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## Coordinate Descent Optimization for One-to-One Correspondence and Supervised Classification of 3D Shapes

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### Abstract

Recent developments in shape analysis and retrieval play an important role in a wide variety of applications that potentially require matching of objects with different geometries. In shape classification, there is no natural way to represent an object but the similarity measure, distance between representations or descriptors, depends heavily on the strategy of computing optimal correspondences. In this paper we introduce a new numerical method for registering surfaces. Thus, finding an optimal one-to-one correspondence between their shapes. Unfortunately, solving this type of optimization problem is generally hard because the solutions space is nonlinear with no natural manifold structure on it. To overcome such limitations we make use of recent methods to represent objects then we find optimal correspondences using a discretized approximation of the search space. The proposed method has the advantage of extending the Riemannian analysis of 3D curves in a natural way for surfaces. We demonstrate the proposed algorithms using different public datasets for 2D and 3D objects matching and classification.

Keywords: Optimization on manifolds ; Coordinate descent ; Parameterized surfaces registration ; Shape classification ; Manifold Learning

### 1 1. Introduction

Due to the importance of finding optimal correspondences among 2D and 3D shapes, there is a great deal of research, analysis and methods that have been introduced for the same purpose [1, 2, 3]. Despite 3 all significant advances that have been achieved (particularly for 2D shapes), the problem of a complete 4 matching between 3D objects is far from being solved. Basically, the main challenges are related to data and 5 the application at hand. At the same time, planar closed curves or boundary surfaces are increasingly being 6 used to represent the outline or shape of objects, enabling them to emerge as a natural choice in shape analysis 7 and their applications. The need for shape analysis arises in many scientific fields, including computer vision, 8 medical imaging, computer graphics, among others [4, 5, 6]. Analyzing shapes and their differences relies on 9 the notion of distance between shapes which is the key element for registration, deformation, and comparison 10 of shapes. For example, in medical imaging, surface registration is always needed for constructing generative 11 models, shape classification and prediction, and supervised segmentation, morphometry, fusion, denoising 12 observed data [7, 8, 9]. In many cases, a surface must match another, while comparing them. Finding an 13 optimal registration that best matches the required constraints is difficult in practice, especially on surfaces 14 undergoing nonlinear deformations [10]. It is therefore necessary to explore efficient numerical methods 15

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to compute optimal re-parametrization of parametrized surfaces. There are several successful approaches 1 for aligning two point clouds, matching non-rigid shapes [11], registering curves to surfaces as a partial 2 matching [12], etc. In this paper, we consider a different problem: Find an optimal one-to-one correspondence 3 between parameterized tubular surfaces as a Riemannian optimization on the space of all re-parametrizations. 4 Solving this type of optimization problem is generally hard, since the space of all re-parametrizations is a complicated infinite dimensional and nonlinear functional space [13]. Considering the actions of the group 6 of diffeomorphisms on the simplest closed manifold, namely the unit circle, of the apparent simplicity, it has 7 been shown that it is highly nontrivial and very extensive [14, Chapter 3]. Thus, developing an efficient opti-8 mization method to register two parametrized surfaces modulo all re-parametrizations is a serious challenge, 9 and a loss of quality is often observed during this step. To overcome such limitations, at least numerically, 10 one can: i) choose a good representation for surfaces, ii) build a metric space of those representations, iii) 11 formulate the registration problem as a minimization of a cost function, and finally iv) discretize the space of 12 re-parameterization and optimize over a finite dimensional space [15, 16]. 13

Optimization procedures as a part of shape analysis play a crucial role in many applications making it an 14 active research area. That is why continuous and discrete optimization procedure for shape analysis has been 15 extensively studied during last decades [17, 18]. Previous methods usually compute the solution by minimiz-16 ing certain energy functionals (e.g. variational formulations). However, the literature on parameterization of 17 tubular objects is by far not as extensive as for open and spherical surfaces. We briefly describe some related 18 methods, a non-exhaustive list, that have been used for similar contexts. For example paper [19] developed 19 a framework to compute large deformations diffeomorphisms between surfaces but their framework based 20 on manually selected points is not applicable for parametrized surfaces. An extension has been presented 21 in [6]. A more complete analysis of shapes of parametrized surfaces has been presented in [20, 21] and the 22 theoretical solutions are considered over the set of all diffeomorphisms and represented by smooth vector 23 fields. However optimization procedures in general are either restrictive to a parametric family of wrapping 24 functions or slow with unsatisfying minimizers in practice. 25

In this work we present a new solution for parametrized tubular surfaces and we show later that this framework can be applied on other surfaces. Finding a one-to-one correspondence is a fundamental problem in mathematics and computation [22, 23, 24] with applications in multiple fields, ranging from medical imaging [25] to computer vision [26], and passing by Computer Aided Design (CAD) [27]. This framework is motivated by shape classification as a supersized learning from a training set of data containing 2D curves (or tubular surfaces) whose classes membership is known. In particular and using new algorithms we show that the accuracies of some standard classifiers are better when the one-to-one correspondence is optimal.

A tubular object is defined to be a surface that is homomorphic to a cylinder, or equivalently, is defined to be a differentiable function  $S : \Omega \in \mathbb{R}^3 : (\xi_1, \xi_2) \mapsto S(\xi_1, \xi_2)$ , where  $\Omega$  denotes  $[0, 1] \times \mathbb{S}^1$  and  $\mathbb{S}^1$  denotes the unit circle. The correspondence can be characterized mathematically by a diffeomorphism. The set of diffeomorphisms, or equivalently called the re-parameterizations group, is defined by

$$\Gamma(\mathbb{D}) = \{ \gamma : \mathbb{D} \to \mathbb{D} \mid \gamma \text{ is an orientation-preserving} \\ \text{diffeomorphism on the domain } \mathbb{D}. \}$$

In particular, for the tubular object, the domain  $\mathbb{D}$  is  $\Omega$  and the boundary is preserved with  $\gamma(\partial \Omega) = \partial \Omega$ . In this setting, looking for a reparameterization leads to an optimization problem:

$$\min_{\gamma \in \Gamma([0,1] \times \mathbb{S}^1)} E(S_1, S_2 \circ \gamma), \tag{1}$$

where E is a real-valued cost function and is usually a distance function in the space of surfaces, and  $S_1$  and  $S_2$  are two tubulars.

Solving the optimization problem (1) efficiently is not easy. For shape of curves, the problem is relatively

simple, and Riemannian optimization method [28], dynamic programming [16] and coordinate descent have

been proposed. These methods have the ability to escape from unsatisfied local minimizers, see their papers
for details. However, the existing methods for the problem (1) is still limited. To the best of our knowledge,
only the steepest descent method with multiple initial iterates have been used. However, gradient-based
algorithms are usually slow and unsatisfied since i) the steepest descent is known to be slow with a local
behavior in nonlinear optimization problems and ii) the number of initial iterates need to be large due to the
large dimension of the domain (infinite in theory and large in practice).

Contributions. We propose a new method to solve this problem for tubular surfaces with an extension
 to other surfaces. For purposes of simplicity, we often use the term surface. The main contributions of this
 work are:

10 1. We use a recent work for elastic shape analysis of surfaces [21] and build a connection with the frame-11 work of elastic shape analysis of curves in [16], see Proposition 3.1.

12 2. We propose a new optimization method (algorithm 2 and algorithm 3) to register surfaces.

3. We use various examples with complicated geometries for illustrations and different benchmarks to
 show the effectiveness of the new methodology. The 3D shape classification is carried out by combining
 the proposed method for correspondence and a Gaussian process autoencoder for classification.

This paper is organized as follows. Section 2 reviews a framework of elastic shape analysis of curves and a recent framework of elastic shape analysis of surfaces. Section 3 shows a relationship between the two frameworks. This motivates the algorithm design and yields Section 4, which presents the proposed optimization algorithms. Section 5 shows that the idea of using curves for surfaces can be applied for other surfaces. The numerical illustrations and experiments are given in Section 6 to show the accuracy of the proposed here it is a first the state of the state of the state of the show the accuracy of the proposed the numerical illustrations and experiments are given in Section 6 to show the accuracy of the proposed

<sup>21</sup> algorithms. Finally, the conclusion is carried out in Section 7.

### 22 2. Elastic Shape Analysis of Curves and Surfaces

Intuitively, a shape is invariant to translation, scaling, rotation and re-parameterization. In elastic shape analysis, a mathematical representation of shape is defined such that it is invariant to the four operations above, and multiple frameworks of elastic shape analysis have been proposed, see e.g., [29, 16, 21]. This paper focuses on the removing reparameterization, which is the most difficult one to be removed. We refer

to [16] and [21] if readers are interested in removing the other three operations.

### 28 2.1. Analyzing Curves

In this paper, the SRVF framework defined in [16] is considered. Specifically, a curve in  $\mathbb{R}^n$  is denoted by a parameterized function  $\beta : \mathbb{D} \to \mathbb{R}^n$ , where  $\mathbb{D}$  is [0, 1] for open curves and  $\mathbb{D}$  is the unit circle  $\mathbb{S}^1$  for closed curves. The square root velocity function of  $\beta$  is

$$q(t) = \begin{cases} \frac{\dot{\beta}(t)}{\sqrt{||\dot{\beta}(t)||_2}}, & \text{if } ||\dot{\beta}(t)||_2 \neq 0; \\ 0, & \text{if } ||\dot{\beta}(t)||_2 = 0. \end{cases}$$
(2)

where  $\|\cdot\|_2$  denote the 2-norm. The curve  $\beta$  can be recovered by  $\beta(t) = \int_0^t q(s)||q(s)||_2 ds + \beta_0$ , where  $\beta_0 \in \mathbb{R}^n$  is the starting point of  $\beta$ .

The group action on curve  $(\beta, \gamma) = \beta \circ \gamma$  is equivalent to the action  $(q, \gamma) = \sqrt{\dot{\gamma}}q \circ \gamma$  on q. Let  $\overline{\mathcal{N}}$  denote the space of all q functions. For open curves,  $\overline{\mathcal{N}}$  is the  $\mathbb{L}^2$  space. For closed curves,  $\overline{\mathcal{N}} = \{q \in \mathbb{L}^2 \mid \int_{\mathbb{S}^1} q(t) \| q(t) \|_2 dt = 0\}$  is a subspace of the  $\mathbb{L}^2$  space. Note that the condition  $\int_{\mathbb{S}^1} q(t) \| q(t) \|_2 dt = 0$ , equivalent to  $\int_{\mathbb{S}^1} \dot{\beta}(t) dt = 0$ , implies that the starting point and end point of the curve  $\beta$  is the same and therefore is a closed curve. A quotient space of  $\overline{\mathcal{N}}$  can be defined as  $\mathcal{N} = \{[q] \mid q \in \overline{\mathcal{N}}\}$ , where  $[q] = \{(q, \gamma) \mid \gamma \in \Gamma(\mathbb{D})\}$  is an orbit. It has been shown in [16] that the group action on q is isometric with respect

to the  $\mathbb{L}^2$  metric. Therefore,  $\mathcal{N}$  with the  $\mathbb{L}^2$  metric is a well defined metric space. The cost function for finding the best reparameterization between two curves therefore is (see [16])

$$C: \Gamma(\mathbb{D}) \to \mathbb{R}: \gamma \to \|q_1 - (q_2, \gamma)\|_{\mathbb{L}^2}^2$$
$$= \int_{\mathbb{D}} \|q_1(t) - \sqrt{\dot{\gamma}} q_2 \circ \gamma(t)\|_2^2 dt$$
(3)

2.2. Analyzing Surfaces

We first give a definition for axial and circular curves.

**Definition 2.1.** Let S be a surface parametrized with  $(\xi_1, \xi_2)$  such that:  $S : (\xi_1, \xi_2) \in [0, 1] \times \mathbb{S}^1 \mapsto S(\xi_1, \xi_2) \in \mathbb{R}^3$ . We adapt the following definitions:

ξ<sub>1</sub>-curves are axial curves on S with ξ<sub>2</sub> constant.
 ξ<sub>2</sub>-curves are circular curves on S with ξ<sub>1</sub> constant.

<sup>4</sup> Suppose  $S : [0,1] \times \mathbb{S}^1 \to \mathbb{R}^3$  is a differentiable surface. Let  $\frac{\partial S}{\partial \xi_1}$  and  $\frac{\partial S}{\partial \xi_2}$  be the two partial derivatives <sup>5</sup> of S along the  $\xi_1$ -curves and  $\xi_2$ -curves. A normal field on S with respect to a positive orientation is defined

6 as follows:

$$N: \quad \Omega \to \mathbb{R}^3: (\xi_1, \xi_2) \mapsto N(\xi_1, \xi_2) = \frac{\partial S}{\partial \xi_1} \times \frac{\partial S}{\partial \xi_2},$$

<sup>7</sup> where  $\times$  denotes the cross product. The Q-field of S is defined as Square Root Normal Field(SRNF) [21]:

**Definition 2.2.** Let S be a surface with a parametrization  $(\xi_1, \xi_2)$  we call Q-field of S the vector field on S along  $\xi_1$  and  $\xi_2$  defined by:

8

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$$Q: \Omega \to \mathbb{R}^{3}$$

$$(\xi_{1}, \xi_{2}) \mapsto Q(\xi_{1}, \xi_{2}) = \frac{N(\xi_{1}, \xi_{2})}{\sqrt{\|N(\xi_{1}, \xi_{2})\|_{2}}}$$
(4)

9

One can see that the framework of SRNF is similar to the framework of SRVF in the sense that SRNF considers taking the square root of the normal field instead of the velocity. This is a natural generalization from 1D curves to 2D surfaces. It is probably the reason that analyzing a curve segment on a 2D surface by SRNF is essentially using the SRVF, see details in Section 3.

Proposition 2.1 gives the group action on Q-fields and proves the isometry of the group action under  $\mathbb{L}^2$ metric. These results have been given for spherical surfaces in [21]. We give them for tubular surfaces and

16 for completeness.

17

**Proposition 2.1.** Let S be a surface and Q its corresponding Q-field, we have the following properties:

- 1. The group action on surface  $(S, \gamma) = S \circ \gamma$  is equivalent to  $(Q, \gamma) = \sqrt{|J_{\gamma}|}Q \circ \gamma$ , where  $J_{\gamma}$  is the Jacobi of  $\gamma \in \Gamma(\Omega)$  and  $|\cdot|$  denotes the determinant.
  - 2. The group action of  $\gamma$  on Q-fields is an isometry under the  $\mathbb{L}^2$  metric.

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Specifically, we have

$$\begin{aligned} \|(Q_{1},\gamma) - (Q_{2},\gamma)\|_{\mathbb{L}^{2}}^{2} &= \left\|\sqrt{|J_{\gamma}|}Q_{1}\circ\gamma - \sqrt{|J_{\gamma}|}Q_{2}\circ\gamma\right\|_{\mathbb{L}^{2}}^{2} \\ &= \int_{\Omega} \left\|\sqrt{|J_{\gamma}|}Q_{1}\circ\gamma(\xi_{1},\xi_{2}) - \sqrt{|J_{\gamma}|}Q_{2}\circ\gamma(\xi_{1},\xi_{2})\right\|_{2}^{2}d\xi_{1}d\xi_{2} \\ &= \int_{\Omega} |J_{\gamma}| \left\|(Q_{1}-Q_{2})\circ\gamma(\xi_{1},\xi_{2})\right\|_{2}^{2}d\xi_{1}d\xi_{2} \\ &= \int_{\Omega} |J_{\gamma}||J_{\gamma^{-1}}| \left\|(Q_{1}-Q_{2})(\xi_{1},\xi_{2})\right\|_{2}^{2}d\xi_{1}d\xi_{2} \\ &= \left\|Q_{1}-Q_{2}\right\|_{\mathbb{L}^{2}}^{2} \end{aligned}$$

The space of Q-fields, denoted by  $\overline{\mathcal{M}}$ , is

$$\left\{ Q \in \mathbb{L}^2 \mid \int_{\mathbb{S}^1} Q(\xi_1, t) \| Q(\xi_1, t) \|_2 dt = 0 \text{ for all } \xi_1 \in [0, 1] \right\},\tag{5}$$

- and  $\mathcal{M}$  denote the quotient space  $\{[Q] \mid Q \in \overline{M}\}$ , where [Q] denotes the orbit of Q under the group action
- <sup>2</sup>  $\Gamma(\Omega)$ , i.e.,  $[Q] = \{(Q, \gamma) \mid \gamma \in \Gamma(\Omega)\}$ . Since the group action is isometric under the  $\mathbb{L}^2$  metric,  $\mathcal{M}$  with  $\mathbb{L}^2$
- <sup>3</sup> is a well-defined metric space.

Given two surfaces  $S_1$  and  $S_2$ , let  $Q_1$  and  $Q_2$  denote their Q-fields, respectively. We define the distance between  $Q_1$  and  $Q_2$  to be  $||Q_1 - Q_2||_{\mathbb{L}^2}$ , since the space of Q is a subspace of  $\mathbb{L}^2$ . The best reparameterization  $\hat{\gamma}$  is therefore the minimizer of the functional:

$$E:\Gamma(\Omega) \to \mathbb{R}: \gamma \mapsto \|Q_1 - (Q_2, \gamma)\|_{\mathbb{L}^2}^2$$
  
=  $\int_{\Omega} \left\| Q_1(\xi_1, \xi_2) - (Q_2, \gamma)(\xi_1, \xi_2) \right\|_2^2 d\xi_1 d\xi_2,$  (6)

4 where  $(Q_2, \gamma)(\xi_1, \xi_2) = \sqrt{|J_\gamma|} Q_2 \circ \gamma(\xi_1, \xi_2).$ 

### 5 3. Optimal Parametrization Between Surfaces

Optimizing over a 2-dimensional domain is usually difficult and expensive due to: *i*) it is relatively simple to parametrize a curve by arc-length using numerical integration, whereas there is no natural way (or directions) to parametrize a surface, and *ii*) the space all re-parametrizations is infinite dimensional function space (or a group) with no natural manifold structure. In this section, we exploit the fact that a surface can be parametrized by a collection of axial and circular curves to show that the cost function (6) can be reduced to a set of the cost function of curves using the SRVF representation.

Proposition 3.1 gives an important fact that motivates Algorithm 2 for finding optimal correspondences
 between two surfaces.

14

**Proposition 3.1.** Let  $\gamma \in \Gamma(\Omega)$ , denoted by  $\gamma : [0,1] \times \mathbb{S}^1 \to [0,1] \times \mathbb{S}^1 : (\xi_1,\xi_2) \mapsto (\gamma_1(\xi_1,\xi_2),\gamma_2(\xi_1,\xi_2))$ , where  $\gamma_1 : [0,1] \times \mathbb{S}^1 \to [0,1]$  and  $\gamma_2 : [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ . Given two surfaces  $S_1$  and  $S_2$  and their corresponding Q-fields  $Q_1$  and  $Q_2$ :

• Let  $\alpha_{\lambda}(\xi_2)$  denote  $\gamma_2(\lambda, \xi_2)$  and therefore  $\alpha_{\lambda} : \mathbb{S}^1 \to \mathbb{S}^1$ . Suppose  $\gamma_1(\xi_1, \xi_2) = \xi_1$  for all  $\xi_2 \in \mathbb{S}^1$ . Then for any given  $\lambda \in [0, 1]$ , the function

$$h_{\lambda} : \Gamma(\mathbb{S}^{1}) \to \mathbb{R} : \alpha_{\lambda} \mapsto h_{\lambda}(\alpha_{\lambda}) = \int_{\mathbb{S}^{1}} \left\| Q_{1}(\lambda,\xi_{2}) - (Q_{2},\gamma)(\lambda,\xi_{2}) \right\|_{2}^{2} d\xi_{2}$$

can be reformulated into the form of (3).

• Likewise, let  $\beta_{\lambda}(\xi_1)$  denote  $\gamma_1(\xi_1, \lambda)$ , and therefore  $\beta_{\lambda} : [0, 1] \to [0, 1]$ . If  $\gamma_2(\xi_1, \xi_2) = \xi_2$  holds for all  $\xi_1 \in [0, 1]$ , the function

$$k_{\lambda} : \Gamma([0,1]) \to \mathbb{R} : \beta_{\lambda} \mapsto k_{\lambda}(\beta_{\lambda}) = \int_{0}^{1} \left\| Q_{1}(\xi_{1},\lambda) - (Q_{2},\gamma)(\xi_{1},\lambda) \right\|_{2}^{2} d\xi_{1}$$

also can be reformulated into the form of (3).

2

1

*Proof.* First of all, using the assumption of  $\gamma$ , the determinant of the Jacobi of  $\gamma$  is

$$\det\left(\begin{bmatrix}\frac{d}{d\xi_1}\gamma_1 & \frac{d}{d\xi_2}\gamma_1\\ \frac{d}{d\xi_1}\gamma_2 & \frac{d}{d\xi_2}\gamma_2\end{bmatrix}\right) = \det\left(\begin{bmatrix}1 & 0\\ \frac{d}{d\xi_1}\gamma_2 & \frac{d}{d\xi_2}\gamma_2\end{bmatrix}\right)$$
$$= \frac{d}{d\xi_2}\gamma_2.$$

Let  $q_{\lambda}(\xi_2)$  and  $p_{\lambda}(\xi_2)$  denote  $Q_1(\lambda, \xi_2)$  and  $Q_2(\lambda, \xi_2)$ , respectively. We have

$$h_{\lambda}(\alpha_{\lambda}) = \int_{\mathbb{S}^{1}} \left\| Q_{1}(\lambda,\xi_{2}) - (Q_{2},\gamma)(\lambda,\xi_{2}) \right\|_{2}^{2} d\xi_{2}$$
$$= \int_{\mathbb{S}^{1}} \left\| Q_{1}(\lambda,\xi_{2}) - \sqrt{|J_{\gamma}|} (Q_{2}\circ\gamma)(\lambda,\xi_{2}) \right\|_{2}^{2} d\xi_{2}$$
$$= \int_{\mathbb{S}^{1}} \left\| q_{\lambda}(\xi_{2}) - \sqrt{\dot{\alpha_{\lambda}}} p_{\lambda} \circ \alpha_{\lambda}(\xi_{2}) \right\|_{2}^{2} d\xi_{2},$$

<sup>3</sup> which completes a proof for h. Likewise we obtain a proof for k.

### **4 4. Optimization Algorithms**

In Section 4.1, we present an optimization algorithm for a general reparameterization problem on  $\Gamma(\Omega)$ . An implementation by discretization is described in Section 4.2 and a technique for handling large deformations is discussed in Section 4.3. 4.1. Optimization over  $\Gamma(\Omega)$ 

The optimization problem (6) can be written as

$$\min_{\gamma \in \Gamma(\Omega)} F(\gamma)$$

Define  $\mathcal{N}_1 = \{\gamma \in \Gamma(\Omega) \mid \gamma_1(\xi_1, \xi_2) = \xi_1 \text{ for all } \xi_2 \in \mathbb{S}^1 \}$  and  $\mathcal{N}_2 = \{\gamma \in \Gamma(\Omega) \mid \gamma_2(\xi_1, \xi_2) = \xi_1 \}$ 

 $\xi_2$  for all  $\xi_1 \in [0,1]$ . We use an optimization algorithm that minimizes F alternatively over  $\mathcal{N}_1$  and  $\mathcal{N}_2$ . 3 The algorithm is stated in Algorithm 1.

Algorithm 1 Alternative Optimization

1: Let  $F^0 \leftarrow F$ ,  $\gamma^0 = \text{id}$ , and  $k \leftarrow 0$ ; 2: for  $k = 0, 1, 2, \dots$  do  $\begin{aligned} &2. \quad \text{for } k = 0, 1, 2, \dots \text{ d} \text{ d} \\ &3: \quad \hat{\gamma}^k = \arg\min_{\gamma \in \mathcal{N}_1} F^k(\gamma); \\ &4: \quad \tilde{\gamma}^k = \arg\min_{\gamma \in \mathcal{N}_2} F^k \circ \hat{\gamma}^k(\gamma); \\ &5: \quad F^{k+1} = F^k \circ \hat{\gamma}^k \circ \tilde{\gamma}^k; \\ &6: \quad \gamma^{k+1} = \gamma^k \circ \hat{\gamma}^k \circ \tilde{\gamma}^k \end{aligned}$ 7: end for

Theorem 4.1 shows that either Algorithm 1 makes progress in the sense of reducing function value or the 5 partial derivative of  $F^k$  is 0.

**Theorem 4.1.** Suppose the solutions  $\hat{\gamma}_k$  and  $\tilde{\gamma}_k$  in Algorithm 1 exist. Then one of the following statements 7 8 hold:

1. the function value strictly decreases, i.e.,  $F(\gamma_{k+1}) < F(\gamma_k)$ ; 2.  $\hat{\gamma}^k = \mathrm{id}, \, \tilde{\gamma}^k = \mathrm{id}, \, i.e., \, \gamma^{k+1} = \gamma^k, \, \frac{\partial}{\partial \gamma_1} F^k|_{\gamma = \mathrm{id}} = 0, \, and \, \frac{\partial}{\partial \gamma_2} F^k|_{\gamma = \mathrm{id}} = 0.$ 10

*Proof.* It can be seen that

$$F(\gamma^k) = F^k(\mathrm{id}) \ge F^k(\hat{\gamma}^k) \ge F^k \circ \hat{\gamma}^k(\hat{\gamma}^k) = F^{k+1}(\mathrm{id}) = F(\gamma^{k+1}),$$

11

where  $F(\gamma^k) = F(\gamma^{k+1})$  if and only if  $\hat{\gamma}^k = \text{id}$  and  $\tilde{\gamma}^k = \text{id}$ . Suppose  $\hat{\gamma}^k = \text{id}$  and  $\tilde{\gamma}^k = \text{id}$ . It follows that  $\gamma^{k+1} = \gamma^k \circ \hat{\gamma}^k \circ \tilde{\gamma}^k = \gamma^k$ . Furthermore, note that  $\text{id} = \arg \min_{\gamma \in \mathcal{N}_1} F^k(\gamma)$  is equivalent to  $\phi = \arg \min_{\gamma_2:\Omega \to \mathbb{R}} F^k(\varphi, \gamma_2)$ , where  $\varphi(\xi_1, \xi_2) = \xi_1$  for all  $\xi_2 \in \mathbb{S}^1$ , and  $\phi(\xi_1, \xi_2) = \xi_2$  for all  $\xi_1 \in [0, 1]$ . It follows that  $\frac{\partial F^k}{\partial \gamma_2}(\varphi, \phi) = 0$  which is equivalent to  $\frac{\partial}{\partial \gamma_2} F^k|_{\gamma = \text{id}} = 0$ . Likewise, it holds that  $\frac{\partial}{\partial \gamma_1} F^k|_{\gamma = \text{id}} = 0$ . 12 13 15

#### 4.2. Discretization 16

A discretization version of Algorithm 1 is stated in Algorithm 2 for minimizing (6). Steps 8 and 14, 17 corresponding to the computations of  $F^k \circ \hat{\gamma}^k$  and  $F^k \circ \hat{\gamma}^k \circ \tilde{\gamma}^k$ , are crucial since using them makes the as-18 sumptions of  $\gamma$  in Proposition 3.1 hold. Steps 5 and 11 in Algorithm 2, the minimization steps, correspond to 19 Steps 3 and 4 of Algorithm 1, respectively. Multiple methods have been proposed to solve the inner problems 20 in Steps 5 and 11, such as dynamic programming [16] and Riemannian methods [28]. The steepest descent 21 method with multiple initial iterates has been proposed to optimize the cost function E. However, this method 22 usually does not give a satisfactory result due to the difficulties given in the introduction. Existing methods 23 of curves works on a grid in  $[0,1]^2$  rather than the continuous space  $\Gamma([0,1])$ . By exhaustively searching over 24

Algorithm 2 Alternative Optimization for *E* 

**Input:** Two Q-functions  $Q_1$  and  $Q_2$ . **Output:** The minimizer  $\gamma^* = \begin{bmatrix} \gamma_1^* \\ \gamma_2^* \end{bmatrix}$ . 1: Discretize [0,1] by n points  $\lambda \in \{0,\ldots,1\}$ , note  $h_{\lambda}$  the circular curve at level  $\lambda$  and  $\alpha_{\lambda}$  its parametrization. Then, each surface has n circular curves. 2: Discretize  $\mathbb{S}^1$  by m points  $\lambda \in \{0, \dots, 2\pi\}$ , note  $k_\lambda$  the axial curve and at level  $\lambda$ , and  $\beta_\lambda$  its parametrization. This leads to a discretization of  $Q_1$  and  $Q_2$  by  $n \times m$  points in  $\mathbb{R}^3$ 3: Set k = 0,  $\gamma_1^*(\xi_1, \lambda) = \xi_1$  and  $\gamma_2^*(\lambda, \xi_2) = \xi_2$  for all  $\lambda \in [0, 1]$ ; 4: for All  $\lambda \in \{0, \ldots, 1\}$  do Optimize  $h_{\lambda}$  for  $Q_1$  and  $Q_2$  to find the minimizer  $\alpha_{\lambda}^* \in \Gamma(\mathbb{S}^1)$ ; 5: 6: end for 7: Define  $\gamma_2(\xi_1, \xi_2) = \alpha_{\xi_1}^*(\xi_2)$ ; 8: Set  $Q_2(\xi_1, \xi_2)$  to be  $\sqrt{\frac{d\gamma_2}{d\xi_2}(\xi_1, \xi_2)}Q_2(\xi_1, \gamma_2(\xi_1, \xi_2))$ ; 9: Set  $\gamma_1^*(\xi_1, \xi_2)$  to be  $\gamma_1^*(\xi_1, \gamma_2(\xi_1, \xi_2))$  and  $\gamma_2^*(\xi_1, \xi_2)$  to be  $\gamma_2^*(\xi_1, \gamma_2(\xi_1, \xi_2))$ 10: for All  $\lambda \in \{0, \dots, 2\pi\}$  do Optimize  $k_{\lambda}$  for  $Q_1$  and  $Q_2$  to find the minimizer  $\beta_{\lambda}^* \in \Gamma([0, 1])$ ; 11: 12: end for 13: Define  $\gamma_1(\xi_1, \xi_2) = \beta_{\xi_2}^*(\xi_1);$ 14: Set  $Q_2(\xi_1,\xi_2) \leftarrow \sqrt{\frac{d\gamma_1}{d\xi_1}(\xi_1,\xi_2)} Q_2(\gamma_1(\xi_1,\xi_2),\xi_2);$ 15: Set  $\gamma_1^{*,k}(\xi_1,\xi_2)$  to be  $\gamma_1^{*}(\gamma_1(\xi_1,\xi_2),\xi_2)$  and  $\gamma_2^{*,k}(\xi_1,\xi_2)$  to be  $\gamma_2^{*}(\gamma_1(\xi_1,\xi_2),\xi_2)$ ; 16: If the algorithm converges, then return  $\begin{bmatrix} \gamma_1^{*,k} \\ \gamma_2^{*,k} \end{bmatrix}$  and stop. Otherwise, k = k + 1 and go to Step 4.

the piece-wise linear functions on the grid with slope constraints, one can find an approximation of a solu-

<sup>2</sup> tion. If the grid is fine, then the approximation can be used as a solution, such as dynamic programming [16].

Otherwise, the approximation can be used as an initial iterate for a gradient method, such as Riemannian
 methods [28].

In practice, every surface is represented by n points along each circular curve (partition of  $[0, 2\pi]$ ) and m points along an axial curve (partition of [0, 1]). Moreover, every surface is a grid of  $n \times m$  points. Each point 6 represents the intersection between a circular and an axial curve. Because of the discretization, all gridpoints are disjoints and should not cross each other during the update. Note that when using a dynamic programming 8 approach to register curves, the complexity is quadratic to the number of points on a curve. Therefore, we have a time complexity  $\mathcal{O}(m \times n^2)$  for reparametrizing circular curves and  $\mathcal{O}(n \times m^2)$  for axial curves. 10 For the Riemannian optimization method, it is shown empirically in [28, Section 6.6] that the complexity 11 is approximately linear to the number of points on curves. Therefore, the complexity in Steps 5 and 11 is 12 approximately O(mn). The complexity in interpolations in Steps 8, 9, 14 and 15 is also O(mn). Therefore, 13 the total complexity of Algorithm 2 with Riemannian optimization method is approximately O(kmn), where 14 k is the number of iterations. In our experiments, the maximum number of iterations is fixed to be 100. 15

### 16 4.3. Successive Infinitesimal Variations

<sup>17</sup> The continuous description of  $\Omega$  involves a space  $\Gamma$  of infinite dimension. Many algorithms based on <sup>18</sup> discretization have difficulties if the deformations between two shapes are large, and therefore, in practice, <sup>19</sup> it is common to model a time-sequence of smaller, more elementary, deformations, rather than a large de-<sup>20</sup> formation, see, e.g., [30, 31]. In this paper, we follow the same idea and propose Algorithm 3 based on <sup>21</sup> Algorithm 2. Concretely, the new methodology can be seen as a spatio-temporal discretization scheme when <sup>22</sup>  $(t, \gamma^t) \in [0, 1] \times \Gamma$  leads to a space discretization.

In the case of a uniform parametrization  $(\xi_1, \xi_2)$  as an equidistant mesh of squares or rectangles, one can compute the initial re-parametrization by solving a one-dimensional problem in  $\xi_1$  direction and subsequently a family of problems in the  $\xi_2$  direction [32]. Consequently, a better strategy would be to re-parametrize the two Q-fields uniformly (by arc-length in each component), fix the parametrization of  $Q_1$  and then search for reparameterization  $\hat{\gamma}$  on  $Q_2$  iteratively. Indeed, at each iteration k we consider a small stepsize  $\rho \in [0, 1]$ , compute  $Q_{\rho}$  in the direction of  $Q_1$ , find  $\gamma^k$  between  $Q_2$  and  $Q_{\rho}$  using Algorithm 2, and then update  $Q_2$  with  $(Q_2, \gamma^k)$ . Therefore, the final solution is given by  $\hat{\gamma} = \gamma^0 \circ \gamma^1 \circ \ldots \gamma^N$ .

Note that the boundary conditions is checked using a projection P (see Line 6 in Algorithm 3) from 30  $\mathbb{L}^2$  to  $\bar{\mathcal{M}}$  to make sure that circular curves are closed and that the optimal infinitesimal re-parametrization 31 is well defined for every iteration. Consequently, using infinitesimal successive variations guarantees the 32 convergence to a diffeomorphic solution [30, Chapter 11]. According to this iterative algorithm, convergence 33 is achieved when no update of  $\hat{\gamma}$  is possible which corresponds to the case where the line between  $Q_1$  and 34  $(Q, \hat{\gamma})$  is orthogonal to  $[Q_2]$ . We illustrate the idea of successive iterations in Figure 1(g): Original  $Q_1$  and 35  $Q_2$  define two equivalent classes  $[Q_1]$  and  $[Q_2]$ . We fix the parametrization of  $Q_1$  to be the identity and we 36 search for the optimal re-parametrization of  $Q_2$ . The algorithm is initialized with  $(Q_2, Id_{\Gamma})$ , iterates over a 37 set of satisfying update  $(Q_2, \gamma^k)$  at iteration k until convergence to  $(Q_2, \gamma^*)$ . 38

To show the quality of registering two surfaces using Algorithm 3, we display an example of registering 39 a cylinder  $S_2$  (b) to an arbitrary surface  $S_1$  (a) in Figure 1. The optimally registered surface  $S_2$  as  $(S_2, \hat{\gamma})$  is 40 given in (c). We also provide three intermediate steps in (d) and the cost function  $||Q_1 - (Q_2, \hat{\gamma}^{iter})||$  in (e) 41 during iterations. For a better evaluation, we consider a linear interpolation  $\alpha$  with  $\alpha_0 = S_1$  (for t = 0) and 42  $\alpha_1 = (S_2, \hat{\gamma})$  (for t = 1) in (f). In-between we show 4 uniform intermediate surfaces  $\alpha_t$  for  $5 * t = 1 \dots 4$ . 43 Note that the linear interpolation  $\alpha$  is an extrinsic geodesic between  $[Q_1]$  and  $[Q_2]$  but may not coincide with 44 a geodesic between them in  $\mathcal{M}$ . Yet, we observe that improved registration corresponds to smooth variations 45 and better preservation of geometric features along  $\alpha$ . 46

<sup>47</sup> 

Algorithm 3 Successive Infinitesimal Variations

**Input:** Two Q-fields  $Q_1$  and  $Q_2$  with  $||Q_1 - Q_2|| > 0$ . **Output:** The minimizer  $\hat{\gamma}$ . 1: Re-parametrize  $Q_1$  and  $Q_2$  uniformly. 2: Set  $\hat{\gamma} = \operatorname{Id}_{\Gamma}$  and  $Q = Q_2$ . 3: Set k = 1 and  $N \ge 3$ . 4: repeat Choose the step size  $\rho = \frac{k}{N}$  where N represents maximum iterations. Compute  $Q^k = P(Q + \rho(Q_1 - Q));$ 5: 6: Compute  $\gamma^k$  from Q to  $Q^k$  using algorithm 2; 7: Update  $Q = (Q, \gamma^k)$ ; Update  $\hat{\gamma} = \hat{\gamma} \circ \gamma^k$ ; 8: 9: Update k = k + 1; 10: 11: **until**  $\rho = 1$  or  $d(\gamma^k, \operatorname{Id}_{\Gamma})$  very small

### 5. Relationship to Open Surfaces

Given two tubular surfaces  $S_1$  and  $S_2$ , if a  $\xi_1$ -curve of  $S_1$  fixedly matches to a  $\xi_1$ -curve of  $S_2$ , then the reparameterization problem between these two tubular surfaces can be viewed as a reparameterization probз lem between two open surfaces. An open surface is a differentiable function  $S : [0,1]^2 \to \mathbb{R}^3 : [\xi_1,\xi_2] \mapsto S(\xi_1,\xi_2)$ . Its normal field is defined as follows:  $N : [0,1]^2 \to \mathbb{R}^3 : (\xi_1,\xi_2) \mapsto N(\xi_1,\xi_2) = \frac{\partial S}{\partial \xi_1} \times \frac{\partial S}{\partial \xi_2}$  and the corresponding Q-field is defined to be  $Q(\xi_1,\xi_2) = \frac{N(\xi_1,\xi_2)}{\sqrt{\|N(\xi_1,\xi_2)\|_2}}$ . It can be shown that the space of all Q-5 6 fields of open surfaces is the  $\mathbb{L}^2$  space and the reparameterization group action on Q-fields is isometric [21]. 7 We point out here without proofs that a connection between SRNF of open surfaces and SRVF of curves 8 can be built similarly (see Lemma 5.1) and Algorithm 2 can be modified slightly for open surfaces (modify 9  $\mathbb{S}^1$  to be [0,1] in Lines 5 and 10 of Algorithm 2). Since the space of all Q-fields of open surfaces forms the 10  $\mathbb{L}^2$  space, the geodesic between any two Q-fields is just the straight line between them, see details in Propo-11 sition 5.1. Note that the geodesics between any Q-fields of tubular surfaces are not the linear combinations 12 since the space  $\bar{\mathcal{M}}$  is a nonlinear subspace of  $\mathbb{L}^2$ 13 **Lemma 5.1.** Given two open surfaces  $S_1$  and  $S_2$  and their corresponding Q-fields  $Q_1$  and  $Q_2$ . Let  $\gamma = (\gamma_1, \gamma_2) \in \Gamma([0, 1]^2). \ \alpha_{\lambda}(\xi_2) \ denote \ \gamma_2(\lambda, \xi_2).$ 

14

1. Suppose  $\gamma_1(\xi_1, \xi_2) = \xi_1$  for all  $\xi_2 \in [0, 1]$ . Then for any given  $\lambda \in [0, 1]$ , the function  $h_\lambda(\alpha_\lambda) = \int_{\mathbb{S}^1} \left\| Q_1(\lambda, \xi_2) - (Q_2, \gamma)(\lambda, \xi_2) \right\|_2^2 d\xi_2$  can be reformulated into the form of (3). 2. Likewise, let  $\beta_\lambda(\xi_1)$  denote  $\gamma_1(\xi_1, \lambda)$ . If  $\gamma_2(\xi_1, \xi_2) = \xi_2$  holds for all  $\xi_1 \in [0, 1]$ , the function

 $k_{\lambda}(\beta_{\lambda}) = \int_{0}^{1} \left\| Q_{1}(\xi_{1},\lambda) - (Q_{2},\gamma)(\xi_{1},\lambda) \right\|_{2}^{2} d\xi_{1} \text{ also can be reformulated into the form of (3).}$ 

15

**Proposition 5.1.** Given two open surfaces  $S_0$  and  $S_1$ , and an optimal parametrization  $\hat{\gamma}$  between them, a geodesic between  $[Q_0]$  and  $[Q_1]$  is given by:  $Q_t(\xi_1, \xi_2) = (1 - t)Q_0(\xi_1, \xi_2) + t(Q_1, \hat{\gamma})(\xi_1, \xi_2)$ .

*Proof for Proposition 5.1:* According to (6),  $\hat{\gamma}$  is a minimizer of

$$d^{2}([Q_{0}], [Q_{1}]) = \int_{\Omega} \|Q_{0}(\xi_{1}, \xi_{2}) - (Q_{1}, \hat{\gamma})(\xi_{1}, \xi_{2})\|_{2}^{2} d\xi_{1} d\xi_{2}.$$

Remind that the space of Q-fields is a linear space, then the geodesic between  $Q_0$  and  $(Q_1, \hat{\gamma})$  is given by a straight line:

$$\alpha_t(\xi_1,\xi_2) = (1-t)Q_0(\xi_1,\xi_2) + t(Q_1,\hat{\gamma})(\xi_1,\xi_2)$$

minimizing the cost function

$$\|Q_0 - (Q_1, \hat{\gamma})\|_{\mathbb{L}^2} = \int_0^1 \sqrt{\left\langle \frac{d}{dt} \alpha_t, \frac{d}{dt} \alpha_t \right\rangle_{\mathbb{L}^2}} dt$$

with  $\alpha_0 = Q_0$  and  $\alpha_1 = (Q_1, \hat{\gamma})$ .

Note that for this special case the projection operator P used in Algorithm 3 in Line 6 is the identity. Except that, the same algorithm can be applied directly without modification to find an optimal parametrization  $\hat{\gamma}$  between any two open surfaces  $S_1$  and  $S_2$ .

### **5 6. Experimental Results**

In this section, we present various examples using both synthetic surfaces and several benchmarks to 6 show the accuracy of the proposed method. Since our framework extends registration of curves to surfaces, we present a comparison when using dynamic programming and a Riemannian optimization method to solve 8 correspondence between curves. Usually parametrizations depend on the application at hand but one can still 9 compute a parametrization of a given surface S as levelset of a differentiable scalar function [33]. Practical 10 examples of functions computations over curved surfaces can be found in [34]. Similarly, all surfaces used 11 in our experiments are represented by a finite set of  $\xi_1$ -curves and  $\xi_2$ -curves. Figure 2(a) gives an example of 12 a parametrization ( $\xi_1, \xi_2$ ) in (d) where  $\xi_1$  is the parameter along axial axis (b) and  $\xi_2$  is the parameter along 13 circular axis (c). Since those curves are represented by 100 points each and therefore, the Q-field is also 14 represented by points that are on some smooth function. 15

In all of the cost functions considered,  $Q_2$  is composed with some function and, therefore, representing  $Q_2$  (all curves) as a set of points is not sufficient and a suitable function must be used. Since the representation requires a  $C^1$  function to compute partial derivatives, an interpolatory cubic spline on  $\xi_1$  and  $\xi_2$  is used.

<sup>19</sup> It should be noted however that there is nothing in the formulation that requires an interpolatory approxi-<sup>20</sup> mation. Finally, all derivatives are calculated with the central difference scheme and all integrals required by <sup>21</sup> the algorithms are approximated by the Composite Trapezoidal Rule.

### 22 6.1. Shape Registration

We show various examples in Figure 4 left. In each row, we present an example of registering a target  $S_2$  in 23 the left column (a) to a reference  $S_1$  in the second column (b) using the proposed method. The reparametrized 24 surface  $(S_2, \hat{\gamma})$  is given in the right column (c). For a better evaluation of the qualitative improvement, results 25 are provided as a linear combination between  $S_1$  and  $(S_2, \hat{\gamma})$ . We note that the resulting registration provides 26 a smooth deformation between surfaces that are "close" (first row) or where  $S_2$  has a simplified geometry 27 (second row). The other example, consider surfaces with more complicated geometries and the outcome 28 reconfirms that the proposed method successfully provides smooth evolutions and better preservations of 29 geometric features along the paths. 30

In order to evaluate the overall quality of our framework when using different methods to register curves, we performed a comparison between dynamic programming (DP) and a Riemannian optimization method (RO) [28]. We began by computing distances using both methods between 1000 random pairwise selections from 30 surfaces, see Figure 4(d) for some samples. Using those distances, we computed the realized gain  $(||Q_1 - Q_2||_{L^2} - ||Q_1 - (Q_2, \hat{\gamma})||_{L^2})$ . We consider the gain as a distribution and report the results as histograms in Figure 4(e): (dark blue) for RO and transparent for DP. We note that the proposed method
provides good results for both methods with an advantage for RO. For example, a gain less than 10 has been
recorded by 37% cases for RO and 50% for DP. In addition, a gain between 30 and 40 has been recorded
approximatively by 18% for RO and by only 8% for DP. Furthermore, in very few cases DP reached a gain

approximatively by 18% for KO and by only 8% for DF. Furthermore, in very rew cases DF rea

<sup>5</sup> superior to 100 whereas the maximum gain realized by RO is 100.

### 6 6.2. 2D Shapes Clustering

Two public datasets are used in the experiments: the Flavia leaf dataset [35] and the MPEG-7 dataset [36]. The Flavia leaf dataset contains images of 1907 leaves from 32 species. Figure 5(a) shows an example leaf from each species. MPEG-7 contains 1400 images in 70 clusters each of which contains 20 shapes. Figure 5(b) shows an example shape from each cluster. The boundary curves of the shapes are extracted and 100 uniformly-spaced points were used to represent the shape.

The two public datasets were used to compare the performances of Riemannian optimization based 12 method [28] and dynamic programming [16]. Indeed, we used k-medoids algorithm to perform cluster-13 ing [37]. It reminds us that k-medoids is a standard and a widely used clustering algorithm. We applied 14 the algorithm on the pre-computed distance matrix. Since k-mediod method may find a local minimizer, 15 we repeated the clustering process 100 times and then take an average of their correctness. The correctness 16 of k-medoids method is reported in Table 1. For completeness, we also provide the average computational 17 time for pairwise distance from [28]. Note that the computational time is dominated by computing the pair-18 wise distances between all shapes. The Riemannian optimization based methods are able to achieve higher 19 correctness using less computational time when compared to the dynamic programming method. 20

### 21 6.3. 3D Shapes Classification

A public 3D objects benchmark [38] is used for the experimental evaluation of the proposed method. The original 763 models were classified into 19 classes. Figure 6 shows several examples from different classes. First, the boundary surfaces of 3D objects were aligned (rotation and translation) to a reference model and uniformly scaled. Then, 50 levelset curves is extracted to represent a surface where each curve is represented by 100 equidistant points. Only the first and the last levelsets were removed to make sure that the resulting representation is well defined. Finally, for all surfaces, the starting points have been chosen as an intersection between each curve and a fixed reference half-plane.

Shape classification has been studied extensively in the literature [39, 40, 41], to cite but a few examples. 29 Nonetheless, whereas the main aim is to show the importance of one-to-one correspondance when comparing 30 shapes, we propose to classify 3D shapes based on 3D geometric representations. For all classification 31 experiments, we extract different features (HOG [42], mean curvature  $H_S$ , Gaussian curvature  $K_S$  [43] and 32 3D wavelet coefficients [44]) and use them separately to represent a surface. We remind that the goal is to use 33 standard descriptors from literature to show the classification accuracy before and after re-parametrization. 34 Thus, the choice of the most performing descriptors is out of the scope of this work. Descriptors' vector 35 are given as input to train a Gaussian Processes Autoencoder (GPAE) [45] with 50 and 60 hidden layers, 36 respectively. We select 75% of the dataset for training and use the rest for test. In order to remove the test 37 bias, we run the methods 10 times. At each run, we randomly select the training set and use the remaining 38 for test. We then average the performance over the 10 runs. To show the effectiveness of our approach, we 39 show the classification accuracy without re-parametrization in Table 2 and after re-parametrization (using our 40 method) in Table 3. 41

Furthermore, in order to objectively evaluate the proposed classification method we compare the GPAE with Support Vector Machine (SVM) classifier [46, 47]. Table 2 and Table 3 show that GPAE outperforms SVM in terms of classification accuracies except for HOG descriptor. Also, we can remark that using a good correspondence improves the accuracy for both GPAE and SVM classifiers. From all experiments, we can note that the classification accuracy is better after re-parametrization regardless of any input (surface representation) or the classifier (GPAE or SVM). In particular, we observe that the best accuracies are achieved when using the mean curvature  $H_S$  as a feature. This result is well illustrated by confusion matrices in Figure 7 where the mean curvature is used to represent shapes and GPAE to classify them. Nevertheless, we should keep in mind that in both cases, we applied a Gaussian filtering, removed rigid transformation and normalized surfaces which certainly improves the accuracy even before re-parametrization.

### 7 7. Conclusion

<sup>8</sup> We have presented a new Riemannian-based method for shape registration and classification. The advan-<sup>9</sup> tage of using the proposed method is that it simplifies the optimization over the space of all re-parametrization <sup>10</sup> by exploiting available frameworks for shape analysis of 3D curves. Beyond the use for finding optimal <sup>11</sup> correspondences, we wanted to show how this framework could be directly applied for 2D and 3D shape <sup>12</sup> registration, clustering or classification. We have tested our method on several benchmarks that demonstrate <sup>14</sup> the effectiveness of the proposed methodology. We used standard descriptors to represent a surface and two <sup>14</sup> different models for classification. From the experiments, we have showed that the classification accuracy is <sup>15</sup> better after re-parametrization regardless of any descriptor or classifier.

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Figure 1: Reparameterization of the target surface  $S_2$  (b) to match the reference surface  $S_1$  (a) using the proposed method. (c) The optimally registered target as  $(S_2, \hat{\gamma})$ . (d) Examples of intermediate steps and (e) the linear combination path between  $S_1$  and  $(S_2, \hat{\gamma})$ . (f) The cost function  $||Q_1 - (Q_2, \hat{\gamma}^{iter})||_{\mathbb{L}^2}$  during iterations and (g) an illustration of the optimization process initialized at  $Q_2^0 = (Q_2, Id_{\Gamma})$  and ending at the optimal solution  $Q_2^* = (Q_2, \hat{\gamma})$ .



Figure 2: Example of a tubular surface S (a) and the corresponding,  $\xi_1$ -curves (axial),  $\xi_2$ -curves (circular), and parametrization ( $\xi_1, \xi_2$ ).



Figure 3: Reparameterization of a surface  $S_2$  (b) as one-to-one correspondence with  $S_1$  (a) using the proposed method. (c) The optimally registered target as  $(S_2, \hat{\gamma})$ . For each row, the linear combination path between  $S_1$  and  $(S_2, \hat{\gamma})$  is given from second column (b) to the right column (c).



Figure 4: (a) Samples of surfaces used for experiments and (b) histograms of the realized gain between the starting distance before re-parametrization  $||Q_1 - Q_2||_{\mathbb{L}^2}$  and the distance after re-parametrization  $||Q_1 - (Q_2, \hat{\gamma})||_{\mathbb{L}^2}$  using the Riemannian method (dark blue) and DP (red dashed).



Figure 5: (a): Samples of leaves from the Flavia leaf dataset. (b) Samples of curves from the MPEG-7 dataset. One sample per cluster is illustrated.



Figure 6: Top: Samples of 3D models from the 3D dataset. Bottom: Examples after rigid alignment and re-parametrization colored with curvature extraction (for each column a-e Gaussian left and mean right).

0.9	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.76	0.0	0.0	0.0	0.1	0.1	0.04	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.5	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.8	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.15	0.0	0.0	0.14	0.6	0.0	0.11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.6	0.0	0.0	0.0	0.0	0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.3	0.0	0.0	0.0	0.0	0.0	0.4	0.0	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.88	0.02	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.2	0.0	0.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.7	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0
0.92	0.02	0.05	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.27	0.0	0.0	0.0	0.0	0.1	0.03	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1	0.0	0.0	0.0	0.0	0.05	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.89	0.0	0.04	0.01	0.06	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.15	0.0	0.0	0.14	0.6	0.0	0.11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.01	0.0	0.0	0.1	0.0	0.89	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.06	0.0	0.0	0.0	0.0	0.0	0.0	0.92	0.04	0.0	0.0	0.0	0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.88	0.02	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.17	0.0	0.0	0.0	0.0	0.0	0.17	0.0	0.66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.03	0.1	0.0	0.0	0.87	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.7	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.05	0.0	0.0	0.0	0.0	0.0	0.03	0.92	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0			10 A		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0

Figure 7: Confusion matrices using the mean curvature with GPAE: (a) before and (b) after re-parametrization.

Table 1: 2D shapes classification rate. Two versions of limited-memory Riemannian BFGS (RO) based methods are used.

	RO-complex	RO-simple	DP
FLAVIA (rate)	68,94%	68.37%	66.53%
FLAVIA (time)	0.088	0.047	0.897
MPEG-7 (rate)	68,94%	68.37%	66.53%
MPEG-7 (time)	0.181	0.134	0.908

Table 2: 3D shapes classification accuracy before re-parametrization.

Method	Results	HOG	Wavelet	$H_S$	$K_S$
CDAE	rate	73.1%	<b>78</b> %	76.8%	73.9%
GFAE	σ	1.1	0.8	1.7	2.0
SVM	rate	74.2%	74.5%	75.6%	70.2%
<b>SVIVI</b>	$\sigma$	1.0	1.2	0.9	1.5

Table 3: 3D shapes classification accuracy after re-parametrization.

Method	Results	HOG	Wavelet	$H_S$	$K_S$	
CDAE	rate	77.8%	79.%	87.5%	80.1%	
GIAL	$\sigma$	1.5	1.4	1.8	1.9	
SVM	rate	<b>79.4</b> %	75.0%	77.3%	78.2%	
<b>SVIVI</b>	σ	1.1	1.0	1.6	2.0	