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**Platonism And Mathematical Explanations:  
Some Critical Remarks On Explanatory Proofs And Debunking  
Arguments (I)<sup>1</sup>**

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Short abstract

Key words

1. *Some widely held assumptions about existence, indispensability and best explanation*
2. *Two nominalization cases: Field and Rizza*
3. *Knowledge (I)*
4. *Knowledge (II)*
5. *Debunking*
6. *Conclusions*

References

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<sup>1</sup> I was the respondent to Mark Colyvan's paper "The Ins and Outs of Mathematical Explanation" at the Indispensability and Explanation workshop I co-organized with Marco Panza at the Institut d'Histoire et de Philosophie des Sciences et des Techniques (IHPST) in Paris in November 2012. Section 1 of this paper builds on my critical remarks. The proceedings of the workshop have since then been published, alas without Colyvan's contribution (see *Molinini, Pataut, and Sereni (eds.) 2016*). At the time the present paper was in print, Colyvan's paper was included in a list of Selected Recent Papers as a piece forthcoming in *Mathematical Intelligencer* (see Colyvan's website at <http://www.colyvan.com>).

A substantially different version of the paper was presented in May 2019 at the Seminar in the philosophy of applied mathematics (MathsApp) at the Maison de la Recherche in Paris. Many thanks are due to Mark Colyvan, Marina Imocrante, Daniele Molinini, Marco Panza, Andrea Sereni and Julien Tricart for their criticisms and comments on these two occasions.

The paper is published in two installments. The second part, containing sections 4 to 6, will appear in the next issue of this journal.

## SHORT ABSTRACT

Ontological parsimony requires that if we can dispense with *A* when best explaining *B*, or when deducing a nominalistically statable conclusion *B* from nominalistically statable premises, we must indeed dispense with *A*. When *A* is a mathematical theory and it has been established that its conservativeness undermines the platonistic force of mathematical derivations (Field), or that a non numerical formulation of some explanans may be obtained so that the platonistic force of the best numerical-based account of the explanandum is also undermined (Rizza), the parsimony principle has been respected.

Since both derivations resorting to conservative mathematics and non numerical best explanations also require abstract objects, concepts and principles, ontological parsimony must also be required of nominalistic accounts. One then might of course complain that such accounts turn out to be as metaphysically loaded as their platonistic counterparts. However, it might prove more fruitful to leave this particular worry on the side, to free oneself, as it were, from parsimony thus construed and to look at other important aspects of the defeating or undermining strategies that have been lavished on the disposal of platonism.

Two aspects are worthy of our attention: epistemic cost and debunking arguments. Our knowledge that good mathematics is conservative is established at a cost, and so is our knowledge that nominalistic proofs play a theoretical role in best explanations. I will suggest that the knowledge one must acquire to show that nominalistic deductions and explanations do play their respective theoretical role involves some question-begging assumptions regarding the nature of proofs. As for debunking, even if the face value content of either conservative or platonistic mathematical claims didn't figure in our explanation of why we hold the mathematical beliefs that we do, we could still be justified in holding them so that the distinction between nominalistic deductions and explanations and platonistic ones turns out to be invidious with respect to the relevant propositional attitude, i.e., with respect to belief.

## KEY WORDS

Abstract Objects; Baker (Alan); Clarke-Doane (Justin); Colyvan (Mark); Debunking Arguments; Explanatory Proof; Field (Hartry); Indispensability Argument; Inference to the Best Explanation; Lange (Mark); Mathematical Explanation; Mathematical Realism; Mathematical Truth; Nominalism; Nominalization; Numbers; Ontological Economy; Platonism; Proof; Quantification (objectual and substitutional); Realism; Rizza (Davide); Safety; Sensitivity.

1. *Some widely held assumptions about existence, indispensability and best explanation*

Ever since philosophers of mathematics have been facing the so-called Quine-Putnam challenge of the late sixties and early seventies (see, e.g., *Quine [1968] 1969* and *Putnam [1971] 1979*), it has been taken for granted by many that if it is shown that mathematics is indispensable in the best explanations of natural phenomena, existence claims of applied mathematics thereby acquire a bona fide philosophical justification.<sup>2</sup> The leading idea is that if mathematics plays a role in these explanations, it is hard to resist a realist reading of it.<sup>3</sup> I understand realism, here, as consisting in an existence claim *simpliciter* with respect to mathematical objects such as, e.g., numbers, involving prima facie no commitment to a further independence claim to the effect that such objects exist *ante rem*, independently of mathematical practice, thinking, structures, principles

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<sup>2</sup> Liggins argues that the phrase “Quine-Putnam challenge” is a misnomer because Quine has another argument for platonism than the one mistakenly attributed to him, and because Putnam isn’t arguing at all in favor of abstract objects. He detects in both Quine and Putnam a stronger argument for platonism than the standard indispensability argument. I shall not discuss this important point here, although, as far the Putnamian point is concerned, the last paragraphs of this section strongly suggest that abstractness, i.e., the non causal, non spatial and non temporal character of the disputed objects is *not* the issue.

Let me however stress the following point. There is a claim about commitment to objects via truth, which comes directly from Quine, and another one about translatability which comes directly from Putnam. These are strictly speaking distinct, and the very idea of there being a Quine-Putnam challenge obviously plays down the distinction. Field, for instance, conflates the two in *Field 1980* : 107n4. For Quine, we must conclude that classical mathematics constitutes a body of truths and this implies that we must “countenance,” as Quine often says, i.e., tolerate, its existential assertions. For Putnam, we must accept the so-called realist reading if only because there is still no translation of the language of science — a language which does indeed involve reference to and quantification over mathematical objects — into a language free of such reference and quantification. Or, if there is one, it disrupts logical relations (*Putnam [1971] 1979* : sect. V).

See *Liggins 2008* for his rejoinder to the prevalent understanding of Quine and Putnam’s stand on the familiar epistemological challenge to platonism.

<sup>3</sup> Colyvan argued verbatim in favor of this view in his presentation at the 2012 workshop.

and language. This isn't a matter of bias with respect to the meaning of the word "realism" which would have one dismiss the independence claim as superfluous, as not being constitutive of the platonist picture, or perhaps as not even being part of it, no matter how small the part. A realist or platonist (I shall be using these terms interchangeably) will of course defend the view that mathematical theories relate to an external realm of abstracta, "cut off from all links with the reflecting subject," as Bernays nicely puts it when contrasting Euclid's postulates, where figures *must* be "constructed," with Hilbert's axioms *stating from the outset the existence, in the system of straight lines, of points, straight lines and planes* (*Bernays [1935] 1983: 259*, emphasis mine).<sup>4</sup>

The point, rather, of insisting on existence *simpliciter*, is that the argument Colyvan is considering when defending platonism rests on the idea that if you need *A* to explain *B*, you should take *A* at face value, and that since what you so take in best explanations of natural phenomena is mathematics, the referents of the mathematical terms involved in the explanation and the values of the variables bound by the objectual existential quantifier of the applied regimented mathematical theory are also to be taken at face value.<sup>5</sup> So, to resort to Baker's celebrated

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<sup>4</sup> See also *Bernays op. cit.*: 267.

For a nice and simple example of construction in Euclid, see *Euclid [ca. 300 BC] 2008*: Book 1, Proposition 1, on how to construct an equilateral triangle on a given finite straight line. There are, of course, many other relevant examples of this kind throughout the *Elements*.

<sup>5</sup> If we were to favor a substitutional reading of the existential quantifier, or introduce a new symbol for a substitutional existential quantifier in the regimented language of applied mathematics, noted  $\Sigma$ , we'd be claiming that a sentence of the form  $(\exists x) Fx$ , or of the form  $(\Sigma x) Fx$ , is true if and only if there is a closed term *t* of the language such that *Ft* is true. If, on the contrary, we were to stick to the standard objectual reading and refrain from such an introduction, we'd be claiming that a sentence of the form  $(\exists x) Fx$  is true if and only if there is some object *x* in the relevant universe of discourse such that *F* is true of that object. In the first case, we would be substituting expressions for the bound variables so as to get true sentences; in the

magicicada exemple (*Baker 2005*), if you need prime numbers to explain why periodical magicicadas crawl out of their subterranean hideouts every 13 or 17 years, depending on the kind they belong to, you should also hold that primes exist.<sup>6</sup>

You might furthermore believe that the mathematical theory that asserts the existence of 13 and 17 and of their properties such as that of being prime, i.e., number theory, describes structures, say à la Shapiro (*Shapiro 1997*) or à la Resnik (*Resnik 1997*), so that these numbers are not truly entities or objects in any (perhaps) naïve, non-Fregean sense, but places that stand in structural relations to one another, i.e., places in the structure of the natural numbers. Were that option chosen, places will be referred to by some appropriate term of the philosophical theory (structuralism, in the case at hand) that discloses what number theory, as we're sometimes fond of saying, is *really* about (not just numbers *tout court* but, more specifically, *numbers qua places*), i.e., quite naturally, "place." Places will then be among the values of the bound variables in expressions of the form  $(\exists x) Fx$ , insofar as the genuine content of our assertions of the existence of numbers and of their properties is disclosed by the aforementioned philosophy of arithmetic, so that  $F$  will be true of places, or of numbers qua places, *under the structuralist reading of the theory*.

But we shouldn't be concerned here with any particular take on the nature of numbers or of any other mathematical object needed in the

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second we would assign values to the bound variables so as to get genuine existence claims.

I won't discuss here the moot point of the relevance of the substitutional reading of existential quantification (or of existential quantification of the substitutional kind) to the platonism issue, when platonism is argued for on the basis of an indispensability claim based on an inference to the best explanation.

<sup>6</sup> I have discussed the so-called "enhanced" version of that argument in a reply to *Baker 2016* in *Pataut 2016*. More on this below in section 3.

application of mathematical theories. The point, rather, is that if the justification we have for the existence assertions, beliefs in the entities and so on and so forth, comes from the success of the application of the mathematics we need, we *must* take the mathematical existence claims *at face value*, i.e., as claims about whatever it is they are quite literally, verbatim, about — say, numbers. To be an entity or an object in whose existence we believe is simply to be referred to by a mathematical term of the theory needed in our best mathematical explanations, or to be among the values of the bound variables of some suitably regimented mathematical theory, given that no paraphrase disposing of such terms and of such values of the bound variables is available, either effectively or in principle.

In other words, the contrasting philosophical mood which might lead you to take mathematical existence assertions and beliefs to be about what a particular philosophical theory claims they are, say places, is *not* available in the present context, either as a solution to the indispensability problem, or as a way out it. You can't say "Oh, but I don't quite believe that primes such as 13 or 17 exist, or literally exist, I just believe in places in the structure of  $\mathbf{N}$ ." And if you wish to say it, your belief in *that* must be supported by an explanation of how it is that places, or numbers *qua* places, play an indispensable role in, say, the best mathematical explanation of why periodical magicicadas crawl out of their subterranean hideouts every 13 or 17 years depending on the kind they belong to. However, to repeat, in the present context of investigation, whatever turns out to be metaphysically the case under a particular philosophical reading, interpretation or construal of the applied theory, need not concern us. The face value claim that numbers are needed and useful must indeed be taken at face value — in earnest.

There will also be disagreements about the details of the argument,

e.g., typically, on whether it is possible to argue for the realist or platonist reading through confirmation holism, i.e., on whether applied mathematics inherits that reading from whatever evidence or justification is available in favor of the empirical theories in which it plays the role of a theoretical apparatus needed for formulations, computations and predictions.<sup>7</sup> So if, say, you obtain some appropriate confirmation for Newton's law of gravitation, a law that involves forces and masses, you'll thereby have a basis on which to secure a realist reading of the real numbers that measure them.

The core contention here is that mathematical explanations "are significant for [the] philosophy of mathematics, especially for the realism /anti-realism debate,"<sup>8</sup> because they justify mathematical existence claims. They do in the sense that it would be *incoherent* to use mathematics *and* to deny the existence of mathematical objects; say, to explain that the life cycles of cicadas is either 13 or 17 years, *and* to claim that these numerical terms fail to refer, or that mathematical existence assertions that have primes among the values of their bound variables are false (or perhaps that they are neither true nor false). Given that double talk is ruled out from the start qua deceptive (and that we really would be in a case of deceptive double talk), inference to the best explanation forces mathematical platonism, or realism, on us.

What is proposed here by the platonist is an inference that relates an indispensability claim with respect to mathematical explanations to a face value claim with respect to mathematicalialia. The inference relies on a number of strong assumptions, and it will be useful to make a note of them.

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<sup>7</sup> See, e.g., *Leng 2010* and *Leng 2016* for a defence of the view that applied mathematics does indeed inherit that reading.

<sup>8</sup> Colyvan's words in his oral presentation.

The strong assumptions are the following:

(A1) If we can dispense with  $A$  when best explaining  $B$ , then we must dispense with  $A$ .

(A2) If we can dispense with  $A$  when best explaining  $B$ , then we must conclude that  $A$  terms fail to refer and that the prima facie (objectual)<sup>9</sup> existence claims of  $A$  are false.

(A3) If we can't dispense with  $A$  when best explaining  $B$ , then we must take  $A$  at face value, conclude that  $A$  terms refer and that the prima facie (objectual) existence claims of  $A$  are true.

(A1), of course, expresses the philosophical motivation behind any attempt at nominalization; it may itself be motivated by a yet more general belief in the virtues of Ockham's razor so that, in case a mathematical explanation and a non mathematical explanation of  $B$  are both on offer, we should prefer the non mathematical one, out of respect for the parsimony principle inherited from Ockham.

This, I think, tells us something important. It tells us that if we are to judge the respective virtues of mathematical explanations and non mathematical or nominalized ones by parity and refrain from invidious distinctions between the two kinds, as we certainly must, we should allow ourselves to yet another inference that likewise relates indispensability claims with respect to best *non* mathematical explanations of  $B$  to face value claims with respect to whatever will have *replaced* mathematicalia in a nominalistically acceptable explanation of  $B$ . The inference is this : if a non mathematical explanation of some natural phenomenon is available, we should conclude that this particular explanation best explains the phenomenon and that its non mathematical claims must be

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<sup>9</sup> Given what has been said in footnote 5, "objectual" is redundant. I resort to it here as a reminder, both in (A2) and in (A3), and shall not resort to it afterwards.

taken at face value. If nominalist explanations have any explanatory virtue, then we must dispense with mathematics and embrace a different ontology. Whether that ontology is genuinely lighter is a problem I shall return to.

The relevant distinction here is between a genuinely mathematical explanation and a genuinely non mathematical one, i.e., between one that applies mathematics and one that applies something else, one that resorts to, say, first order logic and set theory, as in *Field [1980] 2016: ch. 2*, or to axioms of measurement theory as in *Rizza 2011*. Of course, first order logic, set theory and measurement theory are as abstract as can be. In any event, first order logic and set theory are as abstract as arithmetic taken at face value (Field's case), and measurement theory is as abstract as number theory taken at face value (Rizza's case), so that the issue of abstractness and the so-called access problem to abstracta turns out to be pretty much a red herring.<sup>10</sup>

Something else must motivate the choice. Accordingly, parsimony or ontological economy should be construed in such a way that mathematical explanations and non mathematical explanations cannot fare equally with respect to something *not* related to abstractness. In other words, since (A1)-(A3) applies, *mutadis mutandis*, to nominalized explanations, and since this clearly indicates that the success of an

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<sup>10</sup> One might argue, of course, with respect to the Fieldian case, that first order logic is *not* committed to anything. One might object : “there are no objects, in any clear sense of the word ‘object’ of first order logic.” A first order language allows quantifiers to bind only individual variables; or, to put it semantically, first order quantification is quantification over individuals, but in the case of first order *logic*, no ontological commitment is implied via the use of bound individual variables.

Moreover, one might argue that Field's arithmetical example is ill-chosen here because it isn't concerned with what counts as the explanation of a natural phenomenon. Still, we are offered an account of how to reach a conclusion using a mathematical apparatus that's merely conservative so that a dispensability claim is (perhaps only allegedly) established through an argument that relies on substituting logic and set theory for arithmetic. I shall go back to this point in the next section.

explanation, whether platonistic or otherwise, plays the role of a *defeater*, or *underminer* with respect to its competitor, we should be wary of any claim to the effect that a particular explanation has an advantage just because it defeats or undermines a metaphysical or ontological view couched in terms of access to abstracta. *Something else* must be defeated or undermined. What is the *non* metaphysical or ontological benefit of a platonistic or, on the contrary, of a nominalistic explanation?

Suppose, for the sake of argument, that there is nothing in the mathematical world, as we sometimes rather carelessly say, to make mathematical discourse true. Are we thereby losing any justification we might have to look for mathematical explanations? The very same kind of question is in order for competing nominalistic explanations. There is no reason why they should be treated differently in this respect. Suppose there is nothing to make *non* mathematical explanatory discourse true. Suppose, e.g., that there is nothing to make first order logic and set theoretical discourse true (Field's case) or measurement theory true (Rizza's)?<sup>11</sup> Are we thereby losing any justification we might have to look for nominalistic explanations that resort to these theories, i.e., explanations that will justify a dispensability thesis?

Why bother about justifications, one might complain? For the following reason. I'm thinking here of so-called debunking arguments that have been developed in discussions of the evolutionary standpoint in meta-ethics. This takes us far away indeed from mathematics, but it helps us keep a close watch on (A1)-type strategies and this is what counts. Suppose there is nothing in the moral world that makes moral judgments

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<sup>11</sup> Putnam's argument in *Putnam [1967] 1979*: 33-34 is quite different in this respect for he argues that the applicability of number theory requires nothing more than the consistency of mathematics, and consistency involves no commitment whatsoever to existence claims. It involves the claim that nothing logically absurd or self-contradictory is a theorem of mathematics, or that there are no two theorems of mathematics such that one is the negation of the other.

true, or that there is no such thing indeed as a moral world. Does it thereby follow that it would be a mistake to either accept or reject moral judgments on the basis of suitable warrants, or to believe them either true or false? Would it be a waste of time to look for justifications and refutations of a distinctly moral kind?

Another connected question concerns the explanatory role of such beliefs, and that question might obviously be asked about mathematical beliefs as well. It might indeed be asked about beliefs in general. Suppose that the face value content of mathematical claims doesn't figure in our best explanation of why we hold the mathematical beliefs we do indeed hold. Would this undermine these beliefs? Would this then in turn undermine the philosophical view that there are mathematical objects?<sup>12</sup> In meta-ethics, defenders of debunking arguments have argued that if the face value content of ethical claims doesn't figure in our best explanation of why we hold the moral beliefs we do, this would undermine them in the sense that it would threaten or impair our confidence in them. The very same questions are in order for mathematical beliefs. If we still would have the mathematical beliefs we do as a matter of fact have, had they been false, and if we could not have easily had false mathematical beliefs, using the methods that we do use in current mathematical practice, is there any place left for a debunking argument with respect to mathematical beliefs ?

I shall go back to these worries in section 4. The crucial question for now is: how are we to assess the respective values of mathematical explanations and non mathematical ones?

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<sup>12</sup> See *Clarke-Doane 2016* on the twin issues of safety and sensitivity. More on this, with an unpacking of both notions, in section 5.

## 2. Two nominalization cases: Field and Rizza

The question may be asked about the two cases I've briefly mentioned so far, from Field (*Field [1980] 2016 : ch. 2*) and Rizza (*Rizza 2011*). It will be interesting to look at these, in part because it is often argued, or perhaps merely felt, that they take issue with indispensability and ontological commitment in diverging ways.

In Field's case, the arithmetic of natural numbers supplemented by set theory is replaced by the logic of identity supplemented by existential numerical quantifiers (for ease of exposition), so that a particular empirical conclusion to the effect that there are exactly 63 bugs at a given location in space may be reached by means of a deductively valid argument using the arithmetic of natural numbers supplemented by set theory in a non committed way. In Rizza's case, the mathematical results one needs to explain why periodical magicicadas crawl out of their subterranean hideouts every 13 or 17 years are paraphrased with the help of axioms of measurement theory so that the whole cicada case is reformulated without any recourse to mathematics, using only betweenness relations on temporal intervals.

It might be thought that what is shown in Field's arithmetical example is that we may go through a derivation by means of a mathematical system that is conservative but not true, either literally or otherwise, but that *no* genuine explanation has thereby been delivered, let alone a best or nominalistic one. I'll try to dispel the fear that this is all we have in that particular instance, and that, in both cases, although for different reasons, we are in a situation where, if the respective platonistic forces of the explanations to be replaced have been undermined, the advantage thereby yielded *isn't* one of ontological economy.

In Field's example, we start with an argument whose premises constitute a body of "nominalistically stated assertions," i.e., of assertions containing no occurrence of terms or of quantifiers for "abstract objects," we then provide an abstract counterpart for each of the premises, apply a mathematical system "that includes the arithmetic of natural numbers plus some set theory" (*Field op. cit.*: 22) so that the conjunction of the abstract counterparts of the premises and the provable results of the mathematical system "entail" the abstract counterpart of the nominalistic conclusion to be reached. Since the equivalence of nominalistic assertions and their abstract counterparts is provable in the logic of identity supplemented by the numerical quantifiers plus the arithmetic of natural numbers supplemented by set theory, "we have proved the nominalistically stated conclusion" (*Field loc. cit.*).

We may reconstruct the full argument in the following way. ("MS1," etc., stand for the numbered assertions of the mathematical system used in the derivation, "Equiv." stands for "logical equivalence," "CP" stands for "conservative principle," [1\*] and [1\*'] are assertions in which each quantifier has been restricted with a formula " $\neg M(x)$ ," meaning intuitively "x is not a mathematical entity," so that, as far as ontological commitment is concerned, every quantification of the form " $(\exists x) \dots$ " is replaced by one of the form " $\exists x (\neg Mx)$  and...", and "Nom." stands for "Nominalism.")

- (1) There are exactly 21 aardvarks.
- [1\*] [There are exactly 21 aardvarks that are not mathematical objects.]
- (2) On each aardvark there are exactly 3 bugs.
- (3) Each bug is on exactly 1 aardvark.
  
- (1') The cardinality of the set of aardvarks is 21.
- [1\*'] The cardinality of the set of aardvarks that are not mathematical objects is 21.]

- (2') All sets in the range of the function whose domain is the set of aardvarks, and which assigns to each entity in its domain the set of bugs on that entity, have cardinality 3.
- (3') The function in 2' is 1-1 and its range forms a partition of the set of all bugs.

MS1. If all members of a partition of a set  $X$  have cardinality  $\alpha$ , and the cardinality of the set of members of the partition is  $\beta$ , then the cardinality of  $X$  is  $\alpha \times \beta$ .

MS2. The range and domain of a 1-1 function have the same cardinality.

MS3.  $3 \times 21 = 63$ .

(4') The cardinality of the set of all bugs is 63.

(4) There are exactly 63 bugs.

Equiv. The equivalence of (1)-(4) and (1')-(4') is provable in the logic of identity supplemented by numerical quantifiers plus the arithmetic of natural numbers supplemented by set theory.

CP Let  $A$  be any nominalistically storable assertion and  $N$  any body of such assertions ; let  $S$  be any mathematical theory. Then  $A^*$ , that results by restricting each quantifier in  $A$  with the formula " $\neg M(x)$ " (meaning intuitively " $x$  is not a mathematical entity"), isn't a consequence of  $N^*+S+\exists x(\neg Mx)$  unless it is a consequence of  $N$ .

Nom. (4) follows from (1)-(3) alone.

What we have in Field's arithmetical case is an example of how a nominalist may apply mathematics for deducing non mathematical consequences given that mathematics is conservative (see *Field op. cit.*: 14).

What we have in Rizza's take on the Baker example is obviously quite different, namely the application of a system that is *non* mathematical through and through because it resorts to betweenness relations on temporal intervals and that neither may qualify as mathematical objects.

Baker points out in *Baker 2005* that magicicadas spend either 13 or 17

years in larval form before emerging as adult insects because prime periods uniquely maximize the infrequency of overlap with their periodical predators. The explanation in terms of evolutionary advantage has two salient features that are relevant for the platonism issue as it is discussed here: it is mathematical and it is of the optimization kind.

The explanation counts as mathematical because it draws on the number-theoretic result that any natural number  $p$  is coprime with any other natural number  $q$  given that  $q < p$ , and thus maximizes the lowest common multiple with every  $q$  just in case  $p$  is prime. It is of the optimization kind because the primeness of the periods provides a solution to an optimization problem: the target feature, i.e., the avoidance of periodical predators, is optimized given the constraints imposed by ecological factors deemed relevant and a set of life-cycle lengths deemed biologically possible given such constraints.

Baker's claim is that since (i) there is no direct empirical evidence for the entities that feature as a solution to the optimization problem, i.e., for periodical predators of magicicadas, and (ii) prime periods of life cycles of magicicadas are uniquely optimal, there are prime numbers (*Baker 2016*). The platonist conclusion is warranted by the fact that (i) and (ii) conjointly amount to the claim that we are in a genuine case of inference to the best mathematical optimization explanation. Since there is no debating that prime periods are uniquely optimal, the existence of numbers is "implied by the number-theoretical result" (*Baker 2016: 337*). To be precise, the controversial existence claim about numbers *simpliciter* is implied *only* because the result is *applied* in order to obtain an optimization explanation that takes it at face value, given that the

result, taken at face-value, *asserts* (rather than implies) that there are prime and co-prime natural numbers.<sup>13</sup>

The platonistic force of the best optimization explanation is entirely a function of mathematical objects (natural numbers, primes, coprimes and lowest common multiples) featuring in a pure number-theoretical result being applied to concreta (periods) in such a way that the application, by assigning a theoretical role to these objects, justifies a conclusion to the effect that there is an evolutionary advantage to the existence of prime numbers of periodical life cycles. Given the theoretical role that such objects play in an explanation in terms of the minimizing of the intersection of prime periods with any shorter periods, the (Hard Road Piecemeal) nominalist's task is to undermine such controversial platonistic force by showing how that role (the minimizing) may be played by some other entities that do not commit one to the platonist's controversial existence claim but still yield an optimization explanation in terms of evolutionary advantage<sup>14</sup>.

We may formulate the argument in favor of the claim that a mathematically-free explanans that carries the same explanatory load as the mathematical or platonistic one is available for the magicada case in the following way.

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<sup>13</sup> Strictly speaking, the lemma one resorts to in the explanation makes no existential claims. Given Quine's criterion, in order to qualify as a genuine assertion of the existence of mathematical objects that one may furthermore decide to take at face-value, the lemma would have to be rephrased so that natural numbers, prime numbers, coprime numbers and lowest common multiples appear as values of variables bound by the objectual existential quantifier.

<sup>14</sup> The nominalist is dubbed in this instance a "Hard Road Piecemeal Nominalist" because he "respond[s] to the indispensability argument by attempting to undermine the claim that mathematics is explanatorily indispensable for science [...] by attempting some kind of nominalistic paraphrase for, reconstruction of, or alternative to mathematics-laden scientific explanations." What makes that approach specifically piecemeal as opposed to holistic is that it "take[s] individual putative cases of mathematical explanation in science and come[s] up with alternative explanations which are purely nominalistic" (*Baker 2016: 339-340*). See also *Colyvan 2010*.

## 1. Mathematical explanation

*P1.* A cicada subspecies with period  $p$  and a predator species with shorter period  $q$  will coincide at intervals corresponding to the lowest common multiple (LCM) of  $p$  and  $q$ .

*P2.* If  $p$  is prime, then its LCM with any smaller  $q$  is  $p \times q$ , which is as large as possible.

*C.* Prime periods maximize the infrequency of overlap with periodical predators on the assumption that the predators have shorter periods than the cicadas.

*Remark on the mathematical explanation.* The key property in the explanation *P1-C* is co-primeness. (Two numbers are coprime if they have no common factors, which in turn ensures that their LCM is as large as possible.)

*Mathematical result.* Let  $p$  and  $q$  be natural numbers;  $p$  is coprime with every  $q < p$ , and hence maximizes LCM with every  $q < p$ , iff  $p$  is prime.

## 2. Mathematics (or, rather, number) free formulation of the explanans

We may depict life-cycles as segments on a line and then describe divisibility relations between the lengths of sequences of intervals over a segment partitioned into congruent intervals.

(1) The basic results and notions involved in that geometrical description may be given in non-numerical fashion. (2) The properly numerical formulation of these notions and results may be recovered from non-numerical conditions.

(1) and (2) may be established by using axioms of measurement theory.

*Non-committed interpretation of the axioms.* Axioms of measurement theory may be interpreted on relations between time-intervals. The axioms have models that are representable on a numerical structure  $N$  that allows the reduction of the numerical reasoning to the numerical representation of the relevant facts about a geometry of time intervals, so that the numerical reasoning leading to the derivation of the length of magicicada's life cycles depends on the preliminary availability of suitable empirical relations. It is because these empirical relations are in place that it is meaningful to use numbers in order to reason about life-cycles. Once the numerical assignment is in place, it is of course possible to introduce the relevant numerical notions and thus describe the relevant divisibility relations. Although this may be done

numerically, it isn't necessary to appeal to numbers because divisibility relations between life-cycles, i.e. time intervals, may be represented geometrically.<sup>15</sup>

Provided these arguments are conclusive, we have an account of why the mathematics *doesn't have to carry the explanatory load*, either because it is used conservatively, or because it is dispensed with.

The idea that we do have an explanans in Field's arithmetical case is not as preposterous as it might seem at first blush. Of course, one might want to object that the claim that there are exactly 63 bugs at a given location isn't on a par with the claim that magicicadas spend either 13 or 17 years in larval form before emerging as adult insects. One reason is that no regularity is involved in the fact that there are 63 bugs distributed on a limited number of aardvarks, not even an accidental generality. Yet both are empirical facts, facts about the natural world or about natural phenomena, and both claims involved are stated in such a way that the existence of a number of non-mathematical entities (years or life-cycles, bugs) is asserted. The fact that the inference to the best explanation is interesting to biologists because "there are known predators of cicadas, including some parasitic insect species that target cicadas specifically [although none] of [those] known predators or parasites are themselves periodical" (*Baker 2016: 336*), whereas the fortuitous remark about bugs on aardvarks isn't of any interest to anyone, need not worry us.

What should worry us is that the nominalistic proposals play the role of defeaters or underminers with respect to a platonistic competitor, and that the undermining or defeating of mathematical realism supported either by a derivation or by a best explanation in numerical terms taken at face value has a cognitive cost.

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<sup>15</sup> See *Rizza 2011*: sections 4 and 5, and Sam Baron's criticism in *Baron 2016*.

I've said earlier, towards the end of section 1, that something *not* related to the abstractness, remoteness, acausality or inaccessibility of objects of a certain kind must be defeated or undermined so that, provided that we must prefer non mathematical explanations to mathematical ones and abide by (A1), *another* lesson must be drawn from this preference that is different from (A2) and (A3). I will suggest that the lesson cannot be that explanations with mathematics free formulations of the explanantia, or explanations with merely conservative mathematics offer an epistemic advantage.

### 3. Knowledge (I)

Suppose that the mathematics taken at face value doesn't carry the explanatory load so that we know that there are exactly 63 bugs at a given location and why magicicadas spend either 13 or 17 years in larval form before emerging as adult insects *non mathematically*. The two cases are strikingly divergent with respect to the kind of non mathematical knowledge involved. This, of course, is due to the fact that Field's derivation relies on an argument to the effect that mathematics, and arithmetic in particular, is conservative, and thus eschews the prospect of paraphrase and reconstruction, while Rizza's proposal does on the contrary offer a reconstruction of a numerical-based explanation in strictly geometrical fashion so that a numerical reasoning taken at face value may be paraphrased within a non-numerical theory.

Given that we do have an explanans in both cases, we might have two different kinds of worries. To begin with, how do we acquire the knowledge that is required for the undermining of platonism or realism? How do we know that the applied mathematics is conservative, and how do we know, not just the axioms of measurement theory, but that they

may be safely interpreted in terms of relations between time-intervals? Why should we believe in their truth in the first place, either when the axioms are taken at face value or when they are suitably interpreted? It seems that quite a lot has to be known, either to secure conservativeness or the explanatory role of the proposed reconstruction. Secondly, it is unclear how the content of that knowledge may in either case benefit itself from a purely nominalistic account (provided, of course, that one should abide by this further constraint).

In Field's case, we know that a nominalistically stated conclusion follows from nominalistically stated premises *without* having *actually* derived (4) from (1)-(3), or, to be precise, without having derived it directly. It is essential, in that view, that the "intervening" premises one resorts to in the derivation aren't true, either literally or otherwise. It is sufficient that they preserve truth among the nominalistic statements (*Field op. cit.* : 14). How does one know that premises that are not in the market for truth nevertheless preserve truth among nominalistically stated claims?

Field proposes two answers. The weak answer is that "[i]n the conservativeness case, the kind of inductive considerations that are relevant are the knowledge that in the past no one has found counter examples to conservativeness, and also the knowledge that in many actual cases where platonistic devices are used in proofs of nominalistic conclusions from nominalistic premises [as in the arithmetical case], these devices are eliminable in what seems to be a more or less systematic way" (*Field op. cit.* : 109n10).

The stronger answer is that CP follows from another somewhat stronger conservative principle:

(CP') Let  $A$  be any nominalistically statable assertion, let  $A^*$  result

by restricting each quantifier in  $A$  with the formula “ $\neg M(x)$ ” (meaning intuitively “ $x$  is not a mathematical entity”), and let  $S$  be any mathematical theory. Then  $A^*$  isn’t a consequence of  $S$  unless it is logically true.

In Rizza’s case, the explanation that unifies similar instances of periodicity under a common explanatory pattern in terms of betweenness relations on temporal intervals appeals to axioms of measurement theory.

In Baker’s original platonistic account, mathematical objects such as natural numbers, and properties or relations such as being prime and coprime play a role in the explanation (see, e.g., *Baker 2005*: 234). Rizza denies, however, that such properties and relations *must* be “interpreted as properties and relations on certain distinctive mathematical entities” (*Rizza op. cit.* : 105-106). The key idea is that these properties and relations are properties and relations “of time intervals corresponding to life-cycles which [...] can be studied non-numerically” (*Rizza loc. cit.*). One may study them geometrically by depicting respective life-cycles as segments on a line. One then asks how often sequences of consecutive life-cycles of predators and cicadas respectively determine time intervals of equal length. When the life-cycles do determine this, a divisibility relation obtains. Rizza concludes that “a form of divisibility can be recovered in non-numerical terms and so a non-numerical version of primality can be recovered too” (*Rizza loc cit.*).

Of course, as Rizza correctly remarks, although the paraphrase or reconstruction does eliminate reference to numbers “it does not eliminate the mathematical form of the reasoning leading to a derivation of a prime life-cycle length” (*Rizza op. cit.*: 110). The crucial claim in this respect is that “mathematics may play a relevant explanatory role in a non-ontological way, by providing the means to conceptualize the structure of

an empirical phenomenon in such a way that its explanation can be represented as a derivation from certain formal conditions, [with non-numerical counterparts in the reasoning]” (*Rizza op. cit.*: 113).<sup>16</sup>

In the Fieldian case, CP’ is taken to be obvious if only because non-conservative mathematics would turn out to be inconsistent and that a proof of the inconsistency of standard applied classical mathematics is highly unlikely: good applicable useful mathematics is both true in all possible worlds and a priori true (*Field op. cit.*: 13). The a prioricity claim, either couched in terms of truth in all possible worlds or in terms of a knowledge obtained and justified independently of experience is of course a disputable one.<sup>17</sup>

What I want to suggest here is that the problem isn’t so much with the view that mathematics is conservative *as such*, or with the mathematical form of the reconstruction or paraphrase *as such*, as with the kind and breadth of knowledge one must acquire that would justify either the claim that mathematics is conservative and a priori, or the claim that one may recover a non-numerical form of deduction that turns out to be essential to some nominalistically respectable paraphrase.

Field’s claim that good mathematics is logically true and Rizza’s claim that the geometrical concepts that are introduced to study the relevant empirical phenomena constrain the selection of a certain length require, respectively, *further* logical and geometrical knowledge, much more in either case than is needed in standard platonistic accounts. One evaluating the epistemic cost might conclude that too much is required to undermine the platonistic force of the explanations to be replaced so that the cost isn’t really worth our trouble, especially if a nominalization of the content

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<sup>16</sup> Or, again: “the mathematical character of the explanation resides in the form of the reasoning being used and in the concepts invoked, rather than in the reference to [and quantification over] certain particular entities” (*Rizza op. cit.*: 110).

<sup>17</sup> Field doesn’t consider the second, Kantian, construal.

of that knowledge must also be established.

Not only does this further requirement sets the bar very high — presumably way too high —, but problems arise with respect to the proofs at stake both in the paraphrase of the mathematical apparatus appealed to in nominalizations and in the conservativity argument.

To begin with, is it possible, or indeed advisable to provide nominalistically acceptable proofs of all the results appealed to in a nominalistic explanation?

Rizza, for one, proposes to paraphrase any additional mathematical apparatus appealed to in the platonistic proof at stake in the magicada case. Nominalized proofs should presumably be construed as formal derivations (as opposed to, e.g., acts construed à la Brouwer, in orthodox intuitionistic fashion). But how could proofs construed as syntactic objects contribute to a genuinely non-mathematical explanation, whether or not the explanation is of the optimization kind? Proofs, whether construed as acts or derivations, are mathematical objects in their own right. And even if it were established that proofs syntactically construed do contribute to a bona fide nominalistic account or explanation, a further argument would have to be provided that this force is weaker than the platonistic one in spite of the fact that part of the mathematical apparatus that plays a role in nominalistic explanations, such as axioms and postulates, is just assumed to be true.

Moreover, once it is admitted that mathematical proofs play a theoretical role, it looks as though one is committed to some form or other of justification holism, so that *all* proofs with theoretical roles, along with the mathematical apparatus they require to be carried out, will have to be nominalized for the undermining of the platonistic force of platonistic explanations to be successful. Where should one stop? The platonistic force of an explanation varies with the strength and quantity of

mathematics required for a proof of the applied results to be carried out. Not only is it unfair to require that nominalistically acceptable proofs be provided for all the applied number-theoretic results involved in, e.g., the cicada case, but the Hard Road Piecemeal nominalist is now turning into a mathematical holist. By way of a slippery-slope argument, once you've started applying mathematics, there is no bound to the amount of theory you have to nominalize in order to get your Hard Road Piecemeal nominalism going.

Baker's idea, of course, is that it is not enough, in order to commit oneself to the existence of mathematical objects that the theories and statements about them be indispensable in the natural sciences: they must be indispensable *in the right way*, i.e., they must themselves offer the best explanation of some empirical phenomena (*Baker 2009*). Does this restriction help?

Steiner's take on the notion of explanatory proof turns out to be helpful when discussing this particular point. Steiner argues that proofs are explanatory when they directly refer to a characterizing property of the entity mentioned in a theorem such that from the proof, it is evident that the result *depends* on the property (*Steiner 1978*). The role of a proof in the case of a mathematical optimization explanation, such as Baker's cicada case, is strong, or truly explanatory, just in case it is "integral" or "essential" or "central" to it. It is weak just in case it provides a "justification" of the (applied) mathematical claim that is not itself part of the explanation.

Should the nominalist offer paraphrases of proofs that are strong or explanatory in Steiner's sense, i.e., paraphrases that offer *more* than a justification for the truth of the non mathematical claims one must resort to in nominalized optimization explanations? One might think that the most challenging nominalistic approach is one where *both* the

mathematical results at stake *and* strong proofs in Steiner's sense are given a nominalistically acceptable construal.

If that is the case, the reasoning centered on theoretical role puts us in a puzzling situation. It now appears that the platonist accepts proofs as a principal means of justification for useful mathematical claims, so that the business of the nominalist is to paraphrase these in some nominalistic language. This is a strange claim, first of all because platonism, when characterized semantically in terms of truth, must be construed as the claim that mathematical truth is independent from mathematical proof and may indeed go beyond provability in principle (and perhaps even beyond practical or feasible provability)<sup>18</sup>; secondly because the nominalist's task isn't so much to establish that one may dispense with the theoretical *role* of mathematical objects, as previously announced, as to paraphrase the definitions of the mathematical terms occurring in the purely mathematical formulations of the proofs and lemmas one resorts to in platonistic explanations (see *Tallant 2013* on this point).

The crucial point, then, is that the theoretical terms occurring in the formulation of some nominalized optimization explanation must be such that their referents play a strong explanatory role in the explanation. Tallant's idea is that, most of the time, proofs only play a weak role, i.e., justify mathematical claims but aren't strictly speaking part of a best explanation. What needs to be assessed here is whether (and if so, why) proofs, especially proofs that explain, as opposed to proofs that merely prove, would necessarily yield optimization explanations that appeal to the relevant aspects of the world's network of causations rather than to strictly mathematical facts, entities or properties.

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<sup>18</sup> See the beginning of section 1 and footnote 4 on Bernays.

I shall now briefly turn to Mark Lange's take on the issue (*Lange 2013*) before going back to the details of the conservativity argument. Lange claims that the magicicada case is as a matter of fact a case of *non* mathematical optimization explanation. Some scientific explanations are distinctively mathematical because they appeal to mathematical "facts." Although it quantifies over mathematical entities, the biological explanation we have been concerned with so far is an ordinary causal one (*Lange 2013*). What makes it causal and ordinary as opposed to distinctively mathematical is that it relies on a description of the explanandum's causal history and that natural selection is a causal concept *par excellence*. Although the lemmas one uses in the platonistic proof do indeed play an explanatory role, the optimization explanation that relies on them in its explanans *do not* resort to mathematics "in the manner that is exploited by [...] indispensability arguments" (*Lange op. cit.*: 492). Neither the quantification over mathematical entities per se, nor the explanatory role that the entities in the course of values of the bound variables play in an optimization explanation may be used in these arguments.

Even if that were indeed the case, Lange's distinction between mathematical explanations that rely on proofs proper (conceived as acts or otherwise) and mathematical explanations that rely on "facts" alone, i.e., on what proofs establish, as well as on postulates or axioms, doesn't help in way of an explanation of why that is. Lange goes as far as claiming that distinctively mathematical explanations include "mathematical facts [...] but *not* their proofs — *much less* proofs that explain *why* those mathematical facts hold [emphases mine]" (*Lange op. cit.*: 508). Explanations that include mathematical facts (as opposed to proofs), appeal to something that is modally stronger than what explanations that (purportedly) include proofs of those facts provide.

They do because they relate to the framework of *any* causal relation rather than restrict themselves to some specific explanandum's causes and to the particular causal histories that ordinary explanations take into account in their explanans — e.g., typically in the case at hand, specific ecological factors imposing constraints on the avoidance of predators, or some evolutionary advantage assessed in terms of observed past regularities. Moreover, they help answer why questions.

Notice however that in Baker's original case, what makes the (allegedly ordinary) explanation that resorts to Lemma 2 a better explanation of the maximization of the infrequency of overlap of magicicadas with their periodical predators than the (allegedly ordinary) explanation that resorts to the weaker Lemma 1 is precisely what enables the better explanation to answer why questions, in particular why prime periods would *not* be uniquely optimal in certain counterfactual situations (in case there would be, say, only 2-year predators). In the author's view, the increment in explanatory force is indeed cashed out in counterfactual terms. Now, of course, Lange is looking for something yet stronger than that: the explanation must reveal that the explanandum is "more necessary" than the causal laws that govern it, so that optimization explanations in biology may be deemed "distinctively mathematical" only insofar as they are instances of particular mathematical conjectures.<sup>19</sup>

Finally, we must reckon with the fact that the syntactic or formal structure of the proofs one resorts to in both platonistic and nominalistic explanations may count more than expected when one defends the idea that the indispensability argument is primarily concerned with "extra-mathematical" explanations, i.e., with explanations which are given "via

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<sup>19</sup> See my critical comments on this point in *Pataut 2016*.

a ‘spill over’ from within mathematics itself’.<sup>20</sup> After all, explanatoriness is tied to the structure of proofs, if only for the following reason. If you conclude that  $p$  from a contradiction derived from  $\neg p$ , you have a warrant for  $p$ , or for the truth of  $p$ , but you don’t necessarily know *why*  $p$ , or why  $p$  is true. You haven’t always gained what Gowers and Neilson call a “crucial insight” (see *Gowers and Neilson 2009*).

There is yet another and deeper disagreement here, not about what contradictions amount to — we may agree on their logical form — but about how to proceed to get one, a disagreement between advocates of the classical *reductio* and advocates of the intuitionistic *reductio*. The case of negation and of the introduction and elimination rules for that constant is revealing. In the first case, it is sufficient, in order to assert  $\neg p$ , that the mere supposition of the truth of  $p$  would lead to a contradiction. In the second case, we must possess a construction such that, from the supposition that a construction of  $p$  would be obtained, a contradiction may be derived (see *Heyting [1956] 1971* : sections 2.2.2 and 7.1.1).

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<sup>20</sup> Colyvan’s words in his oral presentation.

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