MINIMIZER OF GINZBURG-LANDAU ENERGY ON THE EXTERIOR OF A BALL IN DIMENSION 3 : EXISTENCE, UNIQUENESS AND PROPERTIES

Ramona Anton

MINIMIZER OF GINZBURG-LANDAU ENERGY ON THE EXTERIOR OF A BALL IN DIMENSION 3 : EXISTENCE, UNIQUENESS AND PROPERTIES

by

Ramona Anton

Abstract. — We prove existence and unconditional uniqueness of a positive minimizer for the Ginzburg-Landau energy outside the unit ball in $\mathbb{R}^3$, satisfying Dirichlet boundary conditions. The main ingredient of the proof is a Sturm-Liouville theorem. Due to the structure of the energy space in dimension three, we obtain a strong stability of the minimizer.

1. Introduction

Our interest for this problem came from studying Gross-Pitaevskii equation on an exterior domain [**1**]. Gross-Pitaevskii equation appears in the theory of Bose-Einstein condensates , non-linear optics, superconductivity and in superfluidity e.g. [**30**]. Special solutions, for example travelling waves, have been extensively studied both by physicists and by mathematicians e.g. [**8, 19, 2, 5, 9, 4, 3, 25, 10**].
Gross-Pitaevskii equation is a non-linear Schrödinger equation associated to the Ginzburg-Landau energy

$$\mathcal{E}_{GL}(u) = \int_\Omega \frac{1}{2}|\nabla u|^2(x) + \frac{1}{4}\left(|u|^2(x) - 1\right)^2 dx.$$

We are interested in studying the minimizer of the Ginzburg-Landau energy on the exterior of the unit ball in $\mathbb{R}^3$: $\Omega = \mathbb{R}^3 \backslash B(0,1)$. Due to the non-linear term in the energy, the functions that have finite energy satisfy non-null condition at infinity. More precisely in dimension three they tend to a constant of modulus 1 (see section 2 or [**12**] for a precise statement).
Under Dirichlet boundary conditions on $\mathbb{S}^1 = \partial\Omega$ and non-null condition at infinity, the energy is always **strictly** positive. A natural question is whether there exist a minimal value for the energy $\mathcal{E}_{GL}$ and if there exists a minimizer associated to this minimal value.

In this paper we answer positively to this question :

Theorem 1. — *There exists a unique real positive minimizer of the Ginzburg-Landau energy on the exterior of the ball in* $\mathbb{R}^3$ *with Dirichlet boundary conditions. The minimizer is radial and converges exponentially to 1 at infinity. We also obtain the strong stability of the minimizer (see section 6 for a precise statement).*

The equation satisfied by the minimizer of the Ginzburg-Landau energy on $\Omega = \mathbb{R}^3 \backslash B(0,1)$ is

$$(minGL) \left\{ \begin{array}{rcl} \triangle u & = & (|u|^2 - 1)u, \ on \ \mathbb{R} \times \Omega \\ u_{|_{\mathbb{R} \times \partial\Omega}} & = & 0. \end{array} \right.$$

Due to the non-linear term of the Ginzburg-Landau energy,

$$\mathcal{E}_{GL}(u) = \int_\Omega \frac{1}{2}|\nabla u|^2(x) + \frac{1}{4}\left(|u|^2(x) - 1\right)^2 dx,$$

the energy space is not a vector-space :

$$E = \{u \in H^1_{loc}(\Omega), \ u_{|_{\partial\Omega}} = 0, \ \nabla u \in L^2(\Omega), \ |u|^2 - 1 \in L^2(\Omega)\}.$$

The natural energy space has been studied on $\mathbb{R}^3$ by Patrick Gérard in [**12**] and [**13**]. We adapted the arguments for the exterior of an obstacle in [**1**]. In dimension three every function in the energy space converges to 1 at infinity.

Remark 1. — *The most interesting result of this paper is, in my opinion, the* ***unconditional uniqueness*** *of the minimizer. The fact that every function in the energy space in dimension* 3 *converges to a constant of modulus* 1 *at infinity acts as a constraint that ensures the uniqueness.*

Remark 2. — *However, we can only prove* ***uniqueness of radial*** *minimizers. When the domain has radial symmetry, like the exterior of a ball, we prove that the minimizers also have radial symmetry (see section 4). But for example in the case of the exterior of a non-trapping compact, we cannot conclude. In his paper on a similar problem in* $\mathbb{R}^2$, *Mironescu* [**27**] *proved uniqueness of the minimizers using a very clever method that we exploit also in section 5.5. The problem in implementing this strategy in the general case (like the exterior of a compact obstacle in* $\mathbb{R}^3$*) is making sense of an integration by parts for functions with non-zero condition at infinity. In this paper we used the radial symmetry to obtain extra informations on the behaviour at infinity and thus the method of Mironescu works. It would be interesting to show a similar result on a non-radial domain.*

The plan of the paper is as follows : in section 2 we give the notations and we recall the classical results needed, in section 3 we prove the existence of a real positive minimizer by classical minimizing techniques, in section 4 we prove the radial symmetry of minimizers by a method of Lopez [**22**] and Maris [**24**], [**23**] of symmetrization along a plane, in section 5 we prove uniqueness by a shooting method on the corresponding ODE in the spirit of the classical paper by Kwong [**20**] (we also obtain exponential convergence to 1 at infinity for the radial profile of the minimizer) and in section 6 we prove the stability of the minimizer under the flow of the Gross-Pitaevskii equation.

Acknowledgments. I would like to thank Patrick Gérard for suggesting the problem and for interesting discutions, Mihai Maris for interesting discutions on uniqueness of minimizers, Petru Mironescu for sending me the paper [**27**], as well as Benoît Merlet for the numerical simulation that gives an idea of the profile of the minimizer (see page 18).

2. Function spaces and notations

The natural energy space for the Ginzburg-Landau energy has been studied in $\mathbb{R}^3$ by Patrick Gérard in [**12**] and [**13**]. We adapted the arguments to the case of an exterior domain of $\mathbb{R}^3$ in [**1**]. Here we recall the notations and the classical results needed in the sequel. We denote by

$$\dot{H}^1(\Omega) = \{u \in L^6(\Omega),\ \ \nabla u \in L^2(\Omega)\}.$$

Thanks to the classical Sobolev inequality in dimension 3 : $\|f\|_{L^6} \leq c\|\nabla f\|_{L^2}$, we know that $(\dot{H}^1(\Omega), \|\nabla \cdot \|_{L^2})$ defined above is a complete Hilbert space. Moreover, $C_0^\infty(\mathbb{R}^n)_{|\Omega} = C_0^\infty\left(\bar{\Omega}\right)$, the space of C^∞ function with compact support, restricted to Ω, is dense in $(\dot{H}^1(\Omega), \|\nabla \cdot \|_{L^2})$.
We denote by

$$\dot{H}_0^1(\Omega) = \{u \in L^6(\Omega),\ \ \nabla u \in L^2(\Omega),\ \ u_{|\partial\Omega} = 0\} \subset \dot{H}^1(\Omega)$$

the space of $\dot{H}^1(\Omega)$ functions that satisfy a Dirichlet boundary condition.
The classical trace Sobolev inequality reads in the 3D case as

$$\left(\int_{\partial\mathbb{R}^3_+} f(x)^4 dx\right)^{\frac{1}{4}} \leq c\left(\int_{\mathbb{R}^3_+} \|\nabla f(z)\|_*^2 dz\right)^{\frac{1}{2}}.$$

We have used the notation $\|\nabla f(z)\|_*^2 = \sum_{j=1}^3 |\partial_j f(z)|^2$. See e.g. Bruno Nazaret [**28**].
Let χ be a function in $C_0^\infty(\bar{\Omega})$ such that $\chi \equiv 1$ on an neighbourhood of the boundary of Ω.

Using a partition of unity, a straightening of the boundary and the previous Sobolev trace inequality, we prove the following Sobolev trace inequality for $\chi u \in \dot{H}^1(\Omega)$:

$$||u||_{L^4(\partial\Omega)} = ||\chi u||_{L^4(\partial\Omega)} \leq c||\chi u||_{H^1(\Omega)} = c(||\nabla(\chi u)||_{L^2(\Omega)} + ||\chi u||_{L^2(\Omega)}).$$

As χ is compactly supported, using Hölder inequality, we bound the right hand side term by

$$||\chi u||_{H^1(\Omega)} \leq c(||\nabla u||_{L^2(\Omega)} + ||\chi u||_{L^6(\Omega)}).$$

Thus we obtain

$$||u||_{L^4(\partial\Omega)} \leq c(||\nabla u||_{L^2(\Omega)} + ||u||_{L^6(\Omega)}) \leq c'||\nabla u||_{L^2(\Omega)}.$$

Therefore $(\dot{H}^1_0(\Omega), \|\nabla \cdot \|_{L^2})$ is a complete Hilbert space. Moreover $C_0^\infty(\Omega)$ is dense in $(\dot{H}^1_0(\Omega), \|\nabla \cdot \|_{L^2})$.
We also use the notation

$$\dot{H}^1_{-1}(\Omega) = \{u \in L^6(\Omega),\ \ \nabla u \in L^2(\Omega),\ \ u_{|\partial\Omega} = -1\} = \dot{H}^1_0(\Omega) - \chi,$$

for χ defined above.
In dimension 3 the energy space E has a simple structure, showed by P.Gérard [**12**] on $\mathbb{R}^3$ and that we adapted to the case of an exterior domain in [**1**] :

$$E = \{c(1+v),\ c \in \mathbb{C},\ |c| = 1,\ v \in \dot{H}^1_{-1}(\Omega),\ |v|^2 + 2Re(v) \in L^2(\Omega)\}.$$

It is a complete metric space for the distance δ_E :

$$\delta_E\left(c(1+v), \tilde{c}(1+\tilde{v})\right) = |c-\tilde{c}| + \|\nabla v - \nabla\tilde{v}\|_{L^2} + \||v|^2 + 2Re(v) - |\tilde{v}|^2 - 2Re(\tilde{v})\|_{L^2}.$$

3. Existence of a positive minimizer

To show the existence of a minimizer we proceed classically by considering a minimizing sequence and showing that it has a limit in the energy space.

Proposition 2. — *There exists a positive minimizer for the Ginzburg-Landau energy*

$$\mathcal{E}_{GL}(u) = \int_{\{|x|>1\}} \frac{1}{2}|\nabla u|^2(x) + \frac{1}{4}\left(|u|^2(x) - 1\right)^2 dx$$

in the energy space

$$E = \{u \in H^1_{loc}(\Omega),\ u_{|\partial\Omega} = 0,\ \nabla u \in L^2(\Omega),\ |u|^2 - 1 \in L^2(\Omega)\}.$$

Proof. — Let $E_0 = \inf_{u\in E} \mathcal{E}_{GL}(u)$ and let $(u_n) \subset E$ be a minimizing sequence. From the structure of the energy space we know that u_n is of the form

$$u_n = c_n(1 + v_n),$$

with $c_n \in \mathbb{C}$, $|c_n| = 1$ and $v_n \in \dot{H}^1_{-1}(\Omega)$. Up to a subsequence $c_n \to c$ and $\int |\nabla v_n|^2$ converge. As $\dot{H}^1(\Omega)$ is a Hilbert space, $\exists v \in \dot{H}^1(\Omega)$ and a subsequence of $(v_n)_n$, still noted $(v_n)_n$, such that $v_n \rightharpoonup v$ in $\dot{H}^1(\Omega)$. Moreover, for $\chi \in C_0^\infty(\bar{\Omega})$, $\chi \equiv 1$ on a neighbourhood of $\partial\Omega$ as above, for all n, $v_n + \chi \in \dot{H}^1_0(\Omega)$ and $v_n + \chi \rightharpoonup v + \chi$ in $\dot{H}^1(\Omega)$. As $\dot{H}^1_0(\Omega)$ is closed for the weak convergence we have $v + \chi \in \dot{H}^1_0(\Omega)$. As $v_n \rightharpoonup v$ in $\dot{H}^1(\Omega)$, it follows that

$$\int_\Omega |\nabla v|^2(x)dx \leq \liminf_n \int_\Omega |\nabla v_n|^2(x)dx. \tag{1}$$

Although it is a bit more involved, we show a similar inequality on the non-linear part of the energy :

$$\frac{1}{4}\left(|v|^2(x) + 2Re(v)\right)^2 = \frac{1}{4}(|v|^4 + 4|v|^2 Re(v) + (Re(v))^2).$$

Indeed, the Sobolev imbedding of $\dot{H}^1$ on $\Omega \subset \mathbb{R}^3$ is $\dot{H}^1(\Omega) \hookrightarrow L^6(\Omega)$ and is not sufficient to obtain an inequality on the non-linear term. We proceed as follows : let $R > 1$ and let $\Omega_R = \Omega \cap B(0, R)$. Then the Sobolev imbedding of $\dot{H}^1$ on Ω_R, bounded subset of $\mathbb{R}^3$, is compact for $2 \leq \sigma < 6$ and it reads :

$$\dot{H}^1(\Omega_R) \Subset L^\sigma(\Omega_R), \;\; 2 \leq \sigma < 6.$$

Thus, $v_n \rightharpoonup v$ in $\dot{H}^1(\Omega)$ implies that up to a subsequence (that can be made independ on R by a diagonal procedure),

$$\int_{\Omega_R} |v|^4+4|v|^2Re(v)+(Re(v))^2dx \leq \liminf_{n\to\infty} \int_{\Omega_R} |v_n|^4+4|v_n|^2Re(v_n)+4(Re(v_n))^2dx.$$

On the right hand side we can bound by the same integral on Ω, as the function we integrate is positive :

$$\int_{\Omega_R} |v|^4 + 4|v|^2Re(v) + (Re(v))^2dx \leq \liminf_{n\to\infty} \int_\Omega (|v_n|^2 + 2Re(v_n))^2dx.$$

The integral on the left hand side is increasing as R increases, while the right hand side is fixed. Thus the left hand side has a limit as $R \to +\infty$ and this limit is bounded by :

$$\int_\Omega |v|^4 + 4|v|^2Re(v) + (Re(v))^2dx \leq \liminf_{n\to\infty} \int_\Omega (|v_n|^2 + 2Re(v_n))^2dx \tag{2}$$

By adding inequalities (1) and (2) we obtain

$$\mathcal{E}_{GL}(1 + v) \leq \liminf_{n\to\infty} \mathcal{E}_{GL}(1 + v_n).$$

We recall that $u_n = c_n(1+v_n)$ is a minimizing sequence for the energy in the energy space and therefore $u = c(1+v)$ is in the energy space and moreover u is a minimizer : $\mathcal{E}_{GL}(u) = E_0$. This implies equality in the previous two inequalities (1) and (2). Thus $\nabla v_n \to \nabla v$ in $L^2(\Omega)$ and $|v_n|^2(x) + 2Re(v_n) \to |v|^2(x) + 2Re(v)$ in $L^2(\Omega)$. As a consequence convergence in distance δ holds: $\delta_E(c_n(1+v_n), c(1+v)) \to 0$ as $n \to \infty$, where
(3)
$\delta_E(c_n(1+v_n), c(1+v)) = |c-c_n| + \|\nabla v_n - \nabla v\|_{L^2} + \||v_n|^2 + 2Re(v_n) - |v|^2 - 2Re(v)\|_{L^2}$.

Remark 3. — *We have proved that when $u = c(1+v)$ is a minimizer, taking a minimizing sequence $u_n = c_n(1+v_n) \in E$ such that $\mathcal{E}_{GL}(u_n) \to \mathcal{E}_{GL}(u) = E_0$ there exists a constant $\tilde{c} \in \mathbb{C}$, $|\tilde{c}| = 1$, there exists $\tilde{v} \in \dot{H}^1_{-1}$ and a subsequence such that $\delta_E(c_n(1+v_n), \tilde{c}(1+\tilde{v})) \to 0$ as $n \to \infty$. This is an important fact that will automatically imply, once we prove the uniqueness of a positive minimizer, the stability of the minimizer (see section 6).*

The classical inequality $\|\nabla|u|\|_{L^2} \leq \|\nabla u\|_{L^2}$ (see e.g. [**21**] p164) implies that if u is a minimizer, then $|u|$ is also a minimizer.
This concludes the proof of existence of a positive minimizer. □

Remark 4. — *We have shown previously that if u is a minimizer than $|u|$ is also a minimizer. Because both u and $|u|$ are minimizers and $\|\nabla|u|\|_{L^2} \leq \|\nabla u\|_{L^2}$, then $\|\nabla|u|\|_{L^2} = \|\nabla u\|_{L^2}$. Therefore there exists a constant $c \in \mathbb{C}$ such that $u = c|u|$.*

4. Radial symmetry of minimizers

We follow a method by O.Lopes [**22**] and M.Maris [**24**], [**23**] of symmetrisation along a plane. Using the unique continuation principle we obtain :

Proposition 3. — *Let $u \in E$ be a positive minimizer of the energy $\mathcal{E}_{GL}(u)$ on $\Omega = \mathbb{R}^3 \backslash B(0,1)$. Then u has radial symmetry.*

Proof. — Let $\mathcal{P}$ be a plane of $\mathbb{R}^3$ passing through the origin. Let u_1 and u_2 equal u on each side of $\mathcal{P}$ and u_1 and u_2 be symmetric with respect to $\mathcal{P}$. Then u_1 and u_2 are in $\dot{H}^1(\Omega)$ and also in the energy space E. Moreover

$$\mathcal{E}_{GL}(u_1) + \mathcal{E}_{GL}(u_2) = 2\mathcal{E}_{GL}(u).$$

Therefore $u_1, u_2 \in E$ are also real positive minimizers of the energy. So they satisfy the equation

$$\triangle u = (u^2 - 1)u.$$

As $u \in E$ we know that $u \in L^6$ and therefore the r.h.s. term is in $L^2_{loc}(\Omega)$. Using the elliptic regularity of the laplacian we obtain that $u \in H^2_{loc}(\Omega) \subset L^\infty_{loc}(\Omega)$. The same argument also gives $u_1, u_2 \in H^2_{loc}(\Omega) \subset L^\infty_{loc}(\Omega)$.

Let $w = u - u_1$. Then w satisfies

$$\triangle w + A(x)w = 0,$$

for $A(x) = u^2(x) + u(x)u_1(x) + u_1^2(x) - 1$. As u, u_1 are in $L^\infty_{loc}(\Omega)$, then $A \in L^\infty_{loc}(\Omega)$. Let $\omega \subset \Omega$ be an open and bounded set embedded in the half space where $u = u_1$. Then $w_{|\omega} = 0$. By the unique continuation principle we obtain $w \equiv 0$. This means that u is symmetric with respect to $\mathcal{P}$.
As $\mathcal{P}$ was an arbitrary plane passing trough the origin, we conclude that u is radially symmetric. □

5. Uniqueness of the minimizer

This section is inspired by the classic paper of Kwong [**20**] on uniqueness of the ground state for Schrödinger equation (see also Coffman [**11**], Peletier-Serrin [**29**], McLeod et all [**26**], etc). Due to the radial symmetry, the profile of the minimizer solves an ODE with Dirichlet initial data and $\lim_{r\to\infty} u(r) = 1$. The idea of the proof is to use a **shooting method**, namely we change the condition at infinity by an initial condition $u'(1) = \alpha$ and then we show there exists only one value $\alpha > 0$ such that u is non-negative and the condition at infinity $\lim_{r\to\infty} u(r) = 1$ is satisfied.

$$\left\{ \begin{array}{rcl} u'' + \frac{2}{r}u' + u - u^3 & = & 0, \\ u(1) & = & 0, \\ u'(1) & = & \alpha. \end{array} \right. \tag{4}$$

Remark 5. — *Notice that if $u(r)$ converges to $c \geq 0$ as $r \to \infty$, then necessarily c cancels the nonlinearity, i.e. $c = 0$ or $c = 1$. Let's suppose this was not the case, for example $c^3 - c > 0$. We re-write the equation verified by u as $(ru)'' = r(u^3 - u)$. Then, there exists $c_+ > 0$ and $R_0 > 1$ such that, for $r > R_0$, $u^3 - u > c_+$ and therefore $(ru)'' > rc_+$. This easily implies that $u(r) \to +\infty$ as $r \to \infty$ which contradicts $u(r) \to c$.*

To obtain the uniqueness we do a topological analysis of the influence of $\alpha > 0$ on the behaviour of the corresponding solution u. We define the following sets :

$$S_+ = \{\alpha > 0 \text{ s.t. } \exists r_1 > 1, u(r_1) > 1 \text{ and } u(r) > 0 \ \forall 1 < r < r_1\}$$
$$S_1 = \{\alpha > 0 \text{ s.t. } \lim_{r\to\infty} u(r) = 1 \text{ and } u(r) > 0 \ \forall r > 1\}$$
$$S_0 = \{\alpha > 0 \text{ s.t. } \lim_{r\to\infty} u(r) = 0 \text{ and } u(r) > 0 \ \forall r > 1\}$$
$$S_- = \{\alpha > 0 \text{ s.t. } \exists r_2 > 1, u(r_2) = 0\}.$$

For the moment it is not obvious that $\mathbb{R}^*_+ = S_+ \cup S_1 \cup S_0 \cup S_-$, but the following remarks will ensure that.

Remark 6. — 1. *From the existence of a radial, positive minimizer seen in sections 3 and 4, we know that* S_1 *is nonempty.*

2. *From equation (4), one easily sees that the following situations cannot occur : a local maximum strictly larger than* 1*, a local minimum between* 0 *and* 1.
3. *Due to the uniqueness of the solution of (4), a solution cannot be tangent to* 0 *or to* 1*, which are stationary solutions to (4).*
4. *As the solution of (4) cannot have a local minimum between* 0 *and* 1*, we do not have solutions that oscillate while staying between* 0 *and* 1.
5. *If a solutions gets above* 1*, then it cannot return to* 1*, as in this case it would have a local maximum larger than* 1.
6. *Thus, the sets* S_+, S_1, S_0 *and* S_- *are* ***disjoints***.

From those remarks we have the alternate definition of S_- as :

$$S_- = \{\alpha > 0 \text{ s.t. } \exists r_2 > 1, u(r_2) < 0\}. \tag{5}$$

As there are no solutions oscillating between 0 and 1, we obtain that

$$\mathbb{R}_+^* = S_+ \cup S_1 \cup S_0 \cup S_-.$$

In section 5.2 we show that S_- and S_+ are open sets.

The uniqueness of the radial minimizer in theorem 1 follows from :

Proposition 4. — *There exists an unique* $\alpha_0 > 0$ *such that* u *is solution of* (4) *and* $\lim_{r\to\infty} u(r) = 1$, $u(r) > 0\ \forall r > 1$.

Remark 7. — *Proving proposition 4 is equivalent with proving* $S_1 = \{\alpha_0\}$.

The plan of the proof is as follows : we show that $S_0 = \emptyset$ in section 5.1. Then, in section 5.2, using a Lyapunov function, we obtain that $S_+ \subset \left(\frac{1}{\sqrt{2}}, +\infty\right)$ and S_+, S_- are open sets. Thanks to Sturm-Liouville comparison theorem we prove the monotony with respect to α of solutions with positive values in section 5.3. This fact, combined with the previous analysis, implies that $S_- = (0, \alpha_1)$, $S_+ = (\alpha_2, +\infty)$ and therefore $S_1 = [\alpha_1, \alpha_2]$. The uniqueness of the minimizer follows from $\alpha_1 = \alpha_2$, which we show by contradiction in section 5.5, using an idea from Mironescu [**27**] (see also Brezis-Oswald [**6**]). In order to make this argument work we show in section 5.4 that the minimizer converges exponentially to 1.

5.1. Proof of $S_0 = \emptyset$. —

Proposition 5.1. — *There exists no* $\alpha > 0$ *such that the corresponding solution to equation (4) stays positive* $u(r) > 0$ *for all* $r > 1$ *and* $\lim_{r\to\infty} u(r) = 0$. *In other words* $S_0 = \emptyset$.

Proof. — Let's suppose there exists $\alpha \in S_0$ and we call u the corresponding solution to equation (4). As the solution cannot have a local maximum strictly larger than 1 nor a local minimum between 0 and 1, then necessarily the function u increases to a value strictly less than 1, then it decreases to zero. We denote by $r_1 > 1$ the point of maximum, such that $u'(r_1) = 0$, $0 < u(r_1) < 1$, u is increasing on $(1, r_1)$ and decreasing to zero on $(r_1, +\infty)$. Notice that if there exists $r_2 > r_1$ such that $u''(r) \leq 0$ for all $r > r_2$ than one easily shows that $u(r) \to -\infty$ as $r \to \infty$. Therefore we can choose an $R > 0$ as large as we want such that $u''(R) > 0$.
Considering the derivative of (4), we obtain

$$u'''(r) + u'(r)\left(-\frac{6}{r^2} + 1 - u^2(r)\right) + \frac{2}{r}u(r)\left(u^2(r) - 1\right) = 0.$$

As $u(r) \to 0$ for $r \to \infty$, $u'(r) < 0$ for all $r > r_1$ and $0 < u(r) \leq u(r_1) < 1$ for all $r > 1$, by studding the sign of the terms in the previous equation, we obtain that $u'''(r) > 0$ for all r large enough. This, combined with $u''(R) > 0$ for some R arbitrarily large, implies that, for all $r > R$, $u''(r) > u''(R) > 0$. This implies a contradiction with $\lim_{r\to\infty} u(r) = 0$ and therefore $S_0 = \emptyset$.

□

Remark 8. — *The same proof gives a slightly more general result : if f is a $\mathcal{C}^1$ function such that $f'(0) > 0$ then the equation $u'' + \frac{2}{r}u' + f(u) = 0$ cannot have a decreasing, positive solution that converges to 0 at ∞.*

5.2. Proof of $S_+ \subset \left(\frac{1}{\sqrt{2}}, +\infty\right)$ and S_+, S_- open. — Define the Lyapunov function :

$$L(u)(r) := \frac{u'(r)^2}{2} + \frac{u(r)^2}{2} - \frac{u(r)^4}{4}.$$

Then $\partial_r(L(u))(r) = -\frac{2}{r}u'(r)^2$ so $r \mapsto L(u)(r)$ (strictly) decreasing along the trajectory. Notice that

$$L(u)(1) = \frac{\alpha^2}{2}.$$

If $\alpha \in S_+$ then $\exists r_1 > 1$ s.t. $u(r_1) = 1$. Then

$$L(u)(r_1) = \frac{u'(r_1)^2}{2} + \frac{1}{4} < \frac{\alpha^2}{2} = L(u)(1).$$

Therefore $\alpha > \frac{1}{\sqrt{2}}$ and thus $S_+ \subset \left(\frac{1}{\sqrt{2}}, +\infty\right)$.
Let us show that S_+ is an open set. By continuity of the flow with respect to the initial data, if $\exists \alpha_* > 0$ and $r_* > 1$ s.t. $u(r_*, \alpha_*) > 1$, there exists neighbourhood V of α_* and W of r_* such that

$$\forall \alpha \in V \text{ and } \forall r \in W \quad u(r, \alpha) > 1.$$

The question is : are all those $\alpha \in V$ in S_+ or there exists a solution that is negative at some point and then it gets above 1, in which case α would belong to S_-. The Lyapunov function being decreasing, this cannot happen. Indeed, let's suppose such a solution exists. Then the solution gets above 1 and then it has to "return" below/at 1. This translates as : there exists $r_0 > 1$ such that $u'(r_0) = 0$ and there exists $r_1 > r_0$ such that $u(r_1) = 1$. Then

$$L(u)(r_0) = \frac{u(r_0)^2}{2} - \frac{u(r_0)^4}{4} \leq \frac{1}{4} < L(u)(r_1) = \frac{u'(r_1)^2}{2} + \frac{1}{4}.$$

This contradicts the decreasing monotony of the Lyapunov function. Therefore if $\exists r_2 > 1$ such that $u(r_2) > 1$ then necessarily $u \in S_+$. The conclusion is that $V \subset S_+$ and so S_+ is open.
Using a similar argument we prove that S_- is open.

5.3. Monotony of solutions with respect to α.—

Proposition 5.2. — *For $0 < \alpha_1 < \alpha_2$, the corresponding solutions to equation (4) cannot intersect while having positive values.*

The main ingredient of the proof is a Sturm-Liouville comparison theorem. We use here the theorem as in the classical paper of Kwong [**20**], which is the version found on p.229 of INCE's classical book, Ordinary Differential Equation, Dover, 1956 [**16**].

Sturm-Liouville theorem 1. — *Let U and V be nontrivial solutions of*

$$U''(x) + f(x)U'(x) + g(x)U(x) = 0 \quad \forall x \in (a, b)$$

$$V''(x) + f(x)V'(x) + G(x)V(x) = 0 \quad \forall x \in (a, b)$$

where f, g and G are continous. Let (μ, ν) be a subinterval in which $U(x) \neq 0$ and $V(x) \neq 0$ and in which the comparison condition $G(x) \geq g(x)$ holds $\forall x \in (\mu, \nu)$. Suppose further that

$$\frac{V'(\mu)}{V(\mu)} \leq \frac{U'(\mu)}{U(\mu)}.$$

Then

$$\forall x \in (\mu, \nu) \quad \frac{V'(x)}{V(x)} \leq \frac{U'(x)}{U(x)}.$$

Proof. — Let $0 < \alpha_1 < \alpha_2$ and denote by $u_1(r) = u(r, \alpha_1)$ and $u_2(r) = u(r, \alpha_2)$ the corresponding solutions to equation (4). Then by uniqueness, u_1 verifies

$$\left\{ \begin{array}{rcl} u'' + \frac{2}{r}u' + (1 - u_1^2)u & = & 0, \\ u(1) = 0,\ u'(1) & = & \alpha_1. \end{array} \right.$$

and u_2 verifies

$$\left\{ \begin{array}{rcl} v'' + \frac{2}{r}v' + (1-u_2^2)v & = & 0, \\ v(1) = 0,\ v'(1) & = & \alpha_2. \end{array} \right.$$

Lets suppose that the solutions intersect while taking positive value and we denote by $r_0 > 1$ the first point of intersection : $r_0 > 1$ such that $\forall 1 < r < r_0$ $0 < u_1(r) < u_2(r)$ and $u_1(r_0) = u_2(r_0) > 0$. The Cauchy theory ensures that $u_2'(r_0) \neq u_1'(r_0)$ (or else $u_1 = u_2$). Since $0 < u_1(r) < u_2(r)$ and $u_1(r_0) = u_2(r_0) > 0$, then necessarily

$$u_2'(r_0) < u_1'(r_0).$$

This is an important fact that will help us prove the contradiction.

We want to apply Sturm-Liouville theorem for $U = u_2$ and $V = u_1$. Indeed, $G(r) = 1-u_1^2(r) > g(r) = 1-u_2^2(r)\ \forall r \in (1, r_0)$. Notice that $u_1(1) = u_2(1) = 0$ so we cannot decide if $\frac{V'(1)}{V(1)} \leq \frac{U'(1)}{U(1)}$. We restrict to some interval $(1+\epsilon, r_0)$. From the equation (4) we can compute $u''(1) = -2\alpha$, $u'''(1) = 5\alpha$, $u^{(4)}(1) = -20\alpha$, $u^{(5)}(1) = 106\alpha + 6\alpha^3$. The exact values of those constants do not matter, but rather the linear/nonlinear dependence on α. We obtain

$$\frac{u'(1+\epsilon)}{u(1+\epsilon)} = \frac{\alpha\left(1 - 2\epsilon + c_2'\epsilon^2 + c_3'\epsilon^3 + (c_4' + \frac{6\alpha^2}{4!})\epsilon^4 + o(\epsilon^4)\right)}{\alpha\epsilon\left(1 - \epsilon + c_3\epsilon^2 + c_4\epsilon^3 + (c_5 + \frac{6\alpha^2}{5!})\epsilon^4 + o(\epsilon^4)\right)}.$$

Recall that we look for the monotony of this function with respect to α. The derivatives of u at 1 depend linearly on α up to order 4 and therefore the dependence on α of $\frac{u'(1+\epsilon)}{u(1+\epsilon)}$ cannot be tracked. That's why we need to do an expansion up to order 5 of u solution of (4) in order to have the non-linear dependence on α of u.

The function $\alpha \mapsto \frac{A + \frac{6\alpha^2}{4!}\epsilon^4 + o(\epsilon^4)}{B + \frac{6\alpha^2}{5!}\epsilon^4 + o(\epsilon^4)}$ is increasing iff $\frac{6}{5!}(A + o(\epsilon)) < \frac{6}{4!}(B + o(\epsilon))$, which is true for ϵ small enough, as $A = 1 + o(1)$ and $B = 1 + o(1)$. Therefore $\exists 0 < \epsilon < r_0 - 1$ such

$$\frac{u_1'(1+\epsilon)}{u_1(1+\epsilon)} \leq \frac{u_2'(1+\epsilon)}{u_2(1+\epsilon)}.$$

By Sturm theorem $\frac{u_1'(r)}{u_1(r)} \leq \frac{u_2'(r)}{u_2(r)}$ for $r \in (1+\epsilon, r_0)$. Or $u_1(r_0) = u_2(r_0) > 0$ so we obtain by continuity $u_1'(r_0) \leq u_2'(r_0)$. This contradicts $u_2'(r_0) < u_1'(r_0)$ and the conclusion of the monotony of u with respect to α follows.

□

Remark 9. — *This monotony property has several very important consequences : if $\alpha_0 \in S_+$, the corresponding solution is strictly positive and gets above 1. Therefore, $\forall \alpha > \alpha_0$, the corresponding solution will be strictly*

positive and will get above 1*, that is,* $\alpha \in S_+$*. Moreover* S_+ *is open, thus* $S_+ = (\alpha_2, +\infty)$*. Similarly* $S_- = (0, \alpha_1)$ *and then necessarily* $S_1 = [\alpha_1, \alpha_2]$*.*

5.4. Exponential convergence to 1 of radial minimizers. — The method we follow is inspired by Peletier-Serrin [**29**].

Proposition 5.3. — *Let* $\alpha \in S_1$ *and denote by* u *the corresponding solution to* (4)*. Then*

$$\lim_{r\to\infty} \frac{u'}{1-u} = \sqrt{2}.$$

Moreover, for $0 < \epsilon < \sqrt{2}$*, we have*

$$\limsup_{r\to\infty} (1-u(r))e^{(\sqrt{2}-\epsilon)r} < +\infty.$$

Proof. — We define

$$z = \frac{u'}{1-u}.$$

This is a strictly positive function for all $r > 1$, as $u' > 0$ and $0 < u < 1$. We show that z is bounded using a standard argument that if z is bounded by a certain constant than it remains bounded by that constant. Indeed, z verifies

$$z' = z^2 - \frac{2}{r}z - (u^2 + u).$$

Since $0 < u < 1$, then, $\forall r > 1$, $z' > z^2 - 2z - 2 = \frac{1}{2}z^2 + \left(\frac{1}{2}z^2 - 2z - 2\right)$. We look at the sign of the second term : the equation $\frac{1}{2}z^2 - 2z - 2 = 0$ has roots $2 \pm 2\sqrt{2}$. If for some $r_0 > 1$ the function z gets above $2 + 2\sqrt{2}$ then $\frac{1}{2}z^2 - 2z - 2 > 0$ and the derivative of z will be positive and so $z(r)$ stays above $2 + 2\sqrt{2}$ for all $r \geq r_0$. Then $z' > \frac{1}{2}z^2$ which implies that it exists a constant $c \in \mathbb{R}$ such that for all $r > r_0$, $-\frac{1}{z(r)} > \frac{r}{2} + c$. This contradicts the fact that $z(r) > 0$ for all $r > 1$. Therefore $\forall r > 1$, $0 < z(r) \leq 2 + 2\sqrt{2}$, i.e. z is bounded.
Then by l'Hôpital rule

$$\lim_{r\to\infty} \left(\frac{u'}{1-u}\right)^2 = \lim_{r\to\infty} \frac{-u''}{1-u} = \lim_{r\to\infty} \frac{2}{r}z + (1+u)u = 2.$$

Therefore $\lim_{r\to\infty} \frac{u'}{1-u} = \sqrt{2}$ and we deduce $\forall \epsilon > 0$

$$\limsup_{r\to\infty} (1-u(r))e^{(\sqrt{2}-\epsilon)r} < +\infty. \tag{6}$$

□

5.5. Proof of $S_1 = \{\alpha\}$. — As seen in remark 9, $S_1 = [\alpha_1, \alpha_2]$. We show that $S_1 = \{\alpha\}$ by contradiction, using a very clever idea from Mironescu [**27**] (see also Brezis-Oswald [**6**]). Indeed, he proves uniqueness for the minimizer of the Ginzburg-Landau energy on **bounded** domains without any assumptions on the geometry of the domain. The method consists in supposing there are two solutions, consider their **quotient**, doing an integration by part for a well chosen function and obtaining a side which is positive and one which is negative. Unfortunately, we could only justify similar computations in the **radial** case, using the exponential convergence to 1 of the solution, as r tends to infinity, seen in (6). The problems come from justifying integrations by parts for functions with non-zero boundary conditions at infinity.

Proof. — Suppose $\alpha_1, \alpha_2 \in S_1$ and $\alpha_1 < \alpha_2$. Then, by definition of S_1 and by monotonicity property with respect to α, we have that

$$\forall r > 1, \quad 0 < u_1(r) < u_2(r) < 1.$$

Let

$$w(r) = \frac{u_2(r)}{u_1(r)}.$$

Then

$$div(u_1^2 \nabla w) = u_1 \triangle u_2 - u_2 \triangle u_1 = u_1 u_2 (u_2^2 - u_1^2).$$

Therefore

$$\int_\Omega div(u_1^2 \nabla w)(w-1)dx = \int_\Omega u_1 u_2 (u_2^2 - u_1^2) \left(\frac{u_2}{u_1} - 1 \right) dx > 0. \tag{7}$$

On the other hand, by IBP on $\Omega_R = \Omega \cap B(0,R)$:

$$\int_{\Omega_R} div(u_1^2 \nabla w)(w-1)dx = \int_{\Omega_R} -u_1^2 |\nabla w|^2 dx + \int_{\mathbb{S}^2_R} u_1^2 \nabla w (w-1) \frac{x}{R} d\sigma_R(x). \tag{8}$$

Notice that the boundary term on $|x| = 1$ is equal to 0. The equations (7) and (8) do not use the radial symmetry of the solutions. We want to show that the last integral in (8) tends to 0 as R tends to infinity. We rewrite it as

$$\int_{\mathbb{S}^2} (u_1(R\cdot)\partial_r u_2(R\cdot) - u_2(R\cdot)\partial_r u_1(R\cdot)) \left(\frac{u_2(R\cdot)}{u_1(R\cdot)} - 1 \right) R^2 d\sigma(x).$$

We use the radial symmetry of the minimizers to get rid of the dot and also of the integral. Thus, we only need to prove the convergence to zero of $(u_1(R)\partial_r u_2(R) - u_2(R)\partial_r u_1(R)) \left(\frac{u_2(R)}{u_1(R)} - 1 \right) R^2$. We use the convergence to 1 of the radial solutions to prove that the terms in the first parenthesis tend to 0 :

$$u_1 \partial_r u_2 = u_1 \frac{u_2'}{1-u_2}(1-u_2) \to_{r\to\infty} 1 \times \sqrt{2} \times 0 = 0.$$

From (6) we know there exists $M > 0$ such that for all R large enough :

$$0 < (1 - u_2(R))\, e^{(\sqrt{2}-\epsilon)R} < M$$

and similarly for u_1. Therefore

$$|u_2(R) - u_1(R)|\, R^2 < M \frac{R^2}{e^{(\sqrt{2}-\epsilon)R}}.$$

We obtain thus $(u_2(R) - u_1(R))R^2 \to_{r\to\infty} 0$.
Therefore we can pass to the limit $R \to \infty$ in equation (8) :

$$\int_\Omega div(u_1^2 \nabla w)(w-1)dx = \int_\Omega -u_1^2 |\nabla w|^2 dx < 0. \tag{9}$$

Putting together equations (7) and (9), we obtain a sign contradiction :

$$0 < \int_\Omega u_2(u_2+u_1)(u_2-u_1)^2 dx = \int_\Omega div(u_1^2\nabla w)(w-1)dx = \int_\Omega -u_1^2|\nabla w|^2 dx < 0.$$

The conclusion is that necessarily $\alpha_1 = \alpha_2$ and thus there is only one positive minimizer. □

Remark 10. — *If one can get rid of the boundary term in equation* (8) ***without*** *using the radial symmetry, then one obtains uniqueness in a much more general setting.*

For completeness, we introduce here some pictures, due to Benoit Merlet, showing the approximate value of α corresponding to the minimizer is $\alpha \approx 1.8781$ (see page 18).

6. Stability

We denote by Q the unique positive minimizer of the Ginzburg-Landau energy

$$\mathcal{E}_{GL}(u) = \int_\Omega \frac{1}{2}|\nabla u|^2(x) + \frac{1}{4}\left(|u|^2(x) - 1\right)^2 dx.$$

The Ginzburg-Landau energy is also the preserved energy for the Gross-Pitaevskii equation :

$$(GP) \begin{cases} i\partial_t u + \triangle u &= (|u|^2 - 1)u, \ on\ \mathbb{R}\times\Omega \\ u_{|_{t=0}} &= u_0, \ on\ \Omega \\ u_{|_{\mathbb{R}\times\partial\Omega}} &= 0. \end{cases}$$

After proving existence and uniqueness for the minimizer of the Ginzburg-Landau energy, a natural question is to study the stability of the minimizer under the Gross-Pitaevskii flow. This question is similar to the stability of the ground state for the NLS. As expected, the minimizer is stable under the Gross-Pitaevskii flow and this stability is stronger than the orbital stability.

Theorem 5. — *The minimizer Q is stable under the flow of the Gross-Pitaevskii equation, more precisely : $\forall \epsilon > 0$ $\exists \gamma > 0$ s.t. $\forall u_0 \in E$, $\delta_E(u_0, Q) < \gamma$ then*

$$\sup_t \delta_E(u(t), Q) \leq \epsilon.$$

Remark 11. — *Notice that we have a stability which is stronger than the orbital stability, one reason being the structure of the energy space : if the initial data $u_0 = c_0(1 + v_0)$ is close to the minimizer Q in distance δ_E, than c_0 is close to 1. The constants do not change under the Gross-Pitaevskii flow (see for example* [**12**]*, proposition 4.4 and* [**1**]*). Therefore if the solutions stay close, they do so without the need to consider an orbit.*

We recall the notation

$$\delta_E(u, Q) = |c(u) - 1| + \|\nabla u - \nabla Q\|_{L^2} + \||u|^2 - Q^2\|_{L^2}$$

Proof. — We argue by contradiction : suppose $\exists \epsilon > 0$ s.t. $\forall \gamma_n = \frac{1}{n}$, $\exists u_{0,n} \in E$ and $t_n > 0$ s.t.

$$\delta_E(u_{0,n}, Q) < \frac{1}{n} \text{ and } \delta_E(u_n(t_n), Q) \geq \epsilon.$$

The first inequality implies that $c(u_{0,n}) \to 1$, $\|\nabla u_{0,n}\|_{L^2} \to \|\nabla Q\|_{L^2}$ and $\||u_{0,n}|^2 - 1\|_{L^2} \to \|Q^2 - 1\|_{L^2}$. Thus, $(u_{0,n})$ is a minimizing sequence. As the energy is preserved in time, we also have that $(u_n(t_n))$ is a minimizing sequence. From section 3, remark 3 , we know that there exists a minimizer u and a subsequence of $(u_n(t_n))_n$ that convergences to this minimizer in distance. We thus have that $c(u) = 1$.
The uniqueness of a positive minimizer combined with remark 4 implies there exists a unique minimizer such that $c(u) = 1$ and therefore $u = Q$. Therefore $\delta_E(u_n(t_n), Q) \to 0$, which contradicts $\delta_E(u_n(t_n), Q) \geq \epsilon$. The conclusion of stability follows. □

Remark 12. — ***Scattering*** *- another natural question would be to study the scattering of the solution starting close to the minimizer. In the case of $\mathbb{R}^3$ this was studied by Gustafson-Nakanishi-Tsai* [**14, 15**]*. The classical method is to write $u = Q + v$. Therefore v satisfies the equation*

$$i\partial_t v + (\triangle - Q^2 + 1)v = (|v|^2 + 2QRe(v))v + |v|^2 Q + 2Q^2 Re(v).$$

Notice that the last term is also a linear term in v and a diagonalisation procedure takes it into account. We denote $v_1 = Re(v)$, $v_2 = Im(v)$ and $w = Uv_1 + iv_2$, where $U = (Q^2 - 1 - \triangle)^{\frac{1}{2}}(3Q^2 - 1 - \triangle)^{-\frac{1}{2}}$. Then w satisfies, at least formally, the equation

$$i\partial_t w - Hw = F(w),$$

where $H = (Q^2 - 1 - \triangle)^{\frac{1}{2}}(3Q^2 - 1 - \triangle)^{\frac{1}{2}}$ *and* $F(w)$ *is the non-linear term of the equation. On* $\mathbb{R}^3$ *the equivalent of the operator* H *is* $H_{\mathbb{R}^3} = (-\triangle)^{\frac{1}{2}}(2 - \triangle)^{\frac{1}{2}}$, *which corresponds with replacing "Q" by "1" (the minimizer of the energy on* $\mathbb{R}^3$ *is equal to* 1*). On* Ω *the operators* U *and* H *are not well defined for low frequencies and* r *close to* 1*. In order to study this equation on* $\mathbb{R}^3$*, Gustafson-Nakanishi-Tsai prove a Strichartz inequality for* $e^{itH_{\mathbb{R}^3}}$*. Thanks to the work of Ivanovici* [**17**] *and Ivanovici-Lebeau* [**18**] *on Strichartz estimates for the Schrodinger and wave equations and Burq's work* [**7**] *on the regularising effect, one can hope for a Strichartz inequality for* e^{itH} *on* Ω*, but the difficulties of the zero frequency combined with the variable coefficients of the operator make it for a rather technical analysis. Moreover, in the non-linear part* $F(w)$ *there are quadratic terms that need to be carefully analysed. We postpone this analysis.*

References

[1] Ramona Anton. Global existence for defocusing cubic NLS and Gross-Pitaevskii equations in three dimensional exterior domains. *J. Math. Pures Appl. (9)*, 89(4):335–354, 2008.

[2] Fabrice Bethuel, Haïm Brezis, and Frédéric Hélein. *Ginzburg-Landau vortices*, volume 13 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1994.

[3] Fabrice Béthuel, Philippe Gravejat, and Jean-Claude Saut. Travelling waves for the Gross-Pitaevskii equation. II. *Comm. Math. Phys.*, 285(2):567–651, 2009.

[4] Fabrice Bethuel, Philippe Gravejat, and Didier Smets. Asymptotic stability in the energy space for dark solitons of the Gross-Pitaevskii equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(6):1327–1381, 2015.

[5] Fabrice Bethuel and Jean-Claude Saut. Travelling waves for the Gross-Pitaevskii equation. I. *Ann. Inst. H. Poincaré Phys. Théor.*, 70(2):147–238, 1999.

[6] Haïm Brezis and Luc Oswald. Remarks on sublinear elliptic equations. *Nonlinear Anal.*, 10(1):55–64, 1986.

[7] N. Burq. Smoothing effect for Schrödinger boundary value problems. *Duke Math. J.*, 123(2):403–427, 2004.

[8] P.H. Roberts C.A. Jones, S.J. Putterman. Motions in a bose condensate v. stability of solitary wave solutions of nonlinear schrödinger equations in two and three dimensions. *J. Phys. A, Math. Gen.*, 19:2991–2011, 1986.

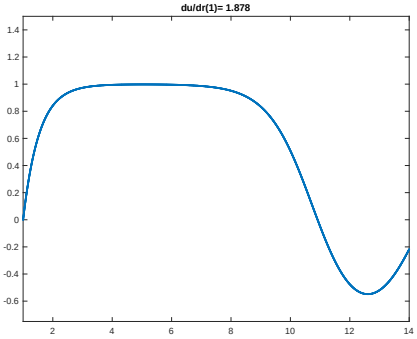
[9] David Chiron. Travelling waves for the Gross-Pitaevskii equation in dimension larger than two. *Nonlinear Anal.*, 58(1-2):175–204, 2004.

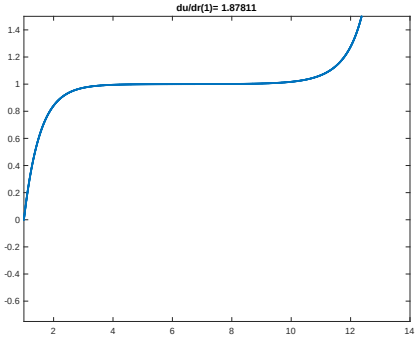
[10] David Chiron and Mihai Mariş. Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity. *Arch. Ration. Mech. Anal.*, 226(1):143–242, 2017.

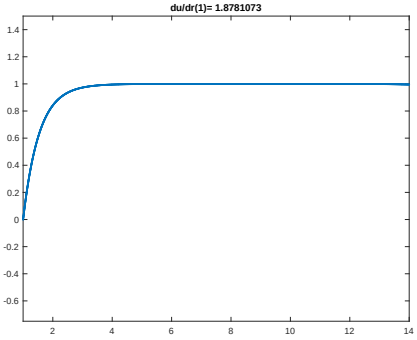
[11] Charles V. Coffman. Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions. *Arch. Rational Mech. Anal.*, 46:81–95, 1972.

[12] P. Gérard. The Cauchy problem for the Gross-Pitaevskii equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(5):765–779, 2006.
[13] Patrick Gérard. The Gross-Pitaevskii equation in the energy space. In *Stationary and time dependent Gross-Pitaevskii equations*, volume 473 of *Contemp. Math.*, pages 129–148. Amer. Math. Soc., Providence, RI, 2008.
[14] Stephen Gustafson, Kenji Nakanishi, and Tai-Peng Tsai. Scattering for the Gross-Pitaevskii equation. *Math. Res. Lett.*, 13(2-3):273–285, 2006.
[15] Stephen Gustafson, Kenji Nakanishi, and Tai-Peng Tsai. Scattering theory for the Gross-Pitaevskii equation in three dimensions. *Commun. Contemp. Math.*, 11(4):657–707, 2009.
[16] E. L. Ince. *Ordinary Differential Equations*. Dover Publications, New York, 1944.
[17] Oana Ivanovici. On the Schrödinger equation outside strictly convex obstacles. *Anal. PDE*, 3(3):261–293, 2010.
[18] Oana Ivanovici and Gilles Lebeau. Dispersion for the wave and the Schrödinger equations outside strictly convex obstacles and counterexamples. *C. R. Math. Acad. Sci. Paris*, 355(7):774–779, 2017.
[19] C.A. Jones and P.H. Roberts. Motions in a bose condensate iv. axisymmetric solitary waves. *J. Phys. A, Math. Gen.*, 15:2599–2619, 1982.
[20] Man Kam Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.*, 105(3):243–266, 1989.
[21] Elliott H. Lieb and Michael Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.
[22] Orlando Lopes. Radial symmetry of minimizers for some translation and rotation invariant functionals. *J. Differential Equations*, 124(2):378–388, 1996.
[23] Orlando Lopes and Mihai Mariş. Symmetry of minimizers for some nonlocal variational problems. *J. Funct. Anal.*, 254(2):535–592, 2008.
[24] Mihai Mariş. On the symmetry of minimizers. *Arch. Ration. Mech. Anal.*, 192(2):311–330, 2009.
[25] Mihai Mariş. Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity. *Ann. of Math. (2)*, 178(1):107–182, 2013.
[26] Kevin McLeod, W. C. Troy, and F. B. Weissler. Radial solutions of $\Delta u + f(u) = 0$ with prescribed numbers of zeros. *J. Differential Equations*, 83(2):368–378, 1990.
[27] Petru Mironescu. Uniqueness of solutions of the Ginzburg-Landau equation. chapter in Habilitation à diriger des recherches, 2000.
[28] Bruno Nazaret. Best constant in Sobolev trace inequalities on the half-space. *Nonlinear Anal.*, 65(10):1977–1985, 2006.
[29] L. A. Peletier and James Serrin. Uniqueness of positive solutions of semilinear equations in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.*, 81(2):181–197, 1983.
[30] Lev Pitaevskii and Sandro Stringari. *Bose-Einstein condensation*, volume 116 of *International Series of Monographs on Physics*. The Clarendon Press, Oxford University Press, Oxford, 2003.

Ramona Anton, Sorbonne Université and Université de Paris, CNRS, IMJ-PRG, F-75006 Paris, France • *E-mail :* ramona.anton@imj-prg.fr


du/dr(1)= 1.878
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
2
4
6
8
10
12
14


du/dr(1)= 1.87811
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
2
4
6
8
10
12
14


du/dr(1)= 1.8781073
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
2
4
6
8
10
12
14