

L^p -trace-free version of the generalized Korn inequality for incompatible tensor fields in arbitrary dimensions

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Abstract

For $n \geq 3$ and $1 < p < \infty$ we prove an L^p -version of the generalized trace-free Korn-type inequality for incompatible, p -integrable tensor fields $P : \Omega \rightarrow \mathbb{R}^{n \times n}$ having p -integrable generalized Curl_n and generalized vanishing tangential trace $P \tau_l = 0$ on $\partial\Omega$, denoting by $\{\tau_l\}_{l=1, \dots, n-1}$ a moving tangent frame on $\partial\Omega$. More precisely, there exists a constant $c = c(n, p, \Omega)$ such that

$$\|P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left(\|\text{dev}_n \text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl}_n P\|_{L^p(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})} \right),$$

where the generalized Curl_n is given by $(\text{Curl}_n P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}$ and $\text{dev}_n X := X - \frac{1}{n} \text{tr}(X) \cdot \mathbb{1}_n$ denotes the deviatoric (trace-free) part of the square matrix X . The improvement towards the three-dimensional case comes from a novel matrix representation of the generalized cross product.

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1 Introduction

The estimate

$$\exists c > 0 \forall u \in W_0^{1,p}(\Omega, \mathbb{R}^n) : \quad \|Du\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \|\text{dev}_n \text{sym } Du\|_{L^p(\Omega, \mathbb{R}^{n \times n})}, \quad (1.1)$$

for $n \geq 2$ and $p \in (1, \infty)$ where $\text{dev}_n X := X - \frac{1}{n} \text{tr}(X) \cdot \mathbb{1}$ denotes the deviatoric (trace-free) part of the square matrix X and its (compatible) generalizations of (1.1) are well known, cf. [3, 4, 14, 16, 17]. In [9] another generalization to the (incompatible) case

$$\begin{aligned} \exists c > 0 \forall P \in W_0^{1,p}(\text{Curl}; \Omega, \mathbb{R}^{3 \times 3}) : \\ \|P\|_{L^p(\Omega, \mathbb{R}^{3 \times 3})} \leq c \left(\|\text{dev}_3 \text{sym } P\|_{L^p(\Omega, \mathbb{R}^{3 \times 3})} + \|\text{Curl } P\|_{L^p(\Omega, \mathbb{R}^{3 \times 3})} \right) \end{aligned} \quad (1.2)$$

has been given. The main objective of the present paper is to extend (1.2) to the trace-free case for $n \geq 3$ dimensions. Such a result was expected, cf. [9, Rem. 3.11] and was already proven to hold true for $p = 2$, cf. [1]. However, the latter used a Helmholtz decomposition and a Maxwell estimate and is not amenable to the L^p -case. On the contrary, the argumentation scheme using the Lions lemma resp. Nečas estimate, known

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from classical Korn inequalities, turned out to be also fruitful in the case of Korn inequalities for incompatible tensor fields, cf. [8, 9, 11, 10] and also [5]. The secret of success is then to determine a linear combination of certain partial derivatives. One such expression in [9] was $D^2(A + \zeta \cdot \mathbb{1}_3) = L(\text{DCurl}(A + \zeta \cdot \mathbb{1}_3))$ denoting by L a constant coefficients linear operator, for a skew-symmetric matrix field A and scalar field ζ . Here we catch up with a corresponding linear expression in all dimensions $n \geq 3$. For that purpose a careful investigation of the generalized cross product, especially a corresponding matrix representation will be given. Indeed, it is this matrix representation which allows us to obtain suitable relations which are not easily visible in index notations. Korn's inequalities in higher dimensions for matrix-valued fields whose incompatibility is a bounded measure and corresponding rigidity estimates were obtained in the recent papers [2, 7], however, without boundary conditions. More precisely, Conti and Garroni [2] obtained as a consequence of a Hodge decomposition with critical integrability due to Bourgain and Brezis for $P \in L^1(\Omega, \mathbb{R}^{n \times n})$ with $\text{Curl}_n P \in L^1(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})$ the sharp geometric rigidity estimate

$$\inf_{R \in \text{SO}(n)} \|P - R\|_{L^{1^*}(\Omega)} \leq c \left(\|\text{dist}(P, \text{SO}(n))\|_{L^{1^*}(\Omega)} + \|\text{Curl}_n P\|_{L^1(\Omega)} \right) \quad (1.3)$$

with a constant $c = c(n, \Omega)$, the Sobolev-conjugate exponent $1^* := \frac{n}{n-1}$ and where the generalized Curl_n is seen without a matrix representation as $(\text{Curl}_n P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}$. Replacing the geometric rigidity by Korn's inequality they deduced from (1.3) furthermore

$$\inf_{A \in \mathfrak{so}(n)} \|P - A\|_{L^{1^*}(\Omega)} \leq c \left(\|\text{sym} P\|_{L^{1^*}(\Omega)} + \|\text{Curl}_n P\|_{L^1(\Omega)} \right). \quad (1.4)$$

This estimates remain true for $\text{Curl}_n P$ being a Radon measure. In that case the L^1 -norm of $\text{Curl}_n P$ has to be substituted by the total variation of the measure $\text{Curl}_n P$, cf. [2]. Lauteri and Luckhaus [7] obtained the rigidity estimate (1.3) in the Lorentz space $L^{1^*, \infty}$. In [11] we have already established the corresponding results in the L^p -setting. Here we focus on the trace-free case showing that the symmetric part can even be replaced by the symmetric deviatoric part.

2 Preliminaries and auxilliary results

By $\cdot \otimes \cdot$ we denote the dyadic product and by $\langle \cdot, \cdot \rangle$ the scalar product, $\mathfrak{so}(n) := \{A \in \mathbb{R}^{n \times n} \mid A^T = -A\}$ is the Lie-Algebra of skew-symmetric matrices and $\text{Sym}(n) := \{X \in \mathbb{R}^{n \times n} \mid X^T = X\}$. Recall that usually the higher dimensional generalization of the curl is an operation $\text{curl}_n : \mathcal{D}'(\Omega, \mathbb{R}^n) \rightarrow \mathcal{D}'(\Omega, \mathbb{R}^{\frac{n(n-1)}{2}})$ given by

$$\text{curl}_n a := \begin{pmatrix} \partial_1 a_2 - \partial_2 a_1 \\ \partial_1 a_3 - \partial_3 a_1 \\ \partial_2 a_3 - \partial_3 a_2 \\ \partial_1 a_4 - \partial_4 a_1 \\ \partial_2 a_4 - \partial_4 a_2 \\ \partial_3 a_4 - \partial_4 a_3 \\ \dots \end{pmatrix}. \quad (2.1)$$

Thus, in order to express this operation using the Hamiltonian formalism as a generalized cross product with ∇ , we focus on the following bijection $\mathfrak{a}_n : \mathfrak{so}(n) \rightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$ given by

$$\mathfrak{a}_n(A) := (A_{12}, A_{13}, A_{23}, \dots, A_{1n}, \dots, A_{(n-1)n})^T \quad \text{for } A \in \mathfrak{so}(n) \quad (2.2a)$$

as well as its inverse $\mathfrak{A}_n : \mathbb{R}^{\frac{n(n-1)}{2}} \rightarrow \mathfrak{so}(n)$, so that

$$\mathfrak{A}_n(\mathfrak{a}_n(A)) = A \quad \text{for all } A \in \mathfrak{so}(n) \quad \text{and} \quad \mathfrak{a}_n(\mathfrak{A}_n(a)) = a \quad \text{for all } a \in \mathbb{R}^{\frac{n(n-1)}{2}} \quad (2.2b)$$

and in coordinates it looks like

$$\mathfrak{A}_n(a) = \begin{pmatrix} 0 & a_1 & a_2 & a_4 & \vdots \\ -a_1 & 0 & a_3 & a_5 & \vdots \\ -a_2 & -a_3 & 0 & a_6 & \vdots \\ -a_4 & -a_5 & -a_6 & 0 & \vdots \\ \dots & \dots & \dots & \dots & 0 \end{pmatrix}. \quad (2.3)$$

Moreover, we will make use of the following notations

$$b = (\bar{b}, b_n)^T \in \mathbb{R}^n \quad \text{with } \bar{b} \in \mathbb{R}^{n-1} \quad (2.4a)$$

and

$$A = \begin{pmatrix} \bar{A} & \vdots & \overline{Ae_n} \\ \dots & \ddots & \dots \\ -(\overline{Ae_n})^T & \vdots & 0 \end{pmatrix} \in \mathfrak{so}(n) \quad \text{with } \bar{A} \in \mathfrak{so}(n-1). \quad (2.4b)$$

2.1 A generalized cross product

Regarding our goal to express curl_n by the Hamiltonian formalism, we apply the following generalization of the cross product for $n \geq 2$ acting as $\times_n : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$ by

$$a \times_n b := \mathfrak{a}_n(a \otimes b - b \otimes a) \quad \text{for } a, b \in \mathbb{R}^n. \quad (2.5a)$$

Since for a fixed $a \in \mathbb{R}^n$ the operation $a \times_n \cdot$ is linear in the second component there exists a unique matrix denoted by $\llbracket a \rrbracket_{\times_n} \in \mathbb{R}^{\frac{n(n-1)}{2} \times n}$ such that

$$a \times_n b =: \llbracket a \rrbracket_{\times_n} b \quad \text{for all } b \in \mathbb{R}^n. \quad (2.5b)$$

The matrices $\llbracket \cdot \rrbracket_{\times_n}$ can be characterized inductively, and for $a = (\bar{a}, a_n)^T$ the matrix $\llbracket a \rrbracket_{\times_n}$ has the form

$$\llbracket a \rrbracket_{\times_n} = \begin{pmatrix} \llbracket \bar{a} \rrbracket_{\times_{n-1}} & \vdots & 0 \\ \dots & \ddots & \dots \\ -a_n \cdot \mathbb{1}_{n-1} & \vdots & \bar{a} \end{pmatrix} \quad \text{where} \quad \llbracket \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rrbracket_{\times_2} = (-a_2, \quad a_1), \quad (2.6)$$

so,

$$\llbracket \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \rrbracket_{\times_3} = \begin{pmatrix} -a_2 & a_1 & \vdots & 0 \\ \dots & \dots & \ddots & \dots \\ -a_3 & 0 & \vdots & a_1 \\ 0 & -a_3 & \vdots & a_2 \end{pmatrix} \quad \text{and} \quad \llbracket \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \rrbracket_{\times_4} = \begin{pmatrix} -a_2 & a_1 & 0 & \vdots & 0 \\ -a_3 & 0 & a_1 & \vdots & 0 \\ 0 & -a_3 & a_2 & \vdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ -a_4 & 0 & 0 & \vdots & a_1 \\ 0 & -a_4 & 0 & \vdots & a_2 \\ 0 & 0 & -a_4 & \vdots & a_3 \end{pmatrix} \quad \text{etc.} \quad (2.7)$$

Remark 2.1. There are many possible identifications of skew-symmetric matrices with vectors. However, it is this matrix representation $\llbracket \cdot \rrbracket_{\times_n}$ of the generalized cross product \times_n which allows us to establish the main identities needed for Lemma 2.9. Indeed, they were not easily visible to us before in index notations. Moreover, with this matrix representation in hand, the discussion of the boundary condition (see Observation 2.7) as well as the partial integration formula (2.56) are more transparent.

Remark 2.2. The entries of the generalized cross product $a \times_3 b$, with $a, b \in \mathbb{R}^3$, are permutations (with a sign) of the entries of the classical cross product $a \times b$. Recall, that also the operation $a \times \cdot$ can be identified with a multiplication with the skew-symmetric matrix

$$\text{Anti}(a) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \quad (2.8)$$

which differs from the expression $\llbracket a \rrbracket_{\times_3}$ and from $\mathfrak{A}_3(a)$ which reads

$$\mathfrak{A}_3(a) = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}. \quad (2.9)$$

Moreover, we have

$$a \times_n b = -b \times_n a = \llbracket -b \rrbracket_{\times_n} a = \left(a^T (\llbracket -b \rrbracket_{\times_n})^T \right)^T \quad (2.10)$$

this allows us to define a generalized cross product of a vector $b \in \mathbb{R}^n$ and a matrix $P \in \mathbb{R}^{n \times m}$ from the left and with a matrix $B \in \mathbb{R}^{m \times n}$ from the right via

$$b \times_n P := \llbracket b \rrbracket_{\times_n} P \in \mathbb{R}^{\frac{n(n-1)}{2} \times m} \quad \text{to be seen as column-wise cross product,} \quad (2.11a)$$

and

$$B \times_n b := B (\llbracket -b \rrbracket_{\times_n})^T \in \mathbb{R}^{m \times \frac{n(n-1)}{2}} \quad \text{to be seen as row-wise cross product.} \quad (2.11b)$$

In such a way we obtain for all $b \in \mathbb{R}^n$:

$$\mathbb{1}_n \times_n b = (\llbracket -b \rrbracket_{\times_n})^T \quad \text{and} \quad b \times_n \mathbb{1}_n = \llbracket b \rrbracket_{\times_n}. \quad (2.12)$$

Furthermore, for $a, b \in \mathbb{R}^n$ it holds

$$\begin{aligned} \llbracket a \rrbracket_{\times_n} \times_n b &= \llbracket a \rrbracket_{\times_n} (\llbracket -b \rrbracket_{\times_n})^T = \begin{pmatrix} \llbracket \bar{a} \rrbracket_{\times_{n-1}} & \vdots & 0 \\ \dots & \dots & \dots \\ -a_n \cdot \mathbb{1}_{n-1} & \vdots & \bar{a} \end{pmatrix} \begin{pmatrix} (\llbracket -\bar{b} \rrbracket_{\times_{n-1}})^T & \vdots & b_n \cdot \mathbb{1}_{n-1} \\ \dots & \dots & \dots \\ 0 & \vdots & -\bar{b}^T \end{pmatrix} \\ &= \begin{pmatrix} \llbracket \bar{a} \rrbracket_{\times_{n-1}} (\llbracket -\bar{b} \rrbracket_{\times_{n-1}})^T & \vdots & b_n \llbracket \bar{a} \rrbracket_{\times_{n-1}} \\ \dots & \dots & \dots \\ a_n (\llbracket \bar{b} \rrbracket_{\times_{n-1}})^T & \vdots & -\bar{a} \otimes \bar{b} - a_n b_n \cdot \mathbb{1}_{n-1} \end{pmatrix} \in \mathbb{R}^{\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}} \end{pmatrix} \quad (2.13)$$

end especially for $a = b$:

$$\llbracket b \rrbracket_{\times_n} \times_n b = \begin{pmatrix} \llbracket \bar{b} \rrbracket_{\times_{n-1}} (\llbracket -\bar{b} \rrbracket_{\times_{n-1}})^T & \vdots & b_n \llbracket \bar{b} \rrbracket_{\times_{n-1}} \\ \dots & \dots & \dots \\ b_n (\llbracket \bar{b} \rrbracket_{\times_{n-1}})^T & \vdots & -\bar{b} \otimes \bar{b} - b_n b_n \cdot \mathbb{1}_{n-1} \end{pmatrix} \in \text{Sym} \left(\frac{n(n-1)}{2} \right) \quad (2.14)$$

Hence, for all $a, b \in \mathbb{R}^n$:

$$\text{tr}(\llbracket a \rrbracket_{\times_n} \times_n b) = \langle \llbracket a \rrbracket_{\times_n}, \llbracket -b \rrbracket_{\times_n} \rangle = -(n-1) \cdot \langle a, b \rangle \quad (2.15)$$

by induction over n , and, especially for $a = b$:

$$\text{tr}(\llbracket b \rrbracket_{\times_n} \times_n b) = -(n-1) \cdot \|b\|^2. \quad (2.16)$$

The entries of $\llbracket b \rrbracket_{\times_n} \times_n b$ are, by definition, linear combinations of $b_i b_j$, the entries of $b \otimes b$. Interestingly, for $n \geq 3$ also the converse holds true, i.e., the entries of $b \otimes b$ are linear combinations of the entries of $\llbracket b \rrbracket_{\times_n} \times_n b$ which will be assertion of the subsequent lemma. Moreover, we will use this as a key observation to achieve the existence of linear combinations in $\mathbb{D}^2(A + \zeta \cdot \mathbb{1}_n) = L(\text{DCurl}_n(A + \zeta \cdot \mathbb{1}_n))$ for $n \geq 3$, so that we can follow the argumentation scheme presented in $n = 3$ dimensions, cf. [9], also in the higher dimensional case.

Lemma 2.3. *For all $b \in \mathbb{R}^n$ with $n \geq 3$ we have:*

$$b \otimes b = L(\llbracket b \rrbracket_{\times_n} \times_n b), \quad (2.17)$$

denoting by L a corresponding constant coefficients linear operator.

Remark 2.4. There are no linear combinations (2.17) in $n = 2$ dimensions. Indeed, we only have

$$\left\llbracket \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\rrbracket_{\times_2} \times_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -b_2 & b_1 \end{pmatrix} \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} = -b_1^2 - b_2^2, \quad (2.18)$$

so that there are no linear expressions of b_1^2 , b_2^2 nor of $b_1 b_2$ in terms of the sole entry of $\left\llbracket \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\rrbracket_{\times_2} \times_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Proof of Lemma 2.3 by induction over $n \geq 3$. For the base case $n = 3$ we have

$$\left\llbracket \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\rrbracket_{\times_3} \times_3 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \stackrel{(2.14)}{=} \begin{pmatrix} -b_1^2 - b_2^2 & \vdots & b_3 \cdot (-b_2 \ b_1) \\ \dots & \dots & \dots \\ b_3 \cdot \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} & \vdots & -\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - b_3^2 \cdot \mathbb{1}_2 \end{pmatrix} \quad (2.19)$$

and

$$\text{tr} \left(\left\llbracket \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\rrbracket_{\times_3} \times_3 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) = -2 \cdot (b_1^2 + b_2^2 + b_3^2). \quad (2.20)$$

Thus, for all $b \in \mathbb{R}^3$

$$b_3^2 = \langle e_1, (\llbracket b \rrbracket_{\times_3} \times_3 b) e_1 \rangle - \frac{1}{2} \text{tr}(\llbracket b \rrbracket_{\times_3} \times_3 b) = L(\llbracket b \rrbracket_{\times_3} \times_3 b) \quad (2.21)$$

and consequently from the expression (2.19) we conclude

$$b \otimes b = L(\llbracket b \rrbracket_{\times_3} \times_3 b) \quad \forall b \in \mathbb{R}^3. \quad (2.22)$$

Now, assume for the inductive step that for all $\bar{b} \in \mathbb{R}^{n-1}$ with $n \geq 4$ we have

$$\bar{b} \otimes \bar{b} = L(\llbracket \bar{b} \rrbracket_{\times_{n-1}} \times_{n-1} \bar{b}). \quad (2.23)$$

For $b \in \mathbb{R}^n$ we have

$$b \otimes b = \begin{pmatrix} \bar{b} \otimes \bar{b} & \vdots & b_n \cdot \bar{b} \\ \dots & \dots & \dots \\ b_n \cdot \bar{b}^T & \vdots & b_n^2 \end{pmatrix}. \quad (2.24)$$

Surely, $b_n \cdot \bar{b} = L(b_n \cdot \llbracket \bar{b} \rrbracket_{\times_{n-1}}) = L(\llbracket b \rrbracket_{\times_n} \times_n b)$ by the expression (2.14). The induction hypothesis gives

$$\bar{b} \otimes \bar{b} = L(\llbracket \bar{b} \rrbracket_{\times_{n-1}} \times_{n-1} \bar{b}) \stackrel{(2.14)}{=} L(\llbracket b \rrbracket_{\times_n} \times_n b), \quad (2.25)$$

hence, also $\|\bar{b}\|^2 = L(\llbracket b \rrbracket_{\times_n} \times_n b)$, so that finally

$$b_n^2 \stackrel{(2.16)}{=} -\frac{1}{n-1} \text{tr}(\llbracket b \rrbracket_{\times_n} \times_n b) - \|\bar{b}\|^2 = L(\llbracket b \rrbracket_{\times_n} \times_n b). \quad \blacksquare$$

By definition (2.11b), the entries of $B \times_n b$ are linear combinations of the entries $B_{ij} b_k$, i.e., of the entries of the matrix B multiplied with the entries of the vector b . For skew-symmetric matrices also the converse holds true. This is the assertion of the next lemma.

Lemma 2.5. *For all $A \in \mathfrak{so}(n)$ and $b \in \mathbb{R}^n$ with $n \geq 2$ we have*

$$(i) \quad A \times_n b = L(\mathfrak{a}_n(A) \otimes b) \qquad (ii) \quad \mathfrak{a}_n(A) \otimes b = L(A \times_n b)$$

denoting by L a corresponding constant coefficients linear operator which can differ in both cases.

Proof. For $A \in \mathfrak{so}(n)$ and $b \in \mathbb{R}^n$ we have

$$\mathfrak{a}_n(A) \otimes b = \begin{pmatrix} \mathfrak{a}_{n-1}(\overline{A}) \\ \overline{A} e_n \end{pmatrix} \otimes \begin{pmatrix} \overline{b} \\ b_n \end{pmatrix} = \begin{pmatrix} \mathfrak{a}_{n-1}(\overline{A}) \otimes \overline{b} & b_n \cdot \mathfrak{a}_{n-1}(\overline{A}) \\ \dots & \dots \\ \overline{A} e_n \otimes \overline{b} & b_n \cdot \overline{A} e_n \end{pmatrix} \quad (2.26)$$

and on the other hand

$$\begin{aligned} A \times_n b &= \begin{pmatrix} \overline{A} & \dots & \overline{A} e_n \\ \dots & \dots & \dots \\ -(\overline{A} e_n)^T & \dots & 0 \end{pmatrix} \begin{pmatrix} (\llbracket -\overline{b} \rrbracket_{\times_{n-1}})^T & b_n \cdot \mathbb{1}_{n-1} \\ \dots & \dots \\ 0 & \dots & -\overline{b}^T \end{pmatrix} \\ &= \begin{pmatrix} \overline{A} (\llbracket -\overline{b} \rrbracket_{\times_{n-1}})^T & \dots & b_n \cdot \overline{A} - \overline{A} e_n \otimes \overline{b} \\ \dots & \dots & \dots \\ (\overline{A} e_n)^T (\llbracket \overline{b} \rrbracket_{\times_{n-1}})^T & \dots & -b_n \cdot (\overline{A} e_n)^T \end{pmatrix} = \begin{pmatrix} \overline{A} \times_{n-1} \overline{b} & \dots & b_n \cdot \overline{A} - \overline{A} e_n \otimes \overline{b} \\ \dots & \dots & \dots \\ -(\overline{A} e_n \times_{n-1} \overline{b})^T & \dots & -b_n \cdot (\overline{A} e_n)^T \end{pmatrix}. \end{aligned} \quad (2.27)$$

Thus, the conclusions of the lemma follow by induction over the dimension n . Indeed, for the base case $n = 2$ we have

$$\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \times_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} -\alpha b_1 \\ -\alpha b_2 \end{pmatrix} = -(\alpha \otimes b)^T \quad (2.28)$$

which establishes (i) and (ii) of the Lemma for $n = 2$.

For the inductive step let us assume that the statement of the Lemma holds for all $\overline{A} \in \mathfrak{so}(n-1)$ and all $\overline{b} \in \mathbb{R}^{n-1}$, i.e., it holds:

$$(i) \quad \overline{A} \times_{n-1} \overline{b} = L(\mathfrak{a}_{n-1}(\overline{A}) \otimes \overline{b}) \quad \text{and} \quad (ii) \quad \mathfrak{a}_{n-1}(\overline{A}) \otimes \overline{b} = L(\overline{A} \times_{n-1} \overline{b}).$$

Thus, returning to $A \in \mathfrak{so}(n)$ and $b \in \mathbb{R}^n$ we have by the expressions (2.26) and (2.27), respectively,

$$(i) \quad \overline{A} \times_{n-1} \overline{b} = L(\mathfrak{a}_n(A) \otimes b) \quad \text{and} \quad (ii) \quad \mathfrak{a}_n(A) \otimes b = L(A \times_n b)$$

and the conclusion of part (i) of the Lemma follows then from the definition of the generalized cross product, indeed,

$$-(\overline{A} e_n \times_{n-1} \overline{b})^T = (\mathfrak{a}_{n-1}(\overline{b} \otimes \overline{A} e_n - \overline{A} e_n \otimes \overline{b}))^T \stackrel{(2.26)}{=} L(\mathfrak{a}_n(A) \otimes b). \quad (2.29)$$

On the other hand we have

$$b_n \cdot \overline{A} - \overline{A} e_n \otimes \overline{b} - (b_n \cdot \overline{A} - \overline{A} e_n \otimes \overline{b})^T + \mathfrak{a}_{n-1}(\overline{A} e_n \times_{n-1} \overline{b}) = 2 b_n \cdot \overline{A}, \quad (2.30)$$

so that, $b_n \cdot \overline{A} = L(A \times_n b)$ and also $b_n \cdot \mathfrak{a}_{n-1}(\overline{A}) = L(A \times_n b)$ which by (2.27) implies that $\overline{A} e_n \otimes \overline{b} = L(A \times_n b)$. This finishes the proof of (ii) since we have shown that all the entries of $\mathfrak{a}_n(A) \otimes b$ can be written as linear combinations of the entries of $A \times_n b$. \blacksquare

Remark 2.6. The identity (2.30) is not a new result, and usually it is expressed using coordinates:

$$(A \times_n b)_{kij} - (A \times_n b)_{kji} + (A \times_n b)_{jik} = 2 A_{ij} b_k \quad \forall i, j, k = 1, \dots, n. \quad (2.31)$$

However, we included the statement as well as the proof not only for the sake of completeness, but also since the use of the matrix representation of the generalized cross product allows us to give a coordinate-free proof and provides a deeper insight in the algebra needed in the present paper.

For a square matrix $P \in \mathbb{R}^{n \times n}$ we can take the generalized cross product with a vector $b \in \mathbb{R}^n$ both left and right, and simultaneously we obtain for the identity matrix

$$b \times_n \mathbb{1}_n \times_n b = \llbracket b \rrbracket_{\times_n} \mathbb{1}_n (\llbracket -b \rrbracket_{\times_n})^T = \llbracket b \rrbracket_{\times_n} (\llbracket -b \rrbracket_{\times_n})^T = \llbracket b \rrbracket_{\times_n} \times_n b \stackrel{(2.14)}{\in} \text{Sym} \left(\frac{n(n-1)}{2} \right) \quad (2.32)$$

and for a general matrix $P \in \mathbb{R}^{n \times n}$

$$(b \times_n P \times_n b)^T = (\llbracket b \rrbracket_{\times_n} P (\llbracket -b \rrbracket_{\times_n})^T)^T = -\llbracket b \rrbracket_{\times_n} P^T (\llbracket b \rrbracket_{\times_n})^T = b \times_n P^T \times_n b. \quad (2.33)$$

Thus, especially for a symmetric matrix $S \in \text{Sym}(n)$ and a skew-symmetric matrix $A \in \mathfrak{so}(n)$ we obtain

$$b \times_n S \times_n b \in \text{Sym} \left(\frac{n(n-1)}{2} \right) \quad \text{and} \quad b \times_n A \times_n b \in \mathfrak{so} \left(\frac{n(n-1)}{2} \right). \quad (2.34)$$

Observation 2.7. Let $A \in \mathfrak{so}(n)$ and $\alpha \in \mathbb{R}$. Then $(A + \alpha \cdot \mathbb{1}_n) \times_n b = 0$ for $b \in \mathbb{R}^n \setminus \{0\}$ implies $A = 0$ and $\alpha = 0$.

Proof. Taking the generalized cross product from the left on both sides of $0 = A \times_n b + \alpha \cdot \mathbb{1}_n \times_n b$ gives

$$0 = \underbrace{b \times_n A \times_n b}_{\text{skew-symmetric}} + \alpha \cdot \underbrace{b \times_n \mathbb{1}_n \times_n b}_{\text{symmetric}}, \quad (2.35)$$

so that taking the trace of the symmetric part on both sides we obtain

$$0 = \alpha \cdot \text{tr}(b \times_n \mathbb{1}_n \times_n b) \stackrel{(2.32)}{=} \alpha \cdot \text{tr}(\llbracket b \rrbracket_{\times_n} \times_n b) \stackrel{(2.16)}{=} -\alpha \cdot (n-1) \cdot \|b\|^2 \quad (2.36)$$

which implies $\alpha = 0$. Consequently we moreover have $A \times_n b = 0$ which by Lemma 2.5 (ii) yields $\mathfrak{a}_n(A) \otimes b = 0$, and thus $\mathfrak{a}_n(A) = 0$ and $A = 0$. \blacksquare

2.2 Considerations from vector calculus

Subsequently we make use of the algebraic behavior of the vector differential operator ∇ as a vector for formal calculations. So, the derivative and the divergence of a vector field $a \in \mathcal{D}'(\Omega, \mathbb{R}^n)$ can be seen as

$$Da = a \otimes \nabla \quad \text{and} \quad \text{div } a = \langle a, \nabla \rangle = \langle Da, \mathbb{1}_n \rangle = \text{tr}(Da). \quad (2.37)$$

In a similar way the generalized curl is related to the generalized cross product

$$\text{curl}_n a := a \times_n (-\nabla) = \nabla \times_n a = \llbracket \nabla \rrbracket_{\times_n} a = (a^T (\llbracket \nabla \rrbracket_{\times_n})^T)^T. \quad (2.38)$$

The latter expression gives a generalized row-wise matrix Curl_n operator for $B \in \mathcal{D}'(\Omega, \mathbb{R}^{m \times n})$ via

$$\text{Curl}_n B := B \times_n (-\nabla) = B (\llbracket \nabla \rrbracket_{\times_n})^T. \quad (2.39)$$

This differential operator kills the Jacobian matrix of a vector field (a compatible field), indeed

$$\text{Curl}_n Da = Da (\llbracket \nabla \rrbracket_{\times_n})^T = (a \otimes \nabla) (\llbracket \nabla \rrbracket_{\times_n})^T = a (\llbracket \nabla \rrbracket_{\times_n} \nabla)^T = a (\nabla \times_n \nabla)^T \equiv 0, \quad (2.40)$$

since $b \times_n b = 0$ for all $b \in \mathbb{R}^n$. Furthermore, for a scalar field $\zeta \in \mathcal{D}'(\Omega, \mathbb{R})$ we obtain

$$\text{Curl}_n(\zeta \cdot \mathbb{1}_n) = \mathbb{1}_n (\llbracket \nabla \zeta \rrbracket_{\times_n})^T = (\llbracket \nabla \zeta \rrbracket_{\times_n})^T. \quad (2.41)$$

For $P \in \mathcal{D}'(\Omega, \mathbb{R}^{n \times m})$ we consider also the column-wise differential operator of first order coming from a cross product form the left, namely

$$\nabla \times_n P = \llbracket \nabla \rrbracket_{\times_n} P, \quad (2.42)$$

which kills the transposed Jacobian $(Da)^T$.

It is clear, that $\text{Curl}_n B = L(DB)$, i.e., that the entries of $\text{Curl}_n B$ are linear combinations of the entries of DB for all $B \in \mathcal{D}'(\Omega, \mathbb{R}^{m \times n})$. However, for skew-symmetric matrix fields also the converse holds true:

Corollary 2.8. *For all $A \in \mathcal{D}'(\Omega, \mathfrak{so}(n))$ with $n \geq 2$ it holds: $DA = L(\text{Curl}_n A)$.*

It is a well known result and follows from the linear expression (ii) in Lemma 2.5 replacing b by $-\nabla$ as well as from its analogous statement written out in coordinates (2.31).

We now catch up with the crucial linear relation needed in our argumentation scheme.

Lemma 2.9. *Let $n \geq 3$, $A \in \mathcal{D}'(\Omega, \mathfrak{so}(n))$ and $\zeta \in \mathcal{D}'(\Omega, \mathbb{R})$. Then the entries of $D^2(A + \zeta \cdot \mathbb{1}_n)$ are linear combinations of the entries of $D\text{Curl}_n(A + \zeta \cdot \mathbb{1}_n)$.*

Remark 2.10. The statement is false in $n = 2$ dimensions. Indeed, with $\alpha, \zeta \in \mathcal{D}'(\Omega, \mathbb{R})$ we have

$$\text{Curl}_2 \begin{pmatrix} \zeta & \alpha \\ -\alpha & \zeta \end{pmatrix} = \begin{pmatrix} \zeta & \alpha \\ -\alpha & \zeta \end{pmatrix} \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} = \begin{pmatrix} \partial_1 \alpha - \partial_2 \zeta \\ \partial_2 \alpha + \partial_1 \zeta \end{pmatrix} \quad (2.43)$$

so that

$$D\text{Curl}_2 \begin{pmatrix} \zeta & \alpha \\ -\alpha & \zeta \end{pmatrix} = \begin{pmatrix} \partial_1 \partial_1 \alpha - \partial_1 \partial_2 \zeta & \partial_1 \partial_2 \alpha - \partial_2 \partial_2 \zeta \\ \partial_1 \partial_2 \alpha + \partial_1 \partial_1 \zeta & \partial_2 \partial_2 \alpha + \partial_1 \partial_2 \zeta \end{pmatrix} \quad (2.44)$$

and we cannot extract $\partial_1 \partial_1 \alpha$ from the components of (2.44).

Proof of Lemma 2.9. The proof is divided into the two observations

1. $D^2 \zeta = L(D\text{Curl}_n(A + \zeta \cdot \mathbb{1}_n))$,
2. $D^2 A = L(D\text{Curl}_n(A + \zeta \cdot \mathbb{1}_n))$

denoting by L a corresponding constant coefficients linear operator which can differ in both cases. To show that the entries of the Hessian $D^2 \zeta$ can be written as linear combinations of the entries of $D\text{Curl}_n(A + \zeta \cdot \mathbb{1}_n)$ we introduce the following second order derivative operator for square matrix fields $P \in \mathcal{D}'(\Omega, \mathbb{R}^{n \times n})$:

$$\mathbf{inc}_n P := \nabla \times_n P \times_n \nabla = -\nabla \times_n (\text{Curl}_n P) \quad (2.45)$$

in the style of the incompatibility operator known from the three-dimensional case. In regard of (2.32) we see

$$\mathbf{inc}_n(\zeta \cdot \mathbb{1}_n) = \nabla \times_n (\zeta \cdot \mathbb{1}_n) \times_n \nabla = (\llbracket \nabla \rrbracket_{\times_n} \times_n \nabla) \zeta \in \text{Sym} \left(\frac{n(n-1)}{2} \right), \quad (2.46)$$

so that substituting b by $-\nabla$ in Lemma 2.3 we obtain

$$D^2 \zeta = (\nabla \otimes \nabla) \zeta = L(\llbracket \nabla \rrbracket_{\times_n} \times_n \nabla) \zeta = L(\mathbf{inc}_n(\zeta \cdot \mathbb{1}_n)). \quad (2.47)$$

Moreover, with regard to (2.34)₂ we have for a skew-symmetric matrix field $A \in \mathcal{D}'(\Omega, \mathfrak{so}(n))$:

$$\mathbf{inc}_n A = \nabla \times_n A \times_n \nabla \in \mathfrak{so} \left(\frac{n(n-1)}{2} \right), \quad (2.48)$$

concluding for the 1. part that

$$D^2\zeta = L(\mathbf{inc}_n(\zeta \cdot \mathbf{1}_n)) = L(\text{sym}(\mathbf{inc}_n(A + \zeta \cdot \mathbf{1}_n))) = L(\mathbf{inc}_n(A + \zeta \cdot \mathbf{1}_n)) = L(\text{DCurl}_n(A + \zeta \cdot \mathbf{1}_n)) \quad (2.49)$$

where in the last step we have used that the entries of $\mathbf{inc}_n P = -\nabla \times_n (\text{Curl}_n P)$ are, of course, linear combinations of the entries from $\text{DCurl}_n P$.

To establish part 2 recall that the entries of DA for a skew-symmetric matrix field are linear combinations of the entries of $\text{Curl}_n A$, giving

$$\begin{aligned} DA &= L(\text{Curl}_n A) = L(\text{Curl}_n(A + \zeta \cdot \mathbf{1}_n)) - L(\text{Curl}_n(\zeta \cdot \mathbf{1}_n)) \\ &\stackrel{(2.41)}{=} L(\text{Curl}_n(A + \zeta \cdot \mathbf{1}_n)) - L\left(\left(\llbracket \nabla \zeta \rrbracket_{\times_n}\right)^T\right). \end{aligned} \quad (2.50)$$

The conclusion follows by taking the ∂_j -derivative of (2.50) together with the observation from the 1. part:

$$\partial_j DA = L(\partial_j \text{Curl}_n(A + \zeta \cdot \mathbf{1}_n)) - L\left(\left(\llbracket \partial_j \nabla \zeta \rrbracket_{\times_n}\right)^T\right) \stackrel{(2.49)}{=} L(\text{DCurl}_n(A + \zeta \cdot \mathbf{1}_n)). \quad \blacksquare$$

In the last result of this section we focus on the kernel of $\text{dev}_n \text{sym}$ and Curl_n :

Lemma 2.11. *Let $n \geq 3$, $A \in L^p(\Omega, \mathfrak{so}(n))$ and $\zeta \in L^p(\Omega, \mathbb{R})$. Then $\text{Curl}_n(A + \zeta \cdot \mathbf{1}_n) \equiv 0$ in the distributional sense if and only if there exist constant $b \in \mathbb{R}^n$, $d \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $\beta \in \mathbb{R}$ such that $A + \zeta \cdot \mathbf{1}_n = \mathfrak{A}_n(-\llbracket b \rrbracket_{\times_n} x + d) + (\langle b, x \rangle + \beta) \cdot \mathbf{1}_n$ almost everywhere in Ω .*

Remark 2.12. The ‘‘only if’’-part is false in $n = 2$ dimensions. To see this, take in (2.43) α and ζ to be the real and imaginary part of a holomorphic function.

Proof of Lemma 2.11. For the ‘‘if’’-part we have

$$\begin{aligned} \text{Curl}_n(\mathfrak{A}_n(-\llbracket b \rrbracket_{\times_n} x) + \langle b, x \rangle \cdot \mathbf{1}_n) &= \text{Curl}_n(\mathfrak{A}_n(-\llbracket b \rrbracket_{\times_n} x)) + \left(\llbracket \nabla \langle b, x \rangle \rrbracket_{\times_n}\right)^T \\ &= (-\llbracket b \rrbracket_{\times_n})^T + (\llbracket b \rrbracket_{\times_n})^T = 0. \end{aligned}$$

Conversely, $\text{Curl}_n(A + \zeta \cdot \mathbf{1}_n) \equiv 0$ in the distributional sense implies by (2.49):

$$D^2\zeta \equiv 0 \quad \text{and} \quad D^2A \equiv 0 \quad \text{in the distributional sense,} \quad (2.51)$$

thus,

$$\zeta(x) = \langle b, x \rangle + \beta \quad \text{and} \quad A = \mathfrak{A}_n(Bx + d) \quad \text{a.e.} \quad (2.52)$$

for some $b \in \mathbb{R}^n$, $\beta \in \mathbb{R}$, $B \in \mathbb{R}^{\frac{n(n-1)}{2} \times n}$ and $d \in \mathbb{R}^{\frac{n(n-1)}{2}}$, and we have

$$\text{Curl}_n(\zeta \cdot \mathbf{1}_n) = (\llbracket \nabla \zeta \rrbracket_{\times_n})^T = (\llbracket b \rrbracket_{\times_n})^T$$

as well as

$$\text{Curl}_n A = \text{Curl}_n(\mathfrak{A}_n(Bx)) = B^T + C^T,$$

where $C \in \mathbb{R}^{\frac{n(n-1)}{2} \times n}$ has only possibly non-zero entries at those positions at which the matrix $\llbracket b \rrbracket_{\times_n}$ has zeros. Hence, the condition, $\text{Curl}_n(A + \zeta \cdot \mathbf{1}_n) \equiv 0$ gives:

$$\left(\llbracket b \rrbracket_{\times_n}\right)^T = \text{Curl}_n(\zeta \cdot \mathbf{1}_n) = -\text{Curl}_n A = -B^T - C^T \quad (2.53)$$

implying that $C \equiv 0$ and $B = \llbracket b \rrbracket_{\times_n}$ almost everywhere in Ω . \blacksquare

Remark 2.13. The expression of the kernel follows also from the consideration for the classical trace-free Korn inequalities. Indeed, on simply connected domains, $\text{Curl}_n P \equiv 0$ implies that $P = Du$ for a vector field $u \in W^{1,p}(\Omega, \mathbb{R}^n)$. Thus, the condition $\text{dev}_n \text{sym} P \equiv 0$ reads $\text{dev}_n \text{sym} Du \equiv 0$, whose solutions are well known as infinitesimal conformal mappings, cf. [1, 3, 6, 12, 14, 15, 16].

2.3 Function spaces

Having above relations at hand we can now catch up the arguments from [9]. Let $\Omega \subseteq \mathbb{R}^n$, we start by defining the space

$$W^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n}) := \{P \in L^p(\Omega, \mathbb{R}^{n \times n}) \mid \text{Curl}_n P \in L^p(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})\} \quad (2.54a)$$

equipped with the norm

$$\|P\|_{W^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n})} := \left(\|P\|_{L^p(\Omega, \mathbb{R}^{n \times n})}^p + \|\text{Curl}_n P\|_{L^p(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})}^p \right)^{\frac{1}{p}} \quad (2.54b)$$

and its subspace $W_0^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n})$ as the completion of $C_0^\infty(\Omega, \mathbb{R}^{n \times n})$ in the $W^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n})$ -norm.

In our proofs we shall use an important equivalence of norms due to Nečas [13, Théorème 1] valid on bounded Lipschitz domains, cf. also discussion in [9, 10] and the references cited therein.

Thus, in what follows $\Omega \subset \mathbb{R}^n$ will be a bounded domain with Lipschitz boundary and we are allowed to define boundary conditions in the distributional sense, so that

$$\begin{aligned} W_0^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n}) &= \{P \in W^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n}) \mid P \times_n \nu = 0 \text{ on } \partial\Omega\} \\ &= \{P \in W^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n}) \mid P \tau_l = 0 \text{ on } \partial\Omega \text{ for all } l = 1, \dots, n-1\}, \end{aligned} \quad (2.55)$$

where ν stands for the outward unit normal vector field and $\{\tau_l\}_{l=1, \dots, n-1}$ denotes a moving tangent frame on $\partial\Omega$, cf. [11]. Here, the generalized tangential trace $P \times_n \nu$ is understood in the sense of $W^{-\frac{1}{p}, p}(\partial\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})$ which is justified by partial integration, so that its trace is defined by

$$\begin{aligned} \forall Q \in W^{1-\frac{1}{p'}, p'}(\partial\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}}) : \\ \langle P \times_n (-\nu), Q \rangle_{\partial\Omega} = \int_{\Omega} \langle \text{Curl}_n P, \tilde{Q} \rangle_{\mathbb{R}^{n \times \frac{n(n-1)}{2}}} + \langle P, \tilde{Q} [\nabla]_{\times_n} \rangle_{\mathbb{R}^{n \times n}} dx \end{aligned} \quad (2.56)$$

having denoted by $\tilde{Q} \in W^{1, p'}(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})$ any extension of Q in Ω , where, $\langle \cdot, \cdot \rangle_{\partial\Omega}$ indicates the duality pairing between $W^{-\frac{1}{p}, p}(\partial\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})$ and $W^{1-\frac{1}{p'}, p'}(\partial\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})$. Indeed, for smooth P and Q on $\bar{\Omega}$ we have

$$\begin{aligned} \int_{\partial\Omega} \langle P \times_n (-\nu), Q \rangle_{\mathbb{R}^{n \times \frac{n(n-1)}{2}}} dS &\stackrel{(2.11b)}{=} \int_{\partial\Omega} \langle P([\nu]_{\times_n})^T, Q \rangle_{\mathbb{R}^{n \times \frac{n(n-1)}{2}}} dS \\ &= \int_{\partial\Omega} \langle ([\nu]_{\times_n})^T, P^T Q \rangle_{\mathbb{R}^{n \times \frac{n(n-1)}{2}}} dS \\ &\stackrel{(*)}{=} \int_{\Omega} \langle ([\nabla]_{\times_n})^T, P^T Q \rangle_{\mathbb{R}^{n \times \frac{n(n-1)}{2}}} dx \\ &= \int_{\Omega} \langle P([\nabla]_{\times_n})^T, Q \rangle_{\mathbb{R}^{n \times \frac{n(n-1)}{2}}} + \langle P, Q [\nabla]_{\times_n} \rangle_{\mathbb{R}^{n \times n}} dx \\ &= \int_{\Omega} \langle \text{Curl}_n P, Q \rangle_{\mathbb{R}^{n \times \frac{n(n-1)}{2}}} + \langle P, Q [\nabla]_{\times_n} \rangle_{\mathbb{R}^{n \times n}} dx, \end{aligned} \quad (2.57)$$

where in (*) we have used the fact that we only deal with linear combinations of partial derivatives and from the classical divergence theorem it holds

$$\int_{\partial\Omega} \nu_i \zeta dS = \int_{\Omega} \partial_i \zeta dx \quad (2.58)$$

for a smooth scalar function ζ on $\bar{\Omega}$.

Further, following [10] we introduce also the space $W_{\Gamma,0}^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n})$ of functions with vanishing tangential trace only on a relatively open (non-empty) subset $\Gamma \subseteq \partial\Omega$ of the boundary by completion of $C_{\Gamma,0}^\infty(\Omega, \mathbb{R}^{n \times n})$ with respect to the $W^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n})$ -norm.

3 Trace-free Korn inequalities for incompatible tensors in higher dimensions

With the auxiliary results in hand we can now catch up with the higher dimensional versions of the results presented in [9].

Lemma 3.1. *Let $n \geq 3$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. Then $P \in \mathcal{D}'(\Omega, \mathbb{R}^{n \times n})$, $\text{dev}_n \text{sym } P \in L^p(\Omega, \mathbb{R}^{n \times n})$ and $\text{Curl}_n P \in W^{-1,p}(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})$ imply $P \in L^p(\Omega, \mathbb{R}^{n \times n})$. Moreover, we have the estimate*

$$\begin{aligned} \|P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left(\|\text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n\|_{W^{-1,p}(\Omega, \mathbb{R}^{n \times n})} \right. \\ \left. + \|\text{dev}_n \text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl}_n P\|_{W^{-1,p}(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})} \right), \end{aligned} \quad (3.1)$$

with a constant $c = c(n, p, \Omega) > 0$.

Proof. We have to show that $\text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n \in L^p(\Omega, \mathbb{R}^{n \times n})$ follows from the assumptions of the lemma. By the linearity of differential operator DCurl_n and the orthogonal decomposition $P = \text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n + \text{dev}_n \text{sym } P$ holding in $\mathcal{D}'(\Omega, \mathbb{R}^{n \times n})$ we obtain

$$\text{DCurl}_n(\text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n) = \text{DCurl}_n P - \text{DCurl}_n \text{dev}_n \text{sym } P \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^{n \times n}). \quad (3.2)$$

Thus, by the assumed regularity of the right hand side, it follows that the left hand side belongs to $W^{-2,p}(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2} \times n})$. Furthermore, we have

$$\begin{aligned} \|\text{DCurl}_n(\text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n)\|_{W^{-2,p}} \leq \|\text{DCurl}_n P\|_{W^{-2,p}} + \|\text{DCurl}_n \text{dev}_n \text{sym } P\|_{W^{-2,p}} \\ \leq c(\|\text{Curl}_n P\|_{W^{-1,p}} + \|\text{dev}_n \text{sym } P\|_{L^p}). \end{aligned} \quad (3.3)$$

By Lemma 2.9 we obtain $\text{D}^2(\text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n) \in W^{-2,p}$ and an application of the Lions lemma resp. Nečas estimate [9, Thm. 2.7 and Cor. 2.8] to $\text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n$ yield the conclusions. \blacksquare

Eliminating the first term on the right hand side of (3.1) gives:

Theorem 3.2. *Let $n \geq 3$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $P \in L^p(\Omega, \mathbb{R}^{n \times n})$ we have*

$$\inf_{T \in K_{dS, C_n}} \|P - T\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left(\|\text{dev}_n \text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl}_n P\|_{W^{-1,p}(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})} \right), \quad (3.4)$$

where the kernel is given by

$$K_{dS, C_n} = \{T : \Omega \rightarrow \mathbb{R}^{n \times n} \mid T(x) = \mathfrak{A}_n(-\llbracket b \rrbracket_{\times_n} x + d) + (\langle b, x \rangle + \beta) \cdot \mathbb{1}_n, b \in \mathbb{R}^n, d \in \mathbb{R}^{\frac{n(n-1)}{2}}, \beta \in \mathbb{R}\}. \quad (3.5)$$

Remark 3.3. This result does not directly extend to $n = 2$, since in that case the condition $\text{dev}_2 \text{sym } Du \equiv 0$ becomes the system of Cauchy-Riemann equations $\{u_{1,x} = u_{2,y} \wedge u_{1,y} = -u_{2,x}\}$ so that the corresponding nullspace is infinite-dimensional.

Proof of Theorem 3.2. The characterization of the kernel of the right-hand side gives

$$K_{dS, C_n} := \{P \in L^p(\Omega, \mathbb{R}^{n \times n}) \mid \text{dev}_n \text{sym } P = 0 \text{ a.e. and } \text{Curl}_n P = 0 \text{ in the dist. sense}\},$$

so that $P \in K_{dS, C_n}$ if and only if $P = \text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n$ and $\text{Curl}_n(\text{skew } P + \frac{1}{n} \text{tr } P \cdot \mathbb{1}_n) = 0$. Hence, (3.5) follows by virtue of Lemma 2.11 and the conclusion follows in a similar way to [9, Thm. 3.8]. \blacksquare

Remark 3.4. For compatible displacement gradients $P = Du$ we get back from (3.4) the quantitative version of the classical trace-free Korn’s inequality, cf. [3, 15, 16].

Finally, we examine the effect of tangential boundary conditions $P \times_n \nu \equiv 0$.

Theorem 3.5. *Let $n \geq 3$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $P \in W_0^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n})$ we have*

$$\|P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left(\|\text{dev}_n \text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl}_n P\|_{L^p(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})} \right). \quad (3.6)$$

Proof. We follow the same argumentation scheme as in the proof of [10, Theorem 3.5] and consider a sequence $\{P_k\}_{k \in \mathbb{N}} \subset W_0^{1,p}(\text{Curl}_n; \Omega, \mathbb{R}^{n \times n})$ converging weakly in $L^p(\Omega, \mathbb{R}^{n \times n})$ to some P^* so that $\text{dev}_n \text{sym } P^* = 0$ almost everywhere and $\text{Curl}_n P^* = 0$ in the distributional sense, i.e. $P^* \in K_{dS, C_n}$. By (2.56) we obtain that $\langle P^* \times_n (-\nu), Q \rangle_{\partial\Omega} = 0$ for all $Q \in W^{1,p'}(\Omega, \mathbb{R}^{n \times \frac{n(n-1)}{2}})$. However, the boundary condition $P^* \times_n \nu = 0$ is also valid in the classical sense, since $P^* \in K_{dS, C_n}$ has an explicit representation. Using the explicit representation of $P^* = \mathfrak{A}_n(-\llbracket b \rrbracket_{\times_n} x + d) + (\langle b, x \rangle + \beta) \cdot \mathbb{1}_n$, we conclude using Observation 2.7 that, in fact, $P^* \equiv 0$:

$$\begin{aligned} & [\mathfrak{A}_n(-\llbracket b \rrbracket_{\times_n} x + d) + (\langle b, x \rangle + \beta) \cdot \mathbb{1}_n] \times_n \nu = 0 \\ & \stackrel{\text{Obs. 2.7}}{\Rightarrow} -\llbracket b \rrbracket_{\times_n} x + d = 0 \quad \text{and} \quad \langle b, x \rangle + \beta = 0 \quad \text{for all } x \in \partial\Omega \\ & \Rightarrow \beta = 0, \quad b = 0, \quad d = 0. \quad \blacksquare \end{aligned}$$

Remark 3.6. Estimate (3.6) should persist also in $n = 2$ dimensions. So, the case $p = 2$ is already contained in [1]. However, for the general case $p \in (1, \infty)$ we need a different approach and it will be the subject of a forthcoming note.

Remark 3.7. For compatible $P = Du$ we recover from (3.6) a tangential trace-free Korn inequality.

Remark 3.8. For $n \geq 3$, the previous results also hold true for tensor fields with vanishing tangential trace only on a relatively open (non-empty) subset $\Gamma \subseteq \partial\Omega$ of the boundary, cf. discussion in [10]. But, this is not the case in $n = 2$ dimensions. Indeed, already the trace-free version of Korn’s first inequality (1.1) with only partial boundary condition is false in the $n = 2$ case, cf. e.g. the counterexample contained in [1, section 6.6].

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