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# ON INVARIANT RANK TWO VECTOR BUNDLES ON $\mathbb{P}^2$

SIMONE MARCHESI AND JEAN VALLÈS

ABSTRACT. In this paper we characterize the rank two vector bundles on  $\mathbb{P}^2$  which are invariant under the action of  $G_p := \text{Stab}_p(\text{PGL}(3))$ , that fixes a point in the projective plane,  $G_L := \text{Stab}_L(\text{PGL}(3))$ , that fixes a line, and  $\mathbf{T} = G_p \cap G_L$ . Moreover, we prove that the geometrical configuration of the jumping locus induced by the invariance does not, on the other hand, characterize the invariance itself. Indeed, we find infinity families that are *almost uniform* but not *almost homogeneous*.

## 1. INTRODUCTION

The description and classification of vector bundles, which are invariant under the action of a determined group, has been widely studied. For instance, rank  $r$  vector bundle over  $\mathbb{P}^n$  which are invariant under the canonical action of  $\text{PGL}(n+1, \mathbb{C})$  are called *homogeneous*. Their complete classification is known up to rank  $n+2$ , see [8] for a reference, and they are given only by direct sums involving line bundles, a twist of the tangent bundle, a twist of the cotangent bundle on  $\mathbb{P}^n$  or their symmetric or anti-symmetric powers.

Furthermore, particular situations have induced to consider the action of specific subgroups of the projective linear group. For example, Ancona and Ottaviani prove in [1] that the Steiner bundles on  $\mathbb{P}^n$  which are invariant under the action of the special linear group  $\text{SL}(2, \mathbb{C})$  are the ones introduced by Schwarzenberger in [15]. Further in this direction, the second named author proves in [16] that any rank 2 stable vector bundle on  $\mathbb{P}^2$  which is invariant under the action of  $\text{SL}(2, \mathbb{C})$  is a Schwarzenberger bundle.

In this paper we will consider rank 2 vector bundles on the projective plane  $\mathbb{P}^2$ , and the chosen subgroups of  $\text{PGL}(3) = \text{PGL}(3, \mathbb{C})$  have been inspired by the following observations. In a previous paper, see [10], both authors have studied *Nearly free vector bundles* coming from line arrangements.

First of all, recall that it is of great interest also the studying and description of the action of a group on an hyperplane arrangement. For example, hyperplane arrangements which are invariants by the action of the group defined by reflections are free, and therefore their associated vector bundle is a direct sum of line bundles and hence homogeneous (see [13] for more details).

Recall moreover that a nearly free vector bundle  $\mathcal{F}$ , which were introduced by Dimca and Sticlaru in [3], can be defined by the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-b-1) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b)^2 \rightarrow \mathcal{F} \rightarrow 0,$$

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with  $(a, b) \in \mathbb{N}^2$  called the exponent of the vector bundle.

In particular, we proved the following two results

- Let  $\mathcal{F}$  be a nearly free vector bundle. Then, there exists a point  $p$  such that a line  $l \in \mathbb{P}^2$  is a jumping line of  $\mathcal{F}$  if and only if  $l$  passes through  $p$ . Moreover, each jumping line has order of jump equal to 1. We call  $p$  the *jumping point* associated to the vector bundle.
- Given a point  $p \in \mathbb{P}^2$  and a couple of integers  $(a, b) \in \mathbb{C}^2$  with  $a \leq b$ , there exists, up to isomorphism, one and only one nearly free vector bundle with exponents  $(a, b)$  whose pencil of jumping lines has  $p$  as base point. Moreover, we can think its defining matrix  $M$  as

$${}^tM = [x, y, z^{b-a+1}].$$

Furthermore, we proved that the geometrical configuration of the jumping locus  $S(\mathcal{F})$ , described in the first item, “almost” characterises nearly free vector bundles (see [10, Theorem 2.8]).

Inspired by the essential nature of the jumping point  $p$ , we focus on the rank 2 vector bundles on  $\mathbb{P}^2$  which are invariant under the action of the subgroup  $G_p \subset \mathrm{PGL}(3)$  that fixes the point  $p$  in the projective plane. Believing that it is probably too optimistic to hope for a complete classification for all subgroups, we focus on the ones which fix linear subspaces of the projective plane. We therefore consider the action of the subgroups  $G_L \subset \mathrm{PGL}(3)$ , which fixes the line  $L$  in the projective space, and  $\mathbf{T} \subset \mathrm{PGL}(3)$  fixing a line  $L$  and a point  $p$  belonging to this line.

Observing that  $\mathbf{T} = G_p \cap G_L$ , we deal first with the  $\mathbf{T}$ -invariant case, deducing the others from this one. The results obtained can be concentrated in the following statement.

**Theorem 1.** *Let  $\mathcal{F}$  an indecomposable rank two vector bundle on  $\mathbb{P}^2$ . Then*

- $\mathcal{F}$  is invariant under the action of  $G_p$  if and only if it is a nearly free vector bundle with jumping point  $p$ ;
- $\mathcal{F}$  is invariant under the action of  $G_L$  if and only if it is homogeneous;
- $\mathcal{F}$  is invariant under the action of  $\mathbf{T}$  if and only if it is a nearly free vector bundle with jumping point  $p$ .

Finally, in Section 7, we investigate a little deeper the relation, for a rank 2 vector bundle  $\mathcal{F}$  on  $\mathbb{P}^2$ , between the invariance for the action of a given group and the geometrical configuration of its jumping locus. In this direction, recall that if we consider the whole  $\mathrm{PGL}(3)$ , hence  $\mathcal{F}$  to be homogeneous, then all the lines  $L \subset \mathbb{P}^2$  induce the same splitting type. Vector bundles satisfying such property are called *uniform*.

It is known, due to the work of many (see for example [2, 5, 7, 8, 14, 17]), that every uniform vector bundle on  $\mathbb{P}^n$  with rank  $r \leq n + 1$  is also homogeneous. On the other hand, it has been of interest to find examples of uniform vector bundles, which are not homogeneous, of the lowest possible rank, see for example [4, 6, 9] and [12, Thm 3.3.2].

We will observe that the equivalence between the invariance and the jumping locus is already broken for the rank 2 case, and we will provide the complete classification of

vector bundles which are *almost uniform*, i.e. whose jumping locus is given by all the lines passing through a fixed point  $p$ , all having the same order of jump.

## 2. ACTION OF $G_p$ , $G_L$ AND $\mathbf{T}$

We consider now the subgroup  $G_p = \text{Stab}_p(\text{PGL}(3, \mathbb{C}))$  that fixes a point  $p \in \mathbb{P}^2$ , the subgroup  $G_L = \text{Stab}_p(\text{PGL}(3, \mathbb{C}))$  that fixes a line  $L \subset \mathbb{P}^2$  and, when  $p \in L$ , we consider also the subgroup defined by the intersection  $\mathbf{T} = G_p \cap G_L$ . In order to have a good description of the matrices representing the elements of the considered groups, let us choose the point  $p = (1 : 0 : 0)$  and the line  $L = \{z = 0\}$  in  $\mathbb{P}^2$ .

In this section we describe the action of these three subgroups of  $\text{PGL}(3)$ . First of all, notice that they can be described as subgroups of matrices in the following way:

$$G_p = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & a & b \\ 0 & c & d \end{bmatrix}, ad - bc \neq 0 \right\}, \quad G_L = \left\{ \begin{bmatrix} a & b & * \\ c & d & * \\ 0 & 0 & 1 \end{bmatrix}, ad - bc \neq 0 \right\} \text{ and}$$

$$T = \left\{ \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix}, abc \neq 0 \right\}.$$

**2.1. Action of  $G_p$ .** First of all, let us describe how the group  $G_p$  acts on the points and lines of the projective plane.

**Lemma 2.1.** *The group  $G_p$  acts transitively on the following sets:*

- (1) *points of  $\mathbb{P}^2 \setminus \{p\}$ ,*
- (2) *lines  $L$  such that  $p \in L$ ,*
- (3) *lines  $L$  such that  $p \notin L$ .*

*Proof.* It is clear that the action of  $G_p$  on these three sets is well defined. We would like to prove that these actions are transitive. In order to prove item (1), we recall that  $G$  acts transitively on the set of quadruples of points of  $\mathbb{P}^2$ , hence the subgroup  $G_p$  acts transitively on the set of triples of  $\mathbb{P}^2 \setminus \{p\}$ . This 4-transitivity of  $G$  implies that  $G$  acts transitively on the pair of lines, which moreover implies the transitivity of the action of  $G_p$  on both sets of lines, proving the last two items.  $\square$

**2.2. Action of  $G_L$ .** Let us focus now on the subgroup  $G_L$ .

**Lemma 2.2.** *The group  $G_L$  acts transitively on the following sets:*

- (1) *points of  $\mathbb{P}^2 \setminus L$ ,*
- (2) *points of the line  $L$ ,*
- (3) *the product  $(\mathbb{P}^2 \setminus L) \times (\mathbb{P}^2 \setminus L)$ .*

*Proof.* All items can be proven directly choosing appropriate matrices.

To prove the first one, notice that the matrix

$$\begin{bmatrix} 1 & * & a \\ 0 & 1 & b \\ 0 & 0 & c \end{bmatrix}$$

sends the point  $(0 : 0 : 1)$  to any point  $(a : b : c)$  with  $c \neq 0$ .

(2) To prove the second one, notice that the matrix

$$\begin{bmatrix} a & b & * \\ c & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

sends the point  $(1 : 0 : 0)$  to any point  $(a : c : 0)$ . Observe that if  $a = 0$ , we ask that  $b \neq 0$ .

(3) The matrix

$$\begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

sends the pair of points  $\{(1 : 0 : 1), (0 : 1 : 1)\}$  to any other pair  $\{(a : b : 1), (c : d : 1)\}$ .  $\square$

**2.3. Action of  $\mathbf{T}$ .** As done for the previous groups, let us prove the transitivity properties of  $\mathbf{T}$  that will be needed.

**Lemma 2.3.** *The group  $\mathbf{T}$  acts transitively on the following sets:*

- (1)  $L \setminus \{p\}$  in the projective plane,
- (2)  $\mathbb{P}^2 \setminus \{p^\vee\}$  in the dual projective plane.
- (3)  $\{p^\vee\} \setminus \{L^\vee\}$  in the dual projective plane.

*Proof.* To prove the first item, it is sufficient to observe that any point  $(u : 1 : 0) \in L \setminus \{p\}$  is the image of  $(0 : 1 : 0)$  by

$$\begin{bmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To prove the second item, it is enough to show that the line  $x = 0$  can be sent on any line  $L_{v,w} = \{x = vy + wz\}$  by an element of the group  $\mathbf{T}$ . Indeed the matrices ( $c \in \mathbb{C}$ )

$$\begin{bmatrix} 1 & v & cv + w \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

send the point  $(0 : \alpha : \beta)$  to  $(\alpha v + \beta(cv + w) : \alpha + c\beta : \beta) \in L_{v,w}$ .

To prove the third item, we have to show that any line  $y + wz = 0$  can be sent to another line of the same type, i.e.  $y + w'z = 0$ . It is enough to show that  $y = 0$  can be sent to the line  $y + wz = 0$ , for any  $w \in \mathbb{C}$ .

The matrices of type

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & -w \\ 0 & 0 & 1 \end{bmatrix},$$

for  $(a, b) \in \mathbb{C}$ , are the required ones.  $\square$

### 3. NEARLY FREE VECTOR BUNDLES

In this section we will focus on the family of Nearly free vector bundles, proving that they are invariant for the action of the groups  $G_p$  and  $\mathbf{T}$ , but not for the action of  $G_L$ .

Consider the family of rank two vector bundles  $\mathcal{E}_n$  parametrized by the positive integers  $n \in \mathbb{N}^*$  and defined by the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-n) \xrightarrow{(y,z,x^n)} \mathcal{O}_{\mathbb{P}^2}^2(1-n) \oplus \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{E}_n \longrightarrow 0.$$

These bundles belong to the family of *Nearly Free vector bundles*, first introduced in [3]. Notice that when  $n = 1$  we have  $\mathcal{E}_1 = T_{\mathbb{P}^2}(-1)$ , when  $n = 2$  the bundle  $\mathcal{E}_2$  is semi-stable and when  $n \geq 3$  the bundle  $\mathcal{E}_n$  is unstable, because in this case  $c_1(\mathcal{E}_n) < 0$  and  $H^0(\mathcal{E}_n) \neq 0$ . In [10], we have proven that, fixing a point  $p \in \mathbb{P}^2$  and supposing that  $p = (1 : 0 : 0)$  up to a change of coordinates, the point  $p$  determines any nearly free vector bundle up to isomorphism and they are all described by the same exact sequence as the one defining  $\mathcal{E}_n$ . The point  $p$  has been denominated *jumping point*, indeed a line is a jumping line for  $\mathcal{E}_n$  when  $n > 1$  if and only if it passes through  $p$ . Moreover, when  $n > 1$ , such point appears as the zero locus of the unique non zero global section of  $H^0(\mathcal{E}_n)$ .

Let us denote by  $NF(p) = \{\mathcal{E}_n, n \in \mathbb{N}^*\}$  the set of all such bundles. Since it is clear that a direct sum of two line bundles is invariant under the action of any subgroup of  $\text{PGL}(3)$ , we can consider only the group action on indecomposable bundles.

We conclude this section studying the action, and the possible invariance, of the considered groups on the bundles in  $NF(p)$ .

**Lemma 3.1.** *The behaviour of  $NF(p)$  under the action of the three considered subgroups is the following:*

- (1) *Any element in  $NF(p)$  is invariant under the action of  $G_p$  and  $\mathbf{T}$ .*
- (2) *The only invariant vector bundle in  $NF(p)$  under the action of  $G_L$  is  $\mathcal{E}_1 = T_{\mathbb{P}^2}(-1)$ .*

*Proof.* — Let us prove first that any element in  $NF(p)$  is  $G_p$ -invariant. Being  $\mathbf{T} = G_p \cap G_L$ , this will also prove the invariance of any bundle in  $NF(p)$  under the action of  $\mathbf{T}$ .

Consider the dual exact sequence

$$0 \longrightarrow \mathcal{E}_n^\vee \longrightarrow \mathcal{O}_{\mathbb{P}^2}^2(n-1) \oplus \mathcal{O}_{\mathbb{P}^2} \xrightarrow{(y,z,x^n)} \mathcal{O}_{\mathbb{P}^2}(n) \longrightarrow 0,$$

and the action, on the sequence, given by the element  $g = \begin{bmatrix} 1 & u & v \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \in G_p$ .

We obtain

$$0 \longrightarrow g^* \mathcal{E}_n^\vee \longrightarrow \mathcal{O}_{\mathbb{P}^2}^2(n-1) \oplus \mathcal{O}_{\mathbb{P}^2} \xrightarrow{(ay+bz,cy+dz,(x+uy+vz)^n)} \mathcal{O}_{\mathbb{P}^2}(n) \longrightarrow 0,$$

which fits into the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & g^* \mathcal{E}_n^\vee & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^2(n-1) \oplus \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{(ay+bz, cy+dz, (x+uy+vz)^n)} & \mathcal{O}_{\mathbb{P}^2}(n) & \longrightarrow & 0 \\
& & \simeq \downarrow & & N \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}_n^\vee & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^2(n-1) \oplus \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{(y,z,x^n)} & \mathcal{O}_{\mathbb{P}^2}(n) & \longrightarrow & 0,
\end{array}$$

with

$$N = \begin{bmatrix} a & c & f_{n-1} \\ b & d & g_{n-1} \\ 0 & 0 & 1 \end{bmatrix}$$

and  $f_{n-1}$  and  $g_{n-1}$  are degree  $n-1$  polynomials verifying

$$(x+uy+vz)^n = x^n + yf_{n-1} + zg_{n-1}.$$

The isomorphism induced in the left map of the diagram proves the invariance.

— Let us prove the second item.

If  $n=1$  the bundle  $\mathcal{E}_1$  is homogeneous and fixed by the whole group  $\mathrm{PGL}(3)$ .

If  $n>1$ , we have recalled that the bundle  $\mathcal{E}_n$ , and therefore the locus of the jumping lines, is determined by its jumping point  $p$ . But this point is not fixed by all the elements in  $G_L$ , proving that  $\mathcal{E}_n$  is not invariant under its action.  $\square$

#### 4. SPLITTING TYPE OF INVARIANT BUNDLES

In this section we will describe the possible splitting types of vector bundles which are invariant under the three subgroups, that is the way their restriction on any line splits.

As we will see better in Section 7, this geometric description of the jumping locus is not equivalent to the invariance.

Let us now consider a two vector bundle  $\mathcal{F}$  which is not uniform. Then the splitting type is constant for any line  $l$  belonging to a non empty open set  $U \subset \mathbb{P}^2$ . This is usually referred to as the *general splitting type*. Let us assume that  $\mathcal{F} \otimes \mathcal{O}_l = \mathcal{O}_l(a) \oplus \mathcal{O}_l(b)$  for  $l \in U$ , denoting by  $\delta = |a-b|$  the gap appearing for such general splitting. This gap is minimal on  $U$ , in other words, for any line  $l$  in the projective plane, we have that  $\delta \leq \delta(l) = |a_l - b_l|$  where  $\mathcal{F} \otimes \mathcal{O}_l = \mathcal{O}_l(a_l) \oplus \mathcal{O}_l(b_l)$ .

Since  $\mathcal{F}$  is not uniform, the set  $S(\mathcal{F})_{\mathrm{set}} = \{l, \delta(l) > \delta\}$ , whose lines are called *jumping lines*, is not empty.

These jumping lines posses a scheme structure that we will denote by  $S(\mathcal{F})$ .

To simplify the description of this scheme let us assume that the bundle  $\mathcal{F}$  is normalized, that is  $c_1 = c_1(\mathcal{F}) \in \{-1, 0\}$ .

- When  $c_1(\mathcal{F}) = 0$  and  $\mathcal{F}$  is stable (resp. semistable) then  $\delta = 0$  and  $S(\mathcal{F})$  is a curve (resp. a union of lines that can have multiplicity) of degree  $c_2(\mathcal{F})$ .
- When  $c_1(\mathcal{F}) = -1$  and  $\mathcal{F}$  is stable then  $\delta = 1$  and  $S(\mathcal{F})$  is, in general, a finite scheme but it could also contain a divisor. If it is a finite scheme then its length is the binomial number  $\binom{c_2(\mathcal{F})}{2}$ .

- When  $\mathcal{F}$  is unstable then  $\delta \geq 2$  and there exists  $n > 0$  such that  $H^0(\mathcal{F}(c_1 - n)) \neq 0$  and  $H^0(\mathcal{F}(c_1 - n - 1)) = 0$ . Then  $h^0(\mathcal{F}(c_1 - n)) = 1$  and the unique non zero section vanishes in codimension 2. The lines meeting this zero scheme form the scheme  $S(\mathcal{F})$  of jumping lines.

#### 4.1. Splitting type of $G_p$ invariant bundles.

**Lemma 4.1.** *Let  $\mathcal{F}$  be a non uniform  $G_p$ -invariant rank two vector bundle on  $\mathbb{P}^2$ . Then,  $S_{\text{set}}(\mathcal{F}) = p^\vee$  and  $\delta(l)$  is constant for any  $l \ni p$ .*

**Remark 4.2.** This is a set-theoretic description. The scheme of jumping lines is then a multiple structure on  $p^\vee$ .

*Proof.* Since  $\mathcal{F}$  is not uniform,  $S(\mathcal{F})$  has dimension at most 1. By Lemma 2.1,  $S(\mathcal{F})$  cannot contain a line  $L$  such that  $p \notin L$  and, because of the invariance combined with the transitivity of the chosen action, it coincides with the whole set  $p^\vee$  of lines through  $p$ . Moreover, any jumping line has the same splitting type.  $\square$

#### 4.2. Splitting type of $G_L$ invariant bundles.

**Lemma 4.3.** *Let  $\mathcal{F}$  be a non uniform  $G_L$ -invariant rank two vector bundle on  $\mathbb{P}^2$ . Then,  $\mathcal{F}$  is stable,  $c_1(\mathcal{F}) = -1$  and  $S(\mathcal{F})$  is a finite scheme of length  $\binom{c_2(\mathcal{F})}{2}$  supported by  $\{L^\vee\}$ .*

*Proof.* Since by Lemma 2.2 the group  $G_L$  acts transitively on  $\check{\mathbb{P}}^2 \setminus \{L^\vee\}$ , the set  $S_{\text{set}}(\mathcal{F})$  cannot contain a line distinct from  $L$ . As we said before, if  $c_1(\mathcal{F}) = 0$  or if  $\mathcal{F}$  is unstable then its scheme of jumping lines contains necessarily a curve. Then,  $\mathcal{F}$  is stable and  $c_1(\mathcal{F}) = -1$ . Since its scheme of jumping lines is finite its length is  $\binom{c_2(\mathcal{F})}{2}$ .  $\square$

#### 4.3. Splitting type of $\mathbf{T}$ invariant bundles.

**Lemma 4.4.** *Let  $\mathcal{F}$  be a non uniform  $\mathbf{T}$ -invariant rank two vector bundle on  $\mathbb{P}^2$ . Then,  $S_{\text{set}}(\mathcal{F}) = p^\vee$  or  $S_{\text{set}}(\mathcal{F}) = \{L^\vee\}$ . The second case cannot occur if  $c_1 = 0$  or if  $\mathcal{F}$  is unstable. More precisely, there are three possible splitting types:*

- $E_l = \mathcal{O}_l(k) \oplus \mathcal{O}_l(-k + c_1)$  with  $k \geq 0$  when  $l^\vee \in \check{\mathbb{P}}^2 \setminus p^\vee$ ,
- $E_l = \mathcal{O}_l(k + h) \oplus \mathcal{O}_l(-k - h + c_1)$  with  $h \geq 0$  when  $l^\vee \in p^\vee \setminus \{L^\vee\}$  and
- $E_l = \mathcal{O}_l(k + h + i) \oplus \mathcal{O}_l(-k - h - i + c_1)$  with  $h \geq 0, i \geq 0$  when  $l = L$ .

*Proof.* Denote the generic splitting of  $\mathcal{F}$  by  $\mathcal{O}_l(k) \oplus \mathcal{O}_l(-k + c_1)$ ; in particular, if  $\mathcal{F}$  is stable (or semi-stable), we have  $k = 0$ . The descriptions given is the second and third item follow directly from the transitivity of the action of  $\mathbf{T}$  described in Lemma 2.3. Indeed, all the (possible) jumping lines are the ones passing through the point  $p$ ; moreover, they must have all the same splitting type except for the line  $L$  fixed by the action, where the gap  $\delta_L$  could be bigger.

Specifically, if  $h > 0$  then the set of jumping lines is the line in the dual projective plane  $S_{\text{set}}(\mathcal{F}) = p^\vee$ ; on the other hand, if  $h = 0$  and  $i > 1$ , this set is just a point  $S_{\text{set}}(\mathcal{F}) = \{L^\vee\}$ .  $\square$

**Remark 4.5.** Under the hypothesis and using the notation of the previous result, notice that the case  $h = 0$  and  $i > 0$ , which gives that  $S_{\text{set}}(\mathcal{F}) = p^\vee$ , can only occur if  $c_1 = -1$  and  $\mathcal{F}$  stable.



5. THE CASE  $c_1(\mathcal{F}_{norm}) = 0$  AND  $S_{\text{set}}(\mathcal{F}) = p^\vee$ .

From the description in the previous section, we have that a non decomposable rank two vector bundle  $\mathcal{F}$  with even  $c_1$  has a curve of jumping lines. Assuming that  $\mathcal{F}$  is normalized and that such curve is a line, eventually with multiplicity, we prove the following result.

**Theorem 5.1.** *A non decomposable rank two vector bundle  $\mathcal{F}$  such that  $c_1(\mathcal{F}) = 0$  and  $S_{\text{set}}(\mathcal{F}) = p^\vee$ , where  $p$  is a point in  $\mathbb{P}^2$ , is either unstable or semistable.*

*Proof.* Assume that  $\mathcal{F}$  is stable. The splitting type on a general line  $l$  through  $p$  is  $\mathcal{O}_l(-h) \oplus \mathcal{O}_l(h)$  with  $h > 0$ . This means that, for a general line  $l$  through  $p$  we have  $h^0(\mathcal{F}|_l(-h)) = 1$ . Thanks to this fact, we can construct a special non zero section of  $\mathcal{F}(n-h)$  for some  $n \geq h$  in the following way. Let us consider the following diagram, constructed blowing up the point  $p$  in  $\mathbb{P}^2$

$$\begin{array}{ccc} \tilde{\mathbb{P}}^2 & \xrightarrow{\tilde{q}} & p^\vee \\ \tilde{p} \downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

Because of the possible splitting types, we get that  $\tilde{q}_*\tilde{p}^*(\mathcal{F}(-h))$  is an invertible sheaf on  $p^\vee$ , that is  $\mathcal{O}_{p^\vee}(-n)$  with  $n > h$  thanks to the stability of  $\mathcal{F}$ . This gives a non zero map

$$\tilde{q}^*\mathcal{O}_{p^\vee} \longrightarrow \tilde{p}^*(\mathcal{F}(-h)) \otimes \tilde{q}^*\mathcal{O}_{p^\vee}(n).$$

Remind that  $\tilde{p}_*\tilde{q}^*\mathcal{O}_{p^\vee}(n) = \mathcal{I}_p^n(n)$ , then taking the direct image on  $\mathbb{P}^2$ , we obtain a short exact sequence

$$(1) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{F}(n-h) \longrightarrow \mathcal{I}_\Gamma(2n-2h+c_1) \longrightarrow 0.$$

with  $\mathcal{I}_\Gamma \subset \mathcal{I}_p^n$ . This gives some numerical conditions. The length of  $\Gamma$  is  $c_2(\mathcal{F}(n-h)) = c_2(\mathcal{F}) + (n-h)^2$  and this length is greater or equal than  $n^2$  by construction. Indeed, locally at  $p = (1:0:0)$ , the zero set  $\Gamma$  is a complete intersection defined by two polynomials in  $\bigoplus_{k \geq 0} \mathbb{H}^0(\mathcal{I}_p^n(n+k))$ , in particular its length is at least  $n^2$ . This means that we have

$$c_2(\mathcal{F}) \geq h(2n-h).$$

Moreover, since  $\mathcal{F}$  is stable,  $c_2(\mathcal{F})$  is the degree of the curve  $S(\mathcal{F})$  which means that, if  $f = 0$  is the linear form defining  $p^\vee$  in  $\mathbb{P}^2$  then  $S(\mathcal{F})$  is defined by  $f^{c_2(\mathcal{F})} = 0$ .

Let  $l$  be a general jumping line.

Using the method implemented by Maruyama in [11] to determine the multiplicity of the

singular point  $l$  of the curve of jumping lines of  $\mathcal{F}$ , we consider an elementary transformation of  $\mathcal{F}$ , given by the jumping line, which induces the following diagram:

$$(2) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(h-n) & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{I}_{\Gamma_1}(n-h-1) \longrightarrow 0 \\ & & \cong \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(h-n) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}_{\Gamma}(n-h) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{O}_l(-h) & \xrightarrow{\cong} & \mathcal{O}_l(-h) \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Recall that Maruyama's method states that, denoting  $(\mathcal{F}_i)_l \simeq \mathcal{O}_l(a_i) \oplus \mathcal{O}_l(b_i)$  with  $a_i \geq b_i$ , the multiplicity is computed as  $\text{mult}_l S(\mathcal{F}) = \sum_{j=0}^k a_j$ , where  $a_k$  is the first integer in the decreasing sequence  $\{a_i\}_{i \geq 0}$  with  $a_k \leq 0$ . Iterating the previous diagram, using subsequent elementary transformations, we get that  $\text{mult}_l S(\mathcal{F}) \leq hn$ . Combining the obtained inequalities, we have that  $h(2n-h) \leq nh$ . The only possibility is  $h = n$  which proves that  $\mathcal{F}$  is semistable but not stable.

The main ingredient to prove the last inequality is to look at the local description of the previous diagram. Locally at the point  $p = (1 : 0 : 0)$ ,  $\Gamma$  is defined by two non homogeneous polynomials  $(f, g)$ , that we describe in terms of their homogeneous components

$$f = \sum_{k=n}^{\deg(f)} f_k, \quad g = \sum_{k=m}^{\deg(g)} g_k.$$

Observe that the lowest degree of the homogeneous components must be  $n$  for one of the two defining polynomials (which we suppose to be  $f$ ) and greater or equal than  $n$  for the other one (in our case  $m \geq n$ ). Else, we would have that  $\Gamma$  contains the fat point defined by a power of  $I_p$  greater than  $n$ , which is impossible. We will mainly focus on the  $n$ -th homogeneous component  $f_n$  of  $f$ , which we describe as

$$f_n = \sum_{i+j=n} \alpha_{i,j} y^i z^j.$$

Because of the hypothesis on the splitting type for the generic jumping line, we can consider a generic change of coordinates of  $\mathbb{P}^2$ , which fixes the point  $p$ , that allows us to suppose that all the coefficients  $\alpha_{i,j}$  are non zero and to consider, as generic jumping line, the one defined by  $y = 0$ .

This implies that, because of Diagram 2, the polynomial

$$f^{(1)} = \frac{f - (\alpha_{0,n} z^n + \sum_{t>n} \beta_t z^t)}{y},$$

with  $\beta_t$  non zero for a finite number of values of  $t$ , belongs to the ideal defining  $\Gamma_1$ . In particular, its homogeneous part of degree  $n - 1$  comes from  $f_n$  and henceforth all the possible monomials  $y^i z^j$ , in this case with  $i + j = n - 1$ , appear with non zero coefficient. Therefore, we have that

$$(\mathcal{F}_1)|_l \simeq \mathcal{O}_l(-h + s_1) \oplus \mathcal{O}_l(h - 1 - s_1), \quad \text{with } s_1 \geq 0.$$

Observe that the integer  $s_1$  appears because, depending on the homogeneous parts of higher degree in the considered polynomials, we could get a lower order, at the point  $p$ , in the intersection of the line  $l$  with  $\Gamma_1$ .

Iterating the previous process, we have that at the  $k$ -th step we must have

$$(\mathcal{F}_k)|_l \simeq \mathcal{O}_l(-h + s_k) \oplus \mathcal{O}_l(h - k - s_k), \quad \text{with } s_k \geq 0.$$

and the iteration end at most at the  $(n - 1)$ -st step. Indeed, we obtain  $f^{(n-1)} = z + \dots$ , which gives a splitting for  $\mathcal{F}_{n-1}$  with both degree less or equal than zero.  $\square$

## 6. MAIN RESULTS

In this section we will finally characterize the vector bundles which are invariant under the action of the considered subgroups.

**Theorem 6.1.** *A non decomposable rank two vector bundle  $\mathcal{F}$  is  $\mathbf{T}$ -invariant if and only if is a Nearly Free vector bundle.*

Before proving this theorem, let us show that it implies the characterization of the invariance for the subgroups  $G_p$  and  $G_L$ .

**Corollary 6.2.** *A non decomposable rank two vector bundle  $\mathcal{F}$  is  $G_p$ -invariant if and only if belongs to  $NF(p)$ .*

*Proof.* We have already seen that any bundle in  $NF(p)$  is  $G_p$ -invariant.

Conversely, we have to prove that these are the only invariant bundles. Since  $\mathbf{T} = G_p \cap G_L$ , the invariant bundles under the action of  $G_p$  must also be invariant under the action of  $\mathbf{T}$ , proving the result.  $\square$

**Corollary 6.3.** *A rank two vector bundle  $\mathcal{F}$  is invariant under the action of  $G_L$  if and only if  $\mathcal{F}$  is homogeneous.*

*Proof.* Theorem 6.1 proves that the invariant bundles under the action of  $\mathbf{T} = G_p \cap G_L$  all belong to  $NF(p)$ . Moreover, Lemma 2.2 shows that a Nearly free bundle  $\mathcal{F}$  that is not homogeneous is not  $G_L$ -invariant, proving the result.  $\square$

*Proof of Theorem 6.1.* Let  $\mathcal{F}$  be a non decomposable normalized rank two vector bundle which is  $\mathbf{T}$ -invariant. Moreover, we will use the notation for the possible splitting types introduced in Section 4.3.

— Assume first that  $\mathcal{F}$  is either unstable or semistable.

Then there is a “negative” section:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k) \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Z(-k + c_1) \longrightarrow 0,$$

where  $k \geq 0$  and  $Z$  is empty or a finite scheme of length  $c_2(\mathcal{F}(-k))$ . This section is unique (except when  $k = c_1 = 0$ ,  $Z = \emptyset$  and  $\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}^2$ ), which implies that  $g(Z) = Z$  for any  $g \in \mathbf{T}$ . This imposes  $Z$  to be a local complete intersection supported on the point  $p$ ; indeed if  $Z_{red} = p + Z'$  with  $Z'$  not supported on  $p$ , then it cannot be invariant according to Lemma 2.3. Hence  $Z$  is a complete intersection supported on  $p$  and containing the fat point defined by  $(y, z)^h$ ; this implies that  $Z$  can be defined by the ideal  $(f_1(y, z), f_2(y, z))$  with  $\deg f_j = h + i_j$ , for  $j = 1, 2$ , and  $0 \leq i_1 \leq i_2$ .

This means that, as done before,  $\mathcal{F}$  admits a resolution of type

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-h - i_1 - i_2 + t) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-i_2 + t) \oplus \mathcal{O}_{\mathbb{P}^2}(-i_1 + t) \oplus \mathcal{O}_{\mathbb{P}^2}(k) \rightarrow \mathcal{F} \rightarrow 0,$$

with  $t = -h - k + c_1$  and  $A = [f_1(y, z), f_2(y, z), f(x, y, z)]$ , where  $f$  denotes a homogeneous form of degree  $2k + 2h + i_1 + i_2 - c_1$ .

Notice that the action given by an element  $g = g_{(a,b,c)} \in \mathbf{T}$  sends the linear forms  $x, y$  and  $z$  to, respectively,  $x - ay + (ac - b)z, y - cz$  and  $z$ . Moreover, denoting by  $A_g$  the matrix obtained from  $A$  by the action of  $g$ , we have, by invariance, the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-h - i_1 - i_2 + t) & \xrightarrow{A} & \mathcal{O}_{\mathbb{P}^2}(-i_2 + t) \oplus \mathcal{O}_{\mathbb{P}^2}(-i_1 + t) \oplus \mathcal{O}_{\mathbb{P}^2}(k) & \longrightarrow & \mathcal{F} \longrightarrow 0 \\ & & \lambda \downarrow & & H \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-h - i_1 - i_2 + t) & \xrightarrow{A_g} & \mathcal{O}_{\mathbb{P}^2}(-i_2 + t) \oplus \mathcal{O}_{\mathbb{P}^2}(-i_1 + t) \oplus \mathcal{O}_{\mathbb{P}^2}(k) & \longrightarrow & g^*\mathcal{F} \longrightarrow 0, \end{array}$$

where

$$H = \begin{bmatrix} \alpha & 0 & 0 \\ h_1 & \beta & 0 \\ h_2 & h_3 & \gamma \end{bmatrix},$$

denoting by  $\lambda, \alpha, \beta$  and  $\gamma$  non vanishing scalars, by  $h_1, h_2$  and  $h_3$  three homogeneous forms of degree respectively  $i_2 - i_1, 2k + h + i_2 - c_1$  and  $2k + h + i_1 - c_1$ .

From the commutativity of the diagram, we have that  $f_1$  must be invariant, up to scalar, for every element of the group. Observing the a monomial of type  $y^\alpha z^\beta$  goes to  $\lambda(y - cz)^\alpha z^\beta$  and, due to the invariance, we must have  $f_1 = z^{h+i_1}$  and  $f_2 = y^{h+i_2}$ .

Considering the relation

$$(y - cz)^{h+i_2} = y^{h+i_2} + h_1 z^{h+i_1}$$

and expanding the power on the left side of the equality, we find that there are no middle terms, in other words  $h + i = 1$ , which means that  $h = 1$  and  $i_1 = i_2 = 0$ . Therefore  $A$  can be written as  $[z, y, x^{2k+c_1+1}]$ , which implies  $\mathcal{F}$  to be Nearly Free.

— Assume now  $\mathcal{F}$  to be stable.

If  $c_1(\mathcal{F}) = 0$ , then because of the  $\mathbf{T}$ -invariance, we would have  $S_{red}(\mathcal{F}) = p^\vee$  and therefore a contradiction by Theorem 5.1; hence,  $c_1(\mathcal{F}) = -1$ . Let  $L \ni p$  the unique line fixed by  $\mathbf{T}$ . Then the equivariant composition

$$\mathcal{F} \longrightarrow \mathcal{F}|_L \longrightarrow \mathcal{O}_L(-h - i - 1)$$

induces an exact sequence, usually called an *elementary transformation*,

$$0 \longrightarrow \mathcal{G}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L(-h-i-1) \longrightarrow 0$$

where  $\mathcal{G}$  is a rank two stable or semistable vector bundle with  $c_1\mathcal{G} = 0$  that is invariant under  $\mathbf{T}$ . Indeed, since  $L$  is fixed and  $\mathcal{F}$  is invariant under the action of  $\mathbf{T}$ , for any  $g \in \mathbf{T}$  we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}(-1) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_L(-h-i-1) & \longrightarrow & 0 \\ & & \downarrow & & \simeq \downarrow & & \parallel & & \\ 0 & \longrightarrow & g^*(\mathcal{G}(-1)) & \longrightarrow & g^*\mathcal{F} & \longrightarrow & \mathcal{O}_L(-h-i-1) & \longrightarrow & 0. \end{array}$$

Completing the diagram, this proves that  $\mathcal{G} \simeq g^*\mathcal{G}$ , in other words  $\mathcal{G}$  is also invariant under the action of  $\mathbf{T}$ .

Again by Theorem 5.1,  $\mathcal{G}$  is semistable. Then a non zero section of  $\mathcal{G}$  gives a non zero section of  $\mathcal{F}(1)$ , fitting in

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{F}(1) \longrightarrow \mathcal{I}_\Gamma(1) \longrightarrow 0,$$

that vanishes along a 0-dimensional scheme  $\Gamma$ .

If  $h^0(\mathcal{F}(1)) = 1$  the scheme  $\Gamma$  is  $\mathbf{T}$ -invariant which proves that  $\Gamma$  is supported exactly on  $p$ . Then  $\Gamma$  is a complete intersection of two polynomials  $(f(y, z), g(y, z))$  respectively of degree  $n + i_1$  and  $n + i_2$  with  $i_1 \leq i_2$ . We obtain a resolution of  $\mathcal{F}$ , now given by

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2n - i_1 - i_2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-n - i_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-n - i_2) \rightarrow \mathcal{F} \rightarrow 0,$$

with  $A = [\ell(x, y, z), f, g]$ . The hypothesis of invariance of  $\mathcal{F}$ , applying analogous computation to the unstable and semistable case described in the previous part, implies that the only possibility is given by  $n = 1$  and  $i_1 = i_2 = 0$  and therefore  $\mathcal{F}$  is nearly free. Moreover, because of its stability, we have  $\mathcal{F} \simeq T_{\mathbb{P}^2}(-2)$ .

If  $h^0(\mathcal{F}(1)) > 1$  then either  $h = i = 0$ , which would mean that  $\mathcal{F}$  is uniform and therefore homogeneous, or  $h^0(\mathcal{G}) > 1$ , which means that  $\mathcal{G} = \mathcal{O}_{\mathbb{P}^2}^2$ , having  $k = 0$  because of the stability of  $\mathcal{F}$ . In the latter case, the exact sequence defining the elementary transformation gives us the resolution of  $\mathcal{F}$ , which is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-n) \xrightarrow{(f, g, L)} \mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(1-n) \longrightarrow \mathcal{F}(1) \longrightarrow 0.$$

where we have denoted by  $n = h + i + 1$ . In particular,  $\Gamma$  is a complete intersection of a curve of degree  $n = c_2(\mathcal{F})$  and the line  $L$ . Exactly as before, the invariance of  $\mathcal{F}$  implies that  $n = 1$  which means that  $\mathcal{F} \simeq T_{\mathbb{P}^2}(-2)$ .  $\square$

## 7. SPECIAL GEOMETRIC CONFIGURATIONS OF THE JUMPING LOCUS

In Section 4.1 we have noticed that if a rank 2 vector bundle  $\mathcal{F}$  on  $\mathbb{P}^2$  is invariant under the action of  $G_p$ , then, if not uniform, its jumping locus is given by all lines passing through  $p$ . Moreover, all jumping lines have the same order.

It is natural to ask ourselves the natural question

*Is the invariance equivalent to the obtained special geometric configuration of the jumping locus?*

From [10, Theorem 2.8], we already know the answer to be negative. Nevertheless, the result can be generalized to any order of jump, finding interesting families of stable bundles.

**Theorem 7.1.** *Let  $p \in \mathbb{P}^2$  be a point and  $\mathcal{F}$  be a rank 2 vector bundle on  $\mathbb{P}^2$  that we can assume to be normalized, i.e.  $c_1(\mathcal{F}) = \{-1, 0\}$ . Assume that  $S(\mathcal{F}) = \{L, L \ni p\}$  and that  $\mathcal{F} \otimes \mathcal{O}_L = \mathcal{O}_L(r) \oplus \mathcal{O}_L(-r + c_1)$ , with  $r \in \mathbb{N}$  and for any  $L \in S(\mathcal{F})$ . Then, we have the following options*

- *If  $\mathcal{F}$  is either unstable or semistable then  $\mathcal{F}$  is defined by the resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2r - k + c_1) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(k) \oplus \mathcal{O}_{\mathbb{P}^2}(-r - k + c_1)^2 \longrightarrow \mathcal{F} \longrightarrow 0,$$

*where  $k \in \mathbb{N}$  such that  $H^0(\mathcal{F}(k)) \neq 0$  and  $H^0(\mathcal{F}(k-1)) = 0$ , i.e. gives a global section of minimal degree. Moreover, the matrix  $A$  is given, assuming  $p = (0 : 0 : 1)$ , by  $f(x, y, z), g(x, y)$  and  $h(x, y)$ , with  $f$  any homogeneous form of degree  $2r + 2k - c_1$  and  $g$  and  $h$  degree  $r$  forms such that*

$$0 = (H^0(\mathcal{O}_L(r-1)) \cdot g) \cap (H^0(\mathcal{O}_L(r-1)) \cdot h) \subset H^0(\mathcal{O}_L(2r-1)).$$

*for any line  $L$  not passing through  $p$ .*

*In particular, if  $r = 1$ ,  $\mathcal{F}$  is a nearly free vector bundle.*

- *If  $\mathcal{F}$  is stable, then  $c_1(\mathcal{F}) = -1$ ,  $c_2(\mathcal{F}) = (r+1)^2$  and the bundle is defined by the resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2r-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-r-1)^2 \longrightarrow \mathcal{F} \longrightarrow 0.$$

*Moreover, after choosing  $p = (0 : 0 : 1)$ ,  $A = (f(x, y, z), g(x, y), h(x, y))$  for any degree  $2r+1$  form  $f$  not passing through  $p$  and  $g$  and  $h$  are degree  $r+1$  forms such that*

$$0 = (H^0(\mathcal{O}_L(r-1)) \cdot g) \cap (H^0(\mathcal{O}_L(r-1)) \cdot h) \subset H^0(\mathcal{O}_L(2r)).$$

*for any line  $L$  not passing through  $p$ .*

*Proof.* Throughout the proof, we will choose  $p = (0 : 0 : 1)$ .

Let us first consider  $\mathcal{F}$  to be unstable or semistable. Hence the bundle  $\mathcal{F}$  is defined by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k) \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_\Gamma(-k + c_1) \longrightarrow 0,$$

being  $\Gamma$  a 0-dimensional scheme supported on  $p$ . Since the jump is constant, we have that  $\mathcal{J}_\Gamma \subset \mathcal{J}_p^r$ . and  $\Gamma$  is locally intersection complete concentrated on one point. This means that  $\Gamma$  is a complete intersection of two forms  $(f, g)$  of degree  $r$  with multiplicity  $r$  at  $p$ , that is two set of  $r$  lines concurrent in  $p$  (with no common component in order to define a codimension two set  $\Gamma$ ). Therefore the resolution of  $\mathcal{F}$  is given by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2r - k + c_1) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(k) \oplus \mathcal{O}_{\mathbb{P}^2}(-r - k + c_1)^2 \longrightarrow \mathcal{F} \longrightarrow 0.$$

where  $A = [f(x, y, z), g(x, y), h(x, y)]$ , with  $f$  homogeneous form of degree  $2r + 2k - c_1$  and  $g$  and  $h$  of degree  $r$ .

A line  $L$  is a jumping line for  $\mathcal{F}$  if and only if  $H^0(\mathcal{F}|_L(-k-1+c_1)) \neq 0$  so, after dualizing

the resolution of  $\mathcal{F}$ , the jumping condition, for a given line  $L$ , is equivalent to the map

$$H^0(\mathcal{O}_L(r-1)^2) \xrightarrow{M} H^0(\mathcal{O}_L(2r-1))$$

not to be injective. Notice that  $M$  is once again defined using the degree  $r$  forms  $g$  and  $h$ . It is straightforward to check that, if  $L \ni p$ , the two forms become independent and therefore we have a jumping line of the required order  $r$ .

On the other hand, when we consider a line  $L \not\ni p$ , we have that  $L$  is not a jumping line for  $\mathcal{F}$  if and only if

$$0 = (H^0(\mathcal{O}_L(r-1)) \cdot g) \cap (H^0(\mathcal{O}_L(r-1)) \cdot h) \subset H^0(\mathcal{O}_L(2r-1)).$$

Let  $\mathcal{F}$  be stable with  $c_1 = 0$ .

Using an analogous construction as in the proof of Theorem 5.1, the dual of the resolution of  $\mathcal{F}$ , shifted by  $n - r - 1$ , is

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n-r-1) \oplus \mathcal{O}_{\mathbb{P}^2}^2(r-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n+r-1) \longrightarrow 0.$$

As before, a line  $L$  is a jumping line for  $\mathcal{F}$  if and only if  $H^0(\mathcal{F}|_L(-1)) \neq 0$ , in other words if and only if the following map, between two vector spaces of dimension  $n+r$

$$H^0(\mathcal{O}_L(n-r-1) \oplus \mathcal{O}_L^2(r-1)) \xrightarrow{M} H^0(\mathcal{O}_L(n+r-1))$$

is not injective.

Let us introduce the following notation. Denote  $g(x, y) = \sum_i \alpha_i x^{n-i} y^i$ ,  $h(x, y) = \sum_i \beta_i x^{n-i} y^i$ , which describe the two  $n$  forms defined by  $n$  lines through  $p = (0 : 0 : 1)$  and let  $f(x, y, z)$  be a form of degree  $2r$ . Our goal is to prove that there is always another line  $L$ , not passing through  $p$ , and therefore given by the equation  $z = ax + by$ , which is a jumping line. This will give a contradiction with the geometric assumption on the jumping locus.

Substituting the equation of the line we have  $f(x, y, ax + by) = \sum_{0 \leq i \leq 2r} \gamma_i(a, b) x^{2r-i} y^i$  where  $\gamma_i(a, b)$  are degree  $2r$  polynomials (non homogeneous).

By linear combination of lines and columns the matrix  $M$  is equivalent to

$$\begin{bmatrix} I_{2r} & 0 \\ 0 & N \end{bmatrix},$$

where  $I_{2r}$  denotes the identity matrix of dimension  $2r$  and  $N$  is a  $(n-r+1) \times (n-r+1)$  matrix of degree  $2r$  polynomials in  $(a, b)$ . Notice that the determinant of  $N$  is polynomial (again not necessarily homogeneous), in the variables  $a$  and  $b$ , of degree  $d = (2r)^{n-r+1}$  and, by our description, the determinant of the matrix  $M$  is equal to the determinant of  $N$ .

The existence of a jumping line for  $\mathcal{F}$  that do not pass through  $p$  is equivalent to the existence of a point  $(a, b)$  with  $\det(N(a, b)) = 0$ .

Consider the ‘‘homogenization’’ of  $N$ , i.e. homogenize each entry of the matrix in order to get a matrix whose entries are homogeneous forms of degree  $2r$ . Suppose the wanted point  $(a, b)$  does not exist, this means that after homogenization, denoted by  $(a : b : c)$ , we have  $\det(N) = c^d$ . In particular  $\det(M(a : b : 0)) = 0$ , which implies that if we fix

$f = z^{2r}$ , all the lines of the stable bundle defined by  $(g, h, z^{2r})$  are jumping lines, which contradicts Grauert-Mulich's theorem.

If  $\mathcal{F}$  is stable with  $c_1 = -1$ , it is possible to adapt the previous argument and find, in an analogous way, a contradiction. □

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