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# A categorical framework for congruence of applicative bisimilarity in higher-order languages

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**Abstract.** Applicative bisimilarity is a coinductive characterisation of observational equivalence in call-by-name lambda-calculus, introduced by Abramsky in 1990. Howe (1989) gave a direct proof that it is a congruence. In previous work with Borthelle (2020), we abstract over this result by proposing a categorical framework for specifying operational semantics, in which we prove that (an abstract analogue of) applicative bisimilarity is automatically a congruence.

However, the framework presents a few infelicities: (1) it requires a non-trivial refinement of the standard approach of Fiore, Plotkin, and Turi (1999) based on *monoid algebras* for specifying syntax with variable binding; (2) it relies on so-called *prebisimulations* instead of the more standard notion of bisimulation by lifting; (3) one of the axioms, called *weak compositionality*, feels *ad hoc*; (4) the proofs involve directed unions of relations leading to quite a few painful inductions. In this paper, we rectify all of these deficiencies.

**Keywords:** operational semantics · category theory · bisimilarity · congruence · Howe’s method.

## 1 Introduction

This paper is a contribution to the search for efficient and high-level mathematical tools to specify and reason about programming languages. The first such tool arguably goes back at least to Turi and Plotkin [37], who coined the name “Mathematical Operational Semantics”, and proved a general congruence theorem for bisimilarity.

In previous work [10], we proved a similar abstract congruence theorem for *applicative* bisimilarity, a behavioural/observational equivalence introduced by Abramsky [1] for big-step, call-by-name  $\lambda$ -calculus. Let us briefly summarise the approach.

- (i) We first axiomatise (labelled) transition systems as objects of a category  $\mathbf{C}$ , in such a way that
- there is a forgetful functor  $\mathbf{C} \rightarrow \mathbf{C}_0$ , intuitively returning the (potentially structured) set of states of a transition system;
  - bisimulation and bisimilarity may be defined for any transition system.

(ii) We then assume that  $\mathbf{C}_0$  is *skew monoidal* [36], and define models of the syntax to be *monoid algebras* for a given *structurally strong* endofunctor  $\Sigma_0$  on  $\mathbf{C}_0$ . This is a generalisation of Fiore, Plotkin, and Turi’s framework, which involves a pointed strong endofunctor on a monoidal category. Monoid algebras, a.k.a.  $\Sigma_0$ -monoids, are  $\Sigma_0$ -algebras equipped with compatible monoid structure – which models capture-avoiding substitution. This yields a category  $\Sigma_0\text{-mon}$  of models, which has an initial object  $\mathbf{Z}_0$ . In the case of call-by-name  $\lambda$ -calculus,  $\mathbf{Z}_0$  is precisely the syntax.

(iii) This category  $\Sigma_0\text{-mon}$  induces by pullback a category  $\Sigma_0\text{-Mon}$  (capital ‘M’!) of transition systems whose states are equipped with  $\Sigma_0$ -monoid structure. We call these *transition monoid algebras*, or *transition  $\Sigma_0$ -monoids*. The relevant notions of bisimulation and bisimilarity for such objects are defined as in (i), but for *substitution-closed* relations.

(iv) We then define models of the dynamics to be certain algebras, called *vertical*, for an endofunctor  $\Sigma_1$  on  $\Sigma_0\text{-Mon}$ . There is an initial vertical algebra  $\mathbf{Z}$ , which in the case of call-by-name  $\lambda$ -calculus is the syntactic transition system. Standard applicative bisimilarity coincides with substitution-closed bisimilarity for this initial vertical algebra.

(v) Finally, following an abstract analogue of Howe’s method, we show that, under suitable hypotheses called *weak compositionality* and *cellularity*, substitution-closed bisimilarity on  $\mathbf{Z}$  is a congruence.

According to one of the referees:

[S]ome ingredients feel ad hoc in the current formulation (e.g., the definition of structural strength, weak compositionality [...]). I expect that [...] some suitable adjustment of the perspective allows for a simpler framework.

In this paper, we rectify these deficiencies, and more.

(a) The starting point is to give a better type to  $\Sigma_1$ , the endofunctor for specifying the dynamics. This small change allows us to work with standard monoid algebras, and even build weak compositionality directly into the framework. We thus solve both problems at once. As a bonus, this allows us to relax the hypothesis that the tensor product is familial.

(b) In [10], because Howe’s closure operates only at the level of states, we work mostly with *prebisimulations*, in the sense of [22, §5.1]. This notion is designed to detect when the state part of a relation underlies a bisimulation, regardless of what it does on transitions. However, it feels more *ad hoc* than the standard definition of bisimulation by lifting [24]. In this paper, we extend Howe’s closure to transitions, thus avoiding prebisimulations entirely.

(c) In [10], we rely on directed unions of relations, which leads to quite a few, rather painful proofs by induction. Here, we use higher-level methods to construct Howe’s closure, essentially through categorification and algebraicisation. Namely:

1. We define bisimilarity as the final object not in some partially-ordered set of relations as usual, but in some category of spans (see also [8]).

2. Furthermore, we define Howe’s closure directly as a free monoid algebra for a suitable pointed strong endofunctor on spans.
  3. More generally, we systematically rely on universal properties, which simplifies a significant number of proofs.
- (d) Finally, we put less emphasis on cellularity, viewing it only as a sufficient condition for a perhaps more natural hypothesis which already appeared in a slightly different form in [35], namely the fact that  $\Sigma_1$  preserves functional bisimulations. We use this to treat the original case of call-by-name  $\lambda$ -calculus, as well as a new example, the non-deterministic  $\lambda$ -calculus of [33, §7].

*Related work* Plotkin and Turi’s *bialgebraic semantics* [37] and its few variants [12,35] prove abstract congruence theorems for bisimilarity. However, they do not cover higher-order languages like the  $\lambda$ -calculus, let alone applicative bisimilarity. This was one of the main motivations for our work. Among more recent work, quite some inspiration was drawn from Ahrens et al. [5,21], notably in the use of vertical algebras. However, a difference is that we do not insist that transitions be stable under substitution. Links with other relevant work by, e.g., Bodin et al. [9] or Dal Lago et al. [25], though desirable, remain unclear, perhaps because of the very different methods used. Furthermore, the cellularity used here is close to but different from the  $\mathbf{T}_s^y$ -familiarity of [22]. It would be instructive to better understand potential links between the two. Finally, let us mention recent work which, just like ours, establishes abstract versions of standard constructions and theorems in programming language theory like type soundness [6] or gluing [16,17].

*Plan* In §2, we start by briefly recalling call-by-name  $\lambda$ -calculus and applicative bisimilarity. We then explain how to view the latter as substitution-closed bisimilarity, and sketch Howe’s method. In §3, we then give a brief overview of the new framework by example, including a recap on monoid algebras and a statement of the main theorem. We then dive into the technical core of the paper by presenting our framework for transition systems and bisimilarity (§4), operational semantics (§5), and then substitution-closed bisimilarity and the main result (Theorem 6.13), together with a high-level proof sketch (§6). In §7, we reformulate the main hypothesis of Theorem 6.13 using cellularity, which allows us to use well-known results from weak factorisation systems as sufficient conditions. The full proof of Theorem 6.13 is given in §8. We then conclude and give some perspectives on future work in §9.

*Notation* We often conflate natural numbers  $n \in \mathbb{N}$  with corresponding sets  $\{1, \dots, n\}$ . For all sets  $X$  and objects  $C$  of a given category, we denote by  $X \cdot C$  the  $X$ -fold coproduct of  $C$  with itself, i.e.,  $\sum_{x \in X} C$ . Let  $\mathbf{Cat}$  denote the category of small categories, and  $\mathbf{CAT}$  the category of locally small categories. Let  $\widehat{\mathbb{C}}$  denote the category of (contravariant) presheaves on  $\mathbb{C}$ , and  $\mathbf{y} : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  the Yoneda embedding, mapping  $c$  to  $\mathbb{C}(-, c)$ . Given a presheaf  $F \in \widehat{\mathbb{C}}$ , an element  $x \in F(c)$ , and a morphism  $c \xrightarrow{f} c'$ , we sometimes denote  $F(f)(x)$  by  $x \cdot f$ . Given

two categories  $C_1$  and  $C_2$ , we denote by  $[C_1, C_2]$  the functor category between them. In a category with products, we interchangeably use spans  $X \leftarrow R \rightarrow Y$  and their pairings  $R \rightarrow X \times Y$ , sometimes also calling the latter spans.

Any finitary endofunctor  $F$  on any cocomplete category admits by [32, Theorem 2.1] an initial algebra, which we denote by  $\mathbf{Z}_F$ . Although this is detailed below, we prefer to avoid confusion and warn the reader that we also use  $\mathbf{Z}_F$  for the initial  $F$ -monoid, for any *pointed strong* endofunctor  $F$  on any nice monoidal category. Throughout the paper, when not explicitly attached to some  $F$ ,  $\mathbf{Z}$  is shorthand for  $\mathbf{Z}_{\mathbb{Z}_1}$  (see, e.g., Proposition 3.16 or Theorem 5.17).

Finally, let us fix some notation about weak factorisation systems. In any category  $\mathcal{C}$ , we say that a morphism  $f : A \rightarrow B$  has the (weak) left lifting property w.r.t.  $g : C \rightarrow D$  when for all commuting squares as on the right there is a lifting  $k$  as shown that makes both triangles commute. Equivalently, we say that  $g$  has the right lifting property w.r.t.  $f$ , and write  $f \pitchfork g$ . Given a fixed set  $\mathbb{J}$  of morphisms, the set of morphisms  $g$  such that  $j \pitchfork g$  for all  $j \in \mathbb{J}$  is denoted by  $\mathbb{J}^\pitchfork$ . Similarly, the set of morphisms  $f$  such that  $f \pitchfork j$  for all  $j \in \mathbb{J}$  is denoted by  $\pitchfork \mathbb{J}$ . In particular, if  $f \in \pitchfork(\mathbb{J}^\pitchfork)$  and  $g \in \mathbb{J}^\pitchfork$ , then  $f \pitchfork g$ . If  $\mathcal{C}$  is locally presentable [2], then  $(\pitchfork(\mathbb{J}^\pitchfork), \mathbb{J}^\pitchfork)$  forms a *weak factorisation system*, in the sense that additionally any morphism  $f : X \rightarrow Y$  factors as  $X \xrightarrow{l} Z \xrightarrow{r} Y$  with  $l \in \pitchfork(\mathbb{J}^\pitchfork)$  and  $r \in \mathbb{J}^\pitchfork$  (see [23, Theorem 2.1.14]). Morphisms in  $\mathbb{J}^\pitchfork$  are generically called *fibrations*, while morphisms in  $\pitchfork(\mathbb{J}^\pitchfork)$  are called *cofibrations*. Let us conclude with the following easy, yet helpful result.

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ f \downarrow & \nearrow k & \downarrow g \\ B & \xrightarrow{v} & D \end{array}$$

**Lemma 1.1.** *If the domains and codomains of maps in  $\mathbb{J}$  are finitely presentable, then fibrations are stable under filtered colimits in the arrow category.*

## 2 A brief review of Howe's method

### 2.1 Applicative bisimilarity

Consider the standard, big-step presentation of call-by-name  $\lambda$ -calculus:

$$\frac{}{\lambda x.e \Downarrow \lambda x.e} \qquad \frac{e_1 \Downarrow \lambda x.e'_1 \quad e'_1[x \mapsto e_2] \Downarrow e_3}{e_1 e_2 \Downarrow e_3}$$

Applicative bisimilarity is standardly introduced in two stages. First, one defines applicative bisimulation on closed terms.

**Definition 2.1.** *A relation  $R$  over closed  $\lambda$ -terms is an applicative bisimulation iff  $e_1 R e_2$  and  $e_1 \Downarrow \lambda x.e'_1$  entail the existence of  $e'_2$  such that  $e_2 \Downarrow \lambda x.e'_2$  and, for all terms  $e$ ,  $e'_1[x \mapsto e] R e'_2[x \mapsto e]$ , and symmetrically.*

Applicative bisimulations are closed under unions, and so there is a largest applicative bisimulation called *applicative bisimilarity* and denoted by  $\sim$ . Then comes the second stage:

**Definition 2.2.** *The open extension of a relation  $R$  on closed terms is the relation  $R^\otimes$  on potentially open terms such that  $e R^\otimes e'$  iff for all closed substitutions  $\sigma$  covering all involved free variables we have  $e[\sigma] R e'[\sigma]$ .*

Let us readily notice the following alternative characterisation of open extension.

**Definition 2.3.** *A relation  $S$  on open terms is substitution-closed iff for all  $e S e'$  and (potentially open) substitutions  $\sigma$ , we have  $e[\sigma] S e'[\sigma]$ .*

**Lemma 2.4.** *The open extension of any relation  $R$  is the greatest substitution-closed relation contained in  $R$  on closed terms.*

*Proof.* Let us first show that  $R^\otimes$  is substitution-closed. For any  $e_1 R^\otimes e_2$  and  $\sigma$ , we want to show  $e_1[\sigma] R^\otimes e_2[\sigma]$ . For this, we in turn need to show that for all closing substitutions  $\gamma$ , we have  $e_1[\sigma][\gamma] R e_2[\sigma][\gamma]$ . But  $e_i[\sigma][\gamma] = e_i[\gamma \circ \sigma]$  and  $\gamma \circ \sigma$  is closing, so because  $e_1 R^\otimes e_2$ , by definition of open extension, we get  $e_1[\sigma][\gamma] R e_2[\sigma][\gamma]$  as desired.

Let us now show that  $R^\otimes$  is the greatest substitution-closed relation containing  $R$  on closed terms. For this, consider any substitution-closed  $R'$  contained in  $R$  on closed terms: for all  $e R' e'$ , by substitution-closedness, we have  $e[\sigma] R' e'[\sigma]$  for all closing  $\sigma$ . So because  $R'$  is contained in  $R$  on closed terms, we further have  $e[\sigma] R e'[\sigma]$ . This proves  $e R^\otimes e'$ , and so thus  $R' \subseteq R^\otimes$  as desired.  $\square$

The result we wish to prove in the abstract setting is the following (see [31] for a historical account).

**Theorem 2.5.** *The open extension  $\sim^\otimes$  of applicative bisimilarity is a congruence: it is an equivalence relation, and furthermore it is context-closed, i.e.,*

- $e_1 \sim^\otimes e_2$  entails  $\lambda x.e_1 \sim^\otimes \lambda x.e_2$  for all  $x$ ;
- $e_1 \sim^\otimes e_2$  and  $e'_1 \sim^\otimes e'_2$  entail  $e_1 e'_1 \sim^\otimes e_2 e'_2$ .

Proving that  $\sim^\otimes$  is an equivalence relation is in fact straightforward. In the following, we focus on the context-closedness property.

## 2.2 Howe's method

Howe's method for proving this consists in considering a suitable context-closed relation  $\sim^\bullet$ , containing  $\sim^\otimes$  by construction, and then to show that it is a bisimulation. By maximality of  $\sim^\otimes$ , we then also have  $\sim^\bullet \subseteq \sim^\otimes$  hence both relations coincide and  $\sim^\otimes$  is context-closed as desired. However, as explained in [10, §5.1], the presence of a substitution in the premises of a transition rule seems to require  $\sim^\bullet$  to be closed under *heterogeneous* substitution, in the sense that, e.g., if  $e_1 \sim^\bullet e'_1$  and  $e_2 \sim^\bullet e'_2$  (for open terms), then  $e_1[x \mapsto e_2] \sim^\bullet e'_1[x \mapsto e'_2]$ . The problem is that building this into the definition of  $\sim^\bullet$  leads to difficulties in the proof that it is a bisimulation. Howe's workaround consists in requiring  $\sim^\bullet$  to be closed under sequential composition with  $\sim^\otimes$  from the outset. Coupling this right

action with context closedness, we thus define  $\sim^\bullet$  as the smallest context-closed relation satisfying the rules

$$\frac{}{x \sim^\bullet x} \qquad \frac{e \sim^\bullet e' \quad e' \sim^\otimes e''}{e \sim^\bullet e''} .$$

By construction,  $\sim^\bullet$  is reflexive and context-closed. By reflexivity and the second rule, it also contains  $\sim^\otimes$ . Finally, we clearly have  $\sim^\bullet; \sim^\otimes \subseteq \sim^\bullet$ . It takes an induction to prove stability under heterogeneous substitution, but to give a feel for it, in the base case where  $e_1 \sim^\otimes e'_1$ , we have

$$e_1[x \mapsto e_2] \sim^\bullet e_1[x \mapsto e'_1] \sim^\otimes e'_1[x \mapsto e'_2]$$

by context closedness of  $\sim^\bullet$  and substitution closedness of  $\sim^\otimes$ , so we conclude by  $\sim^\bullet; \sim^\otimes \subseteq \sim^\bullet$ .

The initial plan was to show that  $\sim^\bullet$  is a bisimulation and deduce that it coincides with  $\sim^\otimes$ . It can in fact be slightly optimised by first showing that  $\sim^\bullet$  is a simulation, and then that its transitive closure  $(\sim^\bullet)^+$  is symmetric. The relation  $(\sim^\bullet)^+$  is also a substitution-closed simulation, hence by symmetry a substitution-closed bisimulation. This entails the last inclusion in the chain  $\sim^\otimes \subseteq \sim^\bullet \subseteq (\sim^\bullet)^+ \subseteq \sim^\otimes$ , showing that all relations coincide. Finally, because  $\sim^\bullet$  is context-closed, so is  $\sim^\otimes$ , as desired.

### 2.3 Non-standard presentation

For technical convenience, we adopt a slightly different presentation of the transition rules, where the evaluation relation relates closed terms to terms with just one potential free variable.

$$\frac{}{\lambda x.e \Downarrow e} \qquad \frac{e_1 \Downarrow e'_1 \quad e'_1[e_2] \Downarrow e_3}{e_1 \ e_2 \Downarrow e_3}$$

Here  $e'_1[e_2]$  denotes substitution of the unique potential free variable in  $e'_1$  by  $e_2$ . We will see below that, with this transition system, the essentially standard notion of bisimulation coupled with the substitution-closedness requirement yields applicative bisimilarity.

## 3 Overview by example

In this section, we describe one particular instance of our framework, which models call-by-name  $\lambda$ -calculus.

### 3.1 Syntax

Let us first define the syntax of  $\lambda$ -calculus, following [18], as an initial<sup>1</sup> object in a suitable category of models. Very roughly, a model of  $\lambda$ -calculus syntax should

<sup>1</sup> This pattern is advocated by the approach of Initial Semantics, where initiality provides a recursion principle.

be something equipped with operations modelling abstraction and application, but also with substitution. Furthermore, certain natural compatibility axioms should be satisfied, e.g.,

$$(e_1 e_2)[\sigma] = e_1[\sigma] e_2[\sigma]. \quad (1)$$

A natural setting for specifying such operations is the functor category  $\mathbf{C}_0 := [\mathbb{F}, \mathbf{Set}]$ , where  $\mathbb{F} \hookrightarrow \mathbf{Set}$  denotes the full subcategory spanning all sets of the form  $n$  (i.e.,  $\{1, \dots, n\}$ , recalling notation from §1). For any  $X \in [\mathbb{F}, \mathbf{Set}]$  and  $n \in \mathbb{F}$ , we think of  $X(n)$  as a set of ‘terms’ with potential free variables in  $\{1, \dots, n\}$ . The action of  $X$  on morphisms  $n \rightarrow n'$  is thought of as variable renaming. Returning to operations, being equipped with abstraction is the same as being a  $\Sigma_\lambda$ -algebra, where  $\Sigma_\lambda : \mathbf{C}_0 \rightarrow \mathbf{C}_0$  is defined by  $\Sigma_\lambda(X)(n) = X(n+1)$ . Similarly, for specifying both application and abstraction, we consider  $\Sigma_0(X)(n) = X(n+1) + X(n)^2$ .

Let us now consider substitution. The idea here is to equip  $\mathbf{C}_0$  with monoidal structure  $(\otimes, I)$ , such that

- elements of  $(X \otimes Y)(n)$  are like *explicit substitutions*  $x(\sigma)$ , where  $x \in X(p)$  and  $\sigma : p \rightarrow Y(n)$  for some  $p$ , considered equivalent up to some standard equations<sup>2</sup>;
- elements of  $I(n) := \{1, \dots, n\}$  are merely variables.

Being equipped with substitution (and variables) is thus the same as being a *monoid* for this tensor product:

- the multiplication  $m_X : X \otimes X \rightarrow X$  maps any formal, explicit substitution  $x(\sigma)$  to an actual substitution  $x[\sigma]$ , and
- the unit  $e_X : I \rightarrow X$  injects variables into terms.

Finally, how do we enforce equations such as (1)? Well, we first describe how substitution is supposed to commute with operations by providing a *pointed strength*, i.e., a natural transformation with components  $st_{X,Y} : \Sigma_0(X) \otimes Y \rightarrow \Sigma_0(X \otimes Y)$ , where  $X \in \mathbf{C}_0$  and  $Y \in I/\mathbf{C}_0$ . For modelling (1), we would in particular define  $st_{X,Y}$  to map any  $(in_2(x_1, x_2))(\sigma)$  to  $in_2(x_1(\sigma), x_2(\sigma))$ , for all  $x_1, x_2 \in X(p)$  and  $\sigma : p \rightarrow Y(n)$ .

Why do we need to take  $Y \in I/\mathbf{C}_0$  instead of  $\mathbf{C}_0$ , you’ll ask. Because of abstraction: supposing that  $Y$  is equipped with a point  $e_Y : I \rightarrow Y$ , we may define  $\sigma^\uparrow : p+1 \rightarrow Y(n+1)$  by copairing

$$p \xrightarrow{\sigma} Y(n) \xrightarrow{Y(in_1)} Y(n+1) \quad \text{and} \quad 1 = I(1) \xrightarrow{(e_Y)_1} Y(1) \xrightarrow{Y(in_2)} Y(n+1).$$

We use this in defining the strength to map any  $in_1(x)(\sigma)$ , where  $x \in X(p+1)$  and  $\sigma : p \rightarrow Y(n)$ , to  $in_1(x(\sigma^\uparrow))$ .

<sup>2</sup> In [10], we instead considered a skew-monoidal variant where the product does not enforce any standard equation.



When we already have monoid and  $\Sigma_0$ -algebra structure, the desired equations such as (1) are then imposed by requiring the following diagram to commute.

$$\begin{array}{ccc}
 \Sigma_0(X) \otimes X & \xrightarrow{st_{X,X}} & \Sigma_0(X \otimes X) & \xrightarrow{\Sigma_0(m_X)} & \Sigma_0(X) \\
 \nu_X \otimes X \downarrow & & & & \downarrow \nu_X \\
 X \otimes X & \xrightarrow{\quad m_X \quad} & & & X
 \end{array} \tag{2}$$

**Definition 3.1.** For any finitary, pointed strong endofunctor  $\Sigma_0$ , a monoid algebra for  $\Sigma_0$ , or a  $\Sigma_0$ -monoid, is a  $\Sigma_0$ -algebra  $(X, \nu_X : \Sigma_0(X) \rightarrow X)$ , equipped with monoid structure  $(m_X : X \otimes X \rightarrow X, e_X : I \rightarrow X)$ , such that (2) commutes. A  $\Sigma_0$ -monoid morphism is a morphism in  $\mathbf{C}_0$  which is both a monoid morphism and a  $\Sigma_0$ -algebra morphism.

Let  $\Sigma_0\text{-mon}$  denote the category of  $\Sigma_0$ -monoids and morphisms between them.

The following result is established in [18,15,10] in slightly different settings. See Proposition 5.1 below for a general and rigorous statement.

**Proposition 3.2.** For any finitary, pointed strong endofunctor  $\Sigma_0$ , under mild hypotheses, the forgetful functor  $\Sigma_0\text{-mon} \rightarrow \mathbf{C}_0$  is monadic, and the free  $\Sigma_0$ -algebra over  $I$  (equivalently the initial  $(I + \Sigma_0)$ -algebra) is an initial  $\Sigma_0$ -monoid.

*Example 3.3.* In the case of  $\lambda$ -calculus, the initial  $\Sigma_0$ -monoid is thus the least fixed point  $\mathbf{Z}_0 := \mu A.(I + \Sigma_0(A))$ , which is isomorphic to the standard, low-level construction of syntax.

### 3.2 Transition systems and bisimilarity

The appropriate notion of transition system  $X$  for  $\lambda$ -calculus is as follows.

**Definition 3.4.** A transition system  $X$  consists of

- a state object  $X_0 \in \mathbf{C}_0 = [\mathbb{F}, \mathbf{Set}]$ ,
- a set  $X_1$  of transitions, and
- maps  $X_0(0) \leftarrow X_1 \rightarrow X_0(1)$  giving the source and target of transitions (cf. §2.3).

Transition systems form a category  $\mathbf{C}$ , whose morphisms  $X \rightarrow Y$  consist of compatible morphisms  $X_0 \rightarrow Y_0$  and  $X_1 \rightarrow Y_1$ .

*Remark 3.5.* Morphisms in  $\mathbf{C}$  may be thought of as graph morphisms, or alternatively functional simulations.

*Example 3.6.* (a) Let  $\mathbf{y}_0$  consist of a single closed state  $k_0$  and its renamings (i.e.,  $(\mathbf{y}_0)_0(n) = 1$  for all  $n$  and for transitions  $(\mathbf{y}_0)_1 = \emptyset$ ).

(b) Let  $\mathbf{y}_\Downarrow$  consist of a closed state  $k_0$ , a state  $k_1$  with one free variable, their renamings, and a transition  $e : k_0 \Downarrow k_1$ .

(c) Let  $\mathbf{y}_{s_\Downarrow} : \mathbf{y}_0 \rightarrow \mathbf{y}_\Downarrow$  denote the morphism mapping  $k_0$  to  $k_0$ .

(d) The syntactic transition system has  $\mathbf{Z}_0 \in \mathbf{C}_0$  from Example 3.3 as state object, and as transitions all derivations following the transition rules. We will come back to this case in Proposition 3.16.

We may define bisimulation and bisimilarity following [24]:

**Definition 3.7.** *A morphism  $X \rightarrow Y$  in  $\mathbf{C}$  is a functional bisimulation when it has the right lifting property w.r.t. the source map  $\mathbf{y}_{s_{\Downarrow}} : \mathbf{y}_0 \rightarrow \mathbf{y}_1$ . A span  $X \leftarrow R \rightarrow Y$  is a simulation when its left leg  $R \rightarrow X$  is a functional bisimulation, and a bisimulation when both legs are.*

*Remark 3.8.*  $\mathbf{C}$  is in fact isomorphic to a presheaf category, and we have a Yoneda lemma:  $\mathbf{C}(\mathbf{y}_0, X) \cong X_0(0)$  and  $\mathbf{C}(\mathbf{y}_{\Downarrow}, X) \cong X_1$ . The right lifting property for a morphism  $f : X \rightarrow Y$  thus says that given any  $e \in Y_1$  whose source  $e \cdot \mathbf{s}$  is  $f(x)$  for some  $x \in X_0(0)$ , there exists  $e' \in X_1$  such that  $f(e') = e$  and  $e' \cdot \mathbf{s} = x$ , as desired.

**Definition 3.9.** *Let  $\mathbf{Bisim}(X, Y)$  denote the category of bisimulations, with span morphisms between them.*

**Proposition 3.10.**  *$\mathbf{Bisim}(X, Y)$  has a terminal object, called bisimilarity and denoted by  $\sim_{X, Y}$ .*

*Example 3.11.* Bisimilarity on the syntactic transition system merely amounts to simultaneous convergence, because evaluation returns an open term, which does not have any further transition. In this case, a more relevant behavioural equivalence is substitution-closed bisimilarity, which we will define below.

### 3.3 Operational semantics

Just as we have defined the syntax as an initial  $\Sigma_0$ -monoid (Example 3.3), let us now define the dynamics by initiality, again starting by finding the right notion of model. First of all, models will be found among transition systems  $X$  whose underlying presheaf  $X_0 \in [\mathbb{F}, \mathbf{Set}]$  is a  $\Sigma_0$ -monoid. Let us give these a name.

**Definition 3.12.** *A transition  $\Sigma_0$ -monoid is a transition system  $X$ , together with  $\Sigma_0$ -monoid structure on its state object  $X_0$ . Transition  $\Sigma_0$ -monoids form a category  $\Sigma_0\text{-Mon}$ .*

The idea is to model the transition rules as an endofunctor on transition  $\Sigma_0$ -monoids, leaving the underlying  $\Sigma_0$ -monoid untouched, i.e., a functor making the triangle

$$\begin{array}{ccc} \Sigma_0\text{-Mon} & \xrightarrow{\Sigma_1} & \Sigma_0\text{-Mon} \\ & \searrow \mathcal{D} & \swarrow \mathcal{D} \\ & \Sigma_0\text{-mon} & \end{array}$$

commute, where  $\mathcal{D}$  denotes the forgetful functor (i.e.,  $\mathcal{D}(X) = X_0$ ).

For call-by-name  $\lambda$ -calculus, the functor  $\Sigma_1 : \Sigma_0\text{-Mon} \rightarrow \Sigma_0\text{-Mon}$  modelling the non-standard rules at the end of §2 is defined as follows.

- On states, commutation of the above triangle imposes  $\Sigma_1(X)_0 = X_0$ .
- On transitions, let  $\Sigma_1(X)_1 = X(1) + A_{\beta}(X)$ , where  $A_{\beta}(X)$  denotes the set of valid premises for the second rule in §2.3, i.e., triples  $(r_1, e_2, r_2)$  such that

- $r_1, r_2 \in X_1$  are transitions,
- $e_2 \in X_0(0)$  is a state, and
- $r_2 \cdot s = (r_1 \cdot \mathbf{t})[e_2]$ , i.e., the source  $r_2 \cdot s$  of  $r_2$  is obtained by substituting  $e_2$  for the unique free variable in the target of  $r_1$ .

Let us notice that substitution here follows from the monoid structure of  $X$ .

- We then define the source and target maps:
  - for the first term  $X(1)$ ,
    - \* the source of any  $in_1(e)$  is  $\lambda_1(e)$ , where  $\lambda_n : X_0(n+1) \rightarrow X_0(n)$  follows from the  $\Sigma_0$ -algebra structure of  $X_0$ ;
    - \* the target is  $e$  itself;
  - for the second term  $A_\beta(X)$ ,
    - \* the source of any  $in_2(r_1, e_2, r_2)$  is  $(r_1 \cdot s) e_2$ , i.e., the application of the source of  $r_1$  to  $e_2$  (again using the  $\Sigma_0$ -algebra structure of  $X_0$ );
    - \* the target is  $r_2 \cdot \mathbf{t}$ .

Accordingly, our notion of model is the following.

**Definition 3.13.** A vertical  $\Sigma_1$ -algebra is a transition  $\Sigma_0$ -monoid  $X$  equipped with a morphism  $v_X : \Sigma_1(X) \rightarrow X$  such that  $\mathcal{D}(v_X) = \text{id}_{X_0}$ , or equivalently a map  $(v_X)_1$  making the triangle

$$\begin{array}{ccc} \Sigma_1(X)_1 & \xrightarrow{\quad} & X_1 \\ & \searrow & \swarrow \\ & X_0(0) \times X_0(1) & \end{array} \quad \text{commute.}$$

In the case of call-by-name  $\lambda$ -calculus, it should be clear that such a vertical algebra is indeed a model of the rules.

However, in order to ensure that the rules are syntax-directed, we want to distinguish, for each rule, the head operator of the source of the conclusion (abstraction for the first rule; application for the second one). Instead of demanding that  $\Sigma_1(X)$  have the form  $\Sigma_1(X)_1 \rightarrow X_0(0) \times X_0(1)$ , we thus rather require something of the form  $\Sigma_1(X)_1 \rightarrow \Sigma_0(X_0)(0) \times X_0(1)$ :

**Definition 3.14.**

- A dynamic signature consists of
  - a finitary functor  $\Sigma_1^F : \Sigma_0\text{-Mon} \rightarrow \mathbf{Set}$ , and
  - a natural transformation  $(\Sigma_1^\partial)_X : \Sigma_1^F(X) \rightarrow \Sigma_0(X_0)(0) \times X_0(1)$ .
- The endofunctor  $\tilde{\Sigma}_1$  induced by a dynamic signature maps any  $X$  to the composite  $\Sigma_1^F(X) \xrightarrow{(\Sigma_1^\partial)_X} \Sigma_0(X_0)(0) \times X_0(1) \rightarrow X_0(0) \times X_0(1)$ .
- A vertical algebra of a dynamic signature is a vertical algebra of the induced endofunctor, in the sense of Definition 3.13.

Concretely, a vertical algebra is a dashed map making the following diagram commute.

$$\begin{array}{ccc} \Sigma_1^F(X) & \dashrightarrow & X_1 \\ \downarrow & & \downarrow \\ \Sigma_0(X_0)(0) \times X_0(1) & \longrightarrow & X_0(0) \times X_0(1) \end{array}$$

*Example 3.15.* For call-by-name  $\lambda$ -calculus, we only need to modify the source components of the above definition of  $\Sigma_1$ , replacing actual operations by formal ones, like so:

- the source of any  $in_1(e) \in X(1) + A_\beta(X)$  is  $in_1(e) \in \Sigma_0(X)(0) = X(1) + X(0)^2$ ;
- the source of any  $in_2(r_1, e_2, r_2)$  is  $in_2((r_1 \cdot s), e_2) \in \Sigma_0(X)(0)$ .

This successfully captures the syntactic transition system:

**Proposition 3.16.** *The initial  $\check{\Sigma}_1$ -algebra  $\mathbf{Z}_{\check{\Sigma}_1}$ , or  $\mathbf{Z}$  for short, is an initial vertical algebra, and is isomorphic to the transition system of (Example 3.6(d)).*

### 3.4 Substitution-closed bisimilarity

There is an obvious notion of bisimulation for transition  $\Sigma_0$ -monoids:

**Definition 3.17.** *A morphism in  $\Sigma_0\text{-Mon}$  is a functional bisimulation iff its underlying morphism in  $\mathbf{C}$  is.*

However, as foreshadowed by Example 3.11, the relevant notion in this case combines bisimulation with substitution-closedness, in the following sense.

**Definition 3.18.** *For any monoid  $M$  in  $\mathbf{C}_0$ , an  $M$ -module is an object  $X$  equipped with algebra structure  $X \otimes M \rightarrow X$  for the monad  $- \otimes M$ . A module morphism is an algebra morphism.*

*Example 3.19.* The monoid  $M$  is itself an  $M$ -module by multiplication, and  $M$ -modules are closed under limits in  $\mathbf{C}_0$ , so in particular  $M^2$  is an  $M$ -module, with action given by the composite  $M^2 \otimes M \xrightarrow{\langle \pi_1 \otimes M, \pi_2 \otimes M \rangle} (M \otimes M)^2 \xrightarrow{m_M^2} M^2$ .

**Definition 3.20.** *For any transition monoid  $M$ , a span of the form  $R \rightarrow M^2$  in  $\mathbf{C}$  is substitution-closed iff  $R_0$  is an  $M_0$ -module and the morphism  $R_0 \rightarrow M_0^2$  is an  $M_0$ -module morphism.*

*Example 3.21.* To see what this definition has to do with substitution-closedness, let us observe that if  $R$  is a relation in  $[\mathbb{F}, \mathbf{Set}]$ , an element of  $(R \otimes M)(n)$  is an explicit substitution  $r(\sigma)$  with  $r \in R(p)$  for some  $p$ , and  $\sigma : p \rightarrow M(n)$ . Now, substitution-closedness amounts to a morphism  $R \otimes M \rightarrow R$  commuting with projections, so if  $R$  is a relation, we indeed get that  $e R e'$  entails  $e[\sigma] R e'[\sigma]$ .

**Proposition 3.22.** *For any transition  $\Sigma_0$ -monoid  $M$ , there is a terminal substitution-closed bisimulation  $\sim_M^\otimes$ , called substitution-closed bisimilarity.*

*Remark 3.23.* Substitution-closed bisimilarity is a relation.

**Proposition 3.24.** *Substitution-closed bisimilarity  $\sim_{\mathbf{Z}}^\otimes$  on the syntactic transition system  $\mathbf{Z}$  coincides with applicative bisimilarity.*

*Proof.* Let us denote the open extension of applicative bisimilarity by  $\sim_{std}^\otimes$ , and recall that applicative bisimilarity is denoted by  $\sim$ . The relation  $\sim_{std}^\otimes$  is straightforwardly a substitution-closed bisimulation, so we have  $\sim_{std}^\otimes \subseteq \sim_{\mathbf{Z}}^\otimes$ . But conversely any substitution-closed bisimulation relation  $R$  (hence  $\sim_{\mathbf{Z}}^\otimes$ ) is in particular a substitution-closed relation contained in  $\sim$  on closed terms. It is thus globally contained in  $\sim_{std}^\otimes$  by Lemma 2.4.  $\square$

Our main result instantiates to the following.

**Theorem 3.25.** *Substitution-closed bisimilarity is context-closed. More precisely, it is a transition  $\Sigma_0$ -monoid, and  $\sim_{\mathbf{Z}}^\otimes \rightarrow \mathbf{Z}^2$  is a transition  $\Sigma_0$ -monoid morphism.*

In particular, there exists a span morphism  $\Sigma_0((\sim_{\mathbf{Z}}^\otimes)_0) \rightarrow (\sim_{\mathbf{Z}}^\otimes)_0$ .

## 4 Transition systems and bisimilarity

In this section, we start to abstract over the situation of §3, by introducing a general framework for transition systems and bisimilarity.

### 4.1 Pre-Howe contexts and transition systems

**Definition 4.1.** *A pre-Howe context<sup>3</sup> consists of*

- a small category  $\mathbb{C}_0$  of state types,
- a small category  $\mathbb{C}_1$  of transition types,
- and two source and target functors  $\mathbf{s}, \mathbf{t} : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ .

*Precomposition by  $\mathbf{s}$  and  $\mathbf{t}$  yields functors  $\Delta_{\mathbf{s}}, \Delta_{\mathbf{t}} : \widehat{\mathbb{C}}_0 \rightarrow \widehat{\mathbb{C}}_1$  mapping any  $X \in \widehat{\mathbb{C}}_0$  to  $X \circ \mathbf{s}$  and  $X \circ \mathbf{t}$ , respectively. Let  $\Delta$  denote the pointwise product  $\Delta_{\mathbf{s}} \times \Delta_{\mathbf{t}}$ .*

We use these functors to define transition systems.

**Definition 4.2.** *Given any pre-Howe context, a transition system  $X$  consists of*

- a state presheaf  $X_0 \in \widehat{\mathbb{C}}_0$ ,
- a transition presheaf  $X_1 \in \widehat{\mathbb{C}}_1$ ,
- and two source and target natural transformations  $X_0 \circ \mathbf{s} \leftarrow X_1 \rightarrow X_0 \circ \mathbf{t}$ , or equivalently a natural transformation  $X_1 \rightarrow \Delta(X_0)$ .

**Proposition 4.3.** *In any pre-Howe context, transition systems are precisely the objects of the lax limit category  $\widehat{\mathbb{C}}_1 / \Delta$  of the functor  $\widehat{\mathbb{C}}_0 \xrightarrow{\Delta} \widehat{\mathbb{C}}_1$  in  $\mathbf{CAT}$ , or equivalently the comma category  $\text{id}_{\widehat{\mathbb{C}}_1} \downarrow \Delta$ .*

<sup>3</sup> The Howe contexts of [10] may be defined similarly. The difference is that for them,  $\mathbf{s}$  and  $\mathbf{t}$  are not necessarily functorial, but  $c_1 \mapsto (\mathbf{s}(c_1), \mathbf{t}(c_1))$  defines a functor  $\mathbb{C}_1 \rightarrow \mathbb{C}_0 \times \mathbb{C}_0$ , where  $\mathbb{C}_0 \times \mathbb{C}_0$  denotes the category whose objects are pairs of elements of  $\mathbb{C}_0$ , and morphisms between  $(a_1, a_2)$  and  $(b_1, b_2)$  consists of pairs of morphisms  $a_1 \rightarrow b_i$  and  $a_2 \rightarrow b_j$  for some  $i, j \in \{1, 2\}$ .

*Proof.* An object of the lax limit is by construction a triple  $(X_1, X_0, \partial)$ , where  $\partial : X_1 \rightarrow \Delta(X_0) = X_0 \mathbf{s} \times X_0 \mathbf{t}$ .  $\square$

**Notation 4.4.** In any pre-Howe context, we let  $\mathbf{C}_0 := \widehat{\mathbf{C}}_0$  and  $\mathbf{C} := \widehat{\mathbf{C}}_1/\Delta$ .

**Lemma 4.5.** The projection functor  $-_0 : \mathbf{C} \rightarrow \mathbf{C}_0$  has a left adjoint mapping any object  $X_0$  to  $\emptyset \rightarrow X_0 \mathbf{s} \times X_0 \mathbf{t}$ .

*Example 4.6.* We can get  $\mathbf{C} \rightarrow \mathbf{C}_0$  to be the forgetful functor  $\mathbf{Gph} \rightarrow \mathbf{Set}$  by taking

- $\mathbf{C}_0 = 1$ , so that  $\mathbf{C}_0 = \widehat{\mathbf{C}}_0 = \widehat{1} \cong \mathbf{Set}$ ,
- $\mathbf{C}_1 = 1$ , so that  $\mathbf{C} = \mathbf{Set}$ , and
- $\mathbf{s}, \mathbf{t} : 1 \rightarrow 1$  to be the unique such functor, i.e., the identity.

A transition system thus consists of sets  $V$  and  $E$  together with a map  $E \rightarrow V^2$ , i.e., a graph.

*Example 4.7.* A (harmless) proof-relevant variant of standard labelled transition systems (over any set  $\mathbb{A}$  of labels) may be obtained as follows. We take

- $\mathbf{C}_0 = 1$  again,
- $\mathbf{C}_1 = \mathbb{A}$  viewed as a discrete category, and
- $\mathbf{s}, \mathbf{t} : \mathbf{C}_1 \rightarrow \mathbf{C}_0$  the unique such functor.

Thus, a transition system  $X$  consists of a set  $X_0$  and sets  $X_a$  for all  $a \in \mathbb{A}$ , together with maps  $X_a \rightarrow X_0^2$  returning the source and target of each  $a$ -labelled edge.

More generally, given any graph  $\mathbb{L}$ , taking  $\mathbf{s}, \mathbf{t} : \mathbf{C}_1 \rightarrow \mathbf{C}_0$  to be the source and target maps  $\mathbb{L}_1 \rightarrow \mathbb{L}_0$  viewed as functors between discrete categories, we obtain for  $\mathbf{C} \rightarrow \mathbf{C}_0$  a functor equivalent to  $\mathbf{Gph}/\mathbb{L} \rightarrow \mathbf{Set}/\mathbb{L}_0$ .

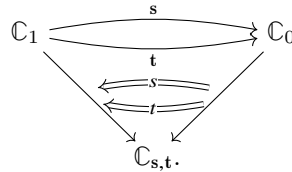
*Example 4.8.* Let  $\mathbf{C}_0 = \mathbb{F}^{op}$  and  $\mathbf{C}_1 = 1$ , with  $\mathbf{s}$  and  $\mathbf{t}$  picking respectively 0 and 1. In particular,  $\widehat{\mathbf{C}}_1 \cong \mathbf{Set}$ . Then,  $\Delta(X_0) = X_0(0) \times X_0(1)$  and we recover the category  $\mathbf{C}$  of §3.2, and its forgetful functor to  $\mathbf{C}_0 = [\mathbb{F}, \mathbf{Set}]$ .

## 4.2 Transition systems as presheaves

Before introducing bisimulation, let us establish an alternative characterisation of the category  $\mathbf{C}$  of transition systems.

**Proposition 4.9.** The lax limit category  $\widehat{\mathbf{C}}_1/\Delta$  of transition systems is isomorphic to a presheaf category  $\widehat{\mathbf{C}}_{\mathbf{s}, \mathbf{t}}$ .

*Proof.* Let  $\mathbf{C}_{\mathbf{s}, \mathbf{t}}$  denote the lax colimit in  $\mathbf{Cat}$  of the parallel pair  $\mathbf{s}, \mathbf{t}$ . By definition, it is the universal category with natural transformations



It thus consists of the coproduct  $\mathbb{C}_1 + \mathbb{C}_0$ , augmented with arrows  $s_L : \mathbf{s}(L) \rightarrow L$  and  $t_L : \mathbf{t}(L) \rightarrow L$  for all  $L \in \mathbb{C}_1$ , naturally in  $L$ . Presheaves on  $\mathbb{C}_{\mathbf{s}, \mathbf{t}}$  coincide with  $\widehat{\mathbb{C}_1}/\Delta$  because the presheaf construction turns lax colimits into lax limits.  $\square$

**Notation 4.10.** We often omit the isomorphism  $\widehat{\mathbb{C}_1}/\Delta \cong \widehat{\mathbb{C}_{\mathbf{s}, \mathbf{t}}}$ , considering it as an implicit coercion. E.g.,  $\mathbf{y}_P$  may be used to denote the transition system  $\underline{P}$  with  $\underline{P}_1 = \emptyset$  and  $\underline{P}_0 = \mathbf{y}_P$ .

Similarly,  $\mathbf{y}_L$  may be used to denote the ‘minimal’ transition system with one transition over  $L$ , say  $\underline{L}$ , i.e.,  $\underline{L}_1 = \mathbf{y}_L$ ,  $\underline{L}_0 = \mathbf{y}_{\mathbf{s}(L)} + \mathbf{y}_{\mathbf{t}(L)}$ , and the map  $\underline{L}_1 \rightarrow \underline{L}_0 \mathbf{s} \times \underline{L}_0 \mathbf{t}$  uniquely determined by the element  $(in_1(\text{id}_{\mathbf{s}(L)}), in_2(\text{id}_{\mathbf{t}(L)})) \in \underline{L}_0(\mathbf{s}(L)) \times \underline{L}_0(\mathbf{t}(L))$ .

Finally,  $\mathbf{y}_{s_L} : \mathbf{y}_{\mathbf{s}(L)} \rightarrow \mathbf{y}_L$  and  $\mathbf{y}_{t_L} : \mathbf{y}_{\mathbf{t}(L)} \rightarrow \mathbf{y}_L$  denote the Yoneda embedding of the canonical morphisms  $s_L$  and  $t_L$  from the proof of Proposition 4.9.

By Yoneda, we thus have:

**Corollary 4.11.** For all  $X$ , we have  $\mathbf{C}(\mathbf{y}_L, X) \cong X_1(L)$  and  $\mathbf{C}(\mathbf{y}_P, X) \cong X_0(P)$ .

**Notation 4.12.** In the case of call-by-name  $\lambda$ -calculus, we call  $\Downarrow$  the unique object coming from  $\mathbb{C}_1 = 1$ .

*Remark 4.13.* Presheaves on  $\mathbb{C}_{\mathbf{s}, \mathbf{t}}$  are intuitively 2-dimensional; the projection functor forgets dimension 1, while the left adjoint (Lemma 4.5) adds an empty dimension 1, thus lifting its 0-dimensional argument to a 1-dimensional object.

### 4.3 Bisimulation and bisimilarity

Morphisms in  $\mathbf{C}$  are a generalisation of graph morphisms, which are a proof-relevant version of functional simulations. The analogue of functional bisimulations is as follows.

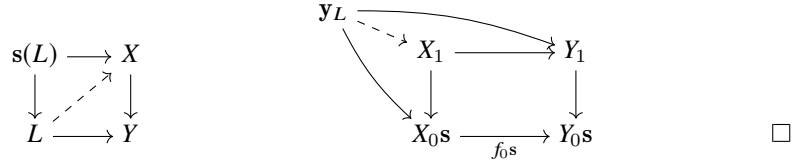
**Definition 4.14.** A morphism  $f : X \rightarrow Y$  in  $\widehat{\mathbb{C}_{\mathbf{s}, \mathbf{t}}}$  is a functional bisimulation, or a fibration, iff it enjoys the (weak) right lifting property w.r.t.  $\mathbf{y}_{s_L} : \mathbf{y}_{\mathbf{s}(L)} \rightarrow \mathbf{y}_L$ , for all  $L \in \mathbb{C}_1$ .

Here is a characterisation of fibrations which will be important. Let us recall that a weak pullback satisfies the same universal property as a pullback, albeit without uniqueness.

**Proposition 4.15.** A morphism  $f : X \rightarrow Y$  is a functional bisimulation iff the following diagram is a pointwise weak pullback.

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow & & \downarrow \\ X_0 \mathbf{s} & \xrightarrow{f_0 \mathbf{s}} & Y_0 \mathbf{s} \end{array}$$

*Proof.* By Yoneda, a lifting problem in  $\mathbf{C}$  as on the left below is the same as a cone in  $\widehat{\mathbb{C}_1}$  as on the right, and a lifting is the same as a mediating morphism to  $X_1$ .



We use this to define general bisimulations. Usually, one considers bisimulation relations. Here, we generalise this a bit and consider arbitrary spans:

**Definition 4.16.** A simulation is a span  $X \leftarrow R \rightarrow Y$  whose left leg is a fibration. A bisimulation is a span of fibrations (equivalently, a simulation whose converse span is also a simulation).

*Remark 4.17.* Of course, the relevant notion in our applications is substitution-closed bisimulation, to which we will come below.

**Lemma 4.18.** Simulation relations and bisimulation relations are stable under unions.

*Proof.* By symmetry, it is enough to deal with the case of simulation relations. Consider any family  $(R_i \hookrightarrow X \times Y)_{i \in I}$  of simulation relations. Their union is the image of their copairing. But because the domain  $\mathbf{y}_{s(L)}$  of  $s_L$  is representable for all  $L \in \mathbf{C}_1$ , any lifting problem  $s_L \rightarrow \bigcup_i R_i$  lifts to a lifting problem  $s_L \rightarrow \sum_i R_i$ , which in turn lifts to a lifting problem so some  $s_L \rightarrow R_{i_0}$  with  $i_0 \in I$ . We then find a lifting for the latter, which yields a lifting for the original.  $\square$

**Proposition 4.19.** For all  $X, Y \in \mathbf{C}$ , the full subcategory  $\mathbf{Bisim}(X, Y)$  of spans between  $X$  and  $Y$  which are bisimulations admits a terminal object  $\sim_{X, Y}$ , called bisimilarity.

*Proof.* As a presheaf category by Proposition 4.9,  $\mathbf{C}$  is well-powered, so we may consider the union  $\sim_{X, Y}$  of all bisimulation relations, which is again a bisimulation by Lemma 4.18. Finally,  $\sim_{X, Y}$  is terminal, because any bisimulation  $R$  factors through its image  $\mathit{im}(R)$ , which is again a bisimulation; as a bisimulation relation,  $\mathit{im}(R)$  thus embeds into  $\sim_{X, Y}$ , hence we obtain a morphism  $R \twoheadrightarrow \mathit{im}(R) \hookrightarrow \sim_{X, Y}$ , which is unique by monicity of  $\sim_{X, Y} \hookrightarrow X \times Y$ .  $\square$

## 5 Howe contexts for operational semantics

Operational semantics is the combination of syntax and transition systems, in the sense that it is about transition systems whose states are models of a certain syntax. Our framework for operational semantics thus combines the frameworks of [18] and §4.

The former is merely an abstract version of §3.1. Here is the important result, which echoes Proposition 3.2.



**Proposition 5.1** (see [18,15,10]). *For any finitary, pointed strong endofunctor  $\Sigma_0$  on a monoidal, cocomplete category  $\mathcal{C}_0$  such that the tensor preserves all colimits on the left and filtered colimits on the right, the forgetful functor  $\Sigma_0\text{-mon} \rightarrow \mathcal{C}_0$  is monadic, and the free  $\Sigma_0$ -algebra over  $I$  (equivalently the initial  $(I + \Sigma_0)$ -algebra) is an initial  $\Sigma_0$ -monoid.*

**Notation 5.2.** *We denote the initial  $\Sigma_0$ -monoid by  $\mathbf{Z}_{\Sigma_0}$ , or  $\mathbf{Z}_0$  for short.*

**Definition 5.3.** *A Howe context consists of a pre-Howe context  $\mathbf{s}, \mathbf{t}: \mathbb{C}_1 \rightarrow \mathbb{C}_0$ , together with a monoidal structure on  $\widehat{\mathbb{C}}_0$ , such that the tensor preserves all colimits on the left and filtered colimits on the right.*

**Notation 5.4.** *As for pre-Howe contexts, we let  $\mathbf{C}_0 = \widehat{\mathbb{C}}_0$  and  $\mathbf{C} := \widehat{\mathbb{C}}_1/\Delta$ .*

Let us assume that some syntax has been specified by a finitary pointed strong endofunctor  $\Sigma_0$  on  $\mathbf{C}_0$ . We then define transition  $\Sigma_0$ -monoids just as in §3.

**Definition 5.5.** *The category  $\Sigma_0\text{-Mon}$  of transition  $\Sigma_0$ -monoids is the following pullback in **CAT**.*

$$\begin{array}{ccc} \Sigma_0\text{-Mon} & \xrightarrow{\mathcal{D}} & \Sigma_0\text{-mon} \\ \downarrow \lrcorner & & \downarrow \\ \mathbf{C} & \longrightarrow & \mathbf{C}_0 \end{array}$$

**Lemma 5.6.** *The adjunction between  $\mathbf{C}$  and  $\mathbf{C}_0$  (Lemma 4.5) lifts to an adjunction<sup>4</sup>  $\mathcal{D} \vdash \mathcal{M} : \Sigma_0\text{-mon} \rightarrow \Sigma_0\text{-Mon}$  with  $\mathcal{D}(\mathcal{M}(X_0)) = X_0$  and  $\mathcal{M}(X_0)_1 = \emptyset$ .*

**Proposition 5.7.** *The forgetful functor  $\mathcal{U} : \Sigma_0\text{-Mon} \rightarrow \mathbf{C}$  is monadic.*

*Proof.* Transition  $\Sigma_0$ -monoids are the algebras of an equational system over  $\mathbf{C}$  in the sense of Fiore and Hur, to which [14, Theorem 6.1] applies.  $\square$

**Notation 5.8.** *We denote by  $\mathcal{L}$  the left adjoint to  $\mathcal{U}$ .*

Similarly, we define abstract dynamic signatures:

**Definition 5.9.** *Given a Howe context  $\mathbf{s}, \mathbf{t}: \mathbb{C}_1 \rightarrow \mathbb{C}_0$  and a pointed strong  $\Sigma_0 : \mathbf{C}_0 \rightarrow \mathbf{C}_0$ , a dynamic signature  $\Sigma_1 = (\Sigma_1^F, \Sigma_1^\partial)$  over  $\Sigma_0$  consists of a finitary functor  $\Sigma_1^F : \Sigma_0\text{-Mon} \rightarrow \widehat{\mathbb{C}}_1$ , together with a natural transformation  $\Sigma_1^\partial$  with components  $\Sigma_1^F(X) \rightarrow \Sigma_0(X_0)\mathbf{s} \times X_0\mathbf{t}$ .*

Let us pack up the static and dynamic notions of signature.

**Definition 5.10.** *An operational semantics signature  $(\Sigma_0, \Sigma_1)$  on a given Howe context  $\mathbf{s}, \mathbf{t}: \mathbb{C}_1 \rightarrow \mathbb{C}_0$  consists of a pointed strong endofunctor  $\Sigma_0$  preserving sifted colimits, together with a dynamic signature  $\Sigma_1$  over it.*

<sup>4</sup> The names,  $\mathcal{M}$  and  $\mathcal{D}$ , stand for “monter” and “descendre”, “go up” and “go down” in French.

In the following, we fix an operational semantics signature  $(\Sigma_0, \Sigma_1)$  on a Howe context  $\mathbf{s}, \mathbf{t} : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ .

*Remark 5.11.* Preservation of sifted colimits [3] is stronger than finitariness for  $\Sigma_0$ . We need it for Lemma 8.42 below. In a presheaf category like  $\mathbf{C}_0$ , if  $\Sigma_0$  preserves pullbacks (for example, by familiarity), it is equivalent to being finitary and preserving all epis, as seen from the proof of [4, Theorem 18.1].

*Remark 5.12.* In [10], we mistakenly only require  $\Sigma_0$  to be finitary, which yields a gap in the proof of [10, Lemma 5.13].

*Example 5.13.* The endofunctor  $\Sigma_0(X)(n) = X(n+1) + X(n)^2$  on  $[\mathbb{F}, \mathbf{Set}]$  preserves sifted colimits. It is easily deduced from the fact that sifted colimits are the ones commuting with products in sets.

Let us now define the category of models of a dynamic signature  $\Sigma_1$ .

**Definition 5.14.** Let  $\check{\Sigma}_1 : \Sigma_0\text{-Mon} \rightarrow \Sigma_0\text{-Mon}$  map any transition  $\Sigma_0$ -monoid  $X$  to the composite

$$\Sigma_1^F(X) \rightarrow \Sigma_0(X_0)\mathbf{s} \times X_0\mathbf{t} \xrightarrow{v_{X_0} \mathbf{s} \times X_0\mathbf{t}} X_0\mathbf{s} \times X_0\mathbf{t},$$

where  $v_{X_0}$  denotes the  $\Sigma_0$ -algebra structure of  $X_0$ .

**Proposition 5.15.** The endofunctor  $\check{\Sigma}_1$  is finitary and makes the following triangle commute.

$$\begin{array}{ccc} \Sigma_0\text{-Mon} & \xrightarrow{\check{\Sigma}_1} & \Sigma_0\text{-Mon} \\ & \searrow & \swarrow \\ & \Sigma_0\text{-mon} & \end{array}$$

**Definition 5.16.** A  $\check{\Sigma}_1$ -algebra  $\check{\Sigma}_1(X) \rightarrow X$  is vertical when its image under the forgetful functor  $\Sigma_0\text{-Mon} \rightarrow \Sigma_0\text{-mon}$  is the identity. Let  $\Sigma_1\text{-alg}_v$  denote the full subcategory of  $\check{\Sigma}_1\text{-alg}$  spanning all vertical algebras.

**Theorem 5.17.** The forgetful functor  $\Sigma_1\text{-alg}_v \rightarrow \Sigma_0\text{-Mon}$  is monadic, and furthermore the initial  $\check{\Sigma}_1$ -algebra  $\mathbf{Z}_{\check{\Sigma}_1}$ , or  $\mathbf{Z}$  for short, may be chosen to be vertical, hence is also initial in  $\Sigma_1\text{-alg}_v$ .

*Proof.* For the first statement, vertical algebras may be specified as an equational system, in the sense of [14]. For the second statement,  $\mathbf{Z}$  is the colimit of the initial chain

$$\mathbf{Z}_0 \rightarrow \check{\Sigma}_1(\mathbf{Z}_0) \rightarrow \dots \rightarrow \check{\Sigma}_1^n(\mathbf{Z}_0) \rightarrow \dots$$

(where  $\mathbf{Z}_0$  is shorthand for  $\mathcal{M}(\mathbf{Z}_0)$ , for readability, recalling Lemma 5.6). The image of this chain in  $\Sigma_0\text{-mon}$  is the everywhere-identity chain on  $\mathbf{Z}_0$ .  $\square$

**Proposition 5.18.** We have  $\mathcal{D}(\mathbf{Z}) = \mathbf{Z}_0$ .

*Remark 5.19.* The composite forgetful functor  $\Sigma_1\text{-alg}_v \rightarrow \Sigma_0\text{-Mon} \rightarrow \mathbf{C}$  appears not to be monadic in general.

## 6 Substitution-closed bisimilarity

We have now defined bisimilarity (Proposition 4.19) for our generalised transition systems, and also introduced operational semantics signatures, which allow us to efficiently specify relevant syntactic transition systems. As in §3.4,  $\Sigma_0\text{-Mon}$  inherits the notion of bisimulation from  $\mathbf{C}$ :

**Definition 6.1.** *A morphism in  $\Sigma_0\text{-Mon}$  is a functional bisimulation iff its underlying morphism in  $\mathbf{C}$  is. A span is a simulation iff its left leg is a functional bisimulation, and a bisimulation iff both of its legs are functional bisimulations.*

However, we have seen in Example 3.11 that bisimilarity is irrelevant for such syntactic transition systems. We thus introduce abstract versions of substitution-closed bisimulation and bisimilarity. For this, let us give the general definition of modules over a monoid.

**Definition 6.2.** *For any monoid  $M$  in a monoidal category  $\mathcal{C}$ , let  $M\text{-Mod}$  denote the category of algebras for the monad  $- \otimes M$ .*

*Example 6.3.*  $M$  itself is an  $M$ -module, with action given by multiplication.

As algebras for the monad  $- \otimes M$ ,  $M$ -modules are closed under all limits in  $\mathcal{C}$ , as well as under all types of colimits preserved by  $- \otimes M$ .

**Definition 6.4.** *In a monoidal category  $\mathcal{C}$  with binary products, given a monoid  $M$  and  $M$ -modules  $X$  and  $Y$ , a span  $R \rightarrow X \times Y$  is substitution-closed iff  $R$  is an  $M$  module and  $R \rightarrow X \times Y$  is an  $M$ -module morphism.*

When  $\mathcal{C}$  has binary products, this is equivalent to commutation of the following diagram.

$$\begin{array}{ccc} R \otimes M & \xrightarrow{\quad\quad\quad} & R \\ \downarrow & & \downarrow \\ (X \times Y) \otimes M & \xrightarrow{\langle \pi_1 \otimes M, \pi_2 \otimes M \rangle} (X \otimes M) \times (Y \otimes M) \xrightarrow{a_X \times a_Y} & X \times Y \end{array}$$

**Definition 6.5.** *Consider any Howe context  $\mathbf{s}, \mathbf{t}: \mathbb{C}_1 \rightarrow \mathbb{C}_0$  and transition monoid  $M \in \mathbf{C}$ . Let  $X, Y \in \mathbf{C}$  be equipped with  $M_0$ -module structure on  $X_0$  and  $Y_0$ . A span  $R \rightarrow X \times Y$  is substitution-closed iff  $R_0 \rightarrow X_0 \times Y_0$  is.*

**Definition 6.6.** *For any Howe context  $\mathbf{s}, \mathbf{t}: \mathbb{C}_1 \rightarrow \mathbb{C}_0$  and monoid  $M \in \mathbf{C}$ , a substitution-closed bisimulation is a substitution-closed span  $R \rightarrow M^2$  (viewing  $M_0$  itself as an  $M_0$ -module) which is a bisimulation.*

Let  $\mathbf{Bisim}^\otimes(M)$  denote the full subcategory of  $\mathbf{C}/M^2$  spanning substitution-closed bisimulations.

It may not be entirely clear that substitution-closed bisimilarity exists. Just as in the particular case of §4.3, we have:

**Lemma 6.7.** *Simulation relations and bisimulations relations are stable under unions.*

**Proposition 6.8.** *For any Howe context  $\mathbf{s}, \mathbf{t}: \mathbb{C}_1 \rightarrow \mathbb{C}_0$  and monoid  $M \in \mathbf{C}$ , the category  $\mathbf{Bisim}^{\otimes}(M)$  of substitution-closed bisimulations over  $M$  admits a terminal object  $\sim_M^{\otimes}$ , called substitution-closed bisimilarity.*

**Notation 6.9.** *When  $M = \mathbf{Z}$ , we abbreviate  $\sim_{\mathbf{Z}}^{\otimes}$  to just  $\sim^{\otimes}$ .*

We now want to state the abstract version of our main theorem, but we need an additional hypothesis, which we now introduce. The idea is essentially that  $\Sigma_1$  should preserve functional bisimulations, which does not quite make sense, because its codomain is  $\widehat{\mathbb{C}}_1$ , where no notion of functional bisimulation has been defined yet. Recalling Proposition 4.15, we work around this as follows.

**Definition 6.10.** *A dynamic signature  $\Sigma_1$  preserves functional bisimulations iff for any functional bisimulation  $R \rightarrow X$  in  $\Sigma_0\text{-Mon}$ , the following square is a pointwise weak pullback.*

$$\begin{array}{ccc} \Sigma_1(R) & \longrightarrow & \Sigma_1(X) \\ \pi_1 \circ (\Sigma_1^{\partial})_R \downarrow & & \downarrow \pi_1 \circ (\Sigma_1^{\partial})_X \\ \Sigma_0(R_0)\mathbf{s} & \longrightarrow & \Sigma_0(X_0)\mathbf{s} \end{array} \quad (3)$$

*Remark 6.11.* It may not be obvious that the dynamic signature for call-by-name  $\lambda$ -calculus preserves functional bisimulations. We will come back to this in §7 by showing that it satisfies a sufficient condition, cellularity.

*Remark 6.12.* It may seem linguistically inappropriate to say that  $\Sigma_1$  preserves functional bisimulations, since we have not even defined fibrations in the codomain category  $\widehat{\mathbb{C}}_1$ . We will justify this in Proposition 7.7, but for now let us move on directly to the main result.

**Theorem 6.13.** *If  $\Sigma_1$  preserves functional bisimulations, then substitution-closed bisimilarity is context-closed. More precisely, it is a transition  $\Sigma_0$ -monoid, and  $\sim^{\otimes} \rightarrow \mathbf{Z}^2$  is a transition  $\Sigma_0$ -monoid morphism.*

*Proof sketch (see §8 for the full proof).*

1. We first define the *Howe closure*  $H_0$  of applicative bisimilarity  $\sim_0^{\otimes}$  on states as the initial  $\Sigma_0^H$ -monoid for the pointed strong endofunctor  $\Sigma_0^H$  on  $\mathbf{C}_0/\mathbf{Z}^2$  defined by  $\Sigma_0^H(R) = \Sigma_0(R) + (R; \sim_0^{\otimes})$ . By construction,  $H_0$  is a  $\Sigma_0$ -monoid and both projections are  $\Sigma_0$ -monoid morphisms.

2. We then define the *transition Howe closure*  $H$  of (the full) applicative bisimilarity  $\sim^{\otimes}$ , as an initial algebra for an endofunctor  $\Sigma_1^H$  on a suitable category  $\mathbf{C}_{\mathbf{Z}}^H$ . Very roughly,  $\mathbf{C}_{\mathbf{Z}}^H$  is the category of spans  $R \rightarrow \mathbf{Z}^2$  whose projection is precisely  $H_0 \rightarrow \mathbf{Z}_0^2$ , and  $\Sigma_1^H(R) = \check{\Sigma}_1(R) + (R; \sim^{\otimes})$ . We show:

**Lemma 6.14.** *There exists a span morphism  $\sim^{\otimes} \rightarrow H$ .*

3. Next comes the key lemma:

**Lemma 6.15.** *If  $\Sigma_1$  preserves functional bisimulations, then the transition Howe closure  $H$  is a substitution-closed simulation.*

*Remark 6.16.* Since  $H_0$  is a  $\Sigma_0$ -monoid by construction,  $H$  is easily seen to be substitution-closed, so the lemma really is about it being a simulation.

The key lemma is proved by characterising  $H$  as an initial algebra for a different endofunctor on a different category, whose initial chain involves iterated applications of  $\Sigma_1$  (preserving simulations by hypothesis) to  $\pi_1 : \mathcal{M}(H_0) \rightarrow \mathcal{M}(\mathbf{Z}_0)$ , which is trivially a simulation.

4. For an appropriate notion of relational transitive closure denoted by  $-\bar{\phantom{x}}$ , we show:

**Lemma 6.17.** *The relational transitive closure  $H_0^{\bar{\phantom{x}}}$  of the Howe closure  $H_0$  on states is symmetric.*

As substitution-closed simulations are closed under transitive closure, we obtain

**Corollary 6.18.**  *$H^{\bar{\phantom{x}}}$  is a substitution-closed simulation which is symmetric on states.*

We then use the following lemma (proved in §8.7).

**Lemma 6.19.** *For any substitution-closed simulation  $R$  such that  $R_0$  is symmetric, there exists a span morphism  $R \rightarrow R'$  such that  $R'$  is a substitution-closed bisimulation.*

By terminality of  $\sim^{\otimes}$ , we thus get a unique morphism  $H^{\bar{\phantom{x}}} \rightarrow \sim^{\otimes}$  over  $\mathbf{Z}^2$ .

5. From the chain  $\sim^{\otimes} \rightarrow H \rightarrow H^+ \rightarrow H^{\bar{\phantom{x}}} \rightarrow \sim^{\otimes}$  we get by terminality that  $\sim^{\otimes}$  is a retract of a transition  $\Sigma_0$ -monoid, namely  $H$ . The result then readily follows from monadicity of transition  $\Sigma_0$ -monoids (Proposition 5.7) and the following result, taking  $X = H$ ,  $Y = \sim^{\otimes}$ , and  $Z = \mathbf{Z}^2$ .  $\square$

**Lemma 6.20.** *Consider a monad  $T : \mathcal{C} \rightarrow \mathcal{C}$  on any category  $\mathcal{C}$ ,  $T$ -algebras  $X$  and  $Z$ , and morphisms  $X \xrightarrow{e} Y \xrightarrow{m} Z$  in  $\mathcal{C}$  such that the composite is a  $T$ -algebra morphism,  $e$  is a split epi, and  $m$  is monic. Then there is a unique  $T$ -algebra structure on  $Y$  such that  $e$  and  $m$  both are  $T$ -algebra morphisms.*

*Proof.* Let  $s : Y \rightarrow X$  denote any section of  $e$ . The desired structure is given by  $T(Y) \xrightarrow{T(s)} T(X) \rightarrow X \xrightarrow{e} Y$ , and the rest follows by monicity of  $m$ .  $\square$

## 7 Preservation of functional bisimulations, and cellularity

Let us now consider the main hypothesis of Theorem 6.13, preservation of functional bisimulations. In §7.1, we rephrase the condition in a way that makes more sense linguistically, i.e., by an actual preservation condition. In §7.2, we then briefly recall *familial* functors [13,11], and show that the dynamic signature for call-by-name  $\lambda$ -calculus induces familial functors. In §7.3, we restrict attention to the case where both components of the dynamic signature are familial, and show that preservation of functional bisimulations is then equivalent to a *cellularity* condition [19,10], which itself comes with a useful sufficient condition.

### 7.1 An alternative characterisation

Let us first give an alternative definition of dynamic signatures.

**Definition 7.1.** Let  $\widehat{\mathbb{C}}_1/\Delta_s$  denote the following lax limit category.

$$\begin{array}{ccc} & \widehat{\mathbb{C}}_1/\Delta_s & \\ & \swarrow \quad \longleftarrow \quad \searrow & \\ \widehat{\mathbb{C}}_0 & \xrightarrow{\Delta_s} & \widehat{\mathbb{C}}_1 \end{array}$$

Concretely, an object consists of presheaves  $X_1$  and  $X_0$ , together with a morphism  $X_1 \rightarrow X_0$ . Just as  $\mathbf{C}$ ,  $\widehat{\mathbb{C}}_1/\Delta_s$  is in fact a presheaf category:

**Proposition 7.2.**  $\widehat{\mathbb{C}}_1/\Delta_s$  is isomorphic to the presheaf category over the lax colimit  $\mathbb{C}_s$  of the functor  $s : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ .

*Remark 7.3.* Concretely,  $\mathbb{C}_s$  is the coproduct of  $\mathbb{C}_0$  and  $\mathbb{C}_1$ , augmented with morphisms  $s_L : s(L) \rightarrow L$  for all  $L \in \mathbb{C}_1$ , naturally in  $L$ .

**Notation 7.4.** In the case of call-by-name  $\lambda$ -calculus, as in  $\mathbf{C}$  (Notation 4.12), we call  $\Downarrow$  the unique object coming from  $\mathbb{C}_1 = 1$ .

**Proposition 7.5.** Any operational semantics signature  $(\Sigma_0, \Sigma_1)$  induces a finitary functor  $\Sigma_1^s : \Sigma_0\text{-Mon} \rightarrow \widehat{\mathbb{C}}_s$  making the following square commute.

$$\begin{array}{ccc} \Sigma_0\text{-Mon} & \xrightarrow{\Sigma_1^s} & \widehat{\mathbb{C}}_s \\ \downarrow & & \downarrow \\ \widehat{\mathbb{C}}_0 & \xrightarrow{\Sigma_0} & \widehat{\mathbb{C}}_0 \end{array} \quad (4)$$

In  $\widehat{\mathbb{C}}_s$ , we may define functional bisimulations by analogy with Definition 4.14.

**Definition 7.6.** A morphism  $f : X \rightarrow Y$  in  $\widehat{\mathbb{C}}_s$  is a functional bisimulation, or a fibration, iff it enjoys the (weak) right lifting property w.r.t.  $\mathbf{y}_{s_L} : \mathbf{y}_{s(L)} \rightarrow \mathbf{y}_L$ , for all  $L \in \mathbb{C}_1$ .

**Proposition 7.7 (Price for our linguistic mischief).** A dynamic signature preserves functional bisimulations (Definition 6.10) iff the induced functor  $\Sigma_0\text{-Mon} \rightarrow \widehat{\mathbb{C}}_s$  does.

*Proof.* The functor maps any functional bisimulation  $f : R \rightarrow Y$  precisely to the square (3), and just as in Proposition 4.15 a morphism in  $\widehat{\mathbb{C}}_1/\Delta_s$  is a functional bisimulation iff the corresponding square is a pointwise weak pullback.  $\square$

Let us recall from Notation 5.8 that  $\mathcal{L} : \mathbf{C} \rightarrow \Sigma_0\text{-Mon}$  is left adjoint to the forgetful functor. The following will be useful in §7.3.

**Proposition 7.8.** A morphism in  $\Sigma_0\text{-Mon}$  is a functional bisimulation (Definition 3.17) iff it has the right lifting property w.r.t. all  $\mathcal{L}(\mathbf{y}_{s_L}) : \mathcal{L}(\mathbf{y}_{s(L)}) \rightarrow \mathcal{L}(\mathbf{y}_L)$ , for  $L \in \mathbb{C}_1$ .

*Proof.* By Definition 6.1, a morphism  $f$  is a functional bisimulation iff  $\mathcal{U}(f)$  is (in  $\widehat{\mathbb{C}}_1/\Delta$ ). We conclude by adjunction.  $\square$

### 7.2 Familiarity

In the previous section, we have shown that functional bisimulations may be defined by lifting both in  $\Sigma_0\text{-Mon}$  and  $\widehat{\mathbb{C}}_1/\Delta_s$ . We now want to exploit this to obtain a characterisation of preservation of functional bisimulations, which will then lead us to useful sufficient conditions.

For this, let us briefly recall familial functors, and show that the functors  $\Sigma_0$  and  $\Sigma_1^s : \Sigma_0\text{-Mon} \rightarrow \widehat{\mathbb{C}}_1/\Delta_s$  induced by the dynamic signature for call-by-name  $\lambda$ -calculus are familial.

Familial functors are a generalisation of polynomial functors on sets, i.e., functors of the form  $F(X) = \sum_{o \in O} X^{n_o}$ , where  $O$  is a set of ‘operations’, and  $n_o \in \mathbb{N}$  is the ‘arity’ of any  $o \in O$ .

We now want to generalise this to presheaf categories. Consider for example the ‘free category’ monad  $T$  on  $\mathbf{Gph}$ . Analysing and abstracting over the definition of  $T$ , we will arrive at the notion of familial functor. Let us first recall that graphs are presheaves over the category  $[0] \begin{smallmatrix} \xrightarrow{s} \\ \xleftarrow{t} \end{smallmatrix} [1]$ .  $T$  does not change the vertex set, and an edge of  $T(G)$  is merely a path in  $G$ . Indexing this by the length of the path, we obtain

$$T(G)[0] = \mathbf{Gph}(y_{[0]}, G) \quad \text{and} \quad T(G)[1] = \sum_n \mathbf{Gph}([n], G),$$

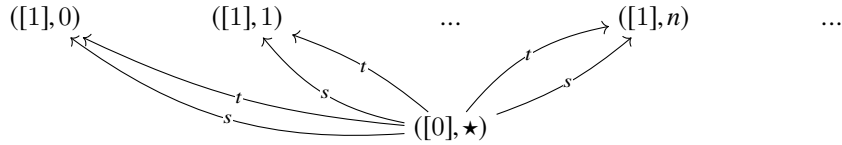
where  $[n]$  denotes the filiform graph  $\bullet \rightarrow \bullet \dots \rightarrow \bullet$  with  $n$  edges (which is consistent with  $[0]$  and  $[1]$  through the Yoneda embedding). Furthermore, the source of a path  $[n] \rightarrow G$  in  $T(G)$  is obtained as the composite

$$[0] \xrightarrow{s_n} [n] \rightarrow G,$$

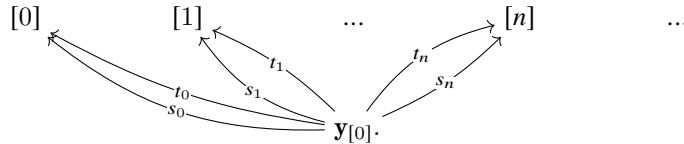
where the first morphism selects the first vertex of the path. Similarly the target is obtained by precomposition with the morphism, say  $t_n$ , selecting the last vertex.

Familial functors abstract over this situation as follows. First, let us recall from MacLane and Moerdijk [29] that the *category of elements*  $\text{el}(X)$  of a presheaf  $X$  over any category  $\mathbb{C}$  has pairs  $(c, x)$  with  $x \in X(c)$  as objects, and a morphism  $f \upharpoonright x' : (c, x) \rightarrow (c', x')$  for all  $f : c \rightarrow c'$  such that  $X(f)(x') = x$ .

The category of elements of  $T(1)$  looks like the following,



and the collection of all source and target maps  $y_{[0]} \rightarrow [n]$  forms a functor  $\text{el}(T(1)) \rightarrow \mathbf{Gph}$ , which we may visualise as



**Definition 7.9.** A functor  $F : \mathcal{A} \rightarrow \widehat{\mathbb{C}}$  to some presheaf category is familial iff we have a natural isomorphism

$$F(X)(c) \cong \sum_{o \in O(c)} \mathcal{A}(E(c, o), X),$$

for some presheaf  $O \in \widehat{\mathbb{C}}$  and functor  $E : \text{el}(O) \rightarrow \mathcal{A}$ . The presheaf  $O$  is called the presheaf of operations, or the spectrum [13] of  $F$ , while  $E$  is called the exponent.

*Remark 7.10.* If  $\mathcal{A}$  has a terminal object, we always have  $O \cong F(1)$ .

*Example 7.11.* Let us show that the endofunctor  $\Sigma_0 : \widehat{\mathbb{C}}_0 \rightarrow \widehat{\mathbb{C}}_0$  for  $\lambda$ -calculus is familial. We have

$$\begin{aligned} \Sigma_0(X)(n) &= X(n+1) + X(n)^2 \\ &\cong \widehat{\mathbb{C}}_0(\mathbf{y}_{n+1}, X) + \widehat{\mathbb{C}}_0(2 \cdot \mathbf{y}_n, X) \end{aligned}$$

Thus, we choose:

$$O(n) = \{\text{abs}, \text{app}\} \quad \begin{aligned} E(n, \text{abs}) &= \mathbf{y}_{n+1} \\ E(n, \text{app}) &= 2 \cdot \mathbf{y}_n. \end{aligned}$$

These definitions can be straightforwardly upgraded to functors  $O \in \widehat{\mathbb{C}}_0$  and  $E : \text{el}(O) \rightarrow \widehat{\mathbb{C}}_0$ , and we get the desired isomorphism.

*Example 7.12.* Let us now show that the functor  $\Sigma_1^s : \Sigma_0\text{-Mon} \rightarrow \widehat{\mathbb{C}}_1/\Delta_s$  for call-by-name  $\lambda$ -calculus is familial. By definition, it maps any  $X$  to the set-map  $X_0(1) + A_\beta(X) \rightarrow \Sigma_0(X_0)(0)$  defined in Example 3.15. Let us transfer this across the isomorphism  $\widehat{\mathbb{C}}_1/\Delta_s \cong \widehat{\mathbb{C}}_s$  (recalling Remark 7.3). On states, we almost may proceed as for  $\Sigma_0$ , except that the domain category has changed (from  $\widehat{\mathbb{C}}_0$  to  $\Sigma_0\text{-Mon}$ ). But recalling that  $\mathcal{L} : \mathbf{C} \rightarrow \Sigma_0\text{-Mon}$  denotes the left adjoint to the forgetful functor  $\mathcal{U}$ , we have

$$\begin{aligned} \Sigma_1^s(X)(n) &= \Sigma_0((\mathcal{U}(X))_0)(n) \\ &\cong \mathbf{C}(\mathbf{y}_{n+1}, \mathcal{U}(X)) + \mathbf{C}(2 \cdot \mathbf{y}_n, \mathcal{U}(X)) \\ &\cong \Sigma_0\text{-Mon}(\mathcal{L}(\mathbf{y}_{n+1}), X) + \Sigma_0\text{-Mon}(\mathcal{L}(2 \cdot \mathbf{y}_n), X), \end{aligned}$$

so we may (partially) define

$$O(n) = \{\text{abs}, \text{app}\} \quad \begin{aligned} E(n, \text{abs}) &= \mathcal{L}(\mathbf{y}_{n+1}) \\ E(n, \text{app}) &= \mathcal{L}(2 \cdot \mathbf{y}_n). \end{aligned} \tag{5}$$

Now, remembering from Notation 7.4 that we call  $\Downarrow \in \mathbb{C}_s$  the unique object of  $\mathbb{C}_s$  coming from  $\mathbb{C}_1 = 1$ , on transitions, we have:

$$\begin{aligned} \Sigma_1^s(X)(\Downarrow) &= X_0(1) + A_\beta(X) \\ &\cong \mathbf{C}(\mathbf{y}_1, \mathcal{U}(X)) + A_\beta(X) \\ &\cong \Sigma_0\text{-Mon}(\mathcal{L}(\mathbf{y}_1), X) + A_\beta(X). \end{aligned}$$

We thus need to find  $E_\beta$  such that  $A_\beta(X) \cong \Sigma_0\text{-Mon}(E_\beta, X)$ , and then we would complete equations (5) with:

$$O(\Downarrow) = \{\lambda\text{-val}, \beta\text{-red}\} \quad \begin{aligned} E(\Downarrow, \lambda\text{-val}) &= \mathcal{L}(\mathbf{y}_1) \\ E(\Downarrow, \beta\text{-red}) &= E_\beta. \end{aligned}$$

And indeed, let  $E_\beta$  denote the following pushout.



$$\begin{array}{ccccc}
\mathcal{L}(\mathbf{y}_0) & \xrightarrow{\bar{\chi}} & \mathcal{L}(\mathbf{y}_0 + \mathbf{y}_1) & \xrightarrow{\mathcal{L}(\mathbf{y}_0 + \mathbf{y}_{\Downarrow})} & \mathcal{L}(\mathbf{y}_0 + \mathbf{y}_{\Downarrow}) \\
\mathcal{L}(\mathbf{y}_{s_{\Downarrow}}) \downarrow & & & & \downarrow \text{in}_2 \\
\mathcal{L}(\mathbf{y}_{\Downarrow}) & \xrightarrow{\text{in}_1} & & & E_{\beta}
\end{array}$$

The morphism  $\bar{\chi}$  is defined to be the mate of some morphism  $\mathbf{y}_0 \rightarrow \mathcal{U}(\mathcal{L}(\mathbf{y}_0 + \mathbf{y}_1))$ , itself corresponding by Yoneda to some element of  $\mathcal{U}(\mathcal{L}(\mathbf{y}_0 + \mathbf{y}_1))(0)$ . The presheaf  $\mathcal{U}(\mathcal{L}(\mathbf{y}_0 + \mathbf{y}_1))$  has as states  $\lambda$ -terms over a closed constant  $k_0$ , and a unary constant  $k_1$ . We pick the element  $k_1(k_0)$ .

Let us now show that  $A_{\beta}(X) \cong \Sigma_0\text{-Mon}(E_{\beta}, X)$  for any  $X$ : as  $\Sigma_0\text{-Mon}(-, X)$  turns colimits into limits, we have the pullback

$$\begin{array}{ccccc}
[E_{\beta}, X] & \xrightarrow{\quad\quad\quad} & [\mathcal{L}(\mathbf{y}_{\Downarrow}), X] & & \\
\downarrow \lrcorner & & \downarrow [\mathcal{L}(\mathbf{y}_{s_{\Downarrow}}), X] & & \\
[\mathcal{L}(\mathbf{y}_0 + \mathbf{y}_{\Downarrow}), X] & \xrightarrow{\quad\quad\quad} & [\mathcal{L}(\mathbf{y}_0 + \mathbf{y}_1), X] & \xrightarrow{[\bar{\chi}, X]} & [\mathcal{L}(\mathbf{y}_0), X]
\end{array}$$

(where we abbreviate  $\Sigma_0\text{-Mon}(-1, -2)$  to  $[-1, -2]$  for readability). By Yoneda, this reduces to

$$\begin{array}{ccccc}
[E_{\beta}, X] & \xrightarrow{\quad\quad\quad} & X(\Downarrow) & & \\
\downarrow \lrcorner & & \downarrow X(s_{\Downarrow}) & & \\
X(0) \times X(\Downarrow) & \xrightarrow{X(0) \times X(t_{\Downarrow})} & X(0) \times X(1) & \xrightarrow{(t, u) \mapsto u[t]} & X(0),
\end{array}$$

which shows that we have  $A_{\beta}(X) \cong [E_{\beta}, X]$  as desired.

We have thus defined the actions of the functors  $O \in \widehat{\mathbb{C}}_s$  and  $E : \text{el}(O) \rightarrow \Sigma_0\text{-Mon}$  on objects. On morphisms, the only non-obvious point is the image of  $s_{\Downarrow} \uparrow \lambda\text{-val}$  and  $s_{\Downarrow} \uparrow \lambda\text{-red}$ . The former morphism is mapped to the identity on  $E(0, \text{abs}) = \mathcal{L}(\mathbf{y}_1) = E(\Downarrow, \lambda\text{-val})$ . The latter is mapped to the composite

$$\mathcal{L}(2 \cdot \mathbf{y}_0) \xrightarrow{\mathcal{L}(\mathbf{y}_0 + \mathbf{y}_{s_{\Downarrow}})} \mathcal{L}(\mathbf{y}_0 + \mathbf{y}_{\Downarrow}) \xrightarrow{\text{in}_2} E_{\beta}. \quad (6)$$

This achieves the desired isomorphism  $\Sigma_1^s(X)(c) \cong \sum_{o \in O(c)} \Sigma_0\text{-Mon}(E(c, o), X)$ .

### 7.3 Cellularity

We now want to exploit familiarity to obtain an alternative characterisation of preservation of functional bisimulations. The starting point is the observation that when a functor  $F : \mathcal{A} \rightarrow \widehat{\mathbb{C}}$  is familial, say as  $F(A)(c) = \sum_{o \in O(c)} \mathcal{A}(E(c, o), A)$ , then any morphism of the form  $f : \mathbf{y}_c \rightarrow F(A)$ , corresponding by Yoneda and familiarity to some pair  $(o, \phi)$  with  $\phi : E(c, o) \rightarrow A$ , factors as

$$\mathbf{y}_c \xrightarrow{(o, \text{id}_{E(c, o)})} F(E(c, o)) \xrightarrow{F(\phi)} F(A).$$

Furthermore, the first component  $(o, \text{id}_{E(c, o)})$  is easily seen to be *generic*, in the following sense.

**Definition 7.13.** Given any functor  $F : \mathcal{A} \rightarrow \mathcal{B}$ , a morphism  $\xi : B \rightarrow F(A)$  is  $F$ -generic (or generic for short) whenever any square of the form below (solid) admits a unique lifting  $k$  (dashed) such that  $F(k) \circ \xi = \chi$  and  $g \circ k = f$ .

$$\begin{array}{ccc} B & \xrightarrow{\chi} & F(C) \\ \xi \downarrow & \nearrow^{F(k)} & \downarrow F(g) \\ F(A) & \xrightarrow{F(f)} & F(D) \end{array}$$

In fact, we have the following important alternative characterisation of familial functors to presheaf categories.

**Theorem 7.14.** For any endofunctor  $F : \mathcal{A} \rightarrow \widehat{\mathbb{C}}$  such that  $\mathcal{A}$  has a terminal object,  $F$  is familial iff all morphisms  $f : X \rightarrow F(A)$  factor as

$$X \xrightarrow{\xi} F(U) \xrightarrow{F(\phi)} F(A),$$

with  $\xi$  generic.

*Proof.* See [38, Theorem 8.1] and [19, §3]. □

*Remark 7.15.* The factorisation is essentially unique.

Now, recalling Proposition 7.8, let us consider a situation in which both categories  $\mathcal{A}$  and  $\widehat{\mathbb{C}}$  are equipped with notions of functional bisimulation defined by lifting: we are given sets  $\mathbb{J}$  and  $\mathbb{K}$  of morphisms in  $\mathcal{A}$  and  $\widehat{\mathbb{C}}$ , respectively. Now for any familial functor  $F : \mathcal{A} \rightarrow \widehat{\mathbb{C}}$  and commuting square

$$\begin{array}{ccc} C & \xrightarrow{u} & F(A) \\ k \downarrow & & \downarrow F(f) \\ D & \xrightarrow{v} & F(B) \end{array} \quad (7)$$

with  $k \in \mathbb{K}$  and  $f \in \mathbb{J}^{\hat{h}}$ , we may take generic factorisations of both horizontal morphisms and use genericity to factor the original square as the solid part of

$$\begin{array}{ccccc} C & \xrightarrow{\xi} & F(X) & \xrightarrow{F(\phi)} & F(A) \\ k \downarrow & & \downarrow F(\delta) & \nearrow^{F(l)} & \downarrow F(f) \\ D & \xrightarrow{\chi} & F(Y) & \xrightarrow{F(\psi)} & F(B) \end{array}$$

Clearly, if  $\delta \in \hat{h}(\mathbb{J}^{\hat{h}})$ , then we find a lifting  $l$  as shown, which makes  $F(l) \circ \chi$  into a lifting for the original square. We have shown the ‘if’ part of

**Lemma 7.16.** Given categories  $\mathcal{A}$  and  $\widehat{\mathbb{C}}$  respectively equipped with sets of morphisms  $\mathbb{J}$  and  $\mathbb{K}$ , then a familial functor  $F : \mathcal{A} \rightarrow \widehat{\mathbb{C}}$  preserves fibrations (i.e., maps  $\mathbb{J}^{\hat{h}}$  to  $\mathbb{K}^{\hat{h}}$ ) iff for all commuting squares

$$\begin{array}{ccc} C & \xrightarrow{k} & D \\ \xi \downarrow & & \downarrow \chi \\ F(X) & \xrightarrow{F(\delta)} & F(Y) \end{array} \quad (8)$$

with  $k \in \mathbb{K}$  and  $\xi$  and  $\chi$  generic, we have  $\delta \in \mathfrak{h}(\mathbb{J}^{\mathfrak{h}})$ .

*Proof.* Conversely, let us assume that  $F$  preserves fibrations, and consider any square of the form (8). We need to show  $\delta \in \mathfrak{h}(\mathbb{J}^{\mathfrak{h}})$ . But for any commuting square as below left

$$\begin{array}{ccc} X & \xrightarrow{\phi} & A \\ \delta \downarrow & & \downarrow f \\ Y & \xrightarrow{\psi} & B \end{array} \qquad \begin{array}{ccccc} C & \xrightarrow{\xi} & F(X) & \xrightarrow{F(\phi)} & F(A) \\ k \downarrow & & \downarrow F(\delta) & \dashrightarrow \downarrow F(l) & \downarrow F(f) \\ D & \xrightarrow{\chi} & F(Y) & \xrightarrow{F(\psi)} & F(B) \end{array}$$

with  $f \in \mathbb{J}^{\mathfrak{h}}$ , by pasting this square with our generic square (8), we obtain the solid part above right. But because  $F$  preserves fibrations, we find a lifting  $\gamma$  as shown, which by genericness of  $\chi$  (and then  $\xi$ ) yields the desired lifting  $l$ .  $\square$

This applies to Howe contexts as follows.

**Corollary 7.17.** *In any Howe context, for any operational semantics signature  $(\Sigma_0, \Sigma_1)$ , if  $\Sigma_1^s$  is familial with exponent  $E : \text{el}(\Sigma_1^s(1)) \rightarrow \Sigma_0\text{-Mon}$ , then it preserves functional bisimulations iff for all  $L \in \mathbb{C}_1$  and  $o \in \Sigma_1^s(1)(L)$ ,  $E(s_L \uparrow o) : E(\mathbf{s}(L), o \cdot s_L) \rightarrow E(L, o)$  is a fibration.*

This characterisation of preservation of functional bisimulations in terms of cofibrations is easier to prove in practice, since cofibrations in turn admit the following well-known characterisation.

**Definition 7.18.** *Consider any set  $\mathbb{J}$  of maps in a given category.*

- A basic relative  $\mathbb{J}$ -cell complex is any morphism obtained by pushing out some morphism from  $\mathbb{J}$  along any morphism, as  $f$  in

$$\begin{array}{ccc} A & \xrightarrow{j \in \mathbb{J}} & B \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow{f} & D. \end{array}$$

- A relative  $\mathbb{J}$ -cell complex is a (potentially transfinite) composite of basic relative  $\mathbb{J}$ -cell complexes.

**Proposition 7.19** ([23, Lemma 2.1.10]). *For any set  $\mathbb{J}$  of maps in a locally presentable category, all relative  $\mathbb{J}$ -cell complexes are cofibrations.*

*Example 7.20.* It is not entirely trivial that the map  $E(s_{\mathbb{J}} \uparrow in_{\mathbb{J},2}(\star))$  corresponding to the second transition rule, defined as the composite (6), is a cofibration. But, as (essentially) noted in [10, Example 5.21], it is a relative complex by construction as both components are pushouts of  $\mathcal{L}(\mathbf{y}_{s_{\mathbb{J}}})$ .

*Example 7.21.* The non-deterministic  $\lambda$ -calculus investigated in [33, §7], may be presented by adding a unary operation  $\wp$  to the syntax of pure  $\lambda$ -calculus, renaming  $\Downarrow$  to  $\Downarrow_\lambda$  (keeping the rules of §2.3), adding a new evaluation relation  $\Downarrow_\tau$  (between closed terms) and adding axioms  $\overline{\wp e \Downarrow_\tau e}$  and  $\overline{\wp e \Downarrow_\tau \Omega}$ , where  $\Omega$  denotes any diverging term, together with rules

$$\frac{e_1 \Downarrow_\lambda e_3 \quad e_3[e_2] \Downarrow_\tau e_4}{e_1 e_2 \Downarrow_\tau e_4} \quad \frac{e_1 \Downarrow_\tau e_3}{e_1 e_2 \Downarrow_\tau e_3 e_2} \quad \frac{}{e \Downarrow_\tau e} \quad \frac{e_1 \Downarrow_\tau e_2 \quad e_2 \Downarrow_\tau e_3}{e_1 \Downarrow_\tau e_3} .$$

By the first rule,  $\Downarrow_\tau$  includes  $\beta$ -reduction; the last two ensure that it is reflexive and transitive, and the second one that it applies the function part of an application. It thus coincides with Sangiorgi's relation  $\Rightarrow$  (albeit with redundant rules). The syntax is easily modelled by taking  $\Sigma_0(X)(n) = X(n+1) + X(n)^2 + X(n)$ , and the rules are easily seen to fit into a  $\Sigma_1$  such that  $\Sigma_1^s$  preserves functional bisimulations by Corollary 7.17. Finally, substitution-closed bisimilarity in the initial model coincides with Sangiorgi's applicative bisimilarity.

## 8 Congruence of substitution-closed bisimilarity

In this section, we elaborate on the proof sketch of Theorem 6.13 given in §6. The overall structure remains the same, and the final part of the proof sketch is complete, so we mainly elaborate on items (1)–(4).

### 8.1 Preliminaries on spans

In this section, we fix a bicomplete category  $\mathcal{C}$ , and develop some tools about spans, including the categorified notions of reflexivity, transitivity, symmetry, and transitive closure. We switch freely from spans  $X \leftarrow S \rightarrow Y$  to their pairing  $S \rightarrow X \times Y$  in  $\mathcal{C}/X \times Y$ . Furthermore, we denote sequential composition (constructed by pullback) by  $;$ .

**Definition 8.1.** *A span  $X \leftarrow S \rightarrow X$  is reflexive if there is a morphism from the diagonal to  $S$  in  $\mathcal{C}/X^2$ . It is transitive if there is a morphism  $S; S \rightarrow S$  over  $X^2$ . Finally, it is symmetric if there is a morphism  $S^\dagger \rightarrow S$ , where  $(-)^{\dagger}$  denotes the functor swapping projections.*

For potentially non-reflexive spans, we will use the following reflexive transitive closure.

**Definition 8.2.** *The reflexive transitive closure  $S^*$  of any span  $X \leftarrow S \rightarrow X$  is the coproduct  $\sum_{n \in \mathbb{N}} S^n$ , or for short  $\sum_{n \in \mathbb{N}} S^n$  when the context is clear, where  $S^n$  denotes iterated span composition of  $S$  with itself, with  $S^0 = X$ .*

At some point, we will also use the relational transitive closure.

**Notation 8.3.** *Given a span  $S$  on  $X$ , we denote by  $\bar{S}$  the induced relation on  $X$  by the image factorisation of  $S \rightarrow X \times X$ .*

**Definition 8.4.** *The relational transitive closure  $S^{\bar{\dagger}}$  of a span  $S$  on  $X$  is the union  $\bigcup_{n>0} \overline{S^n}$ .*

Note that the relational transitive closure commutes with symmetry.

**Lemma 8.5.**  $S^{\dagger\bar{\dagger}} \cong S^{\bar{\dagger}}$

The following lemma will be later used to exploit preservation of sifted colimits by  $\Sigma_0$ .

**Lemma 8.6.** *If  $S$  is a reflexive span on  $X$ , then  $S^{\bar{\dagger}}$  is the (filtered) colimit of the chain*

$$X \rightarrow \overline{S} \cong \overline{S}; X \rightarrow \overline{S}; \overline{S} \cong \overline{S}; \overline{S}; X \rightarrow \overline{S}; \overline{S}; \overline{S} \rightarrow \dots$$

The next result will be useful to show that the relational transitive closure of the Howe closure of substitution-closed bisimilarity is symmetric on states. Let us first observe that if  $R$  is reflexive, then so is  $R^{\bar{\dagger}}$ .

**Lemma 8.7.** *For any reflexive span  $R \rightarrow X^2$ ,  $R^{\bar{\dagger}}$  is symmetric if there is a span morphism  $R \rightarrow R^{\bar{\dagger}}$ .*

*Proof.* Assume given a morphism  $j : R \rightarrow R^{\bar{\dagger}}$ . Consider the composite

$$R^{\bar{\dagger}} \cong \bigcup_{n>0} \overline{R^n} \rightarrow \bigcup_{n>0} \overline{R^{\bar{\dagger};n}} \cong \bigcup_{n>0} \overline{R^{+;n\dagger}} \rightarrow \bigcup_{n>0} \overline{R^{\bar{\dagger}}} \cong \bigcup_{n>0} R^{\bar{\dagger}} \cong R^{\bar{\dagger}},$$

where the first morphism is obtained from  $j$ , and the second one is obtained from morphisms  $R^{\bar{\dagger};n} \rightarrow R^{\bar{\dagger}}$ .  $\square$

## 8.2 Howe closure on states

We fix an operational semantics signature  $(\Sigma_0, \Sigma_1)$  on a Howe context  $\mathbf{s}, \mathbf{t} : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ .

**Definition 8.8.** *Let  $\Sigma_0^H : \mathbb{C}_0/\mathbf{Z}_0^2 \rightarrow \mathbb{C}_0/\mathbf{Z}_0^2$  map any span  $X \rightarrow \mathbf{Z}_0^2$  to*

$$\Sigma_0(X) + (X; \sim_0^\otimes) \rightarrow \Sigma_0(\mathbf{Z}_0)^2 + \mathbf{Z}_0^2 \rightarrow \mathbf{Z}_0^2.$$

**Proposition 8.9.** *The functor  $\Sigma_0^H$  is pointed strong.*

For proving this, we first need the following result.

**Lemma 8.10.** *For any monoid  $X$ , there is a natural transformation with components  $\delta_{U,V,W} : (U; V) \otimes W \rightarrow U \otimes W; V \otimes X$  in  $\mathbb{C}_0/X^2$ .*

*Proof.* Let  $m : X \otimes X \rightarrow X$  denote the multiplication. By tensoring the defining pullback of  $U; V$  with  $W$  we obtain the back face below.

$$\begin{array}{ccccc}
 (U; V) \otimes W & \longrightarrow & V \otimes W & & \\
 \downarrow & \dashrightarrow & \downarrow & \searrow^{V \otimes \pi_2} & \\
 & (U \otimes W); (V \otimes X) & \downarrow \pi_1 \otimes W & \longrightarrow & V \otimes X \\
 U \otimes W & \xrightarrow{\quad} & X \otimes W & \xrightarrow{\quad} & \downarrow m \circ (\pi_1 \otimes X) \\
 \parallel & \downarrow \pi_2 \otimes W & \searrow^{m \circ (X \otimes \pi_2)} & & \\
 U \otimes W & \xrightarrow{\quad} & U \otimes W & \xrightarrow{m \circ (\pi_2 \otimes \pi_2)} & X
 \end{array}$$

By universal property of pullback, we then get the dashed arrow making all faces commute, which gives our candidate  $\delta_{U,V,W}$ . Naturality follows by universal property of pullback.  $\square$

*Proof of Proposition 8.9.* Pointed strong endofunctors are closed under coproducts, so it suffices to show that  $-; \sim_0^\otimes$  and the lifting of  $\Sigma_0$  to  $\mathbf{C}_0/\mathbf{Z}_0^2$  are both pointed strong. The latter inherits the pointed strength of  $\Sigma_0$ , while the pointed strength of the former follows from Lemma 8.10 and substitution-closedness of  $\sim_0^\otimes: (X; \sim_0^\otimes) \otimes Y \rightarrow (X \otimes Y); (\sim_0^\otimes \otimes \mathbf{Z}_0) \rightarrow (X \otimes Y); \sim_0^\otimes$ .  $\square$

Presheaf categories being well-known to be closed under the slice construction, we have the following.

**Lemma 8.11.** *The category  $\mathbf{C}_0/\mathbf{Z}_0^2$  is a presheaf category.*

**Lemma 8.12.** *The endofunctor  $\Sigma_0^H$  is finitary.*

*Proof.* By commutation of filtered colimits with finite limits in presheaf categories.  $\square$

By Proposition 8.9 and the previous lemmas, the following is legitimate.

**Definition 8.13.** *Let  $H_0 = \mathbf{Z}_{\Sigma_0^H}$  denote the initial  $\Sigma_0^H$ -monoid.*

By Proposition 5.1, we also get the following for free.

**Proposition 8.14.** *The object  $H_0 \rightarrow \mathbf{Z}_0^2$  is an initial algebra for the endofunctor  $\mathbf{C}_0/\mathbf{Z}_0^2 \rightarrow \mathbf{C}_0/\mathbf{Z}_0^2$  mapping any  $X \rightarrow \mathbf{Z}_0^2$  to  $I + \Sigma_0^H(X) \rightarrow \mathbf{Z}_0^2$ .*

**Proposition 8.15.** *The underlying object  $H_0$  is a  $\Sigma_0$ -monoid.*

*Proof.* Directly follows from the  $\Sigma_0^H$ -monoid structure.  $\square$

Next, we exhibit an alternative characterisation of  $H_0$ , which relies on the following result.

**Lemma 8.16 (Packing lemma).** *Consider finitary endofunctors  $F$  and  $G$  on a cocomplete category  $\mathbf{C}$ , and let  $F^*(A) \cong \mu S.(A + F(S))$  denote the ‘free  $F$ -algebra’ monad [32, Theorem 2.1]. Then we have  $\mu S.(FS + GS) \cong \mu S.F^*(G(S))$ .*

*Proof.* Indeed, we have

$$\begin{aligned}
\mu S.F^\star(G(S)) &\cong \mu S.\mu U.(G(S) + F(U)) \\
&\cong \mu S.\mu U.(F(U) + G(S)) \\
&\cong \mu S.(F(S) + G(S)) \quad (\text{by the Diagonal rule [7, Theorem 16]}).
\end{aligned}$$

□

**Proposition 8.17.** *The object  $H_0 \rightarrow \mathbf{Z}_0^2$  is an initial algebra for the endofunctor  $\Sigma_0^{H'} : \mathbf{C}_0/\mathbf{Z}_0^2 \rightarrow \mathbf{C}_0/\mathbf{Z}_0^2$  mapping any  $X \rightarrow \mathbf{Z}_0^2$  to  $I; \sim_0^{\otimes*} + \Sigma_0(X); \sim_0^{\otimes*} \rightarrow \mathbf{Z}_0^2$ .*

*Proof.* Taking  $F(S) = S; \sim_0^{\otimes}$  and  $G(S) = \Sigma_0(S)$  in Lemma 8.16, and observing that  $F$  preserves coproducts (because pullback along the first projection  $\sim_0^{\otimes} \rightarrow \mathbf{Z}_0$ , as a left adjoint, preserves colimits), so that  $F^\star(U) \cong \sum_n F^n(U)$ , we have (by commutation of coproducts with  $U; -$ )

$$F^\star(U) \cong \sum_n F^n(U) \cong \sum_n U; (\sim_0^{\otimes})^n \cong U; \sum_n (\sim_0^{\otimes})^n = U; \sim_0^{\otimes*}.$$

Thus,  $F^\star(G(S)) \cong (I + \Sigma_0(S)); \sim_0^{\otimes*} \cong (I; \sim_0^{\otimes*}) + (\Sigma_0(S); \sim_0^{\otimes*})$ , as desired. □

### 8.3 Double categorical notation

Our next goal is to define the Howe closure on transitions. For this, we appeal to Morton's *double bicategories* [30]. They are a refinement of double categories, in which both the horizontal and vertical categories are actually bicategories. We rely in particular on his Theorem 4.1.3, which (when dualised) states that for any category  $\mathcal{C}$  with pullbacks, there is a double bicategory  $2Sp(\mathcal{C})$ :

- objects are objects of  $\mathcal{C}$ ,
- both the vertical and horizontal bicategories are  $Span(\mathcal{C})$ ,
- cells, called *double spans*, are precisely commuting diagrams of the following form.

$$\begin{array}{ccccc}
A & \longleftarrow & B & \longrightarrow & C \\
\uparrow & & \uparrow & & \uparrow \\
A' & \longleftarrow & B' & \longrightarrow & C' \\
\downarrow & & \downarrow & & \downarrow \\
A'' & \longleftarrow & B'' & \longrightarrow & C''
\end{array} \tag{9}$$

We will not need the rest of the structure. All we need to know is that cells compose horizontally and vertically just as in a double category. We will use the double bicategories  $2Sp(\mathbf{C}_0)$  and  $2Sp(\widehat{\mathbf{C}}_1)$ .

**Notation 8.18.** *We use the following notational conventions.*

- We denote cells  $2Sp(\mathbf{C}_0)$  such as (9) above by

$$\begin{array}{ccc}
A & \xrightarrow{\circlearrowright B} & C \\
A' \downarrow & & \downarrow C' \\
A'' & \xrightarrow{\circlearrowleft B''} & C''
\end{array}$$

– Furthermore, cells in  $\widehat{\mathbf{C}}_1$  of the form below left will be denoted as below right.

$$\begin{array}{ccc}
 X_0\mathbf{s} & \leftarrow S_0\mathbf{s} & \longrightarrow Y_0\mathbf{s} \\
 \uparrow & \vdots & \uparrow \\
 X_1 & \leftarrow S_1 & \longrightarrow Y_1 \\
 \downarrow & \vdots & \downarrow \\
 X'_0\mathbf{t} & \leftarrow S'_0\mathbf{t} & \longrightarrow Y'_0\mathbf{t}
 \end{array}
 \qquad
 \begin{array}{ccc}
 X_0 & \xrightarrow{S_0} & Y_0 \\
 x_1 \downarrow & S_1 & \downarrow y_1 \\
 X'_0 & \xrightarrow{S'_0} & Y'_0
 \end{array}
 \quad (10)$$

Explicitly, spans of the form  $X_0\mathbf{s} \leftarrow X_1 \rightarrow X'_0\mathbf{t}$  are denoted by  $X_0 \xrightarrow{X_1} X'_0$ ,

while spans of the form  $X_0 \leftarrow S_0 \rightarrow Y_0$  are still denoted by  $X_0 \xrightarrow{S_0} Y_0$ , but silently coerced by  $\Delta_s$  or  $\Delta_t$  depending on context.

- For both types of cells, we collapse identity borders, as usual.
- When a span is trivial on one side, we use standard arrows for its borders, and a double arrow for its middle arrow, all in the relevant direction. E.g., the diagram below left may be depicted as below right.

$$\begin{array}{ccc}
 A & \leftarrow B & \longrightarrow C \\
 a \uparrow & & \uparrow b & \uparrow c \\
 A' & \leftarrow B' & \longrightarrow C'
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{B} & C \\
 a \uparrow & b \uparrow & \uparrow c \\
 A' & \xrightarrow{B'} & C'
 \end{array}$$

Cells of the form (10) live in  $2Sp(\widehat{\mathbf{C}}_1)$ , hence may be composed horizontally. Relevant examples of vertical composition will be obtained by embedding cells of the form (9) along  $\Delta_s$  (resp.  $\Delta_t$ ), and vertically composing with cells of the form (10) in  $2Sp(\widehat{\mathbf{C}}_1)$ . This yields a top (resp. bottom) action of  $2Sp(\mathbf{C}_0)$ , which we both denote by mere pasting.

**Lemma 8.19.** *Given a composable pasting diagram made of cells of both types, any two parsings agree up to isomorphism.*

*Proof.* By interchange of limits. □

#### 8.4 Howe closure on transitions

Let us now define the Howe closure on transitions. First, we delineate an ambient category  $\mathbf{C}_Z^H$ . The idea is that objects of this category should be transition systems  $S \rightarrow \mathbf{Z}^2$  over  $\mathbf{Z}^2$  whose image under the projection  $\mathbf{C}/\mathbf{Z} \rightarrow \mathbf{C}_0/\mathbf{Z}_0$  is precisely  $H_0 \rightarrow \mathbf{Z}_0^2$ . Thus, object of  $\mathbf{C}_Z^H$  consists of an object  $S_1 \in \mathbf{C}_1$ , equipped with a dashed cone to the outer part of the diagram below.

$$\begin{array}{ccc}
 \mathbf{Z}_0\mathbf{s} & \leftarrow H_0\mathbf{s} & \longrightarrow \mathbf{Z}_0\mathbf{s} \\
 \uparrow & \vdots & \uparrow \\
 \mathbf{Z}_1 & \leftarrow S_1 & \longrightarrow \mathbf{Z}_1 \\
 \downarrow & \vdots & \downarrow \\
 \mathbf{Z}_0\mathbf{t} & \leftarrow H_0\mathbf{t} & \longrightarrow \mathbf{Z}_0\mathbf{t}
 \end{array}
 \quad (11)$$



Equivalently, they are morphisms over the limit, so that we may define  $\mathbf{C}_Z^H$  as a slice category by merely stating the following.

**Definition 8.20.** Let  $R^{\partial H}$  denote the limit of the outer part of (11).

**Definition 8.21.**  $\mathbf{C}_Z^H$  is the category of cones over the outer part of (11), or equivalently, it is the slice category  $\mathbf{C}_1/R^{\partial H}$ .

**Proposition 8.22.** The initial object in  $\mathbf{C}_Z^H$  is the span  $\mathbf{Z} \leftarrow H_0 \rightarrow \mathbf{Z}$ .

**Definition 8.23.** Let  $\Sigma_1^H : \mathbf{C}_Z^H \rightarrow \mathbf{C}_Z^H$  map any object  $\mathbf{Z} \leftarrow S \rightarrow \mathbf{Z}$  to the coproduct of the following two pastings.

$$\begin{array}{ccc}
 \begin{array}{c}
 \mathbf{Z}_0 \xrightarrow{H_0} \mathbf{Z}_0 \\
 \uparrow \quad \uparrow \\
 \Sigma_0(\mathbf{Z}_0) \xrightarrow{\Sigma_0(H_0)} \Sigma_0(\mathbf{Z}_0) \\
 \downarrow \Sigma_1^F(\mathbf{Z}) \quad \downarrow \Sigma_1^F(S) \quad \downarrow \Sigma_1^F(\mathbf{Z}) \\
 \mathbf{Z}_0 \xrightarrow{H_0} \mathbf{Z}_0 \\
 \leftarrow \Sigma_1^F(\mathbf{Z}) \quad \leftarrow \Sigma_1^F(S) \quad \leftarrow \Sigma_1^F(\mathbf{Z}) \rightarrow
 \end{array}
 &
 \begin{array}{c}
 H_0 \\
 \uparrow \\
 \mathbf{Z}_0 \xrightarrow{H_0} \mathbf{Z}_0 \xrightarrow{\sim_0^\otimes} \mathbf{Z}_0 \\
 \downarrow S \quad \downarrow \sim_1^\otimes \\
 \mathbf{Z}_0 \xrightarrow{H_0} \mathbf{Z}_0 \xrightarrow{\sim_0^\otimes} \mathbf{Z}_0 \\
 \downarrow H_0 \\
 H_0
 \end{array}
 & (12)
 \end{array}$$

**Proposition 8.24.** The functor  $\Sigma_1^H : \mathbf{C}_Z^H \rightarrow \mathbf{C}_Z^H$  is finitary.

*Proof.* The forgetful functor  $\mathbf{C}_Z^H \cong \mathbf{C}_1/R^{\partial H} \rightarrow \mathbf{C}_1$  creates colimits, so it suffices to show that the composite  $\mathbf{C}_Z^H \xrightarrow{\Sigma_1^H} \mathbf{C}_Z^H \rightarrow \mathbf{C}_1$  is finitary. This functor maps any  $S$  to  $\Sigma_1^F(S) + S_1; \sim_1^\otimes$ , hence is finitary because  $\Sigma_1$  is and  $-; \sim_1^\otimes$  is cocontinuous.  $\square$

The last result legitimates the following definition.

**Definition 8.25.** Let  $H$  denote the initial (vertical)  $\Sigma_1^H$ -algebra.

We readily can prove the following.

**Lemma 6.14.** There exists a span morphism  $\sim^\otimes \rightarrow H$ .

*Proof.* By construction, the underlying object of  $H$  is in particular a  $\check{\Sigma}_1$ -algebra, so by initiality we obtain a unique span morphism  $\mathbf{Z} \rightarrow H$ . Furthermore, again by construction,  $H$  is an algebra for the endofunctor  $-; \sim^\otimes$  on  $\mathbf{C}/\mathbf{Z}^2$ . We thus may form the composite  $\sim^\otimes \cong \mathbf{Z}; \sim^\otimes \rightarrow H; \sim^\otimes \rightarrow H$ .  $\square$

## 8.5 Alternative characterisations of the Howe closure

In this section, we exhibit a few alternative characterisations of the Howe closure on transitions. The definition in the previous section is convenient for proving that the transitive closure is symmetric, while our final alternative characterisation will enable a conceptual proof of the simulation property.

First of all, by the packing Lemma 8.16, we readily have:

**Lemma 8.27.** *The Howe closure  $H$  is (isomorphic to) the initial algebra of the endofunctor  $\Sigma_{1,pack}^H : \mathbf{C}_Z^H \rightarrow \mathbf{C}_Z^H$  mapping any  $\mathbf{Z} \leftarrow S \rightarrow \mathbf{Z}$  to the following pasting.*

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & H_0 & & \\
 & & \circ & & \\
 & \swarrow & & \searrow & \\
 \mathbf{Z}_0 & \xrightarrow{H_0} & \mathbf{Z}_0 & \xrightarrow{=} & \mathbf{Z}_0 & \xrightarrow{\sim_0^{**}} & \mathbf{Z}_0 \\
 \uparrow & & \uparrow & & \downarrow & & \downarrow \\
 \mathbf{z}_1 \leftarrow \Sigma_0(\mathbf{Z}_0) & \xrightarrow{\Sigma_0(H_0)} & \Sigma_0(\mathbf{Z}_0) & \xrightarrow{=} & \mathbf{z}_1 & \xrightarrow{\sim_1^{**}} & \mathbf{z}_1 \\
 \downarrow \Sigma_1^F(\mathbf{z}) & & \downarrow \Sigma_1^F(S) & & \downarrow \Sigma_1^F(\mathbf{z}) & & \downarrow \Sigma_1^F(\mathbf{z}) \\
 \mathbf{Z}_0 & \xrightarrow{H_0} & \mathbf{Z}_0 & \xrightarrow{=} & \mathbf{Z}_0 & \xrightarrow{\sim_0^{**}} & \mathbf{Z}_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H_0 & & H_0 & & H_0
 \end{array} \\
 \end{array} \tag{13}$$

The next characterisation of  $H$  relies on the following lemma with  $G = \Sigma_{1,pack}^H$ .

**Lemma 8.28.** *Consider any square*

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{J} & \mathcal{D} \\
 F \downarrow & \cong & \downarrow G \\
 \mathcal{C} & \xrightarrow{J} & \mathcal{D}
 \end{array}$$

of finitary functors between cocomplete categories, which commutes up to natural isomorphism, and let  $\mathbf{Z}_F$  denote the initial  $F$ -algebra. If  $J$  preserves the initial object, then  $J(\mathbf{Z}_F)$  is an initial  $G$ -algebra, i.e.,  $J(\mathbf{Z}_F) \cong \mathbf{Z}_G$ .

*Proof.* Let us start from the standard construction of the initial  $G$ -algebra  $\mathbf{Z}_G$  as a colimit of the initial chain to obtain:

$$\begin{aligned}
 \mathbf{Z}_G &\cong \text{colim}_i G^i \emptyset \\
 &\cong \text{colim}_i G^i(J\emptyset) \quad (\text{by hypothesis}) \\
 &\cong \text{colim}_i J(F^i(\emptyset)) \quad (\text{by } GJ = JF) \\
 &\cong J(\text{colim}_i(F^i(\emptyset))) \quad (\text{by finitariness of } J) \\
 &\cong J(\mathbf{Z}_F). \quad \square
 \end{aligned}$$

The category  $\mathbf{C}^H$  we will use in applying the lemma is a relaxation of  $\mathbf{C}_Z^H$ , in which the left-hand object in (11) is only forced to coincide with  $\mathbf{Z}$  on  $\widehat{\mathbf{C}}_0$ .

**Definition 8.29.** *Let  $\mathbf{C}^H$  denote the category whose objects are  $X_1$  and  $S_1$  in  $\mathbf{C}_1$  equipped with dashed arrows in the following diagram, making it commute.*

$$\begin{array}{ccccc}
 \mathbf{Z}_0\mathbf{s} & \longleftarrow & H_0\mathbf{s} & \longrightarrow & \mathbf{Z}_0\mathbf{s} \\
 \uparrow \text{---} & & \uparrow \text{---} & & \uparrow \text{---} \\
 X_1 & \longleftarrow \text{---} & S_1 & \text{---} \longrightarrow & Z_1 \\
 \downarrow \text{---} & & \downarrow \text{---} & & \downarrow \text{---} \\
 \mathbf{Z}_0\mathbf{t} & \longleftarrow & H_0\mathbf{t} & \longrightarrow & \mathbf{Z}_0\mathbf{t}
 \end{array} \tag{14}$$



## 8.6 Simulation property

Our next goal is to prove the following.

**Lemma 6.15.** *If  $\Sigma_1$  preserves functional bisimulations, then the transition Howe closure  $H$  is a substitution-closed simulation.*

For substitution-closedness,  $\sim_0^{\otimes}$  is reflexive and by Lemma 6.14 we have a span morphism  $\sim_0^{\otimes} \rightarrow H_0$ , so we may form the composite

$$H_0 \otimes \mathbf{Z} \rightarrow H_0 \otimes \sim_0^{\otimes} \rightarrow H_0 \otimes H_0 \rightarrow H_0,$$

where the last morphism is the monoid multiplication of  $H_0$ , as established in Proposition 8.15.

For the simulation property, we will use the characterisation of  $H$  as  $\mathbf{Z}_{\Sigma_1^H}$ .

Let us start with the following few results.

**Lemma 8.37.** *A span  $X \leftarrow S \rightarrow Y$  is a simulation iff the top left square in (10) is a pointwise weak pullback.*

*Proof.* By Proposition 4.15. □

**Lemma 8.38.** *Simulations are stable under vertical and horizontal composition in  $2Sp(\widehat{\mathbb{C}}_1)$ .*

*Proof.* Follows from easy pointwise weak pullback lemmas. □

**Lemma 8.39.** *If  $\Sigma_1$  preserves simulations and  $S \in \mathbf{C}^H$  underlies a simulation, then so does  $\Sigma_{1, lax}^H(S)$ .*

*Proof.* The pasting (15) is isomorphic to the following.

$$\begin{array}{ccccc}
 & \mathbf{Z}_0 & \xrightarrow{H_0} & \mathbf{Z}_0 & \\
 & \uparrow & & \uparrow & \\
 \tilde{\Sigma}_1(X)_1 & \Sigma_0(\mathbf{Z}_0) & \xrightarrow{\Sigma_0(H_0)} & \Sigma_0(\mathbf{Z}_0) & \xrightarrow{\sim_0^{\otimes*}} & \mathbf{Z}_0 & \xrightarrow{\sim_0^{\otimes*}} & \mathbf{Z}_0 \\
 & \downarrow \Sigma_1^F(X) & & \downarrow \Sigma_1^F(S) & \Rightarrow & \downarrow \mathbf{Z}_1 & \sim_1^{\otimes*} & \downarrow \mathbf{Z}_1 \\
 & \mathbf{Z}_0 & \xrightarrow{H_0} & \mathbf{Z}_0 & \xrightarrow{=} & \mathbf{Z}_0 & \xrightarrow{\sim_0^{\otimes*}} & \mathbf{Z}_0 \\
 & & & \downarrow & & \downarrow & & \uparrow \\
 & & & \tilde{H}_0 & & & & 
 \end{array} \tag{16}$$

By Lemma 8.38, it suffices to show that all cells in (16) are simulations if  $S$  is. Let us run through them, left-to-right, top-to-bottom:

- The top cell is a simulation because
  - $\mathbf{Z}_0$  is the initial  $(I + \Sigma_0)$ -algebra, so  $I \rightarrow \mathbf{Z}_0 \leftarrow \Sigma_0(\mathbf{Z}_0)$  is a coproduct diagram;
  - similarly, by Proposition 8.17,  $H_0$  is the initial algebra for the endofunctor  $X \mapsto (I; \sim_0^{\otimes*} + \Sigma_0(X); \sim_0^{\otimes*})$ , so  $I; \sim_0^{\otimes*} \rightarrow H_0 \leftarrow \Sigma_0(H_0); \sim_0^{\otimes*}$  is a coproduct diagram;

- so, by extensivity of  $\mathbf{C}_0$ , both squares in

$$\begin{array}{ccccc}
 I; \sim_0^{\otimes*} & \longrightarrow & H_0 & \longleftarrow & \Sigma_0(H_0); \sim_0^{\otimes*} \\
 \downarrow \lrcorner & & \downarrow & & \downarrow \lrcorner \\
 I & \longrightarrow & \mathbf{Z}_0 & \longleftarrow & \Sigma_0(\mathbf{Z}_0)
 \end{array}$$

are pullbacks. The right-hand one is mapped by  $\Delta_s$  (which, as a left adjoint, preserves pullbacks) precisely to the top left square of the top cell.

- The first little cell  $\Sigma_1(S)$  is a simulation by hypothesis.
- The middle little cell is trivially a simulation.
- The third little cell  $\sim_1^{\otimes*}$  is a simulation because simulations are closed under transitive closure.
- Finally, the bottom cell is trivially a simulation.  $\square$

Finally:

*Proof of Lemma 6.15.* Because the domains and codomains of all  $s_L$  are representable, hence finitely presentable, functional bisimulations are closed under filtered colimits in the arrow category by Lemma 1.1. But  $H$  is the initial chain of  $\Sigma_{1, lax}^H$ , so by Lemma 8.39 it suffices to show that the initial object of  $\mathbf{C}^H$  underlies a simulation. By Proposition 8.31, this amounts to showing that the first projection  $H_0 \rightarrow \mathbf{Z}_0$  is a functional simulation, which is trivially the case.  $\square$

## 8.7 Symmetry of transitive closure

In this section, we prove the following.

**Lemma 6.17.** *The relational transitive closure  $H_0^{\bar{\dagger}}$  of the Howe closure  $H_0$  on states is symmetric.*

**Lemma 6.19.** *For any substitution-closed simulation  $R$  such that  $R_0$  is symmetric, there exists a span morphism  $R \rightarrow R'$  such that  $R'$  is a substitution-closed bisimulation.*

By Lemma 8.7, Lemma 6.17 will follow if we construct a span morphism  $H_0 \rightarrow H_0^{\bar{\dagger}}$ . As  $H_0$  is an initial algebra for  $I + \Sigma_0^H$  (Proposition 8.14), it suffices to prove the following lemma.

**Lemma 8.42.** *The span  $(H^{\bar{\dagger}})_0 = H_0^{\bar{\dagger}}$  has an algebra structure for  $(I + \Sigma_0^H)$ .*

This relies on the following lemmas, used in particular with  $F = \Sigma_0$ .

**Lemma 8.43.** *Given an endofunctor  $F$  on some category  $\mathcal{C}$ , the forgetful functor  $F\text{-alg} \rightarrow \mathcal{C}F\text{-algebras}$*

- *creates all limits;*
- *creates colimits that  $F$  preserves.*

**Lemma 8.44.** *Given an endofunctor  $F$  on some category  $\mathcal{C}$  with pullbacks and coequalisers, if  $F$  preserves reflexive coequalisers, then the forgetful functor from the category of  $F$ -algebras creates image factorisations.*

*Proof.* Suppose given an algebra morphism  $A \xrightarrow{f} B$ . The image factorisation is obtained as the (reflexive) coequaliser of the kernel pair  $A \times_f A \rightrightarrows A$ . The diagram of this reflexive coequaliser lifts to  $F$ -alg, hence so does the coequaliser, by the previous lemma.  $\square$

*Proof of Lemma 8.42.* We need to find algebra structures on  $H_0^{\bar{\dagger}}$  for  $I, \Sigma_0$ , and  $-; \sim_0^\otimes$ . For  $I$ , we have the morphism  $I \rightarrow \mathbf{Z}_0 \rightarrow H_0^\dagger \rightarrow H_0^{\bar{\dagger}}$ .

For  $\Sigma_0$ , note that by Lemmas 8.6 and 8.5,  $H_0^{\bar{\dagger}}$  is the colimit of the chain

$$\mathbf{Z}_0 \rightarrow \overline{H_0^\dagger} \cong \overline{H_0^\dagger; \mathbf{Z}_0} \rightarrow \overline{H_0^\dagger; H_0^\dagger} \cong \overline{H_0^\dagger; H_0^\dagger; \mathbf{Z}_0} \rightarrow \overline{H_0^\dagger; H_0^\dagger; H_0^\dagger} \rightarrow \dots$$

As it is filtered and thus sifted, and  $\Sigma_0$  preserves sifted colimits by hypothesis, by Lemma 8.43, it is enough to show that each  $\overline{H_0^\dagger; \dots; H_0^\dagger}$  has a structure of  $\Sigma_0$ -algebra (morphisms in the above chain are then automatically algebra morphisms because the involved spans are relations). But,  $\Sigma_0$  also preserves reflexive coequalisers (which are sifted colimits), thus by Lemma 8.44, the forgetful functor from  $\Sigma_0$ -algebras creates image factorisations, so it is enough to equip  $\overline{H_0^\dagger; \dots; H_0^\dagger}$  with  $\Sigma_0$ -algebra structure, which is straightforward because  $H_0$  is already an algebra and algebras are stable under pullbacks (Lemma 8.43).

It remains to find a suitable morphism  $H_0^{\bar{\dagger}}; \sim_0^\otimes \rightarrow H_0^{\bar{\dagger}}$ , or equivalently, by applying  $-^\dagger$ , a morphism  $\sim_0^{\otimes\dagger}; H_0^\dagger \rightarrow H_0^\dagger$ . But by symmetry of  $\sim_0^\otimes$ , we have the composite

$$\sim_0^{\otimes\dagger}; H_0^\dagger \rightarrow \sim_0^\otimes; H_0^\dagger \rightarrow H_0; H_0^\dagger \rightarrow H_0^\dagger. \quad (17)$$

$\square$

As mentioned in the beginning of the section, we now prove the following result.

**Lemma 6.19.** *For any substitution-closed simulation  $R$  such that  $R_0$  is symmetric, there exists a span morphism  $R \rightarrow R'$  such that  $R'$  is a substitution-closed bisimulation.*

*Proof.* First, consider the relation  $\bar{R}$  induced by  $R$  by the image factorisation  $R \twoheadrightarrow \bar{R} \hookrightarrow \mathbf{Z} \times \mathbf{Z}$ .  $\bar{R}$  is still a substitution-closed simulation and  $\bar{R}_0$  is symmetric. Now, we define  $R'$  as follows:

- $R'_0 = \bar{R}_0$
- $R'_1$  is the limit of the following diagram:

$$\begin{array}{ccccc} \mathbf{Z}_0\mathbf{s} & \longleftarrow & R'_0\mathbf{s} & \longrightarrow & \mathbf{Z}_0\mathbf{s} \\ \uparrow & & \vdots & & \uparrow \\ \mathbf{Z}_1 & \longleftarrow & R'_1 & \dashrightarrow & \mathbf{Z}_1 \\ \downarrow & & \vdots & & \downarrow \\ \mathbf{Z}_0\mathbf{t} & \longleftarrow & R'_0\mathbf{t} & \longrightarrow & \mathbf{Z}_0\mathbf{t}. \end{array}$$

More concretely, an element of  $R'_1(c_1)$  is a pair of transitions at  $c_1$  with related sources and targets. The morphism  $R \rightarrow R'$  is obtained by the composite  $R \rightarrow \bar{R} \rightarrow R'$ , where the last morphism exploits the definition of  $R'_1$  as a limit. It is straightforward to check that  $R'$  is a substitution-closed simulation. Moreover, it is symmetric (even at the level of transitions), so it is a bisimulation.  $\square$

## 9 Conclusion

We have introduced the notion of Howe context, in which we defined transition monoids, an abstract notion of labelled transition system whose states feature some sort of substitution. For them, we have introduced an abstract variant of applicative bisimilarity called substitution-closed bisimilarity.

Furthermore, we have introduced dynamic signatures as a device for specifying syntax with variable binding and operational semantics. We have finally shown that if a dynamic signature preserves functional bisimulations, then substitution-closed bisimilarity on the generated transition monoid is a congruence.

This all follows the pattern of our previous work [10], but simplifying the framework and relaxing some hypotheses, as explained in the introduction.

We hope these simplifications pave the way for more abstract results in the same vein. E.g., congruence for other forms of bisimilarity like normal-form [26], environmental [34], or contextual [27] bisimilarity. We also consider investigating abstract type soundness or compiler correctness results.

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