{H}^1$ there exist unique z and $q_{\mathcal{H}}$ such that $x = z \oplus q_{\mathcal{H}}$. For any $z' \in \mathbb{Z}^3$ we write $z' \oplus x = z' \oplus z \oplus q_{\mathcal{H}}$. Since $z' \oplus z \in \mathbb{Z}^3$ then $q_{\mathcal{H}}(z' \oplus x) = q_{\mathcal{H}}(x)$ and by the definition of $Q_{\mathcal{H}}$ -periodicity we get $f(z' \oplus x) = f(q_{\mathcal{H}}(z' \oplus x)) = f(q_{\mathcal{H}}(x)) = f(x)$, for any $z' \in \mathbb{Z}^3$. \square

Definition 2.5 We denote with $\mathbb{T}_{\mathcal{H}}$ the torus in the Heisenberg group \mathbb{H}^1 , namely $\mathbb{H}^1/\mathbb{Z}^3$ using the following equivalence law: $x \sim y$ if there exists $z \in \mathbb{Z}^3$ such that $z \oplus x = y$. The torus is naturally endowed with the distance induced by $d_{\mathcal{H}}$:

$$d_{\mathbb{T}_{\mathcal{H}}}(x, y) := \inf d_{\mathcal{H}}(x', y') \quad \forall x, y \in \mathbb{T}_{\mathcal{H}}$$

where the infimum is performed over all the couple $(x', y') \in \mathbb{H}^1 \times \mathbb{H}^1$ with $x \sim x'$, $y \sim y'$.

Remark 2.2 Lemma 2.2 ensures that $x \sim x'$ if and only if $q_{\mathcal{H}}(x) = q_{\mathcal{H}}(x')$. It is worth to observe that the Heisenberg torus $\mathbb{T}_{\mathcal{H}}$ does not coincide with the Euclidean torus; especially, $\mathbb{T}_{\mathcal{H}}$ is not obtained identifying the points of two opposite faces of $\overline{Q_{\mathcal{H}}}$ with the same two coordinates. As a matter of facts, this happens between the two faces given by $x_3 = 0$ and $x_3 = 1$. For completeness, let us write the identification of points $(1, x_2, x_3)$ with $(x_2, x_3) \in [0, 1]^2$ with points $(0, x'_2, x'_3)$ with $(x'_2, x'_3) \in [0, 1]^2$: we have

$$(1, x_2, x_3) \sim \begin{cases} (0, x_2, x_3 - x_2) & \text{for } x_3 - x_2 \in [0, 1) \\ (0, x_2, x_3 - x_2 + 1) & \text{for } x_3 - x_2 \in (-1, 0]; \end{cases}$$

actually, for $x_3 - x_2 \in [0, 1)$ there holds $(-1, 0, 0) \oplus (1, x_2, x_3) = (0, x_2, x_3 - x_2)$ while for $x_3 - x_2 \in [-1, 0)$ there holds $(-1, 0, 1) \oplus (1, x_2, x_3) = (0, x_2, x_3 - x_2 + 1)$. Moreover, $(1, 1, x_3) \sim (0, 0, x_3)$ because $(-1, -1, 0) \oplus (1, 1, x_3) = (0, 0, x_3)$ for every $x_3 \in [0, 1)$ and $(1, 1, 1) \sim (0, 0, 0)$ because $(-1, -1, -1) \oplus (1, 1, 1) = (0, 0, 0)$. And similarly for the remaining cases.

Remark 2.3 *With a slight abuse of notations, throughout this paper we shall identify any measure $\eta \in \mathcal{M}(Q_{\mathcal{H}})$ with the same measure on the torus $\mathbb{T}_{\mathcal{H}}$ and also with the measure $\eta' \in \mathcal{M}(\mathbb{H}^1)$ such that $\eta'(z \oplus A) = \eta(A)$ for any measurable set $A \subset Q_{\mathcal{H}}$ and $z \in \mathbb{Z}^3$.*

Remark 2.4 *We recall from [16, Proposition 1.3.21] that the Haar measure associated to the Heisenberg group coincides with the Lebesgue measure.*

2.2 Convolution on Heisenberg group

We define the convolution in Heisenberg group of a function $\psi \in L^1_{loc}(\mathbb{H}^1)$ by a function $\rho \in C_c^\infty(\mathbb{H}^1)$ as

$$(2.8) \quad (\psi * \rho)(x) = \int_{\mathbb{H}^1} \psi(y)\rho(x \ominus y)dy.$$

In the proof of the main result, theorem 3.1, we will use the convolution by the regularizing kernel

$$(2.9) \quad \rho_\epsilon(x) = C(\epsilon)\rho_0(\|x/\epsilon\|_{\mathcal{H}}^4)$$

where $\rho_0(t) = e^{-t}$ and the constant $C(\epsilon)$ is chosen such that $\int_{\mathbb{H}^1} \rho_\epsilon(x)dx = 1$. This convolution has the following properties

Proposition 2.1 *We have*

- (i) $\psi * \rho_\epsilon = \rho_\epsilon * \psi$;
- (ii) *If ψ is $Q_{\mathcal{H}}$ -periodic then also $\psi * \rho_\epsilon$ is $Q_{\mathcal{H}}$ -periodic;*
- (iii) *If ψ is $L^p(\mathbb{H}^1)$ for some $p \geq 1$, then $\psi * \rho_\epsilon$ is $C^\infty(\mathbb{H}^1)$;*
- (iv) *If ψ is $L^1_{loc}(\mathbb{H}^1)$ then $\psi * \rho_\epsilon \rightarrow \psi$ in $L^1_{loc}(\mathbb{H}^1)$ as $\epsilon \rightarrow 0$;*
- (v) *If ψ is differentiable then*

$$X_i\psi * \rho_\epsilon = (X_i\psi) * \rho_\epsilon = \psi * X_i\rho_\epsilon,$$

where the vectors X_i are defined in (2.4);

- (vi) *If $\psi \geq 0$ in \mathbb{H}^1 and $\int_{\mathbb{H}^1} \psi(x)dx = C > 0$ then $\psi * \rho_\epsilon(x) > 0$ for any $x \in \mathbb{H}^1$.*

Proof. All the proofs are easy and standard using the fact that the Haar measure for the Heisenberg group coincides with the Lebesgue measure. For the sake of completeness, we only provide the detailed proof of (v) for X_1 as an example

$$\begin{aligned} X_1(\psi * \rho_\epsilon)(x) &= \partial_{x_1}(\psi * \rho_\epsilon)(x) - x_2\partial_{x_3}(\psi * \rho_\epsilon)(x) = \int \psi(y)[\partial_{x_1}\rho_\epsilon + y_2\partial_{x_3}\rho_\epsilon - x_2\partial_{x_3}\rho_\epsilon]dy \\ &= \int \psi(y)[\partial_{x_1}\rho_\epsilon - (x_2 - y_2)\partial_{x_3}\rho_\epsilon]dy = \int \psi(y)X_1\rho_\epsilon(x \ominus y)dy = \psi * (X_1\rho_\epsilon)(x). \end{aligned}$$

□

3 Definitions, assumptions and main results

In this section, we introduce the functional spaces needed for the definition of solution to system (1.1), our assumptions and we state the main results of this paper. Following [14], we adapt the classical notion of Kantorovich-Rubinstein distance to the set $\mathbb{T}_{\mathcal{H}}$ in terms of the distance $d_{\mathbb{T}_{\mathcal{H}}}$ introduced in Definition 2.5:

$$\mathbf{d}_1(m, m') := \inf_{\pi \in \Pi(m, m')} \int_{\mathbb{T}_{\mathcal{H}} \times \mathbb{T}_{\mathcal{H}}} d_{\mathbb{T}_{\mathcal{H}}}(x, y) d\pi(x, y) \quad \forall m, m' \in \mathcal{P}(\mathbb{T}_{\mathcal{H}})$$

where

(3.1)

$$\Pi(m, m') := \{ \pi \text{ Borel prob. meas. on } \mathbb{T}_{\mathcal{H}} \times \mathbb{T}_{\mathcal{H}} : \pi(A \times \mathbb{T}_{\mathcal{H}}) = m(A), \pi(\mathbb{T}_{\mathcal{H}} \times A) = m'(A) \},$$

where A is any Borel set $A \subset \mathbb{T}_{\mathcal{H}}$.

For the sake of completeness, let us recall that: $\mathbf{d}_1(m, m') = \sup \int_{\mathbb{T}_{\mathcal{H}}} f(x) d(m - m')(x)$, where the supremum is taken over the set of all maps $f : \mathbb{T}_{\mathcal{H}} \rightarrow \mathbb{R}$ which are 1-Lipschitz continuous with respect to $d_{\mathbb{T}_{\mathcal{H}}}$ (see [14, Theorem 1.1.5]).

We set

$$\mathcal{P}_{per}(\mathbb{H}^1) := \left\{ m \in \mathcal{M}(\mathbb{H}^1) : m|_{Q_{\mathcal{H}}} \in \mathcal{P}(Q_{\mathcal{H}}), \quad m \text{ is } Q_{\mathcal{H}}\text{-periodic} \right\}$$

where for “ m is $Q_{\mathcal{H}}$ -periodic” we mean $m(z \oplus A) = m(A)$ for every $z \in \mathbb{Z}^3$ and every measurable $A \subset \mathbb{H}^1$. By Remark 2.3, we identify $\mathcal{P}_{per}(\mathbb{H}^1)$ with $\mathcal{P}(\mathbb{T}_{\mathcal{H}})$. We assume that the set $\mathcal{P}_{per}(\mathbb{H}^1)$ is endowed with the distance \mathbf{d}_1 .

Throughout this paper, unless otherwise explicitly stated, we shall require the following hypotheses:

- (H1) the functions F and G are real-valued function, continuous on $\mathcal{P}_{per}(\mathbb{H}^1) \times \mathbb{H}^1$, moreover, for any fixed $m \in \mathcal{P}_{per}(\mathbb{H}^1)$, $F[m](\cdot)$ and $G[m](\cdot)$ are $Q_{\mathcal{H}}$ -periodic;
- (H2) the map $m \rightarrow F[m](\cdot)$ is Lipschitz continuous from $\mathcal{P}_{per}(\mathbb{H}^1)$ to $C^2(\mathbb{R}^3)$; moreover, there exist $C \in \mathbb{R}$ and $\delta_0 \in (0, 1]$ such that

$$\|F[m](\cdot)\|_{C^{2+\delta_0}(\mathbb{R}^3)}, \|G[m](\cdot)\|_{C^2(\mathbb{R}^3)} \leq C, \quad \forall m \in \mathcal{P}_{per}(\mathbb{H}^1);$$

- (H3) the distribution $m_0 : \mathbb{H}^1 \rightarrow \mathbb{R}$ is a nonnegative C^0 function, $Q_{\mathcal{H}}$ -periodic with $\int_{Q_{\mathcal{H}}} m_0 dx = 1$.

Example 3.1 *Easy examples of F and G are given by the convolution of a regular kernel (as the one defined in (2.9)) with m . In this case, Proposition 2.1 ensures that assumptions (H1) and (H2) are satisfied.*

We now introduce our definitions of solution of the MFG system (1.1) and state the main result concerning its existence.

Definition 3.1 *A couple (u, m) of $Q_{\mathcal{H}}$ -periodic functions on $\mathbb{H}^1 \times [0, T]$ is a solution of system (1.1) if:*

- 1) u belongs to $W^{1,\infty}(\mathbb{H}^1 \times [0, T])$;

- 2) m belongs to $C^0([0, T]; \mathcal{P}_{per}(\mathbb{H}^1))$ and for all $t \in [0, T]$, m_t is absolutely continuous w.r.t. the Lebesgue measure. Let $m(\cdot, t)$ denote the density of m_t . The function $(x, t) \mapsto m(x, t)$ is bounded;
- 3) Equation (1.1)-(i) is satisfied by u in the viscosity sense in $\mathbb{H}^1 \times (0, T)$;
- 4) Equation (1.1)-(ii) is satisfied by m in the sense of distributions in $\mathbb{H}^1 \times (0, T)$.

Remark 3.1 Any solution (u, m) of the MFG system (1.1) is also a solution in $\mathbb{T}_{\mathcal{H}} \times [0, T]$ by the identification of $\mathcal{P}_{per}(\mathbb{H}^1)$ with $\mathcal{P}(\mathbb{T}_{\mathcal{H}})$.

Remark 3.2 From Lemma C.1 in Appendix C, we get that the distributional solution of (1.1)-(ii) stated in point 4) of the definition 3.1 is automatically continuous in the sense of point 2) of the same definition.

In order to give a more detailed description of our solution, it is expedient to use the notion of *mild* solution introduced by [19]. This notion is reminiscent of the Lagrangian approach to MFGs (see [8]) and it relies on replacing probability measures on the state space with probability measures on arcs on the state space.

We define the set of AC arcs in \mathbb{H}^1

$$(3.2) \quad \Gamma := \{\gamma \in AC((0, T), \mathbb{H}^1)\}$$

and the evaluation map $e_t : \Gamma \rightarrow \mathbb{H}^1$ as

$$(3.3) \quad e_t(\gamma) = \gamma(t).$$

For any $x \in \mathbb{H}^1$, we define the set of arcs starting at x

$$\Gamma_0[x] := \{\gamma \in \Gamma, \gamma(0) = x\}$$

and the set of horizontal arcs starting at x with an associated control law

$$\mathcal{A}(x, 0) := \{(\gamma, \alpha) : \gamma \in \Gamma_0[x], \alpha \in L^2([0, T], \mathbb{R}^2), (\gamma, \alpha) \text{ solves (1.5)}\}.$$

Given $m_0 \in \mathcal{P}_{per}(\mathbb{H}^1)$, we define

$$\mathcal{P}_{m_0}(\Gamma) = \{\eta \in \mathcal{M}(\Gamma) : m_0 = e_0 \# \eta \text{ and } e_t \# \eta \in \mathcal{P}_{per}(\mathbb{H}^1) \quad \forall t \in [0, T]\}.$$

For any $\eta \in \mathcal{P}_{m_0}(\Gamma)$ and for any $x \in \mathbb{H}^1$, we consider the cost

$$(3.4) \quad J_x^\eta(\gamma(\cdot), \alpha) := \int_0^T \left[\frac{1}{2} |\alpha(\tau)|^2 + F[e_\tau \# \eta](\gamma(\tau)) \right] d\tau + G[e_T \# \eta](\gamma(T))$$

where $(\gamma, \alpha) \in \mathcal{A}(x, 0)$. For any $\eta \in \mathcal{P}_{m_0}(\Gamma)$ and for any $x \in \mathbb{H}^1$ we define the set of optimal horizontal arcs starting at x

$$\Gamma^\eta[x] := \{\bar{\gamma} : (\bar{\gamma}, \bar{\alpha}) \in \mathcal{A}[x, 0] : J_x^\eta(\bar{\gamma}(\cdot), \bar{\alpha}) = \min_{(\gamma, \alpha) \in \mathcal{A}(x, 0)} J_x^\eta(\gamma, \alpha)\}.$$

Definition 3.2 A measure $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a MFG equilibrium for m_0 if

$$\text{supp } \eta \subseteq \bigcup_{x \in \mathbb{H}^1} \Gamma^\eta[x].$$

This means that the support of η is contained in the set $\cup_{x \in \mathbb{H}^1} \{\gamma \in \Gamma_0[x] : \gamma \text{ is a minimizer of } J_x^\eta\}$ (see also [19]).

Definition 3.3 A couple $(u, m) \in C^0([0, T] \times \mathbb{H}^1) \times C^0([0, T]; \mathcal{P}_{per}(\mathbb{H}^1))$ is called mild solution if there exists a MFG equilibrium η for m_0 such that:

i) $m_t = e_t \# \eta$;

ii) u is given by

$$u(x, t) = \inf_{(\gamma, \alpha) \in \mathcal{A}(x, 0)} \int_t^T \left[\frac{1}{2} |\alpha(\tau)|^2 + F[e_\tau \# \eta](\gamma(\tau)) \right] d\tau + G[e_T \# \eta](\gamma(T)).$$

Now we can state the main result of this paper.

Theorem 3.1 Under the above assumptions:

i) System (1.1) has a solution (u, m) ;

ii) (u, m) is a mild solution.

Remark 3.3 As a matter of fact, from the proof of this theorem we get that any solution, as in Definition 3.1 is a mild solution.

Remark 3.4 Uniqueness holds under classical hypothesis on the monotonicity of F and G as in [22].

4 Formulation of the optimal control problem

In this section, we tackle the optimal control problem associated to the Hamilton-Jacobi equation (1.1)-(i); in particular we shall show that the value function solves this equation, is $Q_{\mathcal{H}}$ -periodic, Lipschitz continuous and semiconcave in x . Throughout this section we shall assume the following hypothesis

Hypothesis 4.1 1. f, g are $Q_{\mathcal{H}}$ -periodic w.r.t. x ;

2. $f \in C^0([0, T], C^2(\mathbb{R}^3))$, $g \in C^2(\mathbb{R}^3)$; so there exists a constant C such that

$$\sup_{t \in [0, T]} \|f(\cdot, t)\|_{C^2(\mathbb{R}^3)} + \|g\|_{C^2(\mathbb{R}^3)} \leq C.$$

Definition 4.1 We consider the following optimal control problem:

$$(4.1) \quad \text{minimize } J_t(x(\cdot), \alpha(\cdot)) := \int_t^T \frac{1}{2} |\alpha(s)|^2 + f(x(s), s) ds + g(x(T))$$

subject to $(x(\cdot), \alpha(\cdot)) \in \mathcal{A}(x, t)$, where

$$(4.2) \quad \mathcal{A}(x, t) := \left\{ (x(\cdot), \alpha(\cdot)) \in AC([t, T]; \mathbb{R}^3) \times L^2([t, T]; \mathbb{R}^2) : (1.5) \text{ holds a.e. with } x(t) = x \right\}.$$

A couple $(x(\cdot), \alpha(\cdot)) \in \mathcal{A}(x, t)$ is said to be admissible. We say that $x^*(\cdot)$ is an optimal trajectory if there is a control $\alpha^*(\cdot)$ such that $(x^*(\cdot), \alpha^*(\cdot)) \in \mathcal{A}(x, t)$ is optimal for the optimal control problem in (4.1). Also, we shall refer to the system (1.5) as to the dynamics of the optimal control problem in (4.1).

Remark 4.1 Notice that, given a control law $\alpha \in L^2([t, T]; \mathbb{R}^2)$ and an initial point x , there is a unique trajectory $x(\cdot)$ such that $(x(\cdot), \alpha) \in \mathcal{A}(x, t)$.

Remark 4.2 Hypothesis 4.1 ensures that, for any $(x, t) \in \mathbb{R}^3 \times (0, T)$, the optimal control problem in definition 4.1 admits a solution $(x^*(\cdot), \alpha^*)$ thanks to the LSC with respect to the weak L^2 topology. Moreover, just testing $J_t(x^*(\cdot), \alpha^*)$ against $J_t(x, 0)$, we get

$$(4.3) \quad \|\alpha^*\|_{L^2(t, T)} \leq C_1 := C[(T - t) + 1],$$

where C is the constant introduced in Hypotheses 4.1. In particular, by Hölder inequality,

$$(4.4) \quad x^* \in C^{1/2}([t, T], \mathbb{H}^1).$$

Definition 4.2 The value function for the cost J_t defined in (4.1) is

$$(4.5) \quad u(x, t) := \inf \{J_t(x(\cdot), \alpha) : (x(\cdot), \alpha) \in \mathcal{A}(x, t)\}.$$

An optimal couple $(x^*(\cdot), \alpha^*)$ for the control problem in definition 4.1 is also said to be optimal for $u(x, t)$.

The following lemma states that, under Hypothesis 4.1, the value function u is $Q_{\mathcal{H}}$ -periodic in x hence we can restrict our study to $Q_{\mathcal{H}}$.

Lemma 4.1 Let u be the value function introduced in (4.5). Then u is $Q_{\mathcal{H}}$ -periodic in x .

Proof. We have to prove that $u(z \oplus x, t) = u(x, t)$ for any $z \in \mathbb{Z}^3$ and for any $x \in \mathbb{H}^1$. Note that if $x(s)$ and $y(s)$ solves (1.5) with the same law of control β and with respectively $x(t) = x$ and $y(t) = z \oplus x$, then $y(s) = z \oplus x(s)$; actually there hold

$$\begin{aligned} y_i(s) &= z_i + x_i + \int_t^s \beta_i(\tau) d\tau = z_i + x_i(s), \quad \text{for } i = 1, 2, \\ y_3(s) &= z_3 + x_3 - z_2 x_1 + z_1 x_2 + \int_t^s (z_2 + x_2(\tau))(-\beta_1(\tau)) + (z_1 + x_1(\tau))\beta_2(\tau) d\tau \\ &= z_3 + \left(x_3 - \int_t^s x_2(\tau)\beta_1(\tau) + x_1(\tau)\beta_2(\tau) d\tau \right) - z_2 \left(x_1 + \int_t^s \beta_1(\tau) d\tau \right) \\ &\quad + z_1 \left(x_2 + \int_t^s \beta_2(\tau) d\tau \right) \\ &= z_3 + x_3(s) - z_2 x_1(s) + z_1 x_2(s). \end{aligned}$$

Taking advantage of the $Q_{\mathcal{H}}$ -periodicity of f and g , we deduce

$$\begin{aligned} u(z \oplus x, t) &= \inf_{\beta} \int_t^T \frac{1}{2} |\beta(s)|^2 + f(y(s), s) ds + g(y(T)) \\ &= \inf_{\beta} \int_t^T \frac{1}{2} |\beta(s)|^2 + f(z \oplus x(s), s) ds + g(z \oplus x(T)) \\ &= \inf_{\beta} \int_t^T \frac{1}{2} |\beta(s)|^2 + f(x(s), s) ds + g(x(T)) = u(x, t) \end{aligned}$$

namely, the value function is $Q_{\mathcal{H}}$ -periodic. \square

The following proposition ensures that we can restrict our study on uniformly bounded controls.

Proposition 4.1 *Let u be the value function introduced in (4.5). Then, there exists a constant C_2 (depending only on T and on the constant C of Hypothesis 4.1) such that there holds*

$$(4.6) \quad u(x, t) = \inf\{J_t(x(\cdot), \alpha) : (x(\cdot), \alpha) \in \mathcal{A}(x, t), \|\alpha\|_\infty \leq C_2\}$$

for any $x = (x_1, x_2, x_3) \in Q_{\mathcal{H}}$ and $t \in [0, T]$. Hence, by the $Q_{\mathcal{H}}$ -periodicity of u , the optimal control α for any point $(x, t) \in \mathbb{H}^1 \times [0, T]$ fulfills: $\|\alpha\|_\infty \leq C_2$.

Proof. The idea of the proof is borrowed from [6, Theorem 2.1]. For $x = (x_1, x_2, x_3) \in Q_{\mathcal{H}}$ and $t \in [0, T]$, let α be an optimal control for $u(x, t)$. For $\mu > 0$, let $I_\mu := \{s \in (t, T) : |\alpha(s)| > \mu\}$. Define

$$(4.7) \quad \alpha^\mu(s) = \begin{cases} \alpha(s) & \text{if } |\alpha(s)| \leq \mu, \\ 0 & \text{if } |\alpha(s)| > \mu. \end{cases}$$

Let $x^\mu(s)$ be the trajectory starting from $x \in Q_{\mathcal{H}}$ associated to the control $\alpha^\mu(s)$. We claim that

$$(4.8) \quad |x^\mu(s) - x(s)| \leq K \int_{I_\mu} |\alpha(\tau)| d\tau \quad \forall s \in [t, T]$$

where K is a constant depending only on C_1 (see (4.3)) and T . Actually, for the first two components of $x^\mu(s) - x(s)$ we have

$$(4.9) \quad |x_i^\mu(s) - x_i(s)| \leq \int_t^s |\alpha_i^\mu(\tau) - \alpha_i(\tau)| d\tau = \int_{I_\mu} |\alpha_i(\tau)| d\tau \quad \forall s \in [t, T], \quad i = 1, 2.$$

For the third component, there holds

$$\begin{aligned} x_3^\mu(s) - x_3(s) &= \int_t^s [-x_2^\mu(\tau)\alpha_1^\mu(\tau) + x_1^\mu(\tau)\alpha_2^\mu(\tau) + x_2(\tau)\alpha_1(\tau) - x_1(\tau)\alpha_2(\tau)] d\tau \\ &= \int_t^s [(x_2(\tau) - x_2^\mu(\tau))\alpha_1^\mu(\tau) + x_2(\tau)(\alpha_1(\tau) - \alpha_1^\mu(\tau)) \\ &\quad + (x_1^\mu(\tau) - x_1(\tau))\alpha_2^\mu(\tau) + x_1(\tau)(\alpha_2^\mu(\tau) - \alpha_2(\tau))] d\tau. \end{aligned}$$

Hence from (4.4) and (4.9), we infer

$$\begin{aligned} |x_3^\mu(s) - x_3(s)| &\leq \int_{I_\mu} |\alpha_2(\tau)| d\tau \int_t^s |\alpha_1^\mu(\tau)| d\tau + [|x_2| + C_1(T-t)^{1/2}] \int_{I_\mu} |\alpha_1(\tau)| d\tau \\ &\quad + \int_{I_\mu} |\alpha_1(\tau)| d\tau \int_t^s |\alpha_2^\mu(\tau)| d\tau + [|x_1| + C_1(T-t)^{1/2}] \int_{I_\mu} |\alpha_2(\tau)| d\tau. \end{aligned}$$

Moreover, by Hölder inequality and (4.3), we have

$$\int_t^s |\alpha_i^\mu(\tau)| d\tau \leq \sqrt{s-t} \|\alpha\|_2 \leq C_1 \sqrt{T-t}, \quad i = 1, 2.$$

Replacing the last inequality in the previous one, since $|x_i| \leq 1$, we accomplish the proof of the claim (4.8).

Now, the definition of the cost $J_t(x(s), \alpha(s))$ in (4.1) and the Lipschitz continuity of f and g yield

$$\begin{aligned}
& J_t(x^\mu(s), \alpha^\mu(s)) - J_t(x(s), \alpha(s)) = \\
& = \int_t^T \frac{1}{2} |\alpha^\mu(s)|^2 + f(x^\mu(s), s) ds + g(x^\mu(T)) - \int_t^T \frac{1}{2} |\alpha(s)|^2 + f(x(s), s) ds - g(x(T)) \\
& \leq - \int_{I_\mu} \frac{1}{2} |\alpha(s)|^2 ds + L_f \int_t^T |x^\mu(s) - x(s)| ds + L_g |x^\mu(T) - x(T)| \\
& \leq \int_{I_\mu} \left(-\frac{1}{2} |\alpha(s)|^2 + K(L_f(T-t) + L_g) |\alpha(s)| \right) ds,
\end{aligned}$$

where the last inequality comes from (4.8). Hence, if I_μ has positive measure for $\mu > 2K(L_f T + L_g)$, the last integrand is negative for every $s \in I_\mu$ which contradicts the optimality of α . This implies that these I_μ have null measure and, in particular, $\|\alpha\|_\infty \leq 2K(L_f T + L_g)$. \square

4.1 Necessary conditions and regularity for the optimal trajectories

The application of the Maximum Principle (see [25, Theorem 22.17]) yields the following necessary conditions.

Proposition 4.2 *Let (x^*, α^*) be optimal for the optimal control problem in (4.1). Then, there exists an arc $p \in AC([t, T]; \mathbb{R}^3)$, hereafter called the costate, such that*

1. The pair (x^*, p) satisfies the system of differential equations for a.e. $s \in [t, T]$

$$(4.10) \quad \begin{cases} x'_1 = p_1 - x_2 p_3, \\ x'_2 = p_2 + x_1 p_3, \\ x'_3 = (x_1^2 + x_2^2) p_3 + x_1 p_2 - x_2 p_1, \\ p'_1 = -(p_2 + x_1 p_3) p_3 + f_{x_1}(x, s) \\ p'_2 = (p_1 - x_2 p_3) p_3 + f_{x_2}(x, s) \\ p'_3 = f_{x_3}(x, s) \end{cases}$$

with the mixed boundary conditions

$$(4.11) \quad x(t) = x, \quad p(T) = -Dg(x(T)).$$

2. The optimal control α^* is given by

$$(4.12) \quad \begin{cases} \alpha_1^*(s) = p_1 - x_2^* p_3, \\ \alpha_2^*(s) = p_2 + x_1^* p_3, \end{cases} \quad a.e \text{ on } [t, T].$$

Remark 4.3 *Let us observe that equations (4.10) and (4.12) can be rewritten in terms of the vector fields as follows*

$$\begin{aligned}
x'_1 &= X_1 p, & x'_2 &= X_2 p, & x'_3 &= x_2 X_1 p - x_1 X_2 p, \\
p'_1 &= -p_3 X_2 p + f_{x_1}(x, s), & p'_2 &= p_3 X_1 p + f_{x_2}(x, s), & p'_3 &= f_{x_3}(x, s)
\end{aligned}$$

and respectively

$$\alpha_1(s) = X_1 p(s), \quad \alpha_2(s) = X_2 p(s).$$

Corollary 4.1 *Let (x^*, α^*) be optimal for the optimal control problem in (4.1). Then:*

1. *The unique solution of the Cauchy problem*

$$\begin{cases} \pi_1' = -(\pi_2 + x_1^* \pi_3) p_3 + f_{x_1}(x^*, s), \\ \pi_2' = (\pi_1 - x_2^* \pi_3) \pi_3 + f_{x_2}(x^*, s), \\ \pi_3' = f_{x_3}(x^*, s), \\ \pi(T) = -Dg(x^*(T)). \end{cases}$$

is the costate p associated to (x^, α^*) as in Proposition 4.2.*

2. *The optimal α^* is a feedback control and it is uniquely expressed by*

$$\begin{cases} \alpha_1^*(s) = p_1 - x_2^* p_3 \\ \alpha_2^*(s) = p_2 + x_1^* p_3 \end{cases}$$

where p is the costate associated to (x^, α^*) .*

3. *The functions x^* and α^* are of class C^1 . In particular equations (4.10) and (4.12) hold for every $s \in [t, T]$.*

4. *Assume that, for some $k \in \mathbb{N}$, $D_x f \in C^k$. Then, the costate p and the control α^* are of class C^{k+1} and x^* is of class C^{k+2} .*

Proof. The proof follows the same lines as in [37, Corollary 2.1] and we refer to that paper for the detailed arguments. \square

Remark 4.4 *The uniqueness of the optimal trajectories after the initial time for a.e. initial data is an open problem. In [37] this result was obtained thanks to the property $\text{meas}\{x : \det B(x)B^T(x) = 0\} = 0$; now, in the Heisenberg setting, this property fails to be true since $\det B(x)B^T(x) = 0$ for any $x \in \mathbb{H}^1$.*

4.2 The Hamilton-Jacobi equation and the value function of the optimal control problem

The aim of this section is to study the Hamilton-Jacobi equation (1.1)-(i) with m fixed, namely

$$(4.13) \quad \begin{cases} -\partial_t u + \frac{1}{2}|D_{\mathcal{H}}u|^2 = f(x, t) & \text{in } \mathbb{H}^1 \times (0, T), \\ u(x, T) = g(x) & \text{on } \mathbb{H}^1. \end{cases}$$

Under Hypothesis 4.1, we shall prove Lipschitz continuity and semiconcavity of u . As a first step, in the next lemma we show that the solution u of (4.13) can be represented as the value function of the control problem defined in (4.5). Hence from Lemma 4.1 we can restrict to study equation (4.13) in $\mathbb{T}_{\mathcal{H}}$.

Lemma 4.2 *Under Hypothesis 4.1, the value function u , defined in (4.5), is the unique continuous bounded viscosity solution to problem (4.13). Moreover u is $Q_{\mathcal{H}}$ -periodic.*

Proof. The proof comes from classical results in viscosity theory, see for example [7, Proposition 3.5], [6, Theorem 3.1] and [27, Corollary 2.1]. \square

In the following lemma we prove the Lipschitz continuity in both variables x and t of the value function.

Lemma 4.3 *Under Hypothesis (4.1), $u(x, t)$ is Lipschitz continuous with respect to the spatial variable x , and the time variable t .*

Proof. In this proof, C_T will denote a constant which may change from line to line but it always depends only on the constants in the assumptions (especially the Lipschitz constants of f and g) and on T .

We study first the Lipschitz continuity w.r.t. x . Let t be fixed. We follow the proof of [22, Lemma 4.7]. From Remark 4.2 we know that there exists $\alpha(\cdot)$ optimal control for $u(x, t)$ and $x(\cdot)$ optimal trajectory i.e.:

$$(4.14) \quad u(x_1, x_2, x_3, t) = \int_t^T \frac{1}{2} |\alpha(s)|^2 + f(x(s), s) ds + g(x(T)).$$

We consider the path $x^*(s)$ starting from $y = (y_1, y_2, y_3)$, with control α . Hence

$$\begin{aligned} x_1^*(s) &= y_1 + \int_t^s \alpha_1(\tau) d\tau = y_1 - x_1 + x_1(s) \\ x_2^*(s) &= y_2 + \int_t^s \alpha_2(\tau) d\tau = y_2 - x_2 + x_2(s) \\ x_3^*(s) &= y_3 - \int_t^s \alpha_1(\tau) x_2^*(\tau) d\tau + \int_t^s \alpha_2(\tau) x_1^*(\tau) d\tau \\ &= y_3 - (y_2 - x_2) \int_t^s \alpha_1(\tau) d\tau + (y_1 - x_1) \int_t^s \alpha_2(\tau) d\tau \\ &\quad + \int_t^s (-\alpha_1(\tau) x_2(\tau) + \alpha_2(\tau) x_1(\tau)) d\tau \\ &= x_3(s) + (y_3 - x_3) - (y_2 - x_2) \int_t^s \alpha_1(\tau) d\tau + (y_1 - x_1) \int_t^s \alpha_2(\tau) d\tau. \end{aligned}$$

Using the Lipschitz continuity of f we get

$$\begin{aligned} f(x^*(s), s) &\leq f(x(s), s) + L(|y_1 - x_1| + |y_2 - x_2| + |y_3 - x_3| + \\ &\quad + |y_2 - x_2| \sqrt{s-t} \|\alpha_1\|_2 + |y_1 - x_1| \sqrt{s-t} \|\alpha_2\|_2) \end{aligned}$$

and from the L^2 uniform estimate for α_1 and α_2 in (4.3) we get

$$f(x^*(s), s) - f(x(s), s) \leq C_T (|y_1 - x_1| + |y_2 - x_2| + |y_3 - x_3|).$$

By the same calculations for g and substituting equality (4.14) in

$$u(y_1, y_2, y_3, t) \leq \int_t^T \frac{1}{2} |\alpha(s)|^2 + f(x^*(s), s) ds + g(x^*(T)),$$

we get

$$u(y_1, y_2, y_3, t) \leq u(x_1, x_2, x_3, t) + C_T (|y_1 - x_1| + |y_2 - x_2| + |y_3 - x_3|).$$

Reversing the role of x and y we get the result.

Let us now prove the Lipschitz continuity of u w.r.t. t . Thanks to the $Q_{\mathcal{H}}$ -periodicity in x of u , it is enough to prove the Lipschitz continuity in t only for $x \in Q_{\mathcal{H}}$. To this end,

taking advantage of the L^∞ -bound for optimal controls established in Proposition 4.1, we can follow the same arguments as those in the proof of [22, Lemma 4.7], noting that

$$|x(s) - x| \leq C(s-t)(\|\alpha_1\|_\infty|x_2| + \|\alpha_2\|_\infty|x_1|) \leq K(s-t).$$

□

In the following lemma we establish the semiconcavity of u w.r.t. x ; we recall here below the definition of semiconcavity with linear modulus and we refer the reader to the monograph [21] for further properties.

Definition 4.3 *Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$. We say that u is semiconcave (with linear modulus) if there exists a constant $C \geq 0$ such that for all $\lambda \in [0, 1]$,*

$$\lambda u(y) + (1-\lambda)u(x) - 2u(\lambda y + (1-\lambda)x) \leq C\lambda(1-\lambda)|y-x|^2 \quad \forall x, y \in \mathbb{R}^d.$$

Lemma 4.4 *Under Hypothesis 4.1, the value function u , defined in (4.5), is semiconcave with respect to the variable x in $Q_{\mathcal{H}}$.*

Proof. For any $x, y \in Q_{\mathcal{H}}$ and $\lambda \in [0, 1]$, consider $x_\lambda := \lambda x + (1-\lambda)y$. Let $\alpha(s)$ and $x_\lambda(s)$ be an optimal control and respectively the corresponding optimal trajectory for $u(x_\lambda, t)$; for $s \in [t, T]$ there holds

$$\begin{aligned} x_{\lambda,i}(s) &= x_{\lambda,i} + \int_t^s \alpha_i(\tau) d\tau, \quad i = 1, 2 \\ x_{\lambda,3}(s) &= x_{\lambda,3} - \int_t^s \alpha_1(\tau)x_{\lambda,2}(\tau) d\tau + \int_t^s \alpha_2(\tau)x_{\lambda,1}(\tau) d\tau. \end{aligned}$$

Let $x(s)$ and $y(s)$ satisfy (1.5) with initial condition respectively x and y still with the same control α , optimal for $u(x_\lambda, t)$. We have to estimate $\lambda u(x, t) + (1-\lambda)u(y, t)$ in terms of $u(x_\lambda, t)$. To this end, arguing as in the proof of [22, Lemma 4.7], we have to estimate the terms $\lambda f(x(s), s) + (1-\lambda)f(y(s), s)$ and $\lambda g(x(T)) + (1-\lambda)g(y(T))$.

We explicitly provide the calculations for the third component $x_3(s)$ since the calculations for $x_1(s)$ and $x_2(s)$ are the same as in [22]. We have

$$\begin{aligned} x_3(s) &= x_3 - \int_t^s \alpha_1(\tau)x_2(\tau) d\tau + \int_t^s \alpha_2(\tau)x_1(\tau) d\tau \\ &= x_3 - x_{\lambda,3} + x_{\lambda,3}(s) - \int_t^s \alpha_1(\tau)(x_2(\tau) - x_{\lambda,2}(\tau)) d\tau + \int_t^s \alpha_2(\tau)(x_1(\tau) - x_{\lambda,1}(\tau)) d\tau. \end{aligned}$$

Since $x_3 - x_{\lambda,3} = (1-\lambda)(x_3 - y_3)$ and

$$(4.15) \quad x_i(\tau) - x_{\lambda,i}(\tau) = (1-\lambda)(x_i - y_i) \quad \text{for } i = 1, 2,$$

we get

$$(4.16) \quad x_3(s) - x_{\lambda,3}(s) = (1-\lambda) \left[x_3 - y_3 - (x_2 - y_2) \int_t^s \alpha_1(\tau) d\tau + (x_1 - y_1) \int_t^s \alpha_2(\tau) d\tau \right].$$

Analogously for $y(s)$: since $y_3 - x_{\lambda,3} = \lambda(y_3 - x_3)$ and

$$(4.17) \quad y_i(\tau) - x_{\lambda,i}(\tau) = \lambda(y_i - x_i) \quad \text{for } i = 1, 2,$$

we get

$$(4.18) \quad y_3(s) - x_{\lambda,3}(s) = \lambda \left[(y_3 - x_3) + (x_2 - y_2) \int_t^s \alpha_1(\tau) d\tau - (x_1 - y_1) \int_t^s \alpha_2(\tau) d\tau \right].$$

For the sake of brevity we provide the explicit calculations only for f omitting the analogous ones for g ; and we write $f(x_1, x_2, x_3) := f(x_1, x_2, x_3, s)$. We have

$$\begin{aligned} & \lambda f(x(s)) + (1 - \lambda) f(y(s)) = \\ & \lambda f(x_1(s), x_2(s), x_{\lambda,3}(s)) + (1 - \lambda)(x_3 - y_3 - (x_2 - y_2) \int_t^s \alpha_1(\tau) d\tau + (x_1 - y_1) \int_t^s \alpha_2(\tau) d\tau) + \\ & + (1 - \lambda) f(y_1(s), y_2(s), x_{\lambda,3}(s)) + \lambda(y_3 - x_3 + (x_2 - y_2) \int_t^s \alpha_1(\tau) d\tau - (x_1 - y_1) \int_t^s \alpha_2(\tau) d\tau). \end{aligned}$$

Since for $i = 1, 2$ there holds

$$\lambda \partial_{x_i} f(x_\lambda(s))(x_i(s) - x_{\lambda,i}(s)) + (1 - \lambda) \partial_{x_i} f(x_\lambda(s))(y_i(s) - x_{\lambda,i}(s)) = 0,$$

the Taylor expansion of f centered in $x_\lambda(s)$ gives:

$$\begin{aligned} & \lambda f(x(s)) + (1 - \lambda) f(y(s)) = \\ & \lambda(f(x_\lambda(s)) + Df(x_\lambda(s))(x(s) - x_\lambda(s)) + R_1) + (1 - \lambda)(f(x_\lambda(s)) + Df(x_\lambda(s))(y(s) - x_\lambda(s)) + R_2) \\ & = \lambda \left(f(x_\lambda(s)) + \partial_{x_3} f(x_\lambda(s))(1 - \lambda)(x_3 - y_3 - (x_2 - y_2) \int_t^s \alpha_1(\tau) d\tau + (x_1 - y_1) \int_t^s \alpha_2(\tau) d\tau) + R_1 \right) \\ & + (1 - \lambda) \left(f(x_\lambda(s)) + \partial_{x_3} f(x_\lambda(s)) \lambda(y_3 - x_3 + (x_2 - y_2) \int_t^s \alpha_1(\tau) d\tau - (x_1 - y_1) \int_t^s \alpha_2(\tau) d\tau) + R_2 \right) = \\ & = f(x_\lambda(s)) + \lambda R_1 + (1 - \lambda) R_2, \end{aligned}$$

where R_1 and R_2 are the error terms of the expansion, namely

$$\begin{aligned} \lambda R_1 + (1 - \lambda) R_2 &= \frac{1}{2} \lambda ((x(s) - x_\lambda(s)) D^2 f(\xi_1) (x(s) - x_\lambda(s)))^T \\ &+ \frac{1}{2} (1 - \lambda) ((y(s) - x_\lambda(s)) D^2 f(\xi_2) (y(s) - x_\lambda(s)))^T, \end{aligned}$$

for suitable $\xi_1, \xi_2 \in Q_{\mathcal{H}}$.

Using relations (4.15)-(4.18) and the L^2 uniform estimate of α in (4.3), we obtain

$$\begin{cases} |x_i(s) - x_{\lambda,i}(s)| |x_j(s) - x_{\lambda,j}(s)| \leq C(1 - \lambda)^2 |x - y|^2 & i, j = 1, 2, 3 \\ |y_i(s) - x_{\lambda,i}(s)| |y_j(s) - x_{\lambda,j}(s)| \leq C\lambda^2 |x - y|^2 & i, j = 1, 2, 3 \end{cases}$$

for some positive constant C . Then, possibly increasing C , we get

$$\lambda R_1 + (1 - \lambda) R_2 \leq C\lambda(1 - \lambda) |x - y|^2,$$

and, in particular,

$$\lambda f(x(s)) + (1 - \lambda) f(y(s)) \leq f(x_\lambda(s)) + C\lambda(1 - \lambda) |x - y|^2$$

which amounts to the semiconcavity of u . □

We state the optimal synthesis principle:

Lemma 4.5 *Let $x(\cdot)$ be an absolutely continuous function such that $x(t) = x \in \mathbb{H}^1$ and for almost every $s \in (t, T)$,*

$$(4.19) \quad u(\cdot, s) \text{ is } \mathcal{H}\text{-differentiable at } x(s),$$

(see Definition A.1 in Appendix A for the precise definition of \mathcal{H} -differentiability and some of its properties) and $x(\cdot)$ satisfies the ODE

$$(4.20) \quad x'(s) = -D_{\mathcal{H}}u(x(s), s)B^T(x(s)), \quad \text{a.e. } s \in (t, T)$$

where u is the value function defined in (4.5). Then the control law $\alpha(s)$, given by

$$(4.21) \quad \alpha(s) = -D_{\mathcal{H}}u(x(s), s),$$

is optimal for $u(x, t)$.

Proof. We adapt the arguments of [37, Lemma 3.6] and [22, Lemma 4.11]. Fix $(x, t) \in \mathbb{H}^1 \times (0, T)$ and consider an absolutely continuous solution $x(\cdot)$ to (4.20); note that this implies that $D_{\mathcal{H}}u$ exists at $(x(s), s)$ for a.e. $s \in (t, T)$. We claim that $x(\cdot)$ is Lipschitz continuous. Indeed system (4.20) reads

$$(4.22) \quad \begin{cases} x'_1(s) = -X_1u(x(s), s) \\ x'_2(s) = -X_2u(x(s), s) \\ x'_3(s) = x_2(s)X_1u(x(s), s) - x_1(s)X_2u(x(s), s) \end{cases}$$

for a.e. $s \in (t, T)$. By Lemma 4.3 and Lemma 4.1, there exists $C > 0$ such that $\|D_{\mathcal{H}}u\|_{\infty} \leq C$; hence, $x_1(\cdot)$ and $x_2(\cdot)$ are both Lipschitz continuous and, in particular they are also bounded. By the third equation in (4.22), we also obtain that $x_3(\cdot)$ is Lipschitz continuous. Hence our claim is proved.

Consequently, from the Lipschitz continuity of u and of $x(\cdot)$ we get that also $u(x(\cdot), \cdot)$ is Lipschitz. For a.e. $s \in (t, T)$ there hold: *i*) $D_{\mathcal{H}}u(x(s), s)$ exists, *ii*) equation (4.20) holds, *iii*) the function $u(x(\cdot), \cdot)$ admits a derivative at s . Fix such a s .

The Lebourg Theorem for Lipschitz function (see [26, Thm 2.3.7] and [26, Thm 2.5.1]) ensures that, for any $h \in \mathbb{R}$ small, there exists (y_h, s_h) in the segment $((x(s), s), (x(s+h), s+h))$ and $(\xi_x^h, \xi_t^h) \in \text{co}D_{x,t}^*u(y_h, s_h)$ such that

$$(4.23) \quad u(x(s+h), s+h) - u(x(s), s) = \xi_x^h \cdot (x(s+h) - x(s)) + \xi_t^h h$$

(here, “co” stands for the convex hull and $D_{x,t}^*u$ is the Euclidean reachable gradient both in x and in t). The Caratheodory theorem (see [21, Thm A.1.6]) guarantees that there exist $(\lambda^{h,i}, \xi_x^{h,i}, \xi_t^{h,i})_{i=1,\dots,5}$ such that $\lambda^{h,i} \geq 0$, $\sum_{i=1}^5 \lambda^{h,i} = 1$, $(\xi_x^{h,i}, \xi_t^{h,i}) \in D_{x,t}^*u(y_h, s_h)$ and $(\xi_x^h, \xi_t^h) = \sum_{i=1}^5 \lambda^{h,i} (\xi_x^{h,i}, \xi_t^{h,i})$. We claim that there holds

$$(4.24) \quad \lim_{h \rightarrow 0} \xi_x^{h,i} B(y_h) = D_{\mathcal{H}}u(x(s), s) \quad \forall i = 1, \dots, 5.$$

Indeed, for any $i = 1, \dots, 5$ fixed, let ξ be any cluster point of $\{\xi_x^{h,i}\}_h$ (which must be finite because u is Lipschitz continuous). Then, by a diagonal extraction, there exist (x_n, t_n) such that u is differentiable at (x_n, t_n) , $(x_n, t_n) \rightarrow (x(s), s)$ and $D_x u(x_n, t_n) \rightarrow \xi$ as $n \rightarrow \infty$. The results in [22, Lemma 4.6], applied to $w_n(\cdot) := u(\cdot, t_n)$ and $w(\cdot) := u(\cdot, s)$, infer: $\xi \in D^+w(s)$. Lemma A.1-(iii) in the appendix ensures $\xi B(x(s)) \in D_{\mathcal{H}}^+w(x(s))$; in

conclusion, by Proposition A.1, since w is \mathcal{H} -differentiable at $x(s)$, we conclude $\xi B(x(s)) = D_{\mathcal{H}}w(x(s)) = D_{\mathcal{H}}u(x(s), s)$ namely our claim (4.24) is completely proved. In particular, we have

$$(4.25) \quad \lim_{h \rightarrow 0} \xi_x^h B(y_h) = D_{\mathcal{H}}u(x(s), s).$$

On the other hand, since u is a viscosity solution to equation (4.13), by [7, Proposition II.1.9], we obtain

$$-\xi_t^{h,i} + \frac{|\xi_x^{h,i} B(y_h)|^2}{2} = f(y_h, s_h);$$

in particular, as $h \rightarrow 0$, we deduce

$$(4.26) \quad \xi_t^h = \frac{1}{2} \sum_{i=1}^5 \lambda^{h,i} |\xi_x^{h,i} B(y_h)|^2 - f(y_h, s_h) \rightarrow \frac{1}{2} |D_{\mathcal{H}}u(x(s), s)|^2 - f(x(s), s).$$

Dividing (4.23) by h and letting $h \rightarrow 0$, by equations (4.20), (4.25) and (4.26), we infer

$$\begin{aligned} \frac{d}{ds} u(x(s), s) &= \lim_{h \rightarrow 0} \xi_x^h \cdot [D_{\mathcal{H}}u(x(s), s) B^T(x(s)) + \frac{x(s+h) - x(s)}{h}] \\ &\quad + \lim_{h \rightarrow 0} \xi_x^h \cdot [D_{\mathcal{H}}u(x(s), s) (B^T(y_h) - B^T(x(s)))] \\ &\quad - \lim_{h \rightarrow 0} \xi_x^h \cdot [D_{\mathcal{H}}u(x(s), s) B^T(y_h)] + \lim_{h \rightarrow 0} \xi_t^h \\ &= -\frac{1}{2} |D_{\mathcal{H}}u(x(s), s)|^2 - f(x(s), s) \\ &= -\frac{1}{2} |\alpha(s)|^2 - f(x(s), s) \quad \text{a.e. } s \in (t, T) \end{aligned}$$

where the last equality is due to our definition of α in (4.21). Integrating this equality on $[t, T]$ and taking into account the final datum of (4.13), we obtain

$$u(x, t) = \int_t^T \frac{|\alpha(s)|^2}{2} + f(x(s), s) ds + g(x(T)).$$

Observe that $x(\cdot)$ satisfies the dynamics (1.5) with the control $\alpha(\cdot)$ defined in (4.21); therefore, the last equality implies that $x(\cdot)$ is an optimal trajectory with optimal control $\alpha(\cdot)$ given by (4.21). \square

5 The continuity equation

This section is devoted to equation (1.1)-(ii), namely

$$(5.1) \quad \begin{cases} \partial_t m - \operatorname{div}_{\mathcal{H}}(m D_{\mathcal{H}}u) = 0 & \text{in } \mathbb{H}^1 \times (0, T) \\ m(x, 0) = m_0(x) & \text{on } \mathbb{H}^1, \end{cases}$$

where u is a solution to problem

$$(5.2) \quad \begin{cases} -\partial_t u + \frac{|D_{\mathcal{H}}u|^2}{2} = F[\bar{m}_t](x) & \text{in } \mathbb{H}^1 \times (0, T) \\ u(x, T) = G[\bar{m}_T](x) & \text{on } \mathbb{H}^1, \end{cases}$$

and the function \overline{m} is fixed in $C^{1/4}([0, T], \mathcal{P}_{per}(\mathbb{H}^1))$. Let us observe that assumptions (H1)-(H3) and Lemma 4.2 ensure that there is a unique bounded solution u to (5.2) which is moreover $Q_{\mathcal{H}}$ -periodic.

Now we deal with the existence, the periodicity and uniform estimates of the solution m of (5.1).

Theorem 5.1 *Under assumptions (H1)-(H3), for any $\overline{m} \in C^{1/4}([0, T], \mathcal{P}_{per}(\mathbb{H}^1))$, problem (5.1) has a bounded solution m in the sense of Definition 3.1. Moreover m belongs to $C^{1/4}([0, T], \mathcal{P}_{per}(\mathbb{H}^1)) \cap \mathbb{L}^\infty(\mathbb{H}^1 \times (0, T))$ and there exists a constant C_1 (independent of \overline{m}) such that*

$$(5.3) \quad \mathbf{d}_1(m_s, m_t) \leq C_1(t - s)^{1/4} \quad \forall 0 \leq s \leq t \leq T.$$

The proof of this Theorem is postponed at the end of this section. It relies on a suitable adaptation of the arguments of the proof of [37, Proposition 3.1] (see also [23, Theorem 5.1] and [22, Theorem 4.20]).

We shall use a vanishing viscosity approach applied to the *whole* MFG system in terms of the *horizontal Laplacian* $\Delta_{\mathcal{H}}$. We need such “degenerate” approximation to ensure that the corresponding solution is still $Q_{\mathcal{H}}$ -periodic in x .

For any $\sigma > 0$, we consider the system

$$(5.4) \quad \begin{cases} (i) & -\partial_t u - \sigma \Delta_{\mathcal{H}} u + \frac{1}{2} |D_{\mathcal{H}} u|^2 = F[\overline{m}_t](x) & \text{in } \mathbb{H}^1 \times (0, T), \\ (ii) & \partial_t m - \sigma \Delta_{\mathcal{H}} m - \operatorname{div}_{\mathcal{H}}(m D_{\mathcal{H}} u) = 0 & \text{in } \mathbb{H}^1 \times (0, T), \\ (iii) & m(x, 0) = m_0(x), u(x, T) = G[\overline{m}_T](x) & \text{on } \mathbb{H}^1. \end{cases}$$

In order to prove Theorem 5.1, it is expedient to establish several properties of the solution (u^σ, m^σ) to system (5.4): the following lemmata collect existence, uniqueness and other properties of u^σ and respectively m^σ .

Let us emphasize some features of equation (5.4)-(ii): the degeneracy of the operator, the unboundedness and the lack of global Lipschitz continuity of the coefficients. These features prevent to apply all the uniqueness result we known in literature. In order to overcome this issue, we shall establish two uniqueness results which are collected in appendix B. Moreover, m_0 is not a probability on \mathbb{H}^1 (but only a nonnegative measure).

For any domain $U \subset \mathbb{H}^1 \times [0, T]$, any $k \in \mathbb{N}$ and any $\delta \in (0, 1]$, we denote $C_{\mathcal{H}}^{k+\delta}(U)$ (resp. $C_{\mathcal{H},loc}^{k+\delta}(U)$) the (resp. local) parabolic Hölder space adapted to the vector fields X_1 and X_2 (for instance, see [17, Section 4] or [18, Definition 10.4]).

Lemma 5.1 *Assume (H1)–(H3) and fix $\overline{m} \in C^{1/4}([0, T], \mathcal{P}_{per}(\mathbb{H}^1))$. The Cauchy problem*

$$(5.5) \quad \begin{cases} -\partial_t u - \sigma \Delta_{\mathcal{H}} u + \frac{1}{2} |D_{\mathcal{H}} u|^2 = F[\overline{m}](x) & \text{in } \mathbb{H}^1 \times (0, T), \\ u(x, T) = G[\overline{m}(T)](x) & \text{on } \mathbb{H}^1 \end{cases}$$

admits exactly one bounded viscosity solution u^σ (with a bound independent of σ). Moreover, the function u^σ fulfills the following properties

(i) u^σ is $Q_{\mathcal{H}}$ -periodic in x , Lipschitz continuous and locally semiconcave in x ,

(ii) there exists a positive constant C , independent of σ , such that:

$$|D_{\mathcal{H}} u^\sigma(x, t)| \leq C \quad \text{and} \quad \Delta_{\mathcal{H}} u^\sigma(x, t) \leq C \quad \forall (x, t) \in \mathbb{H}^1 \times [0, T].$$

(iii) for every $\tau \in [0, T]$ and $\delta \in (0, 1/4]$, there exists a positive constant C (depending on τ , δ and σ) such that

$$\|u^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(\mathbb{H}^1 \times [0, \tau])} + \sum_{i=1}^2 \|X_i u^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(\mathbb{H}^1 \times [0, \tau])} + \sum_{i,j=1}^2 \|X_i X_j u^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(\mathbb{H}^1 \times [0, \tau])} \leq C,$$

(iv) the functions u^σ are $1/4$ -Hölder continuous in time uniformly in σ .

Proof. The differential equation in (5.5) can be written as

$$-\partial_t u - \sigma \operatorname{tr}(D^2 u B(x) B(x)^T) + \frac{1}{2} |Du B(x)|^2 = F[\overline{m}](x);$$

in particular, it fulfills the assumption for the comparison principle established in [27, Theorem 2.1]. Using $w^\pm(x, t) := \pm C(-t + 1)$ as super- and subsolution, we deduce the existence and uniqueness of a viscosity solution u^σ uniformly bounded on σ , i.e. there exists C independent on σ such that

$$(5.6) \quad \|u^\sigma\|_{L^\infty(\mathbb{H}^1 \times [0, T])} \leq C.$$

Let us now prove the several properties of u^σ .

(i). Since the vector fields X_1 and X_2 are left-invariant and $F[\overline{m}](\cdot)$ and $G[\overline{m}](\cdot)$ are $Q_{\mathcal{H}}$ -periodic in x , for any $z \in \mathbb{Z}^3$, also the function $w^\sigma(x, t) := u^\sigma(z \oplus x, t)$ is a solution to (5.5). Again the comparison principle in [27, Theorem 2.1] yields $u^\sigma = w^\sigma$, namely u^σ is $Q_{\mathcal{H}}$ -periodic in x .

Invoking [27, Theorem 2.1], we can represent the solution u^σ as the value function of a stochastic optimal control problem:

$$(5.7) \quad u^\sigma(x, t) = \min \mathbb{E} \left(\int_t^T \left[\frac{1}{2} |\alpha(\tau)|^2 + F[\overline{m}_\tau](Y(\tau)) \right] d\tau + g[\overline{m}_T](Y(T)) \right)$$

where, in $[t, T]$, $Y(\cdot)$ obeys to a stochastic differential equation

$$(5.8) \quad dY = \alpha(t) B(Y_t)^T dt + \sqrt{2\sigma} B(Y_t) dW_t,$$

where $Y(t) = x$ and W_t is a standard 3-dimensional Brownian motion. Arguing as in [22, Theorem 4.20] and following the calculations in the proofs of Lemma 4.3 and Lemma 4.4, we get the Lipschitz continuity and the local semiconcavity (see [22, Theorem 4.20 (proof)] for a similar argument).

(ii). Taking into account of the representation of u^σ (5.7) as the value function of a stochastic optimal control problem, following the procedure used in Lemma 4.3 for the deterministic case, we can prove the uniform Lipschitz continuity of u^σ . Hence $D_{\mathcal{H}} u^\sigma$ is uniformly bounded in $Q_{\mathcal{H}}$ and by the $Q_{\mathcal{H}}$ -periodicity of u^σ we get the first bound of (ii). Still using the representation of u^σ (5.7) we can follow the procedure used in Lemma 4.4 for the deterministic case, (see also [2, Lemma 4.1-(c) (proof)]) to get the uniform local semiconcavity of u^σ , i.e. $D^2 u^\sigma \leq CI$. This implies that $\Delta_{\mathcal{H}} u^\sigma \leq C(1 + x_1^2 + x_2^2)$ and using the periodicity of u^σ we get the second bound of (ii).

(iii). We introduce the Cole-Hopf transformation of u^σ , $w^\sigma(x, t) := \exp\{-u^\sigma(x, t)/(2\sigma)\}$ and we observe that it is bounded and $Q_{\mathcal{H}}$ -periodic in x and it fulfills:

$$X_i u^\sigma = -2\sigma \frac{X_i w^\sigma}{w^\sigma}, \quad X_i^2 u^\sigma = 2\sigma \frac{(X_i w^\sigma)^2}{(w^\sigma)^2} - 2\sigma \frac{X_i^2 w^\sigma}{w^\sigma} \quad (i = 1, 2).$$

Replacing these relations in (5.5), we infer that w^σ is a viscosity solution to the following linear subelliptic parabolic equation

$$(5.9) \quad -\partial_t w^\sigma - \sigma \Delta_{\mathcal{H}} w^\sigma + w^\sigma F[\overline{m}]/(2\sigma) = 0;$$

by the equivalence between distributional solutions and viscosity solutions established by Ishii [32] we deduce that w^σ is also a distributional solution of equation (5.9).

We observe that, by its periodicity, the function $F[\overline{m}]$ belongs to $C_{\mathcal{H}}^{1/4}(\mathbb{H}^1 \times [0, T])$. We consider a bounded domain $Q' \subset \mathbb{H}^1$ such that $\overline{Q_{\mathcal{H}}} \subset Q'$. Classical results for linear subelliptic operators, [18, Theorem 10.7] and [17, Theorem 1.1] ensure that, for every $\tau \in [0, T)$ and $\delta \in (0, 1/4]$, the function w^σ belongs to $C_{\mathcal{H}}^{2+\delta}(Q' \times [0, \tau])$ and there exists a constant C (depending on τ and δ) such that

$$(5.10) \quad \|w^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(Q_{\mathcal{H}} \times [0, \tau])} \leq C.$$

Inverting the Cole-Hopf transformation and using (5.6), we obtain a bound for u^σ as (5.10). Finally, by periodicity of w^σ , we accomplish the proof of: $\|u^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(\mathbb{H}^1 \times [0, \tau])} \leq C$. Moreover, by assumptions (H1) and (H2), also the functions $X_i F[\overline{m}]$ and $X_j X_i F[\overline{m}]$ belong to $C_{\mathcal{H}}^{1/4}(\mathbb{H}^1 \times [0, T])$ for $i, j \in \{1, 2\}$. We observe

$$(5.11) \quad X_1 X_2 w^\sigma - X_2 X_1 w^\sigma = 2\partial_{x_3} w^\sigma, \partial_{x_3} X_i w^\sigma = X_i \partial_{x_3} w^\sigma, \quad i = 1, 2,$$

$$(5.12) \quad X_1(\Delta_{\mathcal{H}} w^\sigma) = \Delta_{\mathcal{H}}(X_1 w^\sigma) + 4X_2 \partial_{x_3} w^\sigma, X_2(\Delta_{\mathcal{H}} w^\sigma) = \Delta_{\mathcal{H}}(X_2 w^\sigma) - 4X_1 \partial_{x_3} w^\sigma.$$

First we remark that the function $W_3 := \partial_{x_3} w^\sigma$ is a distributional solution to

$$\begin{cases} -\partial_t W_3 - \sigma \Delta_{\mathcal{H}} W_3 + W_3 F[\overline{m}]/(2\sigma) = -w^\sigma \partial_{x_3} F[\overline{m}]/(2\sigma) & \text{in } \mathbb{H}^1 \times (0, T) \\ W_3(x, T) = \partial_{x_3}(\exp\{-G[\overline{m}_T]/(2\sigma)\}) & \text{on } \mathbb{H}^1, \end{cases}$$

hence following the same procedure to obtain (5.10), we get

$$(5.13) \quad \|\partial_{x_3} w^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(\mathbb{H}^1 \times [0, \tau])} \leq C.$$

Then the functions $W_i := X_i w^\sigma$, $i = 1, 2$, are distributional solution to

$$(5.14) \quad \begin{cases} -\partial_t W_1 - \sigma \Delta_{\mathcal{H}} W_1 + W_1 F[\overline{m}]/(2\sigma) = 4\sigma X_2 \partial_{x_3} w^\sigma - w^\sigma X_1 F[\overline{m}]/(2\sigma) & \text{in } \mathbb{H}^1 \times (0, T) \\ -\partial_t W_2 - \sigma \Delta_{\mathcal{H}} W_2 + W_2 F[\overline{m}]/(2\sigma) = -4\sigma X_1 \partial_{x_3} w^\sigma - w^\sigma X_2 F[\overline{m}]/(2\sigma) & \text{in } \mathbb{H}^1 \times (0, T) \\ W_i(x, T) = X_i(\exp\{-G[\overline{m}_T]/(2\sigma)\}) & \text{on } \mathbb{H}^1 \end{cases}$$

The uniqueness of bounded viscosity solutions established in [27, Theorem 2.1] and the result in [32] imply the uniqueness of bounded distributional solution of these problems. Using estimate (5.13) in system (5.14) and repeating the same arguments as before, we get $\|X_i u^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(\mathbb{H}^1 \times [0, \tau])} \leq C$ for $i = 1, 2$.

To get the bound for $X_i X_j u^\sigma$ we consider the equation satisfied by $W_{ij} := X_i X_j w^\sigma$, $i, j = 1, 2$. We write it explicitly for $W_{11} = X_1^2 w^\sigma = X_1 W_1$ and $W_{21} = X_2 X_1 w^\sigma = X_2 W_1$, the other cases are similar so we shall omit them. W_{11} is the distributional solution to

$$\begin{cases} -\partial_t W_{11} - \sigma \Delta_{\mathcal{H}} W_{11} + W_{11} F[\overline{m}]/(2\sigma) \\ = 4\sigma X_2 \partial_{x_3} W_1 + 4\sigma X_1 X_2 \partial_{x_3} w^\sigma - W_1 X_1 (F[\overline{m}])/\sigma - w^\sigma X_1^2 F[\overline{m}]/(2\sigma), \\ W_{11}(x, T) = X_1^2(\exp\{-G[\overline{m}_T]/(2\sigma)\}). \end{cases}$$

and W_{21} solves

$$\begin{cases} -\partial_t W_{21} - \sigma \Delta_{\mathcal{H}} W_{21} + W_{21} F[\bar{m}]/(2\sigma) \\ = -4\sigma X_1 \partial_{x_3} W_1 + 4\sigma X_2^2 \partial_{x_3} w^\sigma - W_1 X_2 (F[\bar{m}])/(2\sigma) - w^\sigma X_2 X_1 F[\bar{m}]/(2\sigma) - \\ - X_2 w^\sigma X_1 (F[\bar{m}])/(2\sigma), \\ W_{21}(x, T) = X_2 X_1 (\exp\{-G[\bar{m}_T]/(2\sigma)\}). \end{cases}$$

Taking into account that $X_i \partial_{x_3} W_1 = X_i X_1 W_3$, $i = 1, 2$ and of (5.13), repeating the same arguments as before we get again the uniqueness of bounded distributional solution W_{ij} and $\|X_j X_i u^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(\mathbb{H}^1 \times [0, \tau])} \leq C$ for $i, j = 1, 2$.

(iv). We shall follow the arguments of [37, Lemma 3.4] (see also [23, Theorem 5.1 (proof)]); hence we only provide the main steps of the proof. By our assumptions on G , there exists a constant C_1 , independent of σ , such that the functions $w^\pm(x, t) := G[\bar{m}_T](x) \pm C_1(T - t)$ are respectively a supersolution and a subsolution to (5.5). The comparison principle in [27, Theorem 2.1] entails

$$(5.15) \quad \sup_x |u^\sigma(x, t) - G[\bar{m}_T](x)| \leq C_1(T - t) \quad \forall t \in [0, T].$$

On the other hand, assumption (H2) and the hypothesis on \bar{m} yield

$$\sup_{t \in [h, T]} \|F[\bar{m}_t](x) - F[\bar{m}_{(t-h)}](x)\|_\infty \leq C_2 h^{1/4}.$$

We deduce that the functions $v_h^\sigma(x, t) := u^\sigma(x, t - h) + C_1 h + C_2 h^{1/4}(T - t)$ is a supersolution to the PDE in (5.5) and verifies $v_h^\sigma(x, T) \geq u^\sigma(x, T)$. Thanks to (5.15), again by comparison principle, we get

$$u^\sigma(x, t - h) - u^\sigma(x, t) \geq -C_1 h - C_2 h^{1/4}(T - t).$$

The other inequality can be obtained in a similar way and we shall omit its proof. \square

Lemma 5.2 *Under assumptions (H1) – (H3) we consider*

$$(5.16) \quad \begin{cases} \partial_t m - \sigma \Delta_{\mathcal{H}} m - \operatorname{div}_{\mathcal{H}}(m D_{\mathcal{H}} u^\sigma) = 0 & \text{in } \mathbb{H}^1 \times (0, T), \\ m(x, 0) = m_0(x) & \text{on } \mathbb{H}^1. \end{cases}$$

where u^σ is the solution to problem (5.5) found in Lemma 5.1. Then, problem (5.16) admits exactly one bounded classical solution m^σ . Moreover, m^σ has the following properties:

(i) m^σ is $Q_{\mathcal{H}}$ -periodic and there exists $C_0 > 0$ (independent of σ) such that $0 \leq m^\sigma \leq C_0$,

(ii) for every $\tau \in (0, T]$ and $\delta \in (0, 1/4]$, there exists $C_1 > 0$ (depending on σ , τ and δ) such that

$$\|m^\sigma\|_{C_{\mathcal{H}}^{2+\delta}(\mathbb{H}^1 \times [\tau, T])} \leq C_1.$$

Proof. We observe that the differential equation in (5.16) can be written as

$$\partial_t m^\sigma - \sigma \Delta_{\mathcal{H}} m^\sigma - D_{\mathcal{H}} m^\sigma \cdot D_{\mathcal{H}} u^\sigma - m^\sigma \Delta_{\mathcal{H}} u^\sigma = 0.$$

Lemma 5.1-(iii) ensures that the coefficients of this linear parabolic equation belong to $C_{\mathcal{H}}^\delta(\mathbb{H}^1 \times [0, \tau])$ for any $\delta \in (0, 1)$ and $\tau \in (0, T)$; hence the results in [18] apply

to this equation. In particular, [18, Theorem 10.7] ensures the existence of a bounded distributional solution m^σ to (5.16) with $m^\sigma \in C_{\mathcal{H},loc}^{2+\delta}$. On the other hand, since m^σ satisfies assumption (B.2), then Proposition B.1 in the appendix ensures the uniqueness of a bounded classical solution.

Let us now prove the properties of m^σ .

(i). By the left-invariance of the vector fields generating \mathbb{H}^1 and the $Q_{\mathcal{H}}$ -periodicity of u^σ (see Lemma 5.1-(i)), for any $z \in \mathbb{Z}^3$, the function $\tilde{m}^\sigma(x, t) := m^\sigma(z \oplus x, t)$ is still a solution to (5.16). Applying again Proposition B.1, we have $m^\sigma = \tilde{m}^\sigma$, namely m^σ is $Q_{\mathcal{H}}$ -periodic. Moreover, [18, Theorem 10.7] establishes that the fundamental solution of (5.16) is non-negative; since $m_0 \geq 0$, we get: $m \geq 0$.

Let us now prove the upper bound for m^σ . By Lemma 5.1-(ii) and $m^\sigma \geq 0$, have

$$\partial_t m^\sigma - \sigma \Delta_{\mathcal{H}} m^\sigma - D_{\mathcal{H}} m^\sigma \cdot D_{\mathcal{H}} u - C m^\sigma \leq 0$$

for a constant C independent of σ . By the L^∞ bound of m_0 , using again the comparison principle we obtain the statement.

(ii). It is enough to invoke the results in [18, Theorem 10.7] and in [17, Theorem 1.1] and to use the periodicity of m^σ . \square

As for the Euclidean case (for instance, see [22, Lemma 3.4]) it is expedient to interpret m^σ as the law of a suitable stochastic process. In fact, we shall adapt this approach for the present setting where m_0 is only a nonnegative measure on \mathbb{H}^1 (see assumptions H3) and the coefficients in the SDE are unbounded. To this end, we consider a probability space (Ω, \mathcal{F}, P) , equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$. For any $x \in \mathbb{H}^1$, we introduce the process

$$(5.17) \quad dY_t^x = -D_{\mathcal{H}} u(Y_t^x, t) B^T(Y_t^x) dt + \sqrt{2\sigma} B(Y_t^x) dW_t, \quad Y_0^x = x$$

where $B(x)$ is the matrix introduced in (1.3) and W is a standard 2-dimensional (\mathcal{F}_t) -adapted Wiener process.

Remark 5.1 *By Lemma 5.1-(iii), the drift and the diffusion matrix are locally Lipschitz continuous and have an at most linear growth; hence, by standard theory (for instance, [5, Theorem 8.10 pag. 201] or [4, theorem B.3.1]) there exists a unique solution to (5.17).*

Remark 5.2 *The process Y_t^x fulfills the following translation formula*

$$(5.18) \quad z \oplus Y_t^x = Y_t^{z \oplus x} \quad \forall z \in \mathbb{Z}^3, x \in \mathbb{H}^1, t \in [0, T].$$

Actually, by (5.17) and the periodicity of $X_i u$ (see Lemma 5.1-(i)), the process $Z_t := z \oplus Y_t^x$ satisfies $Z_0 = z \oplus x$ and

$$\begin{aligned} d(Z_t)_i &= d(Y_t^x)_i = X_i u(Y_t^x, t) dt + \sqrt{2\sigma} d(W_t)_i = X_i u(Z_t, t) dt + \sqrt{2\sigma} d(W_t)_i, \quad (i = 1, 2) \\ d(Z_t)_3 &= d(Y_t^x)_3 + z_1 d(Y_t^x)_2 - z_2 d(Y_t^x)_1 \\ &= [(Z_t)_1 X_2 u(Y_t^x, t) - (Z_t)_2 X_1 u(Y_t^x, t)] dt + \sqrt{2\sigma} [(Z_t)_1 d(W_t)_1 - (Z_t)_2 d(W_t)_2] \\ &= [(Z_t)_1 X_2 u(Z_t, t) - (Z_t)_2 X_1 u(Z_t, t)] dt + \sqrt{2\sigma} [(Z_t)_1 d(W_t)_1 - (Z_t)_2 d(W_t)_2] \end{aligned}$$

namely, Z_t solves the SDE in (5.17).

We set

$$(5.19) \quad \eta_t^\sigma := \int_{\mathbb{H}^1} \mathcal{L}(Y_t^x) dm_0(x), \quad t \in [0, T],$$

where $\mathcal{L}(Y_t^x)$ is the law of the process Y_t^x .

In the following lemma we shall prove that η_t^σ is a periodic measure on \mathbb{H}^1 , so using Remark 2.3 we shall denote by η_t^σ also the corresponding probability measure on $Q_{\mathcal{H}}$.

Lemma 5.3 *The function $\eta^\sigma : [0, T] \rightarrow \mathcal{M}(\mathbb{H}^1)$ fulfills the following properties:*

(i) η_t^σ is \mathbb{Z}^3 -periodic, namely

$$\eta_t^\sigma(z \oplus A) = \eta_t^\sigma(A) \quad \forall z \in \mathbb{Z}^3, A \text{ Borel set}, A \subset [0, 1]^3, t \in [0, T];$$

(ii) $\eta_t^\sigma(Q_{\mathcal{H}}) = 1$ for every $t \in [0, T]$ (i.e., $\eta_t^\sigma \in \mathcal{P}(\mathbb{T}_{\mathcal{H}})$);

(iii) there exists $C_1 > 0$, independent of $\sigma \in [0, 1)$ and \bar{m} , such that

$$\mathbf{d}_1(\eta_t^\sigma, \eta_s^\sigma) \leq C_1(t - s)^{1/4} \quad \forall 0 \leq s \leq t \leq T;$$

(iv) η_t^σ is a distributional solution to (5.16), namely it fulfills

$$(5.20) \quad \int_{\mathbb{H}^1} \phi(x, t) \eta_t^\sigma(dx) = \int_{\mathbb{H}^1} \phi(x, 0) m_0(x) dx + \iint_{[0, t] \times \mathbb{H}^1} [\partial_t \phi + \sigma \Delta_{\mathcal{H}} \phi - D_{\mathcal{H}} u^\sigma \cdot D_{\mathcal{H}} \phi] \eta_s^\sigma(dx) ds$$

for every $\phi \in C^{2,1}([0, T] \times \mathbb{H}^1)$; moreover it coincides with m_t^σ .

Proof. (i). Consider z, t and A as in the statement. By the definition (5.19) of η^σ and the translation formula (5.18), we have

$$\begin{aligned} \eta_t^\sigma(z \oplus A) &= \int_{\mathbb{H}^1} P \{Y_t^x \in z \oplus A\} dm_0(x) = \int_{\mathbb{H}^1} P \{Y_t^{(-z) \oplus x} \in A\} dm_0(x) \\ &= \int_{\mathbb{H}^1} P \{Y_t^{x'} \in A\} dm_0(z \oplus x') = \int_{\mathbb{H}^1} P \{Y_t^{x'} \in A\} dm_0(x') = \eta_t(A) \end{aligned}$$

where the second-last equality is due to the periodicity of m_0 .

(ii). By the property of pavage and the periodicity of m_0 , we have

$$\begin{aligned} \eta_t^\sigma(Q_{\mathcal{H}}) &= \sum_{z \in \mathbb{Z}^3} \int_{z \oplus Q_{\mathcal{H}}} P \{Y_t^x \in Q_{\mathcal{H}}\} dm_0(x) = \sum_{z \in \mathbb{Z}^3} \int_{Q_{\mathcal{H}}} P \{Y_t^{z \oplus x'} \in Q_{\mathcal{H}}\} dm_0(x') \\ &= \sum_{z \in \mathbb{Z}^3} \int_{Q_{\mathcal{H}}} P \{Y_t^{x'} \in (-z) \oplus Q_{\mathcal{H}}\} dm_0(x') \\ &= \int_{Q_{\mathcal{H}}} P \{Y_t^{x'} \in \cup_{z \in \mathbb{Z}^3} [(-z) \oplus Q_{\mathcal{H}}]\} dm_0(x') \\ &= \int_{Q_{\mathcal{H}}} P \{Y_t^{x'} \in \mathbb{H}^1\} dm_0(x') = 1. \end{aligned}$$

(iii). First of all observe that, using Remark 2.3, we shall denote by η_t^σ also the corresponding probability measure on $Q_{\mathcal{H}}$.

For each $x \in \mathbb{H}^1$, set

$$(5.21) \quad Y_\tau^{per, x} := q_{\mathcal{H}}(Y_\tau^x) \quad \forall \tau \in [0, T]$$

where $q_{\mathcal{H}}(\cdot)$ is the projection introduced in section 2.1. Fix $0 \leq s \leq t \leq T$ and introduce

$$\tilde{\pi} := \int_{\mathbb{T}_{\mathcal{H}}} \mathcal{L}(Y_s^{per,x}, Y_t^{per,x}) dm_0(x)$$

where $\mathcal{L}(Y_s^{per,x}, Y_t^{per,x})$ is the law of the pair $(Y_s^{per,x}, Y_t^{per,x})$. We claim that

$$(5.22) \quad \tilde{\pi} \in \Pi(\eta_s^\sigma, \eta_t^\sigma)$$

where the set Π is the one introduced in (3.1). Let us assume for the moment that this claim is true. Then, by (5.22), there holds

$$\begin{aligned} \mathbf{d}_1(\eta_t^\sigma, \eta_s^\sigma) &\leq \int_{\mathbb{T}_{\mathcal{H}} \times \mathbb{T}_{\mathcal{H}}} d_{\mathbb{T}_{\mathcal{H}}}(z_1, z_2) \tilde{\pi}(dz_1, dz_2) = \int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E}[d_{\mathbb{T}_{\mathcal{H}}}(Y_s^{per,x}, Y_t^{per,x})] dm_0(x) \\ &\leq \int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E}[d_{\mathbb{T}_{\mathcal{H}}}(Y_s^{per,x}, Y_t^{per,x})] dm_0(x) \\ &\leq \int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E} \left[|Y_s^{per,x} - Y_t^{per,x}|^{1/2} \left(1 + 2|Y_s^{per,x}|^{1/2} + |Y_s^{per,x} - Y_t^{per,x}|^{1/2} \right) \right] dm_0(x). \end{aligned}$$

where the last inequality is due to Remark 2.1. Since now on we denote by C a constant which may change from line to line but which is independent of x, s, t, σ . Since $|Y_s^{per,x}|^{1/2}, |Y_s^{per,x} - Y_t^{per,x}|^{1/2} \leq \sqrt{3}$, we get

$$\begin{aligned} d_1(\eta_t^\sigma, \eta_s^\sigma) &\leq C \int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E} \left[\left| \int_s^t -D_{\mathcal{H}}u(Y_\tau^x, \tau) B^T(Y_\tau^x) d\tau + \sqrt{2\sigma} B(Y_\tau^x) dW_\tau \right|^{1/2} \right] dm_0(x) \\ (5.23) \quad &\leq C \int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E} \left[\left(\int_s^t |D_{\mathcal{H}}u B^T| d\tau \right)^{1/2} \right] dm_0(x) + C\sigma^{1/4} \int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E} \left[\left| \int_s^t B dW_\tau \right|^{1/2} \right] dm_0(x). \end{aligned}$$

By standard theory on SDE (see [5, Theorem 8.10 pag.201]), since $\mathbb{E}[|Y_0^x|^2] = |x|^2$ for every $x \in \mathbb{T}_{\mathcal{H}}$, we obtain that there exists a positive constant K , independent of σ and \overline{m} (by virtue of Lemma 5.1-(ii)), such that:

$$(5.24) \quad \mathbb{E}[|Y_\tau^x|^2] \leq K \quad \forall x \in \mathbb{T}_{\mathcal{H}}, \quad 0 \leq \tau \leq T.$$

By Jensen inequality and by Fubini theorem, there holds

$$\begin{aligned} \int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E} \left[\left(\int_s^t |D_{\mathcal{H}}u B^T| d\tau \right)^{1/2} \right] dm_0(x) &\leq \int_{\mathbb{T}_{\mathcal{H}}} \left(\int_s^t \mathbb{E}[|D_{\mathcal{H}}u(Y_\tau^x, \tau) B^T(Y_\tau^x)|] d\tau \right)^{1/2} dm_0(x) \\ &\leq \int_{\mathbb{T}_{\mathcal{H}}} \left(\int_s^t \mathbb{E}[1 + |Y_\tau^x|] d\tau \right)^{1/2} dm_0(x) \\ &\leq \int_{\mathbb{T}_{\mathcal{H}}} \left(\int_s^t \mathbb{E}[1 + |Y_\tau^x|^2] d\tau \right)^{1/2} dm_0(x) \end{aligned}$$

where the last two inequalities are due to Lemma 5.1-(ii) and the definition of the matrix B in (1.3) and respectively to the Cauchy-Schwarz inequality. Using estimate (5.24) in the previous inequality, we obtain

$$(5.25) \quad \int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E} \left[\left(\int_s^t |D_{\mathcal{H}}u B^T| d\tau \right)^{1/2} \right] dm_0(x) \leq C\sqrt{t-s}.$$

On the other hand, by Jensen inequality and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E} \left[\left| \int_s^t B dW_{\tau} \right|^{1/2} \right] dm_0(x) &\leq \int_{\mathbb{T}_{\mathcal{H}}} \left(\mathbb{E} \left[\left| \int_s^t B(Y_{\tau}^x) dW_{\tau} \right|^2 \right] \right)^{1/2} dm_0(x) \\
&\leq \int_{\mathbb{T}_{\mathcal{H}}} \left(\mathbb{E} \left[\left| \int_s^t B(Y_{\tau}^x) dW_{\tau} \right|^2 \right] \right)^{1/4} dm_0(x) \\
&\leq \int_{\mathbb{T}_{\mathcal{H}}} \left(\mathbb{E} \left[\int_s^t (1 + |Y_{\tau}^x|^2) d\tau \right] \right)^{1/4} dm_0(x)
\end{aligned}$$

where the last inequality is due to standard calculus for Ito's integral. Using again Fubini theorem and estimate (5.24) in the previous inequality, we get

$$\begin{aligned}
\int_{\mathbb{T}_{\mathcal{H}}} \mathbb{E} \left[\left| \int_s^t B dW_{\tau} \right|^{1/2} \right] dm_0(x) &\leq \int_{\mathbb{T}_{\mathcal{H}}} \left(\int_s^t \mathbb{E} \left[(1 + |Y_{\tau}^x|^2) \right] d\tau \right)^{1/4} dm_0(x) \\
(5.26) \qquad \qquad \qquad &\leq C(t-s)^{1/4}.
\end{aligned}$$

Replacing estimates (5.25) and (5.26) in (5.23), taking $\sigma \in [0, 1)$, we obtain the statement.

It only remains to prove our claim (5.22): for any measurable subset $A \subset \mathbb{T}_{\mathcal{H}}$, arguing as in proof of point (i) and using the property of pavage, we have

$$\begin{aligned}
\tilde{\pi}(A \times \mathbb{T}_{\mathcal{H}}) &= \int_{\mathbb{T}_{\mathcal{H}}} P\{Y_s^{per,x} \in A\} dm_0(x) = \sum_{z \in \mathbb{Z}^3} \int_{\mathbb{T}_{\mathcal{H}}} P\{Y_s^x \in z \oplus A\} dm_0(x) \\
&= \sum_{z \in \mathbb{Z}^3} \int_{z \oplus Q_{\mathcal{H}}} P\{Y_s^{x'} \in A\} dm_0(x') = \int_{\mathbb{H}^1} P\{Y_s^{x'} \in A\} dm_0(x') \\
&= \eta_s^{\sigma}(A) = \eta_{s|\mathbb{T}_{\mathcal{H}}}^{\sigma}(A);
\end{aligned}$$

analogously, we have $\tilde{\pi}(\mathbb{T}_{\mathcal{H}} \times A) = \eta_{t|\mathbb{T}_{\mathcal{H}}}(A)$. Hence, our claim (5.22) is completely proved.

(iv). The former part of the statement is due to a standard application of Ito's formula as in the Euclidean setting (for instance, see [22, Lemma 3.3] and also [33, Theorem 5.7.6]). The latter part of the statement is an immediate consequence of Proposition B.2 in the appendix with $b = -D_{\mathcal{H}}u$ and $c = -\Delta_{\mathcal{H}}u$ and of Lemma 5.1-(iii). \square

PROOF OF THEOREM 5.1 We shall follow the arguments of the proof of [23, Theorem 5.1] (see also [22, Theorem 4.20]).

By the estimates in Lemma 5.1-(ii) and (iv), possibly passing to a subsequence (that we still denote by u^{σ}), as $\sigma \rightarrow 0^+$, the sequence $\{u^{\sigma}\}_{\sigma}$ uniformly converges to the function u which solves (5.2), is 1/4-Hölder continuous in time and horizontally Lipschitz continuous in space, with $D_{\mathcal{H}}u^{\sigma} \rightarrow D_{\mathcal{H}}u$ a.e. (by [21, Theorem 3.3.3]).

On the other hand, since $\mathcal{P}_{per}(\mathbb{H}^1)$ can be identified with the space of probabilities on the compact set $\mathbb{T}_{\mathcal{H}}$, the estimates for m^{σ} in Lemma 5.3-(iii) and in Lemma 5.2-(i) ensure that, as $\sigma \rightarrow 0^+$, possibly passing to a subsequence, $\{m^{\sigma}\}_{\sigma}$ converges to some $m \in C^{1/4}([0, T], \mathcal{P}_{per}(\mathbb{H}^1))$ in the $C^0([0, T], \mathcal{P}_{per}(\mathbb{H}^1))$ -topology and in the $L^{\infty}(\mathbb{H}^1 \times (0, T))$ -weak-* topology; m satisfies (5.3) with the same constant C_1 of Lemma 5.3-(iii). In conclusion, we accomplish the proof arguing as in [37, Proposition 3.1(proof)]. \square

6 Proof of Theorem 3.1

PROOF OF THEOREM 3.1

(i) We shall adapt the classical arguments in [22, Theorem 4.1 (proof)]. Let C_1 be the constant introduced in (5.3). Consider the set

$$\mathcal{C} := \left\{ m \in C^{1/4}([0, T]; \mathcal{P}_{per}(\mathbb{H}^1)) : \begin{array}{l} m(0) = m_0 \\ \mathbf{d}_1(m_s, m_t) \leq C_1(t-s)^{1/4} \quad \forall 0 \leq s \leq t \leq T \end{array} \right\}$$

endowed with the norm of $C^0([0, T]; \mathcal{P}_{per}(\mathbb{H}^1))$. Observe that it is a nonempty closed and convex subset of $C^0([0, T]; \mathcal{P}_{per}(\mathbb{H}^1))$. We introduce a map \mathcal{T} on \mathcal{C} as follows: for any $\bar{m} \in \mathcal{C}$, we set $\mathcal{T}(\bar{m}) := m$ where m is the solution to problem (5.16) associated to \bar{m} by means of problem (5.5). Theorem 5.1 ensures that $\mathcal{T}(\bar{m})$ still belongs to \mathcal{C} , namely \mathcal{T} maps \mathcal{C} into itself. Moreover \mathcal{T} is continuous and compact. The proof of the continuity follows by the same arguments as those in [22, Lemma 4.19]. In order to prove the compactness of \mathcal{T} , it suffices to observe that $\mathcal{P}_{per}(\mathbb{H}^1)$ can be identified with $\mathcal{P}_{per}(\mathbb{T}_{\mathcal{H}})$ and, consequently, $\mathcal{T}(\mathcal{C})$ is compact too. Then, Schauder's fixed point theorem guarantees the existence of a fixed point for \mathcal{T} , namely a solution to (5.1).

(ii) Consider the function m found in point (i). Since $t \rightarrow m_t$ is narrowly continuous, applying Theorem C.1, we get that there exists a probability measure η^* in $\mathbb{T}_{\mathcal{H}} \times \Gamma$ which satisfies points (i) and (ii) of Theorem C.1. We denote $\eta \in \mathcal{P}(\Gamma)$ the measure on Γ defined as $\eta(A) := \eta^*(\mathbb{T}_{\mathcal{H}} \times A)$ for every $A \subset \Gamma$ measurable. We claim that η is a MFG equilibrium. Indeed, by (C.18), we have $e_0 \# \eta = m_0$ and $e_t \# \eta \in \mathcal{P}_{per}(\mathbb{H}^1)$, so $\eta \in \mathcal{P}_{m_0}(\Gamma)$. On the other hand, by (C.20), η is supported on the curves solving (C.15). From Lemma 4.5 such curves are optimal, i.e. belong to the set $\Gamma^\eta[x]$, hence our claim is proved.

Let us now prove that (u, m) is a mild solution. By (C.18), we have $m_t = e_t \# \eta$. Moreover, by Lemma 4.2, the function u found in point (i) is the value function associated to m as in Definition 3.3-(ii). In conclusion (u, m) is a mild solution to (1.1). \square

Remark 6.1 *Differently from [2] and [37], in this model we cannot obtain the representation of m as the push-forward of m_0 by the flow associated to the optimal control problem. This is due to the fact that we cannot prove a uniqueness result of the optimal trajectories and then we cannot say that $\Gamma^\eta[x]$ is a singleton, or equivalently that the disintegrated measure η_x (see (C.20)) coincides with the Dirac measure $\delta_{\bar{\gamma}_x}$.*

A \mathcal{H} -differentials

In this appendix we introduce the notions of horizontal generalized differentials extending the Euclidean ones [21, section 3.1] (see also [37, section 6.2] for the Grushin case). We need these notions to study the horizontal regularity of a function u . Still following the same arguments as those in [21, 37] we get the proofs of the properties contained in this appendix.

Definition A.1 *A function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ is \mathcal{H} -differentiable at $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ if there exists $p \in \mathbb{R}^2$ such that*

$$\lim_{\mathbb{R}^2 \ni h=(h_1, h_2) \rightarrow 0} \frac{u(x_1 + h_1, x_2 + h_2, x_3 - x_2 h_1 + x_1 h_2) - u(x_1, x_2, x_3) - p \cdot h}{|h|} = 0,$$

and in this case we denote $p = D_{\mathcal{H}}u(x)$. We define the \mathcal{H} -subdifferential and the lower \mathcal{H} -Dini derivative in the direction $\theta \in \mathbb{R}^2$ respectively as

$$D_{\mathcal{H}}^-u(x) = \left\{ p \in \mathbb{R}^2 \mid \liminf_{\mathbb{R}^2 \ni h \rightarrow 0} \frac{u(x_1 + h_1, x_2 + h_2, x_3 - x_2 h_1 + x_1 h_2) - u(x) - p \cdot h}{|h|} \geq 0 \right\}$$

$$\partial_{\mathcal{H}}^-u(x, \theta) = \liminf_{h \rightarrow 0, \theta' \rightarrow \theta} \frac{u(x_1 + h_1 \theta'_1, x_2 + h_2 \theta'_2, x_3 - x_2 h_1 \theta'_1 + x_1 h_2 \theta'_2) - u(x)}{h}.$$

We define the \mathcal{H} -superdifferential $D_{\mathcal{H}}^+u$ and the upper \mathcal{H} -Dini derivative $\partial_{\mathcal{H}}^+u$ in a similar way.

Remark A.1 $D_{\mathcal{H}}u(x)$ coincides with the horizontal gradient (X_1u, X_2u) when u is sufficiently regular.

Lemma A.1 *i)* If u is \mathcal{H} -differentiable at x then $D_{\mathcal{H}}u(x)$ is a singleton and $D_{\mathcal{H}}^+u(x)$ and $D_{\mathcal{H}}^-u(x)$ are both nonempty.

ii) When u is Lipschitz continuous in a neighbourhood of x the \mathcal{H} -Dini lower derivative reduces to

$$\partial_{\mathcal{H}}^-u(x, \theta) = \liminf_{h \rightarrow 0} \frac{u(x_1 + h_1 \theta_1, x_2 + h_2 \theta_2, x_3 - x_2 h_1 \theta_1 + x_1 h_2 \theta_2) - u(x)}{h}.$$

iii) For any $p = (p_1, p_2, p_3)$ in the Euclidean superdifferential $D^+u(x)$, the vector $pB(x)$ belongs to $D_{\mathcal{H}}^+u(x)$.

Proposition A.1 We have

$$D_{\mathcal{H}}^+u(x) = \{p \in \mathbb{R}^2 : \partial_{\mathcal{H}}^+u(x, \theta) \leq p \cdot \theta, \quad \forall \theta \in \mathbb{R}^2\}$$

$$D_{\mathcal{H}}^-u(x) = \{p \in \mathbb{R}^2 : \partial_{\mathcal{H}}^-u(x, \theta) \geq p \cdot \theta, \quad \forall \theta \in \mathbb{R}^2\}.$$

Moreover, $D_{\mathcal{H}}^+u(x)$ and $D_{\mathcal{H}}^-u(x)$ are both nonempty if and only if u is \mathcal{H} -differentiable at x and in this case they reduce to the singleton $D_{\mathcal{H}}u(x) = D_{\mathcal{H}}^-u(x) = D_{\mathcal{H}}^+u(x)$.

B On the uniqueness for second-order Fokker-Planck equation

In this appendix, for the sake of completeness, we collect some results on the uniqueness of the solution to the Cauchy problem for the second-order Fokker-Planck equation (5.16) with fixed $\sigma > 0$ and dropping the periodicity assumption of the coefficients: for $\sigma > 0$, we consider the Cauchy problem

$$(B.1) \quad \begin{cases} \partial_t m - \sigma \Delta_{\mathcal{H}} m + b \cdot D_{\mathcal{H}} m + cm = 0 & \text{in } \mathbb{H}^1 \times (0, T) \\ m(x, 0) = m_0(x) & \text{on } \mathbb{H}^1. \end{cases}$$

Let us just underline that in Euclidean setting the above differential equation becomes

$$\partial_t m - \sigma \operatorname{tr} \left(D^2 m B B^T \right) + b \cdot (Dm B) + cm = 0$$

which is a degenerate second-order linear equation with unbounded coefficient: the one of the principal part has a quadratic growth while the one of the first-order part has a

linear growth and fails to be globally Lipschitz continuous. Up to our knowledge, the uniqueness for this problem has not been tackled before; however it can be obtained adapting techniques available in literature.

We shall tackle two different settings: in the former the coefficients b and c are bounded and the solution is classical while in the latter the coefficients are possibly unbounded (but more regular) and the solution is weak.

For any domain $U \subset \mathbb{H}^1 \times [0, T]$, any $k \in \mathbb{N}$ and any $\delta \in (0, 1]$, we denote $C_{\mathcal{H}}^{k+\delta}(U)$ (resp. $C_{\mathcal{H},loc}^{k+\delta}(U)$) the (resp. local) parabolic Hölder space adapted to the vector fields X_1 and X_2 (for instance, see [17, Section 4] or [18, Definition 10.4]). For $\delta = 0$ and $k = 0$, we simply denote $C_{\mathcal{H}}^k(U)$ and respectively $C_{\mathcal{H}}^\delta(U)$.

Proposition B.1 *Assume that, in (B.1), b and c are bounded continuous functions defined in $\mathbb{H}^1 \times [0, T]$ and b has a continuous and bounded horizontal gradient. For $i = 1, 2$, let $m_i \in C_{\mathcal{H}}^2(\mathbb{H}^1 \times (0, T]) \cap C^0(\mathbb{H}^1 \times [0, T])$ be two classical solution to (B.1) such that, for some positive constant α ,*

$$(B.2) \quad \iint_{\mathbb{H}^1 \times [0, T]} |m_i(x, t)| \exp\{-\alpha(\|x\|_{\mathcal{H}}^2 + 1)\} dx dt < \infty.$$

Then, $m_1 = m_2$.

Remark B.1 *Estimate (B.2) is verified by any function which is $Q_{\mathcal{H}}$ -periodic and belongs to $\mathbb{L}^1(Q_{\mathcal{H}})$.*

Proof. Without any loss of generality, we assume $c \geq 0$; We shall adapt the techniques of [10, Theorem 1]. To this end, we proceed by contradiction assuming that $m_1 \neq m_2$. Let τ_0 be the first time such that $m(\cdot, t) \neq m_2(\cdot, t)$, namely

$$\tau_0 := \inf\{t \in [0, T] \mid m_1(\cdot, t) \neq m_2(\cdot, t)\}.$$

By our assumption on the continuity of m_i , τ_0 belongs to $[0, T]$. The initial condition of (B.1) (if $\tau_0 = 0$) and the continuity of m_1 and m_2 (if $\tau_0 > 0$) ensure that the function $m := m_1 - m_2$ solves

$$\partial_t m - \sigma \Delta_{\mathcal{H}} m + b \cdot D_{\mathcal{H}} m + cm = 0 \quad \text{in } \mathbb{H}^1 \times (\tau_0, T), \quad m(x, \tau_0) = 0 \quad \text{on } \mathbb{H}^1.$$

For any $\epsilon > 0$, the function $w := \sqrt{m^2 + \epsilon}$ verifies

$$\partial_t w = \frac{m \partial_t m}{w}, \quad X_i w = \frac{m X_i m}{w}, \quad X_i^2 w = \epsilon \frac{(X_i m)^2}{w^3} + \frac{m}{w} X_i^2 m, \quad \Delta_{\mathcal{H}} w = \epsilon \frac{|D_{\mathcal{H}} m|^2}{w^3} + \frac{m}{w} \Delta_{\mathcal{H}} m.$$

We multiply the differential equation by m/w and, by these equalities and the sign of c , we obtain

$$\partial_t w = \sigma \Delta_{\mathcal{H}} w - \sigma \epsilon \frac{|D_{\mathcal{H}} m|^2}{w^3} - b \cdot D_{\mathcal{H}} w - c \frac{m^2}{w} \leq \sigma \Delta_{\mathcal{H}} w - b \cdot D_{\mathcal{H}} w.$$

We deduce that, for any nonnegative test function $v \in C^\infty(\mathbb{H}^1 \times [\tau_0, T])$ with bounded support in space and for every $t \in [\tau_0, T]$, there holds

$$\int_{\mathbb{H}^1} w(x, t) v(x, t) dx - \int_{\mathbb{H}^1} w(x, \tau_0) v(x, \tau_0) dx \leq \iint_{\mathbb{H}^1 \times [\tau_0, t]} w [\partial_t v + \sigma \Delta_{\mathcal{H}} v + \operatorname{div}_{\mathcal{H}}(vb)] dx ds.$$

Since $w(\cdot, \tau_0) = \epsilon$, letting $\epsilon \rightarrow 0^+$, we deduce

$$(B.3) \quad \int_{\mathbb{H}^1} |m(x, t)|v(x, t)dx \leq \iint_{\mathbb{H}^1 \times [\tau_0, t]} |m|[\partial_t v + \sigma \Delta_{\mathcal{H}} v + \operatorname{div}_{\mathcal{H}}(vb)]dxds.$$

Let us state the following technical Lemma whose proof is postponed after this proof. We recall that α is the constant of Proposition B.1.

Lemma B.1 *For $\alpha_1 > \alpha$, the function $\Phi(t, x) := \exp\{-[\alpha_1 + \beta(t - \tau_0)](\|x\|_{\mathcal{H}}^2 + 1)\}$ satisfies*

$$\begin{aligned} i) \quad & \partial_t \Phi + \sigma \Delta_{\mathcal{H}} \Phi + b \cdot D_{\mathcal{H}} \Phi + (\operatorname{div}_{\mathcal{H}} b) \Phi \leq 0 \quad \text{in } (\tau_0, \tau) \times \mathbb{H}^1 \\ ii) \quad & \iint_{\mathbb{H}^1 \times [\tau_0, \tau]} |m_i(x, t)| \Phi(x, t) dx dt < \infty, \quad \iint_{\mathbb{H}^1 \times [\tau_0, \tau]} |m_i(x, t)| D_{\mathcal{H}} \Phi(x, t) dx dt < \infty \end{aligned}$$

for suitable constants $\beta > 0$ and $\tau \in (\tau_0, T]$.

We choose $t \in [\tau_0, \tau]$ and $v = \gamma_R \Phi$ where τ and Φ are respectively the constant and the function introduced in Lemma B.1 while $\gamma_R \in C^\infty(\mathbb{H}^1)$ is a cut-off function such that:

$$\gamma_R(x) = 1 \quad \text{if } |x| \leq R, \quad \gamma_R(x) = 0 \quad \text{if } |x| \geq R + 1, \quad \|D\gamma_R\|_\infty + \|D^2\gamma_R\|_\infty \leq 2.$$

Hence, inequality (B.3) becomes

$$\begin{aligned} \int_{\mathbb{H}^1} |m(x, t)| \gamma_R(x) \Phi(x, t) dx \leq \\ \iint_{[B(0, R+1) \setminus B(0, R)] \times [\tau_0, t]} |m| [\Phi(\sigma \Delta_{\mathcal{H}} \gamma_R + b \cdot D_{\mathcal{H}} \gamma_R) + 2\sigma D_{\mathcal{H}} \gamma_R \cdot D_{\mathcal{H}} \Phi] dx dt. \end{aligned}$$

Letting $R \rightarrow +\infty$, since the dominated convergence theorem and Lemma B.1-(ii) ensure that the right-hand side tends to zero, last inequality yields

$$\int_{\mathbb{H}^1} |m(x, t)| \Phi(x, t) dx \leq 0 \quad \forall t \in [\tau_0, \tau]$$

which entails $m = 0$ in $\mathbb{H}^1 \times (\tau_0, \tau)$ contradicting the definition of τ_0 . \square

PROOF OF LEMMA B.1 The equalities in Lemma 2.1-(i), (ii) and (iv) entail respectively that there hold

$$|X_i(\|x\|_{\mathcal{H}}^2)|^2 \leq C_1 \|x\|_{\mathcal{H}}^2, \quad |D_{\mathcal{H}}(\|x\|_{\mathcal{H}}^2)|^2 \leq C_1 \|x\|_{\mathcal{H}}^2, \quad |\Delta_{\mathcal{H}}(\|x\|_{\mathcal{H}}^2)|^2 \leq C_1$$

for a suitable positive constant C_1 . Taking into account these estimates, denoting by $\alpha_2 := \alpha_1 + \beta(\tau - \tau_0)$, we have

$$\begin{aligned} & \partial_t \Phi + \sigma \Delta_{\mathcal{H}} \Phi + b \cdot D_{\mathcal{H}} \Phi + (\operatorname{div}_{\mathcal{H}} b) \Phi \\ &= \Phi [-\beta(\|x\|_{\mathcal{H}}^2 + 1) + \sigma \alpha_2^2 |D_{\mathcal{H}}(\|x\|_{\mathcal{H}}^2)|^2 - \sigma \alpha_2 \Delta_{\mathcal{H}}(\|x\|_{\mathcal{H}}^2) - \alpha_2 b \cdot D_{\mathcal{H}}(\|x\|_{\mathcal{H}}^2) + \operatorname{div}_{\mathcal{H}} b] \\ &\leq \Phi [-\beta(\|x\|_{\mathcal{H}}^2 + 1) + \sigma \alpha_2^2 C_1 \|x\|_{\mathcal{H}}^2 + \sigma \alpha_2 C_1 + \|b\|_\infty \alpha_2 C_1 \|x\|_{\mathcal{H}} + \|\operatorname{div}_{\mathcal{H}} b\|_\infty]. \end{aligned}$$

Choosing $\tau - \tau_0$ sufficiently small and β sufficiently large, we accomplish the proof of point (i).

Point (ii) is an easy consequence of our choice of α_1 and our assumption (B.2). \square

Let us now establish a uniqueness result for weak solution to problem (B.1). To this end, it is expedient to introduce the following family of test functions

$$(B.4) \quad \mathcal{K}_{t, \beta} := \left\{ \phi \in C^2(\mathbb{H}^1 \times [0, t]) \mid \exists C > 0 : \begin{array}{l} i) \quad |\phi| \leq C \exp\{\beta \|x\|_{\mathcal{H}}^2\} \\ ii) \quad |\mathcal{A}^* \phi| \leq C \exp\{\beta \|x\|_{\mathcal{H}}^2\} \end{array} \right\}$$

where $\mathcal{A}^* \phi := \partial_t \phi + \sigma \Delta_{\mathcal{H}} \phi + \operatorname{div}_{\mathcal{H}}(b\phi) - c\phi$.

Example B.1 It is clear that $C_0^\infty(\mathbb{H}^1) \subset \mathcal{K}_{t,\beta}$ for any $\beta \in \mathbb{R}$. For β nonpositive, the property (i) in (B.4) is satisfied by any $\phi \in \mathbb{L}^1(\mathbb{H}^1)$. For β negative, $\mathcal{K}_{t,\beta}$ contains all the bounded functions $\phi \in C_{\mathcal{H}}^{2,1}$ with $\mathcal{A}^*\phi$ bounded.

Proposition B.2 Assume that, for some $\delta \in (0, 1]$ and some $\beta_0 \in \mathbb{R}$, there hold

I) b, c and their horizontal derivatives up to second order and respectively first order belong to the space $C_{\mathcal{H}}^\delta(\mathbb{H}^1 \times [0, T])$.

II) $\int_{\mathbb{H}^1} |m_0(x)| \exp\{\beta_0 \|x\|_{\mathcal{H}}^2\} dx < \infty$.

Furthermore assume also that, for some fixed constant $\beta \leq \beta_0$, for $i = 1, 2$ the functions $m_i : [0, T] \rightarrow \mathcal{M}(\mathbb{H}^1)$ verify

$$(B.5) \quad \int_{\mathbb{H}^1} \phi(x, t) m_i(t)(dx) = \int_{\mathbb{H}^1} \phi(x, 0) m_0(x) dx + \iint_{\mathbb{H}^1 \times [0, t]} (\mathcal{A}^* \phi) m_i(s)(dx) ds$$

for every $t \in (0, T)$ and every $\phi \in \mathcal{K}_{t,\beta}$. Then, $m_1 = m_2$.

Proof. We shall argue following a classical method going back to Holmgren (see [15, pag.340] and references therein). It suffices to show that, for every $\psi \in C_0^\infty(\mathbb{H}^1)$ with $\|\psi\|_\infty \leq 1$ and $t \in (0, T]$, there holds

$$\int_{\mathbb{H}^1} \psi(x) m_1(t)(dx) = \int_{\mathbb{H}^1} \psi(x) m_2(t)(dx).$$

To this end, we fix such ψ and $t \in (0, T_0]$, where T_0 will be suitably chosen later on and it will only depend on the coefficients b and c , and consider the (backward) Cauchy problem

$$(B.6) \quad \begin{cases} \mathcal{A}^* \phi = \partial_t \phi + \sigma \Delta_{\mathcal{H}} \phi + \operatorname{div}_{\mathcal{H}}(b\phi) - c\phi = 0 & \text{in } \mathbb{H}^1 \times (0, T_0) \\ \phi(T_0, x) = \psi(x) & \text{on } \mathbb{H}^1. \end{cases}$$

Invoking [18, Theorem 10.7-(v)] and [17, Theorem 1.1], we obtain that there exists a function $\phi \in C_{\mathcal{H}}^{2,\delta}$ which is a classical solution to problem (B.6).

Assume for the moment that the function ϕ belongs to $\mathcal{K}_{T_0,\beta}$; then $\int_{\mathbb{H}^1} \phi(x, 0) m_0(x) dx$ is finite. Indeed, by point (i) of (B.4) and since $\beta \leq \beta_0$, we have

$$\begin{aligned} \int_{\mathbb{H}^1} |\phi(x, 0) m_0(x)| dx &\leq C \int_{\mathbb{H}^1} |m_0(x)| \exp\{\beta_0 \|x\|_{\mathcal{H}}^2\} \exp\{(\beta - \beta_0) \|x\|_{\mathcal{H}}^2\} dx \\ &\leq C \int_{\mathbb{H}^1} |m_0(x)| \exp\{\beta_0 \|x\|_{\mathcal{H}}^2\} dx. \end{aligned}$$

Moreover, replacing (B.6) in (B.5) with $i = 1, 2$ we obtain

$$\int_{\mathbb{H}^1} \psi(x) m_1(t)(dx) = \int_{\mathbb{H}^1} \phi(x, 0) m_0(x) dx = \int_{\mathbb{H}^1} \psi(x) m_2(t)(dx).$$

By the arbitrariness of ψ and t , we get $m_1 = m_2$ in $\mathbb{H}^1 \times [0, T_0]$. Iterating this argument on time intervals of length T_0 , we accomplish the proof.

It remains to prove that the function ϕ belongs to $\mathcal{K}_{T_0,\beta}$; in other words, we need to prove that: (a) ϕ verifies the bounds in points (i) and (ii) in definition (B.4), (b) ϕ is a $C^{2,1}$ function.

(a). Let us prove point (i) in (B.4). By [18, Theorem 10.7-(v)], the function ϕ can be written as

$$\phi(s, x) = \int_{\mathbb{H}^1} h(t - s, x; 0, \xi) \psi(\xi) d\xi$$

for a suitable nonnegative kernel h . The final datum ψ in (B.6) belongs to $C_0^\infty(\mathbb{H}^1)$; hence $\text{supp}\psi \subset B_{\mathcal{H}}(0, K_\psi)$ for some positive constant K_ψ . Therefore, taking also advantage of the estimates in [18, Theorem 10.7-(iv)], we deduce that, for some constants C_1 and C_2 (depending only on b and c), there holds

$$\begin{aligned} |\phi(s, x)| &\leq C_1 \int_{B_{\mathcal{H}}(0, K_\psi)} \frac{1}{|B_{\mathcal{H}}(x, C_2(t-s)^{1/2})|} \exp\left\{\frac{-d_{\mathcal{H}}(x, \xi)^2}{C_2^2(t-s)}\right\} d\xi \\ &\leq C_1 \frac{|B_{\mathcal{H}}(0, K_\psi)|}{|B_{\mathcal{H}}(x, C_2(t-s)^{1/2})|} \exp\left\{-\frac{(\|x\|_{\mathcal{H}}^2 - R^2) \vee 0}{C_2^2(t-s)}\right\}. \end{aligned}$$

We fix $T_0 := (C_2\beta)^{-1}$ and we obtain point (i) in (B.4). The requirement (ii) in (B.4) can be obtained in a similar manner (taking advantage of the other estimates for h in [18, Theorem 10.7-(iv)]) so we shall omit its proof.

(ii). We already know: $\partial_t\phi, X_i\phi, X_iX_j\phi \in C^\delta$ so, in particular, they are bounded continuous functions. We shall improve this regularity by a bootstrap argument. By equality (5.11) we get that

$$X_1(\text{div}_{\mathcal{H}}(b\phi)) = \text{div}_{\mathcal{H}}(bX_1\phi) + \text{div}_{\mathcal{H}}(X_1b\phi) + 2\partial_{x_3}(b_2\phi)$$

$$X_2(\text{div}_{\mathcal{H}}(b\phi)) = \text{div}_{\mathcal{H}}(bX_2\phi) + \text{div}_{\mathcal{H}}(X_2b\phi) - 2\partial_{x_3}(b_1\phi)$$

Hence, taking account of (5.12) we get that the functions $\Phi_i := X_i\phi$, $i = 1, 2$ are distributional solution in $\mathbb{H}^1 \times (0, t)$ to

$$\begin{cases} \partial_t\Phi_1 + \sigma\Delta_{\mathcal{H}}\Phi_1 = -\text{div}_{\mathcal{H}}(b\Phi_1) - \text{div}_{\mathcal{H}}((X_1b)\phi) - 2\partial_{x_3}(b_2\phi) - 4\sigma X_2(\partial_{x_3}\phi) - c\Phi_1 - (X_1c)\phi, \\ \partial_t\Phi_2 + \sigma\Delta_{\mathcal{H}}\Phi_2 = -\text{div}_{\mathcal{H}}(b\Phi_2) - \text{div}_{\mathcal{H}}((X_2b)\phi) + 2\partial_{x_3}(b_1\phi) + 4\sigma X_1(\partial_{x_3}\phi) - c\Phi_2 - (X_2c)\phi, \\ \Phi_i(t, x) = X_i\psi(x) \quad \text{on } \mathbb{H}^1 \end{cases}$$

The equation satisfied by $\Phi_3 := \partial_{x_3}\phi$ is

$$\partial_t\Phi_3 + \sigma\Delta_{\mathcal{H}}\Phi_3 = -\text{div}_{\mathcal{H}}(b\Phi_3) - \text{div}_{\mathcal{H}}(\partial_{x_3}b\phi) + c\Phi_3 + (\partial_{x_3}c)\phi.$$

Arguing as in (ii) proof of Lemma 5.1 we get that $\partial_{x_3}\phi \in C_{\mathcal{H}}^{2+\delta}$. Our assumptions and the above bounds for the kernel h and its horizontal derivatives ensures that the right-hand side of the equations satisfied by $\Phi_i := X_i\phi$, $i = 1, 2$ belong to C^δ . Therefore, applying [40, Theorem 18] (see also [40, Theorem 16-(b)]), we get $\Phi_i \in C^{1+\delta}$ and, consequently, $D^2\phi \in C^0$. \square

C Probabilistic representation for the continuity equation

This appendix is devoted to adapt the results in [3, Theorem 8.2.1] to the case of a continuity equation expressed in terms of the vector fields generating the Heisenberg group and with a drift $D_{\mathcal{H}}u$ which is bounded and $Q_{\mathcal{H}}$ -periodic in the sense of section 2.1. As a matter of facts, in our case, the statement of [3, Theorem 8.2.1] does not apply because the sommability assumption [3, equation (8.1.21)] for the drift (which reads $D_{\mathcal{H}}u B^T$

in Euclidean coordinates) does not hold. To get the probabilistic representation of the solution of the continuity equation (1.1) the key ingredient is a “superposition principle” (see (C.19)) which allows to prove that there exists a probability measure concentrated on the solutions of the ODE associated to the optimal synthesis (4.20). To get this superposition principle the key results are Lemma C.3 and Lemma C.4 where we strongly use the properties of the distance associated to the Heisenberg group and of the paving to represent \mathbb{H}^1 .

Throughout this section, we shall only study $Q_{\mathcal{H}}$ -periodic solution m to (1.1)-(ii) and we shall write “a.e.” without specifying the measure when we intend “a.e. with respect to the Lebesgue measure”.

We observe that m is a $Q_{\mathcal{H}}$ -periodic solution of (1.1)-(ii) in the sense of distributions in \mathbb{H}^1 means

$$(C.1) \quad \int_0^T \int_{\mathbb{H}^1} (\partial_t \varphi - D_{\mathcal{H}}u \cdot D_{\mathcal{H}}\varphi) dm_t(x) dt = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{H}^1 \times (0, T)).$$

Choosing $\varphi(t, x) = \eta(t)\zeta(x)$ with $\eta \in C_c^\infty(0, T)$, by density, we get the following equivalent formulation of (C.1):

$$(C.2) \quad \frac{d}{dt} \int_{\mathbb{H}^1} \zeta(x) dm_t(x) = - \int_{\mathbb{H}^1} D_{\mathcal{H}}u \cdot D_{\mathcal{H}}\zeta(x) dm_t(x)$$

for any $\zeta \in C_c^\infty(\mathbb{H}^1)$, in the sense of distribution in $(0, T)$.

Note that, by periodicity, m is a solution of (1.1)-(ii) in the sense of distributions in $(0, T)$ also over $\mathbb{T}_{\mathcal{H}}$, i.e.

$$(C.3) \quad \frac{d}{dt} \int_{\mathbb{T}_{\mathcal{H}}} \zeta(x) dm_t(x) = - \int_{\mathbb{T}_{\mathcal{H}}} D_{\mathcal{H}}u \cdot D_{\mathcal{H}}\zeta(x) dm_t(x), \quad \forall \zeta \in C^\infty(\mathbb{T}_{\mathcal{H}}).$$

The following lemma ensures that any $Q_{\mathcal{H}}$ -periodic distributional solution to (1.1)-(ii) (or, equivalently to (C.2) or to (C.3)) has a representative in $C([0, T], \mathcal{P}_{per}(\mathbb{H}^1))$ which will be always called m .

Lemma C.1 (*Continuous representative*). *Let m_t be a Borel family of probability measures $Q_{\mathcal{H}}$ -periodic satisfying (C.3). Then there exists a narrowly continuous curve $t \in [0, T] \mapsto \tilde{m}_t \in \mathcal{P}(\mathbb{T}_{\mathcal{H}})$ such that $m_t = \tilde{m}_t$ for a.e. $t \in (0, T)$. Moreover, if $\varphi \in C_{\mathcal{H}}^{1,1}(\mathbb{T}_{\mathcal{H}} \times [0, T])$ and $t_1 \leq t_2 \in [0, T]$ we have*

$$(C.4) \quad \int_{\mathbb{T}_{\mathcal{H}}} \varphi(x, t_2) d\tilde{m}_{t_2}(x) - \int_{\mathbb{T}_{\mathcal{H}}} \varphi(x, t_1) d\tilde{m}_{t_1}(x) = \int_{t_1}^{t_2} \int_{\mathbb{T}_{\mathcal{H}}} (\partial_t \varphi + D_{\mathcal{H}}\varphi \cdot D_{\mathcal{H}}u) dm_t(x) dt.$$

Proof. From (C.3) we get that, for any $\zeta \in C^\infty(\mathbb{T}_{\mathcal{H}})$

$$t \mapsto m_t(\zeta) = \int_{\mathbb{T}_{\mathcal{H}}} \zeta(x) dm_t(x) \in W^{1,1}(0, T)$$

with distributional derivative

$$\frac{d}{dt} m_t(\zeta) = - \int_{\mathbb{T}_{\mathcal{H}}} D_{\mathcal{H}}\zeta(x) \cdot D_{\mathcal{H}}u(x, t) dm_t(x) \quad \text{for a.e. } t \in (0, T);$$

so, since m_t is a measure on $\mathbb{T}_{\mathcal{H}}$, by the boundedness of $D_{\mathcal{H}}u$, we deduce

$$\left| \frac{d}{dt} m_t(\zeta) \right| \leq \|D_{\mathcal{H}}u\|_{\infty, \mathbb{T}_{\mathcal{H}}} \|D_{\mathcal{H}}\zeta\|_{\infty, \mathbb{T}_{\mathcal{H}}}.$$

Following the proof of [3, lemma 8.1.2], we get that m_t can be extended in a unique way to a continuous curve $\{\tilde{m}_t\}_{t \in [0, T]}$ in $\mathcal{P}(\mathbb{T}_{\mathcal{H}})$ and also that (C.4) holds. Note that in our case the compactness of $\mathbb{T}_{\mathcal{H}}$ yields directly the tightness of the family m_t . \square

Lemma C.2 *Let $t : s \in [0, T'] \rightarrow t(s) \in [0, T]$ be a strictly increasing absolutely continuous map with absolutely continuous inverse $s := t^{-1}$. Then m_t is a distributional solution of (1.1)-(ii) with drift $D_{\mathcal{H}}u$ if and only if $\hat{m} := m \circ t$, is a distributional solution of (1.1)-(ii) on $(0, T')$ with drift $t' D_{\mathcal{H}}u \circ t$.*

Proof. The proof is analogous to that of [3, Lemma 8.1.3] by replacing $D\hat{\varphi}$ with $D_{\mathcal{H}}\hat{\varphi}$, where $\hat{\varphi} \in C_{\mathcal{H},c}^{1,1}(\mathbb{H}^1 \times (0, T'))$. \square

When the drift v_t in equation (1.1)-(ii) satisfies

$$(C.5) \quad \int_0^T \text{Lip}(v_t, K) dt < +\infty$$

where K is any compact set of \mathbb{H}^1 , we can obtain an explicit solution of (1.1)-(ii) by the classical method of characteristics as proved in Proposition C.2. To obtain the needed regularity we approximate v_t and m_t with v_t^ε and m_t^ε by means of a family of mollifiers (see Section 2.2). For m_t^ε solution of the continuity equation (1.1)-(ii) with drift v_t^ε , we can get a representation formula. The following two Lemma provide the approximation with the needed regularity to obtain the explicit formula proved in Proposition C.2.

Lemma C.3 (*Approximation by regular curves*) *Let m_t be a time continuous solution of (1.1)-(ii). Let $(\rho_\varepsilon) \subset C^\infty(\mathbb{R}^3)$ be the family of strictly positive mollifiers in the x variable, defined in 2.9 and set, by the convolution defined in (2.8)-(2.9)*

$$m_t^\varepsilon := m_t * \rho_\varepsilon, \quad E_t^\varepsilon := (D_{\mathcal{H}}u m_t) * \rho_\varepsilon, \quad v_t^\varepsilon := \frac{E_t^\varepsilon}{m_t^\varepsilon}.$$

Then m_t^ε , E_t^ε and v_t^ε are $Q_{\mathcal{H}}$ -periodic. Moreover m_t^ε is a continuous solution of (1.1)-(ii) with drift v_t^ε :

$$(C.6) \quad \partial_t m_t^\varepsilon - \text{div}_{\mathcal{H}}(v_t^\varepsilon m_t^\varepsilon) = 0, \quad \text{in } \mathbb{H}^1 \times (0, T),$$

where v_t^ε fulfills the regularity property (C.5) and the uniform integrability bound

$$(C.7) \quad \int_{\mathbb{T}_{\mathcal{H}}} |v_t^\varepsilon(x)|^p dm_t^\varepsilon(x) \leq C, \quad \forall t \in (0, T), \quad p \geq 1.$$

Moreover, as $\varepsilon \rightarrow 0^+$, $E_t^\varepsilon \rightarrow v_t m_t$ narrowly and

$$(C.8) \quad \lim_{\varepsilon \rightarrow 0} \|v_t^\varepsilon\|_{L^p(m_t; \mathbb{T}_{\mathcal{H}})} = \|D_{\mathcal{H}}u(\cdot, t)\|_{L^p(m_t; \mathbb{T}_{\mathcal{H}})} \quad \forall t \in (0, T).$$

Proof. Note that, from Proposition 2.1-(i), m_t^ε , E_t^ε and v_t^ε are $Q_{\mathcal{H}}$ -periodic. From Proposition 2.1-(v) and the continuity of $m_t^\varepsilon(x)$ w.r.t. x and t , we get

$$m_t^\varepsilon(x) > 0, \quad \text{for any } x \in \mathbb{T}_{\mathcal{H}} \text{ and any } t \in [0, T].$$

From the definition of ρ_ε , since m_t is bounded then $|m_t^\varepsilon|(t, \cdot)$ is bounded. From the definition of the \mathbb{H}^1 -norm (2.6) we get that

$$D\rho_\varepsilon(x) = C(\varepsilon)e^{-(\|\frac{x}{\varepsilon}\|_{\mathcal{H}}^4)} \left(\frac{4x_1(x_1^2 + x_2^2)}{\varepsilon^4}, \frac{4x_2(x_1^2 + x_2^2)}{\varepsilon^4}, \frac{2x_3}{\varepsilon^4} \right).$$

Hence, in $\mathbb{T}_{\mathcal{H}}$, the spatial gradient of $m_t^\varepsilon(t, \cdot)$ is bounded with a constant depending on ε . Analogously, in $\mathbb{T}_{\mathcal{H}}$, $E^\varepsilon(t, \cdot)$ and its spatial gradient are uniformly bounded in space by the product of $\|D_{\mathcal{H}}u\|_{L^1(m_t)}$ with a constant depending on ε .

Moreover, from the positivity of m_t^ε , the local regularity assumptions (C.5) for $v_t^\varepsilon = E_t^\varepsilon/m_t^\varepsilon$ hold. Lemma C.4 shows that (C.7) holds.

From proposition (2.1)-(v), noting that $X_i((m_t X_i u) * \rho_\varepsilon) = X_i(m_t X_i u) * \rho_\varepsilon$, we get

$$\operatorname{div}_{\mathcal{H}}(v_t m_t) * \rho_\varepsilon = \operatorname{div}_{\mathcal{H}}((v_t m_t) * \rho_\varepsilon) = \operatorname{div}_{\mathcal{H}} E_t^\varepsilon = \operatorname{div}_{\mathcal{H}}(v_t^\varepsilon m_t^\varepsilon).$$

Since m_t solves (1.1)-(ii), then

$$\partial_t(m_t * \rho_\varepsilon) + \operatorname{div}_{\mathcal{H}}(v_t m_t) * \rho_\varepsilon = 0.$$

Hence m_t^ε solves the continuity equation (C.6). Finally, general lower semicontinuity results on integral functionals defined on measures of the form

$$(E, m) \mapsto \int_{\mathbb{T}_{\mathcal{H}}} \left| \frac{E}{m} \right|^p dm$$

and the following Lemma C.4 give (C.8). □

Lemma C.4 *Let $m, E \in \mathcal{P}(\mathbb{T}_{\mathcal{H}})$, $E \in L^\infty(\mathbb{T}_{\mathcal{H}})$ and absolutely continuous with respect to m . Let $p \geq 1$, Then*

$$\int_{\mathbb{T}_{\mathcal{H}}} \left| \frac{E * \rho}{m * \rho} \right|^p m * \rho dx \leq \int_{\mathbb{T}_{\mathcal{H}}} \left| \frac{E}{m} \right|^p dm$$

for any positive convolution kernel ρ (see Section 2.2).

Proof. Arguing as in the proof of [3, Lemma 8.1.10], in particular by the Jensen inequality, for any $x \in \mathbb{H}^1$ we get

$$\begin{aligned} \left| \frac{E * \rho(x)}{m * \rho(x)} \right|^p m * \rho(x) &\leq \int_{\mathbb{H}^1} \left| \frac{E}{m} \right|^p (y) \rho(x \ominus y) dm(y) = \sum_{n \in \mathbb{Z}^3} \int_{n \oplus Q_{\mathcal{H}}} \left| \frac{E}{m} \right|^p (y) \rho(x \ominus y) dm(y) \\ &= \sum_{n \in \mathbb{Z}^3} \int_{Q_{\mathcal{H}}} \left| \frac{E}{m} \right|^p (n \oplus z) \rho(x \ominus n \ominus z) dm(z) \\ &= \int_{Q_{\mathcal{H}}} \left| \frac{E}{m} \right|^p (z) \sum_{n \in \mathbb{Z}^3} \rho(x \ominus n \ominus z) dm(z) \end{aligned}$$

where we used the $\mathbb{T}_{\mathcal{H}}$ -periodicity of m and of E/m . Integrating with respect to x in $\mathbb{T}_{\mathcal{H}}$ we get

$$\begin{aligned} \int_{\mathbb{T}_{\mathcal{H}}} \left| \frac{E * \rho(x)}{m * \rho(x)} \right|^p m * \rho(x) dx &\leq \int_{\mathbb{T}_{\mathcal{H}}} \int_{\mathbb{T}_{\mathcal{H}}} \left| \frac{E}{m} \right|^p (z) \sum_{n \in \mathbb{Z}^3} \rho(x \ominus n \ominus z) dm(z) dx \\ &= \int_{\mathbb{T}_{\mathcal{H}}} \left| \frac{E}{m} \right|^p (z) \left(\sum_{n \in \mathbb{Z}^3} \int_{\mathbb{T}_{\mathcal{H}}} \rho(x \ominus n \ominus z) dx \right) dm(z) = \int_{\mathbb{T}_{\mathcal{H}}} \left| \frac{E}{m} \right|^p (z) dm(z). \end{aligned}$$

The last equality comes from

$$\sum_{n \in \mathbb{Z}^3} \int_{\mathbb{T}_{\mathcal{H}}} \rho(x \ominus n \ominus z) dx = \int_{\mathbb{H}^1} \rho(y) dy = 1$$

and this equality is due to the fact that, fixed $z \in \mathbb{T}_{\mathcal{H}}$,

$$\mathbb{H}^1 = \cup_{n \in \mathbb{Z}^3} \mathbb{T}_{\mathcal{H}} \ominus n \ominus z.$$

To prove it we have to show that for any $y \in \mathbb{H}^1$ there exists a unique $n \in \mathbb{Z}^3$ such that there exists $x \in \mathbb{T}_{\mathcal{H}}$ such that $y = x \ominus n \ominus z$. We recall that, from the property of pavage defined at the beginning of Section 2.1, for any $a \in \mathbb{H}^1$ we denote by $n_{\mathcal{H}}(a)$ the unique $n \in \mathbb{Z}^3$ such that there exists a unique point $x_a \in \mathbb{T}_{\mathcal{H}}$ such that $a = n_{\mathcal{H}}(a) \oplus x_a$. Hence there exists a unique $(n_{\mathcal{H},1}, n_{\mathcal{H},2}, n_{\mathcal{H},3}) = n_{\mathcal{H}}(y \ominus z) \in \mathbb{Z}^3$ such that there exists $x = (x_1, x_2, x_3) \in \mathbb{T}_{\mathcal{H}}$ such that $y \ominus z = n_{\mathcal{H}}(y \ominus z) \oplus x$, i.e. $y_1 - z_1 = n_{\mathcal{H},1} + x_1$, $y_2 - z_2 = n_{\mathcal{H},2} + x_2$, $y_3 - z_3 + y_1 z_2 - y_2 z_1 = n_{\mathcal{H},3} + x_3 - n_{\mathcal{H},1} x_2 + n_{\mathcal{H},2} x_1$. To find the unique $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$ such that $y = x \ominus n \ominus z$ we take $n_1 = -n_{\mathcal{H},1}$, $n_2 = -n_{\mathcal{H},2}$ and $n_3 = -n_{\mathcal{H},3} + 2(n_2 x_1 - n_1 x_2)$. \square

Now using an elementary result of the theory of ODEs, we obtain a maximal existence and uniqueness result for the characteristic system associated to equation (C.6).

Lemma C.5 *Let v^ε be the field introduced in Lemma C.3. Then for any $x \in \mathbb{H}^1$ and $s \in [0, T]$, the ODE*

$$(C.9) \quad \frac{d}{dt} Y_t(x, s) = v_t^\varepsilon(Y_t(x, s)) B^T(Y_t(x, s)), \quad Y_s(x, s) = x$$

admits a unique maximal solution which is defined in $[0, T]$.

Proof. The results in [3, Lemma 8.1.4] ensure that there exists a unique maximal solution to (C.9), defined on some interval I , relatively open in $[0, T]$ and containing s as relatively internal point. Moreover, (C.9) reads

$$Y'_{1,t} = v_{1,t}^\varepsilon, \quad Y'_{2,t} = v_{2,t}^\varepsilon, \quad Y'_{3,t} = -Y_{2,t} v_{1,t}^\varepsilon + Y_{1,t} v_{2,t}^\varepsilon.$$

By the boundedness of v^ε , we get that the first two components of $Y_t^\varepsilon(x, s)$ are bounded in I and, afterwards, we deduce the boundedness of the third component. Applying again [3, Lemma 8.1.4], we conclude that I coincides with the whole interval $[0, T]$. \square

For simplicity, we set $Y_t(x) := Y_t(x, 0)$ in the particular case $s = 0$.

Remark C.1 *Characteristics provide a useful representation formula for classical solutions of the equation which is formally the adjoint to (1.1)-(ii):*

$$(C.10) \quad \partial_t \varphi - v_t^\varepsilon \cdot D_{\mathcal{H}} \varphi = \psi \quad \text{in } \mathbb{H}^1 \times (0, T), \quad \varphi(x, T) = \varphi_T(x) \quad x \in \mathbb{H}^1$$

with $\psi \in C_{b, \mathcal{H}}^1(\mathbb{H}^1 \times (0, T))$, $\varphi_T \in C_{b, \mathcal{H}}^1(\mathbb{H}^1)$. A direct calculation shows that, if $Y_s^\varepsilon(x, t)$ solves (C.9), then

$$(C.11) \quad \varphi(x, t) := \varphi_T(Y_T^\varepsilon(x, t)) - \int_t^T \psi(Y_s^\varepsilon(x, t), s) ds$$

solves (C.10). Indeed $Y_s^\varepsilon(Y_t^\varepsilon(x, 0), t) = Y_s^\varepsilon(x, 0)$ yields

$$\varphi(Y_t^\varepsilon(x, 0), t) = \varphi_T(Y_T^\varepsilon(x, 0)) - \int_t^T \psi(Y_s^\varepsilon(x, 0), s) ds$$

and differentiating both sides with respect to t we obtain

$$\left[\frac{\partial \varphi}{\partial t} - v_t^\varepsilon B^T \cdot D\varphi \right] (Y_t^\varepsilon(x, 0), t) = \psi(Y_t^\varepsilon(x, 0), t).$$

Noting $v_t^\varepsilon B^T \cdot D\varphi = v_t^\varepsilon \cdot D_{\mathcal{H}} \varphi$, by the arbitrariness of x (and then $Y_t(x, 0)$), we conclude that (C.10) is fulfilled.

Now we use characteristics to prove the existence, the uniqueness, and a representation formula of the solution of the continuity equation (C.6).

Lemma C.6 *For any $m_0 \in \mathcal{P}_{per}(\mathbb{H}^1)$, let m_0^ε denote $m_0 * \rho_\varepsilon$ where the kernel ρ_ε has been introduced in (2.9). Let Y_t^ε be the solution of (C.9) (corresponding to $s = 0$). Then $t \mapsto m_t^\varepsilon := Y_t^\varepsilon \# m_0^\varepsilon$ is a continuous (in the topology of $C([0, T], \mathcal{P}_{per}(\mathbb{H}^1))$) solution of (C.6) in $[0, T]$.*

Proof. Note that, from the boundedness of $D_{\mathcal{H}} u$, the velocity field v_t^ε satisfies (C.5) and (C.7). The continuity of m_t^ε follows easily since $\lim_{s \rightarrow t} Y_s^\varepsilon(x) = Y_t^\varepsilon(x)$ for m_0^ε -a.e. $x \in \mathbb{H}^1$: thus for every continuous and bounded function $\zeta : \mathbb{H}^1 \rightarrow \mathbb{R}$ the dominated convergence theorem gives

$$\lim_{s \rightarrow t} \int_{\mathbb{H}^1} \zeta dm_s^\varepsilon = \lim_{s \rightarrow t} \int_{\mathbb{H}^1} \zeta(Y_s^\varepsilon(x)) dm_0^\varepsilon(x) = \int_{\mathbb{H}^1} \zeta(Y_t^\varepsilon(x)) dm_0^\varepsilon(x) = \int_{\mathbb{H}^1} \zeta dm_t^\varepsilon.$$

For any $\varphi \in C_c^\infty(\mathbb{T}_{\mathcal{H}} \times (0, T))$ and for m_0^ε -a.e. $x \in \mathbb{T}_{\mathcal{H}}$ the maps $t \mapsto \phi_t(x) := \varphi(Y_t^\varepsilon(x), t)$ are absolutely continuous in $(0, T)$ and

$$\begin{aligned} \dot{\phi}_t(x) &= \partial_t \varphi(Y_t^\varepsilon(x), t) + \langle D\varphi(Y_t^\varepsilon(x), t), v_t^\varepsilon(Y_t^\varepsilon(x)) B(Y_t^\varepsilon(x)) \rangle = \\ &= \partial_t \varphi(Y_t^\varepsilon(x), t) + \langle D_{\mathcal{H}} \varphi(Y_t^\varepsilon(x), t), v_t^\varepsilon(Y_t^\varepsilon(x)) \rangle = \Lambda(\cdot, t) \circ Y_t^\varepsilon \end{aligned}$$

where $\Lambda(x, t) := \partial_t \varphi(x, t) + \langle D_{\mathcal{H}} \varphi(x, t), v_t^\varepsilon(x) \rangle$. We thus have

$$\begin{aligned} \int_0^T \int_{\mathbb{H}^1} |\dot{\phi}_t(x)| dm_0^\varepsilon(x) dt &= \int_0^T \int_{\mathbb{H}^1} |\Lambda(Y_t^\varepsilon(x), t)| dm_0^\varepsilon(x) dt \\ &= \int_0^T \int_{\mathbb{H}^1} |\Lambda(x, t)| dm_t^\varepsilon(x) dt \\ &\leq \text{Lip}_{\mathcal{H}}(\varphi) \left(T + \int_0^T \int_{\mathbb{H}^1} |v_t^\varepsilon(x)| dm_t^\varepsilon(x) dt \right) < +\infty \end{aligned}$$

where the boundedness of the last integral comes from the fact that we can cover the compact support of φ with a finite number of elements of the pavage where $|v_t^\epsilon|$ is bounded. Therefore

$$\begin{aligned} 0 &= \int_{\mathbb{H}^1} \varphi(x, T) dm_T^\epsilon(x) - \int_{\mathbb{H}^1} \varphi(x, 0) dm_0^\epsilon(x) = \int_{\mathbb{H}^1} (\varphi(Y_T^\epsilon(x), T) - \varphi(x, 0)) dm_0^\epsilon(x) \\ &= \int_{\mathbb{H}^1} \left(\int_0^T \dot{\phi}_t(x) dt \right) dm_0^\epsilon(x) = \int_0^T \int_{\mathbb{H}^1} (\partial_t \varphi + D_{\mathcal{H}} \varphi \cdot v_t^\epsilon) dm_t^\epsilon dt \end{aligned}$$

by a simple application of Fubini's theorem, i.e. (C.6) holds. \square

We want to prove that any solution of (C.6) can be represented as in Lemma C.6.

Proposition C.1 (*Uniqueness and comparison for the continuity equation*). *Let σ_t be a narrowly continuous family of signed $\mathbb{T}_{\mathcal{H}}$ -periodic measures solving $\partial_t \sigma_t + \operatorname{div}_{\mathcal{H}} \cdot (v_t^\epsilon \sigma_t) = 0$ in $\mathbb{H}^1 \times (0, T)$, with $\sigma_0 \leq 0$. Then $\sigma_t \leq 0$ for any $t \in [0, T]$.*

Proof. The proof is the same as the one for [3, Proposition 8.1.7] where we replace \mathbb{R}^d with \mathbb{H}^1 and the Euclidean gradient D with $D_{\mathcal{H}}$. Observe that, from the boundedness of the field v_t^ϵ , we have $\int_0^T \int_{\mathbb{T}_{\mathcal{H}}} |v_t^\epsilon| d|\sigma_t| dt < +\infty$. Moreover covering any compact set C with a finite number of elements of the pavage, we get

$$\int_0^T \left(|\sigma_t|(C) + \sup_C |v_t^\epsilon| + \operatorname{Lip}(v_t^\epsilon, C) \right) dt < +\infty$$

for any bounded closed set $C \subset \mathbb{H}^1$. \square

Proposition C.2 (*Representation formula for the continuity equation*). *Let $m_t^\epsilon \in \mathcal{P}_{\text{per}}(\mathbb{H}^1)$, $t \in [0, T]$, be a family of narrowly continuous measures solving the continuity equation (C.6). Then for m_0 -a.e. $x \in \mathbb{H}^1$ the characteristic system (C.9) admits a globally defined solution $Y_t^\epsilon(x)$ in $[0, T]$ and*

$$(C.12) \quad m_t^\epsilon = Y_t^\epsilon \# m_0^\epsilon, \quad \forall t \in [0, T].$$

Proof. Recall that v_t^ϵ satisfies (C.5). Moreover, by Lemma C.5, Y_t^ϵ is globally defined in $[0, T]$ for m_0 -a.e. in \mathbb{H}^1 . Applying Lemma C.6 and Proposition C.1 we obtain (C.12). \square

Now we want to extend Proposition C.2 to the continuity equation (1.1)-(ii), where the vector field $D_{\mathcal{H}}u$ does not satisfy the local regularity assumptions (C.5) but it is still bounded and $Q_{\mathcal{H}}$ -periodic. In this situation we consider suitable probability measures in the space Γ of the absolutely continuous maps from $[0, T]$ to \mathbb{H}^1 , see definition (3.2).

Our representation formula for the periodic solutions m_t^η of the continuity equation (1.1)-(ii) is given by

$$(C.13) \quad \int_{\mathbb{T}_{\mathcal{H}}} \varphi dm_t^\eta := \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \varphi(\gamma(t)) d\eta(x, \gamma) \quad \forall \varphi \in C^0(\mathbb{T}_{\mathcal{H}}), t \in [0, T],$$

where η is a suitable probability measure in $\mathbb{T}_{\mathcal{H}} \times \Gamma$. With a slight abuse of notations, we denote e_t as in (3.3) also the evaluation map $e_t : \mathbb{T}_{\mathcal{H}} \times \Gamma \rightarrow \mathbb{T}_{\mathcal{H}}$ with $e_t(x, \gamma) = \gamma(t)$. Hence, (C.13) can be written as

$$(C.14) \quad m_t^\eta = e_t \# \eta.$$

Theorem C.1 (Probabilistic representation). *Let $m : [0, T] \rightarrow \mathcal{P}(\mathbb{T}_{\mathcal{H}})$ be a narrowly continuous solution of the continuity equation (1.1)-(ii). Then there exists a probability measure η in $\mathbb{T}_{\mathcal{H}} \times \Gamma$, such that*

(i) η is concentrated on the set of pairs (x, γ) such that $\gamma \in \Gamma$ is a solution of the differential equation

$$(C.15) \quad \dot{\gamma}(t) = -D_{\mathcal{H}}u(\gamma(t), t)B^T(\gamma(t)) \text{ for a.e. } t \in (0, T), \quad \gamma(0) = x.$$

(ii) $m_t = m_t^\eta$ for any $t \in [0, T]$, with m_t^η is defined in (C.13).

Conversely, any η satisfying (i) induces via (C.13) a solution of the continuity equation, with $m_0 = e_0 \# \eta$.

Proof. We adapt the arguments of the proof of [3, Theorem 8.2.1]. We first prove the converse implication. Notice that due to (i), we have

$$\dot{\gamma}(t) = D_{\mathcal{H}}u(\gamma(t), t)B^T(\gamma(t)) \quad \eta - a.e., \text{ for a.e. } t \in (0, T).$$

From (C.13) we deduce that $t \mapsto m_t^\eta$ is narrowly continuous; actually, for every $\varphi \in C^0(\mathbb{T}_{\mathcal{H}})$ and $t \in [0, T]$, there holds

$$(C.16) \quad \int_{\mathbb{T}_{\mathcal{H}}} \varphi dm_t^\eta - \lim_{s \rightarrow t} \int_{\mathbb{T}_{\mathcal{H}}} \varphi dm_s^\eta = \lim_{s \rightarrow t} \left(\int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \varphi(\gamma(t)) d\eta(x, \gamma) - \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \varphi(\gamma(s)) d\eta(x, \gamma) \right) = 0.$$

Now we check that $t \mapsto \int \zeta dm_t^\eta$ is absolutely continuous for $\zeta \in C_{\mathcal{H}}^1(\mathbb{T}_{\mathcal{H}})$ bounded and with a bounded horizontal gradient $D_{\mathcal{H}}\zeta$. Indeed, from (C.16), since $D\zeta \cdot D_{\mathcal{H}}uB^T = D_{\mathcal{H}}\zeta \cdot D_{\mathcal{H}}u$, for $s < t$ in $(0, T)$, we have

$$\begin{aligned} \left| \int_{\mathbb{T}_{\mathcal{H}}} \zeta dm_s^\eta - \int_{\mathbb{T}_{\mathcal{H}}} \zeta dm_t^\eta \right| &\leq \int_s^t \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} |D\zeta(\gamma(\tau)) \cdot \dot{\gamma}(\tau)| d\eta d\tau \\ &= \int_s^t \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} |D\zeta(\gamma(\tau)) \cdot D_{\mathcal{H}}u(\gamma(\tau), \tau)B^T(\gamma(\tau))| d\eta d\tau \\ &= \int_s^t \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} |D_{\mathcal{H}}\zeta(\gamma(\tau)) \cdot D_{\mathcal{H}}u(\gamma(\tau), \tau)| d\eta d\tau \\ &\leq \|D_{\mathcal{H}}\zeta\|_\infty \int_s^t \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} |D_{\mathcal{H}}u(\gamma(\tau), \tau)| d\eta d\tau. \end{aligned}$$

Since $D_{\mathcal{H}}u$ is bounded, the inequality gives the absolute continuity of the map. We have also

$$\frac{d}{dt} \int_{\mathbb{T}_{\mathcal{H}}} \zeta dm_t^\eta = \frac{d}{dt} \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \zeta(\gamma(t)) d\eta = \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} D\zeta(\gamma) \cdot \dot{\gamma}(t) d\eta = \int_{\mathbb{T}_{\mathcal{H}}} D_{\mathcal{H}}\zeta \cdot D_{\mathcal{H}}u dm_t^\eta,$$

for a.e. $t \in (0, T)$. Since this pointwise derivative is also a distributional one, this proves that (C.3) holds for test function φ of the form $\zeta(x)\psi(t)$ and therefore for all test functions.

Conversely, for m_t as in the statement, let us apply Lemma C.3 finding $Q_{\mathcal{H}}$ -periodic approximations $m_t^\varepsilon, v_t^\varepsilon$ satisfying the continuity equation (C.6). Therefore, we can apply Proposition C.2, obtaining the representation formula $m_t^\varepsilon = Y_t^\varepsilon \# m_0^\varepsilon$, where Y_t^ε is the flow of maximal solution of (C.9) with $s = 0$.

Since Y^ε induces naturally a map from $\mathbb{T}_{\mathcal{H}}$ to Γ , we define the measure $\eta^\varepsilon \in \mathcal{P}(\mathbb{T}_{\mathcal{H}} \times \Gamma)$ as $\eta^\varepsilon := (i \times Y^\varepsilon) \# m_0^\varepsilon$ where $(i \times Y^\varepsilon) : \mathbb{T}_{\mathcal{H}} \rightarrow \mathbb{T}_{\mathcal{H}} \times \Gamma$ with $(i \times Y^\varepsilon)(x) := (x, Y^\varepsilon(x, 0))$

where $Y^\varepsilon(x, 0)$ denotes the maximal solution to (C.9) with $Y_0^\varepsilon(x, 0) = x$. In other words, for any Borel function ϕ defined in $\mathbb{T}_{\mathcal{H}} \times \Gamma$, the measure η^ε verifies

$$(C.17) \quad \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \phi(x, \gamma) d\eta^\varepsilon(x, \gamma) = \int_{\mathbb{T}_{\mathcal{H}}} \phi(x, Y^\varepsilon(x, 0)) dm_0^\varepsilon(x).$$

Now we claim that (η^ε) is a relatively compact family of measures on $\mathbb{T}_{\mathcal{H}} \times \Gamma$. Indeed, we set

$$C := \{(x, \gamma) \in \mathbb{T}_{\mathcal{H}} \times \Gamma \mid \gamma(0) = 0, \quad \|\gamma'\|_\infty \leq \beta\}$$

where β is a positive constant such that the solution to (C.9) with $x \in \mathbb{T}_{\mathcal{H}}$ and $s = 0$ satisfies $\|\dot{Y}_t^\varepsilon(x, 0)\|_\infty \leq \beta$. We observe that

$$\eta^\varepsilon(C) = \int_{\mathbb{T}_{\mathcal{H}}} \chi_C(x, Y^\varepsilon(x, 0)) dm_0^\varepsilon(x) = \int_{\{x \in \mathbb{T}_{\mathcal{H}} \mid \|Y^\varepsilon(x, 0)\| \leq \beta\}} dm_0^\varepsilon(x) = 1.$$

Invoking Prokhorov theorem, there exists a subsequence of $\{\eta^\varepsilon\}_{\varepsilon \in (0, 1)}$ which narrowly converges. Hence our claim is completely proved.

Now, let η be a narrow cluster point of $\{\eta^\varepsilon\}_\varepsilon$. We claim $m_t = e_t \# \eta$ and that m_0 is the first marginal of η . Indeed, by the definition of e_t (recall: $e_t : \mathbb{T}_{\mathcal{H}} \times \Gamma \rightarrow \mathbb{T}_{\mathcal{H}}$ with $e_t(x, \gamma) = \gamma(t)$) and (C.17), for every $\phi \in C_b^0(\mathbb{T}_{\mathcal{H}})$ and $t \in [0, T]$, there holds

$$\begin{aligned} \int_{\mathbb{T}_{\mathcal{H}}} \phi(x) d(e_t \# \eta^\varepsilon)(x) &= \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \phi(\gamma(t)) d\eta^\varepsilon(x, \gamma) = \int_{\mathbb{T}_{\mathcal{H}}} \phi(Y_t^\varepsilon(x, 0)) dm_0^\varepsilon(x) \\ &= \int_{\mathbb{T}_{\mathcal{H}}} \phi(x) dm_t^\varepsilon(x) \end{aligned}$$

where the last equality is due to $m_t^\varepsilon = Y_t^\varepsilon \# m_0^\varepsilon$. Passing to the limit in the previous equality, we obtain $m_t = e_t \# \eta$ namely

$$(C.18) \quad \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} (\phi \circ e_t) d\eta(x, \gamma) = \int_{\mathbb{T}_{\mathcal{H}}} \phi(x) dm_t(x), \quad \forall \phi \in C_b^0(\mathbb{T}_{\mathcal{H}}).$$

Moreover, again by (C.17), we have

$$\int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \phi(x) d\eta^\varepsilon(x, \gamma) = \int_{\mathbb{T}_{\mathcal{H}}} \phi(x) dm_0^\varepsilon(x)$$

and, passing to the limit as $\varepsilon \rightarrow 0$, we get

$$\int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \phi(x) d\eta(x, \gamma) = \int_{\mathbb{T}_{\mathcal{H}}} \phi(x) dm_0(x)$$

namely m_0 is the first marginal of η . So our claim is completely proved.

Now we have to show that η is concentrated on solutions of the differential equation (C.15). We claim the following ‘‘superposition principle’’

$$(C.19) \quad \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \left| \gamma(t) - x - \int_0^t D_{\mathcal{H}} u(\gamma(\tau), \tau) B^T(\gamma(\tau)) d\tau \right| d\eta(x, \gamma) = 0 \quad \forall t \in [0, T].$$

If the claim is true then we disintegrate η with respect to its first marginal m_0 (see [3, pag 122] or [22, Theorem 8.5]):

$$(C.20) \quad d\eta(x, \gamma) = d\eta_x(\gamma) dm_0(x)$$

and from (C.19) we get for m_0 -a.e. $x \in \mathbb{T}_{\mathcal{H}}$, η_x -a.e. γ is a solution of the (C.15).

It remains to prove the claim (C.19). First of all we prove

(C.21)

$$\int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \left| \gamma(t) - x - \int_0^t w(\gamma(\tau), \tau) B^T(\gamma(\tau)) d\tau \right| d\eta(x, \gamma) \leq C \int_0^T \int_{\mathbb{T}_{\mathcal{H}}} |(D_{\mathcal{H}}u - w)| dm_t d\tau,$$

where $w(x, t)$ is a $Q_{\mathcal{H}}$ -periodic vector field, bounded and continuous w.r.t. x . We have

$$\begin{aligned} & \int_{\mathbb{T}_{\mathcal{H}} \times \Gamma} \left| \gamma(t) - x - \int_0^t w(\gamma(\tau), \tau) B^T(\gamma(\tau)) d\tau \right| d\eta^\varepsilon(x, \gamma) \\ &= \int_{\mathbb{T}_{\mathcal{H}}} \left| Y_t^\varepsilon(x) - x - \int_0^t w(Y_\tau^\varepsilon(x), \tau) B^T(Y_\tau^\varepsilon(x)) d\tau \right| dm_0^\varepsilon(x) \\ &= \int_{\mathbb{T}_{\mathcal{H}}} \left| \int_0^t (v^\varepsilon - w)(Y_\tau^\varepsilon(x), \tau) B^T(Y_\tau^\varepsilon(x)) d\tau \right| dm_0^\varepsilon(x) \\ &\leq \int_{\mathbb{T}_{\mathcal{H}}} \int_0^t |(v^\varepsilon - w)(Y_\tau^\varepsilon(x), \tau) B^T(Y_\tau^\varepsilon(x))| d\tau dm_0^\varepsilon(x) \\ &= \int_0^t \int_{\mathbb{T}_{\mathcal{H}}} |(v^\varepsilon - w) B^T| dm_\tau^\varepsilon d\tau, \end{aligned}$$

where $Y_t^\varepsilon(x)$ is the solution of (C.9). Setting $w^\varepsilon := \frac{(wm)^* \rho_\varepsilon}{m^\varepsilon}$ we obtain

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}_{\mathcal{H}}} |(v^\varepsilon - w) B^T| dm_\tau^\varepsilon d\tau \\ &\leq C \int_0^t \int_{\mathbb{T}_{\mathcal{H}}} |v^\varepsilon - w^\varepsilon| dm_\tau^\varepsilon d\tau + C \int_0^t \int_{\mathbb{T}_{\mathcal{H}}} |w^\varepsilon - w| dm_\tau^\varepsilon d\tau \\ &\leq C \int_0^T \int_{\mathbb{T}_{\mathcal{H}}} |D_{\mathcal{H}}u - w| dm_\tau d\tau + C \int_0^T \int_{\mathbb{T}_{\mathcal{H}}} \int_{\mathbb{H}^1} \rho_\varepsilon(z) |w(x \oplus z) - w(x)| dz d\tau, \end{aligned}$$

where for the last inequality we used Lemma C.4 with $E = (D_{\mathcal{H}}u - w)m$, $p = 1$ and the definition of convolution (2.8). If $\varepsilon \rightarrow 0$, from the continuity of w we get (C.21). To complete the proof of the claim (C.19) we just take a sequence w_n of $\mathbb{T}_{\mathcal{H}}$ -periodic functions, uniformly bounded continuous w.r.t. x such that $w_n \rightarrow D_{\mathcal{H}}u$ in $L^1(m_t, \mathbb{T}_{\mathcal{H}})$. Applying (C.21) to w_n and noting that $m_t^\eta = m_t$ we get (C.19). \square

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