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Sign choices in the AGM for genus two theta constants

Jean Kieffer

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Abstract

Existing algorithms to compute genus 2 theta constants in quasi-linear time use Borchartd sequences, an analogue of the arithmetic-geometric mean for four complex numbers. In this paper, we show that these Borchartd sequences are given by good choices of square roots only, as in the genus 1 case. This removes the sign indeterminacies in the algorithm without relying on numerical integration.

1 Introduction

Denote by \mathcal{H}_g the Siegel half space for genus g , consisting of all $\tau \in M_g(\mathbb{C})$ such that τ is symmetric and $\text{Im}(\tau)$ is positive definite; for instance, \mathcal{H}_1 is the usual upper half plane. The *theta constants* are the holomorphic functions on \mathcal{H}_g defined by

$$\theta_{a,b}(\tau) = \sum_{m \in \mathbb{Z}^g} \exp \left(i\pi \left(\left(m + \frac{a}{2} \right)^t \tau \left(m + \frac{a}{2} \right) + \left(m + \frac{a}{2} \right)^t \frac{b}{2} \right) \right), \quad (1)$$

where a and b run through $\{0, 1\}^g$. Theta constants have a fundamental importance in the theory of modular forms: if $g = 1$ or 2 , every modular form of level 1 is a polynomial in the theta constants, and hence every modular invariant has an expression in terms of quotients of theta constants.

In numerical algorithms manipulating modular forms, the following operations are therefore very common: first, given (quotients of) theta constants at a given $\tau \in \mathcal{H}_g$, compute τ ; second, given $\tau \in \mathcal{H}_g$, compute the theta constants $\theta_{a,b}(\tau)$. For instance, these operations are important building blocks in algorithms computing modular polynomials or Hilbert class polynomials via complex approximations [4, 5, 6].

Computing theta constants in quasi-linear time. A well-studied strategy to compute theta constants in quasi-linear time in the output precision is to combine the arithmetic-geometric mean (AGM) with Newton iterations. This strategy was first described in [3] in the genus 1 case, [2] in the genus 2 case, and later extended to theta *functions*, in opposition to theta *constants*, in [11, 12]. These references also outline extensions to higher genus.

Let us detail the genus 1 case to convey the general idea. We can usually assume that τ belongs to the classical fundamental domain under the action of $\text{SL}_2(\mathbb{Z})$, denoted by \mathcal{F}_1 .

First assume that theta quotients at $\tau \in \mathcal{F}_1$ are given. Then the sequence

$$B(\tau) = \left(\frac{\theta_{00}^2(2^n \tau)}{\theta_{00}^2(\tau)}, \frac{\theta_{01}^2(2^n \tau)}{\theta_{00}^2(\tau)} \right)_{n \geq 0}$$

is an *AGM sequence*, meaning that each term is obtained from the previous one by means of the transformation

$$(x, y) \mapsto \left(\frac{x+y}{2}, \sqrt{x}\sqrt{y} \right)$$

for some choice of the square roots. This is a consequence of the duplication formula [14, p. 221], the correct square roots being the theta quotients themselves. In the algorithm, the sign ambiguity is easily removed using the fact that \sqrt{x} and \sqrt{y} should lie in a common open quarter plane [3, Thm. 2]: we say that the sequence $B(\tau)$ is given by *good sign choices*. It converges quadratically to $1/\theta_{00}^2(\tau)$, as the series expansion (1) shows.

It turns out that the sequence $B(-1/\tau)$ is also an AGM sequence with good sign choices [3, Prop. 7]. Its first term can be computed from theta quotients at τ using the transformation formulas for theta constants under $\text{SL}_2(\mathbb{Z})$. The limit of $B(-1/\tau)$ is $1/\theta_{00}^2(-1/\tau)$. Finally, we can recover τ using the formula

$$\theta_{00}^2 \left(\frac{-1}{\tau} \right) = -i\tau\theta_{00}^2(\tau). \tag{2}$$

Since AGM sequences with good sign choices converge quadratically, this gives an algorithm to *invert* theta functions on \mathcal{F}_1 with quasi-linear complexity in the output precision, at least for fixed τ . This method was already known to Gauss [7, X.1, pp. 184–206], and we recommend [1, §3C] for a historical exposition of Gauss's works on the AGM and elliptic functions.

In order to compute theta functions at a given $\tau \in \mathcal{F}_1$, the most efficient known method is to build a Newton scheme using this fast inversion algorithm. This yields a quasi-linear algorithm to compute genus 1 theta constants, whose complexity can be made uniform in $\tau \in \mathcal{F}_1$ [3, Thm. 5].

The genus 2 case. A similar strategy can be applied to theta functions in genus 2, using *Borchardt sequences*, a generalization of AGM sequences for four complex numbers. Let us anticipate on §2 for the definition of Borchardt sequences, the numbering of genus 2 theta constants, and the definition of the matrices $\gamma_k \in \text{Sp}_4(\mathbb{Z})$ for $0 \leq k \leq 3$. The Borchardt sequences we consider are the sequences $B(\gamma_k \tau)$ for $1 \leq k \leq 3$, where

$$B(\tau) = \left(\frac{\theta_0^2(2^n \tau)}{\theta_0^2(\tau)}, \frac{\theta_1^2(2^n \tau)}{\theta_0^2(\tau)}, \frac{\theta_2^2(2^n \tau)}{\theta_0^2(\tau)}, \frac{\theta_3^2(2^n \tau)}{\theta_0^2(\tau)} \right)_{n \geq 0}$$

for every $\tau \in \mathcal{H}_2$. It is known that for a given τ , all but a finite number of sign choices in these Borchardt sequences are good, and the other sign choices

can be precomputed using certified computations of hyperelliptic integrals at relatively low precision: see the discussion before Prop. 3.3 in [12], and [13] for an algorithm that provides this input. However, the required precision and the cost of the numerical integration algorithms depend heavily on τ .

Actually, when τ belongs to the usual fundamental domain \mathcal{F}_2 under the action of $\mathrm{Sp}_4(\mathbb{Z})$, practical experiments suggest that *all* sign choices are good in the genus 2 algorithm as well [2, Conj. 9.1]. The goal of this paper is to prove this fact. More precisely, we define in §2 a subset $\mathcal{F}' \subset \mathcal{H}_2$ containing \mathcal{F}_2 , and prove the following result.

Theorem 1.1. *For every $\tau \in \mathcal{F}'$, every $0 \leq k \leq 3$ and every $n \geq 0$, the theta constants*

$$\theta_j(2^n \gamma_k \tau) \quad \text{for } 0 \leq j \leq 3$$

are contained in a common open quarter plane.

As a consequence, we can invert genus 2 theta constants in quasi-linear time using Borchartd sequences only. On the practical side, this result reduces the effort needed to implement a correct algorithm. On the theoretical side, we hope that our result can be a first step towards removing other heuristic assumptions when computing genus 2 theta constants, and obtaining algorithms with uniform complexity in τ .

This document is organized as follows. In Section 2, we introduce our notational conventions. In Section 3, we use the action of the symplectic group to bring the matrices $2^n \gamma_k \tau \in \mathcal{H}_2$ closer to the cusp at infinity: this is critical to obtain accurate information from the series expansion (1). We give estimates on genus 2 theta constants in Section 4, and finish the proof of the main theorem in Section 5.

2 Theta constants and Borchartd sequences

We define a *Borchartd sequence* to be a sequence of complex numbers

$$(s_b^{(n)})_{b \in (\mathbb{Z}/2\mathbb{Z})^2, n \geq 0}$$

with the following property: for every $n \geq 0$, there exist $t_b^{(n)}$ for $b \in (\mathbb{Z}/2\mathbb{Z})^2$ such that $t_b^{(n)}$ is a square root of $s_b^{(n)}$, and

$$s_b^{(n+1)} = \frac{1}{4} \sum_{b_1 + b_2 = b} t_{b_1}^{(n)} t_{b_2}^{(n)} \quad \text{for each } b \in (\mathbb{Z}/2\mathbb{Z})^2.$$

The duplication formula [14, p. 221] states that for every $\tau \in \mathcal{H}_2$, the sequence

$$B(\tau) = (\theta_{0,b}^2(2^n \tau))_{b \in \{0,1\}^2, n \geq 0}$$

is a Borchartd sequence; the choice of square roots at each step is given by the theta constants $\theta_{0,b}(2^n \tau)$ themselves. Moreover, the sequence $B(\tau)$ converges to $(1, 1, 1, 1)$ by the series expansion (1).

We say that a set of complex numbers is *in good position* when it is included in an open quarter plane; this property is invariant by complex scaling. A Borchardt sequence is given by *good sign choices* if for every $n \geq 0$, the complex numbers $t_b^{(n)}$ for $b \in (\mathbb{Z}/2\mathbb{Z})^2$ are in good position.

Let us now detail the algorithm to recover $\tau \in \mathcal{H}_2$ from its theta quotients. We first introduce the matrices $\gamma_k \in \mathrm{Sp}_4(\mathbb{Z})$ alluded to in the introduction. Let

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and define the matrix $\gamma_k \in \mathrm{Sp}_4(\mathbb{Z})$ for $0 \leq k \leq 3$ by

$$\gamma_0 = I_4, \quad \gamma_k = \begin{pmatrix} -I_2 & -S_k \\ S_k & -I + S_k^2 \end{pmatrix} \quad \text{for } k \geq 1.$$

For convenience, we also introduce a numbering of theta constants:

$$\theta_{(a_0, a_1), (b_0, b_1)} =: \theta_j \quad \text{where } j = b_0 + 2b_1 + 4a_0 + 8a_1 \in \llbracket 0, 15 \rrbracket.$$

Assuming that the choices of square roots in the sequences $B(\gamma_k \tau)$ can be determined, we can compute $\tau \in \mathcal{F}_2$ from its theta quotients as follows.

1. For each $0 \leq k \leq 3$, compute theta quotients at $\gamma_k \tau$ using the transformation formulas under $\mathrm{Sp}_4(\mathbb{Z})$, and compute $1/\theta_0^2(\gamma_k \tau)$ as the limit of the Borchardt sequence $B(\gamma_k \tau)/\theta_0^2(\gamma_k \tau)$;
2. Use $\theta_0^2(\gamma_0 \tau) = \theta_0^2(\tau)$ to compute squares of theta *constants* at τ ;
3. Recover $\tau = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ using the relations

$$\theta_0^2(\gamma_1 \tau) = -iz_1 \theta_4^2(\tau), \quad \theta_0^2(\gamma_2 \tau) = -iz_2 \theta_8^2(\tau), \quad \theta_0^2(\gamma_3 \tau) = -\det(\tau) \theta_0^2(\tau).$$

We use the following notational conventions in the sequel. For $\tau \in \mathcal{H}_2$, we write

$$\tau = \begin{pmatrix} z_1(\tau) & z_3(\tau) \\ z_3(\tau) & z_2(\tau) \end{pmatrix} \quad \text{and} \quad \begin{cases} x_j(\tau) = \mathrm{Re} z_j(\tau) \\ y_j(\tau) = \mathrm{Im} z_j(\tau) \end{cases} \quad \text{for } 1 \leq j \leq 3.$$

For $1 \leq j \leq 3$, we also write

$$q_j(\tau) = \exp(-\pi y_j(\tau)).$$

We denote by $\lambda_1(\tau)$ the smallest eigenvalue of $\mathrm{Im}(\tau)$, and define

$$r(\tau) = \min \left\{ \lambda_1(\tau), \frac{y_1(\tau)}{2}, \frac{y_2(\tau)}{2} \right\}.$$

We often omit the argument τ to ease notation. Finally, we define \mathcal{F}' to be the set of all $\tau \in \mathcal{H}_2$ such that

$$\begin{aligned} |x_j(\tau)| &\leq \frac{1}{2} \quad \text{for each } 1 \leq j \leq 3, \\ 2|y_3(\tau)| &\leq y_1(\tau) \leq y_2(\tau), \\ y_1(\tau) &\geq \frac{\sqrt{3}}{2}, \\ |z_j(\tau)| &\geq 1 \quad \text{for } j \in \{1, 2\}. \end{aligned} \tag{3}$$

The domain \mathcal{F}' contains the usual fundamental domain \mathcal{F}_2 for the action of $\mathrm{Sp}_4(\mathbb{Z})$ on \mathcal{H}_2 [10, Prop. 3 p. 33].

3 Other expressions for theta constants at $2^n\gamma_k\tau$

For every $n \geq 0$, we define

$$\begin{aligned} \eta_1^{(n)} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \eta_2^{(n)} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2^n \end{pmatrix}, \\ \eta_3^{(n)} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 2^n & 0 \\ 1 & 0 & 0 & 2^n \end{pmatrix}, & \eta_4^{(n)} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \eta_3^{(n)}. \end{aligned}$$

Lemma 3.1. *Let $n \geq 0$.*

1. *For every $1 \leq k \leq 4$, the matrix $\eta_k^{(n)}$ belongs to $\mathrm{Sp}_4(\mathbb{Z})$.*
2. *For every $\tau = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix} \in \mathcal{H}_2$, we have*

$$\begin{aligned} \tau_1^{(n)} &:= \eta_1^{(n)}(2^n\gamma_1\tau) = \begin{pmatrix} 2^{-n}z_1 & z_3 \\ z_3 & 2^n z_2 \end{pmatrix}, \\ \tau_2^{(n)} &:= \eta_2^{(n)}(2^n\gamma_2\tau) = \begin{pmatrix} 2^n z_1 & z_3 \\ z_3 & 2^{-n} z_2 \end{pmatrix}, \\ \tau_3^{(n)} &:= \eta_3^{(n)}(2^n\gamma_3\tau) = 2^{-n}\tau, \\ \tau_4^{(n)} &:= \eta_4^{(n)}(2^n\gamma_3\tau) = \begin{pmatrix} -2^n/z_1 & -z_3/z_1 \\ -z_3/z_1 & 2^{-n}(z_2 - z_3^2/z_1) \end{pmatrix}. \end{aligned} \tag{4}$$

Proof. 1. The lines of each $\eta_k^{(n)}$ define a symplectic basis of \mathbb{Z}^4 .

2. The matrix $2^n\gamma_k\tau$ is the image of τ under

$$\begin{pmatrix} -2^n I_2 & -2^n S_k \\ S_k & -I + S_k^2 \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{Q}).$$

When we multiply this matrix by $\eta_k^{(n)}$ on the left, we obtain

$$\begin{aligned} \text{Diag}(-1, -2^n, -2^n, -1) & \quad \text{for } k = 1, \\ \text{Diag}(-2^n, -1, -1, -2^n) & \quad \text{for } k = 2, \\ \text{Diag}(-1, -1, -2^n, -2^n) & \quad \text{for } k = 3. \end{aligned} \quad \square$$

We recall the transformation formulas for theta constants in genus 2. For a square matrix m , we denote by m_0 the column vector containing the diagonal of m .

Proposition 3.2 ([9, Thm. 2 p.175 and Cor. p.176]). *Let $a, b \in \{0, 1\}^2$, and let*

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}).$$

Define

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma^t \begin{pmatrix} a - (CD^t)_0 \\ b - (AB^t)_0 \end{pmatrix}.$$

Then, for every $\tau \in \mathcal{H}_2$, we have

$$\theta_{a,b}(\gamma\tau) = \kappa(\gamma) \zeta_8^{\varepsilon(\gamma,a,b)} \det(C\tau + D)^{1/2} \theta_{a',b'}(\tau)$$

where

$$\zeta_8 = e^{i\pi/4}, \quad \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \pmod{2},$$

$$\varepsilon(\gamma, a, b) = 2(B\alpha)^t(C\beta) - (D\alpha)^t(B\alpha) - (C\beta)^t(A\beta) + 2((AB^t)_0)^t(D\alpha - C\beta),$$

and $\kappa(\gamma)$ is an eighth root of unity depending only on γ , with a sign ambiguity coming from the choice of a holomorphic square root of $\det(C\tau + D)$.

Corollary 3.3. *For every $\tau \in \mathcal{H}_2$, we have the following equalities of projective tuples:*

$$\begin{aligned} (\theta_j(2^n \gamma_1 \tau))_{0 \leq j \leq 3} &= \begin{cases} (\theta_4(\tau) : \theta_0(\tau) : \theta_6(\tau) : \theta_2(\tau)) & \text{if } n = 0, \\ (\theta_0(\tau_1^{(n)}) : \theta_4(\tau_1^{(n)}) : \theta_2(\tau_1^{(n)}) : \theta_6(\tau_1^{(n)})) & \text{if } n \geq 1, \end{cases} \\ (\theta_j(2^n \gamma_2 \tau))_{0 \leq j \leq 3} &= \begin{cases} (\theta_8(\tau) : \theta_9(\tau) : \theta_0(\tau) : \theta_1(\tau)) & \text{if } n = 0, \\ (\theta_0(\tau_2^{(n)}) : \theta_1(\tau_2^{(n)}) : \theta_8(\tau_2^{(n)}) : \theta_9(\tau_2^{(n)})) & \text{if } n \geq 1, \end{cases} \\ (\theta_j(2^n \gamma_3 \tau))_{0 \leq j \leq 3} &= (\theta_0(\tau_3^{(n)}) : \theta_8(\tau_3^{(n)}) : \theta_4(\tau_3^{(n)}) : \theta_{12}(\tau_3^{(n)})) \quad \text{for every } n \geq 0, \\ (\theta_j(2^n \gamma_3 \tau))_{0 \leq j \leq 3} &= (\theta_0(\tau_4^{(n)}) : \theta_8(\tau_4^{(n)}) : \theta_1(\tau_4^{(n)}) : \theta_9(\tau_4^{(n)})) \quad \text{for every } n \geq 0, \end{aligned}$$

where the $\tau_{j,n}$ are defined as in Eq. (4).

Proof. Apply Proposition 3.2 to the matrices $\eta_i^{(n)}$. □

When $\tau \in \mathcal{F}'$, the real and imaginary parts of $\tau_k^{(n)}$ for $1 \leq k \leq 3$ are easy to study: for instance, we always have

$$y_3(\tau_k^{(n)})^2 \leq \frac{1}{4}y_1(\tau_k^{(n)})y_2(\tau_k^{(n)}).$$

Such estimates are less obvious for the matrices $\tau_4^{(n)}$.

Lemma 3.4. *Let $\tau \in \mathcal{F}'$. Then, for every $n \geq 0$, we have*

$$\begin{aligned} |y_3(\tau_4^{(n)})| &\leq \frac{3}{2^{n+2}}y_1(\tau_4^{(n)}), \\ y_3(\tau_4^{(n)})^2 &\leq \frac{3}{7}y_1(\tau_4^{(n)})y_2(\tau_4^{(n)}), \\ |x_2(\tau_4^{(n)})| &\leq \frac{9}{2^{n+3}}. \end{aligned}$$

Proof. Write z_1 for $z_1(\tau)$, etc. We have

$$y_3(\tau_4^{(n)}) = \operatorname{Im}(-z_3/z_1) = \frac{1}{|z_1|^2}(x_3y_1 - y_3x_1),$$

so

$$|y_3(\tau_4^{(n)})| \leq \frac{3y_1}{4|z_1|^2}.$$

This proves the first inequality. On the other hand,

$$\operatorname{Im}(\tau_4^{(n)}) = \begin{pmatrix} 2^{-n}z_1 & -2^{-n}z_3 \\ 0 & 1 \end{pmatrix}^{-t} (2^{-n} \operatorname{Im} \tau) \begin{pmatrix} 2^{-n}\bar{z}_1 & -2^{-n}\bar{z}_3 \\ 0 & 1 \end{pmatrix}^{-1}$$

so

$$\det \operatorname{Im}(\tau_4^{(n)}) = \frac{1}{|z_1|^2} \det \operatorname{Im} \tau.$$

Moreover $\det \operatorname{Im} \tau \geq \frac{3}{4}y_1^2$, so

$$\frac{y_3(\tau_4^{(n)})^2}{y_1(\tau_4^{(n)})y_2(\tau_4^{(n)})} \leq \frac{y_3(\tau_4^{(n)})^2}{y_3(\tau_4^{(n)})^2 + \frac{3y_1^2}{4|z_1|^2}} \leq \frac{1}{1 + \frac{4}{3}|z_1|^2} \leq \frac{3}{7}.$$

For the last inequality, we compute that

$$2^n x_2(\tau_4^{(n)}) = x_2 - \frac{1}{|z_1|^2}((x_3^2 - y_3^2)x_1 + 2x_3y_3y_1)$$

and

$$\left| \frac{1}{|z_1|^2}(x_3^2 - y_3^2)x_1 \right| \leq \frac{1}{2} \max\{x_3^2, \frac{y_3^2}{|z_1|^2}\} \leq \frac{1}{8},$$

so

$$|2^n x_2(\tau_4^{(n)})| \leq \frac{1}{2} + \frac{1}{8} + \frac{1}{2} = \frac{9}{8}. \quad \square$$

4 Bounds on theta constants

Typically, when $\tau \in \mathcal{H}_2$ is close enough to the cusp at infinity, useful information on theta constants at τ can be obtained from the series expansion (1). In this section, we gather technical lemmas in this spirit. All our estimates are based on the following key lemma:

Lemma 4.1. *Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be a strictly increasing function, and assume that $f(k+2) - f(k+1) \geq f(k+1) - f(k)$ for every $k \geq 0$. Let $0 < q < 1$. Then*

$$\sum_{k=0}^{\infty} q^{f(k)} \leq \frac{q^{f(0)}}{1 - q^{f(1)-f(0)}}.$$

Proof. Use that $f(k) \geq f(0) + k(f(1) - f(0))$ for all k . □

Lemma 4.2. *Let $k \geq 1$, and let $\tau \in \mathcal{H}_2$ such that*

$$y_3(\tau)^2 \leq \frac{1}{4}y_1(\tau)y_2(\tau), \quad k|y_3(\tau)| \leq y_2(\tau).$$

Define

$$\begin{aligned} \xi_{4,6}(\tau) &= 2 \exp\left(i\pi \frac{z_1(\tau)}{4}\right), \\ \rho_{4,6}^{(k)}(q_1, q_2) &= \frac{q_1^2}{1 - q_1^4} + \frac{q_2^{1-\frac{1}{k}}}{1 - q_2^{3-\frac{1}{k}}} + \frac{q_2^{1+\frac{1}{k}}}{1 - q_2^{3+\frac{1}{k}}} \\ &\quad + \frac{q_1^{7/8} q_2^{1/2}}{(1 - q_2^{3/2})(1 - q_1^2)} + \frac{q_1^{25/8} q_2^{3/2}}{(1 - q_2^{9/2})(1 - q_1^6)}. \end{aligned}$$

Then for $j \in \{4, 6\}$, we have

$$\left| \frac{\theta_j(\tau)}{\xi_{4,6}(\tau)} - 1 \right| \leq \rho_{4,6}^{(k)}(q_1(\tau), q_2(\tau)).$$

Proof. Write $u = (\frac{1}{2}, 0)$. Using the definition, we obtain

$$\left| \frac{\theta_j(\tau)}{\xi_{4,6}(\tau)} - 1 \right| \leq \frac{1}{2} q_1^{-1/4} \sum_{\substack{m \in \mathbb{Z}^2 \\ m \neq (0,0), (-1,0)}} \exp(-\pi(m+u)^t \operatorname{Im}(\tau)(m+u)).$$

We split this sum in two parts, according to whether the first coordinate of m is zero or not. The first part gives

$$q_1^{-1/4} \sum_{m \in \mathbb{N} + \frac{3}{2}} q_1^{m^2} \leq q_1^{-1/4} \frac{q_1^{9/4}}{1 - q_1^4} = \frac{q_1^2}{1 - q_1^4}$$

by Lemma 4.1. The second part is

$$q_1^{-1/4} \sum_{m_1 \in \mathbb{N} + \frac{1}{2}} \sum_{m_2 \geq 1} q_1^{m_1^2} q_2^{m_2^2} \cdot 2 \cosh(2\pi y_3 m_1 m_2).$$

We use the fact that for every $(m_1, m_2) \in \mathbb{R}_+^2$,

$$|2y_3 m_1 m_2| \leq \frac{y_1}{2} m_1^2 + \frac{y_2}{2} m_2^2.$$

When $m_1 = 1/2$, we use the following bound instead:

$$|2y_3 m_1 m_2| = |y_3 m_2| \leq \frac{y_2 m_2}{k}.$$

Therefore the total contribution of the second part is bounded by

$$\begin{aligned} & q_1^{-1/4} \sum_{m_2 \geq 1} q_1^{1/4} q_2^{m_2^2} \cdot 2 \cosh\left(\pi \frac{y_2}{k} m_2\right) \\ & + q_1^{-1/4} \sum_{m_1 \in \mathbb{N} + \frac{3}{2}} \sum_{m_2 \geq 1} q_1^{m_1^2} q_2^{m_2^2} \cdot 2 \cosh\left(\pi \left(\frac{y_1}{2} m_1^2 + \frac{y_2}{2} m_2^2\right)\right) \\ & \leq \frac{q_2^{1-\frac{1}{k}}}{1-q_2^{3-\frac{1}{k}}} + \frac{q_2^{1+\frac{1}{k}}}{1-q_2^{3+\frac{1}{k}}} + \frac{q_1^{7/8} q_2^{1/2}}{(1-q_2^{3/2})(1-q_1^2)} + \frac{q_1^{25/8} q_2^{3/2}}{(1-q_2^{9/2})(1-q_1^6)} \end{aligned}$$

by other applications of Lemma 4.1. \square

Lemma 4.3. *Let $k \geq 1$, and let $\tau \in \mathcal{H}_2$ such that*

$$y_3(\tau)^2 \leq \frac{1}{4} y_1(\tau) y_2(\tau), \quad k |y_3(\tau)| \leq y_1(\tau).$$

Define

$$\begin{aligned} \xi_{8,9}(\tau) &= 2 \exp\left(i\pi \frac{z_2(\tau)}{4}\right), \\ \rho_{8,9}^{(k)}(q_1, q_2) &= \frac{q_2^2}{1-q_2^4} + \frac{q_1^{1-\frac{1}{k}}}{1-q_1^{3-\frac{1}{k}}} + \frac{q_1^{1+\frac{1}{k}}}{1-q_1^{3+\frac{1}{k}}} \\ & \quad + \frac{q_2^{7/8} q_1^{1/2}}{(1-q_1^{3/2})(1-q_2^2)} + \frac{q_2^{25/8} q_1^{3/2}}{(1-q_1^{9/2})(1-q_2^6)}. \end{aligned}$$

Then for $j \in \{8, 9\}$, we have

$$\left| \frac{\theta_j(\tau)}{\xi_{8,9}(\tau)} - 1 \right| \leq \rho_{8,9}^{(k)}(q_1(\tau), q_2(\tau)).$$

Proof. Similar: we essentially exchange q_1 and q_2 in the proof of Lemma 4.2. \square

Lemma 4.4. *Let $\tau \in \mathcal{H}_2$ such that*

$$y_3(\tau)^2 \leq \frac{1}{4}y_1(\tau)y_2(\tau),$$

and define

$$\begin{aligned} \xi_0(\tau) &= 1 + 2 \exp(i\pi z_1(\tau)) + 2 \exp(i\pi z_2(\tau)), \\ \xi_{0,2}(\tau) &= 1 + 2 \exp(i\pi z_1(\tau)), \\ \xi_{0,1}(\tau) &= 1 + 2 \exp(i\pi z_2(\tau)), \\ \rho_0(q_1, q_2) &= \frac{2q_1^4}{1 - q_1^5} + \frac{2q_2^4}{1 - q_2^5} + \frac{2q_1^{1/2}q_2^{1/2}}{(1 - q_1^{3/2})(1 - q_2^{3/2})} + \frac{2q_1^{3/2}q_2^{3/2}}{(1 - q_1^{9/2})(1 - q_2^{9/2})}. \end{aligned}$$

Then we have

$$\begin{aligned} |\theta_0(\tau) - \xi_0(\tau)| &\leq \rho_0(q_1(\tau), q_2(\tau)), \\ |\theta_j(\tau) - \xi_{0,2}(\tau)| &\leq \rho_0(q_1(\tau), q_2(\tau)) + 2q_2(\tau) && \text{for } j \in \{0, 2\}, \\ |\theta_j(\tau) - \xi_{0,1}(\tau)| &\leq \rho_0(q_1(\tau), q_2(\tau)) + 2q_1(\tau) && \text{for } j \in \{0, 1\}, \\ |\theta_j(\tau) - 1| &\leq \rho_0(q_1(\tau), q_2(\tau)) + 2q_1(\tau) + 2q_2(\tau) && \text{for } 0 \leq j \leq 3. \end{aligned}$$

Proof. Similar (but easier). □

Lemma 4.5. *Let $\tau \in \mathcal{H}_2$ such that*

$$|x_3(\tau)| \leq \frac{1}{2} \quad \text{and} \quad 2|y_3(\tau)| \leq \min\{y_1(\tau), y_2(\tau)\},$$

and write

$$\begin{aligned} \xi_{12}(\tau) &= \exp\left(i\pi \frac{z_1(\tau) + z_2(\tau)}{4}\right) \left(\exp\left(i\pi \frac{z_3(\tau)}{2}\right) + \exp\left(-i\pi \frac{z_3(\tau)}{2}\right) \right), \\ \rho_{12}(q_1, q_2) &= \frac{q_1^{3/2}}{1 - q_1^{7/2}} + \frac{q_1^{5/2}}{1 - q_1^{9/2}} + \frac{q_2^{3/2}}{1 - q_2^{7/2}} + \frac{q_2^{5/2}}{1 - q_2^{9/2}} \\ &\quad + \frac{q_1^{7/8}q_2^{7/8}}{(1 - q_1^2)(1 - q_2^2)} + \frac{q_1^{25/8}q_2^{25/8}}{(1 - q_1^6)(1 - q_2^6)}. \end{aligned}$$

Then we have

$$\left| \frac{\theta_{12}(\tau)}{\xi_{12}(\tau)} - 1 \right| \leq \rho_{12}(q_1(\tau), q_2(\tau)).$$

Proof. By definition, we have

$$\theta_{12}(\tau) = 2 \sum_{m_1 \in \mathbb{N} + \frac{1}{2}} \sum_{m_2 \in \mathbb{N} + \frac{1}{2}} \exp(i\pi(m_1^2 z_1 + m_2^2 z_2)) (\exp(2\pi i m_1 m_2 z_3) + \exp(-2\pi i m_1 m_2 z_3)).$$

We leave the term corresponding to $(m_1, m_2) = (\frac{1}{2}, \frac{1}{2})$ aside, and write

$$\begin{aligned} & \left| \frac{\theta_{12}(\tau)}{2 \exp(i\pi(z_1 + z_2)/4)} - (\exp(i\pi z_3/2) + \exp(-i\pi z_3/2)) \right| \\ & \leq \sum_{\substack{(m_1, m_2) \in (\mathbb{N} + \frac{1}{2})^2 \\ (m_1, m_2) \neq (\frac{1}{2}, \frac{1}{2})}} q_1^{m_1^2 - \frac{1}{4}} q_2^{m_2^2 - \frac{1}{4}} \cdot 2 \cosh(2\pi m_1 m_2 y_3). \end{aligned}$$

Since $|x_3| \leq \frac{1}{2}$, the argument of $\exp(i\pi z_3/2)$ is at most $\pi/4$ in absolute value. Therefore

$$|\exp(i\pi z_3/2) + \exp(-i\pi z_3/2)| \geq \exp(\pi |y_3| / 2).$$

We obtain

$$\left| \frac{\theta_{12}(\tau)}{\xi_{12}(\tau)} - 1 \right| \leq \sum_{\substack{(m_1, m_2) \in (\mathbb{N} + \frac{1}{2})^2 \\ (m_1, m_2) \neq (\frac{1}{2}, \frac{1}{2})}} q_1^{m_1^2 - \frac{1}{4}} q_2^{m_2^2 - \frac{1}{4}} \cdot 2 \cosh(2\pi(m_1 m_2 - \frac{1}{4})y_3).$$

We separate the terms corresponding to $m_2 = \frac{1}{2}$. Since $2|y_3| \leq y_1$, their contribution is bounded by

$$\sum_{m_1 \in \mathbb{N} + \frac{3}{2}} \left(q_1^{m_1^2 - \frac{1}{2}m_1} + q_1^{m_1^2 + \frac{1}{2}m_1 - \frac{1}{2}} \right) \leq \frac{q_1^{3/2}}{1 - q_1^{7/2}} + \frac{q_1^{5/2}}{1 - q_1^{9/2}}.$$

Similarly, the contribution from the terms with $m_1 = 1/2$ is bounded by

$$\frac{q_2^{3/2}}{1 - q_2^{7/2}} + \frac{q_2^{5/2}}{1 - q_2^{9/2}}.$$

For the remaining terms, we use the majoration

$$\left| 2\pi(m_1 m_2 - \frac{1}{4})y_3 \right| \leq |2\pi m_1 m_2 y_3| \leq \pi |y_3| (m_1^2 + m_2^2) \leq \frac{\pi}{2} (m_1^2 y_1 + m_2^2 y_2).$$

Thus, the rest of the sum is bounded by

$$\begin{aligned} & \sum_{m_1, m_2 \in \mathbb{N} + \frac{3}{2}} q_1^{m_1^2 - \frac{1}{4}} q_2^{m_2^2 - \frac{1}{4}} \cdot 2 \cosh\left(\frac{\pi}{2}(m_1^2 y_1 + m_2^2 y_2)\right) \\ & \leq \sum_{m_1, m_2 \in \mathbb{N} + \frac{3}{2}} \left(q_1^{\frac{1}{2}m_1^2 - \frac{1}{4}} q_2^{\frac{1}{2}m_2^2 - \frac{1}{4}} + q_1^{\frac{3}{2}m_1^2 - \frac{1}{4}} q_2^{\frac{3}{2}m_2^2 - \frac{1}{4}} \right) \\ & \leq \frac{q_1^{7/8} q_2^{7/8}}{(1 - q_1^2)(1 - q_2^2)} + \frac{q_1^{25/8} q_2^{25/8}}{(1 - q_1^6)(1 - q_2^6)}. \end{aligned}$$

This ends the proof. \square

Finally, we give another version of these estimates that we will use in the case of $\tau_4^{(n)}$.

Lemma 4.6. *Let $k \geq 2$, and let $\tau \in \mathcal{H}_2$ such that*

$$y_3(\tau)^2 \leq \frac{3}{7}y_1(\tau)y_2(\tau), \quad k|y_3(\tau)| \leq y_1(\tau).$$

Let $\alpha = \sqrt{3/7}$, and define

$$\begin{aligned} \rho'_{0,1}(q_1, q_2) &= \frac{2q_2^4}{1-q_2^5} + \frac{2q_1}{1-q_1^3} + \frac{2q_1^{1-\frac{2}{k}}q_2}{1-q_1^{3-\frac{2}{k}}} + \frac{2q_1^{1+\frac{2}{k}}q_2}{1-q_1^{3+\frac{2}{k}}} \\ &\quad + \frac{2q_1^{1-\alpha}q_2^{4(1-\alpha)}}{(1-q_1^{3(1-\alpha)})(1-q_2^{5(1-\alpha)})} + \frac{2q_1^{1+\alpha}q_2^{4(1+\alpha)}}{(1-q_1^{3(1+\alpha)})(1-q_2^{5(1+\alpha)})}, \\ \rho'_{8,9}(q_1, q_2) &= \frac{q_2^2}{1-q_2^4} + \frac{q_1^{1-\frac{1}{k}}}{1-q_1^{3-\frac{1}{k}}} + \frac{q_1^{1+\frac{1}{k}}}{1-q_1^{3+\frac{1}{k}}} \\ &\quad + \frac{q_2^{2-\frac{3}{4}\alpha}q_1^{1-\alpha}}{(1-q_2^{4(1-\alpha)})(1-q_1^{3(1-\alpha)})} + \frac{q_2^{2+\frac{3}{4}\alpha}q_1^{1+\alpha}}{(1-q_2^{4(1+\alpha)})(1-q_1^{3(1+\alpha)})}. \end{aligned}$$

Then we have

$$\begin{aligned} |\theta_j(\tau) - \xi_{0,1}(\tau)| &\leq \rho'_{0,1}(\tau) \quad \text{for } j \in \{0, 1\}, \\ \left| \frac{\theta_j(\tau)}{\xi_{8,9}(\tau)} - 1 \right| &\leq \rho'_{8,9}(\tau) \quad \text{for } j \in \{8, 9\}. \end{aligned}$$

Proof. We bound the cross-product terms using

$$\begin{aligned} |2y_3m_1m_2| &\leq \alpha y_1m_1^2 + \alpha y_2m_2^2, \\ |2y_3m_1m_2| &\leq \frac{1}{k}y_1m_1 && \text{if } m_2 = \frac{1}{2}, \\ |2y_3m_1m_2| &\leq \frac{2}{k}y_1m_1 && \text{if } m_2 = 1. \end{aligned}$$

For $j \in \{0, 1\}$, we separate the terms with $|m_2| \leq 1$ or $m_1 = 0$, and obtain

$$\begin{aligned} |\theta_j(\tau) - \xi_{0,1}(\tau)| &\leq 2 \sum_{m_2 \geq 2} q_2^{m_2^2} + 2 \sum_{m_1 \geq 1} q_1^{m_1^2} + 2 \sum_{m_2 \geq 1} q_2(q_1^{m_1^2 - \frac{2}{k}m_1} + q_1^{m_1^2 + \frac{2}{k}m_1}) \\ &\quad + 2 \sum_{m_1 \geq 1} \sum_{m_2 \geq 2} q_1^{m_1^2} q_2^{m_2^2} \cdot 2 \cosh(\alpha(y_1m_1^2 + y_2m_2^2)) \\ &\leq \frac{2q_2^4}{1-q_2^5} + \frac{2q_1}{1-q_1^3} + \frac{2q_1^{1-\frac{2}{k}}q_2}{1-q_1^{3-\frac{2}{k}}} + \frac{2q_1^{1+\frac{2}{k}}q_2}{1-q_1^{3+\frac{2}{k}}} \\ &\quad + \frac{2q_1^{1-\alpha}q_2^{4(1-\alpha)}}{(1-q_1^{3(1-\alpha)})(1-q_2^{5(1-\alpha)})} + \frac{2q_1^{1+\alpha}q_2^{4(1+\alpha)}}{(1-q_1^{3(1+\alpha)})(1-q_2^{5(1+\alpha)})}. \end{aligned}$$

For $j \in \{8, 9\}$, we separate the terms with $|m_2| = \frac{1}{2}$ or $m_1 = 0$. We obtain

$$\begin{aligned}
\left| \frac{\theta_j(\tau)}{\xi_{8,9}(\tau)} - 1 \right| &\leq q_2^{-1/4} \sum_{m_2 \in \mathbb{N} + \frac{3}{2}} q_2^{m_2^2} + \sum_{m_1 \geq 1} \left(q_1^{m_1^2 - \frac{1}{k}m_1} + q_1^{m_1^2 + \frac{1}{k}m_1} \right) \\
&\quad + q_2^{-1/4} \sum_{m_2 \in \mathbb{N} + \frac{3}{2}} \sum_{m_1 \geq 1} q_2^{m_2^2} q_1^{m_1^2} \cdot 2 \cosh(\alpha(y_1 m_1^2 + y_2 m_2^2)) \\
&\leq \frac{q_2^2}{1 - q_2^4} + \frac{q_1^{1 - \frac{1}{k}}}{1 - q_1^{3 - \frac{1}{k}}} + \frac{q_1^{1 + \frac{1}{k}}}{1 - q_1^{3 + \frac{1}{k}}} \\
&\quad + \frac{q_2^{2 - \frac{3}{4}\alpha} q_1^{1 - \alpha}}{(1 - q_2^{4(1 - \alpha)})(1 - q_1^{3(1 - \alpha)})} + \frac{q_2^{2 + \frac{3}{4}\alpha} q_1^{1 + \alpha}}{(1 - q_2^{4(1 + \alpha)})(1 - q_1^{3(1 + \alpha)})}.
\end{aligned}$$

This ends the proof. \square

Finally, when n is large, we will bound the theta constants $\theta_j(2^n \gamma_k \tau)$ for $0 \leq j \leq 3$ using the following lemma.

Lemma 4.7. *Let $\tau \in \mathcal{H}_2$.*

1. *If $r(\tau) \geq 0.4$, then the $\theta_j(\tau)$ for $0 \leq j \leq 3$ are in good position.*
2. *If $\lambda_1(\tau) \geq 0.6$, then the $\theta_j(\tau)$ for $0 \leq j \leq 3$ are in good position.*

Proof. 1. Write

$$q = \exp(-\pi r(\tau)).$$

For $0 \leq j \leq 3$, we have

$$\begin{aligned}
|\theta_j(\tau) - 1| &\leq 4q^2 + \sum_{n \in \mathbb{Z}^2, \|n\|^2 \geq 2} \exp(-\pi \lambda_1(\tau) \|n\|^2) \\
&\leq 8q^2 + 4q^4 + 8q^5 + 4q^8 + 4 \frac{1+q}{(1-q)^2} q^9.
\end{aligned}$$

by [2, Prop. 6.1]. If $q \leq 0.287$, then this quantity is less than $\sqrt{2}/2$, and the $\theta_j(\tau)$ are contained in a disk which is itself contained in a quarter plane. We have $q \leq 0.287$ when $r(\tau) \geq 0.4$.

2. Write

$$q = \exp(-\pi \lambda_1(\tau)).$$

Then for $0 \leq j \leq 3$, we have

$$|\theta_j(\tau) - 1| \leq 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4 \frac{1+q}{(1-q)^2} q^9.$$

This quantity is less than $\sqrt{2}/2$ when $\lambda_1(\tau) \geq 0.6$. \square

Lemma 4.7 is best combined with lower bounds on r or λ_1 at $\gamma_k \tau$.

Lemma 4.8. For every $\tau \in \mathcal{F}^l$, we have

$$r(\gamma_1\tau) \geq \frac{9y_1(\tau)}{34|z_1(\tau)|^2}, \quad r(\gamma_2\tau) \geq \frac{9y_2(\tau)}{34|z_2(\tau)|^2}, \quad \lambda_1(\gamma_3\tau) \geq \frac{9}{44y_2(\tau)}.$$

Proof. Since $\gamma_1^*\tau = \begin{pmatrix} z_1 & z_3 \\ 0 & -1 \end{pmatrix}$, we have

$$\text{Im}(\gamma_1\tau) = (\gamma_1^*\tau)^{-t} \text{Im}(\tau)(\gamma_1^*\tau)^{-1} = \frac{1}{|z_1|^2} \begin{pmatrix} y_1 & \alpha \\ \alpha & \beta \end{pmatrix}$$

with $\alpha = y_1x_3 - y_3x_1$, so $|\alpha| \leq \frac{3}{4}y_1$. Moreover

$$\det \text{Im}(\gamma_1\tau) = \frac{1}{|z_1|^2} \det \text{Im}(\tau)$$

and $\det \text{Im}(\tau) \geq 9/16$, so

$$\beta \leq \frac{|z_1|^2}{y_1} \det \text{Im}(\tau) + \frac{9}{16}y_1, \quad \beta \geq \frac{|z_1|^2}{y_1} \det \text{Im}(\tau) \geq \frac{9}{16}y_1.$$

Therefore

$$\lambda_1(\gamma_1\tau) \geq \frac{\det \text{Im}(\gamma_1\tau)}{\text{Tr} \text{Im}(\gamma_1\tau)} \geq \frac{y_1}{|z_1|^2} \frac{1}{1 + \frac{25}{16} \frac{y_1^2}{|z_1|^2 \det \text{Im}(\tau)}} \geq \frac{9y_1}{34|z_1|^2}.$$

We did not use the property that $y_1 \leq y_2$, so the same proof works for $\gamma_2\tau$. Finally, we consider $\gamma_3\tau$. We have

$$\gamma_3^*\tau = \begin{pmatrix} z_3 & z_2 \\ z_1 & z_3 \end{pmatrix}, \quad \text{Im}(\gamma_3\tau) = \frac{1}{|\det \tau|^2} \begin{pmatrix} \beta_1 & \alpha \\ \alpha & \beta_2 \end{pmatrix}$$

with

$$\begin{aligned} \beta_1 &= y_1|z_3|^2 + y_2|z_1|^2 - y_3(z_1\bar{z}_3 + z_3\bar{z}_1), \\ \beta_2 &= y_1|z_2|^2 + y_2|z_3|^2 - y_3(z_2\bar{z}_3 + z_3\bar{z}_2). \end{aligned}$$

We compute that

$$|\det \tau|^2 \text{Tr} \text{Im}(\gamma_3\tau) = \beta_1 + \beta_2 \leq y_1y_2^2 + y_1^2y_2 + \frac{1}{2}(y_1 + y_2 + |y_3|) \leq \frac{11}{3}y_1y_2^2$$

because $y_1y_2 \geq 3/4$. Therefore

$$\lambda_1(\gamma_3\tau) \geq \frac{3 \det \text{Im}(\tau)}{11y_1y_2^2} \geq \frac{9}{44y_2}. \quad \square$$

5 Proof of the main theorem

In this final section, we prove Theorem 1.1 by separating different regimes according to the value of n . If n is large enough, then Lemmas 4.7 and 4.8 are enough to conclude; if n is smaller, we apply the theta transformation formula (Proposition 3.2) and the bounds on other theta constants given in §4.

In the proofs, we use numerical calculations, typically in order to show that a given angle $\alpha(q)$ is smaller than $\pi/2$ for certain values of q . Such calculations are easily certified using interval arithmetic, since the functions $\alpha(q)$ we consider are simple: they are either increasing or convex functions of q .

In order to help the reader visualize the different estimates, we created pictures using GeoGebra [8] showing the worst possible cases. One of the quantities is often normalized, so that the point $(1, 0)$ appears as the center of one of the circles. As a rule, the displayed angles are between 89 and 90 degrees.

Proposition 5.1. *Let $\tau \in \mathcal{F}'$. Then for every $n \geq 0$, the theta constants $\theta_j(2^n\tau)$ for $0 \leq j \leq 3$ are in good position.*

Proof. For every $n \geq 0$, we have

$$r(2^n\tau) = 2^n r(\tau) \geq \sqrt{3}/4 \geq 0.4,$$

so the result follows from Lemma 4.7. \square

Lemma 5.2. *Let $\tau \in \mathcal{F}'$.*

1. *For every $n \geq 0$ such that $2^n \leq 8.77y_1(\tau)$, the theta constants $\theta_j(\tau_1^{(n)})$ for $j \in \{0, 2, 4, 6\}$ are in good position.*
2. *For every $n \geq 0$ such that $2^n \leq 8.77y_2(\tau)$, the theta constants $\theta_j(\tau_2^{(n)})$ for $j \in \{0, 1, 8, 9\}$ are in good position.*

Proof. We only prove the first statement, the second one being symmetric. We separate three cases: $n = 0$, $n = 1$, and $n \geq 2$.

Case 1: $n = 0$. Then $\tau_1^{(n)} = \tau$. By [15, Prop. 7.7], we have

$$\begin{aligned} |\theta_j(\tau) - 1| &\leq 0.405 && \text{for } j \in \{0, 1, 2, 3\}, \\ \left| \frac{\theta_j(\tau)}{\xi_{4,6}(\tau)} - 1 \right| &\leq 0.348 && \text{for } j \in \{4, 6\}. \end{aligned}$$

The argument of $\xi_{4,6}(\tau)$ is at most $\pi/8$ in absolute value. Therefore the angle between any two $\theta_j(\tau)$ for $j \in \{0, 1, 2, 3, 4, 6\}$ is at most

$$\frac{\pi}{8} + \arcsin(0.348) + \arcsin(0.405) < \frac{\pi}{2}.$$

Case 2: $n = 1$. We study the relative positions of $\xi_{0,2}$ and $\xi_{4,6}$ at $\tau_1^{(1)}$. Since $|2^{-n}x_1(\tau)| \leq 1/4$, the argument of $\xi_{4,6}(\tau_1^{(1)})$ is bounded by $\pi/16$ in absolute value. Moreover,

$$\left| \xi_{0,2}(\tau_1^{(1)}) \right| \geq 1, \quad \left| \arg(\xi_{0,2}(\tau_1^{(1)})) \right| \leq \arctan\left(\frac{2q_1 \sin(\pi/4)}{1 + 2q_1 \cos(\pi/4)}\right),$$

and the arguments of $\xi_{0,2}$ and $\xi_{4,6}$ have the same sign. Therefore the angle between any two $\theta_j(\tau_1^{(1)})$ for $j \in \{0, 2, 4, 6\}$ is at most

$$\max\left\{\frac{\pi}{16}, \arctan\left(\frac{2q_1 \sin(\pi/4)}{1 + 2q_1 \cos(\pi/4)}\right)\right\} + \arcsin \rho_{4,6}^{(4)}(q_1, q_2) + \arcsin(\rho_0(q_1, q_2) + 2q_2)$$

by Lemmas 4.2 and 4.4. This quantity is less than $\pi/2$ because

$$q_2(\tau_1^{(1)}) \leq \exp(-\pi\sqrt{3}), \quad q_1(\tau_1^{(1)}) \leq \exp(-\pi\sqrt{3}/8).$$

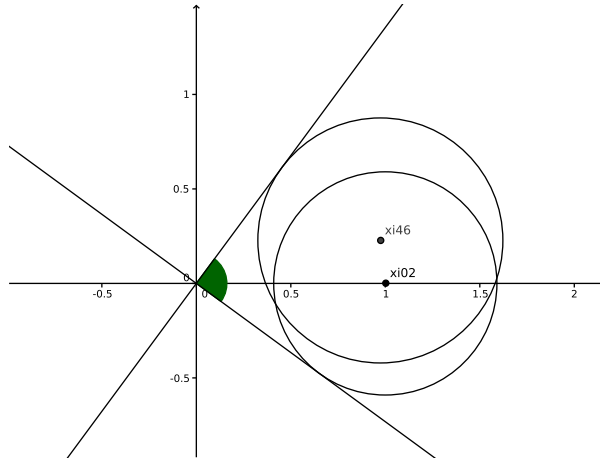
Case 3: $n \geq 2$. We proceed as in Case 2, but we now have

$$q_2(\tau_1^{(n)}) \leq \exp(-2\pi\sqrt{3}), \quad 8 \left| y_3(\tau_1^{(n)}) \right| \leq y_2(\tau_1^{(n)}), \quad \left| x_1(\tau_1^{(n)}) \right| \leq \frac{1}{8}.$$

Therefore the angle between the $\theta_j(\tau_1^{(n)})$ for $j \in \{0, 2, 4, 6\}$ is bounded by

$$\max\left\{\frac{\pi}{32}, \arctan\left(\frac{2q_1 \sin(\pi/8)}{1 + 2q_1 \cos(\pi/8)}\right)\right\} + \arcsin(\rho_0(q_1, \exp(-2\pi\sqrt{3})) + 2\exp(-2\pi\sqrt{3})) + \arcsin \rho_{4,6}^{(8)}(q_1, \exp(-2\pi\sqrt{3})).$$

This angle remains less than $\pi/2$ when $q_1(\tau_1^{(n)}) \leq 0.699$. This is the case when $2^n \geq 8.77y_1(\tau)$. The geometric situation can be represented as follows.



This ends the proof. \square

Proposition 5.3. *Let $\tau \in \mathcal{F}'$.*

1. *For every $n \geq 0$, the theta constants $(\theta_j(2^n \gamma_1 \tau))_{0 \leq j \leq 3}$ are in good position.*
2. *For every $n \geq 0$, the theta constants $(\theta_j(2^n \gamma_2 \tau))_{0 \leq j \leq 3}$ are in good position.*

Proof. By Lemma 4.8, we have

$$r(\gamma_1 \tau) \geq \frac{9 y_1}{34 |z_1|^2} \geq \frac{9 y_1}{34(1/4 + y_1^2)} \geq \frac{0.205}{y_1(\tau)}$$

because $y_1(\tau) \geq \sqrt{3}/2$. By Lemma 4.7, the $\theta_j(2^n \gamma_1 \tau)$ for $0 \leq j \leq 3$ are in good position when $2^n r(\gamma_1 \tau) \geq 0.4$. This is the case when $2^n \geq 1.96 y_1$. On the other hand, Lemma 5.2 applies when $2^n \leq 8.77 y_1$. The second statement is proved in the same way. \square

Lemma 5.4. *Let $\tau \in \mathcal{F}'$. Then, for every $n \geq 0$ such that $2^n \leq 1.66 y_1$, the theta constants $\theta_j(\tau_3^{(n)})$ for $j \in \{0, 4, 8, 12\}$ are in good position.*

Proof. Write $q = q_1(\tau_3^{(n)})$ for short. We separate two cases: $n \geq 1$, and $n = 0$.

Case 1: $n \geq 1$. In this case, we have

$$\left| x_j(\tau_3^{(n)}) \right| \leq 1/4, \quad 1 \leq j \leq 3.$$

Recall that

$$\begin{aligned} \xi_0 &= 1 + 2 \exp(i\pi z_1) + 2 \exp(i\pi z_2), \\ \xi_{4,6} &= 2 \exp\left(i\pi \frac{z_1}{4}\right), \\ \xi_{8,9} &= 2 \exp\left(i\pi \frac{z_2}{4}\right), \\ \xi_{12} &= \exp\left(i\pi \frac{z_1(\tau) + z_2(\tau)}{4}\right) \left(\exp\left(i\pi \frac{z_3(\tau)}{2}\right) + \exp\left(-i\pi \frac{z_3(\tau)}{2}\right) \right). \end{aligned}$$

Therefore, by Lemmas 4.2 to 4.5,

- The angle between $\theta_4(\tau_3^{(n)})$ and $\theta_8(\tau_3^{(n)})$ is bounded by

$$\frac{\pi}{8} + 2 \arcsin \rho_{4,6}^{(2)}(q, q).$$

- The angle between $\theta_4(\tau_3^{(n)})$ (or θ_8) and $\theta_0(\tau_3^{(n)})$ is bounded by

$$\frac{\pi}{16} + \arcsin \rho_{4,6}^{(2)}(q, q) + 2q \sin(\pi/4) + \arcsin \rho_0(q, q).$$

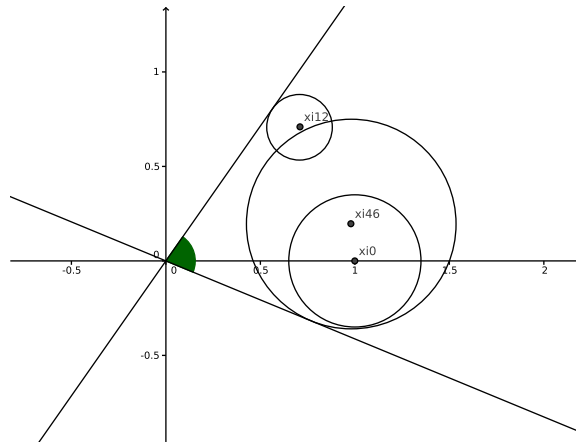
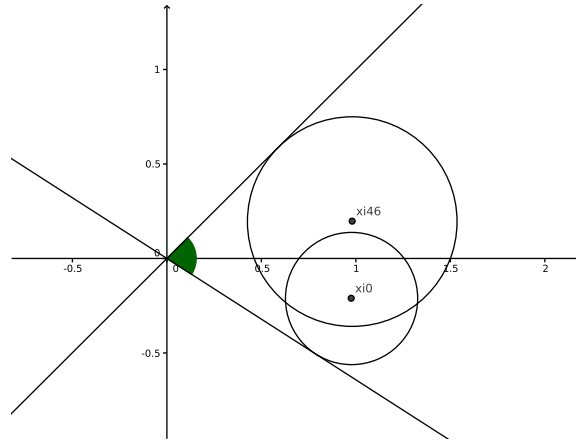
- The angle between $\theta_{12}(\tau_3^{(n)})$ and $\theta_4(\tau_3^{(n)})$ (or θ_8) is bounded by

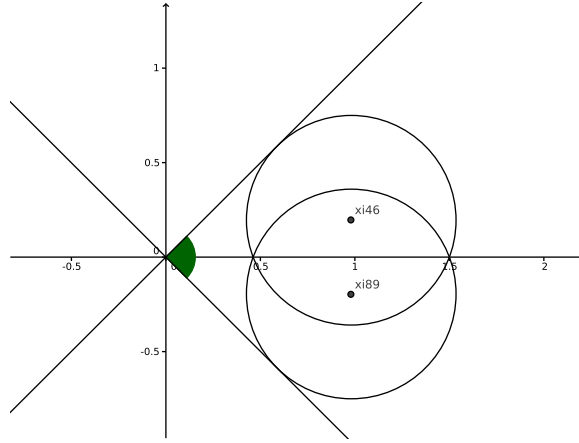
$$\frac{3\pi}{16} + \arcsin \rho_{12}(q, q) + \arcsin \rho_{4,6}^{(2)}(q, q).$$

- The angle between $\theta_{12}(\tau_3^{(n)})$ and $\theta_0(\tau_3^{(n)})$ is bounded by

$$\frac{\pi}{4} + \arcsin \rho_{12}(q, q) + \arcsin \rho_0(q, q).$$

All these quantities remain less than $\pi/2$ when $q \leq 0.151$. This is the case when $2^n \leq 1.66y_1$. Geometrically, these estimations can be represented as follows.





Case 2: $n = 0$. In this case, we have $q \leq \exp(-\pi\sqrt{3}/2)$. Therefore

- The angle between θ_4 and θ_8 is bounded by

$$\frac{\pi}{4} + 2 \arcsin \rho_{4,6}^{(2)}(q, q) < \frac{\pi}{2}.$$

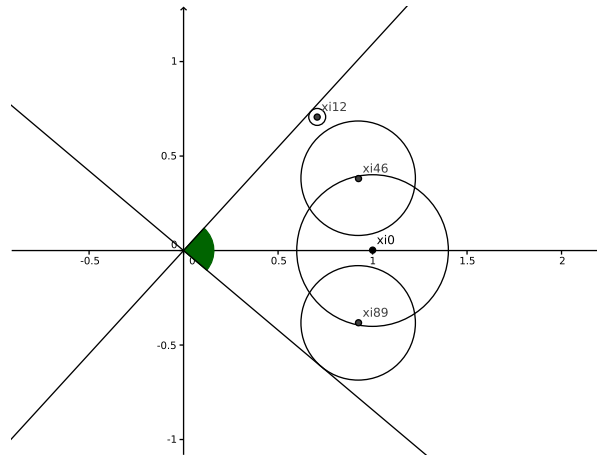
- The angle between θ_4 (or θ_8) and θ_0 is bounded by

$$\frac{\pi}{8} + \arcsin \rho_{4,6}^{(2)}(q, q) + \arcsin(\rho_0(q, q) + 4q) < \frac{\pi}{2}.$$

- The angle between θ_{12} and θ_4 (or θ_8) is bounded by

$$\frac{3\pi}{8} + \arcsin \rho_{12}(q, q) + \arcsin \rho_{4,6}^{(2)}(q, q) < \frac{\pi}{2}.$$

These estimations can be represented as follows:



We finally study the angle between θ_{12} and θ_0 . The argument of $\xi_{12}(\tau)$ is $x_1/4 + x_2/4 + \beta$ with $\beta = \arg(\exp(i\pi z_3/2) + \exp(-i\pi z_3/2))$. Up to conjugating, we may assume that $y_3 \geq 0$ and $x_3 \geq 0$. Then

$$\exp(i\pi z_3/2) + \exp(-i\pi z_3/2) = \exp(-i\pi z_3/2)(1 + \exp(i\pi z_3))$$

so

$$\beta + \frac{\pi x_3}{2} \geq \arctan\left(\frac{q_3 \sin(\pi x_3)}{1 + q_3}\right) \geq \arctan\left(\frac{2x_3 q_3}{1 + q_3}\right).$$

In general, we have

$$|\beta| \leq \frac{\pi}{4} - \arctan\left(\frac{q^{1/2}}{1 + q^{1/2}}\right).$$

On the other hand,

$$1 \leq \operatorname{Re}(\xi_0(\tau)) \leq 1 + 4q, \quad \operatorname{Im}(\xi_0(\tau)) = 2q_1 \sin(\pi x_1) + 2q_2 \sin(\pi x_2).$$

We discuss two cases according to the signs of x_1 and x_2 :

- If x_1 and x_2 have opposite signs, then the angle between θ_{12} and θ_0 is at most

$$\frac{3\pi}{8} + \arctan(2q) + \arcsin \rho_{12}(q, q) + \arcsin \rho_0(q, q).$$

- If x_1 and x_2 have the same sign, say positive, then

$$\frac{x_1 + x_2}{4} - \arg \xi_0(\tau) \leq \frac{x_1 + x_2}{4}.$$

Therefore the angle between θ_{12} and θ_0 is at most

$$\frac{\pi}{2} - \arctan\left(\frac{q^{1/2}}{1 + q^{1/2}}\right) + \arcsin \rho_{12}(q, q) + \arcsin \rho_0(q, q).$$

This function of q is not increasing, but it is convex.

A numerical investigation shows that both quantities remain less than $\pi/2$ when $q \leq \exp(\sqrt{3}/2)$. This ends the proof. \square

Lemma 5.5. *Let $\tau \in \mathcal{F}'$, and let $n_0 \in \mathbb{N}$ such that $2^{n_0} > 1.66y_1$. Then, for every $n \geq n_0$ such that $2^n \leq 4.2y_2(\tau)$, the theta constants $\theta_j(\tau_4^{(n)})$ for $j \in \{0, 1, 8, 9\}$ are in good position.*

Proof. By assumption, we have $y_1(\tau_4^{(n)}) \geq \frac{3}{4} \cdot 1.66 \geq 1.24$, so $q_1(\tau_4^{(n)}) \leq 0.021$. Moreover we must have $n \geq 1$, so by Lemma 3.4, $|x_2(\tau_4^{(n)})| \leq 9/16$, and

$$|y_3(\tau_4^{(n)})| \leq \frac{3}{8}y_1(\tau_4^{(n)}).$$

Therefore, we can apply Lemma 4.6 with $k = 8/3$: we have

$$\begin{aligned} \left| \theta_j(\tau_4^{(n)}) - \xi_{0,1}(\tau_4^{(n)}) \right| &\leq \rho_{0,1}'^{(8/3)}(0.021, q_2(\tau_4^{(n)})) \quad \text{for } j \in \{0, 1\}, \\ \left| \frac{\theta_j(\tau_4^{(n)})}{\xi_{8,9}(\tau_4^{(n)})} - 1 \right| &\leq \rho_{8,9}'^{(8/3)}(0.021, q_2(\tau_4^{(n)})) \quad \text{for } j \in \{8, 9\}. \end{aligned}$$

Let us investigate the difference between the arguments of $\xi_{8,9}(\tau_4^{(n)})$ and $\xi_{0,1}(\tau_4^{(n)})$. Both have the sign of $x_2(\tau_4^{(n)})$, which we may assume to be positive. If the argument of $\xi_{8,9}$ is largest, then the difference is bounded by

$$\arg \xi_{8,9}(\tau_4^{(n)}) \leq \frac{9\pi}{64}.$$

If the argument of $\xi_{0,1}$ is largest, we distinguish two cases. If $x_2(\tau_4^{(n)}) \geq \frac{3\pi}{8}$, then

$$\arg \xi_{0,1}(\tau_4^{(n)}) - \arg \xi_{8,9}(\tau_4^{(n)}) \leq \arctan\left(\frac{2q_2}{1 + 2q_2 \cos(9\pi/16)}\right) - \frac{3\pi}{32}.$$

On the other hand, if $x_2(\tau) \leq 3\pi/8$, then

$$\arg \xi_{0,1}(\tau_4^{(n)}) - \arg \xi_{8,9}(\tau_4^{(n)}) \leq \arg \xi_{0,1}(\tau_4^{(n)}) \leq \arctan\left(\frac{2q_2 \sin(3\pi/8)}{1 + 2q_2 \cos(3\pi/8)}\right)$$

Note that $\left| \xi_{0,1}(\tau_4^{(n)}) \right|$ is always greater than $\cos(\pi/16)$. Therefore the angle between the $\theta_j(\tau_4^{(n)})$ for $j \in \{0, 1, 8, 9\}$ is at most

$$\begin{aligned} \max \left\{ \frac{9\pi}{64}, \arctan\left(\frac{2q_2}{1 + 2q_2 \cos(9\pi/16)}\right) - \frac{3\pi}{32}, \arctan\left(\frac{2q_2 \sin(3\pi/8)}{1 + 2q_2 \cos(3\pi/8)}\right) \right\} \\ + \arcsin \rho_{8,9}'^{(8/3)}(0.021, q_2) + \arcsin \frac{\rho_{0,1}'^{(8/3)}(0.021, q_2)}{\cos(\pi/16)}. \end{aligned}$$

This quantity is less than $\pi/2$ when $q_2(\tau_4^{(n)}) \leq 0.38$. Since $y_2(\tau_4^{(n)}) \geq \frac{3}{2^{n+2}}y_2(\tau)$ by Lemma 3.4, this is the case when $2^n \leq 2.43y_2(\tau)$.

On the other hand, if $2^n > 2.43y_2(\tau)$, then we must have $n \geq 2$. Moreover

$$y_1(\tau_4^{(n)}) > 2.43 \frac{y_1(\tau)y_2(\tau)}{|z_1(\tau)|^2} > 1.82,$$

so $q_1(\tau_4^{(n)}) < 0.0033$. Then, the angle bound improves to

$$\begin{aligned} \max \left\{ \frac{9\pi}{128}, \arctan\left(\frac{2q_2 \sin(9\pi/32)}{1 + 2q_2 \cos(9\pi/32)}\right) \right\} \\ + \arcsin \rho_{8,9}'^{(16/3)}(0.0033, q_2) + \arcsin \rho_{0,1}'^{(16/3)}(0.0033, q_2). \end{aligned}$$

This quantity is less than $\pi/2$ when $q_2(\tau_4^{(n)}) \leq 0.571$, and the latter inequality holds when $2^n \leq 4.2y_2(\tau)$. \square

Proposition 5.6. *Let $\tau \in \mathcal{F}'$. Then, for every $n \geq 0$, the theta constants $\theta_j(2^n \gamma_3 \tau)$ for $0 \leq j \leq 3$ are in good position.*

Proof. By Lemma 4.8, we have

$$\lambda_1(\gamma_3 \tau) \geq \frac{9}{44y_2(\tau)}.$$

Therefore, by Lemma 4.7, the theta constants are in good position as soon as

$$2^n \frac{9}{44y_2(\tau)} \geq 0.6, \quad \text{or} \quad 2^n \geq 2.94y_2(\tau).$$

When n is smaller, we use the transformation formulas. Lemma 5.4 applies when $2^n \leq 1.66y_1(\tau)$, and Lemma 5.5 applies when $1.66y_1(\tau) < 2^n \leq 4.2y_2(\tau)$. \square

Put together, Propositions 5.1, 5.3 and 5.6 complete the proof of our main theorem.

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