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Camille Carvalho, Zoïs Moitier

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# ASYMPTOTICS FOR METAMATERIAL CAVITIES AND THEIR EFFECT ON SCATTERING

CAMILLE CARVALHO AND ZOS MOITIER

**ABSTRACT.** It is well-known that optical cavities can exhibit localized phenomena responsible for numerical instabilities in approximating scattering problems. In classic optical cavities, those localized phenomena concentrate at the inner boundary of the cavity and are called whispering gallery modes. In this paper we consider cavities made of Negative-Index Metamaterials (NIM). Those manufactured materials can exhibit unusual optical properties, leading to the appearance of exotic waves at the interface of metamaterial-dielectric interface, such as surface plasmons. There is a great interest in controlling and guiding surface plasmons to design the next-generation of nano-antennas, sensors, and others. Surface plasmons have been usually characterized in the context of the specific quasi-static case. Using asymptotic analysis, we provide a systematic mathematical characterization of emerging surface plasmons for general metamaterial cavities for the full wave problem, and answer the following questions: can surface plasmons be always excited in scattering problems? Can they create numerical instabilities? Our asymptotic analysis reveals that the nature of surface plasmons changes depending on the metamaterial's properties, leading to different effects on scattering problems. Numerical results for several metamaterial cavities are provided.

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## 1. INTRODUCTION

Optical micro-resonators, commonly involving dielectric cavities, play an important role in many applications in photonics and others [27]. Optical cavities have been extensively studied over the past decades as they offer the opportunity to confine light in small volumes. This is possible in particular when the resonator supports the so-called Whispering Gallery Modes (WGM) [37]. With the development of metamaterials, such as the Negative-Index Metamaterials (NIM) which exhibit unusual optical

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properties (for instance a negative effective permittivity  $\varepsilon$  and/or a negative effective permeability  $\mu$ ), there is a great interest in modeling metamaterial cavities to confine and control light. In particular, at optical frequencies, localized surface waves called surface plasmons can arise at dielectric-metamaterial interfaces (as well for some dielectric-metal interfaces) [31]. The field of plasmonics is very active as surface plasmons offer strong light enhancement, with applications to next-generation sensors, antennas, high-resolution imaging, cloaking and other [38]. However surface plasmons are very sensitive to the geometry and therefore challenging to capture, experimentally and numerically [8, 28].

Mathematically, surface plasmons are solutions of the homogeneous Maxwell's equations, they are oscillatory waves along the dielectric-metamaterial interface while exponentially decreasing in the transverse direction. For simple interface geometries their expression is known explicitly, however in general surface plasmons have been mainly characterized and investigated in the context of the quasi-static approximation (e.g. [12, 25, 8, 4, 16, 14, 13]). While the quasi-static case offers valuable insights into their behavior, the connection with exciting the characterized surface plasmons in the context of light scattering by metamaterials is less clear.

It is well-known that the approximation of light scattering in dielectric optical microcavities can be drastically affected by WGM, in particular if the excitation wavenumber of the source is close to a WGM resonance [32, 6].

In the spirit of [6], we carry out in this paper an asymptotic analysis to systematically characterize surface plasmons arising in general two-dimensional metamaterial cavities (arbitrary smooth shape, arbitrary permittivity function) for the full wave problem (time-harmonic Maxwell's equations in Transverse Magnetic polarization), and we answer the following questions: Can surface plasmons be always excited in scattering problems? Can we numerically observe surface plasmons in practice? Can they create numerical instabilities? Our asymptotic analysis reveals that the spectral nature of surface plasmons changes depending on the metamaterial's properties, leading to different effects on scattering problems. Additionally, this analysis provides guidance about when surface plasmons can be excited in practice.

The paper is organized as follows. We present the considered scattering problem and known results in [Section 2](#). [Section 3](#) presents how to characterize surface plasmons and their effect on scattering problems for the model case of a circular metamaterial cavity. [Section 4](#) provides the general asymptotic analysis of emerging surface plasmons. [Section 5](#) presents numerical results based on the asymptotic analysis, and [Section 6](#) presents our concluding remarks. [Appendix A](#) provides theoretical results regarding well-posedness of the scattering problem and [Appendix B](#) provides additional results and proofs needed in [Section 4](#).

## 2. MATHEMATICAL PROBLEM AND PRELIMINARIES

**2.1. Scattering by a metamaterial cavity.** We consider a *cavity*  $\Omega \subset \mathbb{R}^2$ , an open bounded connected set with smooth boundary  $\Gamma$ . The optical cavity  $\Omega$  is characterized by a permeability  $\mu = 1$ , and a permittivity  $\varepsilon := \varepsilon_c < 0$  with  $\varepsilon_c \in \mathcal{C}^\infty(\overline{\Omega})$ . The cavity is embedded in an homogeneous background characterized by  $\mu = 1$ , and  $\varepsilon = 1$ . We consider the scattering problem of a plane wave by the cavity  $\Omega$ . Let  $u^{\text{in}} : x \mapsto e^{ik \cdot d \cdot x}$ , the plane wave with wavenumber  $k > 0$  and direction  $d \in \mathbb{S}^1$ , be the incident wave.

The scattering problem writes: Find  $u \in H_{\text{loc}}^1(\mathbb{R}^2)$  such that  $u = u^{\text{in}} + u^{\text{sc}}$  and

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1} \nabla u) - k^2 u = 0 & \text{in } \mathbb{R}^2 \\ [u]_{\Gamma} = 0 \quad \text{and} \quad [\varepsilon^{-1} \partial_n u]_{\Gamma} = 0 & \text{across } \Gamma \\ u^{\text{sc}} \text{ } k\text{-outgoing} \end{cases} \quad (2.1)$$

with  $H_{\text{loc}}^1(\mathbb{R}^2) := \{u \in L_{\text{loc}}^2(\mathbb{R}^2) \mid \forall \chi \in \mathcal{C}_{\text{comp}}^{\infty}(\mathbb{R}^2), \chi u \in H^1(\mathbb{R}^2)\}$ ,  $n : \Gamma \rightarrow \mathbb{S}^1$  is the unit normal vector outward to  $\Omega$ , and the function  $\varepsilon \in L^{\infty}(\mathbb{R}^2)$  is given by

$$\varepsilon(x) = \begin{cases} \varepsilon_c(x) < 0 & \text{if } x \in \bar{\Omega} \\ 1 & \text{if } x \in \mathbb{R}^2 \setminus \bar{\Omega} \end{cases}. \quad (2.2)$$

Given  $X$ , we denote  $[X]_{\Gamma}(\gamma) = \lim_{x \rightarrow \gamma^+} X(x) - \lim_{x \rightarrow \gamma^-} X(x)$ , for any  $\gamma \in \Gamma$ , the jump condition across  $\Gamma$ . The jump conditions  $[u]_{\Gamma} = 0$  and  $[\varepsilon^{-1} \partial_n u]_{\Gamma} = 0$  will be referred to as the *transmission conditions*. We say that  $v$  is *k-outgoing* if it satisfies the outgoing wave condition:

$$v(r, \theta) = \sum_{m \in \mathbb{Z}} w_m(r) e^{im\theta} = \sum_{m \in \mathbb{Z}} c_m H_m^{(1)}(kr) e^{im\theta} \quad (2.3)$$

with polar coordinates  $(r, \theta)$  such that  $r > \sup_{x \in \Omega} |x|$ ,  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ ,  $H_m^{(1)}$  the Hankel function of the first kind of order  $m$ , and  $(c_m)_{m \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ .

*Remark 2.1.* The condition  $v$  is *k-outgoing* defined in [Eq. \(2.3\)](#) is equivalent to  $v$  satisfying the so-called Sommerfeld radiation condition, if and only if  $k > 0$ . This outgoing condition is more general, and will be also used for the associated spectral problem, where one can have  $k \in \mathbb{C}$ .

In classic scattering problem (i.e. when  $\varepsilon(x) > 0$ ,  $\forall x \in \mathbb{R}^2$ ), it is well known that Problem [Eq. \(2.1\)](#) is well-posed [\[15, 21\]](#). In our case  $\varepsilon$  defined as [Eq. \(2.2\)](#) is *sign-changing*, and the problem may be ill-posed. Using the T-coercivity theory [\[10, 8, 9\]](#), one can establish the following Lemma (see [Appendix A](#) for details):

**Lemma 2.2.** *Problem (2.1) is well-posed if and only if  $\varepsilon_c|_{\Gamma} \neq -1$ . Moreover there exists a constant  $C(k) > 0$  such that*

$$\|u^{\text{sc}}\|_{L^2(\mathbb{D}(0, \rho))} \leq C(k) \|u^{\text{in}}\|_{L^2(\mathbb{D}(0, \rho))} \quad (2.4)$$

where  $\mathbb{D}(0, \rho) \ni \Omega$  is the disk of center 0 and radius  $\rho$ .

**2.2. Resonances of a dielectric-metamaterial open cavity.** As done for classical cavities, it is essential to study the spectral problem associated to Problem (2.1) to identify if resonances appear and study their affect if one chooses  $k$  “close” to a resonance [\[41, 22, 32\]](#). We define the operator  $P := -\operatorname{div}(\varepsilon^{-1} \nabla)$  on  $L^2(\mathbb{R}^2)$  with domain  $D(P) := \{u \in L^2(\mathbb{R}^2) \mid \operatorname{div}(\varepsilon^{-1} \nabla u) \in L^2(\mathbb{R}^2)\}$ . The spectral problem associated to (2.1) writes: Find  $\ell \in \mathbb{C}^{\frac{1}{2}}$  and  $u \in D_{\text{loc}}(P)$ ,  $u \neq 0$  such that

$$\begin{cases} Pu = \ell^2 u & \text{in } \mathbb{R}^2 \\ [u]_{\Gamma} = 0 \quad \text{and} \quad [\varepsilon^{-1} \partial_n u]_{\Gamma} = 0 & \text{across } \Gamma, \\ u \text{ } \ell\text{-outgoing} \end{cases} \quad (2.5)$$

where  $\mathbb{C}^{\frac{1}{2}} := \{z \in \mathbb{C} \mid \arg(z) \in (-\frac{\pi}{2}, \frac{\pi}{2}]\}$ . We set the branch cut to be  $\mathbb{R}_-$ , from now on complex roots are uniquely defined. In order to capture eigenvalues and resonances, we study Problem (2.5) in an extended framework  $D_{\text{loc}}(P) \supsetneq D(P)$ , with

$$D_{\text{loc}}(P) := \left\{ u \in L_{\text{loc}}^2(\mathbb{R}^2) \mid \forall \chi \in \mathcal{C}_{\text{comp}}^{\infty}(\mathbb{R}^2), \chi u \in D(P) \right\}.$$

*Remark 2.3.* With the chosen branch cut, the outgoing condition gives that  $u$  is a solution of [Eq. \(2.5\)](#) in  $L^2(\mathbb{R}^2)$  if, and only if,  $\ell \in \mathbb{C}_*^{i+} := \{z \in \mathbb{C}^{\frac{1}{2}} \mid \Im(z) > 0\}$ . This is coming from the fact that, for  $m \in \mathbb{Z}$ , the asymptotic behavior of  $H_m^{(1)}$  as  $r \rightarrow +\infty$  is given by  $|H_m^{(1)}(\ell r)| \sim \sqrt{\frac{2}{\pi|\ell|r}} e^{-\Im(\ell)r}$ . We also define  $\mathbb{C}^{i-} := \{z \in \mathbb{C}^{\frac{1}{2}} \mid \Im(z) \leq 0\}$ . As a consequence we have  $u \in D(P)$  when associated to  $\ell \in \mathbb{C}_*^{i+}$ , and  $u \in D_{\text{loc}}(P)$  when associated to  $\ell \in \mathbb{C}^{i-}$ .

Using T-coercivity arguments [[18](#), [16](#)], one can show that, if  $\varepsilon_c|_\Gamma \neq -1$ , then the operator  $(P, D(P))$  is self-adjoint on  $L^2(\mathbb{R}^2)$  and its essential spectrum is  $\mathbb{R}_+$  (we refer to [[17](#)] for details). However, contrary to the case with classical cavities [[6](#)],  $P$  doesn't admit a lower bound, it can exhibit negative eigenvalues, as well as *interface modes* related to surface plasmons [[16](#)]. The rest of the paper is dedicated to: (i) investigate what types of resonances holds Problem [Eq. \(2.5\)](#) (explicitly or asymptotically), and (ii) identify if the computations can be deteriorated close to resonances (related to surface plasmons, or others). We start with a simple case where calculations are explicit.

*Remark 2.4.* In this paper, we consider the Helmholtz equation case, we do not assume the quasi-static approximation. The considered spectral problem, where the spectral parameter is the wavenumber  $\ell$ , differs from the quasi-static spectral problem, where the spectral parameter is the permittivity  $\varepsilon_c$  [[25](#), [13](#)].

### 3. CASE OF A CIRCULAR CAVITY

We consider  $\Omega = \mathbb{D}(0, R)$  the disk of center 0 and radius  $R > 0$ . We set  $\varepsilon \equiv 1$  outside the cavity, and for simplicity we fix  $\varepsilon = \varepsilon_c \equiv -\eta^2$  inside, with constant  $\eta > 0$  (in other words  $\eta := \sqrt{-\varepsilon_c}$ ).

**3.1. Scattering by a disk.** We take  $d = (0, 1)^\top$ . Note that we can get the solution for a direction  $d'$  by doing a rotation that transforms  $d$  into  $d'$ . In the following,  $J_m$  denotes the Bessel function of the first kind of order  $m$ , and  $I_m$  denotes the modified Bessel function of the first kind of order  $m$ . Taking advantage of the geometry, we look for solution of Problem [Eq. \(2.1\)](#) of the form:

$$u(x) = u(r, \theta) = \sum_{m \in \mathbb{Z}} u_m(r, \theta) = \sum_{m \in \mathbb{Z}} w_m(r) e^{im\theta}, \quad (3.1)$$

with  $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$  the polar coordinates corresponding to the Cartesian coordinates  $x$ , and  $w_m(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) e^{-im\theta} d\theta$ ,  $m \in \mathbb{Z}$ . Similarly we write  $u^{\text{in}}(x) = \sum_{m \in \mathbb{Z}} w_m^{\text{in}} e^{im\theta}$  and  $u^{\text{sc}}(x) = \sum_{m \in \mathbb{Z}} w_m^{\text{sc}}(r) e^{im\theta}$ .

*Remark 3.1.* The Jacobi-Anger expansion [[35](#), Eq. 10.12.1] states that

$$u^{\text{in}}(x) = e^{ik \cdot x} = \sum_{m \in \mathbb{Z}} J_m(kr) e^{im(\theta - \phi + \frac{\pi}{2})}, \quad \text{with } d = (\cos(\phi), \sin(\phi))^\top. \quad (3.2)$$

The series in [\(3.2\)](#) converges absolutely on every compact set of  $\mathbb{R}^2$ . With  $d = (0, 1)^\top$ , we obtain  $w_m^{\text{in}}(r) = J_m(kr)$ ,  $\forall m \in \mathbb{Z}$ .

Plugging in Problem Eq. (2.1), we obtain a family of 1D problems indexed by  $m \in \mathbb{Z}$ : Find  $w_m \in H_{\text{loc}}^1(\mathbb{R}_+, r dr)$  such that

$$\begin{cases} -\frac{1}{r} \partial_r (r \partial_r w_m) + \frac{m^2}{r^2} w_m - \varepsilon k^2 w_m = 0 & \text{in } (0, R) \cup (R, +\infty) \\ [w_m]_{\{R\}} = 0 \quad \text{and} \quad [\varepsilon^{-1} w'_m]_{\{R\}} = 0 & \text{across } \{R\} \\ w'_0(0) = 0 \quad \text{and} \quad w_m^{\text{sc}}(r) \propto H_m^{(1)}(kr) & r > R \end{cases} \quad (3.3)$$

with  $\propto$  meaning “up to a constant”. For  $m \neq 0$ , the term  $\frac{m^2}{r^2} w_m$  imposes a homogeneous Dirichlet boundary condition at zero [7]. Since the solution is continuous at  $r = 0$ , using the outgoing wave condition and the transmission conditions at  $r = R$ , we write

$$w_m^{\text{sc}}(r) = \begin{cases} \alpha_m I_m(\eta k r) - J_m(k r) & \text{if } r \leq R \\ \beta_m H_m^{(1)}(k r) & \text{if } r > R \end{cases} \quad (3.4)$$

with  $(\alpha_m, \beta_m)$  solution of the system  $M_m(\eta, k R)(\alpha_m, \beta_m)^\top = (J_m(k R), J'_m(k R))^\top$  where

$$M_m(\eta, k R) = \begin{pmatrix} I_m(\eta k R) & -H_m^{(1)}(k R) \\ -\eta^{-1} I'_m(\eta k R) & -H_m^{(1)'}(k R) \end{pmatrix}. \quad (3.5)$$

*Remark 3.2.* Since  $k > 0$  and the problem is well-posed for  $\eta \neq 1$ , coefficients  $(\alpha_m, \beta_m)$  are uniquely defined and  $\det(M_m(\eta, k R)) \neq 0$ , with

$$\det(M_m(\eta, z)) := -\eta^{-1} I'_m(\eta z) H_m^{(1)}(z) - I_m(\eta z) H_m^{(1)'}(z), \quad \forall z \in \mathbb{C}^*. \quad (3.6)$$

Now that we have an explicit expression of  $u^{\text{sc}}$ , we can analyze its behavior for various wavenumbers  $k$  and permittivities  $\varepsilon_c$ . For numerical purposes, we define  $u_k$  the approximate solution of Problem Eq. (2.1) associated to the wavenumber  $k$ , of order  $M$ :  $u_k = \sum_{m=-M}^M w_m(r) \mathbf{e}^{im\theta}$ , and we define the sequence  $(u_k)_k := (u_k^{\text{in}})_k + (u_k^{\text{sc}})_k$ . We choose here  $M = 32$ , so that  $(u_k)_k$  converges to order  $10^{-16}$ . We define  $L_\rho^2 := L^2(\mathbb{D}(0, \rho))$  for  $\rho > R$  and the function  $N_{\varepsilon, \rho}$  defined by

$$N_{\varepsilon, \rho}(k) = \frac{\|u_k^{\text{sc}}\|_{L_\rho^2}}{\|u_k^{\text{in}}\|_{L_\rho^2}}.$$

The function  $k \mapsto N_{\varepsilon, \rho}(k)$  characterizes the well-posedness constant as explained in Remark A.2. We consider the scattering of a plane wave by the unit disk ( $R = 1$ ) and choose  $\rho = 2$ . Figure 1 represents the log plot of  $N_{\varepsilon, 2}$  with respect to  $k$ , for various values of  $\varepsilon_c$ . One observes that

- For  $-1 < \varepsilon_c < 0$ ,  $N_{\varepsilon, 2}$  remains bounded.
- For  $\varepsilon_c < -1$ , there exists a sequence  $k_m$  such that  $N_{\varepsilon, 2}(k_m)$  peaks and the sequence  $(N_{\varepsilon, 2}(k_m))_{m \geq 1}$  grows exponentially.

To understand the observed instabilities when  $\varepsilon_c < -1$ , let us take a closer look at  $u_k^{\text{sc}}$  for  $k = k_m$  and  $k \neq k_m$ , for a given  $m$ . Figure 2 represents the modulus of  $u_k^{\text{sc}}$  for  $k = \{k_{12} - 0.01, k_{12}, k_{12} + 0.01\}$  and  $\varepsilon_c \equiv -1.1$ . One observes for  $k = k_m$  an intensely localized interface behavior for the scattered field with roughly four orders of magnitude compared to the amplitude's field when  $k \neq k_m$ . Moreover the localized behavior is oscillatory along the interface contrary to when  $k \neq k_m$ . This localized behavior indicates the appearance of a surface plasmon.

The above results provide the following:

- Even if the scattering problem is well-posed for all  $k > 0$ , we can have strong instabilities for a sequence of wavenumbers  $(k_m)_m$ .

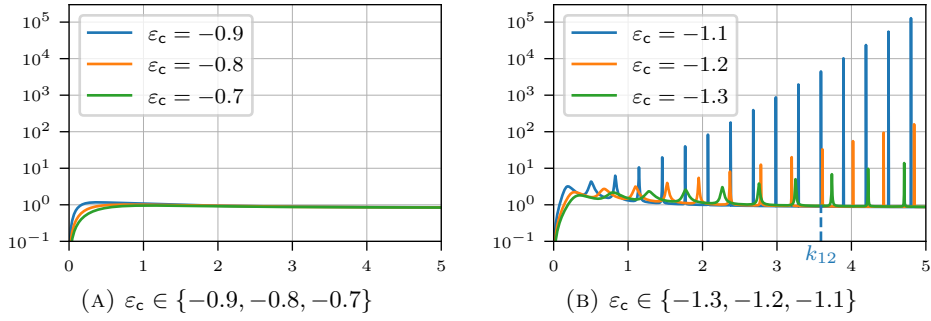


FIGURE 1. Log plot of  $k \mapsto N_{\epsilon,2}(k)$  with respect to  $k$  for  $\epsilon_c \in \{-0.9, -0.8, -0.7\}$  (left), for  $\epsilon_c \in \{-1.3, -1.2, -1.1\}$  (right). The value  $k_{12}$  marked on the graphs corresponds to the reference value used in Figure 2.

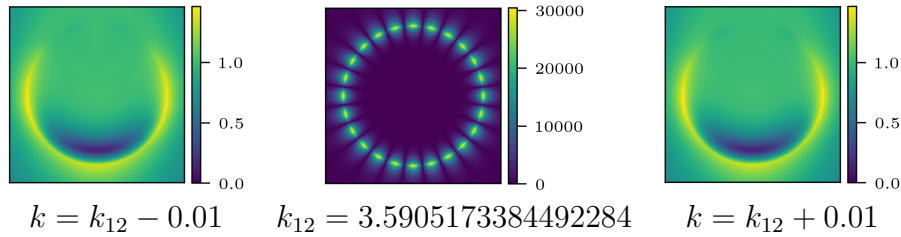


FIGURE 2. Modulus of the scattered field  $u^{sc}$  for a disk of radius 1 with  $\epsilon_c = -1.1$  ( $\eta = \sqrt{1.1}$ ), and  $k = k_{12} - 0.01$  (left column),  $k_{12}$  (middle), and  $k_{12} + 0.01$  (right column).

- Those instabilities arise only for  $\epsilon_c < -1$ , and are due to surface plasmons.

Can we characterize the sequence  $(k_m)_{m \geq 1}$ ? Can we justify that the instabilities are caused by surface plasmons? In what follows we investigate the associated spectral problem and use semi-classical analysis to answer those questions.

**3.2. Eigenvalues and resonances for the disk.** Proceeding similarly as in Section 3.1, the spectral problem Eq. (2.5) set on a disk can be rewritten as a family of 1D problems indexed by  $m \in \mathbb{Z}$ : Find  $(\ell, w_m) \in \mathbb{C}^{\frac{1}{2}} \times H_{\text{loc}}^1(\mathbb{R}_+, r \, dr) \setminus \{0\}$ , such that

$$\begin{cases} -\frac{1}{r} \partial_r (r \partial_r w_m) + \frac{m^2}{r^2} w_m = \varepsilon \ell^2 w_m & \text{in } (0, R) \cup (R, +\infty) \\ [w_m]_{\{R\}} = 0 \quad \text{and} \quad [\varepsilon^{-1} w'_m]_{\{R\}} = 0 & \text{across } \{R\} \\ w'_0(0) = 0 \quad \text{and} \quad w_m(r) \propto H_m^{(1)}(\ell r) & r > R \end{cases} \quad (3.7)$$

Similarly, we write

$$w_m(r) = \begin{cases} \alpha_m \mathfrak{l}_m(\eta \ell r) & \text{if } r \leq R \\ \beta_m H_m^{(1)}(\ell r) & \text{if } r > R \end{cases} \quad (3.8)$$

however this time, the pair  $(\ell, w_m)$  is solution of Eq. (3.7) if, and only if, there exists  $(\alpha_m, \beta_m)^\top \in \ker(M(\eta, \ell R)) \setminus (0, 0)^\top$ , with  $M(\eta, \ell R)$  defined in Eq. (3.5). The values  $\ell^2 \in \mathbb{C}$  such that  $(\ell, w_m)$  is a solution of Eq. (3.7) are called resonances, and  $w_m$  is called the associated resonant mode. To distinguish eigenvalues from resonances, we will use the fact that a resonant mode associated to an eigenvalue is in  $H^1(\mathbb{R}_+, r \, dr)$ . Given  $m \in \mathbb{Z}$ , and using Eq. (3.6), we define the set of resonances

$$\mathcal{R}[\varepsilon_c, R](m) = \left\{ \ell^2 \in \mathbb{C} \mid \det(M_m(\eta, \ell R)) = 0 \text{ and } \ell \in \mathbb{C}^{\frac{1}{2}} \right\}. \quad (3.9)$$



Finally we define the set of resonances of Problem Eq. (2.5)

$$\mathcal{R}[\varepsilon_c, R] := \bigcup_{m \in \mathbb{Z}} \mathcal{R}[\varepsilon_c, R](m). \quad (3.10)$$

*Remark 3.3.* Given  $\ell^2 \in \mathcal{R}[\varepsilon_c, R](m)$ , one finds  $\alpha_m = c$  and  $\beta_m = c \frac{l_m(\eta \ell R)}{H_m^{(1)}(\ell R)}$  with  $c \in \mathbb{C}^*$  since the resonant modes are defined up to some normalization.

*Remark 3.4.* Since  $l_{-m} = l_m$  and  $H_{-m}^{(1)} = (-1)^m H_m^{(1)}$ ,  $\forall m \in \mathbb{Z}$ , [35, Eq. 10.27.1 and 10.4.2], by symmetry all the resonances  $\ell^2$ , corresponding to  $m \neq 0$ , are of multiplicity 2, and the two associated modes are conjugate, given by  $u_m(r, \theta) := w_m(r) e^{\pm im\theta}$ . It turns out  $\mathcal{R}[\varepsilon_c, R] := \bigcup_{m \in \mathbb{N}} \mathcal{R}[\varepsilon_c, R](m)$ .

The resonances sets  $(\mathcal{R}[\varepsilon_c, R](m))_m$  defined in Eq. (3.9) cannot be computed analytically, however one can use contour integration techniques on Eq. (3.6) to compute a subset  $\mathcal{R}_N[\varepsilon_c, R] := \bigcup_{m=0}^N \mathcal{R}[\varepsilon_c, R](m) \subset \mathcal{R}[\varepsilon_c, R]$  (see [30, 36]). Figure 3 represents the set  $\mathcal{R}_{64}[\varepsilon_c, 1]$  for the unit disk and for various permittivities  $\varepsilon_c$ . The color bar indicates the value of  $m$ .

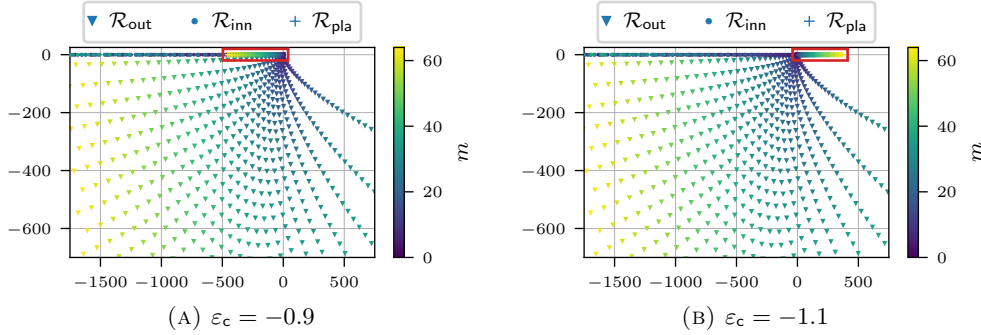


FIGURE 3. Graph of the sets  $\mathcal{R}_{64}[-0.9, 1]$  (left) and  $\mathcal{R}_{64}[-1.1, 1]$  (right) in the complex plane  $(\Re(\ell^2), \Im(\ell^2))$ , computed using complex contour integration [36] on the analytic function Eq. (3.6).

In classical cavities, resonances of Problem Eq. (2.5) (at least for  $\varepsilon_c > 1$ ) [6] are split into two categories: inner resonances  $\mathcal{R}_{\text{inn}}[\varepsilon_c, R]$  associated to resonant modes essentially supported inside the cavity  $\Omega$ , and outer resonances  $\mathcal{R}_{\text{out}}[\varepsilon_c, R]$  associated to resonant modes essentially supported in the exterior of the cavity  $\mathbb{R}^2 \setminus \bar{\Omega}$ . The inner resonance category includes the so-called *Whispering Gallery Modes* (WGM), associated to resonances  $\ell_{\text{WGM}}$  such that  $-1 \ll \Im(\ell_{\text{WGM}}) < 0$  [19]. In particular the approximation of Eq. (2.1) can be deteriorated if one chooses  $k = \Re(\ell_{\text{WGM}})$ , where those modes can be excited [32, Sec. 6.2]. When  $\varepsilon_c < 0$  we split the resonances into three categories. From Fig. 3, and using Remark 2.3 we conclude:

- The outer resonances  $\mathcal{R}_{\text{out}}[\varepsilon_c, 1]$  (represented as triangles in Fig. 3) are resonances with a negative imaginary part.
- The inner resonances  $\mathcal{R}_{\text{inn}}[\varepsilon_c, 1]$  (represented as dots in Fig. 3) are negative real eigenvalues of the operator  $P$  on  $L^2(\mathbb{R}^2)$ . They contain whispering gallery modes.
- The resonances represented at '+' inside the red rectangles in Fig. 3 are associated to resonant modes essentially supported on the interface  $\Gamma$  (see Fig. 4 for an example). We refer to those modes as surface plasmons, and we call this family the interface resonances  $\mathcal{R}_{\text{pla}}[\varepsilon_c, 1]$ . We denote the interface resonances  $(\ell_m^2)_m$  so that  $\mathcal{R}_{\text{pla}}[\varepsilon_c, 1] = \{\ell_m^2 \mid m \in \mathbb{N}^*\}$ .



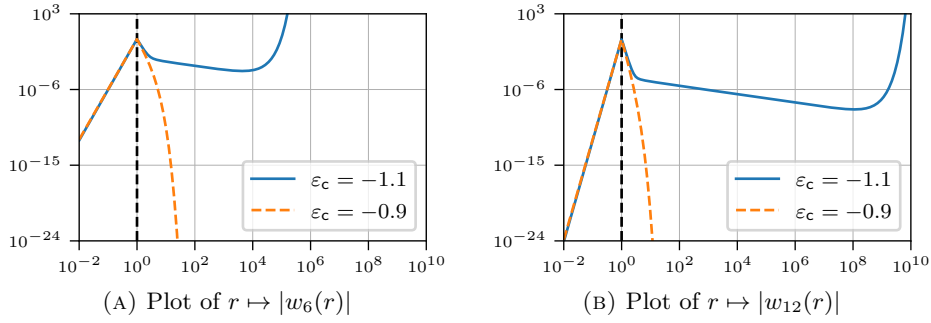


FIGURE 4. Log-Log plots of the radial component  $r \mapsto w_m(r)$  of the interface resonant mode  $(r, \theta) \mapsto u_m(r, \theta) = w_m(r)e^{im\theta}$  for  $\varepsilon_c \in \{-1.1, -0.9\}$  and for  $m = 6$  (c), for  $m = 12$  (f). When  $\varepsilon_c < -1$ ,  $w_m(r)$  tends to  $\infty$  as  $r \rightarrow \infty$ , therefore  $u_6 \in L^2_{\text{loc}}(\mathbb{R}^2)$  ( $\ell_6 \in \mathbb{C}^{i-}$  corresponds to a resonance). When  $-1 < \varepsilon_c < 0$ ,  $w_6(r)$  tends to 0 as  $r \rightarrow \infty$ , therefore  $u_6 \in L^2(\mathbb{R}^2)$  ( $\ell_6 \in \mathbb{C}^{i+}$  corresponds to an eigenvalue).

In the end we write  $\mathcal{R}[\varepsilon_c, R] = \mathcal{R}_{\text{out}}[\varepsilon_c, R] \cup \mathcal{R}_{\text{inn}}[\varepsilon_c, R] \cup \mathcal{R}_{\text{pla}}[\varepsilon_c, R]$ . The interface resonances are quite peculiar as their nature changes depending on  $\varepsilon_c$ . As illustrated in Fig. 3, they correspond to negative real eigenvalues when  $-1 < \varepsilon_c < 0$ , while they correspond to complex resonances in  $\mathbb{C}^{i-}$  when  $\varepsilon_c < -1$ . One observes that the negative eigenvalues diverge to  $-\infty$  as  $m \rightarrow \infty$ , while the real part of the complex resonances diverge toward  $+\infty$  as  $m \rightarrow \infty$ , and their negative imaginary part tends to 0 exponentially fast as  $m \rightarrow \infty$ . Additionally, a closer observation gives us that  $\Re(\ell_m^2) \propto m^2$ . Figure 4 represents the behavior of  $w_m(r)$  far from the boundary for  $m = \{6, 12\}$ ,  $\varepsilon_c = \{-1.1, -0.9\}$ . One can see that the modes are locally exponentially decreasing moving away from the interface (and oscillatory along the interface in  $e^{im\theta}$ ), which is the mathematical characterization of surface plasmons [31, 8]. Going back to the scattering problem, it turns out that the dashed blue lines in Fig. 1 correspond to the real part of the interface resonances:  $(\Re(\ell_m))_{m \geq 1}$  with  $\ell_m^2 \in \mathcal{R}_{\text{pla}}[-1.1, 1]$ . In other words, the instabilities observed in the scattering problems are caused by plasmonic resonances close to the real axis. In the next section, we characterize this interface resonances family  $(\ell_m)_{m \geq 1}$  by performing asymptotic expansion as  $m \rightarrow \infty$ . In particular we will confirm that  $\ell_m^2 \propto m^2$ .

**3.3. Interpretation with Schrödinger operator for the disk.** From Section 3.2 we found that the nature of plasmonic resonances changes depending on  $\varepsilon_c$  (i.e.  $\eta$ ). In this section we use asymptotic expansions to explain this change of behavior. To do so we provide an analogy with the Schrödinger operator. One can rewrite Problem Eq. (3.7) as

$$\begin{cases} -m^{-2} \frac{1}{r} \partial_r (r \partial_r w_m^\pm) + \frac{1}{r^2} w_m^\pm = \varepsilon \check{\lambda} w_m^\pm & \text{in } (0, R) \text{ and } (R, +\infty) \\ w_m^-(R) = w_m^+(R) \text{ and } -\eta^{-2} \partial_r w_m^-(R) = \partial_r w_m^+(R) & \text{across } \{R\} \\ w_0^{-'}(0) = 0 \text{ and } w_m^+ \in \mathcal{S}([R, +\infty)) \end{cases} \quad (3.11)$$

with  $\check{\lambda} = m^{-2} \ell^2$  the new spectral parameter,  $w_m^\pm$  restrictions of  $w_m$  in each material, and  $\mathcal{S}(\mathbb{R}_+)$  denoting the Schwartz space. We replace the outgoing wave condition by the Schwartz in order to characterize exponentially decreasing behaviors from both sides of the interface (a.k.a surface plasmons). To identify this behavior, first we rescale the problem Eq. (3.11) by  $\xi = r/R - 1$  such that  $r = R$  corresponds to  $\xi = 0$ . We then

define  $v_m^\pm(\xi) = w_m^\pm(R(1 + \xi))$ , and rewrite Problem Eq. (3.11) as

$$\begin{cases} -m^{-2} \mathcal{L} v_m^\pm + V v_m^\pm = \varepsilon R^2 \check{\lambda} v_m^\pm & \text{in } (-1, 0) \text{ and } (1, +\infty) \\ v_m^-(0) = v_m^+(0) \text{ and } -\eta^{-2} \partial_\xi v_m^-(0) = \partial_\xi v_m^+(0) & \text{across } \{0\} \\ v_0'^-(-1) = 0 \text{ and } v_m^+ \in \mathcal{S}(\mathbb{R}_+) \end{cases} \quad (3.12)$$

where  $\mathcal{L}(\xi, \partial_\xi) = \frac{1}{1+\xi} \partial_\xi((1+\xi) \partial_\xi)$  is a positive elliptic operator (Laplacian like) and  $V(\xi) = \frac{1}{(1+\xi)^2}$  is a potential. In that sense, the operator  $v \mapsto (-m^{-2} \mathcal{L} + V)v$  in (3.12) can be interpreted as a Schrdinger operator. Consequently, depending on the sign of  $\check{\lambda}$  one can find an eigenvalue or a resonance:  $\check{\lambda} < 0$  will correspond to an eigenvalue, and  $\check{\lambda} > 0$  will correspond to a resonance. Note that the choice to set  $\check{\lambda} = m^{-2} \ell^2$  implies that we look for the leading order only. In other words we look for  $\ell^2 \in \mathbb{R}$  at leading order. Figure 5 illustrates both situations. To construct localized modes at the

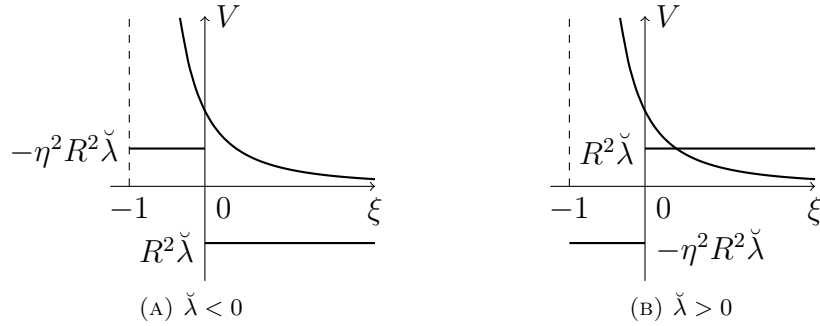


FIGURE 5. Graphs of the potential  $V$  and the spectral parameter  $\check{\lambda}$ : for  $\check{\lambda} < 0$  (a), for  $\check{\lambda} > 0$  (b). In case (a)  $V - R^2 \check{\lambda} > 0$  leads to an eigenvalue, in case (b)  $V(\xi) - R^2 \check{\lambda} < 0$  for  $\xi > \xi_0$  leads to a resonance.

interface, we consider the principal part of  $-m^2 \mathcal{L} + V$  with its coefficients frozen at  $\xi = 0$ , corresponding to  $-m^{-2} \partial_\xi^2 + 1$ . It is then natural to rescale by  $\sigma = m\xi$ , and the leading order behavior of Problem Eq. (3.12) becomes

$$\begin{cases} -\partial_\sigma^2 \varphi^- + \varphi^- = -\eta^2 R^2 \check{\lambda} \varphi^- & \text{in } (-\infty, 0) \\ -\partial_\sigma^2 \varphi^+ + \varphi^+ = R^2 \check{\lambda} \varphi^+ & \text{in } (0, +\infty) \\ \varphi^-(0) = \varphi^+(0) \text{ and } \eta^{-2} \partial_\sigma \varphi^-(0) = \partial_\sigma \varphi^+(0) & \text{across } \{0\} \\ \varphi^\pm \in \mathcal{S}(\mathbb{R}_\pm) \end{cases} \quad (3.13)$$

with  $\varphi^\pm(\sigma) = v_m^\pm(\xi)$ . Note that the condition  $v_m^-(-1) = \varphi^-(-m) = 0$  becomes  $\varphi^- \in \mathcal{S}(\mathbb{R}_-)$  to keep a localized behavior as  $m \rightarrow +\infty$ . Solutions of (3.13) are given by  $(\check{\lambda}, \varphi^\pm) = (R^{-2}(1 - \eta^{-2}), e^{-\eta^{\mp 1} |\sigma|})$ , where the modes are exponentially decreasing on both sides of the interface  $\sigma = 0$ . Back to the initial problem, the leading behavior corresponds to

$$\ell_m^2 = \frac{m^2}{R^2} (1 - \eta^{-2}), \quad \text{and} \quad \underline{w}_m^\pm(r) = \exp\left(-\eta^{\mp 1} m \left|\frac{r}{R} - 1\right|\right). \quad (3.14)$$

We conclude:

- when  $\varepsilon_c < -1$  ( $\eta > 1$ ), surface plasmons are associated to  $\ell^2 > 0$  (at first order), which corresponds to a resonance;
- when  $-1 < \varepsilon_c < 0$  ( $0 < \eta < 1$ ), surface plasmons are associated to  $\ell^2 < 0$  (at first order), which corresponds to an eigenvalue.

We have then characterized the asymptotic behavior of surface plasmons by building pairs  $(\ell_m^2, \underline{w}_m)_{m \geq 1}$ . The obtained results match the observed behaviors in previous sections, and provide accurate predictions of the instabilities in the scattering problem for  $\varepsilon_c < -1$ .

The case of the circular cavity with constant  $\varepsilon_c$  is quite intuitive, and the leading order computations are explicit. In the next section we generalize the approach, to any order, for the general case: arbitrary shaped smooth boundary, and varying coefficients  $\varepsilon_c \in \mathcal{C}^\infty(\Omega)$ . To that aim, we will use semi-classical WKB (Wentzel–Kramers–Brillouin) expansions in a tubular neighborhood of the interface, and matched asymptotic expansions along the interface. The higher order terms allow to show a super-algebraic behavior of the peaks seen in Fig. 1, explaining the exponential increase asymptotically.

*Remark 3.5.* In this paper we focus on the construction of the pairs  $(\ell_m^2, \underline{w}_m)_{m \geq 1}$  to characterize the asymptotic behavior of surface plasmons. The proper justification of the observed instabilities in the scattering problem (corresponding to the real part of the resonances  $\ell_m$ ) is not addressed in this paper. This involves technical details of spectral theory for problems with sign-changing coefficients, and we refer the reader to [17] for details. However we discuss the connection between the constructed pairs and the well-posedness constant  $N_{\varepsilon, \rho}$  in Section 5.

#### 4. ASYMPTOTIC FOR METAMATERIAL CAVITIES

To asymptotically characterize the interface resonances and their different behaviors in the general case, we construct a sequence of approximate solutions  $(\lambda_m, \underline{u}_m)_{m \geq 1}$  of Problem Eq. (2.5), called quasi-pair.

**Definition 4.1.** A *quasi-pair* for Problem Eq. (2.5) is formed by a sequence  $(\lambda_m)_{m \geq 1}$  of real numbers called quasi-resonances, and a sequence  $(\underline{u}_m)_{m \geq 1}$  of complex valued functions called quasi-modes that satisfy the following conditions:

- (1) For any  $m \geq 1$ , the functions  $\underline{u}_m$  are uniformly compactly supported and

$$\underline{u}_m \in D(P), \quad \text{with} \quad \|\underline{u}_m\|_{L^2(\mathbb{R}^2)} = 1;$$

- (2) We have the following quasi-pair estimate as  $m \rightarrow +\infty$ ,

$$\|P\underline{u}_m - \lambda_m \underline{u}_m\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(m^{-\infty}), \quad (4.1)$$

with the notation  $a_m = \mathcal{O}(m^{-\infty})$  to indicate that for all  $N \in \mathbb{N}$ , there exists  $C_N > 0$  such that  $|a_m| \leq C_N m^{-N}$ , for all  $m \geq 1$ .

**4.1. Statement.** Recall  $\Omega \subset \mathbb{R}^2$  be a cavity of smooth boundary  $\Gamma$  (see Section 2.1). Let  $L$  be the length of  $\Gamma$ , and we denote  $\varepsilon_c = -\eta^2$  with  $\eta := \sqrt{-\varepsilon_c} \in \mathcal{C}^\infty(\bar{\Omega}, (0, +\infty))$  a positive smooth function.

**Theorem 4.2.** *If  $\eta|_\Gamma \neq 1$  ( $\varepsilon_c|_\Gamma \neq -1$ ), there exists a quasi-pair  $(\lambda_m, \underline{u}_m)_{m \geq 1}$  of Problem Eq. (2.5). Moreover, we have  $\lambda_m \propto m^2$ , and*

- $\lambda_m > 0$ ,  $\lim_{m \rightarrow \infty} \lambda_m = +\infty$ , if  $\eta|_\Gamma > 1$ ;
- $\lambda_m < 0$ ,  $\lim_{m \rightarrow \infty} \lambda_m = -\infty$ , if  $\eta|_\Gamma < 1$ .

*Remark 4.3.* The proof of Theorem 4.2 relies on  $\lambda_m$  being constructed via a function  $\Lambda \in \mathcal{C}^\infty([0, \frac{L}{2\pi}])$  so that  $\lambda_m = (\frac{2\pi m}{L})^2 \Lambda(\frac{L}{2\pi m})$ , with  $\Lambda$  (see Lemma 4.6 and Lemma 4.11)

admitting the Taylor expansion

$$\Lambda(h) = \varsigma \left\langle \left| 1 - \frac{1}{\eta|_{\Gamma}^2} \right|^{-\frac{1}{2}} \right\rangle^{-2} + \mathcal{O}(h) \text{ as } h \rightarrow 0, \quad \text{with } \langle f \rangle := \frac{1}{L} \int_{\Gamma} f \, d\Gamma, \quad \forall f \in L^1(\Gamma).$$

Above,  $\varsigma = \pm 1$  is the sign of  $1 - \eta|_{\Gamma}^{-2}$  and  $\langle \cdot \rangle$  is the mean along the interface. From the above construction, one can easily check the properties from  $\underline{\lambda}_m$  stated in [Theorem 4.2](#).

**4.2. Proof.** The proof of [Theorem 4.2](#) follows the same structure as the proof of Theorem 4.A in [6], which asymptotically characterizes quasi-pairs for WGM in classic cavities. We make use of a different scaling to capture surface plasmons.

**4.2.1. Formal expansions.** First, we define a tubular neighborhood  $\mathcal{V}_{\delta}$  of the interface  $\Gamma$ . Let  $\gamma : \mathbb{T}_L \rightarrow \Gamma$  be a counterclockwise curvilinear parameterization of  $\Gamma$  with the notation  $\mathbb{T}_L := \mathbb{R}/L\mathbb{Z}$ . Let  $n = (\gamma'_2, -\gamma'_1)^{\top}$  be the unit exterior normal to  $\Omega$  and  $\kappa : \mathbb{T}_L \rightarrow \mathbb{R}$  be the signed curvature. We define the tubular neighborhood  $\mathcal{V}_{\delta} := \{\gamma(s) + \xi n(s) \mid (s, \xi) \in \mathbb{T}_L \times (-\delta, \delta)\}$  [33], see [Fig. 6](#).

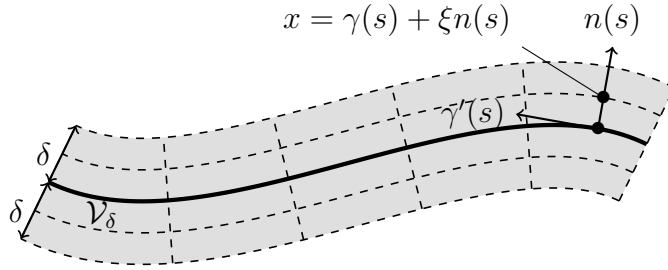


FIGURE 6. Tubular neighborhood and notations:  $s$  denotes an arclength parametrization of the curve  $\gamma$ , and  $\xi$  is the normal variable.

We now consider the problem:

$$\begin{cases} Pu = \lambda u & \text{in } \Omega \cap \mathcal{V}_{\delta} \text{ and } (\mathbb{R}^2 \setminus \overline{\Omega}) \cap \mathcal{V}_{\delta} \\ [u]_{\Gamma} = 0 \text{ and } [\varepsilon^{-1} \partial_n u]_{\Gamma} = 0 & \text{across } \Gamma \\ u = 0 & \text{on } \partial \mathcal{V}_{\delta} \end{cases} \quad (4.2)$$

where  $P = -\operatorname{div}(\varepsilon^{-1} \nabla)$  with  $\varepsilon$  defined in [Eq. \(2.2\)](#). By [Definition 4.1](#), the quasi-pairs are compactly supported therefore the outgoing condition does not play a role in their construction. We replace in particular the outgoing wave condition by an homogeneous Dirichlet boundary condition in order to construct localized quasi-pairs. The formal construction relies on: (i) an initialization where we setup the equation and the expansions in the tubular neighborhood, (ii) the leading order term computation, and (iii) a recurrence to compute higher order terms.

**Initialization.** The change of variables from the tubular coordinates  $(s, \xi) \in \mathbb{T}_L \times (-\delta, \delta)$  to the Cartesian coordinates  $x \in \mathcal{V}_{\delta}$  is a smooth diffeomorphism for  $0 < \delta < (\max_{\mathbb{T}_L} |\kappa|)^{-1}$ . In this tubular coordinate system the operator  $P$  becomes

$$P = -g^{-1} \operatorname{div}_{s,\xi} (\varepsilon^{-1} G \nabla_{s,\xi}) \quad (4.3)$$

where  $g(s, \xi) = 1 + \xi \kappa(s) > 0$  and  $G(s, \xi) = \begin{pmatrix} g(s, \xi)^{-1} & 0 \\ 0 & g(s, \xi) \end{pmatrix}$ .

For the general case we use a WKB framework [5] in order to provide an asymptotic expansion of the spectral parameter as the number of oscillations around the interface,

denoted  $m$  in [Section 3.3](#), goes to infinity. We introduce a small parameter  $h > 0$  that will later be linked to  $m$  and the ansatz for the quasi-pair  $(\lambda, u)$ :

$$u(s, \xi) = w(s, \xi) \exp\left(\frac{i}{h} \theta(s)\right) \quad \text{and} \quad \lambda = h^{-2} \check{\lambda} \quad (4.4)$$

where  $w : \mathbb{T}_L \times (-\delta, \delta) \rightarrow \mathbb{C}$ ,  $\theta : [0, L] \rightarrow \mathbb{C}$ , and  $\check{\lambda} \in \mathbb{C}$ . We add the constrain  $s \mapsto e^{\frac{i}{h} \theta(s)} \in \mathcal{C}^\infty(\mathbb{T}_L)$  so that the function  $u$  in [Eq. \(4.4\)](#) is a smooth function in  $\mathcal{V}_\delta \setminus \Gamma$ . Following [\[5\]](#) we formally expand the unknowns  $w$ ,  $\theta$ , and  $\check{\lambda}$  as

$$w(s, \xi) = \sum_{n \geq 0} w_n(s, \xi) h^n, \quad \theta(s) = \sum_{n \geq 0} \theta_n(s) h^n, \quad \text{and} \quad \check{\lambda} = \sum_{n \geq 0} \check{\lambda}_n h^n. \quad (4.5)$$

System [Eq. \(4.2\)](#) with the new unknowns [Eq. \(4.4\)](#) becomes

$$\begin{cases} \mathcal{L}_h[\varepsilon](w, \theta) = \check{\lambda} w & \text{in } \mathbb{T}_L \times ((-\delta, \delta) \setminus \{0\}) \\ [w]_{\mathbb{T}_L \times \{0\}} = 0 \text{ and } [\varepsilon^{-1} \partial_\xi w]_{\mathbb{T}_L \times \{0\}} = 0 & \text{across } \mathbb{T}_L \times \{0\} \\ w = 0 & \text{on } \mathbb{T}_L \times \{-\delta, \delta\} \end{cases} \quad (4.6)$$

Above,  $\mathcal{L}_h[\varepsilon](w, \theta) = h^2 e^{-\frac{i}{h} \theta} P(w e^{\frac{i}{h} \theta})$ , and it can be decomposed as

$$\mathcal{L}_h[\varepsilon](w, \theta) = \mathcal{L}_h^3[\varepsilon](w, \theta, \theta) + \mathcal{L}_h^2[\varepsilon](w, \theta) + \mathcal{L}_h^1[\varepsilon](w) \quad (4.7)$$

where  $\mathcal{L}_h^j[\varepsilon]$  are  $j$ -linear for  $j \in \{1, 2, 3\}$  and

$$\mathcal{L}_h^3[\varepsilon](w, \theta, \vartheta) = g^{-2} \varepsilon^{-1} w \partial_s \theta \partial_s \vartheta, \quad (4.8a)$$

$$\mathcal{L}_h^2[\varepsilon](w, \theta) = -h i (g^{-2} \varepsilon^{-1} \partial_s w \partial_s \theta + g^{-1} \partial_s (g^{-1} \varepsilon^{-1} w \partial_s \theta)), \quad (4.8b)$$

$$\mathcal{L}_h^1[\varepsilon](w) = -h^2 g^{-1} (\partial_\xi (g \varepsilon^{-1} \partial_\xi w) + \partial_s (g^{-1} \varepsilon^{-1} \partial_s w)). \quad (4.8c)$$

In the above decomposition, only  $\mathcal{L}_h^1[\varepsilon]$  involves derivatives with respect to  $\xi$ . Since  $g$  (resp.  $\eta = \sqrt{-\varepsilon_c} > 0$ ) is a smooth function on  $\mathbb{T}_L \times (-\delta, \delta)$  (resp.  $\mathbb{T}_L \times (-\delta, 0]$ ), then  $G$  is smooth and we write the formal Taylor expansions about  $\xi = 0$ :

$$g(s, \xi) = 1 + \xi \kappa(s), \quad G(s, \xi) = \sum_{n \geq 0} \frac{\partial_\xi^n G(s, 0)}{n!} \xi^n, \quad \eta(s, \xi) = \sum_{n \geq 0} \frac{\eta_n(s)}{n!} \xi^n, \quad (4.9)$$

where  $\eta_n(s) = \partial_\xi^n \eta(s, 0)$ . Since  $g$  and  $\eta$  do not vanish on  $\mathbb{T}_L \times \{0\}$ , the formal expansions of  $g^{-1}$ ,  $g^{-2}$ , and  $\eta^{-2}$  about  $\xi = 0$  can be computed with [Eq. \(4.9\)](#).

Like in [Section 3.3](#), we introduce the scaled variable  $\sigma = h^{-1} \xi$  for the normal variable  $\xi \in (-\delta, \delta)$ , and we define

$$\varphi^\pm(s, \sigma) = w(s, h\sigma) \quad \text{for } (s, \sigma) \in \mathbb{T}_L \times \mathbb{R}_\pm. \quad (4.10)$$

Then with  $g = g(s, h\sigma)$  we rewrite

$$\mathcal{L}_h^1[\varepsilon](\varphi^\pm) = -g^{-1} \partial_\sigma (\varepsilon^{-1} g \partial_\sigma \varphi^\pm) - h^2 g^{-1} \partial_s (\varepsilon^{-1} g^{-1} \partial_s \varphi^\pm). \quad (4.11)$$

Problem [Eq. \(4.6\)](#) becomes the formal problem: Find  $(\varphi_n^\pm)_{n \in \mathbb{N}} \in \mathcal{C}^\infty(\mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))^\mathbb{N}$ ,  $(\exp(i h^{-1} \theta_n))_{n \in \mathbb{N}} \in \mathcal{C}^\infty(\mathbb{T}_L)^\mathbb{N}$ , and  $(\check{\lambda}_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$  such that

$$\begin{cases} \mathcal{L}_h[\varepsilon](\sum_n \varphi_n^\pm h^n, \sum_n \theta_n h^n) = (\sum_n \check{\lambda}_n h^n) (\sum_n \varphi_n^\pm h^n) & \text{in } \mathbb{T}_L \times \mathbb{R}_\pm^* \\ \sum_n \varphi_n^-(s, 0) h^n = \sum_n \varphi_n^+(s, 0) h^n & \text{on } \mathbb{T}_L \times \{0\} \\ -\eta_0(s)^{-2} \sum_n \partial_\sigma \varphi_n^-(s, 0) h^n = \sum_n \partial_\sigma \varphi_n^+(s, 0) h^n & \text{on } \mathbb{T}_L \times \{0\} \end{cases} \quad (4.12)$$

Note that for simplicity we extend the scaled domain  $\mathbb{T}_L \times (-\frac{\delta}{h}, \frac{\delta}{h})$  to the domain  $\mathbb{T}_L \times \mathbb{R}$  in order to be independent of  $h$  in [Eq. \(4.12\)](#), and we replace the homogeneous Dirichlet boundary condition on  $\mathbb{T}_L \times \{-\frac{\delta}{h}, \frac{\delta}{h}\}$  by the conditions  $\sigma \mapsto \varphi^\pm(s, \sigma) \in \mathcal{S}(\mathbb{R}_\pm)$  for all  $s \in \mathbb{T}_L$ . One can always multiply the quasi-mode by a cutoff function  $\xi \mapsto \chi(\xi)$  to be in the domain  $\mathbb{T}_L \times (-\frac{\delta}{h}, \frac{\delta}{h})$ , as done later in [Eq. \(4.18\)](#). With [Eq. \(4.7\)](#) and [Eq. \(4.9\)](#), we can formally expand the operators  $\mathcal{L}_h^j[-\eta^2] = \sum_{n \geq 0} \mathbf{L}_n^{j,-} h^n$  and  $\mathcal{L}_h^j[1] = \sum_{n \geq 0} \mathbf{L}_n^{j,+} h^n$  where  $\mathbf{L}_n^{j,\pm}$  are independent of  $h$ , for  $j \in \{1, 2, 3\}$ . From Problem [Eq. \(4.12\)](#) we obtain the family of problems  $(\mathcal{P}_n)_{n \in \mathbb{N}}$  by identifying powers of  $h$ :

$$\begin{cases} \sum_{p \in \mathbb{N}_n^4} \mathbf{L}_{p_1}^{3,\pm} (\varphi_{p_2}^\pm, \theta_{p_3}, \theta_{p_4}) + \sum_{p \in \mathbb{N}_n^3} \mathbf{L}_{p_1}^{2,\pm} (\varphi_{p_2}^\pm, \theta_{p_3}) + \sum_{p \in \mathbb{N}_n^2} \mathbf{L}_{p_1}^{1,\pm} (\varphi_{p_2}^\pm) = \sum_{p \in \mathbb{N}_n^2} \check{\lambda}_{p_1} \varphi_{p_2}^\pm \\ \varphi_n^-(s, 0) = \varphi_n^+(s, 0) \quad \text{and} \quad -\eta_0(s)^{-2} \partial_\sigma \varphi_n^-(s, 0) = \partial_\sigma \varphi_n^+(s, 0) \end{cases} \quad (4.13)$$

with the notation  $\mathbb{N}_n^d = \{p \in \mathbb{N}^d \mid p_1 + \dots + p_d = n\}$ .

Leading order terms. From [Eq. \(4.7\)](#) and [Eq. \(4.9\)](#) one obtains

**Lemma 4.4.** *The first terms of the expansions of  $\mathcal{L}_h^3[\varepsilon]$ ,  $\mathcal{L}_h^2[\varepsilon]$ , and  $\mathcal{L}_h^1[\varepsilon]$ , are given by*

$$\mathbf{L}_0^{3,-}(\phi, \theta, \vartheta) = -\eta_0^{-2} \phi \partial_s \theta \partial_s \vartheta, \quad \mathbf{L}_0^{2,-}(\phi, \theta) = 0, \quad \mathbf{L}_0^{1,-}(\phi) = \eta_0^{-2} \partial_\sigma^2 \phi, \quad (4.14a)$$

$$\mathbf{L}_0^{3,+}(\phi, \theta, \vartheta) = \phi \partial_s \theta \partial_s \vartheta, \quad \mathbf{L}_0^{2,+}(\phi, \theta) = 0, \quad \mathbf{L}_0^{1,+}(\phi) = -\partial_\sigma^2 \phi. \quad (4.14b)$$

Using [Lemma 4.4](#), we rewrite Problem  $(\mathcal{P}_0)$  as: Find  $\varphi_0^\pm \in \mathcal{C}^\infty(\mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$ ,  $\exp(i h^{-1} \theta_0) \in \mathcal{C}^\infty(\mathbb{T}_L)$ , and  $\check{\lambda}_0 \in \mathbb{C}$  such that  $(\varphi_0^-, \varphi_0^+) \not\equiv (0, 0)$  and

$$\begin{cases} \partial_\sigma^2 \varphi_0^- - (\theta_0'^2 + \eta_0^2 \check{\lambda}_0) \varphi_0^- = 0 & \text{in } \mathbb{T}_L \times \mathbb{R}_- \\ \partial_\sigma^2 \varphi_0^+ - (\theta_0'^2 - \check{\lambda}_0) \varphi_0^+ = 0 & \text{in } \mathbb{T}_L \times \mathbb{R}_+ \\ \varphi_0^-(s, 0) = \varphi_0^+(s, 0) & \text{on } \mathbb{T}_L \times \{0\} \\ -\eta_0(s)^{-2} \partial_\sigma \varphi_0^-(s, 0) = \partial_\sigma \varphi_0^+(s, 0) & \text{on } \mathbb{T}_L \times \{0\} \end{cases} \quad (4.15)$$

**Notation 4.5.** We recall that  $\eta_0 = \eta|_\Gamma$  and, we define the scalar  $\varsigma = \pm 1$  to be the sign of  $1 - \eta_0^{-2}$ , the functions  $\tau_0 = |1 - \eta_0^{-2}|^{-\frac{1}{2}}$  and  $\hat{\tau}_0 = \frac{\tau_0}{\langle \tau_0 \rangle}$ .

**Lemma 4.6.** *One can choose  $h = \frac{L}{2\pi m}$  for  $m \in \mathbb{N}^*$  so that  $(\varphi_0^\pm, \theta_0, \check{\lambda}_0)$  given by*

$$\check{\lambda}_0 = \frac{\varsigma}{\langle \tau_0 \rangle^2}, \quad \theta_0(s) = \int_0^s \hat{\tau}_0(t) dt, \quad \text{and} \quad \varphi_0^\pm(s, \sigma) = \alpha(s) \exp(-|\sigma| \hat{\tau}_0(s) \eta_0(s)^{\mp 1}),$$

with  $\alpha \in \mathcal{C}^\infty(\mathbb{T}_L, \mathbb{C}^*)$ , is solution of Problem  $(\mathcal{P}_0)$  defined in [Eq. \(4.15\)](#).

The proof is detailed in [Appendix B.1](#).

*Remark 4.7.* If we unravel the scaling and return to tubular coordinates, for  $m \geq 1$  and  $(s, \xi) \in \mathbb{T}_L \times \mathbb{R}$ , we formally have a pair  $(\underline{\lambda}_m, \underline{u}_m)$

$$\begin{aligned} \underline{\lambda}_m &= \left( \frac{2\pi m}{L} \right)^2 [\check{\lambda}_0 + \mathcal{O}(m^{-1})], \\ \underline{u}_m(s, \xi) &= e^{i \frac{2\pi m}{L} [\theta_0(s) + \mathcal{O}(m^{-1})]} \left[ \begin{cases} \varphi_0^-(s, \frac{2\pi m}{L} \xi) & \text{if } \xi \leq 0 \\ \varphi_0^+(s, \frac{2\pi m}{L} \xi) & \text{if } \xi > 0 \end{cases} + \mathcal{O}(m^{-1}) \right], \end{aligned}$$

which characterize surface plasmons at leading order.

*Remark 4.8.* The construction relies on several choices that are not unique.

- One can choose the main phase to satisfy  $\theta'_0 = \widehat{\tau}_0$  or  $\theta'_0 = -\widehat{\tau}_0$ . Then one can construct two modes corresponding to  $\underline{u}_m$  and  $\overline{\underline{u}_m}$  (see [Remark 4.16](#)).
- The function  $\theta_0$  is defined up to a constant  $c$ . Then  $\underline{u}_m$  in [Remark 4.7](#) is defined up to  $e^{i\frac{2\pi m}{L}c}$ . For simplicity we consider  $c = 0$  as we normalize in the end.
- The functions  $\varphi_0^\pm$  are defined up to a function  $\alpha : \mathbb{T}_L \rightarrow \mathbb{C}^*$ , which contributes to the phase of  $\underline{u}_m$  and therefore affects the number of oscillations along the interface. One can always shift indices so that  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1 - q_\alpha}$ , for some  $q_\alpha \in \mathbb{Z}$ , corresponds to a wave with  $m$  oscillations along the interface.

Recurrence. From [Eq. \(4.13\)](#), [Lemma 4.4](#), and [Lemma 4.6](#), for  $n \geq 1$ , we can rewrite Problem  $(\mathcal{P}_n)$  as: Find  $\varphi_n^\pm \in \mathcal{C}^\infty(\mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$ ,  $\exp(i h^{n-1} \theta_n) \in \mathcal{C}^\infty(\mathbb{T}_L)$ , and  $\check{\lambda}_n \in \mathbb{C}$  such that

$$\begin{cases} \partial_\sigma^2 \varphi_n^- - \widehat{\tau}_0^2 \eta_0^2 \varphi_n^- = \left(2\widehat{\tau}_0 \theta'_n + \eta_0^2 \check{\lambda}_n\right) \varphi_0^- + \eta_0^2 S_{n-1}^- & \text{in } \mathbb{T}_L \times \mathbb{R}_- \\ \partial_\sigma^2 \varphi_n^+ - \widehat{\tau}_0^2 \eta_0^{-2} \varphi_n^+ = \left(2\widehat{\tau}_0 \theta'_n - \check{\lambda}_n\right) \varphi_0^+ - S_{n-1}^+ & \text{in } \mathbb{T}_L \times \mathbb{R}_+ \\ \varphi_n^-(s, 0) = \varphi_n^+(s, 0) & \text{on } \mathbb{T}_L \times \{0\} \\ -\eta_0(s)^{-2} \partial_\sigma \varphi_n^-(s, 0) = \partial_\sigma \varphi_n^+(s, 0) & \text{on } \mathbb{T}_L \times \{0\} \end{cases} \quad (4.16)$$

where

$$\begin{aligned} S_{n-1}^\pm &= \sum_{p=1}^{n-1} \check{\lambda}_{n-p} \varphi_p^\pm - \sum_{p=0}^{n-1} \mathbf{L}_{n-p}^{1,\pm}(\varphi_p^\pm) - \sum_{p \in \mathbb{N}_{n-1}^3} \mathbf{L}_{p_1}^{2,\pm}(\varphi_{p_2}^\pm, \theta_{p_3}) \\ &\quad - \mathbf{L}_n^{2,\pm}(\varphi_0^\pm, \theta_0) - \sum_{p \in \mathbb{N}_{n-1}^4} \mathbf{L}_{p_1}^{3,\pm}(\varphi_{p_2}^\pm, \theta_{p_3}, \theta_{p_4}) - \mathbf{L}_n^{3,\pm}(\varphi_0^\pm, \theta_0, \theta_0). \end{aligned} \quad (4.17)$$

**Lemma 4.9.** Define  $(\varphi_0^\pm, \theta_0, \check{\lambda}_0)$  according to [Lemma 4.6](#). For  $n \geq 1$ , there exists  $(\varphi_n^\pm, \theta_n, \check{\lambda}_n) \in \mathcal{C}^\infty(\mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm)) \times \mathcal{C}^\infty(\mathbb{T}_L) \times \mathbb{C}$  solution of Problem  $(\mathcal{P}_n)$  defined in [Eq. \(4.16\)](#). In particular,  $\varphi_n^\pm$  is given by

$$\varphi_n^\pm(s, \sigma) = P_n^\pm(s, \sigma) \exp(-|\sigma| \widehat{\tau}_0(s) \eta_0(s)^{\mp 1}),$$

with polynomials  $P_n^\pm \in \mathcal{C}^\infty(\mathbb{T}_L, \mathbb{P})^1$ .

The proof is detailed in [Appendix B.2](#).

*Remark 4.10.* In addition to [Remark 4.8](#),  $(\theta_n)_{n \geq 0}$  and  $(\varphi_n)_{n \geq 0}$  are not uniquely defined at each step of the construction. However, the sequence  $(\lambda_n)_{n \geq 0}$  will be unique in the sense of [Corollary 4.15](#).

**4.2.2. Quasi-pairs.** Based on the formal series  $\sum_{n \in \mathbb{N}} \varphi_n^\pm h^n$ ,  $\sum_{n \in \mathbb{N}} \theta_n h^n$ , and  $\sum_{n \in \mathbb{N}} \check{\lambda}_n h^n$  with  $h = \frac{L}{2\pi m}$ , we now construct quasi-pairs in the sense of [Definition 4.1](#). First we use Borel's Lemma [[29](#), Thm. 1.2.6] for  $\check{\lambda}$  and  $\theta$ , and a direct generalization on the Fréchet space  $\mathcal{C}^\infty(\mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$  [[6](#), Lem. A.5] for  $\varphi^\pm$  to establish the

<sup>1</sup>We denote  $\mathbb{P}$  the space of polynomial of a single variable with complex coefficients.



**Lemma 4.11.** *There exists  $\Phi^\pm \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$ ,  $\Theta \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L)$ , and  $\Lambda \in \mathcal{C}^\infty([0, \frac{L}{2\pi}])$  such that, for  $N \geq 1$ ,  $h \in [0, \frac{L}{2\pi}]$ ,  $s \in \mathbb{T}_L$ , and  $\sigma \in \mathbb{R}_\pm$ , we have*

$$\begin{aligned}\Phi^\pm(h; s, \sigma) &= \sum_{n=0}^{N-1} \varphi_n^\pm(s, \sigma) h^n + h^N R_N^\pm(h; s, \sigma), \\ \Theta(h; s) &= \sum_{n=0}^{N-1} \theta_n(s) h^n + h^N R_N^\Theta(h; s), \quad \text{and} \quad \Lambda(h) = \sum_{n=0}^{N-1} \check{\lambda}_n h^n + h^N R_N^\Lambda(h)\end{aligned}$$

where  $R_N^\pm \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$ ,  $R_N^\Theta \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L)$ ,  $R_N^\Lambda \in \mathcal{C}^\infty([0, \frac{L}{2\pi}])$ .

From those functions, we now define the quasi-resonances  $\underline{\lambda}_m$  and the quasi-modes  $\underline{u}_m$  in the tubular neighborhood as

$$\begin{aligned}\underline{\lambda}_m &= \left(\frac{2\pi m}{L}\right)^2 \Lambda\left(\frac{L}{2\pi m}\right) \\ \underline{u}_m(s, \xi) &= \chi(\xi) \exp\left(i\frac{2\pi m}{L} \Theta\left(\frac{L}{2\pi m}; s\right)\right) \begin{cases} \Phi^-\left(\frac{L}{2\pi m}; s, \frac{2\pi m}{L}\xi\right) & \text{if } \xi \leq 0 \\ \Phi^+\left(\frac{L}{2\pi m}; s, \frac{2\pi m}{L}\xi\right) & \text{if } \xi > 0 \end{cases},\end{aligned}\tag{4.18}$$

where  $\chi$  is a cutoff function,  $\chi \in \mathcal{C}_{\text{comp}}^\infty((-\delta, \delta))$  and  $\chi \equiv 1$  on  $[-\frac{\delta}{2}, \frac{\delta}{2}]$ . In what follows, we establish that [Eq. \(4.18\)](#) is a quasi-pair. First we have

**Lemma 4.12.** *The pair  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$  defined in [Eq. \(4.18\)](#) satisfies the following:*

- (i)  $\underline{u}_m$  is uniformly compactly supported and smooth in  $\Omega$  and  $\mathbb{R}^2 \setminus \overline{\Omega}$ .
- (ii)  $\underline{u}_m$  satisfies  $[\underline{u}_m]_\Gamma = \mathcal{O}(m^{-\infty})$  and  $[\varepsilon^{-1} \partial_n \underline{u}_m]_\Gamma = \mathcal{O}(m^{-\infty})$ .
- (iii)  $\underline{u}_m$  admits the norm expansion  $\|\underline{u}_m\|_{L^2(\mathbb{R}^2)} = a m^{-\frac{1}{2}} + \mathcal{O}(m^{-\frac{3}{2}})$ ,  $a > 0$ .
- (iv) Let  $\underline{R}_m := P \underline{u}_m - \underline{\lambda}_m \underline{u}_m$ , then  $\|\underline{R}_m\|_{L^2(\Omega)} + \|\underline{R}_m\|_{L^2(\mathbb{R}^2 \setminus \overline{\Omega})} = \mathcal{O}(m^{-\infty})$ .
- (v) If two quasi-pairs  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$ ,  $(\underline{\mu}_m, \underline{v}_m)_{m \geq 1}$  satisfy (i)–(iv), and the quasi-modes have the same leading phase  $\theta_0(s) = \int_0^s \widehat{\tau}_0(t) dt$  then:  
 $\int_{\mathbb{R}^2} \underline{u}_m \overline{\underline{v}_m} dx = z_0 m^{-1} + \mathcal{O}(m^{-2})$ ,  $z_0 \in \mathbb{C}^*$ , and  $\int_{\mathbb{R}^2} \underline{u}_m \underline{v}_m dx = \mathcal{O}(m^{-\infty})$ .

*Remark 4.13.* Items (iii) and (v) of [Lemma 4.12](#) give us

$$\int_{\mathbb{R}^2} \frac{\underline{u}_m}{\|\underline{u}_m\|_{L^2(\mathbb{R}^2)}} \frac{\overline{\underline{v}_m}}{\|\underline{v}_m\|_{L^2(\mathbb{R}^2)}} dx = z'_0 + \mathcal{O}(m^{-1}), \quad \text{with } z'_0 \in \mathbb{C}^*.$$

*Proof.* Recall that we set  $h = \frac{L}{2\pi m}$ , and to simplify notations we denote  $\chi_h : \sigma \mapsto \chi(\sigma h)$ ,  $\Phi_h^\pm : (s, \sigma) \mapsto \Phi^\pm(h; s, \sigma)$ ,  $\Theta_h : s \mapsto \Theta(h; s)$ , and  $\Lambda_h = \Lambda(h)$ .

(i) By definition of  $(\underline{u}_m)_{m \geq 1}$ , (i) is satisfied.

(ii) Using [Lemma 4.11](#), one can show that  $[\underline{u}_m]_\Gamma = \mathcal{O}(m^{-N})$  and  $[\varepsilon^{-1} \partial_n \underline{u}_m]_\Gamma = \mathcal{O}(m^{-N})$  for all  $N \geq 0$ , which is the definition of  $\mathcal{O}(m^{-\infty})$ .

(iii) We introduce the weighted  $L^2$  semi-norm on  $\mathbb{T}_L \times \mathbb{R}_\pm$

$$\|f\|_{L_\pm^2[h]}^2 = \int_{\mathbb{T}_L} \int_{\mathbb{R}_\pm \cap (-\frac{\delta}{h}, \frac{\delta}{h})} |f(s, \sigma)|^2 h(1 + \kappa(s)\sigma h) d\sigma ds.\tag{4.19}$$

Form [Eq. \(4.18\)](#), we obtain

$$\|\underline{u}_m\|_{L^2(\mathbb{R}^2)}^2 = \left\| \chi_h \Phi_h^- e^{\frac{i}{h} \Theta_h} \right\|_{L_-^2[h]}^2 + \left\| \chi_h \Phi_h^+ e^{\frac{i}{h} \Theta_h} \right\|_{L_+^2[h]}^2.$$

From [Lemma 4.6](#) and [Lemma 4.11](#) for  $N = 1$ , we have

$$\Theta_h(s) = \int_0^s \tau_0(t) dt + \theta_1(s)h + h^2 R_2^\Theta(h; s)$$

$$\Phi_h^\pm(s, \sigma) = \alpha(s) \exp(-|\sigma| \widehat{\tau}_0(s) \eta_0(s)^{\mp 1}) + h R_1^\pm(h; s, \sigma)$$

where  $R_2^\Theta \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L)$  and  $R_1^\pm \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$ . We deduce that

$$\left| \left\| \chi_h \Phi_h^\pm e^{\frac{i}{h} \Theta_h} \right\|_{L_\pm^2[h]}^2 - \left\| \chi_h \alpha e^{-|\sigma| \widehat{\tau}_0(s) \eta_0(s)^{\mp 1}} e^{i \theta_1} \right\|_{L_\pm^2[h]}^2 \right| \leq C_1^\pm h^2$$

for  $C_1^\pm$  some positive constant. We write  $\left\| \chi_h \alpha e^{-|\sigma| \widehat{\tau}_0(s) \eta_0(s)^{\mp 1}} e^{i \theta_1} \right\|_{L_\pm^2[h]}^2 = I_1^\pm + I_2^\pm + I_3^\pm$ ,

$$I_1^\pm = h \int_{\mathbb{T}_L} \int_{\mathbb{R}_\pm} |\alpha(s)|^2 e^{\mp 2\sigma \widehat{\tau}_0(s) \eta_0(s)^{\mp 1}} e^{-2\Im \theta_1(s)} d\sigma ds = h \int_{\mathbb{T}_L} \frac{|\alpha(s)|^2 e^{-2\Im \theta_1(s)}}{2\widehat{\tau}_0(s) \eta_0(s)^{\mp 1}} ds,$$

$$I_2^\pm = h \int_{\mathbb{T}_L} \int_{\mathbb{R}_\pm} (|\chi(\sigma h)|^2 - 1) |\alpha(s)|^2 e^{\mp 2\sigma \widehat{\tau}_0(s) \eta_0(s)^{\mp 1}} e^{-2\Im \theta_1(s)} d\sigma ds,$$

$$I_3^\pm = h^2 \int_{\mathbb{T}_L} \int_{\mathbb{R}_\pm} |\chi(\sigma h) \alpha(s)|^2 e^{\mp 2\sigma \widehat{\tau}_0(s) \eta_0(s)^{\mp 1}} e^{-2\Im \theta_1(s)} \kappa(s) \sigma d\sigma ds.$$

One can show that  $I_2^\pm = \mathcal{O}(h^\infty)$  using [Lemma B.1](#). Since  $\chi$  is bounded and the function  $(h; s, \sigma) \mapsto |\alpha|^2 e^{\mp 2\sigma \widehat{\tau}_0(s) \eta_0(s)^{\mp 1}} e^{-2\Im \theta_1(s)} \kappa \sigma$  is in  $\mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$  there exists a constant  $C_3^\pm$  such that  $|I_3^\pm| \leq C_3^\pm h^2$ . Combining the results we get

$$\|u_m\|_{L^2(\mathbb{R}^2)}^2 = a^2 m^{-1} + \mathcal{O}(m^{-2})$$

with

$$a^2 = \frac{2\pi}{hL} (I_1^+ + I_1^-) = \frac{2\pi}{L} \int_{\mathbb{T}_L} |\alpha(s)|^2 e^{-2\Im(\theta_1(s))} \frac{\eta_0(s)^{-1} + \eta_0(s)}{2\widehat{\tau}_0(s)} ds > 0.$$

(iv) Revisiting the change of variables in tubular coordinates and the scaling, we get

$$\|\underline{R}_m\|_{L^2(\Omega)} = h^{-2} \left\| e^{i h^{-1} \Theta_h} (\mathcal{L}_h[\varepsilon](\cdot, \Theta_h) - \Lambda_h) (\chi_h \Phi_h^-) \right\|_{L_-^2[h]}, \quad (4.20a)$$

$$\|\underline{R}_m\|_{L^2(\mathbb{R}^2 \setminus \overline{\Omega})} = h^{-2} \left\| e^{i h^{-1} \Theta_h} (\mathcal{L}_h[\varepsilon](\cdot, \Theta_h) - \Lambda_h) (\chi_h \Phi_h^+) \right\|_{L_+^2[h]} \quad (4.20b)$$

with  $\mathcal{L}_h[\varepsilon]$  defined in [Eq. \(4.6\)](#). [Lemma 4.11](#) with  $N = 1$  and [Lemma 4.6](#) give the estimation  $\Im \Theta_h = \mathcal{O}(h)$  so there exists  $c_\Theta > 0$  such that  $|e^{i h^{-1} \Theta_h}| \leq c_\Theta$ . Introducing the commutator  $[\mathcal{L}_h[\varepsilon](\cdot, \Theta_h), \chi_h]$  of the differential operator  $\Phi \mapsto \mathcal{L}_h[\varepsilon](\Phi, \Theta_h)$  with the scaled cutoff function  $\chi_h$ , we deduce from [Eq. \(4.20\)](#)

$$\|\underline{R}_m\|_{L^2(\Omega)} \leq c_\Theta h^{-2} (\mathcal{N}_- + \mathcal{N}'_-) \quad \text{and} \quad \|\underline{R}_m\|_{L^2(\mathbb{R}^2 \setminus \overline{\Omega})} \leq c_\Theta h^{-2} (\mathcal{N}_+ + \mathcal{N}'_+) \quad (4.21)$$

where  $\mathcal{N}_\pm = \|\chi_h (\mathcal{L}_h[\varepsilon](\cdot, \Theta_h) - \Lambda_h) \Phi_h^\pm\|_{L_\pm^2[h]}$ ,  $\mathcal{N}'_\pm = \|[\mathcal{L}_h[\varepsilon](\cdot, \Theta_h), \chi_h] \Phi_h^\pm\|_{L_\pm^2[h]}$ . Let's start with  $\mathcal{N}_\pm$ . We write for  $N \geq 1$ ,

$$\begin{aligned} \mathcal{L}_h[\varepsilon](\Phi_h^\pm, \Theta_h) &= \sum_{n=0}^{N-1} h^n (\mathbf{L}_n^{\pm,3}(\Phi_h^\pm, \Theta_h, \Theta_h) + \mathbf{L}_n^{\pm,2}(\Phi_h^\pm, \Theta_h) + \mathbf{L}_n^{\pm,1}(\Phi_h^\pm)) \\ &\quad + h^N (\mathbf{R}_N^{\pm,3}(h; \Phi_h^\pm, \Theta_h, \Theta_h) + \mathbf{R}_N^{\pm,2}(h; \Phi_h^\pm, \Theta_h) + \mathbf{R}_N^{\pm,1}(h; \Phi_h^\pm)) \end{aligned}$$

where  $\mathbf{R}_N^{\pm,j}(h)$  are  $j$ -linear second order differential operators such that all the coefficients in  $\chi_h \mathbf{R}_N^{\pm,j}(h)$  are smooth bounded functions for  $j \in \{1, 2, 3\}$ . We use [Lemma 4.11](#) with different  $N$  for each occurrence of  $\Phi_h^\pm$  and  $\Theta_h$ , and we obtain

$$\begin{aligned} & \mathcal{L}_h[\varepsilon](\Phi_h^\pm, \Theta_h) - \Lambda_h \Phi_h^\pm \\ &= h^N \left[ \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}_{N-n}^3} \mathbf{L}_n^{\pm,3}(R_{p_1}^\pm(h), R_{p_2}^\Theta(h), R_{p_3}^\Theta(h)) + \mathbf{R}_N^{\pm,3}(h; R_0^\pm(h), R_0^\Theta(h), R_0^\Theta(h)) \right. \\ & \quad + \sum_{n=0}^{N-1} \sum_{p \in \mathbb{N}_{N-n}^2} \mathbf{L}_n^{\pm,2}(R_{p_1}^\pm(h), R_{p_2}^\Theta(h)) + \mathbf{R}_N^{\pm,2}(h; R_0^\pm(h), R_0^\Theta(h)) \\ & \quad \left. + \sum_{n=0}^{N-1} \left( \mathbf{L}_n^{\pm,1} - \check{\lambda}_n \right) R_{N-n}^\pm(h) + \mathbf{R}_N^{\pm,1}(h; R_0^\pm(h)) - R_N^\Lambda(h) R_0^\pm(h) \right] \end{aligned} \quad (4.22)$$

where we used the relations in [Eq. \(4.13\)](#), giving us that for all  $Q \in \mathbb{N}$

$$\sum_{p \in \mathbb{N}_Q^4} \mathbf{L}_{p_1}^{3,\pm}(\varphi_{p_2}^\pm, \theta_{p_3}, \theta_{p_4}) + \sum_{p \in \mathbb{N}_Q^3} \mathbf{L}_{p_1}^{2,\pm}(\varphi_{p_2}^\pm, \theta_{p_3}) + \sum_{p \in \mathbb{N}_Q^2} \left( \mathbf{L}_{p_1}^{1,\pm} - \check{\lambda}_{p_1} \right) (\varphi_{p_2}^\pm) = 0.$$

The coefficients in the operator  $\chi_h \mathcal{L}_h[\varepsilon](\cdot, \Theta_h)$  are smooth bounded functions in  $\mathbb{T}_L \times \mathbb{R}_\pm$  (see [Eq. \(4.8a\)](#), [Eq. \(4.8b\)](#), [Eq. \(4.11\)](#)). From [Eq. \(4.22\)](#), we get  $\mathcal{N}_\pm \leq h^N \|F^\pm(h)\|_{L_\pm[h]}$  where  $F^\pm \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$  so we have  $\mathcal{N}_\pm \leq C_N h^N$  for  $C_N$  a constant independent of  $h$  as  $h \rightarrow 0$ . Now, we consider the two commutator norms  $\mathcal{N}'_\pm$ . We observe that the coefficients of the operators  $[\mathcal{L}_h[\varepsilon](\cdot, \Theta_h), \chi_h]$  are zero in  $\mathbb{T}_L \times (-\frac{\delta}{2h}, 0)$  and  $\mathbb{T}_L \times (0, \frac{\delta}{2h})$ . From this observation, we deduce that

$$\mathcal{N}'_\pm{}^2 = \int_{\mathbb{T}_L} \int_{I_\pm(h)} |G^\pm(h; s, \sigma)|^2 d\sigma ds$$

where  $G^\pm \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$  and  $I_\pm(h)$  are as in [Lemma B.1](#) for  $\rho = \frac{\delta}{2}$ . We deduce that  $\mathcal{N}'_\pm = \mathcal{O}(h^\infty)$ , and we get  $\|\underline{R}_m\|_{L^2(\Omega)} + \|\underline{R}_m\|_{L^2(\mathbb{R}^2 \setminus \bar{\Omega})} = \mathcal{O}(h^{N-2})$  for all  $N > 1$ .

(v) Let  $(\theta_n)_{n \geq 0}$  (resp.  $(\vartheta_n)_{n \geq 0}$ ) be a sequence of phases constructed for  $\underline{u}_m$  (resp.  $\underline{v}_m$ ) and  $\alpha$  (resp.  $\beta$ ) the function in [Lemma 4.6](#). A similar computation as in (iii) gives that  $\int_{\mathbb{R}^2} \underline{u}_m \overline{\underline{v}_m} dx = z_0 h + \mathcal{O}(h^2)$  where

$$\begin{aligned} z_0 &= \sum_{\pm} \int_{\mathbb{T}_L} \alpha(s) \overline{\beta(s)} e^{i\theta_1(s) - i\vartheta_1(s)} \int_{\mathbb{R}_\pm} e^{\mp 2\sigma \hat{\tau}_0(s) \eta_0(s) \mp 1} d\sigma ds \\ &= \int_{\mathbb{T}_L} \alpha(s) \overline{\beta(s)} e^{i\theta_1(s) - i\vartheta_1(s)} \frac{\eta_0(s)^{-1} + \eta_0(s)}{2\hat{\tau}_0(s)} ds. \end{aligned}$$

Using [Lemma B.2](#), we get  $i\theta_1(s) - i\vartheta_1(s) = -2f(s) - \int_0^s \frac{\alpha'(t)}{\alpha(t)} + \frac{\overline{\beta'(t)}}{\overline{\beta(t)}} dt$  where  $f$  is a real function independent of  $\alpha$  and  $\beta$ . A derivative computation shows that the functions  $\alpha e^{-\int \frac{\alpha'}{\alpha}} \equiv \alpha_0 \in \mathbb{C}^*$ ,  $\beta e^{-\int \frac{\beta'}{\beta}} \equiv \beta_0 \in \mathbb{C}^*$  are constant so  $z_0 = \alpha_0 \overline{\beta_0} \int_{\mathbb{T}_L} \frac{\eta_0(s)^{-1} + \eta_0(s)}{2\hat{\tau}_0(s)} e^{-2f(s)} ds \neq 0$ . Denoting  $R$  (resp.  $S$ ) the remainder in the construction of  $\underline{u}_m$  (resp.  $\underline{v}_m$ ), we have

$$\int_{\mathbb{R}^2} \underline{u}_m \overline{\underline{v}_m} dx = \int_{\mathbb{T}_L} F(h; s) e^{i\frac{4\pi m}{L}\theta_0(s)} ds$$

where

$$F(h; s) = e^{iR_1^\Theta(h; s) + iS_1^\Theta(h; s)} \sum_{\pm} \int_{\mathbb{R}_\pm} \chi_u(h\sigma) \chi_v(h\sigma) R_0^\pm(h; s, \sigma) S_0^\pm(h; s, \sigma) h(1 + \sigma\kappa(s)h) d\sigma.$$

Note that  $F \in \mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L)$ . Since  $\theta'_0 = \widehat{\tau}_0 > 0$ ,  $\theta_0$  is a smooth diffeomorphism from  $\mathbb{T}_L$  to  $\mathbb{T}_L$ , we perform the change of variable  $x = \theta_0(s)$

$$\int_{\mathbb{T}_L} F(h; s) e^{i\frac{4\pi m}{L}\theta_0(s)} ds = \int_{\mathbb{T}_L} (\theta_0^{-1})'(x) F(h; \theta_0^{-1}(x)) e^{i\frac{4\pi}{L}mx} dx.$$

From the fact that the function  $(h; x) \mapsto (\theta_0^{-1})'(x) F(h; \theta_0^{-1}(x)) \in \mathcal{C}^\infty([0, \frac{2\pi}{L}] \times \mathbb{T}_L)$  and the Riemann–Lebesgue lemma, we get

$$\int_{\mathbb{T}_L} (\theta_0^{-1})'(x) F(h; \theta_0^{-1}(x)) e^{i\frac{4\pi}{L}mx} dx = \mathcal{O}(m^{-\infty}).$$

□

To prove [Theorem 4.2](#), one just needs to establish that  $\underline{u}_m$  satisfies the first condition in [Definition 4.1](#).

*Proof of Theorem 4.2.* Consider  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$  in [Eq. \(4.18\)](#), satisfying [Lemma 4.12](#). We now define

$$\check{\underline{u}}_m(s, \xi) = \chi(\xi) \begin{cases} 0 & \text{if } \xi \leq 0 \\ [\underline{u}_m]_{\mathbb{T}_L \times \{0\}}(s) + \xi [\varepsilon^{-1} \partial_\xi \underline{u}_m]_{\mathbb{T}_L \times \{0\}}(s) & \text{if } \xi > 0 \end{cases}.$$

Using [Lemma 4.12](#), we have  $\|\check{\underline{u}}_m\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(m^{-\infty})$  therefore  $\underline{u}_m - \check{\underline{u}}_m \in D(P)$  and  $(P - \underline{\lambda}_m)(\underline{u}_m - \check{\underline{u}}_m) = \mathcal{O}(m^{-\infty})$ . We then replace  $\underline{u}_m$  by  $\underline{u}_m = \frac{\underline{u}_m - \check{\underline{u}}_m}{\|\underline{u}_m - \check{\underline{u}}_m\|_{L^2(\mathbb{R}^2)}}$  which now makes  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$  a quasi-pair in the sense of [Definition 4.1](#). □

We finally show uniqueness of the quasi-resonances in the following sense:

**Lemma 4.14.** *Let  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$  and  $(\underline{\mu}_m, \underline{v}_m)_{m \geq 1}$  two quasi-pairs in the sense of [Definition 4.1](#) corresponding to the same integer  $m$  and having the same leading order phase  $\theta_0 : s \mapsto \int_0^s \widehat{\tau}_0(t) dt$ . Then we have the following estimate*

$$\underline{\lambda}_m - \overline{\underline{\mu}_m} = \mathcal{O}(m^{-\infty}).$$

*Proof.* Let  $\underline{R}_m, \underline{S}_m$  be the residuals  $\underline{R}_m = P\underline{u}_m - \underline{\lambda}_m \underline{u}_m$ ,  $\underline{S}_m = P\underline{v}_m - \underline{\mu}_m \underline{v}_m$ . By definition, the residuals satisfy  $\|\underline{R}_m\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(m^{-\infty})$  and  $\|\underline{S}_m\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(m^{-\infty})$ . Using the symmetry of the operator  $P$ , we get

$$(\underline{\lambda}_m - \overline{\underline{\mu}_m}) \int_{\mathbb{R}^2} \underline{u}_m \overline{\underline{v}_m} dx = \int_{\mathbb{R}^2} \underline{u}_m \overline{\underline{S}_m} dx - \int_{\mathbb{R}^2} \underline{R}_m \overline{\underline{v}_m} dx = \mathcal{O}(m^{-\infty}).$$

From [Remark 4.13](#) one can show that there exists  $z_0 \in \mathbb{C}^*$  such that  $\int_{\mathbb{R}^2} \underline{u}_m \overline{\underline{v}_m} dx = z_0 + \mathcal{O}(m^{-1})$ . Then  $\underline{\lambda}_m - \overline{\underline{\mu}_m} = \mathcal{O}(m^{-\infty})$  as  $m \rightarrow +\infty$ . □

**Corollary 4.15.** *The quasi-resonances  $(\underline{\lambda}_m)_{m \geq 1}$  are real, and unique in the sense that if  $(\underline{\lambda}_m)_{m \geq 1}$  and  $(\underline{\mu}_m)_{m \geq 1}$  are two quasi-resonances,  $\underline{\lambda}_m - \underline{\mu}_m = \mathcal{O}(m^{-\infty})$ .*

*Proof.* By applying [Lemma 4.14](#) to  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$  and  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$ , we get  $\Im \underline{\lambda}_m = \mathcal{O}(m^{-\infty})$  which implies that  $\Im \check{\lambda}_n = 0$  for all  $n \in \mathbb{N}$ . Then taking  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$  and  $(\underline{\mu}_m, \underline{v}_m)_{m \geq 1}$  two quasi-pairs in the sense of [Definition 4.1](#) having the same leading phase  $\theta_0 : s \mapsto \int_0^s \widehat{\tau}_0(t) dt$ , [Lemma 4.14](#) and the fact that the quasi-resonances are real gives us  $\underline{\lambda}_m - \underline{\mu}_m = \mathcal{O}(m^{-\infty})$ .  $\square$

*Remark 4.16.* With [Corollary 4.15](#), given a quasi-pair  $(\underline{\lambda}_m, \underline{u}_m)_{m \geq 1}$ , we have a second quasi-orthogonal quasi-pair  $(\underline{\lambda}_m, \overline{\underline{u}_m})_{m \geq 1}$  with the same quasi-resonance in the sense that, from (v) in [Lemma 4.12](#),

$$\int_{\mathbb{R}^2} \underline{u}_m \overline{\underline{u}_m} dx = \mathcal{O}(m^{-\infty}).$$

The quasi-resonances have an asymptotic multiplicity of 2, related to the chosen sign of the leading phase  $\theta_0$  (see [Remark 4.8](#)).

## 5. BACK TO THE SCATTERING PROBLEM

Now that we have constructed quasi-pairs characterizing surface plasmons, let's investigate their effect on scattering.

**5.1. Instabilities and well-posedness.** Let  $\underline{\ell}_m := \sqrt{\underline{\lambda}_m} \in \mathbb{C}^{\frac{1}{2}}$  for  $m \geq 1$ . The following theorem explains the instabilities observed in [Section 3.1](#).

**Theorem 5.1.** *If  $\varepsilon_c|_{\Gamma} < -1$ , there exists  $m_0$  such that for all  $m \geq m_0$ , we have  $\underline{\ell}_m > 0$ . Then along the quasi-resonances  $(\underline{\ell}_m)_{m \geq m_0}$ , constructed in [Section 4](#), the stability constant  $C(\underline{\ell}_m)$  of Problem (2.1) (see [Eq. \(A.2\)](#)) explodes super-algebraically, in the sense that for all  $N \geq 0$ , there exists  $c_N > 0$  such that*

$$C(\underline{\ell}_m) \geq c_N m^N, \quad \forall m \geq 1.$$

*Remark 5.2.* From the definition of the function  $N_{\varepsilon, \rho}$  and stability constant  $C(k)$ , we have the estimate  $N_{\varepsilon, \rho}(k) \leq C(k) \leq \sup_{u^{\text{in}} \in \mathcal{F}_{\rho}^{\text{in}} \setminus \{0\}} N_{\varepsilon, \rho}(k)$ , where  $\mathcal{F}_{\rho}^{\text{in}} = \{v \in H^1(\mathbb{D}(0, \rho)) \mid -(\Delta + k^2)v = 0\}$ .

*Proof.* From [Theorem 4.2](#), we have  $-\operatorname{div}(\varepsilon^{-1} \nabla \underline{u}_m) - \underline{\ell}_m^2 \underline{u}_m = r_m$  with the remainder estimate  $\|r_m\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(m^{-\infty})$ . [Lemma A.1](#), with  $u^{\text{in}} = 0$  and  $f = r_m$ , gives us

$$\|\underline{u}_m\|_{L^2(\mathbb{R}^2)} \leq C(\underline{\ell}_m) \|r_m\|_{L^2(\mathbb{R}^2)}.$$

Since  $\|\underline{u}_m\|_{L^2(\mathbb{R}^2)} = 1$  by definition and for all  $N \geq 1$ , there exists  $\tilde{c}_N > 0$  such that  $\|r_m\|_{L^2(\mathbb{R}^2)} \leq \tilde{c}_N m^{-N}$  then  $\tilde{c}_N^{-1} m^N \leq C(\underline{\ell}_m)$ , for all  $m \geq 1$ .  $\square$

We have now a systematic way to characterize asymptotically where are instabilities created by surface plasmons. In practice we compute the first order terms of the quasi-resonances and provide intervals where those instabilities arise. In what follows we provide several numerical examples using Finite Element Method (FEM) to illustrate this result.

**5.2. Numerical results.** Using results from previous sections, we compute the first three terms of the quasi-resonances expansions, and compute related *plasmonic* intervals. From the expansion  $\underline{\lambda}_m = \left(\frac{2\pi m}{L}\right)^2 \sum_{n=0}^2 \check{\lambda}_n \left(\frac{L}{2\pi m}\right)^n$ , we deduce an expansion  $\underline{\ell}_m := \sqrt{\underline{\lambda}_m} = \frac{2\pi m}{L} \sum_{n=0}^2 \check{\ell}_n \left(\frac{L}{2\pi m}\right)^n$ . In the case  $\varepsilon_c < -1$ , all the coefficients  $\check{\ell}_n$  are real and from the expansion of  $\underline{\ell}_m$ , we denote the interval  $I_m = [a_m, b_m]$  centered at  $\underline{\ell}_m$  with

$$a_m = \frac{2\pi m}{L} \check{\ell}_0 + \check{\ell}_1 + \min\left(0, 2\check{\ell}_2\right) \frac{L}{2\pi m} \quad \text{and} \quad b_m = \frac{2\pi m}{L} \check{\ell}_0 + \check{\ell}_1 + \max\left(0, 2\check{\ell}_2\right) \frac{L}{2\pi m}.$$

We consider  $\mathbb{D}(0, 1.5)$  as computational domain and various cavities  $\Omega \subset \mathbb{D}(0, 1.5)$ . We compute the function  $k \mapsto N_{\varepsilon, 1.5}$  from Problem (2.1), for various  $\varepsilon_c$ , with  $u^{\text{in}}(x, y) = e^{iky}$  using FEM, and we check if instabilities can be captured in the intervals  $(I_m)_{m \geq 1}$ . The outgoing condition is imposed using a Dirichlet-to-Neumann map (DtN) [34] that considers 65 Fourier modes. All FEM computations are done with XLiFE++ [40], we use finite elements of order 7 on quadrangular structured meshes of order 3 constructed with GMSH [24], with embedded tubular neighborhood as defined in Section 4.2.1 (see Fig. 7 for some examples). Note that optimal FEM convergence is guaranteed as long as the mesh is locally symmetric along the interface  $\Gamma$  [9]. We consider three different cases summarized in Fig. 7:

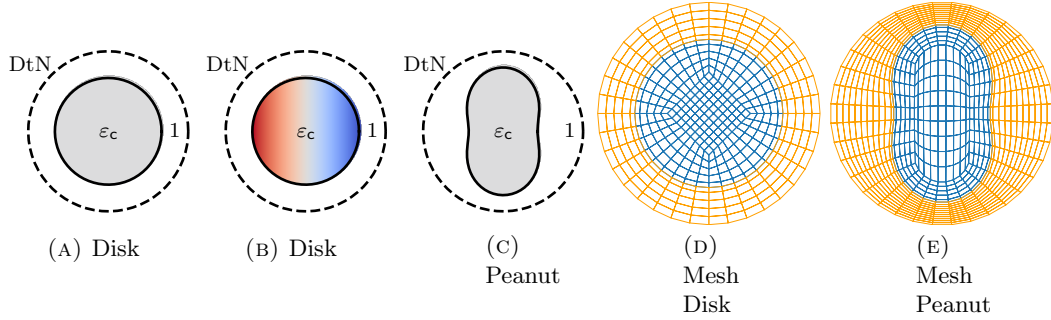


FIGURE 7. Sketch representing the three considered configurations (a), (b), (c), for the numerical examples, and associated structured meshes: circular cavity (d), peanut shape cavity (e).

Case (a). Circular cavity of radius 1 with constant  $\varepsilon_c$  as represented in Fig. 7a with associated mesh in Fig. 7d. For the numerical examples we consider  $\varepsilon_c = -1.1$ , a grid of 239 points for  $k$ , and 21736 degree of freedoms (dofs) for the FEM computations.

Case (b). Circular cavity of radius 1 with linearly varying permittivity  $\varepsilon_c^{\varepsilon_m, \varepsilon_M} : (x, y) \mapsto \frac{\varepsilon_m + \varepsilon_M}{2} + \frac{\varepsilon_M - \varepsilon_m}{2}x$  as represented in Fig. 7b with associated mesh in Fig. 7d. We consider  $(\varepsilon_m, \varepsilon_M) = (-1.2, -1.1)$ , a grid of 216 points for  $k$ , and 21736 dofs for the FEM computations.

Case (c). Peanut cavity with constant  $\varepsilon_c$  as represented in Fig. 7c with associated mesh in Fig. 7e. The peanut boundary is parameterized by  $r(\theta) = (1 - \frac{3}{10} \cos(2\theta))/L$ ,  $\theta \in [0, 2\pi]$ , with  $L$  such that  $\Gamma$  as length  $2\pi$ . For the numerical examples we consider  $\varepsilon_c = -1.1$ , a grid of 237 points for  $k$ , and 27028 dofs for the FEM computations.

Figure 8 represents the function  $k \mapsto N_{\varepsilon, 1.5}(k)$  for the cases (a), (b), and (c) from left to right. The dotted orange lines correspond to FEM computations, the solid blue lines correspond to the analytic computation done in Section 3.1 (valid for case (a)). The purple zones correspond to the intervals  $(I_m)_{m \geq 1}$  and the orange ‘x’ correspond to local maxima. We observe:

- Figures 8a, 8b, and 8c present, as predicted by Theorem 5.1, instabilities manifested by sharp peaks where the scattering field is big.
- The observed peaks lie within the intervals  $I_m$ , for  $m$  sufficiently large (due to the asymptotic nature of the estimates). In other words, the intervals  $I_m$  are good estimates for spotting instabilities due to surface plasmons, for  $m$  large enough.
- While FEM captures instabilities, it fails to capture the peak’s intensities: in particular in Fig. 8a FEM doesn’t match analytic results, and in Fig. 8c FEM captures



sometimes local maxima so small it is not noticeable. In other words, while one can identify where surface plasmons arise using the intervals  $I_m$ , in practice FEM is unable to accurately capture them.

The provided asymptotic method allows us to identify regions where surface plasmons arise (responsible for large stability constants  $C(k)$ ) for various metamaterial cavities, and FEM computations confirm expected instabilities. The constructed quasi-resonances provide guidance in order to avoid those instabilities. In practice, FEM is unable to accurately compute the emerging surface plasmons, extracting the asymptotic characterization (quasi-modes) could help the numerical method in that case.

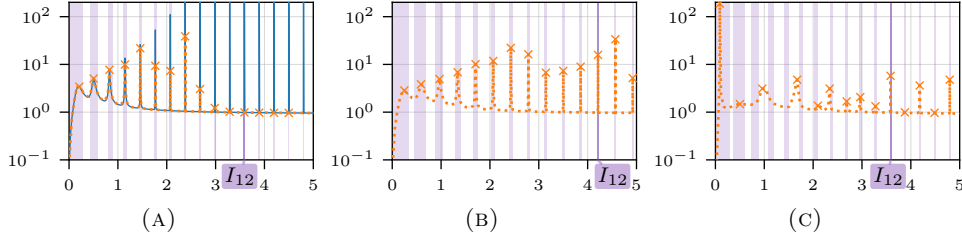


FIGURE 8. Plot of the function  $k \mapsto N_{\varepsilon,1.5}$  for the three cases (a), (b), and (c). The dotted orange lines correspond to FEM computations, the solid blue lines correspond to the analytic computation done in Section 3.1 (valid for case (a)). The purple zones correspond to the intervals  $I_m$  for  $1 \leq m \leq 12$  and we have highlighted  $I_{12}$ . The 'x' correspond to local maxima. We consider a uniform grid in  $k$  with geometric refinements in the intervals  $I_m$  centered at  $\ell_m$ .

## 6. CONCLUSIONS

Similar to classical optical cavities, the scattering by metamaterial cavities can be significantly affected by localized waves. Those localized waves correspond here to emerging surface plasmons, which we have asymptotically characterized for various cavity properties (shape, permittivity, etc.). This study has been carried out without reducing to the quasi-static case, and the considered spectral parameter is the wavenumber by opposition to [25, 39, 1, 2]. Our asymptotic analysis revealed that the spectral nature of surface plasmons depends on whether  $\varepsilon_c < -1$ , or not: they are associated to negative eigenvalues when  $-1 < \varepsilon_c < 0$  which can't be excited in practice, while they are associated to complex resonances when  $\varepsilon_c > -1$ . Those resonances affect the stability constant and are responsible for numerical instabilities (large scattered field). FEM computations confirm that the asymptotic analysis provides good estimates where surface plasmons emerge, however FEM fails to accurately capture them. Similar to the singular complement method [20], one could consider extracting the asymptotic plasmonic behavior from the problem. Our asymptotic characterization of emerging surface plasmons can be applied to dispersive materials, namely while considering permittivity models depending on the wave number  $k$ , such as Drude's or Lorentz' model. In that context, our analysis confirms that surface plasmons can be excited only for frequencies lower than the surface plasmon frequency [31]. Extensions of the approach to polygonal metamaterial cavities and dispersive materials will be considered. In the quasi-static case, the spectral analysis for this case reveals hypersingular plasmonic behaviors and has been well investigated [26, 13]. The proposed approach is valid for arbitrary negative permittivity  $\varepsilon_c$ , one could also consider arbitrary negative permeability  $\mu_c$  and work with the double-negative PDE  $-\operatorname{div}(\varepsilon^{-1} \nabla u) - \mu k^2 u = 0$



(e.g. [11, 23, 3]). In that case the associated spectral problem becomes difficult because the operator is no longer self-adjoint. All the derivation has been provided for two-dimensional problems, one could consider three dimensional cavities. We have established that the existence of quasi-pairs implies the explosion of the stability constant when  $\varepsilon_c < -1$ . Ongoing work focuses on proving that those quasi-pairs imply the existence of resonances close to the real axis via the *Black Box Scattering* [22].

## APPENDIX A. WELL-POSEDNESS OF THE SCATTERING PROBLEM

Let  $\mathbb{D}(0, \rho)$  be a disk a radius  $\rho$  such that  $\Omega \Subset \mathbb{D}(0, \rho)$  and  $f \in L^2(\mathbb{D}(0, \rho))$ . Following [8], we use a Dirichlet-to-Neumann map, denoted  $\mathcal{S}$ , to rewrite a generalized version of Problem Eq. (2.1) in  $\mathbb{D}(0, \rho)$  (with source  $f$ ): Find  $u^{\text{sc}} \in H^1(\mathbb{D}(0, \rho))$  such that  $u = u^{\text{in}} + u^{\text{sc}}$  and

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1} \nabla u) - k^2 u = f & \text{in } \mathbb{D}(0, \rho) \\ [u]_{\Gamma} = 0 \quad \text{and} \quad [\varepsilon^{-1} \partial_n u]_{\Gamma} = 0 & \text{across } \Gamma \\ \partial_r u - \mathcal{S}u = \partial_r u^{\text{in}} - \mathcal{S}u^{\text{in}} =: g^{\text{in}} & \text{across } \partial\mathbb{D}(0, \rho) \end{cases} \quad (\text{A.1})$$

Problem Eq. (A.1) with  $f \equiv 0$  is equivalent to Problem Eq. (2.1). The above general version will be useful in Section 4.

**Lemma A.1.** *Problem (A.1) is well-posed if and only if  $\varepsilon_c|_{\Gamma} \neq -1$ . Moreover there exists a stability constant  $C(k) > 0$  such that*

$$\|u^{\text{sc}}\|_{L^2(\mathbb{D}(0, \rho))} \leq C(k) \left( \|u^{\text{in}}\|_{L^2(\mathbb{D}(0, \rho))} + \|f\|_{L^2(\mathbb{D}(0, \rho))} \right). \quad (\text{A.2})$$

*Proof.* Consider  $\varepsilon_c < 0$  constant. It has already been established that Problems Eq. (2.1)-Eq. (A.1) is well-posed (in Hadamard's sense) if and only if  $\varepsilon_c \neq -1$  [8, Section 2]. The proof relies on T-coercivity arguments [10, 8, 9]. Consider now  $\varepsilon_c \in \mathcal{C}^\infty(\Omega)$  non constant. Lemma 1 in [8] establishes that problem Eq. (A.1) admits at most one solution in  $H_{\text{loc}}^1(\mathbb{R}^2)$ . One simply needs to establish that the operator  $u \mapsto -\operatorname{div}(\varepsilon^{-1} \nabla u) - k^2 u$  is Fredholm if and only if  $\varepsilon_c|_{\Gamma} \neq -1$  to conclude. Since  $\partial\Omega$  is a smooth interface, it can always be seen as locally straight with  $\varepsilon_c$  locally constant, then Theorems 4.3 and 6.2 in [10] apply and provide the needed results. Moreover, well-posedness gives us that there exists  $\tilde{C}(k) > 0$  such that

$$\|u\|_{H^1(\mathbb{D}(0, \rho))} \leq \tilde{C}(k) \left( \|g^{\text{in}}\|_{L^2(\mathbb{D}(0, \rho))} + \|f\|_{L^2(\mathbb{D}(0, \rho))} \right),$$

which leads to

$$\|u^{\text{sc}}\|_{L^2(\mathbb{D}(0, \rho))} \leq C(k) \left( \|u^{\text{in}}\|_{L^2(\mathbb{D}(0, \rho))} + \|f\|_{L^2(\mathbb{D}(0, \rho))} \right) \quad (\text{A.3})$$

using properties of the Dirichlet-to-Neumann map and Poincaré's inequality.  $\square$

*Remark A.2.* Equation (A.3) can also be rewritten as  $N_{\varepsilon, \rho} = \frac{\|u^{\text{sc}}\|_{L_p^2}}{\|u^{\text{in}}\|_{L_p^2}} \leq C(k)$ . In Section 3.1 we compute an approximation of  $N_{\varepsilon, \rho}$ . Computed results are then directly related to the well-posedness and the stability of the problem.

## APPENDIX B. PROOFS AND ADDITIONAL RESULTS FOR THE ASYMPTOTIC EXPANSIONS

### B.1. Proof of Lemma 4.6.

*Proof.* We solve Equation (4.15) as ordinary differential equations with  $s \in \mathbb{T}_L$  as a parameter. The conditions  $\varphi_0^\pm(s, \cdot) \in \mathcal{S}(\mathbb{R}_\pm)$  give the following restrictions  $\theta'_0(s)^2 + \eta_0(s)^2 \check{\lambda}_0 \in \mathbb{C} \setminus \mathbb{R}_-$  and  $\theta'_0(s)^2 - \check{\lambda}_0 \in \mathbb{C} \setminus \mathbb{R}_-$ . If one of the above restrictions is false, then there are no solutions  $\varphi^\pm(s, \cdot)$  in  $\mathcal{S}(\mathbb{R}_\pm)$ . Under those restrictions, there exists  $\alpha(s), \beta(s) \in \mathbb{R}$  such that  $\alpha(s)\beta(s) \neq 0$  and

$$\begin{aligned}\varphi_0^-(s, \sigma) &= \alpha(s) \exp \left( \sigma \sqrt{\theta'_0(s)^2 + \eta_0(s)^2 \check{\lambda}_0} \right), \\ \varphi_0^+(s, \sigma) &= \beta(s) \exp \left( -\sigma \sqrt{\theta'_0(s)^2 - \check{\lambda}_0} \right),\end{aligned}$$

where the square roots are chosen to be in  $\mathbb{C}^{\frac{1}{2}}$ . The first transmission condition  $\varphi_0^-(s, 0) = \varphi_0^+(s, 0)$  implies that  $\alpha(s) = \beta(s) \neq 0$ . Then the second transmission condition  $-\eta_0(s)^{-2} \partial_\sigma \varphi_0^-(s, 0) = \partial_\sigma \varphi_0^+(s, 0)$  gives us

$$-\eta_0(s)^{-2} \sqrt{\theta'_0(s)^2 + \eta_0(s)^2 \check{\lambda}_0} = -\sqrt{\theta'_0(s)^2 - \check{\lambda}_0},$$

leading to the eikonal equation

$$\theta'_0(s)^2 = \frac{\check{\lambda}_0}{1 - \eta_0(s)^{-2}}. \quad (\text{B.1})$$

The right-hand side needs to be positive, leading to

$$\theta_0(s) = \int_0^s \sqrt{\check{\lambda}_0 (1 - \eta_0(t)^{-2})^{-1}} dt$$

and from the condition  $\exp(ih^{-1}\theta_0) \in \mathcal{C}^\infty(\mathbb{T}_L)$ , we deduce that  $\exp(ih^{-1}\theta_0(L)) = \exp(ih^{-1}\theta_0(0))$  which implies that there exists  $m \in \mathbb{N}$  such that

$$2\pi m = h^{-1} (\theta_0(L) - \theta_0(0)) = h^{-1} \int_0^L \sqrt{\check{\lambda}_0 (1 - \eta_0(s)^{-2})^{-1}} ds.$$

By choosing  $h = \frac{L}{2\pi m}$  for  $m \in \mathbb{N}^*$ , we get

$$\left\langle \sqrt{\check{\lambda}_0 (1 - \eta_0^{-2})^{-1}} \right\rangle = \left\langle \sqrt{\check{\lambda}_0 \varsigma |1 - \eta_0^{-2}|^{-1}} \right\rangle = 1$$

which gives  $\check{\lambda}_0 = \varsigma \langle \tau_0 \rangle^{-2}$ . Then with the relation  $\tau_0^2 = \varsigma (1 - \eta_0^{-2})^{-1}$  we obtain that

$$\sqrt{\theta'_0(s)^2 + \eta_0(s)^2 \check{\lambda}_0} = \widehat{\tau}_0(s) \eta_0(s) > 0 \quad \text{and} \quad \sqrt{\theta'_0(s)^2 - \check{\lambda}_0} = \widehat{\tau}_0(s) \eta_0(s)^{-1} > 0,$$

which concludes the proof.  $\square$

## B.2. Proof of Lemma 4.9.

*Proof.* For  $(s, \sigma) \in \mathbb{T}_L \times \mathbb{R}_\pm$ , we define  $\mathbf{e}^\pm(s, \sigma) = \exp(-|\sigma| \widehat{\tau}_0(s) \eta_0(s)^{\mp 1})$ . We proceed by induction on  $n$ . For  $n = 0$ , Lemma 4.6 gives  $(\varphi_0^\pm, \theta_0, \check{\lambda}_0)$  the solution of  $(\mathcal{P}_0)$  defined in Eq. (4.15). Let  $n \geq 1$ , from the definition of  $S_{n-1}^\pm$  in Eq. (4.17), there exists  $Q_{n-1}^\pm \in \mathcal{C}^\infty(\mathbb{T}_L, \mathbb{P})$  such that  $S_{n-1}^\pm = Q_{n-1}^\pm \mathbf{e}^\pm$  and using Lemma A.1 in [6], there exists  $\tilde{P}_n^\pm \in \mathcal{C}^\infty(\mathbb{T}_L, \mathbb{P})$  such that  $\tilde{\varphi}_n^\pm = \sigma \tilde{P}_n^\pm \mathbf{e}^\pm$ ,  $\partial_\sigma^2 \tilde{\varphi}_n^- - \widehat{\tau}_0^2 \eta_0^2 \tilde{\varphi}_n^- = \eta_0^2 S_{n-1}^-$ , and

$\partial_\sigma^2 \tilde{\varphi}_n^+ - \hat{\tau}_0^2 \eta_0^{-2} \tilde{\varphi}_0^+ = -S_{n-1}^+$ . For  $(s, \sigma) \in \mathbb{T}_L \times \mathbb{R}_\pm$ , we obtain

$$\begin{aligned}\varphi_n^-(s, \sigma) &= \alpha(s) \sigma \left( \frac{\eta_0(s) \check{\lambda}_n}{2\hat{\tau}_0(s)} + \frac{\theta'_n(s)}{\eta_0(s)} + \frac{\tilde{P}_n^-(s, \sigma)}{\alpha(s)} \right) \mathbf{e}^-(s, \sigma), \\ \varphi_n^+(s, \sigma) &= \alpha(s) \sigma \left( \frac{\eta_0(s) \check{\lambda}_n}{2\hat{\tau}_0(s)} - \eta_0(s) \theta'_n(s) + \frac{\tilde{P}_n^+(s, \sigma)}{\alpha(s)} \right) \mathbf{e}^+(s, \sigma).\end{aligned}$$

The first transmission condition  $\varphi_n^-(\cdot, 0) = \varphi_n^+(\cdot, 0)$  is satisfied because  $\varphi_n^\pm(\cdot, 0) = 0$ . Using the second transmission condition  $-\eta_0^{-2} \partial_\sigma \varphi_n^-(\cdot, 0) = \partial_\sigma \varphi_n^+(\cdot, 0)$  and the condition  $\exp(i h^{n-1} \theta_n) \in \mathcal{C}^\infty(\mathbb{T}_L)$ , we can choose

$$\theta_n(s) = \int_0^s \frac{\check{\lambda}_n}{2\hat{\tau}_0(t)(1 - \eta_0(t)^{-2})} + \frac{\eta_0(t) \tilde{P}_n^-(t, 0) + \eta_0(t)^3 \tilde{P}_n^+(t, 0)}{\alpha(t) (\eta_0(t)^4 - 1)} dt \quad (\text{B.2})$$

and with the relation  $\tau_0(t)^2 (1 - \eta_0^{-2}) = \varsigma$  this give us

$$\check{\lambda}_n = -\frac{2\varsigma}{\langle \tau_0 \rangle^2} \left\langle \frac{\eta_0 \tilde{P}_n^-(\cdot, 0) + \eta_0^3 \tilde{P}_n^+(\cdot, 0)}{\alpha (\eta_0^4 - 1)} \right\rangle. \quad (\text{B.3})$$

Setting  $P_n^\pm(s, \sigma) = \alpha(s) \sigma \left( \frac{\eta_0(s) \check{\lambda}_n}{2\hat{\tau}_0(s)} \mp \eta_0(s)^{\pm 1} \theta'_n(s) + \frac{\tilde{P}_n^\pm(s, \sigma)}{\alpha(s)} \right)$  finishes the proof.  $\square$

### B.3. Additional results for Schwartz functions.

**Lemma B.1.** *Consider  $F : (h; s, \sigma) \mapsto F(h; s, \sigma)$  in  $\mathcal{C}^\infty([0, \frac{L}{2\pi}] \times \mathbb{T}_L, \mathcal{S}(\mathbb{R}_\pm))$ ,  $\rho > 0$ , and the intervals  $I_-(h) = (-\infty, -\frac{\rho}{h})$  and  $I_+(h) = (\frac{\rho}{h}, +\infty)$ . Then*

$$\int_{\mathbb{T}_L} \int_{I_\pm(h)} |F(h; s, \sigma)|^2 d\sigma ds = \mathcal{O}(h^\infty) \quad \text{as } h \rightarrow 0.$$

*Proof.* Notice that, for any  $N \geq 1$ , there exists a constant  $C_N > 0$  such that  $|\sigma^N F(h; s, \sigma)| \leq C_N$  for all  $(h; s, \sigma) \in [0, \frac{L}{2\pi}] \times \mathbb{T}_L \times \mathbb{R}_\pm$ . Hence,

$$\int_{\mathbb{T}_L} \int_{I_\pm(h)} |F(h; s, \sigma)|^2 d\sigma ds \leq \frac{C_N L}{2N \rho^{2N-1}} h^{2N-1},$$

which finishes the proof.  $\square$

### B.4. Additional results used in Section 4.

**Lemma B.2.** *For  $s \in \mathbb{T}_L$ ,*

$$\begin{aligned}\theta_1(s) &= \int_0^s \frac{\check{\lambda}_1}{\check{\lambda}_0} \hat{\tau}_0(t) + \frac{(\eta_0(t)^2 - 1) \kappa(t)}{2 \eta_0(t)} + \frac{\eta_1(t)}{2 \eta_0(s)^2 (\eta_0(t)^2 - 1)} \\ &\quad + i \frac{(\eta_0(t)^4 + 3) \eta'_0(t)}{2 \eta_0(t) (\eta_0(t)^4 - 1)} + i \frac{\alpha'(t)}{\alpha(t)} dt.\end{aligned}$$

*Proof.* This follows from Eq. (4.16) and Eq. (4.17).  $\square$

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DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF CALIFORNIA, MERCED, 5200 NORTH LAKE ROAD, MERCED, CA 95343, USA

*E-mail address:* ccarvalho3@ucmerced.edu

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF CALIFORNIA, MERCED, 5200 NORTH LAKE ROAD, MERCED, CA 95343, USA

*E-mail address:* zmoitier@ucmerced.edu