



HAL
open science

Fourier series analysis of the (pseudodifferential) Dirichlet to Neumann operator for a layer of dielectric material

Olivier Lafitte

► **To cite this version:**

Olivier Lafitte. Fourier series analysis of the (pseudodifferential) Dirichlet to Neumann operator for a layer of dielectric material. 2020. hal-02969222v1

HAL Id: hal-02969222

<https://hal.science/hal-02969222v1>

Preprint submitted on 16 Oct 2020 (v1), last revised 14 Dec 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fourier series analysis of the (pseudodifferential) Dirichlet to Neumann operator for a layer of dielectric material

Olivier Lafitte^{a,b}

^aLAGA, Institut Galilée, Université Paris 13, 99 avenue J.B. Clément, F-93430 Villetaneuse

^bCRM, UMI3457, Université de Montréal

Contents

1	Introduction	2
2	General results	5
3	Exact resolution of the Calderòn problem for a ring and an elliptic ring using special functions	6
3.1	The case of the infinite cylindrical ring	6
3.2	The exact representation of solutions of the Helmholtz equation for elliptic coordinates	8
4	Analysis of the asymptotic regimes for cylindrical coordinates	10
4.1	Estimates on $\frac{d_p(k_3r)}{d_p(k_3R)}$ using asymptotic results.	10
4.2	Bessel functions toolbox	10
4.3	Eikonal equations.	16
4.4	Estimates on the solution	19
5	The Dirichlet to Neumann operator	22
5.1	Proof of the result of the Introduction for the plane boundary	22
5.2	The Dirichlet to Neumann operator for the cylindrical layer	23
5.3	High frequency expansion of the DTN Fourier multiplier for a fixed mode in the hyperbolic regime	24
5.4	High frequency analysis for the exact solution in cylindrical coordinates in the high frequency regime in (k_z, p) : the hyperbolic region.	26
5.5	Dirichlet to Neumann operator for the mixed- hyperbolic-elliptic case	28
5.6	Dirichlet to Neumann operator in the elliptic region	30
6	Dirichlet to Neumann operator for elliptic layers	31
6.1	Calderon operator	31
6.2	Dirichlet to Neumann operator for an elliptic-type layer with same focal points for the two boundaries	32
6.3	Elliptical cylinder with homothetic boundaries	37
7	The Calderòn operator for a cylindrical layer in 3D for the Maxwell equations	40

8 Annex	45
8.1 Pseudodifferential discrete operators	45
8.2 Proof of Lemma 2	46

1. Introduction

One considers a general cylinder $\Omega = \Theta \times (-\infty, +\infty)$, where Θ^c is characterized by $\{n > 0, M(s) + n\vec{n}(s)\}$, $M(s)$ describes, for $s \in [0, L]$, a closed convex curve, characterized by its curvilinear absciss s and the unit outgoing normal vector $\vec{n}(s)$. The body Θ is constituted of a perfectly conducting core supplemented by a layer of dielectric material $\{-l < n < 0\}$. The Helmholtz operator inside the layer¹ is

$$\frac{1}{1 + \kappa(s)n} \frac{\partial}{\partial s} \left[\frac{1}{1 + \kappa(s)n} \frac{\partial u}{\partial s} \right] + \frac{1}{1 + \kappa(s)n} \frac{\partial}{\partial n} \left[(1 + \kappa(s)n) \frac{\partial u}{\partial n} \right] + (\omega^2 \epsilon \mu - k_z^2) u = 0. \quad (1)$$

A similar problem has been already studied in [9], using microlocal analysis to get the impedance boundary condition, in the case of a bounded convex obstacle. The microlocal analysis gave the exact behavior, in term of the size of the layer, in the high frequency regime, using a local approximation of the surface, which enabled the author to obtain the diffracted wave. Theorem 1 p 1042 of [9] gives the leading order term of the impedance operator, which corresponds to the 'tangent plane approximation'.

Throughout this paper, we assume (by analogy with the classical conductivity approach where $\epsilon' = \epsilon_0 - i\frac{\sigma}{\omega}$) that

$$\Im \epsilon \mu < 0. \quad (2)$$

This convention of sign comes from the time convention of a solution with $e^{i\omega t}$. We treat (as in [9]) the case where $\Im \epsilon \mu$ does not depend on ω . We also consider the case where $\omega \Im \epsilon \mu$ is independent on ω .

it was obtained in [9] the following approximate relation $\partial_n u = Op(\omega \frac{\sqrt{\epsilon \mu - g_{11}(x)\eta_2^1 - 2g_{12}(x)\eta_1\eta_2 - g_{22}(x)\eta_2^2}}{\tan l\omega \sqrt{\epsilon \mu - g_{11}(x)\eta_2^1 - 2g_{12}(x)\eta_1\eta_2 - g_{22}(x)\eta_2^2}} + r)(u)$, where g is the metric induced by the Euclidean metric on the boundary $\partial\Omega$. It is then straightforward to deduce from this result, under the assumption (2), that $\tan l\omega \sqrt{\epsilon \mu - g_{11}(x)\eta_2^1 - 2g_{12}(x)\eta_1\eta_2 - g_{22}(x)\eta_2^2} \rightarrow -i$, exponentially, hence the leading order term of the Dirichlet to Neumann operator is a pseudodifferential operator of principal symbol $i\omega \sqrt{\epsilon \mu - g_{11}(x)\eta_2^1 - 2g_{12}(x)\eta_1\eta_2 - g_{22}(x)\eta_2^2}$.

We are interested in this paper on the behavior of the lower order term of the impedance operator (in the scalar case) for measuring the effect of the curvature of the body in the high frequency regime, that is the leading order term of r , which is of order 0 in ω .

For simplicity, we restrict ourselves to an infinite cylinder or ellipse, for which one does not have, in the case $\epsilon \mu \in \mathbb{R}_+^*$, any solution of the problem (see Lemma 1) in this Introduction for an explanation), but we can also consider a cylinder of finite length $\Theta \times [0, L]$ with periodic boundary conditions in z for which $k_z \in \frac{2\pi}{L}\mathbb{Z}$.

Using the ideas developed in the thesis of P. Payen, along with results of B. Stupfel, where, for a cylinder, the solution was expressed in terms of Bessel functions (hence taking into account the polar coordinates system), we derive here, using Fourier modes in the θ variable (with θ denoting the Euler angle, see [2] for example), a more complete version of the Calderòn operator.

The main feature of this paper is to consider, in the case of cylindrical coordinates, k_z and $\frac{p}{R}$ as **high frequency variables**, and to consider as well the case where the waves are evanescent, meaning that we

¹which is the operator acting on the components of the electromagnetic field for the harmonic Maxwell's equations

consider the division in elliptic, hyperbolic, and glancing regions for the high frequency Helmholtz operator.

In this Introduction, we recall the classical case of a plane layer $B = \mathbb{R}^2 \times [-l, 0]$ to outline the main features of the cases we shall look at. We will construct uniquely the solution of a boundary problem in this layer, and use this solution to derive a relation between u and $\partial_{x_3} u$ on $x_3 = 0$. The first notion is the notion of **resonances**.

- Lemma 1.** 1. *The resonances of $-\Delta$ in B are the values of ω such that there exists $(k_1, k_2, n) \in \mathbb{R}^2 \times \mathbb{N}^*$ such that $\omega^2 \epsilon \mu = k_1^2 + k_2^2 + \frac{n^2 \pi^2}{\rho^2}$. There are no resonances if and only if $\epsilon \mu \notin \mathbb{R}_+^*$.*
2. *If one replaces B by its finite counterpart $[0, L_1] \times [0, L_2] \times [-l, 0]$, and if one assumes $\epsilon \mu \in \mathbb{R}_+^*$, then the resonances are the values of (n_1, n_2) such that $\omega^2 \epsilon \mu = n_1^2 \frac{4\pi^2}{L_1^2} + n_2^2 \frac{4\pi^2}{L_2^2} + \frac{n^2 \pi^2}{\rho^2}$.*

Note that there is no resonance when $\epsilon \mu \notin \mathbb{R}$. In this case, for all $(k_1, k_2) \in \mathbb{R}^2$, denote by

$$k_{\perp} = \sqrt{\omega^2 \epsilon \mu - k_1^2 - k_2^2}, \Im k_{\perp} < 0. \quad (3)$$

One checks, in the case $\Im \epsilon \mu < 0$ independent on ω , that there exists $\alpha_0 > 0$ such that

$$\forall (k_1, k_2) \in \mathbb{R}^2, \Im k_{\perp} \leq -\omega \alpha_0. \quad (4)$$

One notices as well that the case $k_1^2 + k_2^2 > \Re \epsilon \mu \omega^2$ is covered in this expression. Note that in this case, $k_{\perp} = a_*(k_1, k_2) + ib_*(k_1, k_2)$, where $2a_* b_* = \Im \epsilon \mu \omega^2$, and $0 < a_*(k_1, k_2) = (\frac{1}{2}(|\epsilon \mu \omega^2 - k_1^2 - k_2^2|^2 + \Re \epsilon \mu \omega^2 - k_1^2 - k_2^2))^{\frac{1}{2}}$, a_* goes to 0 when $k_1^2 + k_2^2 \rightarrow +\infty$ and $b_* = O(\sqrt{k_1^2 + k_2^2})$.

In the case $\epsilon \mu \in \mathbb{R}_+^*$, we adopt the same notation for the equivalent following quantity:

$$k_{\perp} = \sqrt{\omega^2 \epsilon \mu - k_1^2 - k_2^2}, \omega^2 \epsilon \mu - k_1^2 - k_2^2 \geq 0, i \sqrt{k_1^2 + k_2^2 - \omega^2 \epsilon \mu}, k_1^2 + k_2^2 - \omega^2 \epsilon \mu > 0,$$

but, evidently, (4) is not true anymore.

- Lemma 2.** 1. *Assume $\epsilon \mu \in \mathbb{R}_+^*$. Provided that $\omega^2 \epsilon \mu$ is not a resonance of $-\Delta$ on B , for all $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$, there exists a unique solution in B of*

$$\begin{cases} (\Delta + \omega^2 \epsilon \mu)u = 0 \\ u(., ., -l) = 0 \\ u(., ., 0) = u_0. \end{cases} \quad (5)$$

- *The quantity $\omega^2 \epsilon \mu$ is always an element of the spectrum of $-\Delta$ (continuous spectrum).*
- *However, fix (k_1, k_2) and consider now the ODE on $[-l, 0]$. Assume $\epsilon \mu \omega^2 - k_1^2 - k_2^2 > 0$. Denote by $s_0 = \min_{n \in \mathbb{Z}} |\frac{\pi}{l} \sqrt{\epsilon \mu \omega^2 - k_1^2 - k_2^2} - n|$,*

$$|\hat{u}(k_1, k_2, x_3)| \leq \frac{1}{\sin \pi s_0}.$$

This case (and similar cases) will be called throughout the paper the hyperbolic case. It is the case where a wave can propagate inside the layer without attenuation.

- *Similarly, if we fix (k_1, k_2) and if $\epsilon \mu \omega^2 - k_1^2 - k_2^2 < 0$, $|\hat{u}(k_1, k_2, x_3)| \leq 1$. This case can be called the elliptic case (there is no $k_3 \in \mathbb{R}$ such that $\epsilon \mu \omega^2 = k_1^2 + k_2^2 + k_3^2 = 0$.)*

2. For $\epsilon\mu \notin \mathbb{R}_+^*$, u satisfies the pointwise estimate:

$$|\hat{u}(k_1, k_2, x_3)| \leq \frac{2}{1 - e^{-\omega\alpha_0}}.$$

3. There exists a linear operator \mathcal{DTN} , called the Dirichlet to Neumann operator, such that

$$\partial_{x_3} u(\cdot, \cdot, 0) = \mathcal{C}(u(\cdot, \cdot, 0)).$$

It is given by

$$\partial_{x_3} \hat{u}(k_1, k_2, 0) = ik_\perp \frac{1 + e^{-2ik_\perp l}}{1 - e^{-2ik_\perp l}} \hat{u}(k_1, k_2, 0) = ik_\perp \frac{1 + e^{-2ia_* l + 2b_* l}}{1 - e^{-2ia_* l + 2b_* l}}.$$

The Dirichlet to Neumann operator is characterized by the Fourier multiplier

$$\frac{k_\perp}{\tan(k_\perp l)} = l \frac{\cos k_\perp l}{\text{sinc}_\perp l} = \frac{ik_\perp}{\tanh(b_* - ia_*)l} = \frac{ib_* - a_*}{\tanh(b_* - ia_*)l} = (ib_* - a_*) \frac{\cosh(b_* - ia_*)l}{\sinh(b_* - ia_*)l}, \quad (6)$$

the latter equality being valid also for $k_\perp = 0$. Note that, when $\omega \rightarrow +\infty$, the Fourier multiplier satisfies

$$ik_\perp \frac{1 + e^{-2ik_\perp l}}{1 - e^{-2ik_\perp l}} = ik_\perp (1 + O(e^{2l\Im k_\perp})),$$

which remainder term is exponentially decaying in ω .

For $u_0 \in \mathcal{S}(\mathbb{R}^2)$,

$$\partial_{x_3} u(0, x_1, x_2) = \frac{1}{(2\pi)^2} \int \int il \frac{\cos k_\perp l}{\text{sinc}_\perp k_\perp l} \hat{u}_0(k_1, k_2) dk_1 dk_2 = \frac{1}{(2\pi)^2} \int \int ik_\perp \frac{\cos k_\perp l}{\sin k_\perp l} \hat{u}_0(k_1, k_2) dk_1 dk_2.$$

4. If one considers the case of $B_{L_1, L_2} = \{(0 \leq x \leq L_1, 0 \leq y \leq L_2, -l \leq z \leq 0)\}$, $(k_1, k_2) \in \frac{2\pi}{L_1}\mathbb{Z} \times \frac{2\pi}{L_2}\mathbb{Z}$ and the previous results are true in the discrete Fourier operators set-up.

This result (proven in Annex 1), which is straightforward to obtain, gives the aim of the present paper.

It has been proven [4] that, outside resonances, the problem of inhomogeneous boundary conditions has a unique solution. This defines the Calderòn operator. Our aim in this paper is to obtain, for $\epsilon\mu \notin \mathbb{R}_+^*$, an estimate on the exact solutions in a layer which prove that these solutions are in $\mathcal{S}'(\mathbb{R}^2) \times C^2([-l, 0])$ and deduce explicit versions of the Calderòn operator for more complicated geometries than the plane layer. Section 3 solves explicitly the Dirichlet problem in the cylindrical and in the elliptical geometry. The study of the solution and of the Dirichlet to Neumann operator for the cylindrical geometry, where the formal solutions after separation of variables are known is also well known, is the aim of Sections 4 and 5 of this paper. Section 6 gives the expression and the asymptotics of the Dirichlet to Neumann, in the case of two elliptical cylinders: in one case, the Calderòn is a Fourier multiplier operator, diagonal on each Fourier mode, while in the other case, it is a convolution Fourier operator. In Section 7, we study the asymptotics of the Calderòn operator for one layer of dielectric material for the Maxwell equations.

Remark that, if $\Omega - C$ is not bounded and if $\epsilon\mu \in \mathbb{R}_+^*$, the continuous spectrum of the operator prevents the existence of the Calderòn operator. It is interesting to define a weaker form of the Calderòn analysis in this case.

2. General results

Consider C a 2d bounded domain with a smooth boundary, and assume $\partial C = \Gamma_- \cup \Gamma_+$, such that there exists an ellipse $E \subset \mathbb{R}^2$ such that $\Gamma_- \subset E$ and $\Gamma_+ \subset E^c$. An annulus, or a layer on a perfectly conducting body of cylinder shape is a model for this. The following Theorem is classical:

- Theorem 1.** 1. *There exists a sequence λ_n of eigenvalues of $-\Delta$ in $H_0^1(C)$ (this operator being denoted by $-\Delta_D$), each eigenvalue is of multiplicity 1, they are strictly positive and normalized eigenfunctions form an orthonormal basis of $L^2(C)$.*
2. *If k^2 is not an eigenvalue of $-\Delta_D$, the problem*

$$\begin{cases} (\Delta + k^2)u = 0 \text{ in } C \\ u = u_0 \text{ on } \partial C \end{cases} \quad (7)$$

has a unique solution through Fredholm alternative.

3. *When $\epsilon\mu \notin \mathbb{R}$, and for $\omega \in \mathbb{R}$, the problem*

$$\begin{cases} (\Delta + \omega^2 \epsilon\mu)u = 0 \text{ in } C \\ u = 0 \text{ on } \Gamma_- \\ u = f \text{ on } \Gamma_+ \end{cases}$$

has a unique solution in $H^1(C)$ and the operator $f \rightarrow \partial_n u|_{\Gamma_+}$ is called the Dirichlet to Neumann operator.

The proof of this Theorem can be, in particular, found in Cessenat [4]. We do not reproduce this proof here.

Another result we recall here is the following (proven in [9])

Theorem 2. *Let (x_1, x_2) be a local system of coordinates on the boundary, and $g \in \mathcal{M}_2(\mathbb{R}^2)$ the metric induced on the boundary near this point. Let $\omega(\eta_1, \eta_2)$ be the cotangent coordinates near this point. The Dirichlet to Neumann operator has the following principal symbol:*

$$\omega \frac{v(x, \eta)}{\tan l\omega v(x, \eta)}$$

where $v(x, \eta) = \sqrt{\epsilon\mu - g_{11}(x)\eta_1^2 - 2g_{12}(x)\eta_1\eta_2 - g_{22}(x)\eta_2^2}$ of strictly negative imaginary part.

One interprets in the case of a cylinder the previous result as

Remark 1. *The principal part of the Dirichlet to Neumann operator reduces to $i\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}$ in the case of the cylinder in the hyperbolic regime.*

Our aim in this paper is to improve the general result obtained in [9] under the assumption ωl finite to two particular geometries: the cylindrical layer and the elliptic layer, in order to obtain the two first terms in ω of the expansion of the Dirichlet to Neumann symbol, in the Fourier-discrete Fourier space.

In the infinite cylindrical layer geometry ($r_0 \leq r \leq R$), denote by k_z the wave number in z , p the Fourier mode, and assume

1. the dielectric constants satisfy $\Im\epsilon\mu < 0$, independent on ω

2. the wave numbers $\frac{k_z}{\omega}$, $\frac{p}{\omega R}$ are bounded when $\omega \rightarrow +\infty$ (high frequency regime), and in addition $\Re \epsilon \mu - \frac{k_z^2}{\omega^2} - \frac{p^2}{R^2 \omega^2} > 0$ (hyperbolic regime)

Define

$$k_3 = \sqrt{\epsilon \mu \omega^2 - k_z^2} := a_0 + ib_0, a_0 > 0, b_0 < 0, \forall k_z \in \mathbf{R}, \quad (8)$$

$$k_\perp = \sqrt{\epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2}}, \Im k_\perp < 0, \forall p, \forall k_z \in \mathbf{R}. \quad (9)$$

We have the Theorem (a more precise statement is given below)

Theorem 3. 1. *In the case of the cylindrical layer, the leading order term in ω of the Dirichlet to Neumann operator is $i\omega \sqrt{\epsilon \mu - \frac{k_z^2}{\omega^2} - \frac{p^2}{R^2 \omega^2}} = ik_\perp$. The symbol of the Dirichlet to Neumann operator is, up to lower order terms, $ik_\perp - \frac{1}{2R} \frac{k_3^2}{k_\perp^2}$.*

2. *In the infinite elliptical layer geometry (characterized by $\mathcal{E} = \{(\rho \cosh u \cos v, \sinh u \sin v, z), u_0 \leq u \leq u_1, z \in \mathbb{R}\}$), let $a_n(\sqrt{\epsilon \mu \omega^2 - k_z^2} \rho)$ be the n -th Floquet eigenvalue for the Mathieu equation (edge of the n -th band of the spectrum).*

Assume that one is in the following regime (hyperbolic regime): there exists $\delta_0 > 0$ such that $\delta_0 \leq \left| \frac{a_n(\sqrt{\epsilon \mu \omega^2 - k_z^2} \rho)}{\omega^2} \right| \leq \frac{1}{\delta_0}$ bounded independently of ω , $\Re \epsilon \mu - \frac{k_z^2}{\omega^2} - \frac{a_n(\sqrt{\epsilon \mu \omega^2 - k_z^2} \rho)}{\rho^2 \omega^2} > 0$ (which means n of order ω).

The leading order term in ω of the Dirichlet to Neumann operator is

$$i\omega \frac{\sqrt{\frac{1}{2}(\epsilon \mu \omega^2) \cosh^2 u_1 - k_z^2 - \frac{a_n(\sqrt{\epsilon \mu \omega^2 - k_z^2} \rho)}{\rho^2}}}{\sqrt{\cosh^2 u_1 \cos^2 v + \sinh^2 u_2 \sin^2 v}}$$

and the term of order 0 is 0 (which means that the leading order term contains the contribution of the radius of curvature at each point).

3. Exact resolution of the Calderòn problem for a ring and an elliptic ring using special functions

3.1. The case of the infinite cylindrical ring

The program of this section is to perform the following analysis:

After Fourier transform² in z, θ , we obtain a formal solution $u(r, \theta, z) = \sum_p (\alpha_p(k_z) J_p(k_3 r) + \beta_p(k_z) Y_p(k_3 r)) e^{ip\theta}$, J_p, Y_p being the classical Bessel functions, k_3 is the square root of $\epsilon \mu \omega^2 - k_z^2$. This sum has a meaning when it is finite in p , but nothing is known about the behavior in k_z .

After using the Dirichlet boundary condition at $r = r_0$, we obtain, still formally, all solutions of the Helmholtz equation in the annulus which satisfy the Dirichlet boundary condition at $r = r_0$. Its formal expression is $\sum_p a_p(k_z) u_p(r, k_z) e^{ip\theta}$, $u_p(R, k_z) = 1$.

a reecrire

We obtain an inequality of the form $|u_p(r, r_0, k_z)| \leq e^{\Psi(R, r_0, p, k_z)}$ for all $r \in [r_0, R]$. If one assumes $|a_p(k_z)| \leq (1 + k_z^2 + \frac{p^2}{R^2})^{-2} e^{-\Psi(R, r_0, p, k_z)}$, this sum defines an element of $L^2(\mathbb{R} \times \mathbb{N})$, which gives a rigorous framework for identifying a class of solutions (not all solutions).

²Note that it is a Fourier series in θ and a Fourier transform in z

For this class of solutions, we are able (in Section 4.1) to derive the Fourier multiplier which transforms $u|_{r=R}$ into $\partial_r u|_{r=R}$. Using properties of this Fourier multiplier, we show that the associated operator extends to an operator which sends $H^1(\mathbb{R} \times [0, 2\pi])$ to $L^2(\mathbb{R} \times [0, 2\pi])$.

Introduce the following Definition

Definition 1. If ν is real and $\lambda > 0$, the roots of $J_\nu(z)Y_\nu(\lambda z) - J_\nu(z)Y_\nu(\lambda z) = 0$ are real and simple. They are denoted by the increasing sequence $z_n(\nu, \lambda)$. It is stated in (9.5.27) of [1].

Similar to the result obtained in the Introduction, we have (the formulae here are classical, see Stupfel [16] for example), but not the conditions on ω .

Lemma 3. The resonances of the operator $-\Delta$ are the values of ω such that there exists at least a n , and a couple $(p, k_z) \in \mathbb{Z} \times \mathbb{R}$ such that $r_0^2(\epsilon\mu\omega^2 - k_z^2) = z_n(p, \frac{R}{r_0})$, where $z_n(\lambda)$ is defined in Definition 1. Every value of ω is a resonance in this case. If k_z is fixed, we have an infinite sequence of resonances of the operator $-\Delta + k_z^2$. If $\epsilon\mu \notin \mathbb{R}$, there are no resonances. If one restricts to the 'periodic cylinder' $\{(r, \theta, z), 0 \leq r_0, \theta \in [0, 2\pi], z \in [0, L]\}$ with periodicity conditions in z , we have $k_z \in \frac{2\pi}{L}\mathbb{Z}$, hence a dispersion relation, in the case $\epsilon\mu \in \mathbb{R}_+$, $\epsilon\mu\omega^2 = \frac{z_n}{r_0^2} + (\frac{2\pi}{L})^2 q^2$, $(n, q) \in \mathbb{N} \times \mathbb{Z}$.

add a proof and comments One has the following

Proposition 1. Provided that $\omega^2\epsilon\mu$ is not a resonance of $-\Delta + k_z^2$ on C , for all $u_0 \in H^{\frac{1}{2}}(S_R) \simeq H^{\frac{1}{2}}([0, 2\pi])$ there exists a unique solution in $H^1(C)$ of

$$\begin{cases} (\Delta + \omega^2\epsilon\mu)u = 0 \\ u(r_0, \cdot, \cdot) = 0 \\ u(R, \cdot, \cdot) = u_0. \end{cases} \quad (10)$$

It is given by its partial Fourier transform in z as a Fourier series in θ

$$\hat{u}(r, \theta, k_z) = \sum_p a^p(k_z) \frac{J_p(k_3 r)Y_p(k_3 r_0) - Y_p(k_3 r)J_p(k_3 r_0)}{J_p(k_3 R)Y_p(k_3 r_0) - Y_p(k_3 R)J_p(k_3 r_0)} e^{ip\theta}, \quad (11)$$

where $a^p(k_z)$ is the p -th Fourier coefficient of $\hat{u}_0(k_z, \theta)$.

Analysis: After partial Fourier transform in x_3 , the equation reads

$$(\Delta + k_3^2)\hat{u} = 0,$$

where we kept the notation Δ for the Laplace operator in \mathbb{R}^2 . The expansion $\hat{u}(r, \theta) = \sum a_p(r)e^{ip\theta}$, where $\sum |a_p|^2 < +\infty$ (at least) yields the classical Bessel equation on a_p :

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{da_p}{dr} \right) + \left(\omega^2\epsilon\mu - k_z^2 - \frac{p^2}{r^2} \right) a_p = \frac{1}{r} \frac{d}{dr} \left(r \frac{da_p}{dr} \right) + \left(k_3^2 - \frac{p^2}{r^2} \right) a_p = 0. \quad (12)$$

This Bessel equation implies that there exists (α_p, β_p) depending on k_z such that $a_p(r) = \alpha_p J_p(k_3 r) + \beta_p H_p^{(2)}(k_3 r)$. Note, as in the Introduction, that it is a necessary (but not necessary and sufficient condition).

Define the functions of $r \in [r_0, R]$:

$$d_p(k_3 r) := J_p(k_3 r_0)Y_p(k_3 r) - Y_p(k_3 r_0)J_p(k_3 r), \quad dd_p(k_3 r) := J'_p(k_3 r_0)Y_p(k_3 r) - Y'_p(k_3 r_0)J_p(k_3 r). \quad (13)$$

The first one is a solution of the Bessel equation which vanishes at $r = r_0$ and the second quantity is a solution of the Bessel equation which derivative vanishes at $r = r_0$. It is a pair of fundamental solutions of the Bessel equation, under the condition:

$$J'_p(k_3 r_0)Y_p(k_3 r_0) - Y'_p(k_3 r_0)J_p(k_3 r_0) \neq 0. \quad (14)$$

Remark that, if one picks any pair of independent solutions of the Bessel equations $\{f_p, g_p\}$, there exists a constant $C(f_p, g_p)$ such that $d_p(k_3 r) = C(f_p, g_p)[f_p(k_3 r_0)g_p(k_3 r) - g_p(k_3 r_0)f_p(k_3 r)]$ and $dd_p(k_3 r) = C(f_p, g_p)[f'_p(k_3 r_0)g_p(k_3 r) - g'_p(k_3 r_0)f_p(k_3 r)]$. Hence we can choose any pair of independent solutions instead of J_p and of Y_p for the representation of d_p and dd_p .

3.2. The exact representation of solutions of the Helmholtz equation for elliptic coordinates

In this Section, we rely on the classical approach using Mathieu and modified Mathieu functions. Mathieu functions are periodic functions which form a basis of $L^2([0, 2\pi])$ and modified Mathieu functions are the analogous of the Bessel and Hankel functions. B. Stupfel [16] already used such an approach to compute a good estimate

The (transversal) change of variable in which the Laplacian operator is diagonal is

$$(x, y) = \rho(\cosh u \cos v, \sinh u \sin v).$$

If one wants to use this on the ellipse $\Omega = \{(x, y), \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$. of constants $a, b, a > b$, one defines $\rho = \sqrt{a^2 - b^2}$ and u_0 such that $\tanh u_0 = \frac{b}{a}$. The open set Ω is characterized by $\{(u, v), (\frac{\cosh u}{\cosh u_0})^2 \cos^2 v + (\frac{\sinh u}{\sinh u_0})^2 \sin^2 v \leq 1, \}$, its boundary being $\partial\Omega = \{u = u_0, v \in (0, 2\pi)\}$.

The normal derivative and the radius of curvature for the ellipse are expressed as follows

Lemma 4. *The normal unit vector to the ellipse at the point $\rho(\cosh u_1 \cos v, \sinh u_1 \sin v)$ is*

$$\vec{n} = \frac{1}{\sqrt{\sinh^2 u_1 \cos^2 v + \cosh^2 u_1 \sin^2 v}} (\sinh u_1 \cos v, \cosh u_1 \sin v)$$

and the radius of curvature of the ellipse at this point is

$$\rho(s) = \rho \frac{(\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v)^{\frac{3}{2}}}{\cosh u_1 \sinh u_1}.$$

Proof. A tangent vector to the boundary $u = u_1$ is $(\cosh u_1 \sin v, -\sinh u_1 \cos v)$. The curvilinear absciss is given through $\rho \|\frac{d}{ds}((\cosh u_1 \cos v, \sinh u_1 \sin v))\| = 1$, that is $\frac{dv}{ds} \sqrt{\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v} = \rho^{-1}$. The normal unit vector is

$$\frac{1}{\sqrt{\sinh^2 u_1 \cos^2 v + \cosh^2 u_1 \sin^2 v}} (\sinh u_1 \cos v, \cosh u_1 \sin v).$$

One knows that, for a unit tangent vector $\frac{1}{\sqrt{a^2+b^2}}(a(s), b(s))$, its derivative being $\frac{a'b-ab'}{(a^2+b^2)^{\frac{3}{2}}}(b, -a)$, the radius of curvature is $\frac{a^2+b^2}{a'b-ab'}$. With $\frac{dv}{ds} = (\rho \sqrt{\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v})^{-1}$, along with

$$a'b - ab' = \frac{dv}{ds} \left(\frac{da}{dv} b - \frac{db}{dv} a \right) = \frac{dv}{ds} (-\cosh u_1 \sinh u_1 \cos^2 v - \sinh u_1 \cosh u_1 \sin^2 v) = -\frac{dv}{ds} \cosh u_1 \sinh u_1,$$

one gets

$$\rho(s) = \frac{\rho}{\cosh u_1 \sinh u_1} (\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v)^{\frac{3}{2}}$$

which ends the proof. □

All solutions of $(\Delta + \omega^2 \epsilon \mu)u = 0$ are of the form

$$u(x, y, z) = F(u)G(v)\phi(z),$$

where ϕ satisfies $\phi''(z) + c\phi(z) = 0$ and F and G are solutions of

$$F''(u) + \left(\frac{\omega^2 \epsilon \mu - c}{2}\rho^2 \cosh 2u - a\right)F = 0, \quad (15)$$

$$G''(v) + \left(a - \frac{\omega^2 \epsilon \mu - c}{2}\rho^2 \cos 2v\right)G = 0, \quad (16)$$

thanks to the transformation of the Helmholtz equation into

$$\left[\frac{\partial^2}{\partial z^2} + \frac{2}{\rho^2(\cosh 2u - \cos 2v)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \epsilon \mu \omega^2\right]u = 0.$$

The natural choice in our analysis is to consider $c = k_z^2$. Let $U(x, y) = F(u)G(v)$. For all $(a, k_3\rho)$, there exists (through Floquet theory, as mentioned in [1], 20.3.1) $\nu(a, k_3\rho)$ such that $g_{\pm}(v) = e^{\pm i\nu(a, k_3\rho)} P_{a, k_3\rho}(\pm v)$ is a pair of independent solutions of (16), where $P_{a, k_3\rho}$ are periodic functions.

Let $a_n(k_3\rho)$ and $b_n(k_3\rho)$ be the unique solutions, respectively, of $\nu(a, k_3\rho) = n$ and $\nu(a, k_3\rho) = -n$.

Lemma 5. *The eigenvalues of the Mathieu operator $-\frac{d^2}{dv^2} + \frac{k_3^2 \rho^2}{2} \cos 2v$ on $L^2([0, 2\pi])$ are $a_n(k_3\rho), b_n(k_3\rho)$, and the associated eigenfunctions, with a suitable normalization are denoted by $ce_n(v, k_3\rho)$ and $se_n(v, k_3\rho)$.*

The notation recalling that each of which is closely related to the cosine and sine (namely for $k_3\rho = 0$, $a_n(0) = b_n(0) = n^2$ and $ce_n(v) = \cos nv$, $se_n(v) = \sin nv$).

Let $c_n(k_3\rho) = a_n(k_3\rho)$ for $n \geq 1$, $c_n(k_3\rho) = b_{-n}(k_3\rho)$ for $n \leq -1$.

Lemma 6. *The equation*

$$F'' - \left(c_n(k_3\rho) - \frac{k_3^2 \rho^2}{2} \cosh 2u\right)F = 0 \quad (17)$$

has a pair of canonical solutions, even and odd respectively, denoted by $C_{|n|}^{k_3\rho}, S_{|n|}^{k_3\rho}$. where n positive stands for $c_n(k_3\rho) = a_n(k_3\rho)$ and $n \geq 1$, $c_n(k_3\rho) = b_n(k_3\rho)$ and $n \leq -1$.

Lemma 7. *The periodic in v solutions of the Helmholtz equations $(\Delta + k_3^2)U = 0$ in $\mathbb{C}^2(\mathcal{E}(A, B))$ are*

$$U_n(x, y) = (AC_{|n|}^{k_3\rho} + BS_{|n|}^{k_3\rho})(u)g_n(v), n \in \mathbb{Z}.$$

All the items of these Lemma are well known through [1] or through [6]. The item on the growth of the canonical solutions is a consequence of the positivity of the solutions of $\phi'' = q(u)\phi$ with $(\phi, \phi')(u_0) = (1, 0)$ and $(\phi, \phi')(u_0) = (0, 1)$ when $q(u) > 0$.

4. Analysis of the asymptotic regimes for cylindrical coordinates

4.1. Estimates on $\frac{d_p(k_3 r)}{d_p(k_3 R)}$ using asymptotic results.

The aim of this Section is to prove the fundamental result

Proposition 2. *There exists a constant K such that*

$$\forall r \in [r_0, R], \forall (k_z, p) \in \mathbb{R} \times \mathbb{N}, \left| \frac{d_p(k_3 r)}{d_p(k_3 R)} \right| \leq K.$$

The proof of this Proposition is the aim of the next section. We shall use the known behavior of the Bessel functions to derive an result identical to (2), thanks to precise estimates on Bessel functions. The precise estimates are listed in Section 4.2, in five cases.

4.2. Bessel functions toolbox

In this subsection, we describe the representations of the Bessel functions that will be used in this paper. We concentrate mainly on $J_p + iY_p = H_p^{(1)}$ and $J_p - iY_p = H_p^{(2)}$. The variables z, β, a are defined in (29). We have the representations:

1. In the case of p fixed and of the argument of the Bessel functions large, let $\chi_p(Z) = Z - (\frac{1}{2}p + \frac{1}{4}\pi)$. There exists four functions P^*, Q^*, R^*, S^* , where one gets estimates of remainder terms in the asymptotic expansions, such that

$$H_p^{(1)}(Z) = \frac{2}{\pi Z} [P^*(p, Z) + iQ^*(p, Z)] e^{i\chi_p(Z)}, (H_p^{(1)})'(Z) = \frac{2}{\pi Z} [iR^*(p, Z) - S^*(p, Z)] e^{i\chi_p(Z)}, \quad (18)$$

and the conjugate solution which is $H_p^{(2)}$. We have in addition

$$P^*(p, Z) = 1 + O(Z^{-2}), 8ZQ^*(p, Z) = 4p^2 - 1 + O(Z^{-2}), R^*(p, Z) = 1 + O(Z^{-2}), 8ZS^*(p, Z) = 4p^2 + 3 + O(Z^{-2}).$$

This one was used already in [3] and one notices that all the constants in $O(z^{-2})$ depend on p . More precisely, the expansion of P in inverse powers of Z is $\sum_{j \geq 0} Z^{-2j} a_j(p)$, where $a_j(p)$ is a polynomial of degree $2j$.

2. In the hyperbolic case $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2} > 0$, let $\Psi(p, \beta) = p(\tan \beta - \beta)$. From [1], (9.3.15) to (9.3.19), there exists four functions L, M, N, O depending on p and β such that

$$J_p\left(\frac{p}{\cos \beta}\right) + iY_p\left(\frac{p}{\cos \beta}\right) = \sqrt{\frac{2}{\pi p \tan \beta}} (L - iM) e^{i\Psi}, J_p'\left(\frac{p}{\cos \beta}\right) + iY_p'\left(\frac{p}{\cos \beta}\right) = i \sqrt{\frac{\sin 2\beta}{\pi p}} (N + iO) e^{i\Psi}, \quad (19)$$

where L, N, pM, pO have an asymptotic expansion in inverse powers of p^2 when p large, and $L(p, \beta) = 1 + O(p^{-2})$, $N(p, \beta) = 1 + O(p^{-2})$, $pM(p, \beta) = a(\beta) + O(p^{-1})$, $pO(p, \beta) = b(\beta) + O(p^{-1})$, where $a(\beta) = \frac{1}{8 \tan \beta} + \frac{5}{24 \tan^3 \beta}$, $b(\beta) = \frac{3}{8 \tan \beta} + \frac{7}{24 \tan^3 \beta}$.

3. In the elliptic case, namely $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2} < 0$, we use 10.20 in [6]. Note that z and ζ introduced in (29) can be extended to $z \in \mathbb{C}$. There exists A, B, C, D , continuous functions on ζ , depending on p , such that

$$\begin{cases} H_p^{(1)}(pz) \simeq 2e^{-\frac{i\pi}{3}} \left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}} [p^{-\frac{1}{3}} Ai(e^{\frac{2i\pi}{3}} p^{\frac{2}{3}} \zeta) A(\zeta, p) - p^{-\frac{5}{3}} e^{\frac{2i\pi}{3}} Ai'(e^{\frac{2i\pi}{3}} p^{\frac{2}{3}} \zeta) \zeta^{-\frac{1}{2}} B(\zeta, p)] \\ (H_p^{(1)})'(pz) \simeq \frac{4e^{-\frac{2i\pi}{3}}}{z} \left(\frac{1-z^2}{\zeta}\right)^{\frac{1}{4}} [-p^{-\frac{4}{3}} e^{-\frac{2i\pi}{3}} Ai(e^{\frac{2i\pi}{3}} p^{\frac{2}{3}} \zeta) \zeta^{\frac{1}{2}} C(\zeta, p) + p^{-\frac{2}{3}} Ai'(e^{\frac{2i\pi}{3}} p^{\frac{2}{3}} \zeta) D(\zeta, p)], \end{cases} \quad (20)$$

where

$$A(\zeta, p) = \sum_{k=0}^{+\infty} \frac{A_k(\zeta)}{p^{2k}},$$

and similar uniform expansions for B, C, D (the notations for B and C are slightly different to the ones appearing in 10.20 in [6], $A_0(\zeta) = D_0(\zeta) = v_0, B_0(\zeta) = C_0(\zeta) = u_0$).

First notice that $a(p, \zeta) = \frac{Ai(p^{\frac{2}{3}}\zeta)}{Bi(p^{\frac{2}{3}}\zeta)}$ is exponentially small as $\omega \rightarrow +\infty$ when $\Re\zeta^{\frac{3}{2}} > 0$ and that

$b(p, \zeta) = \frac{Bi(p^{\frac{2}{3}}\zeta)}{Ai(p^{\frac{2}{3}}\zeta)}$ is exponentially small as $\omega \rightarrow +\infty$ when $\Re\zeta^{\frac{3}{2}} < 0$.

Introduce

$$\begin{aligned} k_{\pm}(p, \zeta) &= -\zeta^{-\frac{1}{2}} p^{-\frac{4}{3}} e^{\pm 2i\frac{\pi}{3}} \frac{Ai'}{Ai}(e^{\pm \frac{2}{3}i\pi} p^{\frac{2}{3}}\zeta) \frac{B(\zeta, p)}{A(\zeta, p)}, \\ \tau_{\pm}(p, \zeta) &= -\zeta^{\frac{1}{2}} p^{-\frac{2}{3}} e^{\mp 2i\frac{\pi}{3}} \frac{Ai}{Ai'}(e^{\pm \frac{2}{3}i\pi} p^{\frac{2}{3}}\zeta) \frac{C(\zeta, p)}{D(\zeta, p)}, \end{aligned}$$

which both satisfy uniformly $k_{\pm} = O(p^{-1}), \tau_{\pm} = O(p^{-1})$. More precisely one has

$$k_{\pm}(p, \zeta) = -p^{-1} \frac{u_0}{v_0} (1 + O(p^{-1})), \tau_{\pm}(\zeta, p) = -p^{-1} \frac{u_0}{v_0} (1 + O(p^{-1})).$$

Estimates (20) yield

$$H_p^{(1)}(pz) = 2e^{-i\frac{\pi}{3}} p^{-\frac{1}{3}} \left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}} Ai(e^{\frac{2i\pi}{3}} p^{\frac{2}{3}}\zeta) A(p, \zeta) [1 + k_+(p, \zeta)],$$

and a similar estimate on $H_p^{(2)}(pz)$.

We also observe that

$$Ai(e^{\frac{2i\pi}{3}} p^{\frac{2}{3}}\zeta) = \frac{1}{2} e^{i\frac{\pi}{3}} Bi(p^{\frac{2}{3}}\zeta) (-i + a(\zeta)) = \frac{1}{2} e^{i\frac{\pi}{3}} Ai(p^{\frac{2}{3}}\zeta) (1 - ib(\zeta)), \quad (21)$$

from which one deduces the behavior of $H_p^{(j)}(pz)$ according to the sign of $\Re\zeta^{\frac{3}{2}}$:

$$\begin{aligned} H_p^{(1)}(pz) &= \left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}} p^{-\frac{1}{3}} A(\zeta, p) (1 + k_+(p, \zeta)) Bi(p^{\frac{2}{3}}\zeta) (-i + a(\zeta)) \\ &= \left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}} p^{-\frac{1}{3}} A(\zeta, p) (1 + k_+(p, \zeta)) Ai(p^{\frac{2}{3}}\zeta) (1 - ib(\zeta)), \end{aligned} \quad (22)$$

not writing the similar estimate on $H_p^{(2)}(pz)$ (one needs to replace k_+ by k_- , and all the conjugate coefficients), stressing the role of the sign of $\Re\zeta^{\frac{3}{2}}$: the equivalent of $H_p^{(1)}(pz)$ for $\Re\zeta^{\frac{3}{2}} > 0$ is $-i \left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}} p^{-\frac{1}{3}} A(\zeta, p) Bi(p^{\frac{2}{3}}\zeta)$ and is $\left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}} p^{-\frac{1}{3}} A(\zeta, p) Ai(p^{\frac{2}{3}}\zeta)$ for $\Re\zeta^{\frac{3}{2}} < 0$.

Introducing also $\tilde{a}(\zeta) = \frac{Ai'(p^{\frac{2}{3}}\zeta)}{Bi'(p^{\frac{2}{3}}\zeta)}, \tilde{b}(\zeta) = \frac{Bi'(p^{\frac{2}{3}}\zeta)}{Ai'(p^{\frac{2}{3}}\zeta)}$, one gets

$$\begin{aligned} (H_p^{(1)})'(pz) &= \frac{2e^{-\frac{2i\pi}{3}}}{z} \left(\frac{1-z^2}{\zeta}\right)^{\frac{1}{4}} p^{-\frac{2}{3}} D(\zeta, p) (1 + \tau_+(p, \zeta)) Bi'(p^{\frac{2}{3}}\zeta) (-i + \tilde{a}(\zeta)) \\ &= \frac{2e^{-\frac{2i\pi}{3}}}{z} \left(\frac{1-z^2}{\zeta}\right)^{\frac{1}{4}} p^{-\frac{2}{3}} D(\zeta, p) (1 + \tau_+(p, \zeta)) Ai'(p^{\frac{2}{3}}\zeta) (1 - i\tilde{b}(\zeta)), \end{aligned} \quad (23)$$

It will be useful as well to obtain the following equalities on $Y_p(pz)$ and $J_p(pz)$, using (10.20.4) and (10.20.5) of [6] and the expressions (different from the definitions mentioned) $A(\zeta, p) = v_0 + O(p^{-2}), B(\zeta, p) = u_0 + O(p^{-2}), C(\zeta, p) = v_0 + O(p^{-2}), D(\zeta, p) = u_0 + O(p^{-2}), u_0 = v_0 = 1$:

$$\left\{ \begin{array}{l} J_p(pz) = \left(\frac{4\zeta}{1-\zeta^2}\right)^{\frac{1}{4}} p^{-\frac{1}{3}} [A(\zeta, p) Ai(p^{\frac{2}{3}} \zeta) - p^{-\frac{4}{3}} \zeta^{-\frac{1}{2}} B(\zeta, p) Ai'(p^{\frac{2}{3}} \zeta)] \\ \quad = \left(\frac{4\zeta}{1-\zeta^2}\right)^{\frac{1}{4}} p^{-\frac{1}{3}} A(\zeta, p) (1 + k(p, \zeta)) Ai(p^{\frac{2}{3}} \zeta) \\ Y_p(pz) = -\left(\frac{4\zeta}{1-\zeta^2}\right)^{\frac{1}{4}} p^{-\frac{1}{3}} [A(\zeta, p) Bi(p^{\frac{2}{3}} \zeta) - p^{-\frac{4}{3}} \zeta^{-\frac{1}{2}} B(\zeta, p) Bi'(p^{\frac{2}{3}} \zeta)] \\ \quad = -\left(\frac{4\zeta}{1-\zeta^2}\right)^{\frac{1}{4}} p^{-\frac{1}{3}} A(\zeta, p) (1 + \tilde{k}(p, \zeta)) Bi(p^{\frac{2}{3}} \zeta) \\ J'_p(pz) = -\frac{2}{\zeta} \left(\frac{1-\zeta^2}{4\zeta}\right)^{\frac{1}{4}} p^{-\frac{2}{3}} [D(\zeta, p) Ai'(p^{\frac{2}{3}} \zeta) - p^{-\frac{2}{3}} \zeta^{\frac{1}{2}} C(\zeta, p) Ai(p^{\frac{2}{3}} \zeta)] \\ \quad = -\frac{2}{\zeta} \left(\frac{1-\zeta^2}{4\zeta}\right)^{\frac{1}{4}} p^{-\frac{2}{3}} D(\zeta, p) (1 + \tau(p, \zeta)) Ai'(p^{\frac{2}{3}} \zeta) \\ Y'_p(pz) = \frac{2}{\zeta} \left(\frac{1-\zeta^2}{4\zeta}\right)^{\frac{1}{4}} p^{-\frac{2}{3}} [D(\zeta, p) Bi'(p^{\frac{2}{3}} \zeta) - p^{-\frac{2}{3}} \zeta^{\frac{1}{2}} C(\zeta, p) Bi(p^{\frac{2}{3}} \zeta)] \\ \quad = \frac{2}{\zeta} \left(\frac{1-\zeta^2}{4\zeta}\right)^{\frac{1}{4}} p^{-\frac{2}{3}} D(\zeta, p) (1 + \tilde{\tau}(p, \zeta)) Bi'(p^{\frac{2}{3}} \zeta), \end{array} \right. \quad (24)$$

where this equality defines $k, \tilde{k}, \tau, \tilde{\tau}$ satisfy $k(\zeta, p) = p^{-1} + O(p^{-2})$, $\tilde{k}(\zeta, p) = -p^{-1} + O(p^{-2})$, $\tau(p, \zeta) = p^{-1} + O(p^{-2})$, $\tilde{\tau}(p, \zeta) = -p^{-1} + O(p^{-2})$, in order to deal with the totally elliptic case. We have

Lemma 8. *Let $\delta_* > 0$ be arbitrary. For all ζ , $|\zeta| \geq 1$ and $|\arg(\zeta)| \leq \pi - \delta_*$*

- *The function $Ai(p^{\frac{2}{3}} \zeta) (p^{\frac{2}{3}} \zeta)^{\frac{1}{4}} e^{\frac{2}{3} p \zeta^{\frac{3}{2}}}$ is bounded uniformly,*
- *The function $Bi(p^{\frac{2}{3}} \zeta) (p^{\frac{2}{3}} \zeta)^{\frac{1}{4}} e^{-\frac{2}{3} p \zeta^{\frac{3}{2}}}$ is bounded uniformly,*
- *The function $(p^{\frac{2}{3}} \zeta)^{\frac{1}{2}} \frac{Ai}{Ai'}(p^{\frac{2}{3}} \zeta)$ is bounded uniformly,*
- *The function $(p^{\frac{2}{3}} \zeta)^{\frac{1}{2}} \frac{Bi}{Bi'}(p^{\frac{2}{3}} \zeta)$ is bounded uniformly.*

All these quantities have an uniform expansion in powers in $(p^{\frac{2}{3}} \zeta)^{-\frac{3}{2}}$.

All these terms have a finite limit $p \rightarrow +\infty$.

4. In the glancing case (namely $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2} \simeq 0$), there exists four functions P, Q, R, S depending on a, p , defined in 10.19.9 of [6] such that

$$P(a, p) = \sum_{k=0}^{+\infty} \frac{P_k(a)}{p^{\frac{2k}{3}}}$$

and similar equalities for Q, R, S such that one has the following equalities (and the conjugate ones), recalling $Ai(-X) - iBi(-X) = \frac{1}{2} e^{-i\frac{\pi}{3}} Ai(e^{-i\frac{\pi}{3}} X)$ and $Ai'(-X) - iBi'(-X) = \frac{1}{2} e^{i\frac{\pi}{3}} Ai'(e^{-i\frac{\pi}{3}} X)$:

$$\left\{ \begin{array}{l} H_p^{(1)}(p + ap^{\frac{1}{3}}) = \frac{2^{\frac{1}{3}}}{p^{\frac{1}{3}}} (Ai - iBi)(-2^{\frac{1}{3}} a) P + \frac{2^{\frac{2}{3}}}{p} (Ai' - iBi')(-2^{\frac{1}{3}} a) Q \\ \quad = \frac{2^{\frac{4}{3}}}{p^{\frac{1}{3}}} e^{-i\frac{\pi}{3}} Ai(e^{-i\frac{\pi}{3}} 2^{\frac{1}{3}} a) P + \frac{2^{\frac{5}{3}}}{p} e^{i\frac{\pi}{3}} Ai'(e^{-i\frac{\pi}{3}} 2^{\frac{1}{3}} a) Q \\ (H_p^{(1)})'(p + ap^{\frac{1}{3}}) = -\frac{2^{\frac{1}{3}}}{p^{\frac{2}{3}}} (Ai'(-2^{\frac{1}{3}} a) - iBi'(-2^{\frac{1}{3}} a)) R + \frac{2^{\frac{1}{3}}}{p^{\frac{1}{3}}} (Ai(-2^{\frac{1}{3}} a) - iBi(-2^{\frac{1}{3}} a)) S \\ \quad = -\frac{2^{\frac{5}{3}}}{p^{\frac{2}{3}}} e^{i\frac{\pi}{3}} Ai'(e^{-i\frac{\pi}{3}} 2^{\frac{1}{3}} a) R + \frac{2^{\frac{1}{3}}}{p^{\frac{1}{3}}} e^{-i\frac{\pi}{3}} Ai(e^{-i\frac{\pi}{3}} 2^{\frac{1}{3}} a) S. \end{array} \right. \quad (25)$$

One has the following asymptotic estimates on P, Q, R, S :

$$P = 1 - \frac{a}{5p^{\frac{2}{3}}} + O(p^{-\frac{4}{3}}), Q = \frac{3}{10} a^2 + O(p^{-\frac{2}{3}}), R = 1 - \frac{4a}{5p^{\frac{2}{3}}} + O(p^{-\frac{4}{3}}), S = \frac{3}{5} a^2 - \frac{1}{5} + O(p^{-\frac{2}{3}}).$$

The first case, which is the most classical one, will nevertheless be of less use than the others in this analysis. It is the case of a fixed p and of $k_3 r$ large. It relies on the equalities (18), which yield:

$$\frac{d_p(k_3 r)}{d_p(k_3 R)} = \frac{R}{r} e^{-ik_3(R-r)} \frac{e^{-2ik_3(r-r_0)}(P_* - iQ_*)(k_3 r_0)(P_* + iQ_*)(k_3 r) - (P_* + iQ_*)(k_3 r_0)(P_* - iQ_*)(k_3 r)}{e^{-2ik_3(R-r_0)}(P_* - iQ_*)(k_3 r_0)(P_* + iQ_*)(k_3 R) - (P_* + iQ_*)(k_3 r_0)(P_* - iQ_*)(k_3 R)}.$$

As $\Im k_3 < 0$, one has $|e^{-2ik_3(R-r_0)}| = e^{2\Im k_3(R-r_0)} \leq 1$ and $|e^{-2ik_3(r-r_0)}| \leq 1$, along with $|e^{-ik_3(R-r)}| \leq 1$. Hence there exists $C(p)$ such that, for $\omega \geq 1$

$$\left| \frac{d_p(k_3 r)}{d_p(k_3 R)} \right| \leq C(p), \forall k_z, \forall r \in [r_0, R].$$

This bound is not uniform in p , hence this result shall only be able to control $\sum_{p=0}^P a_p(k_z) e^{ip\theta}$ in k_z in order to perform an inverse Fourier transform, but cannot be used to obtain a behavior in $(p; k_z)$ for the regularity when $(p, k_z) \rightarrow +\infty$.

We need, for dealing with (p, k_z) , to consider p AND k_z large. In classical situations, this corresponds to what can be called the **high frequency regime**, where both k_z and $\frac{p}{r}$ are of order of magnitude ω . It is however necessary to have estimates for all $(k_z, p) \in \mathbb{R} \times \mathbb{N}$,

We rely on the following definition, based on $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2} > \Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r_0^2}$ for all (p, k_z) , under the condition

$$\Re \epsilon \mu \omega^2 > k_z^2 : \quad (26)$$

Definition 2. *One has the following classification on (p, k_z) (it is a classification on the wave vector or dual vector)*

1. *One is in the totally elliptic region when $0 > \Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2}$.*
2. *One is in the mixed-elliptic-hyperbolic region when $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2} > 0 > \Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r_0^2}$.*
3. *One is in the totally hyperbolic region when $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r_0^2} > 0$.*

In cases 2 and 3, one has

$$k_z^2 + \frac{p^2}{R^2} \leq C\omega^2. \quad (27)$$

One denotes by k_\perp , $\Im k_\perp < 0$ (as in the plane layer)

$$k_\perp = \sqrt{\epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2}} \quad (28)$$

An explanation of the names of the zones is provided below in Section 4.3.

Let us prove (27).

In the case 'mixed elliptic hyperbolic', it is enough to choose $C = \Re \epsilon \mu$.

In the case 'totally hyperbolic', one has

$$k_z^2 + \frac{p^2}{r_0^2} < k_z^2 + \frac{\Re \epsilon \mu \omega^2 - k_z^2}{r_0^2} R^2 \leq \Re \epsilon \mu \frac{R^2}{r_0^2} \omega^2,$$

hence $C = \Re \epsilon \mu \frac{p^2}{r_0^2}$ is possible. We will thus introduce the following variables, which replace the variable r in the asymptotic analysis in (ω, k_z, p) :

$$\begin{aligned} k_z^2 + \frac{p^2}{r^2} > \Re \epsilon \mu \omega^2 : \quad z = \frac{k_3 r}{p}, \zeta = \left(\frac{3}{2} \int_z^1 \frac{\sqrt{1-t^2}}{t} dt \right)^{\frac{2}{3}}, \int_z^1 \frac{\sqrt{1-t^2}}{t} dt = \ln \frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2} \\ k_z^2 + \frac{p^2}{r^2} < \Re \epsilon \mu \omega^2 : \quad \frac{p}{\cos \beta} = k_3 r, \sin \beta = \frac{\sqrt{\epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2}}}{k_3}, \Re \sin \beta > 0 \\ k_z^2 + \frac{p^2}{r^2} \simeq \Re \epsilon \mu \omega^2 : \quad k_3 r = p + ap^{\frac{1}{3}}. \end{aligned} \quad (29)$$

Remark 2. In the hyperbolic, mixed, or elliptic regimes, β, z, ζ , when introduced, have a finite limit when $\omega \rightarrow +\infty$. In the glancing regime and $\Im \epsilon \mu$ independent on ω , $a = O(\omega^{\frac{2}{3}})$ and we are not in the range of application of this regime. If $\Im \epsilon \mu = \frac{\sigma}{i\omega}$, we are in the range $a = O(\omega^{\frac{1}{6}})$ for $\Re \epsilon \mu - \frac{k_z^2}{\omega^2} - \frac{p^2}{r^2 \omega^2} = 0$.

Remark 3. For $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2} \geq 0$, introduce a_0 and b_0 such that $k_3 = a_0 + ib_0$, $\alpha_0(r)$ and $\beta_0(r)$ such that $k_3 \sin \beta = \alpha_0(r) + i\beta_0(r)$, and $a(r)$ and $b(r)$ such that $\sin \beta = a(r) + ib(r)$.

For $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2} \leq 0$, introduce $p(r), q(r)$ such that $p\zeta^{\frac{1}{2}} \frac{d\zeta}{dr} = -(p(r) + iq(r))$.

They are uniquely determined through $b_0 < 0$, $\alpha_0(r) > 0$ (condition crucial for the analysis, coming from the assumption $\sin \beta > 0$ when $\epsilon \mu \in \mathbf{R}$ (see [6])) and $p(r) > 0$ (choice of the determination of the square root as indicated in [6]), and one has

$$\beta_0(r) = \frac{p^2 \Im \epsilon \mu \omega^2}{r^2 |k_3|^4 r \alpha_0(r)} < 0, q(r) = -\frac{\Im \epsilon \mu \omega^2}{2p(r)} > 0.$$

Proof. Let $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2} > 0$. One has $k_3^2 \sin^2 \beta = k_3^2 - k_3^2 \cos^2 \beta = k_3^2 - \frac{p^2}{r^2} = k_\perp^2$. Moreover $\sin^2 \beta = 1 - \frac{p^2}{r^2 k_3^2} = 1 - \frac{p^2 k_3^2}{r^2 |k_3|^4}$, from which one deduces (imaginary part)

$$2a(r)b(r) = \frac{2a_0 b_0 p^2}{r^2 |k_3|^4}.$$

As $2a_0 b_0 = \Im \epsilon \mu \omega^2 < 0$, one has $a(r)b(r) < 0$ hence $b(r) < 0$. Observe then that $\Im k_3 \sin \beta = b_0 a(r) + a_0 b(r)$, which yields $\beta_0 = \Im k_3 \sin \beta < 0$.

Let $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2} < 0$. From $p z = k_3 r$, and $\frac{2}{3} \zeta^{\frac{3}{2}} = \int_z^1 \frac{\sqrt{1-t^2}}{t} dt$, one deduces $\zeta^{\frac{1}{2}} \frac{d\zeta}{dr} = -\frac{\sqrt{1-z^2}}{z} \frac{dz}{dr} = -\frac{\sqrt{1-z^2}}{z} \frac{k_3}{p}$.

One deduces (observe that one uses $\lambda \sqrt{e} = \sqrt{\lambda^2 e}$ for all $\lambda > 0$)

$$p\zeta^{\frac{1}{2}} \frac{d\zeta}{dr} = -p \frac{\sqrt{1 - \frac{k_3^2 r^2}{p^2}}}{\frac{k_3 r}{p}} = -p \frac{\sqrt{1 - \frac{k_3^2 r^2}{p^2}}}{r} = -\sqrt{\frac{p^2}{r^2} - k_3^2} = -\sqrt{\frac{p^2}{r^2} + k_z^2 - \epsilon \mu \omega^2}.$$

Hence $2p(s)q(s) = -\Im \epsilon \mu \omega^2$, which gives the second result. □

Let $r_e > 0$ solve

$$\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r^2} = 0, \quad (30)$$

id est $r_e = \frac{\sqrt{\Re\epsilon\mu\omega^2 - k_z^2}}{p} R$. Fixing $\eta = \frac{k_z}{\omega}$ and $q = \frac{p}{\omega R}$ (dimensionless numbers), we observe

$$r_e = \frac{\sqrt{\Re\epsilon\mu - \eta^2}}{q} R.$$

Lemma 9. 1. *The identity holds true*

$$i\Psi(r_e) = -\frac{2}{3}p\zeta^{\frac{3}{2}}(r_e).$$

2. *We have $\Re\frac{2}{3}p\zeta^{\frac{3}{2}}(r_e) < 0$.*

3. *$\Re i\Psi(\beta(r)) > 0$ for all $r > r_e$, and is an increasing function on $r > r_e$,*

4. *There exists $r_- < r_e$ such that $\Re\frac{2}{3}p\zeta_e^{\frac{3}{2}} = \int_{r_-}^{r_e} p(s)ds$ for $r < r_-$, and $\Re\frac{2}{3}\zeta^{\frac{3}{2}} > 0$ for $r > r_-$, and it is a decreasing function of r .*

Remark that the point $r = r_e$ is the only point where the limit of Φ and the limit of ζ can be computed simultaneously according to our definitions.

Proof. Let $\psi(r) = \Psi(\beta(r))$. One has $\psi'(r) = \frac{d\beta}{dr}\Psi'(\beta(r)) = \frac{d\beta}{dr}p \tan^2 \beta(r)$. Using $-\sin\beta(r)\frac{d\beta}{dr} = -\frac{k_3}{pr^2}$, one gets

$$\psi'(r) = k_3 \sin\beta(r) = \alpha(r) + i\beta(r).$$

Let $g(r) = \frac{2}{3}p\zeta^{\frac{3}{2}}(z(r))$. One already obtained

$$g'(r) = -(p(r) + iq(r)).$$

• For all r such that $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2} \geq 0$, $\alpha^2(r) - \beta^2(r) = \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2}$, $2\alpha\beta = \Im\epsilon\mu\omega^2$, $\alpha > 0$, $\beta < 0$.

• For all r such that $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r^2} \leq 0$, $p^2(r) - q^2(r) = \frac{p^2}{r^2} + k_z^2 - \Re\epsilon\mu\omega^2$, $2pq = -\Im\epsilon\mu\omega^2$, $p > 0$, $q > 0$. One deduces $\alpha^2 - \beta^2 = q^2 - p^2$, $\alpha\beta = -pq$.

One checks that $\alpha^2(r_e) = q^2(r_e)$, $\beta^2(r_e) = q^2(r_e)$, $2\alpha(r_e)\beta(r_e) = -2p(r_e)q(r_e)$, hence, using the sign conditions, $\alpha(r_e) = q(r_e)$, $\beta(r_e) = -p(r_e)$ and $i(\alpha(r_e) + i\beta(r_e)) = p(r_e) + iq(r_e)$.

The function Ψ satisfies $\Psi(0) = 0$, hence the complex continuation of ψ satisfies $\psi(r_*) = 0$ for $r_* \in \mathbb{C}$ such that $\beta(r_*) = \alpha(r_*) = 0$, that is r_* given by $k_3 r_* = p$. One deduces $i\psi(r_e) = \int_{r_*}^{r_e} (-\beta(s) + i\alpha(s))ds$. Similarly, the function ζ satisfies $\frac{2}{3}p\zeta^{\frac{3}{2}}(1) = 0$, hence $g(r) := \frac{2}{3}p\zeta^{\frac{3}{2}}(z(r))$ satisfies $g(r_*) = 0$ (point such that $z = 1$ id est $p = k_3 r_*$ as well). The equality obtained above shows $g(r_e) = -\int_{r_*}^{r_e} (p(s) + iq(s))ds$. Using the identity $p(s) = -\beta(s)$, $q(s) = \alpha(s)$, one obtains

$$g(r_e) = -i\psi(r_e).$$

This summarizes as the first item of Lemma 9.

For the second item, introduce $\alpha = \frac{-\Im\epsilon\mu\omega^2}{\sqrt{\Re\epsilon\mu\omega^2 - k_z^2}} > 0$. From $\frac{2}{3}\zeta_e^{\frac{3}{2}} = \int_{z_e}^1 \frac{\sqrt{1-t^2}}{t} dt = \int_{\sqrt{1-i\alpha}}^1 \frac{\sqrt{1-t^2}}{t} dt$, and denoting by $iu = 1 - t^2$, id est $dt = -\frac{idu}{2t}$, one obtains

$$\frac{2}{3}\zeta_e^{\frac{3}{2}} = i^{\frac{3}{2}} \int_0^\alpha \frac{\sqrt{u}}{2(1-iu)} du = i^{\frac{3}{2}} \int_0^\alpha \frac{\sqrt{u}(1+iu)}{2(1+u^2)} du = i^{\frac{3}{2}} \int_0^\alpha \frac{\sqrt{u}}{2(1+u^2)} du + i^{\frac{5}{2}} \int_0^\alpha \frac{u\sqrt{u}}{2(1+u^2)} du$$

where $i^{\frac{3}{2}}$ and $i^{\frac{5}{2}}$ are both of negative real part, hence the inequality.

For the expressions of $\alpha(r)$ and $\beta(r)$, one has

$$\alpha^2(r) - \beta^2(r) = \frac{p^2}{r_e^2} - \frac{p^2}{r^2}, \alpha^2(r) + \beta^2(r) = |i\mathfrak{I}\epsilon\mu\omega^2 + \frac{p^2}{r_e^2} - \frac{p^2}{r^2}|$$

which extends the expression of $\alpha(r), \beta(r)$ for $r < r_e$, and one observes that, extending also the same expression for $p(r)$ and $q(r), i(\alpha(r) + i\beta(r)) = p(r) + iq(r)$. We may thus notice that one could formally deduce the identity $i\Psi(\beta(r)) = -\frac{2}{3}p\zeta^{\frac{3}{2}}(z(r))$ for all r , by extending the equalities defining p, q, α, β .

The third item uses $\Re i\Psi(r) = \Re i\Psi(r_e) - \int_{r_e}^r \beta(s)ds$, with $\beta < 0, r > r_e$ and $\Re i\Psi(r_e) > 0$, which implies that, for all $r > r_e, \Re i\Psi(r) \geq \Re i\Psi(r_e) > 0$.

For the last item, use $\Re \frac{2}{3}p\zeta^{\frac{3}{2}}(z(r)) = \Re \frac{2}{3}p\zeta_e^{\frac{3}{2}} - \int_{r_e}^r p(s)ds = \Re \frac{2}{3}p\zeta_e^{\frac{3}{2}} + \int_r^{r_e} p(s)ds$. As there exists $\delta_0 > 0$ such that $\delta_0 \leq sp(s) \leq 2\delta_0$ for s large enough, $\int_r^{r_e} p(s)ds \rightarrow +\infty$ when $r \rightarrow 0_+$, there is a unique solution to $\Re \frac{2}{3}p\zeta_e^{\frac{3}{2}} + \int_r^{r_e} p(s)ds = 0$, that one calls r_- . As $\Re \frac{2}{3}p\zeta^{\frac{3}{2}}$ is strictly decreasing, $\Re \frac{2}{3}p\zeta^{\frac{3}{2}} > 0$ for $r < r_-$ and $\Re \frac{2}{3}p\zeta^{\frac{3}{2}} < 0$ for $r > r_-$.

Remark 4. We could also have deduced the equalities above, using the fact that $\zeta(z)$ is solution of the ODE

$$\zeta\left(\frac{d\zeta}{dz}\right)^2 = \frac{1 - z^2}{z^2},$$

which transforms, with $z = \frac{k_3 r}{p}$, into

$$p\zeta\left(\frac{d\zeta}{dr}\right)^2 = \frac{p^2}{r^2} + k_z^2 - \epsilon\mu\omega^2 = (p(r) + iq(r))^2.$$

□

It is useful to introduce and recall the following notations:

$$k_3 = \sqrt{\epsilon\mu\omega^2 - k_z^2}, k_\perp = \sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}, k_l = \frac{p}{R}, \cos\beta_* = \frac{k_l}{k_3}, \sin\beta_* = \frac{k_\perp}{k_3}, \tan\beta_* = \frac{k_\perp}{k_l}. \quad (31)$$

4.3. Eikonal equations.

We rely on the previous asymptotics of the Bessel and Hankel functions in the case where there exists $K > 0$ such that $k_z^2 + \frac{p^2}{r^2} \leq K\omega^2$. It is always the case, as it was noticed, when one is in the hyperbolic, top glancing, bottom glancing, and mixed hyperbolic elliptic regimes on $[r_0, R]$, and we enforce it additionally when $\frac{p^2}{R^2} + k_z^2 > \Re\epsilon\mu\omega^2$. This is called the asymptotic regime. The PDO that one considers is

$$r^{-1}\partial_r(r\partial_r) + \partial_z^2 + r^{-2}\partial_\theta^2 + \omega^2\epsilon\mu.$$

After a change of unknown of the form $v = r^{\frac{1}{2}}u$ and a change of notation $r = R + n$, it writes, in the high-frequency regime, as the action of the operator $\partial_{n^2}^2 + R(n, i^{-1}\omega^{-1}\partial_z, i^{-1}\partial_\theta)$ (k_z as the dual variable of z and p as the (discrete) dual variable of θ):

$$R(n, k_z, p) = \epsilon\mu\omega^2 - k_z^2 - \frac{p^2 - \frac{1}{4}}{(n + R)^2}.$$

The eikonal equation associated with this PDE is thus

$$(\partial_r \Phi)^2 + (R+n)^{-2} (\partial_\theta \Phi)^2 + \epsilon \mu \omega^2 - k_z^2 = 0. \quad (32)$$

In the classical analysis of operators of the form $\partial_n^2 + R(n, x, i^{-1} \partial_x)$, in the case $\epsilon \mu \in \mathbb{R}_+^*$, the elliptic region is characterized as the set of points $(x_0, \xi_0) \in T^* \partial \Omega$ for which $R(0, x_0, \xi_0) < 0$. In the case of the Helmholtz equation in cylindrical coordinates, we observe that, considering p as the discrete Fourier variable in θ , this corresponds, when $\epsilon \mu \in \mathbb{R}_+^*$, to $\epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2} < 0$. In a similar fashion, the hyperbolic region is characterized by $\epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2} > 0$, and the glancing region is characterized by $\epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2} = 0$. In the latter case, as $\partial_n R(0, k_z, p) = \frac{2p^2}{R^3} > 0$, we fall in the category of **strictly diffractive points** (which is not surprising, because this corresponds to the model problem of the strictly diffractive analysis).

The classical decomposition of the space of (k_z, p) into elliptic, hyperbolic, and glancing region respectively for $R(0, k_z, p) < 0$, $R(0, k_z, p) > 0$ and $R(0, k_z, p) = 0$, where $n = 0$ is equivalent to $r = R$ is valid for a real principal symbol.

If one assumes that we keep the same division in regions when $\epsilon \mu \notin \mathbb{R}$ is replaced by $\Re(\epsilon \mu)$, we keep the same division. We shall study the asymptotics of the Dirichlet to Neumann operator in the three regions and adopt the terminology 'elliptic-hyperbolic-glancing' even if it is not mathematically true (except if we use a limiting absorption principle with $\Im \epsilon \mu \rightarrow 0^-$).

- Elliptic region: $\frac{\Re k_3 R}{p} < 1$. One has thus to study asymptotics of $H_p^{(j)}(pz)$, where z is defined in (29).

We use the estimates (22) of Section 4.2. The asymptotic solutions use the Airy functions $Ai(e^{\pm \frac{2i\pi}{3}} p^{\frac{2}{3}} \zeta)$, associated with the phase

$$G(\theta, r) = p\theta \pm i \frac{2}{3} p \zeta^{\frac{3}{2}},$$

where ζ is defined in (29).

- Hyperbolic region: we consider β defined in (29).

We deduce from the asymptotic expansion of the Hankel functions an equivalent for the Dirichlet to Neumann operator is a consequence of the analysis of $H_p^{(1)}(k_3 r)$ when p and k_3 are of the same order. Following Abramovitz and Stegun [1], one has the following phase for the Bessel functions (estimates (19) of Section 4.2)

$$\Psi(p, \beta) = p(\tan \beta - \beta).$$

and the total phase is

$$\Phi(\theta, r) = \Psi(p, \beta) + p\theta.$$

Lemma 10. For each (k_z, p) , define r_e such that $\Re \epsilon \mu \omega^2 - k_z^2 = \frac{p^2}{r_e^2}$.

1. For $r > r_e$, use (19). The phase $\Phi : (\theta, r) \rightarrow \Psi(p, \beta) + p\theta$ is a solution of the eikonal equation

$$(\partial_r \Phi(r, \theta))^2 + \left(\frac{\partial_\theta \Phi(r, \theta)}{r} \right)^2 = k_3^2 = \omega^2 \epsilon \mu - k_z^2.$$

2. For $r < r_e$, use (22). The phase $G : (\theta, r) \rightarrow p\theta - \frac{2}{3} \zeta^{\frac{3}{2}}$ is solution of the eikonal equation as well:

$$(\partial_r G(r, \theta))^2 + \left(\frac{\partial_\theta G(r, \theta)}{r} \right)^2 = k_3^2 = \omega^2 \epsilon \mu - k_z^2.$$

3. For $r \simeq r_e$, use (25). The phase $h : (\theta, r) \rightarrow p\theta \pm \frac{2}{3}(2^{\frac{1}{3}}a)^{\frac{3}{2}}$, with $k_3r = p + ap^{\frac{1}{3}}$ solves as well

$$(\partial_r h)^2 + \frac{1}{r^2}(\partial_\theta h)^2 = \omega^2 \epsilon \mu - k_z^2.$$

Proof. Consider

$$\Phi(r, \theta) = p(\theta + \tan \beta - \beta), \cos \beta = \frac{p}{k_3 r}.$$

One has

$$\partial_r \Phi = p \partial_r \beta \tan^2 \beta, \partial_\theta \Phi = p.$$

Using $k_3 r = \frac{p}{\cos \beta}$, $\partial_r f = \frac{1}{r \tan \beta} \partial_\beta f$, hence

$$(\partial_r \Phi(r, \theta))^2 + \left(\frac{\partial_\theta \Phi(r, \theta)}{r}\right)^2 = k_3^2.$$

This is the eikonal equation associated with the PDE. In the case of $r = R + n$ (where the description corresponds to a circle of radius R and n is the algebraic distance to the circle), one finds

$$(\partial_n \phi(n, \theta))^2 = k_3^2 - \frac{p^2}{(n + R)^2}.$$

The first eikonal equation is obtained.

Consider now G . One notices

$$\partial_r G = \pm i p \partial_r \left(\frac{2}{3} \zeta^{\frac{3}{2}}\right), \partial_\theta G = p$$

Using $p \partial_r = k_3 \partial_z$ and the expression of $\frac{2}{3} \zeta^{\frac{3}{2}}$, one obtains

$$\partial_r G = \pm i k_3 \frac{\sqrt{1 - z^2}}{z}.$$

One then obtains

$$(\partial_r G)^2 + r^{-2}(\partial_\theta G)^2 = -k_3^2 \frac{1 - z^2}{z^2} + \frac{p^2}{r^2} = k_3^2 + \frac{p^2}{r^2} - k_3^2 \frac{p^2}{k_3^2 r^2} = k_3^2.$$

This phase also solves the eikonal equation.

Finally, consider the case where

$$k_3 r = p + ap^{\frac{1}{3}},$$

which corresponds to the glancing case.

The relation $\partial_r h = \partial_r a \partial_a h$ yields $\partial_r h = \frac{k_3}{\frac{1}{3}} \partial_a h$, hence one is left with $\frac{k_3^2}{p^{\frac{2}{3}}} (\partial_a h)^2 + \frac{k_3^2}{(p + ap^{\frac{1}{3}})^2}$. As $h(a) = \pm \frac{2}{3}(2^{\frac{1}{3}}a)^{\frac{3}{2}}$, $\partial_a h(a) = k_3^2(1 + O(a^2 p^{-\frac{2}{3}}))$. \square

Remark 5. One deduces p in terms of a and k_3 through Cardan formulae on $\tau = \left(\frac{p}{\omega R}\right)^{\frac{1}{3}}$.

The complex number τ is a solution of

$$\tau^3 + a(\omega R)^{-\frac{2}{3}}\tau = \frac{k_3}{\omega} = \sqrt{\epsilon\mu - \eta^2}, \eta = \frac{k_z}{\omega}.$$

which leads to

$$\tau = \left(\frac{1}{2}\right)^{\frac{1}{3}} \left[\left(\sqrt{\epsilon\mu - \eta^2 + \frac{4a^3}{27(R\omega)^2}} + \sqrt{\epsilon\mu - \eta^2} \right)^{\frac{1}{3}} - \left(\sqrt{\epsilon\mu - \eta^2 + \frac{4a^3}{27(R\omega)^2}} - \sqrt{\epsilon\mu - \eta^2} \right)^{\frac{1}{3}} \right],$$

and with the choice $\eta^2 < \Re\epsilon\mu$, one obtains, under $\frac{a^3}{R^2\omega^2} \ll 1$, an approximate value for τ^3 equal to $\sqrt{\epsilon\mu - \eta^2}$, and the inequality

$$|\tau^6 - \epsilon\mu + \eta^2| \leq D_3 \frac{a^3}{(\omega R)^2}$$

where D_3 depends also on $\sqrt{\epsilon\mu - \eta^2}$. In this case $a(r)$ can be introduced such that $k_3 r = p + a(r)p^{\frac{1}{3}}$.

We will continue this section by using the classification of Definition 2. One needs to observe the following thing

Remark 6. *The Dirichlet to Neumann operator uses the asymptotics of the Hankel functions and their derivatives at $r = r_0$ and at $r = R$. This means that it uses the same phase function at the numerator and at the denominator, denoted by $l(r_0, R, k_z, p, \omega)$, $\Re l(r_0, R, k_z, p, \omega) > 0$ and the exact DTN for the mode p is of the form*

$$\frac{n_+ e^l - n_- e^{-l}}{d_+ e^l - d_- e^{-l}} = \frac{n_+ - n_- e^{-2l}}{d_+ - d_- e^{-2l}}.$$

we will thus only take care of the coefficients n_{\pm}, d_{\pm} in the analysis.

4.4. Estimates on the solution

We have the estimate

Proposition 3. *Assume $\Re\epsilon\mu\omega^2 > k_z^2$. For all M large enough, there exists a constant D such that, for all $\omega \geq M$, for all p, k_z , for all $r \in [r_0, R]$*

$$\left| \frac{d_p(k_3 r)}{d_p(k_3 R)} \right| \leq D.$$

Our aim is to obtain the estimate of Proposition 3. We will use the following equalities of the toolbox:

1. In the hyperbolic case, we use (19) of Section 4.2,
2. In the elliptic case, we use (22) of Section 4.2,
3. In the mixed-elliptic-hyperbolic case, for all $r \in [r_0, R]$, we use (22) for the behavior at r_0 , (19) for the behavior at R .

This proposition 3 is detailed above, so that we can see the different cases that are involved here. We state two detailed results.

Proposition 4. • *For all (p, k_z) satisfying $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} > 0$ we are in the totally hyperbolic region. There exists a constant $D_0 > 0$ such that*

$$\left| \frac{d_p(k_3 r)}{d_p(k_3 R)} \right| \leq D_0 e^{\Im(\Psi_* - \Psi(\beta(r)))} \leq D_0. \quad (33)$$

- For all (p, k_z) such that

$$\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2} < 0 < \Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{r_0^2}$$

assume that r_e does not depend on ω . One is in the mixed elliptic-hyperbolic region.

- ★ For $r_+ < r_0 < r < r_e$, there exists a constant K such that

$$\left| \frac{J_p(k_3 r) Y_p(k_3 r_0) - Y_p(k_3 r) J_p(k_3 r_0)}{J_p(k_3 R) Y_p(k_3 r_0) - Y_p(k_3 R) J_p(k_3 r_0)} \right| \leq K e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}} - \Re(\frac{2}{3} p \zeta^{\frac{3}{2}} + i \Psi_*)} \leq K. \quad (34)$$

- ★ For $r_0 < r < r_+$, there exists a constant K_0 such that

$$\left| \frac{J_p(k_3 r) Y_p(k_3 r_0) - Y_p(k_3 r) J_p(k_3 r_0)}{J_p(k_3 R) Y_p(k_3 r_0) - Y_p(k_3 R) J_p(k_3 r_0)} \right| \leq G_0 e^{\frac{2}{3} p \Re(\zeta^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}})} \leq K_0 e^{-\Re(\frac{2}{3} p \zeta^{\frac{3}{2}} + i \Psi_*)} \leq K_0. \quad (35)$$

One notes that, as $|k_z^2 r^2 - p^2| \geq |\Im \epsilon \mu| \omega^2$, one is never in the glancing case.

Proof. We consider the totally hyperbolic regime. Recall that $\Re i(\Psi(\beta(r)) - \Psi(\beta_0)) > 0$ because $\Re i \Psi$ is increasing. For notational simplicity, we write Ψ_0 instead of $\Psi(p, \beta(r_0)) = \Psi(p, \beta_0)$, L_0, M_0, L, M, L_*, M_* instead of, respectively, $L(p, \beta_0), M(p, \beta_0)$ and so on.

$$\frac{J_p(k_3 r) Y_p(k_3 r_0) - Y_p(k_3 r) J_p(k_3 r_0)}{J_p(k_3 R) Y_p(k_3 r_0) - Y_p(k_3 R) J_p(k_3 r_0)} = \frac{e^{i(\Psi(\beta(r)) - \Psi_*)} (L_0 - iM_0)(L + iM) - (L_0 + iM_0)(L - iM) e^{2i(\Psi_0 - \Psi(\beta(r)))}}{(L_0 - iM_0)(L_* + iM_*) - (L_0 + iM_0)(L_* - iM_*) e^{2i(\Psi_0 - \Psi_*)}},$$

As $|e^{2i(\Psi_0 - \Psi_*)}| \leq 1$, $|e^{2i(\Psi_0 - \Psi(\beta(r)))}| \leq 1$, using precise uniform bounds on L and M , one checks that there exists a constant $D_0 > 0$, **independent on $p; k_z$** , such that

$$\left| \frac{J_p(k_3 r) Y_p(k_3 r_0) - Y_p(k_3 r) J_p(k_3 r_0)}{J_p(k_3 R) Y_p(k_3 r_0) - Y_p(k_3 R) J_p(k_3 r_0)} \right| \leq D_0 e^{\Im(\Psi_* - \Psi(\beta))} \leq D_0.$$

As k_z and $\frac{p}{r}$ are controlled by ω , we get an uniform control of β, β_0, β_* hence of L, M and other quantities above.

Let us look now at the elliptic-hyperbolic case.

For $r_0 < r < r_e$, it is natural to use the previous result (because we use the elliptic representation at r_0 and at r). Assume that $r_0 > r_-$ first. Note that $\Re(i\Psi_* + \frac{2}{3} p \zeta^{\frac{3}{2}}) > 0$ and recall that $\Re \zeta_0^{\frac{3}{2}} < 0$.

$$\frac{J_p(pz_0) Y_p(pz) - J_p(pz) Y_p(pz_0)}{J_p(pz_0) Y_p(\frac{p}{\cos \beta_*}) - J_p(\frac{p}{\cos \beta_*}) Y_p(pz_0)} = \frac{e^{\frac{2}{3} p(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})} J_p(pz_0) e^{\frac{2}{3} p \zeta_0^{\frac{3}{2}}} Y_p(pz) e^{-\frac{2}{3} p \zeta^{\frac{3}{2}}} e^{-\frac{4}{3} p \Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})} - J_p(pz) e^{\frac{2}{3} p \zeta^{\frac{3}{2}}} Y_p(pz_0) e^{-\frac{2}{3} p \zeta_0^{\frac{3}{2}}}}{e^{i\Psi_* - \frac{2}{3} p \zeta_0^{\frac{3}{2}}} J_p(pz_0) e^{\frac{2}{3} p \zeta_0^{\frac{3}{2}}} Y_p(\frac{p}{\cos \beta_*}) e^{-i\Psi_*} - J_p(\frac{p}{\cos \beta_*}) e^{-i\Psi_*} [Y_p(pz_0) e^{-\frac{2}{3} p \zeta_0^{\frac{3}{2}}} e^{\frac{4}{3} p \zeta_0^{\frac{3}{2}}}]}$$

Using $J_p(pz_0) e^{\frac{2}{3} p \zeta_0^{\frac{3}{2}}}$, $Y_p(pz) e^{-\frac{2}{3} p \zeta^{\frac{3}{2}}}$, $J_p(pz) e^{\frac{2}{3} p \zeta^{\frac{3}{2}}}$, $Y_p(pz_0) e^{-\frac{2}{3} p \zeta_0^{\frac{3}{2}}}$ bounded, $|e^{\frac{4}{3} p \zeta_0^{\frac{3}{2}}}| < 1$, $|e^{-\frac{4}{3} p \Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})}| < 1$, one gets, with the equality

$$\sqrt{\frac{\pi p \tan \beta_*}{2}} J_p\left(\frac{p}{\cos \beta_*}\right) = \frac{1}{2} [(L - iM) e^{i\Psi_*} + (L + iM) e^{-i\Psi_*}], \quad \sqrt{\frac{\pi p \tan \beta_*}{2}} Y_p\left(\frac{p}{\cos \beta_*}\right) = \frac{1}{2i} [(L - iM) e^{i\Psi_*} - (L + iM) e^{-i\Psi_*}], \quad (36)$$

$$\left| \frac{d_p(k_3 r)}{d_p(k_3 R)} \right| \leq K |e^{\frac{2}{3} p (\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}}) - i \Psi_* + \frac{2}{3} \zeta_0^{\frac{3}{2}}} | = K e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}} - \Re(\frac{2}{3} p \zeta^{\frac{3}{2}} + i \Psi_*)} \leq K.$$

Assume now $r_0 < r < r_-$,

$$\frac{J_p(pz_0)Y_p(pz) - J_p(pz)Y_p(pz_0)}{J_p(pz_0)Y_p(\frac{p}{\cos \beta_*}) - J_p(\frac{p}{\cos \beta_*})Y_p(pz_0)} = \frac{e^{\frac{2}{3} p (\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})} J_p(pz_0) e^{\frac{2}{3} p \zeta_0^{\frac{3}{2}}} Y_p(pz) e^{-\frac{2}{3} p \zeta^{\frac{3}{2}}} e^{-\frac{4}{3} p \Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})} - J_p(pz) e^{\frac{2}{3} p \zeta^{\frac{3}{2}}} Y_p(pz_0) e^{-\frac{2}{3} p \zeta_0^{\frac{3}{2}}}}{e^{i \Psi_* + \frac{2}{3} p \zeta_0^{\frac{3}{2}}} J_p(pz_0) e^{\frac{2}{3} p \zeta_0^{\frac{3}{2}} - \frac{4}{3} p \zeta_0^{\frac{3}{2}}} Y_p(\frac{p}{\cos \beta_*}) e^{-i \Psi_*} - J_p(\frac{p}{\cos \beta_*}) e^{-i \Psi_*} Y_p(pz_0) e^{-\frac{2}{3} p \zeta_0^{\frac{3}{2}}}}.$$

Use again the estimate (36), the inequality $\Re \frac{2}{3} \zeta_0^{\frac{3}{2}} > 0$ (because $r_0 < r_+$), $\Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}}) > 0$, we obtain

$$\left| \frac{d_p(k_3 r)}{d_p(k_3 R)} \right| \leq K_0 |e^{-\frac{2}{3} p \zeta^{\frac{3}{2}} - i \Psi_*}| = K e^{-\Re(\frac{2}{3} p \zeta^{\frac{3}{2}} + i \Psi_*)} \leq K.$$

□

Proposition 5 deals with the totally elliptic case (that is $k_z^2 + \frac{p^2}{R^2} > \Re \epsilon \mu \omega^2$). It enables to treat the case where $\frac{k_z^2 + \frac{p^2}{R^2}}{\omega^2} \rightarrow +\infty$ when $\omega \rightarrow +\infty$.

Proposition 5. For all (p, k_z) satisfying $\Re \epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2} < 0$, we are in the totally elliptic region. Let $r_0 < r \leq R$. There exists a constant K_0 such that one has, for ω large enough

$$\left| \frac{J_p(k_3 r)Y_p(k_3 r_0) - Y_p(k_3 r)J_p(k_3 r_0)}{J_p(k_3 R)Y_p(k_3 r_0) - Y_p(k_3 R)J_p(k_3 r_0)} \right| \leq e^{-\frac{2}{3} p \Re(\zeta^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})} K_0 \leq K_0. \quad (37)$$

More precisely, for any $\delta > 0$ small enough, there exists ω_δ such that there exists K_δ such that (37) is uniformly true for all $r \in [r_0 + \delta, R]$.

Proof. In the totally elliptic regime, introduce the notations z, z_0, z_* and ζ, ζ_0, ζ_* corresponding to $pz = k_3 r$, $pz_0 = k_3 r_0$, $pz_* = k_3 R$.

One notices that the functions $H_p^{(j)}(pz)$ contains $e^{\frac{2}{3} p \zeta^{\frac{3}{2}}}$ as well as $e^{-\frac{2}{3} p \zeta^{\frac{3}{2}}}$, because it contains $Ai(p^{\frac{2}{3}} \zeta)$ as well as $Bi(p^{\frac{2}{3}} \zeta)$. It is then more relevant to distinguish the role of Ai and Bi , and use the Bessel functions as in (24)

$$\frac{d_p(k_3 r)}{d_p(k_3 R)} = \frac{J_p(pz)Y_p(pz_0) - J_p(pz_0)Y_p(pz)}{J_p(pz_*)Y_p(pz_0) - J_p(pz_0)Y_p(pz_*)} = \frac{J_p(pz)}{J_p(pz_*)} \frac{1 - \tau}{1 - \tau_*},$$

where

$$\tau = \frac{Y_p(pz)J_p(pz_0)}{J_p(pz)Y_p(pz_0)}, \tau_* = \frac{Y_p(pz_*)J_p(pz_0)}{J_p(pz_*)Y_p(pz_0)}.$$

We use (24) to deduce estimates for τ and τ_* . We then have to study estimates for $\frac{Bi(p^{\frac{2}{3}} \zeta) Ai(p^{\frac{2}{3}} \zeta_0)}{Ai(p^{\frac{2}{3}} \zeta) Bi(p^{\frac{2}{3}} \zeta_0)}$. As the classical estimates for the Airy functions ([6] 9.7(ii)) are valid uniformly in the sector $|\arg z| \leq \pi - \delta_*$, one concludes that

$$\frac{Bi(p^{\frac{2}{3}} \zeta) e^{-\frac{2}{3} p \zeta^{\frac{3}{2}}} Ai(p^{\frac{2}{3}} \zeta_0) e^{\frac{2}{3} p \zeta_0^{\frac{3}{2}}}}{Ai(p^{\frac{2}{3}} \zeta) e^{\frac{2}{3} p \zeta^{\frac{3}{2}}} Bi(p^{\frac{2}{3}} \zeta_0) e^{-\frac{2}{3} p \zeta_0^{\frac{3}{2}}}}$$

is bounded, hence there exists M such that

$$\left| \frac{Bi(p^{\frac{2}{3}}\zeta)e^{-\frac{2}{3}p\zeta^{\frac{3}{2}}} Ai(p^{\frac{2}{3}}\zeta_0)e^{\frac{2}{3}p\zeta_0^{\frac{3}{2}}}}{Ai(p^{\frac{2}{3}}\zeta)e^{\frac{2}{3}p\zeta^{\frac{3}{2}}} Bi(p^{\frac{2}{3}}\zeta_0)e^{-\frac{2}{3}p\zeta_0^{\frac{3}{2}}}} \right| = \left| \frac{Bi(p^{\frac{2}{3}}\zeta)Ai(p^{\frac{2}{3}}\zeta_0)}{Ai(p^{\frac{2}{3}}\zeta)Bi(p^{\frac{2}{3}}\zeta_0)} \right| e^{-\frac{4}{3}p(\zeta^{\frac{3}{2}} - \zeta_0^{\frac{3}{2}})} \leq M$$

This shows that

$$|\tau| \leq Me^{-\frac{4}{3}\Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})}, |\tau_*| \leq Me^{-\frac{4}{3}\Re(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})}$$

and one uses $Me^{-\frac{4}{3}\Re(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})} < \frac{1}{2}$ and $Me^{-\frac{4}{3}\Re(\zeta_0^{\frac{3}{2}} - \zeta^{\frac{3}{2}})} < \frac{1}{2}$ for $r > r_0$ for large ω to conclude that there exists K such that $|\frac{1-\tau}{1-\tau_*}| \leq K$ for all $r, r_0 < r \leq R$ fixed, which imply Proposition 5. The precise statement at the end of this proposition comes from the fact that for all (p, k_z) satisfying $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} < 0, p \geq C\omega, \tau \rightarrow 0$ exponentially in ω . \square

The proof of Proposition 2 is a consequence of the two propositions stated above.

5. The Dirichlet to Neumann operator

The first subsection is devoted to the result stated for the plane layer on the form and the regularity of the Dirichlet to Neumann operator (which is an operator acting on functions on $\{z = 0\}$).

5.1. Proof of the result of the Introduction for the plane boundary

Lemma 11. *The Dirichlet to Neumann multiplier is a Fourier multiplier from $H^s(\mathbb{R}^2)$ to $H^{s-1}(\mathbb{R}^2)$, in particular for $s = \frac{1}{2}$ whenever $\epsilon\mu \notin \mathbb{R}_+^*$. It is not defined when $\epsilon\mu \in \mathbb{R}_+^*$. However, if $\epsilon\mu \in \mathbb{R}_+^*$, for each (k_1, k_2) given, there exists a sequence $(\omega_n(\sqrt{k_1^2 + k_2^2}))_n$ of values of ω such that the Fourier multiplier is not defined for ω belonging to this sequence. When ω does not belong to this sequence, the Fourier multiplier is well defined.*

Proof. Begin with the last item. In the case $\epsilon\mu \in \mathbb{R}_+^*$, there exists at least $(k_1, k_2) \in \mathbb{R}^2$ such that $\epsilon\mu\omega^2 = k_1^2 + k_2^2$, hence there exists at least a resonance, hence no uniqueness of the solution. The Dirichlet to Neumann multiplier is never defined as an operator on $\mathcal{S}'(\mathbb{R}^2)$. This is a consequence of the fact that the domain is unbounded, hence continuous spectrum exist and, in particular, for all (k_1, k_2) there exists at least an associated point of the continuous spectrum. However, for (k_1, k_2) given, define $\omega_n(\sqrt{k_1^2 + k_2^2}) = \sqrt{\frac{k_1^2 + k_2^2 + \pi^2 n^2}{\epsilon\mu}}$. These are values for which $\sin k_\perp l = 0$, hence non uniqueness of the solution. For a fixed value of ω which is not in this list, there exists α_1 such that $|\sin k_\perp l| \geq \alpha_1$, which ensures the existence and uniqueness of the solution of the ODE on $[-l, 0]$.

Consider now $\epsilon\mu \notin \mathbb{R}$ and assume $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$ (that is $(1 + k_1^2 + k_2^2)^{\frac{1}{4}} \hat{u}_0(k_1, k_2) \in L^2(\mathbb{R}^2)$), introduce $U(k_1, k_2, x_3) = \frac{\sin k_\perp(x+l)}{\sin k_\perp l} \hat{u}_0(k_1, k_2)$. Estimate (59) shows that $U \in L^2(\mathbb{R}^2 \times [-l, 0])$, hence

$$u(x_1, x_2, x_3) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\sin k_\perp(x_3 + l)}{\sin k_\perp l} \hat{u}_0(k_1, k_2) e^{ik_1 x_1 + ik_2 x_2} dk_1 dk_2$$

is a function of $L^2(\mathbb{R}^2 \times [-l, 0])$. Its Fourier transform solves (58). Hence U is a L^2 solution of

$$\begin{cases} (\Delta + \omega^2 \epsilon \mu)U = 0 \\ U(-l) = 0 \\ U(0) = u_0. \end{cases}$$

It is straightforward to check that $U \in H^1(\mathbb{R}^2 \times [-l, 0])$.

The trace of the normal derivative is thus given through

$$\partial_{x_3} \hat{u}(0, k_1, k_2) = k_\perp \frac{\cos(k_\perp l)}{\sin k_\perp l} \hat{u}_0(k_1, k_2)$$

from which one deduces that $\partial_{x_3} \hat{u}(0, k_1, k_2) = \hat{C}(k_1, k_2) \hat{u}_0(k_1, k_2)$ and that $|\hat{C}| \leq M \sqrt{k_1^2 + k_2^2}$. One has thus $(1 + k_1^2 + k_2^2)^{-\frac{1}{4}} \partial_{x_3} \hat{u}(0, k_1, k_2) \leq M(1 + k_1^2 + k_2^2)^{\frac{1}{4}} \hat{u}_0(k_1, k_2)$, hence \hat{C} is a Fourier multiplier, and $\partial_{x_3} u(., ., 0) = C u(., ., 0)$ where C goes from $H^{\frac{1}{2}}(\mathbb{R}^2)$ to $H^{-\frac{1}{2}}(\mathbb{R}^2)$. \square

5.2. The Dirichlet to Neumann operator for the cylindrical layer

Proposition 6. *The Dirichlet to Neumann operator C is a Fourier multiplier, which expression is, for each (p, k_z) such that $k_3 \neq 0$ and k_3 is not a resonance of the problem*

$$C(p, k_z) = k_3 \frac{H_p^{(2)}(k_3 r_0) (H_p^{(1)})'(k_3 R) - H_p^{(1)}(k_3 r_0) (H_p^{(2)})'(k_3 R)}{H_p^{(2)}(k_3 r_0) H_p^{(1)}(k_3 R) - H_p^{(1)}(k_3 r_0) H_p^{(2)}(k_3 R)}.$$

In the case $k_3 = 0$, one obtains, for $\tau = \frac{R}{r_0}$

$$C(p, k_z) = \frac{p \tau^p + \tau^{-p}}{R \tau^p - \tau^{-p}}.$$

Proof. Consider $a^p(k_z) \in L^2(\mathbf{R})$. Thanks to Proposition 3 and Proposition ??, the formal solution (11) written in Proposition 1 satisfies

$$(k_z, r) \rightarrow a^p(k_z) \frac{H_p^{(2)}(k_3 r_0) H_p^{(1)}(k_3 r) - H_p^{(1)}(k_3 r_0) H_p^{(2)}(k_3 r)}{H_p^{(2)}(k_3 r_0) H_p^{(1)}(k_3 R) - H_p^{(1)}(k_3 r_0) H_p^{(2)}(k_3 R)} \in L^2([r_0, R] \times \mathbf{R})$$

and if $a^p(k_z) \in H^1(\mathbf{R}_{k_z}) \times h^1([0, 2\pi])$,

$$(k_z, r, \theta) \rightarrow a^p(k_z) \frac{H_p^{(2)}(k_3 r_0) H_p^{(1)}(k_3 r) - H_p^{(1)}(k_3 r_0) H_p^{(2)}(k_3 r)}{H_p^{(2)}(k_3 r_0) H_p^{(1)}(k_3 R) - H_p^{(1)}(k_3 r_0) H_p^{(2)}(k_3 R)} e^{ip\theta} \in H^1([r_0, R] \times [0, 2\pi] \times \mathbf{R}_{k_z}).$$

The derivative in r is indeed in L^2 thanks to $|k_3| \leq |k_z|$, and one uses $k_z \rightarrow k_3 a^p(k_z)$. Similarly, the derivative in θ is in L^2 thanks to $p a^p(k_z) \in \dot{L}^2$. One can consider $\partial_r u(R, ., .)$, which is a Fourier multiplier of the corresponding mode of $u(R, ., .)$, which defines the Dirichlet to Neumann operator C from $H^1(S_R)$ to $L^2(S_R)$.

This lemma complements the results of (13), (14) of [16] by characterizing the values where (16) is not fulfilled as resonances of the problem, and by asserting estimates on the solution. Of course, one has also

$$C(p, k_z) = k_3 \frac{Y_p(k_3 r_0) J'_p(k_3 R) - J_p(k_3 r_0) Y'_p(k_3 R)}{Y_p(k_3 r_0) J_p(k_3 R) - J_p(k_3 r_0) Y_p(k_3 R)}. \quad (38)$$

Let us prove finally the result for $k_3 = 0$ (that is $\omega^2 \epsilon \mu - k_z^2 = 0$). The Helmholtz equation reads, for $U(r, \theta, z) = e^{ik_z z + ip\theta} u(r)$

$$\frac{1}{r} \partial_r (r \partial_r u) - \frac{p^2}{r^2} u = 0.$$

Solutions of this ODE are $u(r) = Ar^p + Br^{-p}$, hence the Dirichlet boundary condition at $r = r_0$ yields $u(r) = a(r^p r_0^{-p} - r_0^p r^{-p})$, hence at $r = R$ one deduces $u(R) = a_p(R^p r_0^{-p} - r_0^p R^{-p})$ and $\partial_r u(R) = p a_p(R^{p-1} r_0^{-p} + r_0^p R^{-p-1})$, from which one deduces the Dirichlet to Neumann operator as a Fourier multiplier

$$C(p, k_z) = p \frac{R^{p-1} r_0^{-p} + r_0^p R^{-p-1}}{R^p r_0^{-p} - r_0^p R^{-p}},$$

which rewrites, denoting by $\tau = \frac{r_1}{r_0}$

$$C(p, k_z) = \frac{p}{R} \frac{\tau^p + \tau^{-p}}{\tau^p - \tau^{-p}},$$

and the Dirichlet to Neumann operator is the operator such that

$$\sum_p a_p e^{ip\theta + i\sqrt{\epsilon\mu}\omega z} \rightarrow R^{-1} \sum_p p a_p \frac{\tau^p + \tau^{-p}}{\tau^p - \tau^{-p}} e^{ip\theta + i\sqrt{\epsilon\mu}\omega z}.$$

It is worth noticing that $\sum |p| a_p^2 < +\infty$ implies $\sum_{p \neq 0} |p|^{-1} (p a_p)^2 \left(\frac{\tau^p + \tau^{-p}}{\tau^p - \tau^{-p}}\right)^2 < +\infty$ hence one checks this expression sends $h^{\frac{1}{2}}([0, 2\pi])$ onto $h^{-\frac{1}{2}}([0, 2\pi])$. □

5.3. High frequency expansion of the DTN Fourier multiplier for a fixed mode in the hyperbolic regime

In this section, for p fixed, we describe a high frequency equivalent ($\omega \rightarrow +\infty$, or $|k_3| \rightarrow +\infty$, the latter being for ω fixed and $|k_z| \rightarrow +\infty$) of each mode of the Dirichlet to Neumann operator.

We added 'hyperbolic regime' through the assumption that in the limit $\Im \epsilon \mu \rightarrow 0$, $k_3 \in \mathbb{R}_+^*$.

As it is easier to manipulate the oscillating phases $e^{\pm i\Psi}$ hence we shall use $(H_p^{(1)}, H_p^{(2)})$ as the suitable pair of independent solutions for this analysis and use the equality (18) of Section 4.2. One gets

Proposition 7. *Let $C > 0$ and $|p| \leq M$ is fixed. Assume $\frac{|k_z|}{\omega} < \Re \epsilon \mu$.*

- *The leading order term, for $\omega \rightarrow +\infty$ of the Fourier multiplier for the Dirichlet to Neumann operator is $C(p, k_z) \simeq k_3 \frac{\cos k_3 l}{\sin k_3 l}$. Note that the influence of the Fourier mode has just disappeared in this equivalent; it is of lower order.*
- *One has the estimate*

$$C(p, k_z) = k_3 \frac{\cos k_3 l + \frac{1}{8k_3} \left(\frac{4p^2-1}{r_0} - \frac{4p^2+3}{R} \right) \sin k_3 l}{\sin k_3 l - \frac{1}{8k_3} \left(\frac{4p^2-1}{r_0} - \frac{4p^2-1}{R} \right) \cos k_3 l} (1 + O(\omega^{-2})).$$

- In the case $\Im\epsilon\mu \neq 0$, ($\Im k_3 \rightarrow -\infty$ when $\omega \rightarrow +\infty$),

$$C(p, k_z) = (ik_3 - \frac{1}{2R} + O(\omega^{-1}))(1 + O(e^{2\Im k_3 l})).$$

Remark 7. Note that the Item 2 of Proposition 7 shows the dependency of the Fourier multiplier in terms of the radius of curvature. This dependency is not uniform in p because the remainder term is not.

Proof. Let us prove first the last item. Using the equivalence $\cos k_3 l \simeq \frac{1}{2}e^{-ik_3 l}$, $\sin k_3 l \simeq \frac{i}{2}e^{-ik_3 l}$, thanks to $\Re k_3 > 0$, $\Im k_3 < 0$, one gets

$$C(p, k_z) \simeq k_3 \frac{1 + \frac{i}{8k_3}(\frac{4p^2-1}{r_0} + \frac{4p^2+3}{R})}{-i + \frac{1}{8k_3}(\frac{4p^2-1}{r_0} - \frac{4p^2+3}{R})} \simeq ik_3(1 + \frac{i}{8k_3}(\frac{4p^2-1}{r_0} + \frac{4p^2+3}{R}) - \frac{i}{8k_3}(\frac{4p^2-1}{r_0} - \frac{4p^2+3}{R})),$$

hence the result.

For proving the first item, it is enough to replace P by 1 and Q by 0. In this case $H_p^{(1)}(z) \simeq \frac{2}{\pi z}e^{i\chi_p(z)}$, $(H_p^{(1)})'(z) \simeq \frac{2}{\pi z}ie^{i\chi_p(z)}$, and one has thus

$$C(p, k_z) \simeq k_3 \frac{e^{i\chi_p(k_3 r_0)}(-ie^{-i\chi_p(k_3 R)}) - e^{-i\chi_p(k_3 r_0)}ie^{i\chi_p(k_3 R)}}{e^{i\chi_p(k_3 r_0)}ie^{-i\chi_p(k_3 R)} - e^{-i\chi_p(k_3 r_0)}e^{i\chi_p(k_3 R)}},$$

that is

$$C(p, k_z) \simeq k_3 \frac{\cos k_3(R - r_0)}{\sin k_3(R - r_0)} = ik_3 \frac{\cos k_3 l}{\sin k_3 l}, l = R - r_0.$$

The expansion at a higher order yields the following term of the expansion in k_3 , it depends on p . One uses

$$\begin{aligned} H_p^{(1)}(kr_0)(H_p^{(2)})'(kR) &= \frac{2}{\pi\sqrt{r_0 R}}e^{i\chi_p(r_0)-i\chi_p(R)}(1 + i\frac{4p^2-1}{8k_3 r_0})(-i - \frac{4p^2+3}{8k_3 R})(1 + O(k_3^{-2})) \\ &= \frac{2}{\pi\sqrt{r_0 R}}e^{i\chi_p(r_0)-i\chi_p(R)}(-i + \frac{1}{8k_3}(\frac{4p^2-1}{r_0} - \frac{4p^2+3}{R}) + O(k_3^{-2})), \end{aligned}$$

as well as the conjugate term $H_p^{(2)}(kr_0)(H_p^{(1)})'(kR)$, and

$$\begin{aligned} H_p^{(1)}(kr_0)H_p^{(2)}(kR) &= \frac{2}{\pi\sqrt{r_0 R}}e^{i\chi_p(r_0)-i\chi_p(R)}(1 + i\frac{4p^2-1}{8k_3 r_0})(1 - i\frac{4p^2-1}{8k_3 R})(1 + O(k_3^{-2})) \\ &= \frac{2}{\pi\sqrt{r_0 R}}e^{i\chi_p(r_0)-i\chi_p(R)}(1 + i\frac{(4p^2-1)}{8k_3}(\frac{1}{r_0} - \frac{1}{R}) + O(k_3^{-2})), \end{aligned}$$

and its conjugate expression $H_p^{(2)}(kr_0)H_p^{(1)}(kR)$.

Collecting, one deduces

$$\begin{aligned} C(p, k_z) &= k_3 \frac{e^{i\chi_p(r_0)-i\chi_p(R)}(-i + \frac{1}{8k_3}(\frac{4p^2-1}{r_0} - \frac{4p^2+3}{R})) - e^{-i\chi_p(r_0)+i\chi_p(R)}(i + \frac{1}{8k_3}(\frac{4p^2-1}{r_0} - \frac{4p^2+3}{R})) + O(k_3^{-2})}{e^{i\chi_p(r_0)-i\chi_p(R)}(1 + \frac{i(4p^2-1)}{8k_3}(\frac{1}{r_0} - \frac{1}{R})) - e^{-i\chi_p(r_0)+i\chi_p(R)}(1 - \frac{i(4p^2-1)}{8k_3}(\frac{1}{r_0} - \frac{1}{R})) + O(k_3^{-2})} \\ &= k_3 \frac{-2i \cos k_3(r_0 - R) + \frac{1}{8k_3}(\frac{4p^2-1}{r_0} - \frac{4p^2+3}{R})(2i \sin k_3(r_0 - R)) + O(k_3^{-2})}{2i \sin k_3(r_0 - R) + 2i \cos k_3(r_0 - R)\frac{(4p^2-1)}{8k_3}(\frac{1}{r_0} - \frac{1}{R}) + O(k_3^{-2})} \\ &= k_3 \frac{\cos k_3 l + \frac{1}{8k_3}(\frac{4p^2-1}{r_0} - \frac{4p^2+3}{R}) \sin k_3 l + O(k_3^{-2})}{\sin k_3 l - \frac{(4p^2-1)}{8k_3}(\frac{1}{r_0} - \frac{1}{R}) \cos k_3 l + O(k_3^{-2})}. \end{aligned}$$

This proves the second Item of Proposition 7 when p is fixed and ω goes to $+\infty$ (including the case $k_z = \eta\omega$, $\eta \in \mathbb{R}$).

Finally $\frac{\cos k_3 l}{\sin k_3 l} = i \frac{e^{ik_3 l} + e^{-ik_3 l}}{e^{ik_3 l} - e^{-ik_3 l}} = i \frac{e^{-2ik_3 l} + 1}{1 - e^{-2ik_3 l}} \rightarrow i$, $\omega \rightarrow +\infty$. \square

We shall now study a complete asymptotic regime (p, k_z, ω going to ∞).

5.4. High frequency analysis for the exact solution in cylindrical coordinates in the high frequency regime in (k_z, p) : the hyperbolic region.

The case studied in this section is

$$\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} > 0.$$

In the high frequency regime we are able to find the equivalent of Proposition 7.

Proposition 8. Assume $\frac{k_z}{\omega} = \eta$, $|\eta| < \sqrt{\Re\epsilon\mu}$, and $\omega^2\Re\epsilon\mu - k_z^2 - \frac{p^2}{r_0^2} > 0$. Let β_*, β_0 be given by $\cos\beta_* = \frac{k_3}{\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}} = \frac{\sqrt{\epsilon\mu\omega^2 - k_z^2}}{\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}}$ (and a similar definition for β_0). Denote by $\Psi_* = p(\tan\beta_* - \beta_*)$, $\Psi_0 = p(\tan\beta_0 - \beta_0)$. The Dirichlet to Neumann operator is equal to

$$C(p, k_z) = -ik_3 \sin\beta_* \frac{(L_0 - iM_0)(N_* - iO_*) + (L_0 + iM_0)(N_* + iO_*)e^{-2i(\Psi_* - \Psi_0)}}{(L_0 - iM_0)(L_* + iM_*) - (L_0 + iM_0)(L_* - iM_*)e^{-2i(\Psi_* - \Psi_0)}}.$$

We interpret easily k_\perp as the (complex) normal transmitted wave number, the tangent waves vectors are (k_l, k_z) , and β_* is an angle of incidence.

Proposition 9. The following asymptotic expansions hold:

1. For $|p| \rightarrow +\infty$ (which is equivalent to $\omega \rightarrow +\infty$)

$$C(p, k_z) = k_\perp \frac{\cos(\Psi_* - \Psi_0) - \frac{a_0+b}{p} \sin(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0) - \frac{a-a_0}{p} \cos(\Psi_* - \Psi_0)} (1 + O(p^{-2})).$$

2. One has

$$C(p, k_z) = \sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}} \frac{\cos(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0)} - \frac{\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}}{p} \frac{(a_* - a_0) \cos^2(\Psi_* - \Psi_0) - (a_0 + b_*) \sin^2(\Psi_* - \Psi_0)}{\sin^2(\Psi_* - \Psi_0)} + O(\omega^{-1}).$$

3. One has, consequently,

$$\begin{aligned} C(p, k_z) &= [i \sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}} - \frac{1}{R} \frac{\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}}{\frac{p}{R}} (a_* + b_*) + O(\omega^{-1})] (1 + O(e^{-2\Im(\Psi_* - \Psi_0)})) \\ &= [ik_\perp - \frac{1}{2R} \frac{k_3^2}{k_\perp^2} + O(\omega^{-1})] (1 + O(e^{-2\omega\delta_*})) \end{aligned}$$

Proof. Using Bessel functions toolbox, one obtains

$$C(p, k_z) = k_3 \frac{(H_p^{(2)})'(\frac{p}{\cos\beta_*})H_p^{(1)}(\frac{p}{\cos\beta_0}) - (H_p^{(1)})'(\frac{p}{\cos\beta_*})H_p^{(2)}(\frac{p}{\cos\beta_0})}{H_p^{(2)}(\frac{p}{\cos\beta_*})H_p^{(1)}(\frac{p}{\cos\beta_0}) - H_p^{(1)}(\frac{p}{\cos\beta_*})H_p^{(2)}(\frac{p}{\cos\beta_0})},$$

$$C(p, k_z) = -ik_3 \frac{\sqrt{\frac{\sin 2\beta_*}{\pi p}} \sqrt{\frac{2}{\pi p \tan\beta_0}} (e^{i\Psi_0 - i\Psi_*} (L_0 - iM_0)(N_* - iO_*) + (N_* + iO_*)(L_0 + iM_0)) e^{i\Psi_* - i\Psi_0}}{\sqrt{\frac{2}{\pi p \tan\beta_0}} \sqrt{\frac{2}{\pi p \tan\beta_*}} (e^{i\Psi_0 - i\Psi_*} (L_0 - iM_0)(L_* + iM_*) - (L_0 + iM_0)(L_* - iM_*) e^{i\Psi_* - i\Psi_0})}.$$

The expansion in inverse powers of ω is given through

$$\begin{aligned} C(p, k_z) &= -ik_3 \sin \beta_* \frac{e^{i\Psi_0 - i\Psi_*} (1 - i\frac{a_0}{p})(1 - i\frac{b_*}{p}) + (1 + i\frac{b_*}{p})(1 + i\frac{a_0}{p}) e^{i\Psi_* - i\Psi_0}}{e^{i\Psi_0 - i\Psi_*} (1 - i\frac{a_0}{p})(1 + i\frac{a_*}{p}) - (1 + i\frac{a_0}{p})(1 - i\frac{a_*}{p}) e^{i\Psi_* - i\Psi_0}} \\ &= k_3 \sin \beta_* \frac{\cos(\Psi_* - \Psi_0) - \frac{a_0 + b_*}{p} \sin(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0) - \frac{a_* - a_0}{p} \cos(\Psi_* - \Psi_0)} (1 + O(p^{-2})). \end{aligned}$$

We then have

$$\begin{aligned} C(p, k_z) - k_3 \sin \beta_* \frac{\cos(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0)} &= -\frac{k_3 \sin \beta_*}{p} \left[\frac{(a_0 + b_*) \sin^2(\Psi_* - \Psi_0) - (a_* - a_0) \cos^2(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0)(\sin(\Psi_* - \Psi_0) - \frac{a_* - a_0}{p} \cos(\Psi_* - \Psi_0))} \right] + O(\omega^{-1}), \\ &= -\frac{k_3 \sin \beta_*}{p} \left[(a_* + b_*) - \frac{(a_* - a_0)}{\sin^2(\Psi_* - \Psi_0)} \right] + O(\omega^{-1}). \end{aligned} \quad (39)$$

As $a_* + b_* = \frac{1}{2} \left(\frac{1}{\tan \beta_*} + \frac{1}{\tan^3 \beta_*} \right) = \frac{1}{2} \frac{1 + \tan^2 \beta_*}{\tan^3 \beta_*}$ one gets $\frac{k_3 \sin \beta_*}{p} (a_* + b_*) = \frac{1}{R} \tan \beta_* \frac{1}{2} \frac{1 + \tan^2 \beta_*}{\tan^3 \beta_*}$ hence $\frac{k_3 \sin \beta_*}{p} (a_* + b_*) = \frac{1}{2R \sin^2 \beta_*} = \frac{1}{2R} \frac{k_z^2}{k_\perp^2}$.

When $\Im \epsilon \mu < 0$ independent on ω , Lemma 9 shows that $\Re i(\Psi(\beta_*) - \Psi(\beta_0)) > 0$, hence there exists δ_0 such that $e^{-2i(\Psi(\beta_*) - \Psi(\beta_0))} = O(e^{-2\delta_0 \omega})$. Using

$$C(p, k_z) = -ik_3 \sin \beta_* \frac{e^{i\Psi_0 - i\Psi_*} (L_0 - iM_0)(N_* - iO_*) + (N_* + iO_*)(L_0 + iM_0) e^{i\Psi_* - i\Psi_0}}{e^{i\Psi_0 - i\Psi_*} (L_0 - iM_0)(L_* + iM_*) - (L_0 + iM_0)(L_* - iM_*) e^{i\Psi_* - i\Psi_0}},$$

one obtains

$$C(p, k_z) = ik_3 \sin \beta_* \frac{N_* + iO_*}{L_* - iM_*} (1 + O(e^{-2\Im(\Psi_* - \Psi_0)})).$$

from which one deduces

$$C(p, k_z) = (ik_3 \sin \beta_* - k_3 \sin \beta_* \frac{a(\beta_*) + b(\beta_*)}{p} + O(\omega^{-1}))(1 + O(e^{-2\Im(\Psi_* - \Psi_0)})).$$

As

$$k_3 \sin \beta_* \frac{a(\beta_*) + b(\beta_*)}{p} = pR \frac{1}{2p \tan \beta_*} \left(1 + \frac{1}{\tan^2 \beta_*} \right) = \frac{1}{2R \sin^2 \beta_*},$$

we get that the leading order term of the Dirichlet to Neumann operator is

$$i\omega \sqrt{\epsilon \mu - \frac{k_z^2}{\omega^2} - \frac{p^2}{\omega^2 R^2}}$$

and that the lower order term is

$$-\frac{1}{2R \sin^2 \beta_*} = -\frac{1}{2R} \frac{\epsilon \mu \omega^2 - k_z^2}{\epsilon \mu \omega^2 - k_z^2 - \frac{p^2}{R^2}}.$$

This can be observed directly in (39) by using the equality

$$\frac{\cos(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0)} = -i \frac{1 + e^{2i(\Psi_* - \Psi_0)}}{1 + e^{2i(\Psi_* - \Psi_0)}} \rightarrow -i,$$

and $\frac{\cos^2(\Psi_* - \Psi_0)}{\sin^2(\Psi_* - \Psi_0)} \rightarrow -1$.

□

Remark 8. • Assume $\epsilon\mu = \Re\epsilon\mu + i\frac{\sigma}{\omega}$, σ independent of ω . The leading order term in p is

$$k_{\perp} \frac{\cos(\Psi_* - \Psi_0)}{\sin(\Psi_* - \Psi_0)} + O(1).$$

- Assume again $\epsilon\mu = \Re\epsilon\mu + i\frac{\sigma}{\omega}$, σ independent of ω and $l := R - r_0$ is small. The leading order term in p of $C(p, k_3)$ is equal to

$$\frac{\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}}{\tan l \sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}} = \frac{k_{\perp}}{\tan lk_{\perp}},$$

which is the impedance of the tangent plane approximation and is identical to the result of Theorem 1 of [9], stated in Theorem 2, in the regime $l\omega$ small.

- Assume $l = R - r_0$, the width of the layer, small. The leading order term of the operator satisfies

$$C_0(p, k_z, \omega) = \frac{\sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}}}{\tan(l \sqrt{\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2}})} + O(l^2). \quad (40)$$

This expression is exactly the expression (6) in the Introduction.

Proof. As the leading order term is $C_0(p, k_z, \omega) = \frac{k_{\perp}}{\tan(\Psi_* - \Psi_0)}$, Assume l small. From

$$p(\Psi_* - \Psi_0) = (p \tan^2 \beta)(\beta - \beta_0) + O(l^2), \frac{p}{k_3} \left(\frac{1}{r_0} - \frac{1}{R} \right) = \cos \beta_0 - \cos \beta_* = (\sin \beta_*)(\beta_* - \beta_0) + O(l^2),$$

one deduces $p(\Psi_* - \Psi_0) = k_3 \sin \beta_* l + O(l^2) = \sqrt{\omega^2 \epsilon\mu - k_z^2 - \frac{p^2}{R^2}} l + O(l^2) = k_{\perp} l + O(l^2)$. This proves the last item of the Remark. \square

5.5. Dirichlet to Neumann operator for the mixed- hyperbolic-elliptic case

We are in the case

$$\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > 0 > \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2}.$$

Note that, contrary to the hyperbolic regime, an exponentially growing term appears both in $H_p^{(1)}(k_3 r_0)$ and in $H_p^{(2)}(k_3 r_0)$. This corresponds to the branches identified in [11] for the decomposition near a turning point of the solution of a second order ODE.

Proposition 10. Assume $\Im\epsilon\mu < 0$, independent on ω and $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} > 0 > \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2}$.

The Dirichlet to Neumann operator is given by

$$\begin{aligned} C(p, k_z) &= ik_3 \sin \beta_* \frac{N_* + iO_*}{L_* - iM_*} (1 + O(e^{-2\Re(i\Psi_*)})) (1 + O(e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}}})) \\ &= (ik_{\perp} - \frac{k_3^2}{2Rk_{\perp}^2} + O(p^{-2})) (1 + O(e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}}})) (1 + O(e^{-2\Re(i\Psi_*)})). \end{aligned}$$

Proof. Use

$$C(p, k_z) = k_3 \frac{(H_p^{(1)})'(k_3 R) H_p^{(2)}(k_3 r_0) - (H_p^{(2)})'(k_3 R) H_p^{(1)}(k_3 r_0)}{(H_p^{(1)})'(k_3 R) H_p^{(2)}(k_3 r_0) - (H_p^{(2)})'(k_3 R) H_p^{(1)}(k_3 r_0)} = k_3 \frac{(H_p^{(1)})'(k_3 R) \frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)} - (H_p^{(2)})'(k_3 R)}{(H_p^{(1)})'(k_3 R) \frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)} - (H_p^{(2)})'(k_3 R)}.$$

We use the expressions (24) which are more convenient. We assume that .

Assume first that $\Re \zeta_0^{\frac{3}{2}} < 0$. In this case, $Ai(p^{\frac{2}{3}} \zeta)$ is exponentially growing and $Bi(p^{\frac{2}{3}} \zeta)$ is exponentially decaying. Through

$$\frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)} = \frac{J_p(pz_0) - iY_p(pz_0)}{J_p(pz_0) + iY_p(pz_0)}$$

one observes that this term is of order of magnitude 1. Indeed, one observes first that

$$A(p, \zeta_0) Ai(p^{\frac{2}{3}} \zeta_0) - \zeta_0^{-\frac{1}{2}} p^{-\frac{4}{3}} B(p, \zeta_0) Ai'(p^{\frac{2}{3}} \zeta_0) = A(p, \zeta_0) Ai(p^{\frac{2}{3}} \zeta_0) [1 - \frac{Ai'}{Ai}(p^{\frac{2}{3}} \zeta_0) p^{-\frac{4}{3}} \zeta_0^{-\frac{1}{2}} \frac{B}{A}(p, \zeta_0)],$$

and one has $\frac{B}{A}(p, \zeta_0) = 1 + O(p^{-2})$ and $\frac{Ai'}{Ai}(p^{\frac{2}{3}} \zeta_0) p^{-\frac{4}{3}} \zeta_0^{-\frac{1}{2}} = -p^{-1}(1 + O(p^{-1}))$ (the last estimate coming from the expansion of the Airy function with terms of order $X^{-\frac{3n}{2}}$).

Hence

$$A(p, \zeta_0) Ai(p^{\frac{2}{3}} \zeta_0) - \zeta_0^{-\frac{1}{2}} p^{-\frac{4}{3}} B(p, \zeta_0) Ai'(p^{\frac{2}{3}} \zeta_0) = A(p, \zeta_0) Ai(p^{\frac{2}{3}} \zeta_0) [1 + \frac{1}{p} + O(p^{-2})]. \quad (41)$$

In a similar fashion, thanks to $\frac{Bi'}{Bi}(p^{\frac{2}{3}} \zeta_0) p^{-\frac{4}{3}} \zeta_0^{-\frac{1}{2}} = p^{-1}(1 + O(p^{-1}))$,

$$A(p, \zeta_0) Bi(p^{\frac{2}{3}} \zeta_0) - \zeta_0^{-\frac{1}{2}} p^{-\frac{4}{3}} B(p, \zeta_0) Bi'(p^{\frac{2}{3}} \zeta_0) = A(p, \zeta_0) Bi(p^{\frac{2}{3}} \zeta_0) [1 - \frac{1}{p} + O(p^{-2})]. \quad (42)$$

One deduces, thanks to $\frac{Bi}{Ai}(p^{\frac{2}{3}} \zeta_0) = O(e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}}})$,

$$\frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)} = \frac{J_p(pz_0) - iY_p(pz_0)}{J_p(pz_0) + iY_p(pz_0)} = \frac{1 + \frac{1}{p} + O(p^{-2}) - iO(e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}}})}{1 + \frac{1}{p} + O(p^{-2}) + iO(e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}}})} = 1 + O(e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}}}).$$

As

$$C(p, k_z) = -ik_3 \sin \beta_* \frac{(N_* - iO_*)e^{-i\Psi_*} + (N_* + iO_*)e^{i\Psi_*} \frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)}}{(L_* + iM_*)e^{-i\Psi_*} - (L_* - iM_*)e^{-i\Psi_*} \frac{H_p^{(2)}(pz_0)}{H_p^{(1)}(pz_0)}},$$

one obtains the same estimate as before in the case $\Im \epsilon \mu < 0$ independent on ω , which is the estimate based on $\Re(i\Psi_*) > 0$:

$$ik_3 \sin \beta_* \frac{N_* + iO_*}{L_* - iM_*} (1 + O(e^{-2\Re(i\Psi_*)})) (1 + O(e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}}})) = ik_3 \sin \beta_* (1 + i \frac{a(\beta_*) + b(\beta_*)}{p} + O(p^{-2})) (1 + O(e^{-2\Re(i\Psi_*)})) (1 + O(e^{\frac{4}{3} p \Re \zeta_0^{\frac{3}{2}}}))$$

This ends the proof of Proposition 10. □

5.6. Dirichlet to Neumann operator in the elliptic region

The case studied in this Section is

$$\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{R^2} < \Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} < 0.$$

One introduces, according to the Bessel functions toolbox:

$$z_0 = \frac{k_3 r_0}{p}, z_* = \frac{k_3 R}{p}, \zeta_*, \zeta_0.$$

Denote by A_0, B_0 the quantities $A(\zeta_0, p), B(\zeta_0, p)$, and A_*, B_*, C_*, D_* the quantities $A(\zeta_*, p), B(\zeta_*, p), C(\zeta_*, p), D(\zeta_*, p)$.

Proposition 11. 1. *The Dirichlet to Neumann operator is*

$$C(p, k_z) = k_3 \frac{2}{z_*} \sqrt{\frac{1 - z_*^2}{4\zeta_*}} p^{-\frac{1}{3}} \frac{A_0 D_*(1 + \tau_*)(1 + \tilde{k}_0) Ai'(p^{\frac{2}{3}} \zeta_*) Bi(p^{\frac{2}{3}} \zeta_0) - A_0 D_*(1 + \tilde{\tau}_*)(1 + k_0) Bi'(p^{\frac{2}{3}} \zeta_*) Ai(p^{\frac{2}{3}} \zeta_0)}{A_0 A_*(1 + k_*)(1 + \tilde{k}_0) Ai(p^{\frac{2}{3}} \zeta_*) Bi(p^{\frac{2}{3}} \zeta_0) - A_0 D_*(1 + \tilde{k}_*)(1 + k_0) Bi(p^{\frac{2}{3}} \zeta_*) Ai(p^{\frac{2}{3}} \zeta_0)}.$$

2. *Assume $\Im\epsilon\mu < 0$, independent on ω and assume $\Re\epsilon\mu\omega^2 - k_z^2 - \frac{p^2}{r_0^2} < 0$. One has*

$$C(p, k_z) = \left[\sqrt{k_z^2 + \frac{p^2}{R^2} - \epsilon\mu\omega^2} - \frac{1}{R} \sqrt{k_z^2 + \frac{p^2}{R^2} - \epsilon\mu\omega^2} \right] \zeta_*^{-\frac{3}{2}} (d_1 - c_1) + O(p^{-1}) \left(1 + O(e^{-\frac{4}{3} p \Re(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})}) \right).$$

Proof. Instead of using the Hankel functions in this case, we use the expressions (24). Indeed, the behavior in the elliptic case is rather an exponentially growing or decaying solution instead of oscillatory representations. One has

$$C(p, k_z) = k_3 \frac{J'_p(pz_*) Y_p(pz_0) - Y'_p(pz_*) J_p(pz_0)}{J_p(pz_*) Y_p(pz_0) - Y_p(pz_*) J_p(pz_0)}.$$

Assume $\Im\epsilon\mu < 0$ independent on ω . We use Lemma 8.

One has $\frac{2}{3} p \Re(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}}) > 0$ hence $Ai(p^{\frac{2}{3}} \zeta_*) Bi(p^{\frac{2}{3}} \zeta_0)$ is the leading term in these equalities, hence $J_p(pz_*) Y_p(pz_0)$ and $J'_p(pz_*) Y_p(pz_0)$ are the two dominant terms, hence

$$C(p, k_z) = k_3 \frac{J'_p(pz_*) Y_p(pz_0) \frac{1 - \frac{Y'_p(pz_*) J_p(pz_0)}{J'_p(pz_*) Y_p(pz_0)}}{1 - \frac{Y_p(pz_*) J_p(pz_0)}{J'_p(pz_*) Y_p(pz_0)}}}{J_p(pz_*) Y_p(pz_0) \frac{1 - \frac{Y'_p(pz_*) J_p(pz_0)}{J'_p(pz_*) Y_p(pz_0)}}{1 - \frac{Y_p(pz_*) J_p(pz_0)}{J'_p(pz_*) Y_p(pz_0)}}} = k_3 \frac{J'_p(pz_*)}{J_p(pz_*)} (1 + O(e^{-\frac{4}{3} p \Re(\zeta_0^{\frac{3}{2}} - \zeta_*^{\frac{3}{2}})})).$$

One has

$$\frac{J'_p(pz_*)}{J_p(pz_*)} = -\frac{2}{z_*} \left(\frac{1 - z_*^2}{4\zeta_*} \right)^{\frac{1}{2}} p^{-\frac{1}{3}} \frac{D(\zeta_*, p) Ai'(p^{\frac{2}{3}} \zeta_*)}{A(\zeta_*, p) Ai(p^{\frac{2}{3}} \zeta_*)} \frac{1 - p^{-\frac{2}{3}} \zeta_*^{\frac{1}{2}} \frac{C}{D} \frac{Ai}{Ai'}(p^{\frac{2}{3}} \zeta_*)}{1 - p^{-\frac{4}{3}} \zeta_*^{-\frac{1}{2}} \frac{B}{A} \frac{Ai'}{Ai}(p^{\frac{2}{3}} \zeta_*)}.$$

We observe that

$$\begin{aligned} -p^{-\frac{2}{3}} \zeta_*^{\frac{1}{2}} \frac{C}{D} \frac{Ai}{Ai'}(p^{\frac{2}{3}} \zeta_*) &= \frac{C}{D} p^{-1} (-p^{\frac{2}{3}} \zeta_*)^{\frac{1}{2}} \frac{Ai}{Ai'}(p^{\frac{2}{3}} \zeta_*) \simeq p^{-1}, \\ -p^{-\frac{4}{3}} \zeta_*^{-\frac{1}{2}} \frac{B}{A} \frac{Ai'}{Ai}(p^{\frac{2}{3}} \zeta_*) &= \frac{B}{A} p^{-1} (-p^{\frac{2}{3}} \zeta_*)^{-\frac{1}{2}} \frac{Ai'}{Ai}(p^{\frac{2}{3}} \zeta_*) \simeq p^{-1}, \end{aligned}$$

hence

$$\frac{1 - p^{-\frac{2}{3}} \zeta_*^{\frac{1}{2}} \frac{C}{D} \frac{Ai}{Ai'}(p^{\frac{2}{3}} \zeta_*)}{1 - p^{-\frac{4}{3}} \zeta_*^{-\frac{1}{2}} \frac{B}{A} \frac{Ai'}{Ai}(p^{\frac{2}{3}} \zeta_*)} = 1 + O(p^{-2}).$$

One has also $\frac{D}{A}(\zeta_*, p) = 1 + \frac{D_1(\zeta_*) - A_1(\zeta_*)}{p^2} + O(p^{-4}) = 1 + O(p^{-2})$. One is thus left with the estimate of $\frac{Ai}{Ai'}(p^{\frac{2}{3}} \zeta_*)$, where we use 10.4.59 of [1], which yields

$$-(p^{\frac{2}{3}} \zeta_*)^{-\frac{1}{2}} \frac{Ai'}{Ai}(p^{\frac{2}{3}} \zeta_*) = 1 - \frac{d_1 - c_1}{(p^{\frac{2}{3}} \zeta_*)^{\frac{3}{2}}} + O(p^{-2}) = 1 - \frac{d_1 - c_1}{p} \zeta_*^{-\frac{3}{2}} + O(p^{-2}),$$

hence

$$\frac{J'_p(pz_*)}{J_p(pz_*)} = \left(\frac{1 - z_*^2}{z_*}\right)^{\frac{1}{2}} \left(1 - \frac{d_1 - c_1}{p} \zeta_*^{-\frac{3}{2}}\right) (1 + O(p^{-2})).$$

One has $k_3 \frac{\sqrt{1 - z_*^2}}{z_*} = \sqrt{k_z^2 + \frac{p^2}{R^2} - \epsilon \mu \omega^2}$, hence the result of the proposition. \square

Remark 9. Surprisingly, the expression of the lower order term in $\frac{1}{R}$ is not the same for the hyperbolic and the elliptic regime.

6. Dirichlet to Neumann operator for elliptic layers

6.1. Calderon operator

The last step is solve (rather explicitly) the Dirichlet problem with homogeneous Dirichlet condition on $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\} := \partial\Omega$. Using the notation ρ, u_0 as above, one gets

Lemma 12. Let C be a bounded open set of \mathbb{R}^2 . Assume that $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\} \subset C$, with $d(\partial\Omega, \partial C) > 0$. Define $\rho = \sqrt{a^2 - b^2}$ and u_0 such that $\tanh u_0 = \frac{b}{a}$. Define u_1 such that $C \subset \{(x, y), \frac{x^2}{\cosh^2 u_1} + \frac{y^2}{\sinh^2 u_1} \leq 1\}$. The function U is a $C^2(C)$ solution of the Helmholtz equation which is equal to 0 on the ellipse if and only if there exists $\alpha_n, n \in \mathbb{Z}$ such that

$$U(x, y) = \sum_{n \in \mathbb{Z}} \alpha_n (C_{|n|}(u) S_{|n|}(u_0) - C_{|n|}(u_0) S_{|n|}(u)) g_n(v),$$

provided that $\gamma_n : U \rightarrow \frac{1}{2\pi} \int_0^{2\pi} U(x, y) g_n(v) dv$ is continuous for each n . Note that $e^{ik_z z} u$ is NOT the Fourier transform of a solution of $(\Delta + \omega^2 \epsilon \mu)U = 0$ because one cannot assert, for the moment, that each term in in S' as estimates on α_n are not provided.

The first item is a consequence of the continuity (note that the space where U lives is not given here, and such equalities are only formal if the sum is infinite) of the application γ_n for calculation the Dirichlet boundary condition.

Once all these results are proven, we are ready to study two cases for evaluating the Dirichlet to Neumann multiplier, the first subsection below deals with the case of the cofocal ellipses (where the parameter ρ is the same for both ellipses) and the second subsection deals with the case of homothetic ellipses.

6.2. Dirichlet to Neumann operator for an elliptic-type layer with same focal points for the two boundaries

B. Stupfel [16] already used such an approach to study the following problem: Ω is as usual a perfectly conducting body, supplemented with a layer which is also with an elliptic boundary $\Gamma_1 = \{(x, y), \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1\}$, under the assumption that $a_1^2 - b_1^2 = a^2 - b^2$, $a_1 \geq a$, $b_1 \geq b$. This means that the two elliptic boundaries $\partial\Omega$ and $\Gamma - 1$ are ellipses sharing the same focal points. In this set-up, if one denotes by u_1 such that $\tanh u_1 = \frac{b_1}{a_1}$, then $\{(x, y), \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} \leq 1\} - \Omega = \{u_0 \leq u \leq u_1, v \in [0, 2\pi]\}$. In this case, the result reads exactly as in the case of the cylinder. It is stated in Section 6 of [16], using expressions (67), (70) and all expressions that follow. The coefficients of the Calderòn operator are given by the results of [16], but we rephrase it in the language of operators.

Lemma 13. *The Dirichlet to Neumann multiplier acts on the orthonormal Hilbert base of $L^2([0, 2\pi]) \{g_m\}$ as:*

$$\begin{aligned} C(k_z)(U) &= \sum_n \frac{C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S_{|n|}(u_1)} \frac{g_n(v)}{\rho \sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} U_n \\ &= \sum_{n,m} \frac{C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S_{|n|}(u_1)} \rho^{-1} K_{nm} U_n g_m(v) \end{aligned}$$

for $U = \sum_n U_n g_n(v)$, or

$$\int_0^{2\pi} C(k_z)(U)(w) g_m(w) dw = \sum_{n,m} \frac{C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S_{|n|}(u_1)} K_{nm} \int_0^{2\pi} U(v') g_n(v') dv'.$$

It is a discrete pseudo-differential operator.³

Remark 10. *This result is easier to obtain, when one considers the Dirichlet boundary condition on the inner boundary, when one is able to decompose the solution on periodic modes.*

Proof. Let us start by identifying the normal derivative, which is the crucial point of this first result. A tangent vector to the boundary is $\vec{t} = (-\rho \cosh u \sin v, \rho \sinh u \cos v)$, hence

$$\vec{n} = \frac{1}{\sqrt{\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v}} (\sinh u \cos v, \cosh u \sin v),$$

hence

$$\partial_n f = \frac{1}{\rho \sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} \partial_u f.$$

We thus deduce

$$\begin{aligned} \partial_n U &= \frac{1}{\rho \sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} \sum_{n \in \mathbb{Z}} \alpha_n (C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1)) g_n(v) \\ &= \rho^{-1} \sum_{n \in \mathbb{Z}} \alpha_n (C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1)) \frac{g_n(v)}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} \end{aligned}$$

We then define the matrix

$$K_{mn} = \frac{1}{2\pi} \int_0^{2\pi} \frac{g_n(v) g_m(v)}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} dv$$

³The action of a discrete pseudo-differential operator expresses as $Op(a(v, n))(U)(v) = \sum_n a(v, n) g_n(v) U_n$, by comparison with $Op(a(v, \xi))f(x) = \frac{1}{2\pi} \int e^{ix \cdot \xi} a(v, \xi) \hat{f}(\xi) d\xi$.

and use

$$\frac{g_n(v)}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} = \sum_m K_{nm} g_m(v)$$

to obtain the result of Lemma 13. □

As in the case of the Bessel functions, we have a classification of the regions to be studied. The following classification applies:

1. totally elliptic region if $\Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_0 > \Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_1 > 0$,
2. mixed elliptic-hyperbolic region if $\Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_0 > 0 > \Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_1$,
3. hyperbolic region if $0 > \Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_0 > \Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_1$,
4. top glancing region if $\Re a_n(k_3\rho) - \frac{\Re\epsilon\mu\omega^2 - k_z^2}{2} \cosh 2u_0 \approx 0$.

Proposition 12. *The asymptotics of the coefficient $M_n = \frac{C'_{|m|}(u_1)S_{|m|}(u_0) - C_{|m|}(u_0)S'_{|m|}(u_1)}{C_{|m|}(u_1)S_{|m|}(u_0) - C_{|m|}(u_0)S_{|m|}(u_1)}$ is, in the hyperbolic regime (and $\omega\eta = k_z < \sqrt{\Re\epsilon\mu\omega}$)*

$$M_n = \pm i \sqrt{a_n(k_3\rho)} \sqrt{\frac{k_3^2 \rho^2}{2a_n(k_3\rho)} \cosh 2u_1 - 1(1 + O(\omega^{-2}))} = \pm \omega \rho \sqrt{\frac{\epsilon\mu - \eta^2}{2} \cosh 2u_1 - \frac{a_n(k_3\rho)}{\rho^2 \omega^2} (1 + O(\omega^{-2}))}.$$

Remark 11. *We notice that the Helmholtz equation (after Fourier transform in z) writes*

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{1}{2}(\epsilon\mu\omega^2 - k_z^2)\rho^2(\cosh 2u - \cos 2v) \right] u = 0,$$

and the mode analysis of this equation yields

$$\left[\frac{\partial^2}{\partial u^2} + \frac{1}{2}(\epsilon\mu\omega^2 - k_z^2)\rho^2 \cosh 2u - a_n(k_3\rho) \right] u = 0,$$

which principal symbol leads exactly to $\pm i\omega\rho \sqrt{\frac{1}{2}(\epsilon\mu - \eta^2) \cosh 2u - \frac{a_n(k_3\rho)}{\rho^2 \omega^2}}$.

Remark 12. *The coefficient obtained is thus*

$$\pm i \frac{\sqrt{\frac{\epsilon\mu - \eta^2}{2} \cosh 2u_1 - \frac{a_n(k_3\rho)}{\rho^2 \omega^2}}}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}} = \pm i \frac{\sqrt{\frac{\epsilon\mu - \eta^2}{2} \cosh^2 u_1 + \left(\frac{\epsilon\mu - \eta^2}{2} - \frac{a_n(k_3\rho)}{\rho^2 \omega^2}\right) \sinh^2 u_1}}{\sqrt{\cosh^2 u_1 \sin^2 v + \sinh^2 u_1 \cos^2 v}},$$

which shows the influence of the radius of curvature at each point of the boundary (characterized by v).

Proof. We use the asymptotic expansions of the modified Mathieu functions described in Sharples [18], namely the following UNIFORM asymptotic expansion of

$$W_j(u, \theta, x), \quad \frac{dW_j}{dx}(u, \theta, x)$$

where u, θ, x are defined in [18], and will be recalled here. They are obtained through the modified Mathieu functions $M_p^{(j)}(z, h)$ described in [6]. Of use will be also the seminal paper of Olver [13], where uniform

asymptotic expressions for solution of ordinary differential equations with a large parameter are discussed in the elliptic, hyperbolic or glancing zones.

Collecting the notations of Sharple, the equation (4) is $y'' = (\lambda - 2h^2 \cosh 2z)y$, which yields $u = z$, $\lambda = a_n(k_3\rho)$, $h^2 = \frac{k_3^2\rho^2}{4}$, that is $h = \frac{k_3\rho}{2}$.

It is said in [17] that the classical expansion of $a_n(k_3\rho)$ when $k_3\rho \rightarrow 0$ is $n^2 + F(n, h)$, where $F(n, h) = O(n^{-2}h^4)$. However, this classical expansion will **not be** of use here, because $|k_3\rho| \rightarrow +\infty$.

We need for this to use the seminar result of Harrell [8], which studies the band spectrum of the operator

$$Q = -\epsilon^2 \frac{d^2}{dx^2} + q(x)$$

where q is periodic of period 2, symmetric with respect to $x = 1$, C^2 , and has $2n$ as strict minimums (absolute) on the whole line. The Floquet modes a_n and b_n are then the edges of the bands for $-\frac{d^2}{dv^2} + 2s \cos 2v$.

The result of [8] is the following Theorem (stated in a sloppy way for the moment)

Theorem 4. For each fixed n , the n -th band of $-\frac{d^2}{dX^2} + \kappa^{-2}q(\kappa X)$, $\kappa^2 = \epsilon$ concentrates, when $\epsilon \rightarrow 0$, around $2n + 1$, and its width is exponentially decaying in $e^{-\frac{Dn}{n}}$.

Lemma 14. Under the hypothesis $k_z = \omega\eta$, η of order of magnitude 1, and $\Re\epsilon\mu - \eta^2 > 0$, the Floquet modes of the Mathieu operator $a_n(k_3\rho)$, $b_n(k_3\rho)$ have the following approximation for n large:

$$a_n(k_3\rho) = -\frac{k_3^2\rho^2}{2} + k_3\rho(2n + 1), b_{n+1}(k_3\rho) = -\frac{k_3^2\rho^2}{2} + k_3\rho(2n + 1).$$

Proof. The operator considered here is $-\frac{d^2}{dv^2} + \frac{k_3^2\rho^2}{2}(\cos 2v + 1)$, which transforms into

$$-\frac{4}{\pi^2} \frac{d^2}{dT^2} - \frac{k_3^2\rho^2}{2}(\cos \pi T - 1).$$

The model problem associated is $\frac{4}{\pi^2} \frac{d^2}{dT^2} + \frac{k_3^2\rho^2}{4}\pi^2 T^2$, which corresponds to $E_n = k_3\rho(2n + 1)$, where, for $n = 0$, the associated mode is $e^{-k_3\rho \frac{\pi^2}{4} T^2}$.

To $2n + 1$ are associated the two edges of the bands E_n^+, E_{n+1}^- , such that $E_{n+1}^- - E_n^+ \simeq e^{-d/h}$ and $E_n^+ \simeq k_3\rho(2n + 1)$, hence an approximation of $a_n(k_3\rho)$ and $b_{n+1}(k_3\rho)$ by $-\frac{k_3^2\rho^2}{2} + k_3\rho(2n + 1)$, which allows to deduce that the high frequency regime is n of order of magnitude ω . \square

The modified Mathieu equation is thus $y''(z) = [a_n(k_3\rho)(1 - \frac{2h^2}{a_n(k_3\rho)} \cosh 2z)]y$. Recall that $k_3^2 = \epsilon\mu\omega^2 - k_z^2$, which means that

$$\frac{2h^2}{a_n(k_3\rho)} = \frac{k_3^2\rho^2}{a_n(k_3\rho)} = \frac{\epsilon\mu\omega^2 - k_z^2}{\rho^2}.$$

Sharple treats the case $h \notin \mathbb{R}$ as well, hence the case $h = \frac{k_3\rho}{\sqrt{a_n(k_3\rho)}}$ is covered. The angle θ is defined by $h = |h|e^{i\theta}$, hence $k_z^2 < \Re\epsilon\mu\omega^2$ allows to follow the branch associated with $\theta \in] -\frac{\pi}{4}, 0]$. One notes that $\frac{k_3\rho}{n}$ corresponds exactly to $\frac{1}{\sin\beta}$ introduced for the hyperbolic region for the cylinder.

In the elliptic case, we define the variable ξ such that

$$\left(\frac{d\xi}{du}\right)^2 = 1 - \frac{2h^2}{a_n(k_3\rho)} \cosh 2u, \quad (43)$$

for an asymptotic representation with exponentials $e^{\pm \sqrt{a_n(k_3\rho)}\xi}$, where $\sqrt{a_n(k_3\rho)}\xi$ represents the phase solution of the reduced eikonal equation.

In the hyperbolic case, we define the variable Ξ such that

$$\left(\frac{d\Xi}{du}\right)^2 = \frac{2h^2}{a_n(k_3\rho)} \cosh 2u - 1, \quad (44)$$

In the glancing case, we define ζ such that

$$\zeta \left(\frac{d\zeta}{du}\right)^2 = 1 - \frac{2h^2}{a_n(k_3\rho)} \cosh 2u, \quad (45)$$

for an asymptotic representation with Airy functions, where $(a_n(k_3\rho))^{\frac{1}{3}}\zeta$ is the argument of the Airy functions considered.

We denote, for simplification, by $N = \sqrt{a_n(k_3\rho)}$. It is NOT an approximation of n , and, in the high frequency regime, $N \simeq \omega \sqrt{\frac{(2n+1)\rho}{\omega}} \sqrt{\epsilon\mu - \eta^2}$ (dimension of the inverse of a velocity? or dimensionless?).

In the hyperbolic case, we check that the modified Mathieu equation becomes

$$y'' = (iN)^2 \left(\frac{d\Xi}{dz}\right)^2 y, \quad (46)$$

which corresponds in Sharples' analysis to the case $h = iN$, $|h| = |N|$, and $h = |h|e^{i\frac{\pi}{2}+\theta}$, which yields the change of variable $u = xe^{-i\frac{\pi}{2}-\theta}$, leading to $hu = iNu = ie^{i\theta}|h|u = |h|x$.

In the case of the Airy function analysis, $W = \left(\frac{d\zeta}{dz}\right)^{-\frac{1}{2}}y$ solves the equation $\frac{d^2W}{d\zeta^2} = (N^2\zeta + f(\zeta))W$, which yields solutions as

$$\begin{aligned} W_j(N, \theta, u) &= P_j(N^{\frac{2}{3}}\zeta)P_j(N, u) + P'_j(N^{\frac{2}{3}}\zeta)Q_j(N, u) \\ \frac{d}{d\zeta}W_j(N, \theta, u) &= N^{\frac{2}{3}}P'_j(N^{\frac{2}{3}}\zeta)R_j(N, u) + P_j(N^{\frac{2}{3}}\zeta)S_j(N, u) \end{aligned}$$

where $P_1(X) = Ai(X)$, $P_2(X) = Ai(e^{-\frac{2i\pi}{3}}X)$, $P_3(X) = Ai(e^{\frac{2i\pi}{3}}X)$, $P_4(X) = Bi(X)$.

The analysis of the conditions on which one can apply the method of Olver [13] and the results of Sharples [18], relies on the inequality, for the ODE $\frac{d^2F}{dz^2} = (h^2 + f(\theta, z, u))F$ is essentially $|f(\theta, z, u)| \leq \frac{C}{1+|z|^{1+\sigma}}$ and additional conditions under the hypothesis that f has an expansion in u^{-1} . This is the case when one introduces k_z, n of order ω , and we transform the modified Mathieu ODE (46) by using Ξ as new variable. Indeed it writes

$$\frac{d\Xi}{du} \frac{d}{d\Xi} \left(\frac{d\Xi}{du} \frac{dy}{d\Xi}\right) = (iN)^2 \left(\frac{d\Xi}{dz}\right)^2 y,$$

hence

$$\frac{d}{d\Xi} \left(\frac{d\Xi}{du}\right) \frac{dy}{d\Xi} + \frac{d\Xi}{du} \frac{d^2y}{d\Xi^2} = (iN)^2 \frac{d\Xi}{dz} y.$$

Introducing $W = \left(\frac{d\Xi}{du}\right)^{\frac{1}{2}}y$, one deduces

$$\frac{d^2W}{d\Xi^2} = \left[(iN)^2 + \left(\frac{d\Xi}{du}\right)^{-\frac{1}{2}} \frac{d^2}{d\Xi^2} \left(\left(\frac{d\Xi}{du}\right)^{\frac{1}{2}}\right)\right]W,$$

where the bound on $(\frac{d\Xi}{du})^{-\frac{1}{2}} \frac{d^2}{d\Xi^2} ((\frac{d\Xi}{du})^{\frac{1}{2}})$ is fulfilled (we are on compact sets).

Application of Theorem A of [13] leads to

$$W_{\pm}(\Xi, n) = e^{\pm iN\Xi} \left(\sum_s \frac{(\pm)^s A_s^{\pm}(\xi)}{N^s} \right), \quad \frac{d}{d\Xi} W_{\pm}(\Xi, N) = \pm iN e^{\pm iN\Xi} \left(\sum_s \frac{(\pm)^s B_s^{\pm}(\xi)}{N^s} \right),$$

and we deduce in this case that, with $y_{\pm}(u, N) = (\frac{d\Xi}{du})^{-\frac{1}{2}} W_{\pm}(\Xi(u), N)$,

$$\frac{d}{du}(y_{\pm})(u, N) = \left(\frac{d\Xi}{du}\right)^{-\frac{1}{2}} \left[\frac{d\Xi}{du} \frac{dW_{\pm}}{d\Xi} + \left(\frac{d\Xi}{du}\right)^{\frac{1}{2}} \frac{d}{du} \left(\left(\frac{d\Xi}{du}\right)^{-\frac{1}{2}}\right) W_{\pm} \right],$$

which becomes

$$\frac{d}{du}(y_{\pm})(u, N) = \left(\frac{d\Xi}{du}\right)^{-\frac{1}{2}} e^{iN\Xi} \left[\pm \frac{d\Xi}{du} iN B^{\pm}(\Xi, N) + \left(\frac{d\Xi}{du}\right)^{\frac{1}{2}} \frac{d}{du} \left(\left(\frac{d\Xi}{du}\right)^{-\frac{1}{2}}\right) A^{\pm}(\Xi, N) \right],$$

where A^{\pm} and B^{\pm} are the coefficients whose expansion in inverse powers of n is uniform for n large enough. Note that $A^{\pm} = 1 \pm \frac{A_1}{iN} + O(N^{-2})$, $B^{\pm} = 1 \pm \frac{B_1}{iN} + O(N^{-2})$, A_1 and B_1 are given by the expressions of [17].

The evaluation of the Dirichlet to Nemann passes through the computation

$$\frac{y'_+(u_1)y_-(u_0) - y'_-(u_1)y_+(u_0)}{y_+(u_1)y_-(u_0) - y_-(u_1)y_+(u_0)}$$

which amounts to, after all the needed simplifications

$$iN \frac{d\Xi}{du}(u_1) \frac{e^{iN(\Xi(u_1) - \Xi(u_0))} B^+(u_1) A^-(u_0) + e^{-iN(\Xi(u_1) - \Xi(u_0))} B^-(u_1) A^+(u_0)}{e^{iN(\Xi(u_1) - \Xi(u_0))} A^+(u_1) A^-(u_0) - e^{-iN(\Xi(u_1) - \Xi(u_0))} A^-(u_1) A^+(u_0)}.$$

In the case $\Im(\Xi(u_1) - \Xi(u_0)) > 0$, one has the estimate

$$\frac{y'_+(u_1)y_-(u_0) - y'_-(u_1)y_+(u_0)}{y_+(u_1)y_-(u_0) - y_-(u_1)y_+(u_0)} = -iN \frac{d\Xi}{du}(u_1) \left(1 + \frac{A_1 - B_1}{iN} + O(N^{-2}) \right) (1 + e^{-2N\Im(\Xi(u_1) - \Xi(u_0))}),$$

and in the case $\Im(\Xi(u_1) - \Xi(u_0)) < 0$

$$\frac{y'_+(u_1)y_-(u_0) - y'_-(u_1)y_+(u_0)}{y_+(u_1)y_-(u_0) - y_-(u_1)y_+(u_0)} = iN \frac{d\Xi}{du}(u_1) \left(1 - \frac{A_1 - B_1}{iN} + O(N^{-2}) \right) (1 + e^{-2N\Im(\Xi(u_1) - \Xi(u_0))}).$$

Using $A_1 = B_1$, one gets the estimate. □

The second case of interest is the case called above mixed-elliptic-hyperbolic. In this case, the previous analysis is used at $u = u_1$, while new uniform expansions are used at $u = u_0$.

Proposition 13. *The asymptotics of the coefficient $M_n = \frac{C'_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S'_{|n|}(u_1)}{C_{|n|}(u_1)S_{|n|}(u_0) - C_{|n|}(u_0)S_{|n|}(u_1)}$ is, in the mixed elliptic-hyperbolic regime (and $\omega\eta = k_z < \sqrt{\Re\epsilon\mu\omega}$)*

$$M_n = \pm iN \sqrt{\frac{k_3^2 \rho^2}{2a_n(k_3\rho)}} \cosh 2u_1 - 1 (1 + O(N^{-2})) = \pm \omega\rho \sqrt{\frac{\epsilon\mu - \eta^2}{2}} \cosh 2u_1 - \frac{N^2}{\rho^2 \omega^2} (1 + O(N^{-2})).$$

Proof. One begins with the analysis of the solutions near $u = u_0$.

In this case, one recalls that one introduced ξ above, and there exists two solutions, respectively denoted by $E_{\pm}(u)$, such that

$$E_{\pm}(\xi) = \left(\frac{d\xi}{du}\right)^{\frac{1}{2}} y_{\pm}(u)$$

solution of

$$\frac{d^2 E_{\pm}}{d\xi^2} = (N^2 + \tau(\xi, N))E_{\pm},$$

such that

$$E_{\pm}(\xi) = e^{\pm N\xi} \left(1 + \sum_{j \geq 1} \frac{A_j(\xi)}{N}\right),$$

as well as $\frac{dE_{\pm}}{d\xi} = \pm N e^{\pm N\xi} \left(1 + \sum_{j \geq 1} \frac{B_j(\xi)}{N}\right)$.

However, the relation between the solutions denoted by W_{\pm} in the proof of the previous proposition and the solutions described above has to be carefully deduced from the branch cuts analysis of the solutions. The situation was different in the case of the Bessel functions, where the phase analysis either for the Hankel solutions and for the Bessel solution has been studied before. \square

6.3. Elliptical cylinder with homothetic boundaries

We concentrate in this Section on a slightly different problem, where the two ellipses $\partial\Omega$ and Γ_1 , instead of having the same focal points, are homothetic: one considers Ω as a perfectly conducting body, $\gamma = \partial\Omega$ and $\Gamma := \{(x, y), \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1\}$, where $\frac{B}{A} = \frac{b}{a}$. We are able to prove a more explicit result than the general theorem of this Section, but however less explicit than the results stated by Stupfel [16]:

Proposition 14. *Assume that $\{(x, y, z), 1 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq \frac{A^2}{a^2}, z \in \mathbb{R}\}$ is a dielectric material, of dielectric constants ϵ and μ . Let k_z be a wave number, with the assumption $\Re(\epsilon\mu\omega^2) - k_z^2 > 0$. Denote by $k_3 = \sqrt{\epsilon\mu\omega^2 - k_z^2}$. We call g_m the family of periodic solutions of the Mathieu equation, and (C_m, S_m) the associated solutions of the modified Mathieu equation.*

If $\epsilon\mu \notin \mathbb{R}$, k_3 is not a resonance of (7), the Dirichlet to Neumann multiplier is well defined as an operator on $\ell^2(\{g_m, m \in \mathbb{Z}\})$ where this space is the set of limits of $\sum_{m \in K} \alpha_m g_m$, $\sum_{m \in K} |\alpha_m|^2 < C$, C independent on K . Its matrix (infinite) is given using (50) below by

$$C_{ml} = \sum_{n \in \mathbb{Z}} r_{nl} [D_{nm} S_n(u_0) - E_{nm} C_n(u_0)].$$

where $(r_{nl})_{n,l} (I_{nm} S_n(u_0) - J_{nm} C_n(u_0))_{n,m} = \delta_{nm}$.

Proof. Assume in this proof that $\rho = \sqrt{a^2 - b^2}$, which fixes the system of variables (u, v) . The same formal analysis as before yields $U = \sum_n \alpha_n (C_{|n|}(u) S_{|n|}(u_0) - S_{|n|}(u) C_{|n|}(u_0)) g_n(v)$, where the Mathieu and modified Mathieu functions are associated with $k_3 \rho$. In order to obtain the Dirichlet to Neumann operator, it is needed to find $(\alpha_n)_n$ in terms of $U|_{\Gamma}$, more precisely its expansion on $(g_n)_n$. Let us call K the application which yields $(\frac{1}{2\pi} \int_0^{2\pi} U|_{\Gamma}(v) g_n(v) dv)_n$ in terms of α_n . One needs to invert K . As we are in an Hilbert separable space of infinite dimension, it is sufficient to be able to write K (or any operator deduced from K as $Id - T$, where T satisfies $\|T\| < 1$). This can be achieved if one can prove $\sup \frac{|T(\alpha)|_{l_2}}{|\alpha|_{l_2}} < 1$. Hence the choice of coefficients $(\alpha_n)_n$ is not the best choice.

Note that, for $(x, y) \in \Gamma$, $\frac{\rho^2 \cosh^2 u \cos^2 v}{A^2} + \frac{\rho^2 \sinh^2 u \sin^2 v}{B^2} = 1$, which is equivalent to $\sinh^2 u = \frac{1 - \frac{\rho^2}{A^2} \cos^2 v}{\frac{\rho^2}{A^2} \cos^2 v + \frac{\rho^2}{B^2} \sin^2 v}$,

or

$$\tanh^2 u = \frac{1 - \frac{\rho^2}{A^2} \cos^2 v}{1 + \frac{\rho^2}{B^2} \sin^2 v}.$$

It is easy to see that

$$\tanh^2 u_0 < \frac{1}{1 + \frac{\rho^2}{B^2}} \leq \tanh^2 u \leq 1 - \frac{\rho^2}{A^2}, \forall v \in [0, 2\pi].$$

We denote by u_1 and u_2 such that $\tanh^2 u_1 = \frac{1}{1 + \frac{\rho^2}{B^2}}$, $\tanh^2 u_2 = 1 - \frac{\rho^2}{A^2}$, such that $u_0 < u_1 < u_2$. Use then the estimate (??) and choose $u_3 > u_2$. Denote by

$$\beta_n = \alpha_n \kappa_n, \kappa_n = (C'_n(u_0)S_n(u_0) - S'_n(u_0)C_n(u_0)) \frac{\sinh(k_n(u_0)(u_3 - u_0))}{k_n(u_0)}. \quad (47)$$

One has

$$U(\rho \cosh u \cos v, \rho \sinh u \sin v) = \sum_n \beta_n \frac{C_n(u)S_n(u_0) - S_n(u)C_n(u_0)}{\kappa_n} g_n(v).$$

Define

$$T((\beta_n)_n) = \left(\frac{1}{2\pi} \int_0^{2\pi} \sum_n \beta_n \left(1 - \frac{C_n(u)S_n(u_0) - S_n(u)C_n(u_0)}{\kappa_n} \right) g_n(v) g_m(v) dv \right)_m.$$

One checks that

$$U|_{\Gamma} = \sum_n \beta_n \frac{C_n(\psi(v))S_n(u_0) - S_n(\psi(v))C_n(u_0)}{\kappa_n} g_n(v),$$

and that, denoting by \mathcal{T} the operator

$$\mathcal{T}\left(\sum_n \beta_n g_n\right) = \sum_n \beta_n g_n - U|_{\Gamma},$$

the coefficients of $\mathcal{T}(\sum_n \beta_n g_n)$ on the Mathieu complete orthonormal family (g_n) of $L^2([0, 2\pi])$ is $T((\beta_n)_n)$. In addition, estimate (??) proves that

$$\int_0^{2\pi} \left(\sum_n \beta_n g_n - U|_{\Gamma} \right)^2 dv \leq \left| 1 - \frac{C_n(u)S_n(u_0) - S_n(u)C_n(u_0)}{\kappa_n} \right|_{\infty} \sum \beta_n^2.$$

As one has $1 - \frac{C_n(u)S_n(u_0) - S_n(u)C_n(u_0)}{\kappa_n} \leq 1 - \frac{\sinh k_n(u_0)(u_2 - u_0)}{\sinh k_n(u_0)(u_3 - u_0)}$, one deduces that \mathcal{T} satisfies $\|\mathcal{T}\| < 1$, hence its inverse is $\sum_{p \geq 0} (\mathcal{T})^p$. This proof relies only on modes $|n| \geq n_0$, hence on a subspace of $L^2([0, 2\pi])$ of finite codimension.

We thus proved that $(\beta_n)_n$ is known when $U|_{\Gamma}$ is known. This yields

$$\beta_n = \sum_m G_{nm} \mu_m, U|_{\Gamma} = \sum_n \mu_n g_n. \quad (48)$$

Note that one can compute G_{nm} for each n by considering the problem $(\Delta + k_3^2)U = 0, U|_{\partial\Omega} = 0, U|_{\Gamma} = g_n$, which has a unique solution in $H^1(C - \Omega)$. □

Assume for now on that k_3 does not lead to a resonance of the problem (7). Assume that, on the surface Γ (concentric ellipse), we impose $U_\Gamma \in L^2([0, 2\pi])$, as a function of ϕ such that $x = A \cos \phi, y = B \sin \phi$. This surface has not a simple expression in terms of u, v . The equation of Γ is $u = \Psi(v)$ such that

$$\sinh u = \sqrt{\frac{1 - \frac{\rho^2}{A^2} \cos^2 v}{\frac{\rho^2}{B^2} - (\frac{1}{B^2} - \frac{1}{A^2})\rho^2 \cos^2 v}} \Leftrightarrow u = \psi(v). \quad (49)$$

The trace of u on Γ is

$$U|_\Gamma = U_\Gamma = \sum_{n \in \mathbb{Z}} \alpha_n (C_{|n|}(\psi(v))S_{|n|}(u_0) - C_{|n|}(u_0)S_{|n|}(\psi(v)))g_n(v),$$

where the variable v describes $[0, 2\pi]$. Two equivalent possibilities are thus available:

- assume that the expansion of $u|_\Gamma$ uses Fourier series expansions,
- assume that the expansion of $u|_\Gamma$ follows Mathieu series expansion.

The normal derivative on Γ is $\vec{n} \cdot \nabla$, and as the equation of Γ writes $\sinh^2 u = f(\cos^2 v)$, where $f(X) = \frac{A^2 - \rho^2 X}{e^2 \rho^2 - \rho^2 (e^2 - 1)X}$, $e = \frac{b}{a} = \frac{B}{A}$, a normal vector is given through $\sinh 2u du = -\sin 2v f'(\cos^2 v) dv$, that is $N = (\sinh 2u, \sin 2v f'(\cos^2 v))$, hence the normal outgoing unit vector is $n = (\sin^2 2u + (f'(\cos^2 v) \sin^2 2v)^{-\frac{1}{2}} N|_{u=\psi(v)})$.

One then uses the following coefficients, from which one deduces a linear transform in $L^2([0, 2\pi])$, using the orthogonality of the Mathieu functions:

$$\begin{aligned} I_{nm} &= \frac{1}{2\pi} \int_0^{2\pi} C_{|n|}(\psi(v))g_n(v)g_m(v)dv, \\ J_{nm} &= \frac{1}{2\pi} \int_0^{2\pi} S_{|n|}(\psi(v))g_n(v)g_m(v)dv, \\ D_{nm} &= \frac{1}{2\pi} \int_0^{2\pi} \vec{n}(v) \cdot [C'_{|n|}(\psi(v))g_n(v), C_{|n|}(\psi(v))g'_n(v)]g_m(v)dv, \\ E_{nm} &= \frac{1}{2\pi} \int_0^{2\pi} \vec{n}(v) \cdot [S'_{|n|}(\psi(v))g_n(v), S_{|n|}(\psi(v))g'_n(v)]g_m(v)dv. \end{aligned} \quad (50)$$

These quantities are just diagonal matrices when we have the Bessel functions and the usual Fourier modes because in this case $\psi(v)$ does not depend on v (for the cylinder).

Let U be the unique solution of the problem (apart for k_3 in the set of resonances of the Dirichlet problem)

$$\begin{cases} (\Delta + k_3^2)U = 0, 1 < \frac{x^2}{a^2} + \frac{y^2}{b^2} < \frac{A^2}{a^2}, \\ U|_{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} = 0, \\ U|_{\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{A^2}{a^2}} = F \in L^2([0, 2\pi]). (x, y) = \frac{A}{a}(c \cos \theta, b \sin \theta). \end{cases}$$

One has (denoting $C_{-n} = C_n$ and $S_{-n} = S_n$ for all $n \in \mathbb{N}$),

$$U = \sum_{n \in \mathbb{Z}} \alpha_n (C_n(u)S_n(u_0) - S_n(u)C_n(u_0))g_n(v),$$

which imply that

$$\frac{1}{2\pi} \int_0^{2\pi} U|_\Gamma g_{-m}(v)dv = \sum_{n \in \mathbb{Z}} \alpha_n (I_{nm}S_n(u_0) - J_{nm}C_n(u_0)).$$

Let $F \in L^2([0, 2\pi])$. As the family g_n is a complete family of $L^2(\Gamma)$, there exists a sequence F_n such that $F = \sum_{n \in \mathbb{Z}} F_n g_n$. As this system has a unique solution there exists $(r_{nm})_{n,m}$ such that

$$\alpha_n = \sum_{m \in \mathbb{Z}} r_{nm} F_m.$$

On the other side,

$$\partial_n U = \vec{n} \cdot \nabla U|_{\Gamma} = \sum_{n \in \mathbb{Z}} \alpha_n \vec{n} \cdot [(C'_n(u)S_n(u_0) - S'_n(u)C_n(u_0))g_n(v), C_n(u)S_n(u_0) - S_n(u)C_n(u_0)]g'_n(v)|_{u=\psi(v)}$$

which imply

$$\frac{1}{2\pi} \int_0^{2\pi} \partial_n U g_{-m}(v) dv = \sum_{n \in \mathbb{Z}} \alpha_n [D_{nm}S_n(u_0) - E_{nm}C_n(u_0)].$$

Finally, one has

$$\frac{1}{2\pi} \int_0^{2\pi} \partial_n U g_{-m}(v) dv = \sum_{n \in \mathbb{Z}, l \in \mathbb{Z}} r_{nl} F_l [D_{nm}S_n(u_0) - E_{nm}C_n(u_0)],$$

which imply that, in the basis (g_n) , the Dirichlet to Neumann multiplier is characterized by the matrix

$$C_{ml} = \sum_{n \in \mathbb{Z}} r_{nl} [D_{nm}S_n(u_0) - E_{nm}C_n(u_0)].$$

7. The Calderòn operator for a cylindrical layer in 3D for the Maxwell equations

This section is a follow-up of one of the chapters of the thesis of Pierre Payen [15]. Indeed, in Stupfel [16] the case of a 2D cylindrical layer is thoroughly described, with exact solutions. But, as this was a 2D problem, the case of oblique incidence (*id est* a wave vector with a non zero z component) was not accounted for. In Payen [15] a procedure for constructing the Calderòn operator is described, but an explicit formula for this operator has to be deduced from the expressions therein. This is the aim of this Section. We use there the same set-up as the set-up used in [10]. Denote by

$$K(r, n, k_z) = \begin{pmatrix} \frac{k_z n}{r} & \epsilon \mu \omega^2 - k_z^2 \\ -(\epsilon \mu \omega^2 - k_z^2) & -\frac{k_z n}{r} \end{pmatrix}.$$

Lemma 15. *The system of Maxwell equations is equivalent to*

$$\left\{ \begin{array}{l} \frac{in}{r} E_z - ik_z E_\theta = i\omega \mu H_r \\ \frac{in}{r} H_z - ik_z H_\theta = -i\omega \epsilon E_r \\ \left(\frac{1}{r} \frac{d}{dr} (r H_\theta) \right) = \frac{i}{\mu \omega} K \begin{pmatrix} E_z \\ E_\theta \end{pmatrix}, \left(\frac{1}{r} \frac{d}{dr} (r E_\theta) \right) = -\frac{i}{\epsilon \omega} K \begin{pmatrix} H_z \\ H_\theta \end{pmatrix}. \end{array} \right.$$

The functions E_z and H_z are solution of the Bessel equation (12).

Proof. The system of Maxwell equations writes

$$\left\{ \begin{array}{l} \frac{in}{r} E_z - ik_z E_\theta = i\omega \mu H_r \\ \frac{in}{r} H_z - ik_z H_\theta = -i\omega \epsilon E_r \\ ik_z E_r - \frac{dE_z}{dr} = i\omega \mu H_\theta \\ \frac{1}{r} \left[\frac{d}{dr} (r E_\theta) - in E_r \right] = i\omega \mu H_z \\ ik_z H_r - \frac{dH_z}{dr} = -i\omega \epsilon E_\theta \\ \frac{1}{r} \left[\frac{d}{dr} (r H_\theta) - in H_r \right] = -i\omega \epsilon E_z \end{array} \right. \quad (51)$$

The two first equations yield H_r in terms of (E_θ, E_z) and E_r in terms of (H_θ, H_z) , hence the first result of Lemma 15. Note that this result decouples the ODE system, allowing it to be written with a matrix $\begin{pmatrix} 0 & \frac{i}{\mu\omega}K \\ -\frac{i}{\epsilon\omega}K & 0 \end{pmatrix}$. Note also that, if one introduces $\eta = \frac{k}{\omega}$ and $\delta(r) = \frac{n}{r\omega}$, one has

$$\frac{1}{\omega^2}K = \begin{pmatrix} \eta\delta(r) & \epsilon\mu - \eta^2 \\ -(\epsilon\mu - \eta^2) & -\eta\delta(r) \end{pmatrix}.$$

As for the Bessel equations on (E_z, H_z) , it comes from the vectorial Helmholtz equation that E and H solve, and we observe that, while the equations on (E_θ, E_r) on one side and on (H_θ, H_r) are coupled, the equation on E_z or on H_z is a scalar equation. □

The Calderòn operator given by what follows ($C(r_0, R) \times R$ denoting the infinite cylindrical annulus between $r = r_0$ and $r = R$, $S(r)$ denoting the surface of the infinite cylinder of radius r , e_r is the normal unit outgoing vector to a point of the boundary $S(r)$ or $S(R)$):

Theorem 5. *If there is no resonance, the problem of Maxwell equations in $C(r_0, R) \times \mathbb{R}$, supplemented with $E|_{S(r_0)} = 0$ on $S(r_0)$ and with $-(E \wedge e_r) \wedge e_r|_{S(R)} = \vec{E}_0 \in L^2(S(R), T(S(R)))$ has a unique solution in $C(r_0, R) \times \mathbb{R}$, denoted by $\vec{E}(\vec{E}_0)$. One has thus $e_r \wedge \vec{H}_{S(R)} = C(E_0)$, where C is the interior Calderòn operator.*

This Theorem is a theorem of [4].

Lemma 16. 1. *If the domain is $C(r_0, R) \times R$ and if $\epsilon\mu \notin \mathbb{R}_+^*$, there is no resonance.*
2. *If the domain is $C(r_0, R) \times [0, L]$, and $d_n(k_3R)(dd_n)'(k_3R) \neq 0$, there is no resonance.*

Introduce now, in the case $d_n(k_3R)dd_n(k_3R) \neq 0$, the functions

$$\begin{aligned} S_n(k_3r) &= \frac{J'_n(k_3r_0)Y_n(k_3r) - Y'_n(k_3r_0)J_n(k_3r)}{J'_n(k_3r_0)Y'_n(k_3R) - Y'_n(k_3r_0)J'_n(k_3R)}, \\ T_n(k_3r) &= \frac{J_n(k_3r_0)Y'_n(k_3r) - Y_n(k_3r_0)J'_n(k_3r)}{J_n(k_3r_0)Y'_n(k_3R) - Y_n(k_3r_0)J'_n(k_3R)}, \\ Q_n(k_3r) &= \frac{J_n(k_3r_0)Y_n(k_3r) - Y_n(k_3r_0)J_n(k_3r)}{J_n(k_3r_0)Y'_n(k_3R) - Y_n(k_3r_0)J'_n(k_3R)}, \\ R_n(k_3r) &= \frac{J'_n(k_3r_0)Y'_n(k_3r) - Y'_n(k_3r_0)J'_n(k_3r)}{J'_n(k_3r_0)Y'_n(k_3R) - Y'_n(k_3r_0)J'_n(k_3R)}. \end{aligned} \tag{52}$$

We notice that these functions do not depend on the choice of the pair of fundamental solutions of the Bessel equation considered.

Lemma 17. *The Calderòn operator for the cylindrical annulus is given by*

$$\begin{pmatrix} -H_\theta \\ H_z \end{pmatrix} = \begin{pmatrix} -\frac{i\omega\epsilon}{k_3} [T_n(k_3R) - \frac{ik_z^2 n^2}{k_3^2 R^2 \omega^2 \epsilon \mu} S_n(k_3R)] & \frac{ik_z n}{k_3 R \omega \mu} S_n(k_3R) \\ \frac{ik_z n}{k_3 R \omega \mu} S_n(k_3R) & \frac{ik_3}{\omega \mu} S_n(k_3R) \end{pmatrix} \begin{pmatrix} E_z \\ E_\theta \end{pmatrix}.$$

Remark 13. *One hoped that the leading order term of this operator is $\sqrt{\frac{\epsilon}{\mu}}$, at the first order, which is the case because the approximation of $S_n(k_3R)$ is $-\frac{i}{\sin\beta_*}$ while the approximation of $T_n(k_3R)$ is $i \sin\beta_*$ as it is stated in Lemma 18.*

Remark 14. The impedance operator, which expresses $n \wedge E$ in terms of $-n \wedge n \wedge H$, is

$$\begin{pmatrix} -E_\theta \\ E_z \end{pmatrix} = \begin{pmatrix} \frac{i\omega\mu}{k_3} [(S_n(k_3R))^{-1} - \frac{ik_z^2 n^2}{k_3^2 R^2 \omega^2 \epsilon \mu} (T_n(k_3R))^{-1}] & -\frac{ik_z n}{k_3 R \omega \epsilon} (T_n(k_3R))^{-1} \\ -\frac{ik_z n}{k_3 R \omega \epsilon} (T_n(k_3R))^{-1} & -\frac{ik_3}{\omega \epsilon} (T_n(k_3R))^{-1} \end{pmatrix} \begin{pmatrix} H_z \\ H_\theta \end{pmatrix}.$$

At normal incidence it is diagonal and equal to

$$\sqrt{\frac{\mu}{\epsilon}} \begin{pmatrix} i(S_n(\sqrt{\epsilon\mu}\omega R))^{-1} & 0 \\ 0 & -i(T_n(\sqrt{\epsilon\mu}\omega R))^{-1} \end{pmatrix}$$

For $n = 0$ it is also diagonal and equal to

$$\begin{pmatrix} \frac{i}{\sqrt{\frac{\mu}{\epsilon} - \frac{k_z^2}{\omega^2}}} (S_0(k_3R))^{-1} & 0 \\ 0 & -i \sqrt{\frac{\mu}{\epsilon} - \frac{k_z^2}{\omega^2}} (T_0(k_3R))^{-1} \end{pmatrix}.$$

One recognizes a classical form for the impedance matrix.

Proof. Consider for the moment formal solutions of the Helmholtz equations obtained on E_z, H_z . After Fourier transform in z and expanding in Fourier series in θ , one has, for $k_3 \neq 0$

$$\begin{cases} E_z = aJ_n(k_3r) + bY_n(k_3r) \\ H_z = cJ_n(k_3r) + dY_n(k_3r). \end{cases}$$

We now deduce H_θ and E_θ through the equalities 3 and 5 of (51), after replacing E_r, H_r using the equalities 1 and 2 of (51) for $k_3 \neq 0$:

$$\begin{cases} H_\theta = \frac{i\omega\epsilon}{k_3^2} \left[\frac{dE_z}{dr} + \frac{ik_z n}{\omega \epsilon r} H_z \right] = \frac{i\omega\epsilon}{k_3^2} \frac{dE_z}{dr} - \frac{k_z n}{k_3^2 r} H_z \\ E_\theta = -\frac{i\omega\mu}{k_3^2} \left[\frac{dH_z}{dr} - \frac{ink_z}{\omega \mu r} E_z \right] = -\frac{i\omega\mu}{k_3^2} \frac{dH_z}{dr} - \frac{nk_z}{k_3^2 r} E_z \end{cases}$$

that is

$$\begin{cases} H_\theta = \frac{i\omega\epsilon}{k_3} (aJ'_n + bY'_n) - \frac{k_z n}{k_3^2 r} (cJ_n + dY_n) \\ E_\theta = -\frac{i\omega\mu}{k_3} (cJ'_n + dY'_n) - \frac{k_z n}{k_3^2 r} (aJ_n + bY_n). \end{cases} \quad (53)$$

Any solution of the Cauchy problem on $(E_z, E_\theta, H_z, H_\theta)$ stated in Lemma 15, in the case $k_3 \neq 0$ can be written

$$\begin{pmatrix} E_z \\ E_\theta \\ H_z = \frac{ik_3}{\omega\mu} (E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0) S_n(R) \frac{d_n(r)}{d_n(R)} \\ H_\theta \end{pmatrix} = a \begin{pmatrix} J_n(k_3r) \\ -\frac{nk_z}{k_3^2 r} J_n(k_3r) \\ 0 \\ \frac{i\omega\epsilon}{k_3} J'_n(k_3r) \end{pmatrix} + b \begin{pmatrix} Y_n(k_3r) \\ -\frac{nk_z}{k_3^2 r} Y_n(k_3r) \\ 0 \\ \frac{i\omega\epsilon}{k_3} Y'_n(k_3r) \end{pmatrix} + c \begin{pmatrix} 0 \\ -\frac{i\omega\mu}{k_3} J'_n(k_3r) \\ J_n(k_3r) \\ -\frac{k_z n}{k_3^2 r} J_n(k_3r) \end{pmatrix} + d \begin{pmatrix} 0 \\ -\frac{i\omega\mu}{k_3} Y'_n(k_3r) \\ Y_n(k_3r) \\ -\frac{k_z n}{k_3^2 r} Y_n(k_3r) \end{pmatrix},$$

which defines a fundamental basis of the Cauchy problem.

For one annulus, the boundary conditions $E_\theta(r_0) = 0, E_z(r_0) = 0$ yield

$$\begin{cases} aJ_n(k_3r_0) + bY_n(k_3r_0) = 0 \\ cJ'_n(k_3r_0) + dY'_n(k_3r_0) = 0. \end{cases}$$

The boundary conditions at $r = R$ yield

$$\begin{cases} aJ_n(k_3R) + bY_n(k_3R) = E_z^0 \\ cJ'_n(k_3R) + dY'_n(k_3R) = \frac{ik_z n}{k_3 R \omega \mu} E_z^0 + \frac{ik_3}{\omega \mu} E_\theta^0 = \frac{ik_3}{\omega \mu} [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0]. \end{cases}$$

Collecting, we have

$$\begin{cases} aJ_n(k_3r_0) + bY_n(k_3r_0) = 0 \\ aJ_n(k_3R) + bY_n(k_3R) = E_z^0 \\ cJ'_n(k_3r_0) + dY'_n(k_3r_0) = 0 \\ cJ'_n(k_3R) + dY'_n(k_3R) = \frac{ik_3}{\omega \mu} [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0], \end{cases}$$

which gives two decoupled systems. Hence, under $d_n(k_3R)(dd_n)'(k_3R) \neq 0$ there is existence and uniqueness of a solution of this system. Note that the condition $d_n(k_3R)(dd_n)'(k_3R) \neq 0$ do not depend on the pair of independent solutions of the Bessel equation chosen because there exists, for any pair $\{f, g\}$ of independent solutions of Equation (12), a constant $C(f, g)$ such that $f(r_0)g(R) - f(R)g(r_0) = C(f, g)d_n$. The condition $d_n(R)(dd_n)'(R) = 0$ was obtained in Pierre Payen's thesis, as well as the Fourier multiplier (with an equivalent method).

The solution of the system on (a, b, c, d) is:

$$\begin{cases} a = -\frac{Y_n(k_3r_0)}{d_n(k_3R)} E_z^0 \\ b = \frac{J_n(k_3r_0)}{d_n(k_3R)} E_z^0 \\ c = -\frac{Y'_n(k_3r_0)}{D_n(k_3R)} \frac{ik_3}{\omega \mu} [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] \\ d = \frac{J'_n(k_3r_0)}{D_n(k_3R)} \frac{ik_3}{\omega \mu} [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0]. \end{cases} \quad (54)$$

One deduces

$$E_z(r) = E_z^0 S_n(k_3r), \quad H_z(r) = \frac{ik_3}{\omega \mu} [\frac{k_z n}{k_3^2 R} E_z^0 + E_\theta^0] S_n(k_3r).$$

We use then (53) to deduce the two other components. In particular

$$\frac{dE_z}{dr}(r) = k_3 E_z^0 T_n(k_3r), \quad \frac{dH_z}{dr}(r) = \frac{ik_3^2}{\omega \mu} [E_\theta^0 + \frac{ik_z n}{k_3^2 R} E_z^0] R_n(k_3r),$$

which yields

$$\begin{cases} H_\theta = \frac{i\omega \epsilon}{k_3} E_z^0 T_n(k_3r) - \frac{ik_z n}{k_3 r \omega \mu} [E_\theta^0 + \frac{ik_z n}{k_3^2 R} E_z^0] S_n(k_3r), \\ E_\theta = [E_\theta^0 + \frac{ik_z n}{k_3^2 R} E_z^0] R_n(k_3r) - \frac{k_z n}{k_3^2 r} E_z^0 Q_n(k_3r). \end{cases}$$

Collecting, the electromagnetic fields that were obtained here are

$$\begin{cases} E_z(r) = Q_n(k_3r) E_z^0, \\ E_\theta(r) = [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] R_n(k_3r) - \frac{nk_z}{k_3^2 r} Q_n(k_3r) E_z^0 \\ H_z(r) = \frac{ik_3}{\omega \mu} [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] S_n(k_3r) \\ H_\theta(r) = -\frac{ik_z n}{k_3 \omega \mu r} [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] S_n(k_3r) + \frac{i\omega \epsilon}{k_3} E_z^0 T_n(k_3r) \end{cases} \quad (55)$$

As before, these expressions do not depend on the pair of independent solutions of the Bessel equation chosen.

Finally,

$$\begin{cases} H_z(R) = \frac{ik_3}{\omega\mu} [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] S_n(R) \\ H_\theta(R) = -\frac{ik_z n}{k_3 \omega \mu R} [E_\theta^0 + \frac{k_z n}{k_3^2 R} E_z^0] S_n(R) + \frac{i\omega\epsilon}{k_3} E_z^0 T_n(R) = \frac{i\omega\epsilon}{k_3} E_z^0 [T_n(R) - \frac{k_z^2 n^2}{k_3^2 R^2 \omega^2 \epsilon \mu} S_n(R)] - \frac{ik_z n}{k_3 \omega \mu R} E_\theta^0, \end{cases}$$

which yields the Calderòn operator and ends the proof of Lemma 17. \square

Lemma 18. *Let*

$$F = \Psi - \Psi_0 = p(\tan\beta_* - \beta_* - \tan\beta_0 + \beta_0).$$

The following asymptotic expansions hold, under the hypotheses described in Proposition 9 and the notation $F = \Psi - \Psi_0$

$$T_n(k_3 R) = -i \sin\beta_* \frac{(L_0 - iM_0)(N - iO)e^{-iF} + (L_0 + iM_0)(N + iO)e^{iF}}{(L_0 - iM_0)(L + iM)e^{-iF} - (L_0 + iM_0)(L - iM)e^{iF}},$$

or

$$T_n(k_3 R) = \sin\beta_* \frac{\cos F - \frac{a_0 + b}{p} \sin F}{\sin F - \frac{a - a_0}{p} \cos F} (1 + O(p^{-2})).$$

Similarly,

$$S_n(k_3 R) = \frac{i}{\sin\beta_*} \frac{(N_0 + iO_0)(L_* + iM_*)e^{-iF} + (N_0 - iO_0)(L_* - iM_*)e^{-iF}}{(N_0 + iO_0)(N_* - iO_*)e^{-iF} - (N_0 - iO_0)(N_* + iO_*)e^{iF}},$$

which yields

$$S_n(k_3 R) = -\frac{1}{\sin\beta_*} \frac{\cos F + \frac{b_0 + a_*}{p} \sin F}{\sin F + \frac{b_* - b_0}{p} \cos F} (1 + O(p^{-2})).$$

and its counterpart when $\Im\epsilon\mu$ does not depend on ω :

Lemma 19. *Assume that $\Im\epsilon\mu < 0$ is a constant independent on ω . Depending on the sign of $\Im F$, one obtains, up to terms which are $O(e^{-2|\Im F|})$:*

$$\begin{aligned} S_n(k_3 R) &= \pm \frac{i}{\sin\beta_*} - (b + a) \frac{k_3}{p \sin\beta_*} + O(\omega^{-1}), \\ T_n(k_3 R) &= \pm i \sin\beta_* + (b + a) \frac{k_3 \sin\beta_*}{p} + O(\omega^{-1}). \end{aligned}$$

Remark 15. *The conditions for existence and uniqueness of a solution for a multi-layer annulus is not the intersection of the conditions for each layer. It is, for example, for a double layer $(\det M_E)(\det M_H) \neq 0$, where M_E, M_H are given by (56), (57). This condition does not depend, again, on the choice of the pair of independent solutions of the Bessel equation chosen in each layer.*

For example, for two layers of dielectric materials, of dielectric constants $(\epsilon, \mu), r_0 < r < r_1$ and $(\epsilon_1, \mu_1), r_1 < r < R$, introduce $k_3^1 = \sqrt{\epsilon_1 \mu_1 \omega^2 - k_z^2}$. The condition $\det M_E \neq 0$ rewrites

$$\begin{aligned} & (k_3^1)^{-1} (J_n(k_3 r_0) Y_n(k_3 r_1) - J_n(k_3 r_1) Y_n(k_3 r_0)) [J_n(k_3^1 R) Y_n'(k_3^1 r_1) - Y_n(k_3^1 R) J_n'(k_3^1 r_1)] \\ & \neq \\ & (k_3)^{-1} (J_n(k_3 r_0) Y_n'(k_3 r_1) - J_n'(k_3 r_1) Y_n(k_3 r_0)) [J_n(k_3^1 R) Y_n(k_3^1 r_1) - Y_n'(k_3^1 R) (k_3^1 r_1)]. \end{aligned}$$

The proof of Remark 15 comes from the expression of the continuity at the interface $r = r_1$ of the fields (E_z, H_θ) , that is the continuity of $(E_z, \frac{dE_z}{dr})$, and the Dirichlet boundary conditions on E_z at $r = r_0$ and $r = R$, leading to the determinant of the matrix M :

$$M_E := \begin{pmatrix} J_n(k_3 r_0) & Y_n(k_3 r_0) & 0 & 0 \\ J_n(k_3 r_1) & Y_n(k_3 r_1) & -J_n(k_3^1 r_1) & -Y_n(k_3^1 r_1) \\ (k_3)^{-1} J_n'(k_3 r_1) & (k_3)^{-1} Y_n'(k_3 r_1) & -(k_3^1)^{-1} J_n'(k_3^1 r_1) & (k_3^1)^{-1} Y_n'(k_3^1 r_1) \\ 0 & 0 & J_n(k_3^1 R) & Y_n(k_3^1 R) \end{pmatrix}, \quad (56)$$

the system to be solved being

$$M_E \begin{pmatrix} a \\ b \\ a_1 \\ b_1 \end{pmatrix} = E_z^0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

In a similar fashion, the system on (c, d, c_1, d_1) reads

$$M_H \begin{pmatrix} c \\ d \\ c_1 \\ d_1 \end{pmatrix} = \left(\frac{(k_3^1)^2}{i\omega\mu_1} E_\theta^0 + \frac{ikn}{i\omega\mu_1 R} E_z^0 \right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{k_z n}{r_1} \left[\frac{1}{(k_3^1)^2} (a_1 J_n'(k_3^1 r_1) + b_1 Y_n'(k_3^1 r_1)) - \frac{1}{k_3^2} (a J_n'(k_3 r_1) + b Y_n'(k_3 r_1)) \right] \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$M_H := \begin{pmatrix} J_n'(k_3 r_0) & Y_n'(k_3 r_0) & 0 & 0 \\ J_n(k_3 r_1) & Y_n(k_3 r_1) & -J_n(k_3^1 r_1) & -Y_n(k_3^1 r_1) \\ \frac{i\omega\mu_1}{k_3^2} J_n'(k_3 r_1) & \frac{i\omega\mu_1}{k_3^2} Y_n'(k_3 r_1) & \frac{i\omega\mu_1}{(k_3^1)^2} J_n'(k_3^1 r_1) & \frac{i\omega\mu_1}{(k_3^1)^2} Y_n'(k_3^1 r_1) \\ 0 & 0 & J_n'(k_3^1 R) & Y_n'(k_3^1 R) \end{pmatrix}. \quad (57)$$

No attempt to solve these systems is to be done here. See [15] for a procedure leading in most cases to the values of the traces on $r = R$, even if it does not rely on the global system.

8. Annex

8.1. Pseudodifferential discrete operators

We define in this Section what is a pseudodifferential operator acting on $\mathbb{R} \times [0, 2\pi]$. Consider the usual algebra of symbols on $T^*(\mathbb{R}^2)$, and assume this symbol is periodic in θ , of period 2π , satisfying the classical estimates in $S_{1,0}^m$. The action of such a symbol $a(z, \theta, \xi, \eta)$ is, as usual

$$Op(a)f(z, \theta) = \frac{1}{(2\pi)^2} \int \int a(z, \theta, \xi, \eta) \hat{f}(\xi, \eta) e^{ix\xi + i\theta\eta} d\eta d\xi$$

If F is a function in $L^2(\mathbb{R} \times [0, 2\pi])$, periodic in θ , consider the associated function on \mathbb{R}^2 where $\tilde{F}(z, \theta) := F(z, \theta - 2\pi[\frac{\theta}{2\pi}])$. Using the Fourier series of F :

$$F(z, \theta) = \sum_n F_n(z) e^{in\theta}$$

one defines directly $\tilde{F}(z, \theta) = \sum_n F_n(z) e^{in\theta}$, and $\tilde{F} \in \mathcal{S}'(\mathbb{R}^2)$, and

$$\mathcal{F}(\tilde{F})(\xi, \eta) = 2\pi \sum_n \hat{F}_n(\xi) \delta_n.$$

One has thus

$$\begin{aligned} Op(a)\tilde{F}(z, \theta) &= \frac{1}{2\pi} \langle \sum_n \hat{F}_n(\xi) \delta_n, a(z, \theta, \xi, \eta) e^{iz\xi + i\theta\eta} \rangle \\ &= \sum_n \frac{1}{2\pi} \langle \hat{F}_n(\xi), a(z, \theta, \xi, n) e^{iz\xi} e^{in\theta} \rangle \\ &= \sum_n e^{in\theta} \left(\frac{1}{2\pi} \int a(z, \theta, \xi, n) e^{iz\xi} \hat{F}_n(\xi) d\xi \right). \end{aligned}$$

If, in addition, the symbol does not depend on z (that is the operator does not depend on z , as it will be the case here)

$$Op(a)\tilde{F}(z, \theta) = \sum_n e^{in\theta} \left(\frac{1}{2\pi} \int a(\theta, \xi, n) e^{iz\xi} \hat{F}_n(\xi) d\xi \right),$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} Op(a)\tilde{F}(z, \theta) d\theta &= \sum_n \frac{1}{2\pi} \int \left(\frac{1}{2\pi} \int_0^{2\pi} a(\theta, \xi, n) e^{i(n-m)\theta} d\theta \right) e^{iz\xi} \hat{F}_n(\xi) d\xi \\ &= \sum_n \frac{1}{2\pi} \int a_{m-n}(\xi, n) e^{iz\xi} \hat{F}_n(\xi) d\xi \end{aligned}$$

where $a_p(\xi, n)$ is the p -th Fourier coefficient of $a(\theta, \xi, n)$. As this definition comes from the classical definition of pseudodifferential calculus, the pseudodifferential calculus extends to this case.

8.2. Proof of Lemma 2

Proof. Begin by a necessary condition. For this purpose, consider the partial Fourier transform in (x_1, x_2) of $u \in \mathcal{S}'(\mathbb{R}^2, C^2([-l, 0]))$. One has the following ODE problem on $\hat{u} \in \mathcal{S}'(\mathbb{R}^2, C^2([-l, 0]))$

$$\begin{cases} \left(\frac{d^2}{dx_3^2} + \omega^2 \epsilon \mu - k_1^2 - k_2^2 \right) \hat{u} = 0 \\ \hat{u}(-l) = 0 \\ \hat{u}(0) = \hat{u}_0(k_1, k_2). \end{cases} \quad (58)$$

All solutions in $C^\infty(\mathbb{R})$ of this problem read

$$\hat{u}(x_3) = a e^{ik_\perp x_3} + b e^{-ik_\perp x_3}.$$

It is important to check that, for all $K > 0$, one has the implication

$$\forall (k_1, k_2) \in \mathbb{R}^2, |a| \leq K |e^{ik_\perp l}|, |b| \leq K \Rightarrow |\hat{u}(x_3)| \leq 2K, -l \leq x_3 \leq 0$$

ensuring that the conditions on a and b are sufficient conditions for u being in $L^\infty(\mathbb{R}^2 \times [-l, 0]) \subset \mathcal{S}'(\mathbb{R}^2, L^\infty([-l, 0]))$, which is a sufficient condition for showing that \hat{u} is the Fourier transform of a tempered distribution.

Under such conditions, one gets $a + b = \hat{u}_0$, $a e^{-ik_\perp l} + b e^{ik_\perp l} = 0$, from which one gets

$$a = \frac{\hat{u}_0}{1 - e^{-2ik_\perp l}}, b = -\frac{\hat{u}_0 e^{-2ik_\perp l}}{1 - e^{-2ik_\perp l}}.$$

The solution, if it belongs to $\mathcal{S}'(\mathbb{R}^2, C^2([-l, 0]))$ is then

$$\hat{u}(x_3) = \hat{u}_0 \left[\frac{e^{ik_\perp x_3}}{1 - e^{-2ik_\perp l}} - \frac{e^{-ik_\perp x_3 - 2ik_\perp l}}{1 - e^{-2ik_\perp l}} \right] = \hat{u}_0 \frac{\sin(k_\perp (x_3 + l))}{\sin(k_\perp l)}.$$

As $\Im k_\perp \leq -\alpha_0 \omega$, we have

$$\frac{\sin(k_\perp (x_3 + l))}{\sin(k_\perp l)} = e^{-\Im k_\perp x_3} \frac{e^{ik_\perp^0 (x+l)} - e^{-ik_\perp^0 (x_3+l) + 2k_\perp^1 (x_3+l)}}{e^{ik_\perp^0 l} - e^{-ik_\perp^0 l + 2k_\perp^1 l}}$$

hence, as $-l \leq x_3 \leq 0$, this expression satisfies the estimate

$$\left| \frac{\sin(k_{\perp}(x_3 + l))}{\sin(k_{\perp}l)} \right| \leq \frac{2}{1 - e^{-2\Im k_{\perp}l}} \leq \frac{2}{1 - e^{-2\alpha_0 \omega l}}, \quad (59)$$

and similar estimates hold for \hat{u}' and \hat{u}'' , hence $\hat{u}(x_3)$ belongs to $\mathcal{S}'(\mathbb{R}^2, C^2([-l_0, 0]))$ for all $a^0 \in \mathcal{S}(\mathbb{R}^2)$. \square

Assume now that $\epsilon\mu \in \mathbf{R}$.

Proof. Two cases occur only: $k_{\perp} \in \mathbb{R}^*$ (when $\epsilon\mu\omega^2 - k_1^2 - k_2^2 > 0$) or $\Re k_{\perp} = 0$ (otherwise).

In the first case, that is for $k_1^2 + k_2^2 < \omega^2\epsilon\mu$, the system on (a, b) is $a + b = \hat{u}_0$, $ae^{-ik_{\perp}l} + be^{ik_{\perp}l} = 0$, which yields $a(1 - e^{-2ik_{\perp}l}) = \hat{u}_0$. The condition $(1 - e^{-2ik_{\perp}l}) \neq 0$ grants uniqueness of the solution. If there exists n such that $\omega^2\epsilon\mu = k_1^2 + k_2^2 + \frac{n^2\pi^2}{l^2}$, the function $A \sin \frac{n\pi}{l}x_3$ is a non zero solution of the system (58) for $\hat{u}_0 = 0$. This yields the resonances of the problem. However, this only ensures that the distribution $A \sin \frac{n\pi}{l}x_3$ in $\mathcal{S}'(\mathbb{R}^2, C^2([-l, 0]))$ where \hat{A} is supported on $k_1^2 + k_2^2 = \omega^2\epsilon\mu - \frac{n^2\pi^2}{l^2}$, is a nonzero solution of (58) with $u_0 = 0$.

If (k_1, k_2) is such that $\omega^2\epsilon\mu - k_1^2 - k_2^2 \neq \frac{n^2\pi^2}{l^2}$, the notation s_0 imply that

$$n_0 \in \left\{ \frac{l}{\pi} \sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}, \frac{l}{\pi} \sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} + 1 \right\}.$$

In this case, for n_0 even, $l \sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} - n_0\pi \geq s_0\pi$ or $(n_0 + 1)\pi - l \sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2} \geq s_0\pi$, hence $\sin(l \sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}) \geq \sin \pi s_0$ (and similar inequalities for n_0 odd). This means that

$$\left| \frac{\sin(l + x_3) \sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}}{\sin l \sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}} \right| \leq \frac{1}{\sin \pi s_0}.$$

In the case $k_{\perp} = 0$, the equation reads $(\hat{u})'' = 0$, hence $\hat{u}(x_3) = a(k_1, k_2)x_3 + b(k_1, k_2)$, the Dirichlet boundary conditions yields $\hat{u}(x_3) = a(k_1, k_2)(x_3 + l)$, and $\partial_{x_3} \hat{u}(0) = a(k_1, k_2)$, hence $\hat{u}(x_3) = \hat{u}_0 \frac{x_3 + l}{l}$.

In the second case, that is for $k_1^2 + k_2^2 > \omega^2\epsilon\mu$, $\Im k_{\perp} = \sqrt{-\epsilon\mu\omega^2 + k_1^2 + k_2^2}$ and the formal⁴ solution is

$$\hat{u}(k_1, k_2, x_3) = a(k) [e^{\Im k_{\perp}(x_3 + l)} - e^{-\Im k_{\perp}(x_3 + l)}],$$

and if one computes the value at $x_3 = 0$, one finds $a(k)(2 \sinh l \Im k_{\perp}) = \hat{u}(k_1, k_2, 0)$. One then deduces

$$\hat{u}(k_1, k_2, x_3) = \hat{u}(k_1, k_2, 0) \frac{\sinh(x_3 + l) \Im k_{\perp}}{\sinh l \Im k_{\perp}},$$

and $\frac{\sinh(x_3 + l) \Im k_{\perp}}{\sinh l \Im k_{\perp}}$ is increasing, hence the estimate $|\hat{u}(k_1, k_2, x_3)| \leq |\hat{u}(k_1, k_2, 0)|$, $\forall (k_1, k_2)$, $k_1^2 + k_2^2 > \omega^2\epsilon\mu$. This shows that this formal solution belongs to $C^2(\mathcal{S}'(\mathbb{R}^2), [-l, 0])$ hence it was possible to use the Fourier transform.

This proves the first item of Lemma 2 and explains the term resonance for values of (k_1, k_2) associated with n through Lemma 1. \square

⁴because this is not an element of $\mathcal{S}'(\mathbb{R}^2)$ because of the exponential growth in $\sqrt{k_1^2 + k_2^2}$ for at least one term

- [1] M.A. Abramovitz, I.A. Stegun: Handbook of Mathematical Functions, 10th printing, 1972 (National Bureau of Standards)
- [2] D. Bouche, O. Lafitte: Comparaison couche limite analyse microlocale *Asymptotic Analysis*
- [3] Boumaza, H., Lafitte, O. The band spectrum of the periodic Airy-Schrödinger operator on the real line. *J. Differential Equations* 264 (2018), no. 1, 455–505
- [4] Cessenat, Michel Mathematical methods in electromagnetism. Linear theory and applications. *Series on Advances in Mathematics for Applied Sciences*, 41. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
- [5] David Colton and Peter Monk: The Scattering of Electromagnetic Waves by a Perfectly Conducting Infinite Cylinder *Mathematical Methods in the Applied Sciences*, Vol. 12, 503-518 (1990)
- [6] Digital Library of Mathematical Functions: <https://dlmf.nist.gov>
- [7] On the effect of the boundary conditions on the eigenvalues of ordinary differential equations *Contributions to analysis and geometry: John Hopkins University Press, Baltimore, 1981*, 139-150
- [8] Harrell: The band-structure of a one-dimensional, periodic system in a scaling limit. *Ann. Physics* 119 (1979), no. 2, 351–369
- [9] O. Lafitte: Diffraction in the high frequency regime by a thin layer of dielectric material I, II: the equivalent boundary condition *SIAM J. Appl. Maths* 59 (3) 1020-1052, 1999
- [10] O. Lafitte: Hybrid singularity for the Oblique Incidence Response of a Cold Plasma *Indiana University Math. Journal* 69 (4), 1317-1363, 2020
- [11] O. Lafitte, M. Williams, K. Zumbrun: The Erpenbeck high frequency instability theorem for Zeldovitch-von Neumann-Döring detonations. *Arch. Ration. Mech. Anal.* 204 (2012), no. 1, 141–187
- [12] Samuel Nosal, Paul Soudais, and Jean-Jacques Greffet: Integral Equation Modeling of Doubly Periodic Structures With an Efficient PMCHWT Formulation *IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION*, VOL. 60, NO. 1, JANUARY 2012
- [13] F.J. Olver: Uniform Asymptotic Expansions of Solutions of Linear Second-Order Differential Equations for Large Values of a Parameter *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 1958, 250 (984), 479-517
- [14] F.J. Olver: Asymptotics and special functions, AK Peters, Wellesley, MA, USA, 1997.
- [15] P. Payen: Thèse de doctorat de l'Université Paris-XIII, 2020.
- [16] B. Stupfel: Improved transmission conditions for a one-dimensional domain decomposition method applied to the solution of the Helmholtz equation *Journal of Computational Physics* 229 (2010) 851–874
- [17] A. Sharples: Uniform asymptotic expansions of modified Mathieu functions. *J. Reine Angew. Math.* 247 (1971), 1–17
- [18] A. Sharples: Uniform asymptotic forms of modified Mathieu functions. *Quart. J. Mech. Appl. Math.* 20 (1967), 365–380.