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SEMI-CLASSICAL PROPAGATION OF SINGULARITIES FOR THE STOKES SYSTEM

CHENMIN SUN

ABSTRACT. We study the quasi-mode of Stokes system posed on a smooth bounded domain Ω with Dirichlet boundary condition. We prove that the semi-classical defect measure associated with a sequence of solutions concentrates on the bicharacteristics of Laplacian as a matrix-valued Radon measure. Moreover, we show that the support of the measure is invariant under the Melrose-Sjöstrand flow.

Key Words: Stokes system; Propagation of singularities; Semi-classical analysis.

AMS Classification: 35P20, 35S15, 35Q35.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. Consider the eigenvalue problem of the Stokes operator

$$\begin{cases} -\Delta u_k + \nabla P_k = \lambda_k^2 u_k, & \text{in } \Omega \\ \operatorname{div} u_k = 0, & \text{in } \Omega \\ u_k|_{\partial\Omega} = 0 \end{cases} \quad (1.1)$$

where $u_k \in (H^2(\Omega))^d \cap V$, $\|u_k\|_{L^2} = 1$, are \mathbb{R}^d -valued normalized eigenfunctions and

$$V = \{u \in (H_0^1(\Omega))^d : \operatorname{div} u = 0\}.$$

We collect several facts which are well-known in functional analysis:

- u_k forms a orthonormal basis of

$$H = \{u \in (L^2(\Omega))^d : \operatorname{div} u = 0, u \cdot \nu|_{\partial\Omega} = 0\}$$

The canonical projector $\Pi : (L^2(\Omega))^d \rightarrow H$ is called Leray projector.

- The pressure $P_k \in L^2(\Omega)/\mathbb{R}$ satisfies $\int_{\Omega} P_k = 0$.
- $\|\nabla u_k\|_{L^2}^2 = \lambda_k^2$, $\|u_k\|_{H^2} \leq C\lambda_k^2$, $\|\nabla P_k\|_{L^2} \leq C\lambda_k^2$, $\|P_k\|_{L^2} \leq C\lambda_k^2$.

We rephrase the system (1.1) by semi-classical reduction. Taking $h_k = \lambda_k^{-1}$ and $q_k = \lambda_k^{-1} P_k$, dropping the sub-index, we obtain the following h -dependent system

$$\begin{cases} -h^2 \Delta u - u + h \nabla q = 0, & \text{in } \Omega \\ h \operatorname{div} u = 0, & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

In this article, we will study the following generalization by adding a quasi-mode:

$$\begin{cases} -h^2 \Delta u - u + h \nabla q = f, & \text{in } \Omega \\ h \operatorname{div} u = 0, & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \quad (1.2)$$

with the following conditions:

$$\begin{aligned} \|u\|_{L^2} = 1, \|h\nabla u\|_{L^2} = O(1), \|h^2\nabla^2 u\|_{L^2} = O(1), \\ \|h\nabla q\|_{L^2} = O(1), f \in H, \|f\|_{L^2} = o(h). \end{aligned}$$

When h is small, the corresponding solution $u = u(h)$ can be interpreted as high-frequency quasi-mode as its mass, i.e., the L^2 norm, is essentially concentrated on the frequency scale h^{-1} .

Before stating the main result, it is worth mentioning the eigenvalue problem of the Laplace operator in semi-classical version:

$$\begin{cases} -h^2\Delta u - u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \quad (1.3)$$

One method to capture the high-frequency behavior of the solutions of (1.3) is to use semi-classical defect measure associated to a bounded sequence (u_k) of $L^2(\Omega)$ and to a sequence of positive scales h_k converging to zero. This measure is aimed to describe quantitatively the oscillations of (u_k) at the frequency scale h_k^{-1} . More precisely, for any bounded sequence (w_k) of $L^2(\mathbb{R}^d)$, there exists a subsequence of (w_k) and a non-negative Radon measure μ on $T^*\mathbb{R}^d$ such that for any $a(x, \xi) \in \mathcal{S}(\mathbb{R}^{2d})$,

$$\lim_{k \rightarrow \infty} (a(x, h_k D_x) w_k | w_k)_{L^2(\mathbb{R}^d)} = \langle \mu, a \rangle.$$

When Ω is a bounded domain, the precise definition of defect measure corresponding to the boundary value problem will be given later.

Let us mention that a counterpart of semi-classical defect measure, micro-local defect measure, was introduced by P. Gérard [6] and L. Tartar [16] independently. These objects are widely used in the study of control and stabilization, scattering theory and quantum ergodicity, see for example [3], [2], [7].

In the context of semi-classical defect measure, the classical theorem of Melrose-Sjöstrand about propagation of singularities ([12],[13]) for hyperbolic equation can be rephrased as follows:

Theorem 1.1 ([7]). *Assume that Ω is a smooth, bounded domain with no infinite order of contact on the boundary. Suppose μ is the semi-classical defect measure associated to the pair (u_k, h_k) where (u_k) is a sequence of solutions to (1.3) (with $h = h_k$) which are bounded in $L^2(\Omega)$. Then μ is invariant under the Melrose-Sjöstrand flow.*

We will give the precise definition of the Melrose-Sjöstrand flow and the associated concept of the order of contact in the second section. Intuitively, these flows are the generalization of geometric optics. No infinite order of contact means that the trajectory of the flow can not tangent to the boundary with an infinite order.

The main result of this paper is as follows.

Theorem 1.2. *Assume that Ω is a smooth, bounded domain with no infinite order of contact on the boundary. Suppose (u_k) is a sequence of solutions to the quasi-mode problem (1.2) with semi-classical parameters $h = h_k$. Assume that $f_k \in H$, $\|f_k\|_{L^2(\Omega)} = o(h_k)$ and u_k converges weakly to 0 in $L^2(\Omega)$. Assume that μ is a semi-classical measure associated to some subsequence of (u_k, h_k) , then $\text{supp}(\mu)$ is invariant under the Melrose-Sjöstrand flow.*

We make some comments about the result. Firstly, the measure μ is Hermitian matrix-valued, and we have no information so far on the precise propagation for μ except for $\text{supp}(\mu)$. Secondly, since the eigenfunctions of Stokes operator converge weakly to 0 in $L^2(\Omega)$, our results includes this special case.

The Propagation theorem for a given quasi-mode has many applications, in particular, it leads to the stabilization of the associated damped evolution system. In the context of the damped wave equation, it was shown that (see [15], [1], [10]) under the geometric control condition, the energy decays exponentially. An application of Theorem 1.2 is the stabilization of a hyperbolic Stokes system, a model in the theory of linear elasticity introduced in [11], under the geometric control condition. More precisely, consider the damped hyperbolic-Stokes system:

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p + a(x)\partial_t u = 0 & \text{in } \mathbb{R} \times \Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R} \times \Omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ (u(0, x), \partial_t u(0, x)) = (u_0, v_0) \in V \times H, \end{cases} \quad (1.4)$$

The energy

$$E[u](t) = \frac{1}{2} \int_{\Omega} (|\partial_t u|^2 + |\nabla u|^2) dx$$

is dissipative. In [5], we use propagation Theorem 1.2 to show that the energy decays exponentially in time.

Let us describe briefly our strategy for the proof of Theorem 1.2. The pressure term q is harmonic and in heuristic, it can only have the influence to the solution near the boundary. Hence we will prove that the measure μ_i is propagated in the same way as Laplace quasi-mode (semi-classical analogue of wave equation) along the rays inside the domain. When a ray reaches the boundary, we need a more careful analysis between the wave-like propagation phenomenon and the concentration phenomenon of the pressure. It is difficult to get a simple propagation formula near the boundary, comparing to the treatment of quasi-mode problem of Laplace operator as in [3],[7]. We partition the phase space into elliptic region \mathcal{E} , hyperbolic region \mathcal{H} and glancing surface \mathcal{G} . It turns out that no singularity accumulates near the elliptic region. For the hyperbolic region, we prove the propagation by the standard energy estimate, with an additional treatment when the incidence of the ray is right. Near the glancing surface, we will follow the arguments of Ivrii and Melrose-Sjöstrand. The main difference is that we will encounter two new cross terms essentially of the form $(q|u)_{L^2}$ after certain micro-localization. To overcome this difficulty, we further micro-localize the solution according to the distance to the glancing surface \mathcal{G} and treat them separately. For the part nearing \mathcal{G} , we use the fact that the pressure decays fast away from the boundary while the solution can not concentrate too much near the boundary, provided that it is micro-localized close enough to the glancing surface. For the part away from \mathcal{G} , it can be well-controlled by induction argument, using geometric properties of the generalized bicharacteristic flow.

2. PRELIMINARY

2.1. Notations. We will sometimes drop the sub-index k for a sequence of functions (u_k) and semi-classical parameters h_k . In this circumstance, the notion $\|u\|_X = O(1), o(1)$ as $h \rightarrow 0$ should be understood as $\|u_k\|_X = O(1), o(1)$ as $k \rightarrow \infty$ (thus $h_k \rightarrow 0$) up to certain subsequence.

As in the introduction, we follow the notation in the context of the analysis of Stokes system (see [17])

$$V = \{u \in H_0^1(\Omega)^N : \operatorname{div} u = 0\}$$

and

$$H = \{u \in L^2(\Omega)^N : \operatorname{div} u = 0, u \cdot \nu|_{\partial\Omega} = 0\}.$$

In this paper we always use ν to denote the outward normal vector on $\partial\Omega$.

For a manifold M , we let TM be its tangent bundle and T^*M be the cotangent bundle with canonical projection

$$\pi : TM(\text{ or } T^*M) \rightarrow M.$$

We will identify system (1.2) as a system on differential form

$$\begin{cases} h^2 \Delta_H u - u + hdq = f \text{ in } \Omega \\ hd^*u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \quad (2.1)$$

where the unknown $u \in \Lambda^1(\Omega)$ is 1-form, and

$$d : \Lambda^p(\Omega) \rightarrow \Lambda^{p+1}(\Omega), d^* : \Lambda^{p+1}(\Omega) \rightarrow \Lambda^p(\Omega)$$

are exterior differential and divergence operator on forms, with respectively. Recall also that the Hodge Laplace operator is defined by

$$\Delta_H = dd^* + d^*d = (d + d^*)^2.$$

In the tubular neighborhood of boundary, we can identify Ω locally as one side of the tubular neighborhood denoted by $Y_+ = [0, \epsilon_0) \times X$, $X = \{x' \in \mathbb{R}^{d-1} : |x'| < 1\}$. We denote by $\partial Y_+ = Y_+|_{y=0}$ and $Y_+^0 = Y_+|_{y>0}$. For $x \in \bar{\Omega}$, we note $x = (y, x')$, where $y \in [0, \epsilon_0)$, $x' \in X$, and $x \in \partial\Omega$ if and only if $x = (0, x')$. In this coordinate system, the Euclidean metric dx^2 can be written as matrices

$$\bar{g} = \begin{pmatrix} 1 & 0 \\ 0 & g(y, x') \end{pmatrix}, \bar{g}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & g^{-1}(y, x') \end{pmatrix},$$

with $|\xi'|_{g^{-1}(y, x')}^2 = \langle \xi', g^{-1}(y, x')\xi' \rangle_{\mathbb{R}^{d-1}} = g^{jk}\xi'_j\xi'_k$ be the induced metric on $T^*\partial\Omega$, parametrized by y . Note that $|\xi'|_{g^{-1}(0, x')}^2 = \langle \xi', g^{-1}(0, x')\xi' \rangle_{\mathbb{R}^{d-1}} = g^{jk}\xi'_j\xi'_k$ is the natural norm on $T^*\partial\Omega$, dual of the norm on $T\partial\Omega$, induced by the canonical metric on $\bar{\Omega}$. Write $(x, \xi) = (y, x', \eta, \xi')$ and denote by $|\xi|$ the Euclidean norm on $T^*\mathbb{R}^d$. For $u, v \in \Lambda^1(Y_+)$ with support in the local chart of turbulence neighborhood, we define the L^2 norms and inner product on $[0, \epsilon_0) \times X$ via

$$\begin{aligned} \|u\|_{L^2(Y_+)}^2 &:= \int_0^{\epsilon_0} \int_X \langle u|u \rangle \sqrt{\det(\bar{g})} dx' dy, \\ (u|v)_{L^2(Y_+)} &:= (u|v)_{Y_+} := \int_0^{\epsilon_0} \int_X \langle u|v \rangle \sqrt{\det(\bar{g})} dx' dy, \\ \|u(y, \cdot)\|_{L^2(\partial Y_+)}^2 &:= \int_X \langle u(y, \cdot)|u(y, \cdot) \rangle \sqrt{\det(g)} dx', \\ (u|v)_{L^2(\partial Y_+)}(y) &:= \int_X \langle u(y, \cdot)|v(y, \cdot) \rangle \sqrt{\det(g)} dx', \end{aligned}$$

where for $u = u_0 dy + u_j dx'^j$, $v = v_0 dy + v_j dx'^j$,

$$\langle u|v \rangle = u_0 \bar{v}_0 + u_j \bar{v}_k g^{jk}.$$

In certain situations we also use global notation for L^2 inner product:

$$(u|v)_\Omega := \int_\Omega u \cdot \bar{v} dx, \quad (f|g)_{\partial\Omega} := \int_{\partial\Omega} f \cdot \bar{g} d\sigma(x).$$

We will identify the unknown vector fields u, v , etc. with their dual 1-forms. Formulation of differential form will simplify some calculations. In the tubular neighborhood, we write a vector field

$$L = L_\perp \frac{\partial}{\partial y} + L_\parallel, \quad L_\parallel = \sum_{j=1}^{d-1} L_{\parallel,j} \frac{\partial}{\partial x'_j}$$

and we write $L = (L_\perp, L_\parallel)$. The normal component obeys the following convention: $(a, 0) = -a\nu$.

Following [4], we will write down system (1.2) in the tubular neighborhood. For $u = (u_\perp, u_\parallel)$, equation (1.2) can be written as:

$$\begin{cases} (-h^2 \Delta_\parallel - 1)u_\parallel + h \nabla_{x'} q = f_\parallel, \\ (-h^2 \Delta_g - 1)u_\perp + h \partial_y q = f_\perp, \\ h \operatorname{div}_\parallel u_\parallel + \frac{h}{\sqrt{\det g}} \partial_y (\sqrt{\det g} u_{\perp}) = 0 \end{cases} \quad (2.2)$$

where

$$\begin{aligned} h^2 \Delta_\parallel &= h^2 \partial_y^2 - \Lambda^2(y, x', hD_{x'}) + hM_\parallel(y, x', hD'_x) + hM_1(y, x')h\partial_y, \\ h^2 \Delta_g &= h^2 \partial_y^2 - \Lambda^2(y, x', hD_{x'}) + hM_\perp(y, x', hD'_x) + hN_1(y, x')h\partial_y, \\ h \operatorname{div}_\parallel u_\parallel &= \frac{h}{\sqrt{\det g}} \sum_{j=1}^{N-1} \partial_{x'_j} (\sqrt{\det g} u_{\parallel,j}). \end{aligned}$$

$h^2 \Lambda^2(y, x', hD_{x'})$ has the symbol $\lambda^2 = |\xi'|_{\alpha(y,\cdot)}^2$, and $M_{\parallel,\perp}$ are both first-order matrix-valued semi-classical differential operators.

2.2. Geometric Preliminaries. Denote by ${}^b T\bar{\Omega}$ the vector bundle whose sections are the vector fields $X(p)$ on $\bar{\Omega}$ with $X(p) \in T_p \partial\Omega$ if $p \in \partial\Omega$. Moreover, denote by ${}^b T^* \bar{\Omega}$ the Melrose's compressed cotangent bundle which is the dual bundle of ${}^b T\bar{\Omega}$. Let

$$j : T^* \bar{\Omega} \rightarrow {}^b T^* \bar{\Omega}$$

be the canonical map. In our geodesic coordinate system near $\partial\Omega$, ${}^b T\bar{\Omega}$ is generated by the vector fields $\frac{\partial}{\partial x'_1}, \dots, \frac{\partial}{\partial x'_{d-1}}, y \frac{\partial}{\partial y}$ and thus j is defined by

$$j(y, x'; \eta, \xi') = (y, x'; v = y\eta, \xi').$$

The principal symbol of operator $P_h = -(h^2 \Delta + 1)$ is

$$p(y, x', \eta, \xi') = \eta^2 + |\xi'|_{g^{-1}(y, x')}^2 - 1.$$

By $\operatorname{Car}(P)$ we denote the characteristic variety of p :

$$\operatorname{Car}(P) := \{(x, \xi) \in T^* \mathbb{R}^d|_{\bar{\Omega}} : p(x, \xi) = 0\}, \quad Z := j(\operatorname{Car}(P)).$$

By writing in another way

$$p = \eta^2 - r(y, x', \xi'), \quad r(y, x', \xi') = 1 - |\xi'|_{g^{-1}(y, x')}^2,$$

we have the decomposition

$$T^* \partial\Omega = \mathcal{E} \cup \mathcal{H} \cup \mathcal{G},$$

according to the value of $r_0 := r|_{y=0}$ where

$$\mathcal{E} = \{r_0 < 0\}, \mathcal{H} = \{r_0 > 0\}, \mathcal{G} = \{r_0 = 0\}.$$

The sets $\mathcal{E}, \mathcal{H}, \mathcal{G}$ are called elliptic, hyperbolic and glancing, with respectively.

For a symplectic manifold S with local coordinate (z, ζ) , a Hamiltonian vector field associated with a real function f is given by

$$H_f = \frac{\partial f}{\partial \zeta} \frac{\partial}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial}{\partial \zeta}.$$

Now for $(x, \xi) \in \Omega$ far away from the boundary, the Hamiltonian vector field associated to the characteristic function p is given by

$$H_p = 2\xi \frac{\partial}{\partial x}.$$

We call the trajectory of the flow

$$\phi_s : (x, \xi) \mapsto (x + s\xi, \xi)$$

bicharacteristic or simply ray, provided that the point $x + s\xi$ is still in the interior.

To classify different situations as a ray reaching the boundary, we need more accurate decomposition of the glancing set \mathcal{G} . Let $r_1 = \partial_y r|_{y=0}$ and define

$$\mathcal{G}^{k+3} = \{(x', \xi') : r_0(x', \xi') = 0, H_{r_0}^j(r_1) = 0, \forall j \leq k; H_{r_0}^{k+1}(r_1) \neq 0\}, k \geq 0$$

$$\mathcal{G}^{2,\pm} := \{(x', \xi') : r_0(x', \xi') = 0, \pm r_1(x', \xi') > 0\}, \mathcal{G}^2 := \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}.$$

Denote by $\mathcal{G}^\infty = \mathcal{G} \setminus (\cup_{j \geq 2} \mathcal{G}^j)$. We say that there is no infinite order of contact on the boundary if $\mathcal{G}^\infty = \emptyset$.

By setting

$$H_p^G := H_p + \frac{H_p^2 y}{H_y^2 p} H_y,$$

H_p^G is a well-defined vector field tangent to \mathcal{G} which is called the gliding vector field. Given a ray $\gamma(s)$ with $\pi(\gamma(0)) \in \Omega$ and $\pi(\gamma(s_0)) \in \partial\Omega$ be the first point that reaches the boundary. If $\gamma(s_0) \in \mathcal{H}$, then $\eta_\pm(\gamma(s_0)) = \pm \sqrt{r_0(\gamma(s_0))}$ are the two different roots of $\eta^2 = r_0$ at this point. Notice that the ray starting with direction η_- will leave Ω , while the ray with direction η_+ will enter the interior of Ω . This motivates the following definition of broken bicharacteristic:

Definition 2.1 ([8]). *A broken bicharacteristic arc of p is a map:*

$$s \in I \setminus B \mapsto \gamma(s) \in T^*\Omega \setminus \{0\},$$

where I is an interval on \mathbb{R} and B is a discrete subset, such that

- If J is an interval contained in $I \setminus B$, then $s \in J \mapsto \gamma(s)$ is a bicharacteristic of p over Ω .
- If $s \in B$, then the limits $\gamma(s^+)$ and $\gamma(s^-)$ exist and belongs to $T_x^* \bar{\Omega} \setminus \{0\}$ for some $x \in \partial\Omega$, and the projections in $T_x^* \partial\Omega \setminus \{0\}$ are the same hyperbolic point.

When a ray $\gamma(s)$ reaches a point $\rho_0 \in \mathcal{G}$, there are several situations. If $\rho_0 \in \mathcal{G}^{2,+}$, then the ray passes transversally over ρ_0 and enters $T^*\Omega$ immediately. If $\rho_0 \in \mathcal{G}^{2,-}$ or $\rho_0 \in \mathcal{G}^k$ for some $k \geq 3$, then we can continue it inside $T^*\partial\Omega$ as long as it can not leave the boundary along the trajectory of the Hamiltonian flow of H_{-r_0} . We now give the precise definition.

Definition 2.2 ([8]). *A generalized bicharacteristic ray of p is a map:*

$$s \in I \setminus B \mapsto \gamma(s) \in (T^*\bar{\Omega} \setminus T^*\partial\Omega) \cup \mathcal{G}$$

where I is an interval on \mathbb{R} and B is a discrete set of I such that $p \circ \gamma = 0$ and the following:

- (1) $\gamma(s)$ is differentiable and $\frac{d\gamma}{ds} = H_p(\gamma(s))$ if $\gamma(s) \in T^*\bar{\Omega} \setminus T^*\partial\Omega$ or $\gamma(s) \in \mathcal{G}^{2,+}$.
- (2) Every $s \in B$ is isolated, $\gamma(s) \in T^*\bar{\Omega} \setminus T^*\partial\Omega$ if $s \neq t$ and $|s - t|$ is small enough, the limits $\gamma(s^\pm)$ exist and are different points in the same fibre of $T^*\partial\Omega$.
- (3) $\gamma(s)$ is differentiable and $\frac{d\gamma}{ds} = H_p^G(\gamma(s))$ if $\gamma(s) \in \mathcal{G} \setminus \mathcal{G}^{2,+}$.

Remark 2.3. *The definition above does not depend on the choice of local coordinate, and in the geodesic coordinate system, the map*

$$s \mapsto (y(s), \eta^2(s), x'(s), \xi'(s))$$

is always continuous and

$$s \mapsto (x'(s), \xi'(s))$$

is always differentiable and satisfies the ordinary differential equations

$$\frac{dx'}{dt} = -\frac{\partial r}{\partial \xi'}, \quad \frac{d\xi'}{dt} = \frac{\partial r}{\partial x'}$$

the map $s \mapsto y(s)$ is left and right differentiable with derivative $2\eta(s^\pm)$ for any $s \in B$ (hyperbolic point).

Moreover, there is also the continuous dependence with the initial data, namely the map

$$(s, \rho) \mapsto (y(s, \rho), \eta^2(s, \rho), x'(s, \rho), \xi'(s, \rho))$$

is continuous. We denote the flow map by $\gamma(s, \rho)$.

Remark 2.4. *Under the map $j : T^*\bar{\Omega} \rightarrow {}^bT^*\bar{\Omega}$, one could regard $\gamma(s)$ as a continuous flow on the compressed cotangent bundle ${}^bT^*\bar{\Omega}$, and it is called the Melrose-Sjöstrand flow. We will also call each trajectory generalized bicharacteristic or simply ray in the sequel.*

From the classical result of Melrose-Sjöstrand, a generalized bicharacteristic that does not meet \mathcal{G}^∞ is uniquely defined (see Corollary 24.3.10 in [8]). Then, having $\mathcal{G}^\infty = \emptyset$, meaning $\mathcal{G}^j = \emptyset$ for some $j \geq 2$, implies the uniqueness of all generalized bicharacteristics and thus the existence of the Melrose-Sjöstrand flow. We refer to Example 24.3.11 in [8] where nonuniqueness occurs precisely in the case of a point in \mathcal{G}^∞ .

2.3. definition of defect measure. We follow closely as in [2] and one can find in [7] for a little different but comprehensive introduction.

We denote by S^m the usual symbol class. Define the partial symbol class S_∂^m and the class of boundary h -pseudo-differential operators \mathcal{A}_h^m as follows

$$S_\partial^m := \{a(y, x', \xi') : \sup_{\alpha, \beta, y \in [0, \epsilon_0]} |\partial_x^\alpha \partial_{\xi'}^\beta a(y, x', \xi')| \leq C_{m, \alpha, \beta} (1 + |\xi'|)^{m-\beta}\}.$$

$$\mathcal{A}_h^m := \text{Op}_h^{\text{comp}}(S^m) + \text{Op}_h(S_{\xi'}^m) := \mathcal{A}_{h, i}^m + \mathcal{A}_{h, \partial}^m.$$

Consider functions of the form $a = a_i + a_\partial$ with $a_i \in C_c^\infty(\Omega \times \mathbb{R}^d)$ which can be viewed as a symbol in S^0 , and $a_\partial \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$ can be viewed as a symbol in

$S_{\xi'}^0$. We quantize a as follows: Take $\varphi_i \in C_c^\infty(\Omega)$, equal to 1 near the x -projection of $\text{supp}(a_i)$ and $\varphi_\partial \in C_c^\infty(\mathbb{R}^d)$, equal to 1 near the x -projection of $\text{supp}(a_\partial)$. Define

$$\begin{aligned} \text{Op}_h^{\varphi_i, \varphi_\partial}(a)f(y, x') &= \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} e^{\frac{i(x-z)\xi}{h}} a_i(x, \xi) \varphi_i(z) f(z) dz d\xi \\ &\quad + \frac{1}{(2\pi h)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{\frac{i(x'-z')\xi'}{h}} a_\partial(y, x', \xi') \varphi_\partial(y, z') f(y, z') dz' d\xi'. \end{aligned}$$

According to the symbolic calculus, the operator $\text{Op}_h^{\varphi_i, \varphi_\partial}(a)$ does not depend on the choice of functions $\varphi_i, \varphi_\partial$, modulo operators of norms $O_{L_{\text{loc}}^2 \rightarrow L_{\text{comp}}^2}(h^\infty)$, and we will use the notion $\text{Op}_h(a)$ in the sequel. Notice that the acting of tangential operator $\text{Op}_h(a_\partial)$ can be viewed as pseudo-differential operator on the manifold $\partial\Omega$, parametrized by the parameter $y \in [0, \epsilon_0)$. The bounded family of operators $\mathcal{A}_{h, \partial}^m$ is defined uniquely up to a family of operators with norms uniformly dominated by Ch , as $h \rightarrow 0$. Moreover, for any family (A_h) , such that

$$\|A_h - \text{Op}_h(a_\partial)\|_{L^2 \rightarrow L^2} = O(h),$$

the principal symbol $\sigma(A)$ is determined uniquely as a function on $T^*\partial\Omega$, smoothly depending on y , i.e. $\sigma(A) \in C^\infty([0, \epsilon_0) \times T^*\partial\Omega)$.

When we deal with vector-valued functions, we could require the symbol a to be matrix-valued. Now for any sequence of vector-valued function w_k , uniformly bounded in $L^2(\Omega)$, there exists a subsequence (still use w_k for simplicity), and a nonnegative definite Hermitian matrix-valued Radon measure μ_i on $T^*\Omega$ such that

$$\lim_{k \rightarrow 0} (\text{Op}_{h_k}(a_i)w_k|w_k)_{L^2} = \langle \mu_i, a_i \rangle := \int_{T^*\Omega} \text{tr}(a_i d\mu_i).$$

For a proof, see for example [2] or the textbook [18], and the micro-local version was appeared in [6].

From now on the symbols and operators will be scalar-valued unless otherwise specified. Suppose u_k be a sequence of solutions to

$$\begin{cases} -h_k^2 \Delta u_k - u_k + h_k \nabla q_k = f_k, & (u_k, f_k) \in (H^2(\Omega) \cap V) \times H, \\ h_k \text{div} u_k = 0, & \text{in } \Omega \end{cases} \quad (2.3)$$

under the assumptions below:

$$\begin{aligned} \|u_k\|_{L^2(\Omega)} &= O(1), \quad f_k \in H \text{ and } \|f_k\|_{L^2(\Omega)} = o(h_k), \\ \|h \nabla q_k\|_{L^2(\Omega)} &= O(1), \quad \int_{\Omega} q_k dx = 0. \end{aligned} \quad (2.4)$$

Let μ_i be the interior measure associated with $(u_k)_k$. Then the following result shows that μ_i is supported on $\text{Car}(P)$.

Proposition 2.5. *Let $a_i \in C_c^\infty(\Omega \times \mathbb{R}^d)$ be equal to 0 near $\text{Car}(P)$, then we have*

$$\lim_{k \rightarrow \infty} (\text{Op}_{h_k}(a_i)u_k|u_k)_{L^2} = 0.$$

Proof. Note that the symbol $b(x, \xi) = \frac{a_i(x, \xi)}{|\xi|^2 - 1} \in S^0$ is well-defined from the assumption on a_i . From symbolic calculus, we have

$$\text{Op}_{h_k}(a_i) = B_{h_k}(-h_k^2 \Delta - 1) + O_{L^2 \rightarrow L^2}(h_k).$$

Therefore

$$\begin{aligned}
 (B_{h_k}(-h_k^2\Delta - 1)u_k|u_k)_{L^2} &= (B_{h_k}f_k|u_k)_{L^2} - (B_{h_k}h_k\nabla q_k|u_k)_{L^2} \\
 &= o(1) + ([h_k\nabla, B_{h_k}]q_k|u_k)_{L^2} - (h_k\nabla B_{h_k}q_k|u_k)_{L^2} \\
 &= o(1), \text{ as } k \rightarrow \infty,
 \end{aligned}$$

where in the last line we have used the symbolic calculus, integration by part, and Lemma 3.3. \blacksquare

Now we denote by $Z = j(\text{Car}(P))$. Proposition 2.5 indicates that the interior defect measure μ_i is supported on Z . To define the defect measure near the boundary, we have to check that if $a_\partial \in C_c^\infty(U \times \mathbb{R}^{d-1})$ vanishing near Z (i.e. a_∂ is supported in the elliptic region for all y small) then

$$\lim_{k \rightarrow \infty} (\text{Op}_{h_k}(a_\partial)u_k|u_k)_{L^2} = 0.$$

Indeed, this can be ensured by the analysis of the boundary value problem in the elliptic region, which will be given later. Now for any family of operators $A_h \in \mathcal{A}_h^0$, let $a = \sigma(A_h)$ be the principal symbol of A_h and we define $\kappa(a) \in C^0(Z)$ via $\kappa(a)(\rho) := a(j^{-1}(\rho))$. Note that Z is a locally compact metric space and the set

$$\{\kappa(a) : a = \sigma(A_h), A_h \in \mathcal{A}_h^0\}$$

is a locally dense subset of $C^0(Z)$. We then have the following proposition, which guarantees the existence of a Radon measure on Z :

Proposition 2.6. *There exists a subsequence of u_k, h_k and a nonnegative definite Hermitian matrix-valued Radon measure μ on Z , such that*

$$\lim_{k \rightarrow \infty} (A_{h_k}u_k|u_k)_{L^2} = \langle \mu, \kappa(a) \rangle, a = \sigma(A_h), \forall A_h \in \mathcal{A}_h^0.$$

The proof of this result can be found in [2], see also [3] and [6] for its micro-local counterpart. Notice that if we write $a = a_i + a_\partial$, then

$$(A_k u_k | u_k) \rightarrow \int_{T^*\Omega} \text{tr}(a_i(\rho) d\mu_i(\rho)) + \int_Z \text{tr}(a_\partial(\rho) d\mu(\rho)).$$

The following result shows that information about frequencies higher than the scale h_k^{-1} is not lost, and the measure μ contains the relevant information of the sequence (u_k) .

Proposition 2.7. *The sequence of solution (u_k) is h_k -oscillating in the following sense:*

$$\begin{aligned}
 \lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_{|\xi| \geq Rh_k^{-1}} |\widehat{\psi u_k}(\xi)|^2 d\xi &= 0, \forall \psi \in C_c^\infty(\Omega), \\
 \lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_0^{\epsilon_0} dy \int_{|\xi'| \geq Rh_k^{-1}} |\widehat{\psi u_k}(y, \xi')|^2 d\xi' &= 0, \forall \psi \in C_c^\infty(\bar{\Omega}),
 \end{aligned}$$

where in the second formula, the Fourier transform is only taken for the x' direction. Consequently, the total mass $\|u_k\|_{L^2(\Omega)}^2$ converges to $\langle \mu_i, T^*\Omega \rangle + \langle \mu, \mathbf{1}_Z \rangle$.

The proof will be given in appendix.

3. A PRIORI INFORMATION ABOUT THE SYSTEM

3.1. Information about the trace. We consider the semi-classical Stokes system

$$\begin{cases} -h^2\Delta u - u + h\nabla q = f, & (u, f) \in (H^2(\Omega) \cap V) \times H \\ h \operatorname{div} u = 0, & \text{in } \Omega \end{cases} \quad (3.1)$$

Assume that $\|u\|_{L^2(\Omega)} = O(1)$, $\|f\|_{L^2(\Omega)} = o(h)$. Taking inner product with u and doing integration by part, we have $\|h\nabla u\|_{L^2(\Omega)} = O(1)$. Since $q \in L^2(\Omega)/\mathbb{R}$, we may assume that $\int_{\Omega} q dx = 0$. From the regularity theory of the steady Stokes system, (see [17], page 33) and Poincaré's inequality, we have

$$\|h^2\nabla^2 u\|_{L^2(\Omega)} = O(1), \|q\|_{L^2(\Omega)} = O(h^{-1}), \|h\nabla q\|_{L^2(\Omega)} = O(1).$$

The following is a direct consequence of trace theorem for $q_0 = q|_{\partial\Omega}$.

Lemma 3.1. $\|q_0\|_{H^{1/2}(\partial\Omega)} = O(h^{-1})$.

We also have the hidden regularity property for the normal derivative.

Lemma 3.2. $h\partial_{\nu}u|_{\partial\Omega} = (h\partial_{\nu}u_{\parallel}, 0)$ and $\|h\partial_{\nu}u|_{\partial\Omega}\|_{L^2(\partial\Omega)} = O(1)$.

The proof of this lemma will be given in appendix. We will recover some information for low frequencies from the following lemma:

Lemma 3.3. *Suppose $u \rightharpoonup 0$ in $L^2(\Omega)$. Then after extracting to subsequences, we have $h\nabla q \rightharpoonup 0$ weakly in $L^2(\Omega)$ and $hq \rightarrow 0$ strongly in $H^{1/2}(\Omega)$.*

Proof. We may assume that $h\nabla q \rightharpoonup r$ weakly in $L^2(\Omega)$. Then the Rellich theorem implies that $hq \rightarrow P$ strongly in $L^2(\Omega)$, and thus $\nabla P = r$ with the property $\int_{\Omega} P = 0$. Moreover $\Delta P = 0$ in Ω , in the distributional sense. Since the sequence $(h^2\nabla^2 u)$ is bounded in L^2 , then up to a subsequence, $h^2\nabla^2 u \rightharpoonup W$ weakly in L^2 . From the Rellich theorem, the sequence (h^2u) converges strongly in L^2 and the strong limit must be 0, due to the fact that $u \rightharpoonup 0$, weakly in L^2 . Thus $W = 0$ and this implies that $\nabla P = 0$. Finally, we must have $P = 0$ since it has zero mean value. The last assertion follows from the Rellich theorem. \blacksquare

3.2. Semi-classical parametrix of the pressure term. In system (3.1), the family of pressures q satisfy the boundary value problem of the Laplace equation

$$-h^2\Delta q = 0, \text{ in } \Omega, q|_{\partial\Omega} = q_0$$

with unknown boundary data q_0 . We denote by $\text{PI}(q_0)$ the Poisson integral of the corresponding harmonic function with trace q_0 . Let \mathcal{N} be the Dirichlet-Neumann operator satisfying

$$\mathcal{N}q_0 = \partial_{\nu}\text{PI}(q_0)|_{\partial\Omega}.$$

Next we study the behaviour of the sequence of pressures q in the regime of frequency scale h^{-1} . We always fix the notation

$$\lambda(y, x', \xi') = |\xi'|_{g^{-1}(y, x')} \sim |\xi'|.$$

Let $Y = (-\epsilon_0, \epsilon_0)_y \times X_x$ and $Y_+ = [0, \epsilon_0)_y \times X_x$. We first have the L^2 bound of q , micro-locally away from $\xi' = 0$.

Lemma 3.4. *Let $(f_h)_{0 < h < 1}$ be a h -dependent family of distributions such that $\|f_h\|_{L^2(\mathbb{R}^n)} = O(h^{-N})$ for some $N \in \mathbb{N}$. Assume that for any $\chi \in C_c^\infty(\mathbb{R}^{2n})$, vanishing near $\xi = 0$, we have $\|\chi(x, hD_x)f_h\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} = O(h^{-1})$. Then*

$$\|\chi(x, hD_x)f_h\|_{L^2(\mathbb{R}^n)} = O(h^{-\frac{1}{2}}).$$

Proof. Assume that $\{|\xi| \leq 2\delta_0\} \cap \text{supp}(\chi) = \emptyset$. Take $\Phi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\Phi(\xi) = 1, |\xi| \leq \delta_0, \quad \Phi(\xi) = 0, |\xi| > 2\delta_0.$$

We write

$$\chi(x, hD_x)f = \Phi(hD_x)\chi(x, hD_x)f + (1 - \Phi(hD_x))\chi(x, hD_x)f.$$

From the support property we have $\Phi(hD_x)\chi(x, hD_x)f = O_{H^\infty}(h^\infty)$. Thus $(1 - \Phi(hD_x))\chi(x, hD_x)f = O_{H^{\frac{1}{2}}}(h^{-1})$. Let $b(\xi) = |\xi|^{1/2}(1 - \Phi(\xi))$, and we have $b(hD_x)\chi(x, hD_x)f = O_{L^2}(h^{-\frac{1}{2}})$. Since $b(\xi) \neq 0$ on $\text{supp}(\chi)$, we have

$$\|\chi(x, hD_x)f\|_{L^2(\mathbb{R}^n)} \leq C\|b(hD_x)\chi(x, hD_x)f\|_{L^2(\mathbb{R}^n)} + Ch\|\chi(x, hD_x)f\|_{L^2(\mathbb{R}^n)} \leq Ch^{-\frac{1}{2}}. \quad \blacksquare$$

Lemma 3.5. *Given $\delta_0 > 0$ and $\varphi, \tilde{\varphi} \in C_c^\infty(Y_+)$. For any $\chi_{\delta_0} \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$ such that χ_{δ_0} vanishes if $\lambda(y, x', \xi') \leq 2\delta_0$, we have*

$$\|\tilde{\varphi}\text{Op}_h(\chi_{\delta_0})(\varphi q)\|_{L^2(\mathbb{R}_+^d)} + h^{1/2}\|(\tilde{\varphi}\text{Op}_h(\chi_{\delta_0})(\varphi q))|_{y=0}\|_{L^2(\mathbb{R}^{d-1})} \leq C\delta_0, \varphi, \tilde{\varphi}.$$

Proof. Write $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, we have $\|hD_j(\varphi q)\|_{L^2(\mathbb{R}_+^d)} = O(1)$. Since $\frac{\xi'_j}{|\xi'|^2}\chi_{\delta_0}(y, x', \xi') \in S_\partial^0$, then for $\chi_j = \frac{\xi'_j}{|\xi'|^2}\chi_{\delta_0}$, we have

$$\tilde{\varphi}\chi_{\delta_0}(y, x', hD_{x'}) (\varphi q) = \sum_{j=1}^{d-1} \tilde{\varphi}\chi_j(y, x', hD_{x'}) hD_j(\varphi q) + O_{L^2(\mathbb{R}_+^d)}(1) = O_{L^2(\mathbb{R}_+^d)}(1),$$

where the implicit bound in big O depends on $\delta_0, \varphi, \tilde{\varphi}$. For the boundary term, we observe that $\tilde{\varphi}\text{Op}_h(\chi_{\delta_0})(\varphi q)|_{y=0} = O_{H^{1/2}(\mathbb{R}^{d-1})}(h^{-1})$ from trace theorem. Thus from Lemma 3.4, $\tilde{\varphi}\text{Op}_h(\chi_{\delta_0})(\varphi q)|_{y=0} = O_{L^2(\mathbb{R}^{d-1})}(h^{-1/2})$. \blacksquare

We express semi-classical Laplace operator $h^2\Delta_g$ in the geodesic coordinates of tubular neighborhood Y by

$$P_0 = h^2\partial_y^2 + g^{ij}\partial_i\partial_j + hM_j(y, x')h\partial_j + hH(y, x')h\partial_y$$

where $\partial_j = \partial_{x'_j}$. We make the ansatz

$$\tilde{q}(y, x') := \frac{1}{(2\pi h)^{d-1}} \int a(y, h, x', \xi') e^{\frac{ix'\xi'}{h}} \theta(\xi') d\xi',$$

then we calculate

$$\begin{aligned} P_0(\tilde{q})(y, x', \xi') &= \frac{1}{(2\pi h)^{d-1}} \int (h^2\partial_y^2 a + g^{jk}(h^2\partial_j\partial_k a - g^{jk}\xi'_j\xi'_k a)) e^{\frac{ix'\xi'}{h}} \theta(\xi') d\xi' \\ &\quad + \frac{1}{(2\pi h)^{d-1}} \int (ihg^{jk}\xi'_k\partial_j a) e^{\frac{ix'\xi'}{h}} \theta(\xi') d\xi' \\ &\quad + \frac{1}{(2\pi h)^{d-1}} \int ((h^2M_j\partial_j a + ihM_j\xi'_j a) + h^2H\partial_y a) e^{\frac{ix'\xi'}{h}} \theta(\xi') d\xi'. \end{aligned}$$

We next look for the formal semi-classical expansion

$$a(y, h, x', \xi') \simeq \sum_{j \geq 0} h^j a_j(y, h, x', \xi')$$

with $a_j \in S_{\partial}^{-j}$ and $h^k \partial_y^k a_j \in S_{\partial}^{-j+k}$. We obtain

$$\begin{aligned} P_0 \tilde{q} \simeq & \frac{1}{(2\pi h)^{d-1}} \int ((h^2 \partial_y^2 a_0 - g^{ij} \xi_i \xi_j a_0) \\ & + h(i g^{jk} \xi'_k \partial_j a_0 + i M_j \xi'_j a_0 + h^2 H \partial_y a_0) \\ & + h(h^2 \partial_y^2 a_1 - g^{jk} \xi'_j \xi'_k a_1) \\ & + h^2(g^{jk} \partial_j \partial_k a_0 + M_j \partial_j a_0) \\ & + h^2(i g^{jk} \xi'_k \partial_j a_1 + i M_j \xi'_j a_1 + h^2 H \partial_y a_1) \\ & + h^2(h^2 \partial_y^2 a_2 - g^{jk} \xi'_j \xi'_k a_2) \\ & + \dots) e^{\frac{i x' \xi'}{h}} \theta(\xi') d\xi'. \end{aligned}$$

Pick $\varphi_{1,0} = \varphi_1|_{\partial\Omega}$, $\varphi_1 \in C_c^\infty(Y)$. For $\tilde{q}_0 = \varphi_{1,0} q_0$, we put

$$\theta(\xi') = \mathcal{F}_h(\tilde{q}_0(\xi')) = (2\pi h)^{-(d-1)} \int_{\mathbb{R}^{d-1}} \tilde{q}_0(x') e^{-i x' \xi' / h} dx',$$

$$a_0(0, \cdot) \equiv 1, \quad a_j(0, \cdot) \equiv 0, \forall j \geq 1,$$

and we define the functions a_j inductively as follows: firstly we define a_0

$$a_0(y, x', \xi') = e^{-\frac{y \lambda(y, x', \xi')}{h}}, \quad \lambda(y, x', \xi') =: \sqrt{g^{ij} \xi'_i \xi'_j} \sim |\xi'|,$$

and the quantity

$$(h^2 \partial_y^2 - \lambda^2) a_0 = h \left(\frac{h^2}{\lambda^2} \frac{y^2 \lambda^2}{h^2} (\partial_y \lambda)^2 + \frac{2y\lambda}{h} \partial_y \lambda - 2\partial_y \lambda \right) e^{-\frac{y\lambda}{h}}$$

can be viewed as of order h . Next we set $a_j, j \geq 1$ implicitly by solving a sequence of linear ODEs:

$$\begin{aligned} h^2 \partial_y^2 a_1 - \lambda^2 a_1 &= -h^{-1} (h^2 \partial_y^2 - \lambda^2) a_0 - (i g^{jk} \xi'_k \partial_j a_0 + i M_j \xi'_j a_0 + h^2 H \partial_y a_0). \\ h^2 \partial_y^2 a_n - g^{ij} \xi_i \xi_j a_n &= -(g^{ij} \partial_i \partial_j a_{n-2} + M_j \partial_j a_{n-2}) \\ &\quad - (i g^{jk} \xi'_k \partial_j a_{n-1} + i M_j \xi'_j a_{n-1} + h^2 H \partial_y a_{n-1}), n \geq 2. \end{aligned}$$

Unfortunately, the functions a_j constructed above are not symbols, since they have singularities when $\xi' = 0$. This indicates that we can only obtain information of $q(h)$ from such parametrix away from $\xi' = 0$. We modify the construction above by setting

$$A_0(y, x', \xi') = e^{-\frac{y\lambda}{h}} \psi_{\delta_0}(\lambda) \varphi_2(y, x'), \quad (y, x', \xi') \in \mathbb{R}_+^d \times \mathbb{R}^{d-1},$$

with $\psi_{\delta_0} = \psi(\delta_0^{-1} \cdot)$, $\psi \in C^\infty(\mathbb{R}_+)$ satisfying $\psi(s) \equiv 1$ when $s \geq 1$ and $\psi(s) = 0$ when $0 < s \leq \frac{1}{2}$. We next modify other A_j in the same manner. Indeed, the ODEs which define A_j are linear ODEs in y variable. Thus for $j \geq 1$, $A_j(y, x', \xi') \equiv 0$ when $\lambda(y, x', \xi') \leq \frac{\delta_0}{2}$. We define the particular class of symbols in S_{∂}^j .

Definition 3.6.

$$\mathcal{E}_{\partial}^j := \left\{ a \in S_{\partial}^j : |(h \partial_y)^l \partial_{x', \xi'}^\alpha a(y, x', \xi')| \leq C_{l, \alpha} e^{-\frac{C'_{l, \alpha} y}{h}} \right\}.$$

Lemma 3.7. *The symbols constructed above can be chosen to satisfy $A_j \in \mathcal{E}_\partial^{-j}$ for all $j \in \mathbb{N}$.*

The proof will be given in appendix.

In summary, we have $A \simeq \sum_{j \geq 0} h^j A_j$, and a tangential symbol $B_{\delta_0}(y, x', \xi')$ compactly supported in $\lambda(y, x', \xi') \leq \frac{\delta_0}{2}$, such that

$$\begin{aligned} \varphi P_0 A(y, x', hD_{x'}) (\varphi_{1,0} q_0) &= \varphi B_{\delta_0}(y, x', hD_{x'}) (\varphi_{1,0} q_0) + O_{H^\infty}(h^\infty), \\ \varphi_0 A(0, x', hD_{x'}) (\varphi_{1,0} q_0) &= \varphi_0 \text{Op}_h(\psi_\delta(\lambda)) (\varphi_{1,0} q_0) + O_{H^\infty}(h^\infty). \end{aligned}$$

The following proposition states that the parametrix constructed above is an approximation of the pressure q in the relevant semi-classical scale.

Proposition 3.8. *There exists $A \in S_\partial^0$ with principal symbol*

$$A_0(y, x', \xi') = \exp\left(-\frac{y\lambda(y, x', \xi')}{h}\right) \psi_{\delta_0}(\lambda(y, x', \xi')) \varphi_1(y, x'),$$

which satisfies asymptotic expansion $A \sim \sum_{j \geq 0} h^j A_j$, $A_j \in \mathcal{E}_\partial^{-j}$. Moreover, for any

$$\varphi, \varphi_1 \in C_c^\infty(Y_+), \varphi_1|_{\text{supp}(\varphi)} \equiv 1,$$

we have $\varphi \text{Op}_h(\chi_{\delta_0} A_j) (\varphi_{1,0} q_0) = O_{L^2(\mathbb{R}_+^d)}(1)$ for all j , and

$$\begin{aligned} \varphi \text{Op}_h(\chi_{\delta_0}) (\varphi_1 q) &= \varphi \text{Op}_h(\chi_{\delta_0} A) (\varphi_{1,0} q_0) + O_{L^2(\mathbb{R}_+^d)}(h^{3/4}), \\ \varphi \text{Op}_h(\chi_{\delta_0}) h \partial_y (\varphi_1 q) &= \varphi \text{Op}_h(\chi_{\delta_0} \lambda A) (\varphi_{1,0} q_0) + O_{L^2(\mathbb{R}_+^d)}(h^{3/4}), \\ \varphi \text{Op}_h(\chi_{\delta_0}) h \partial_y (\varphi_1 q) &= \varphi \text{Op}_h(\chi_{\delta_0} \lambda A) (\varphi_{1,0} q_0) + O_{H^{\frac{2}{3}}(\mathbb{R}_+^d)}(h^{1/4}), \end{aligned}$$

where $\varphi_0 = \varphi|_{\partial\Omega}$, $\varphi_{1,0} = \varphi_1|_{\partial\Omega}$, $\chi_{\delta_0,0} = \chi_{\delta_0}|_{y=0}$.

We postpone the proof of this proposition in appendix. A direct consequence is that the singularities of the family of pressures (q_h) must concentrate in a very thin strip near the boundary.

Lemma 3.9. *With the same $\chi_{\delta_0} \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$ and $\varphi_1, \varphi \in C_c^\infty(Y_+)$, for any $0 < y_0 < \epsilon_0$, we have*

$$\int_{y_0}^{\epsilon_0} \|\varphi \text{Op}_h(\chi_{\delta_0}) (\varphi_1 q)\|_{L^2(\mathbb{R}^{d-1})}^2 dy \leq C_{\delta_0} (e^{-\frac{cy_0}{h}} + h),$$

where the constant C_{δ_0} only depends on δ_0 and is independent of y_0 and h .

Proof. The second term appearing on the right hand side comes from all the possible remainder terms. It suffices to estimate the term

$$\int_{y_0}^{\epsilon_0} \|\varphi \text{Op}_h(\chi_{\delta_0} A_0) (\varphi_1 q_0)\|_{L^2(\mathbb{R}^{d-1})}^2 dy.$$

Since $\varphi_{1,0} q_0 = O_{L^2(\mathbb{R}^{d-1})}(h^{-1/2})$, micro-locally, we have for each fixed $y > 0$ that

$$\begin{aligned} \|\varphi \text{Op}_h(\chi_{\delta_0} A_0) (\varphi_{1,0} q_0)\|_{L^2(\mathbb{R}^{d-1})} &\leq Ch^{-1/2} \sum_{|\beta| \leq Cd} h^{\frac{|\beta|}{2}} \sup_{y>0, (x', \xi')} |\partial_{x', \xi'}^\beta (\chi_{\delta_0} A_0)| + O(h^\infty) \\ &\leq Ch^{-1/2} e^{-\frac{cy}{h}} \left(1 + \sum_{1 \leq m, n \leq Cd} h^{m/2} \left(\frac{y}{h}\right)^n\right) + O(h^\infty). \end{aligned}$$

Squaring and Integrating the right hand side in y variable yields the desired conclusion. \blacksquare

4. MAIN STEPS OF THE PROOF

The proof of Theorem 1.2 can be divided into several steps according to different geometric situations. We want to show that for any given point $\rho_0 \in {}^b T^* \bar{\Omega}$, if $\rho_0 \notin \text{supp } \mu$, then $\gamma(s, \rho_0) \notin \text{supp}(\mu)$ for any $s > 0$. The first step is to show that if $\rho_0 \in T^* \Omega$, $\rho_0 \notin \text{supp}(\mu)$, then $\gamma(s, \rho_0) \notin \text{supp}(\mu)$ for all $s > 0$ provided that $\pi(\gamma(\cdot, \rho_0)|_{[0, s]}) \cap \partial \Omega = \emptyset$. This can be summarized by the following proposition, in which we have stronger conclusion that the measure is also invariant under the flow.

Proposition 4.1. *For any real-valued scalar function $a \in C_c^\infty(\Omega \times \mathbb{R}^d)$ vanishing near $\xi = 0$, we have*

$$\frac{d}{ds} \langle \mu, a \circ \gamma(s, \cdot) \rangle = 0.$$

Proof. Let $A = \text{Op}_h(a)$ and $P = -h^2 \Delta - 1$. Applying the equation and Lemma 3.3, we have

$$\begin{aligned} \frac{i}{h} ([P, A]u|u)_\Omega &= \frac{i}{h} (Au|Pu)_\Omega - \frac{i}{h} (APu|u)_\Omega \\ &= \frac{i}{h} (Au|f - h\nabla q)_\Omega - \frac{i}{h} (A(f - h\nabla q)|u)_\Omega \\ &= -\frac{i}{h} (Au|h\nabla q)_\Omega + \frac{i}{h} (Ah\nabla q|u)_\Omega + o(1) \\ &= -\frac{i}{h} ([A, h\text{div}]u|q)_\Omega + \frac{i}{h} ([A, h\nabla]q|u)_\Omega + o(1) \\ &= i(\text{Op}_h(\nabla a) \cdot u|q)_\Omega - i(\text{Op}_h(\nabla a)q|u)_\Omega + o(1) \\ &= i(u|\text{Op}_h(\nabla \bar{a})q)_\Omega - i(\text{Op}_h(\nabla a)q|u)_\Omega + o(1). \end{aligned} \tag{4.1}$$

where we have used integration by part freely, thanks to the fact that A has compact support in $x \in \Omega$. Now we claim that for any $\chi = a$ or \bar{a} , vanishing near $\xi = 0$, we have

$$(u|\text{Op}_h(\nabla \chi)q)_\Omega = o(1).$$

Indeed, $q = O_{L^2(\Omega)}(1)$ micro-locally away from $\xi = 0$ since $h\nabla q = O_{L^2}(1)$. Moreover, $h^2 \Delta(\text{Op}_h(\nabla \chi)q) = O_{L^2(\Omega)}(h)$ and this implies that $\text{Op}_h(\nabla \chi)q = o_{L^2}(1)$ since the symbol of $h^2 \Delta \text{Op}_h(\nabla \chi)$ vanishes near $\xi = 0$ as well as x near the boundary. In view of the definition of μ , this completes the proof. \blacksquare

For the second step, we need prove that if $\rho_0 \in \mathcal{E}$, then $\mu = 0$ in a neighborhood of ρ_0 .

Proposition 4.2. $\mu \mathbf{1}_\mathcal{E} = 0$. *Furthermore, for ν , the semi-classical defect measure of the sequence $(h_k \partial_\nu u_k|_{\partial \Omega}, h_k)$, we have $\nu \mathbf{1}_\mathcal{E} = 0$.*

The third step consists of proving that after reflection near a hyperbolic point, the measure μ is still zero.

Proposition 4.3. *Suppose that $\rho_0 \notin \text{supp}(\mu)$ and there exists $s_0 > 0$ such that $\gamma(s_0, \rho_0) \in \mathcal{H}$ and $\pi(\gamma(s, \rho_0)) \in \Omega$ for all $0 \leq s < s_0$. Then there exists $\delta > 0$ such that*

$$\pi(\gamma(\cdot, \rho_0)|_{[s_0, s_0 + \delta]}) \cap \text{supp}(\mu) = \emptyset.$$

Next step is to prove the propagation near a diffractive point.

Proposition 4.4. *Suppose that $\rho_0 \notin \text{supp}(\mu)$ and there exists $s_0 > 0$ such that $\gamma(s_0, \rho_0) \in \mathcal{G}^{2,+}$ and $\pi(\gamma(s, \rho_0)) \in \Omega$ for all $0 \leq s < s_0$. Then $\gamma(s_0, \rho_0) \notin \text{supp}(\mu)$.*

To deal with higher order contact, we will use induction. First let us introduce

Definition 4.5 (*k*-propagation property). *For $k \geq 2$, we say that *k*-propagation property holds, if along generalized ray $\gamma(s, \rho_0)$, the following statement is true: For some $\sigma_0 > 0$, if $\gamma(\cdot, \rho_0)|_{[0, \sigma_0]} \cap \text{supp}(\mu) = \emptyset$ (or $\gamma(\cdot, \rho_0)|_{(-\sigma_0, 0]} \cap \text{supp}(\mu) = \emptyset$) and $\gamma(\sigma_0, \rho_0) \in \bigcup_{2 \leq j \leq k} \mathcal{G}^j$ (or $\gamma(-\sigma_0, \rho_0) \in \bigcup_{2 \leq j \leq k} \mathcal{G}^j$), then $\gamma(\sigma_0, \rho_0) \notin \text{supp}(\mu)$ (or $\gamma(-\sigma_0, \rho_0) \notin \text{supp}(\mu)$).*

The last step for the proof of Theorem 1.2 can be reduced to

Proposition 4.6. *k*-propagation property holds for all $k \geq 2$.

5. NEAR \mathcal{E}

This section is devoted to the proof of Proposition 4.2. We set $Q(y, x', \xi') := \sqrt{\lambda^2 - 1}$ and define $Q_h = \varphi \text{Op}_h(Q\chi_0)\varphi_1$ with $\chi_0 \in C_c^\infty(\mathbb{R}_{\xi'}^{d-1})$ with support near \mathcal{E} in which $1 + \delta < \lambda < C$. With a bit abuse of notation, we refer q_0, q to be $\varphi \text{Op}_h(\chi_0)\varphi_1 q_0, \varphi \text{Op}_h(\chi_0)\varphi_1 q$ and u to be $\varphi \text{Op}_h(\chi_0)\varphi_1 u$. In this manner, we can combine the parametrix in last section to write the system (1.2) as

$$\begin{cases} (-h^2 \partial_y^2 + Q_h^2)u_{\parallel, j} + g^{jk} h \partial_{x'_k} (\text{Op}_h(A_0)q_0) = R_{\parallel, j} = O_{L^2(\mathbb{R}_+^d)}(h), \\ (-h^2 \partial_y^2 + Q_h^2)u_{\perp} + h \partial_y (\text{Op}_h(A_0)q_0) = R_{\perp} = O_{L^2(\mathbb{R}_+^d)}(h). \end{cases} \quad (5.1)$$

Note that the symbol $A_0(y, x', \xi')$ is defined in last section which equals to $e^{-\frac{y\lambda}{h}}$ since $\lambda > 1$. Take $\psi \in C^\infty(\mathbb{R}_+)$, with $\psi|_{[0, \epsilon_0]} \equiv 1, \psi|_{[2\epsilon_0, \infty)} \equiv 0$. Denoting the extended distributions of u by $w = (w_{\parallel}, w_{\perp}) = (u_{\parallel}, u_{\perp})\psi(y)\mathbf{1}_{y \geq 0}$, we have from standard elliptic parametrix construction (see Appendix A) that modulo $O_{H^\infty(\mathbb{R}_+^d)}(h^\infty)$,

$$\begin{cases} w_{\parallel, j} = E(y, x', hD_y, hD_{x'})(-\psi(y)g^{jk} h \partial_{x'_k} (\text{Op}_h(A_0)q_0) + hv_j \otimes \delta_{y=0} + \psi(y)R_{\parallel, j}), \\ w_{\perp} = E(y, x', hD_y, hD_{x'})(-h\psi(y)\partial_y (\text{Op}_h(A_0)q_0) + \psi(y)R_{\perp}). \end{cases} \quad (5.2)$$

where $v = h \partial_y u_{\parallel}|_{y=0} = O_{L^2_{x'}}(1)$. Recall that the principal symbol of E is given by

$$E^0 := \frac{\chi_0(\xi')\varphi(y, x')}{\eta^2 + \lambda(y, x', \xi')^2 - 1},$$

Now we need a lemma which deals with the trace of error terms:

Lemma 5.1. *Assume that $R = \varphi \text{Op}_h(\chi_0)\varphi_1 R + O_{H^\infty}(h^\infty)$, then if $\|\psi(y)R\|_{L^2(\mathbb{R}_+^d)} = O(h)$, we have*

$$\|E(y, x', hD_y, hD_{x'}) (\psi(y)R)|_{y=0}\|_{L^2(\mathbb{R}_{x'}^{d-1})} = O(h^{1/3}).$$

Proof. From the parametrix construction above, we know that

$$|\partial_{y, x', \eta, \xi'}^\alpha E(y, x'; \eta, \xi')| \leq \frac{C_\alpha}{\eta^2 + Q(y, x', \xi')^2}.$$

Therefore, the symbols $\eta E(y, x'; \eta, \xi')$ and $\lambda(y, x', \xi')E(y, x'; \eta, \xi')$ are uniformly bounded in S^0 . Thus $E(y, x'; hD_y, hD_{x'}) (\psi R) = O_{L^2(\mathbb{R}_+^d)}(h) = O_{H^1(\mathbb{R}_+^d)}(1)$, and

from interpolation, we have $E(y, x'; hD_y, hD_{x'}) (\psi R) = O_{H^{2/3}(\mathbb{R}_+^d)}(h^{1/3})$. The conclusion then follows from the trace theorem that $H^s(\mathbb{R}_+^d) \rightarrow H^{s-1/2}(\mathbb{R}^{d-1})$ is bounded for $s > 1/2$. \blacksquare

Proof of Proposition 4.2. Denote by $\mathcal{F}_h(q_0) = \theta$ the semi-classical Fourier transform of q_0 , we calculate

$$\begin{aligned}
& E(y, x', hD_y, hD_{x'}) (\psi(y) h \partial_y \text{Op}_h(A_0) q_0) \\
&= \frac{1}{(2\pi h)^d} \iint e^{\frac{i(y-z)\eta}{h}} dz d\eta \int \frac{e^{\frac{ix'\xi'}{h}} \theta(\xi') \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q^2(y, x', \xi')} \psi(z) h \partial_z (e^{-\frac{z\lambda(z, x', \xi')}{h}}) d\xi' + R_1 \\
&= -\frac{h}{(2\pi h)^d} \iint \frac{\langle \xi' \rangle \theta(\xi') e^{\frac{i(y\eta + x'\xi')}{h}} B_1(\eta, x', \xi') \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q^2(y, x', \xi')} d\eta d\xi' \\
&\quad - \frac{h^2}{(2\pi h)^d} \int e^{\frac{ix'\xi'}{h}} \theta(\xi') d\xi' \int \frac{e^{\frac{i y \eta}{h}} B_0(\eta, x', \xi') \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q(y, x', \xi')^2} d\eta + R_1,
\end{aligned} \tag{5.3}$$

with reminder term $R_1 = O_{L^2(\mathbb{R}^d)}(h)$, where $\lambda_0 = \lambda|_{y=0}$,

$$B_1(\eta, x', \xi') = \int_0^\infty \psi(z) e^{-\left(\frac{i\eta + \lambda(z, x', \xi')}{h}\right)z} \frac{\lambda(z, x', \xi')}{\langle \xi' \rangle} \frac{1}{h} dz,$$

and

$$B_0(\eta, x', \xi') = \int_0^\infty \psi(hz) z (\partial_z \lambda)(hz, x', \xi') e^{-(i\eta + \lambda(hz, x', \xi'))z} dz.$$

We notice that

$$K_0(y, x', \xi') := \int \frac{e^{\frac{i y \eta}{h}} B_0(\eta, x', \xi') \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q(y, x', \xi')^2} d\eta$$

is a bounded symbol in $S_{\xi'}^0$. Thus the second term on the right hand side of (5.3) is equal to $R_2 = O_{C^0(\mathbb{R}_y; L^2(\mathbb{R}_{x'}^{d-1}))}(h)$ and we may concentrate on the first term. Write

$$B_1(\eta, x', \xi') = \int_0^\infty \psi(hz) e^{-(i\eta + \lambda(hz, x', \xi'))z} \frac{\lambda(hz, x', \xi')}{\langle \xi' \rangle} dz.$$

Taylor expansion gives

$$\begin{aligned}
e^{-\lambda(hz, x', \xi')z} \lambda(hz, x', \xi') \psi(hz) &= e^{-\lambda_0(x', \xi')z} \lambda_0(x', \xi') + \int_0^1 \frac{d}{dt} \left(e^{-\lambda(htz, x', \xi')z} \lambda(htz, x', \xi') \psi(htz) \right) dt \\
&= e^{-\lambda_0(x', \xi')z} \lambda_0(x', \xi') + h \int_0^1 P_t(z, x', \xi') e^{-\lambda(htz, x', \xi')z} dt
\end{aligned}$$

with

$$P_t(z, x', \xi') = -z^2 (\lambda \partial_y \lambda)(htz, x', \xi') \psi(htz) + z (\partial_y \lambda)(htz, x', \xi') \psi(htz) + z \lambda(htz, x', \xi') \psi'(htz).$$

Thus we have $B_1(\eta, x', \xi') = \frac{\lambda_0(x', \xi')}{(i\eta + \lambda_0(x', \xi')) \langle \xi' \rangle} + h \tilde{B}_1(\eta, x', \xi')$, where

$$\tilde{B}_1(\eta, x', \xi') = \int_0^1 \int_0^\infty e^{-(i\eta + \lambda(htz, x', \xi'))z} \frac{1}{\langle \xi' \rangle} P_t(z, x', \xi') dz dt.$$

Note that near a point in \mathcal{E} , $|\partial_{x'}^\alpha \partial_{\xi'}^\beta P_t(z, x', \xi')| \leq C_{\alpha\beta} z^2$, independent of t, h , hence the symbol

$$\tilde{K}_1(y, x', \xi') = \int \frac{e^{\frac{i y \eta}{h}} \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q(y, x', \xi')^2} \tilde{B}_1(\eta, x', \xi') d\eta \in S_{\xi'}^0.$$

Therefore, the symbol in the principal term of $E(y, x', hD_y, hD_{x'}) (\psi(y) h \partial_y \text{Op}_h(A_0) q_0)$ equals to

$$\begin{aligned} K_1(y, x', \xi') &= \lambda_0(x', \xi') \int \frac{e^{\frac{i y \eta}{h}} \varphi(y, x') \chi_0(\xi')}{(\eta^2 + Q(y, x', \xi')^2)(i\eta + \lambda_0(x', \xi'))} d\eta + h \tilde{K}_1(y, x', \xi') \\ &= 2\pi \lambda_0 \varphi(y, x') \chi_0(\xi') \left(\frac{e^{-\frac{yQ}{h}}}{2(\lambda_0 - Q)Q} - \frac{e^{-\frac{y\lambda_0}{h}}}{\lambda_0^2 - Q^2} \right) + h \tilde{K}_1(y, x', \xi') \\ &= 2\pi \lambda_0 \varphi(y, x') \chi_0(\xi') \left(\frac{e^{-\frac{yQ}{h}} - e^{-\frac{y\lambda_0}{h}}}{2(\lambda_0 - Q)Q} + \frac{e^{-\frac{y\lambda_0}{h}}}{2Q(\lambda_0 + Q)} \right) + h \tilde{K}_1(y, x', \xi'). \end{aligned}$$

Note that

$$E_1(y, x', \xi') = 2\pi \lambda_0 \varphi(y, x') \chi_0(\xi') \left(\frac{e^{-\frac{yQ}{h}} - e^{-\frac{y\lambda_0}{h}}}{2(\lambda_0 - Q)Q} + \frac{e^{-\frac{y\lambda_0}{h}}}{2Q(\lambda_0 + Q)} \right) > 0$$

near \mathcal{E} , we have

$$E(y, x', hD_y, hD_{x'}) (\psi(y) h \partial_y \text{Op}_h(A_0) q_0) = E_1(y, x', hD_{x'}) q_0 + R_1 + R_2,$$

with $R_2 = O_{C(\mathbb{R}_y; L^2(\mathbb{R}_{x'}^{d-1}))}(h)$. We claim that $R_1 = O_{H^1(\mathbb{R}^d)}(1)$. Once it is justified, by interpolation, we have $\|R_1\|_{H^{2/3}(\mathbb{R}^d)} = O(h^{1/3})$. To verify the claim, we note that the symbol of the reminder term R_1 is of the form hS^{-1} (in both η and ξ' variables), hence the symbolic calculus yields $\partial_y R_1 = O_{L^2(\mathbb{R}^d)}(1)$, and $\partial_{x'} R_1 = O_{L^2(\mathbb{R}^d)}(1)$.

We next calculate the parallel component

$$\begin{aligned} & \frac{1}{(2\pi h)^d} \iint \psi(z) e^{\frac{i(y-z)\eta}{h}} dz d\eta \int \frac{e^{\frac{i x' \xi' - z \lambda(z, x', \xi')}{h}} \theta(\xi') g^{jk}(z, x') \xi'_k \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q(y, x', \xi')^2} d\xi' \\ &= \frac{1}{(2\pi h)^d} \iint \frac{e^{\frac{i(x' \xi' + y\eta)}{h}} \xi'_k \theta(\xi') \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q(y, x', \xi')^2} d\eta d\xi' \int_0^\infty \psi(z) e^{-\frac{i\eta + \lambda(z, x', \xi')}{h} z} g^{jk}(z, x') dz \\ &= \frac{h}{(2\pi h)^d} \iint \frac{e^{\frac{i(x' \xi' + y\eta)}{h}} \xi'_k \theta(\xi') \varphi(y, x') \chi_0(\xi')}{(\eta^2 + Q(y, x', \xi')^2)} B_{2,jk}(\eta, x', \xi') d\eta d\xi' \\ &=: E_2(y, x', hD_{x'}) q_0. \end{aligned}$$

where

$$B_{2,jk}(\eta, x', \xi') = \int_0^\infty \psi(z) e^{-\frac{i\eta + \lambda(z, x', \xi')}{h} z} g^{jk}(z, x') \frac{1}{h} dz.$$

Define

$$K_{2,jk}(y, x', \xi') = \int \frac{e^{\frac{i y \eta}{h}} B_{2,jk}(\eta, x', \xi') \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q(y, x', \xi')^2} d\eta,$$

and from similar argument we have

$$K_{2,k}(y, x', \xi') = g^{jk}(0, x') \int \frac{e^{\frac{i y \eta}{h}} \varphi(y, x') \chi_0(\xi')}{(\eta^2 + Q(y, x', \xi')^2)(i\eta + \lambda_0(x', \xi'))} d\eta + h \tilde{K}_2(y, x', \xi')$$

and the principal symbol of $E_2(y, x', hD_{x'})$ is elliptic if $\lambda_0(\xi') > 1$ and y small enough. Finally,

$$\begin{aligned}
& E(y, x', hD_y, hD_{x'})(hv \otimes \delta_{y=0}) \\
&= \frac{h}{(2\pi h)^d} \iint \frac{\mathcal{F}_h(v)(\xi') e^{\frac{i(y\eta + x'\xi')}{h}} \varphi(y, x') \chi_0(\xi')}{\eta^2 + Q(y, x', \xi')^2} d\xi' d\eta + O_{L^2(\mathbb{R}^d)}(h) \\
&= \frac{h}{(2\pi h)^d} \int \mathcal{F}_h(v)(\xi') e^{\frac{ix'\xi'}{h}} \frac{\pi e^{-\frac{yQ(y, x', \xi')}{h}} \varphi(y, x') \chi_0(\xi')}{Q(y, x', \xi')} d\xi' + O_{L^2(\mathbb{R}^d)}(h) \\
&=: E_3(y, x', hD_{x'})v + O_{L^2(\mathbb{R}^d)}(h),
\end{aligned} \tag{5.4}$$

and again, $E_3(y, x', hD_{x'})$ is elliptic near $\lambda_0(\xi') > 1$. Moreover, we deduce from the same argument as for R_1 that the reminder terms are indeed of $O_{H^{2/3}(\mathbb{R}^d)}(h^{1/3})$. Now the boundary condition $(w_\perp, w_\parallel)|_{y=0} = 0$ and the trace theorem yield

$$E_1(0, x', hD_{x'})q_0 = O_{L^2(\mathbb{R}_{x'}^{d-1})}(h^{1/3}),$$

$$E_2(0, x', hD_{x'})q_0 + E_3(0, x', hD_{x'})v = O_{L^2(\mathbb{R}_{x'}^{d-1})}(h^{1/3}).$$

Therefore, from the ellipticity of E_1, E_2, E_3 , the measure of pressure in the elliptic region vanishes, so does the measure of v , namely $\nu|_{\mathcal{E}} = 0$. The proof of Proposition 4.2 is complete. \blacksquare

6. NEAR \mathcal{H}

We take $\varphi_1, \varphi \in C_c^\infty(Y_+)$ such that $\varphi_1|_{\text{supp}(\varphi)} \equiv 1$. For any tangential symbol $b \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$, we define the pseudo-differential operator $B_h = \varphi \text{Op}_h(b) \varphi_1$, with compact support in Y_+ . We will change the notation of tangential variables (x', ξ') to (x, ξ) . We always work in local coordinate (y, x) and sometimes abuse the notation $u = \varphi_1 u, q = \varphi_1 q$ as compactly supported functions in Y_+ . Note that q_0 , the trace of q is not bounded in L^2 in priori. Fortunately, it turns out that $q_0 = O_{L^2}(1)$, micro-locally near a point in \mathcal{H} .

6.1. L^2 bound of boundary datum. Take $b(y, x, \xi), b_1(y, x, \xi) \in C_c^\infty([0, \epsilon_0] \times \mathcal{H})$, such that $b_1|_{[0, \epsilon_0/2] \times \text{supp}(b)} \equiv 1$. Let $Q(y, x, \xi) = \sqrt{1 - \lambda(y, x, \xi)^2} b_1(y, x, \xi)$. We will first factorize the operator $(-h^2 \Delta - 1)$ near a hyperbolic point.

Lemma 6.1. *For $0 \leq y < \epsilon_0$, we have*

$$B_h(-h^2 \Delta - 1) = -(hD_y - Q_h^+)(hD_y - Q_h^-) + R' = -(hD_y - \tilde{Q}_h^-)(hD_y - \tilde{Q}_h^+) + R'',$$

where $R', R'' \in C^\infty([0, \epsilon_0], h^\infty \Psi^{-\infty}(\partial\Omega))$, and Q_h^\pm, \tilde{Q}_h^\pm have principal symbols $\pm Q(y, x, \xi)$.

Proof. The proof is quite standard, and we follow the construction in [3] by translating word by word to the semi-classical setting. In local coordinate, we have

$$B_h(-h^2 \Delta - 1) = h^2 D_y^2 + R(y, x, hD_x) + hM_1(y, x')hD_y + hM_0(y, x)hD_x$$

with $\sigma(R) = Q^2$. Set $q_1^+ = \sqrt{Q}(y, x', \xi')$, $Q_1^+ = \text{Op}_h(q_1^+)$ and $Q_1^- = -Q_1^+ - hM_1$. Direct calculation gives

$$\begin{aligned}
(hD_y - Q_1^+)(hD_y - Q_1^-) &= h^2 D_y^2 - (Q_1^+)^2 - hQ_1^+ M_1 - (Q_1^+ + Q_1^-)hD_y - \frac{h}{i} \partial_y(Q_1^-) \\
&= h^2 D_y^2 - (Q_1^+)^2 + hM_1 hD_y - h(Q_1^+ M_1 - i\partial_y(Q_1^-)).
\end{aligned}$$

Thus $B_h(-h^2\Delta - 1) - (hD_y - Q_1^+)(hD_y - Q_1^-) = hT_1$, with some operator T_1 , bounded in L^2 . Now for $j \geq 1$, suppose that we have

$$B_h(-h^2\Delta - 1) - (hD_y - Q_j^+)(hD_y - Q_j^-) = h^j T_j,$$

by setting $Q_{j+1}^\pm := Q_j^\pm + h^j S_{j+1}^\pm$ with $S_{j+1}^+ + S_{j+1}^- = 0$ and $\sigma(S_{j+1}^+) = \frac{\sigma(T_j)}{2\sigma(Q_j^+)}$, we obtain that

$$\begin{aligned} & B_h(-h^2\Delta - 1) - (hD_y - Q_{j+1}^+)(hD_y - Q_{j+1}^-) \\ &= h^j T_j + h^j (S_{j+1}^+ Q_j^- + Q_j^+ S_{j+1}^-) - h^j (S_{j+1}^+ + S_{j+1}^-) hD_y - \frac{h^{j+1}}{i} \partial_y (S_{j+1}^-) + h^{2j} S_{j+1}^+ S_{j+1}^- \\ &=: h^{j+1} T_{j+1}, \end{aligned}$$

for some operator T_{j+1} bounded in L^2 . Thus the proof can be completed by induction. \blacksquare

Define $w = \varphi_1 u - h\nabla(\varphi_1 q)$, $w^\pm = B_h(hD_y - Q_h^\pm)w$ and their boundary values $w_0^\pm := w^\pm|_{y=0}$. Note that $\varphi P_h w = \varphi f$.

Proposition 6.2. $\|B_h h \partial_y w_\perp\|_{L^2(\mathbb{R}_+^d)} = O(1)$, and consequently, $\|w_\perp^\pm\|_{L^2(\mathbb{R}_+^d)} = O(1)$.

Proof. From $h \operatorname{div} u = 0$, we have $\varphi h \operatorname{div} w = 0$, hence

$$\varphi(h \partial_y w_\perp + h \operatorname{div}_\parallel w_\parallel) = O_{L^2(\mathbb{R}_+^d)}(h),$$

where in local coordinates,

$$\operatorname{div}_\parallel w_\parallel = \frac{1}{\sqrt{\det(g)}} \sum_{j=1}^{d-1} \partial_{x_j} (\sqrt{\det(g)} w_{\parallel,j}).$$

Therefore,

$$\|B_h h \partial_y w_\perp\|_{L^2(\mathbb{R}_+^d)} \leq O(h) + \|B_h h \operatorname{div}_\parallel w_\parallel\|_{L^2(\mathbb{R}_+^d)} = O(1). \quad \blacksquare$$

Now we recall the following hyperbolic energy estimate.

Lemma 6.3. *Suppose $A_h = \operatorname{Op}_h(a)$ is elliptic (with real-valued symbol a smoothly depending on t) of order 0 on a compact manifold M and w are solutions of the h -dependent equations*

$$(hD_t \pm A_h)w = g, \quad (t, x) \in \mathbb{R} \times M.$$

Assume that for any compact time interval I and small h ,

$$\|w\|_{L^2(I \times M)} \leq C(I), \quad \|g\|_{L^2(I \times M)} \leq C(I)h,$$

then we have for all small h ,

$$\sup_{t \in I'} \|w(t)\|_{L^2(M)} \leq C(I'), \quad \forall I' \subset I \text{ compact.}$$

Proof. By symmetry, it suffices to treat the case $hD_t - A_h$. Take $\chi(t) \in C_c^\infty(I')$, and we may assume that $0 \in I'$ with $\chi(0) = 1$. Multiplying by $\chi(t)$ to the equation, we have

$$(hD_t - A_h)(\chi w) = \chi g + [\chi, hD_t - A_h]w =: r = O_{L^2(\mathbb{R} \times M)}(h).$$

We now calculate

$$\begin{aligned} h \frac{d}{dt} (\chi w | \chi w)_{L^2(M)} &= (ihD_t \chi w | \chi w)_{L^2(M)} + (\chi w | ihD_t \chi w)_{L^2(M)} \\ &= i(A_h(\chi w) + r | \chi w)_{L^2(M)} - i(\chi w | A_h(\chi w) + r)_{L^2(M)} \\ &= i((A_h - A_h^*)\chi w | \chi w)_{L^2(M)} + i(r | \chi w)_{L^2(M)} - i(\chi w | r)_{L^2(M)} \end{aligned}$$

Integrating the formula above from 0 to $\sup I'$, we finally have $\|w(0)\|_{L^2(M)}^2 = O(1)$. \blacksquare

Lemma 6.4. $\|w_0^\pm\|_{L^2(\mathbb{R}^{d-1})} = O(1)$.

Proof. From Proposition 6.2, we have $(hD_y - Q_h^\mp)w_\perp^\pm = O_{L^2(\mathbb{R}_\pm^d)}(h)$. Applying the previous lemma to w_\perp^\pm , we have $\|w_{0,\perp}^\pm\|_{L^2(\mathbb{R}_x^{d-1})} = O(1)$. Combining the boundary condition, we have

$$B_h(Q_h^+ - Q_h^-)(h\partial_y q)|_{y=0} = -B_h(Q_h^+ - Q_h^-)h\mathcal{N}q_0 = w_{0,\perp}^+ - w_{0,\perp}^- = O_{L^2(\mathbb{R}^{d-1})}(1).$$

Remark that in priori, \mathcal{N} is a classical first order pseudo-differential operator, and we only have

$$\|B_h h\mathcal{N}q_0\|_{L^2(\mathbb{R}^{d-1})} \leq \|B_h\|_{H^{-1} \rightarrow L^2} h \|\mathcal{N}q_0\|_{H^{-1}(\mathbb{R}^{d-1})} = O(h^{-1}).$$

From the exact principal symbol of Q_h^\pm , we have $\|B_h h\mathcal{N}q_0\|_{L^2(\mathbb{R}^{d-1})} = O(1)$, and the constant in big O depends on the micro-local cut-off $b(y, x', \xi')$. As a consequence, $\|w_{0,\perp}^\pm\|_{L^2(\mathbb{R}^{d-1})} = O(1)$.

It remains to study w_\parallel^\pm . Notice that their boundary values are

$$w_{0,\parallel}^\pm = B_h(v - (hD_y h\nabla_\parallel q)|_{y=0}) - B_h Q_h^\pm h\nabla_\parallel q_0,$$

where $v = (h\partial_y u)|_{y=0} = O_{L^2(\mathbb{R}^{d-1})}(1)$. All terms are obviously bounded in $L^2(\mathbb{R}^{d-1})$ except the trace of $B_h h\nabla_\parallel hD_y q$. To bound it, we use the support property of b and Proposition 3.8, hence $B_h h\nabla_\parallel hD_y q|_{y=0} = -B_h h\nabla_\parallel h\mathcal{N}q_0 = O_{L^2(\mathbb{R}_x^{d-1})}(1)$. \blacksquare

Again by hyperbolic estimates, we have the following result:

Proposition 6.5. $\|w^\pm\|_{L^2(\mathbb{R}_\pm^d)} = O(1)$. In particular,

$$\|B_h hD_y w\|_{L^2(\mathbb{R}_\pm^d)} + \|B_h h\mathcal{N}q_0\|_{L^2(\mathbb{R}_x^{d-1})} + \|B_h h^2 \Delta_0 q_0\|_{L^2(\mathbb{R}_x^{d-1})} = O(1),$$

where $\Delta_0 = \Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on $\partial\Omega$.

Proof. It remains to prove $\|B_h h^2 \Delta_0 q_0\|_{L^2(\mathbb{R}_x^{d-1})} = O(1)$. Indeed,

$$\begin{aligned} B_h h\partial_y w_\perp &= B_h h\partial_y u_\perp - h^2 B_h \partial_y^2 q \\ &= h^2 B_h \partial_y u_\perp + h^2 B_h \left(\frac{1}{\sqrt{\det(g)}} \sum_{1 \leq j, k \leq d-1} \partial_j (g^{jk} \partial_k q_0) + \frac{\partial_y \sqrt{\det(g)}}{\sqrt{\det(g)}} \partial_y q \right). \end{aligned}$$

Thus

$$B_h h\partial_y w_\perp|_{y=0} = B_h h^2 \Delta_0 q_0 + O_{L^2(\mathbb{R}^{d-1})}(1),$$

thanks to $h\partial_y u_\perp = 0$ and $B_h h \mathcal{N} q_0 = O_{L^2(\mathbb{R}^{d-1})}(h^{-1})$. From $w_{0,\perp}^\pm = B_h h D_y w_\perp|_{y=0} + B_h Q_h^\pm h \mathcal{N} q_0 = O_{L^2(\mathbb{R}^{d-1})}(1)$, we deduce that $B_h h D_y w_\perp|_{y=0} = O_{L^2(\mathbb{R}^{d-1})}(1)$, and these yield $\|B_h h^2 \Delta_0 q_0\|_{L^2(\mathbb{R}^{d-1})} = O(1)$. \blacksquare

Corollary 6.6. $\|B_h h \nabla q\|_{L^2(\mathbb{R}_+^d)} = o(1)$.

Proof. We will go back to the global notation in this calculation. It suffices to show that $B_h h \nabla q = \varphi \text{Op}_h(b) \varphi_1 h \nabla q = o_{L^2(\Omega)}(1)$ since there are only change of bounded weight in the integral with respect to the measure $\sqrt{\det(g)} dy dx$ and $dy dx$ in local coordinate, and the former allows us to apply integration by part and the structure of the equation in a simple way. We calculate

$$\begin{aligned} (B_h h \nabla q | B_h h \nabla q)_{L^2(\Omega)} &= ([B_h, h \nabla] q | B_h h \nabla q)_{L^2(\Omega)} + (h \nabla B_h q | B_h h \nabla q)_{L^2(\Omega)} \\ &= o(1) - (A_h B_h q | h \text{div} B_h h \nabla q)_{L^2(\Omega)} \\ &\quad + (B_h h q_0 | B_h (h \partial_\nu q) |_{\partial\Omega})_{L^2(\partial\Omega)} \\ &= o(1) + (B_h h q_0 | B_h h \mathcal{N} q_0)_{L^2(\partial\Omega)}, \end{aligned}$$

where we have used the fact that $h q = o_{L^2(\Omega)}(1)$ and $\Delta q = 0$ in the calculation. Now from Lemma 3.3, we know that $h q \rightharpoonup 0$ weakly in $H^1(\Omega)$ and $h q_0 \rightarrow 0$ strongly in $L^2(\partial\Omega)$. The last term is $o(1)$ since $B_h h \mathcal{N} q_0 = O_{L^2(\partial\Omega)}(1)$. \blacksquare

6.2. propagation estimate. In this subsection, we prove Proposition 4.3. We factorize $-h^2 \Delta - 1$ as $(h D_y - Q_h^\pm)(h D_y - Q_h^\mp) + R^\pm$ near $z_0 \in \mathcal{H}$ and choose Q_h^\pm with principal symbols $\pm Q(y, x, \xi) = \sqrt{1 - \lambda^2 b_1(y, x, \xi)}$, as in the previous subsection. Take $\psi \in C_c^\infty([0, \epsilon_0])$ with $\psi \equiv 1$ in a neighborhood of $y = 0$. By an abuse of notation, we introduce

$$w^\pm = B_h^\pm (h D_y - Q_h^\mp) w,$$

where B_h^\pm have principal symbols $\psi(y) b^\pm(y, x, \xi)$. Here, b^\pm are solutions of

$$\frac{\partial b^\pm}{\partial y} \mp H_{Q(y, x, \xi)} b^\pm = 0, \quad b^\pm|_{y=0} = b_0, \quad (6.1)$$

where b_0 is another micro-localization near z_0 with $b_1|_{\text{supp}(b_0)} = 1$, and $H_Q b = \{Q, b\}$. Note that the compact support of $\psi(y) b^\pm$ can be chosen arbitrarily close to the semi-bicharacteristic curves γ^\pm corresponding to the principal symbol p . Moreover, b^\pm are invariant along γ^\pm . Under these notations, Proposition 4.3 can be rephrased as follows

Proposition 6.7. *Let μ be the defect measure of u . If*

$$b^+ \mu \mathbf{1}_{0 < y \leq \epsilon_0} = 0 \quad (b^- \mu \mathbf{1}_{0 < y \leq \epsilon_0} = 0),$$

then we have

$$b^- \mu \mathbf{1}_{0 < y \leq \epsilon_0} = 0 \quad (b^+ \mu \mathbf{1}_{0 < y \leq \epsilon_0} = 0).$$

Moreover, we have in fact $b^+ \mu = b^- \mu = 0$ in this case.

The proof will be divided into several lemmas. First we calculate

$$(h D_y - Q_h^\pm) w^\pm = [h D_y - Q_h^\pm, B_h^\pm] (h D_y - Q_h^\mp) w + B_h^\pm (h D_y - Q_h^\pm) (h D_y - Q_h^\mp) w,$$

and

$$[h D_y - Q_h^\pm, B_h^\pm] = \frac{h}{i} \text{Op}_h(\partial_y b^\pm \mp H_Q b^\pm) \psi(y) + \frac{h}{i} \psi'(y) B_h^\pm + R''.$$

The first operator vanishes thanks to the definition of b^\pm , and the remainder term $R'' = O_{L^2(\mathbb{R}_+^d)}(h^2)$. Therefore we have

$$\|R''(hD_y - Q_h^\mp)w\|_{L^2(\mathbb{R}_+^d)} = O(h^2),$$

and consequently

$$(hD_y - Q_h^\pm)w^\pm = \frac{h}{i}\psi'(y)w^\pm + g^\pm,$$

with $g^\pm = o_{L^2(\mathbb{R}_+^d)}(h)$.

Lemma 6.8. *Let μ^\pm be the semi-classical defect measure of w^\pm and b is defined as above. Suppose that $b^\pm\mu^\pm\mathbf{1}_{0 < y \leq \epsilon_0} = 0$, then we must have $b^\pm\mu^\pm \equiv 0$ and $\mu_0^\pm = 0$, where μ_0^\pm is the defect measure of $w_0^\pm = w^\pm|_{y=0}$.*

Proof. For $y_0 = \epsilon_0/2$, we first claim that $\|w^\pm(y_0)\|_{L_x^2} = o(1)$. Indeed, from the assumption and compactness, the measure μ^\pm vanishes in a small neighborhood of semi-bicharacteristic curve γ^\pm . Thus $\|w^\pm\|_{L^2([y_0, \epsilon_0] \times \mathbb{R}^{d-1})} = o(1)$, provided that we choose $\text{supp}(b_0)$ small enough in the definition of w^\pm . Finally, repeating the argument in the proof of Lemma 6.3, we have

$$-h\|w^\pm(y_0)\|_{L^2(\mathbb{R}^{d-1})}^2 = i \int_{y_0}^{\epsilon_0} ((Q_h^\pm - (Q_h^\pm)^*)\chi w^\pm | \chi w^\pm)_{L^2(\mathbb{R}^{d-1})}(y) dy + o(h).$$

The claim then follows.

Integrating the identity

$$\begin{aligned} h \frac{d}{dy} (w^\pm | w^\pm)_{L^2(\mathbb{R}^{d-1})} &= (i(Q_h^\pm - (Q_h^\pm)^*)w^\pm | w^\pm)_{L^2(\mathbb{R}^{d-1})} + 2h(\psi'(y)w^\pm | w^\pm)_{L^2(\mathbb{R}^{d-1})} \\ &\quad + 2\text{Im}(w^\pm | g^\pm)_{L^2(\mathbb{R}^{d-1})} \end{aligned}$$

from $y = z < y_0$ to $y = y_0$, we have

$$\|w^\pm(z)\|_{L^2(\mathbb{R}^{d-1})}^2 \leq C \int_z^{y_0} \|w^\pm(y)\|_{L^2(\mathbb{R}^{d-1})}^2 dy + o(1).$$

Using $\int_0^{y_0} \|w^\pm(y)\|_{L^2(\mathbb{R}^{d-1})}^2 dy = o(1)$, we obtain that $\|w_0^\pm\|_{L^2(\mathbb{R}^{d-1})} = o(1)$. This completes the proof of Lemma 6.8. \blacksquare

Remark 6.9. *Away from the boundary, the defect measure of u equals to the defect measure of w , and it propagates along the bicharacteristic curves γ^\pm . Since we can decompose w into w^+ and w^- near a hyperbolic point, we call w^+ (w^-) the incoming wave and the out-coming wave. Thus the above proposition asserts that if we have no singularity of w^+ (w^-) along incoming wave (out-coming wave) near the boundary but strictly away from the boundary, then there is no singularity of the boundary data of incoming wave (out-coming wave).*

Changing the role of $y = y_0$ and $y = 0$ in the proof of Lemma 6.8, we conclude that if $\mu_0^\pm = 0$, then $b^\pm\mu^\pm = 0$. To finish the proof of Proposition 6.7, we need understand how the singularity transfers from boundary data of in-coming wave to the boundary data of out-coming wave.

Lemma 6.10. $\mu_0^\pm = 0$ implies that $\mu_0^\mp \mathbf{1}_{\xi \neq 0} = 0$. Consequently, $\mu^\mp \mathbf{1}_{\xi \neq 0} = 0$.

Proof. By symmetry, we only need to deduce $\mu_0^- \mathbf{1}_{\xi \neq 0} = 0$ from $\mu_0^+ = 0$. For $\delta > 0$, we define

$$b_{0,\delta}(x, \xi) = b_0(x, \xi) \left(1 - \tilde{\psi} \left(\frac{\lambda(0, x, \xi)}{\delta} \right) \right),$$

with some $\tilde{\psi} \in C_c^\infty(\mathbb{R})$, $\tilde{\psi}|_{[-2,2]} \equiv 1$. We define $b_\delta^\pm(y, x, \xi)$ by solving ODE (6.1) with initial data $b_{0,\delta}$. Let $B_{\delta,h}^\pm$ be the associated semi-classical PdO of b_δ^\pm . From compactness and continuous dependence of the initial data, we have that $\delta < \lambda(y, x, \xi) < c_0 < 1$ on $\text{supp}(b_\delta(y))$ for $0 \leq y \leq \epsilon_0$, since on $Y_+ \times \mathbb{R}^{d-1}$, $\frac{\lambda(y, x, \xi)}{|\xi|} \sim 1$. Note that the solutions of the transport equation (6.1) are given by

$$b^\pm(y, x, \xi) = b_0 \circ \gamma^\pm(y)^{-1}(x, \xi), \quad b_\delta^\pm(y, x, \xi) = b_{0,\delta} \circ \gamma^\pm(y)^{-1}(x, \xi),$$

we have that $\frac{b_\delta^\pm}{b_\delta^\pm}$ is a smooth function with compact support in $Y_+ \times \mathbb{R}^{d-1}$. Denote by $w_\delta^\pm = B_{\delta,h}^\pm(hD_y - Q_h^\mp)w$, and μ_δ^\pm its semi-classical defect measure, we have $\mu_\delta^\pm = \mu^\pm \left(\frac{b_\delta^\pm}{b_\delta^\pm} \right)^2$. In particular, $\mu_{\delta,0}^\pm = \mu_0^\pm \left(1 - \tilde{\psi} \left(\frac{\lambda_0}{\delta} \right) \right)^2$ and $\text{supp}(\mu_\delta^\pm) \subset \text{supp}(\mu^\pm)$. On the boundary, $B_{\delta,h}^+$ and $B_{\delta,h}^-$ coincide and will be denoted by $B_{\delta,h}^0$. Taking the trace of w_δ^\pm , we have

$$\begin{cases} w_{\delta,0,\parallel}^+ = -iB_{\delta,h}^0 v + ih^2 B_{\delta,h}^0 \partial_y(\nabla q)|_{y=0} + B_{\delta,h}^0 Q_h^+ h \nabla_\parallel q_0, \\ w_{\delta,0,\perp}^+ = iB_{\delta,h}^0 h^2 \partial_y^2 q|_{y=0} + B_{\delta,h}^0 Q_h^+ h \partial_y q|_{y=0}, \end{cases}$$

where $v = h \partial_y u|_{y=0} = O_{L_x^2}(1)$. Similarly, we have

$$\begin{cases} w_{\delta,0,\parallel}^- = -iB_{\delta,h}^0 v + ih^2 B_{\delta,h}^0 \partial_y(\nabla q)|_{y=0} + B_{\delta,h}^0 Q_h^- h \nabla_\parallel q_0, \\ w_{\delta,0,\perp}^- = iB_{\delta,h}^0 h^2 \partial_y^2 q|_{y=0} + B_{\delta,h}^0 Q_h^- h \partial_y q|_{y=0}. \end{cases}$$

Notice that $\sigma(Q_h^+) = -\sigma(Q_h^-)$, we write $\alpha = -B_{\delta,h}^0 h^2 \Delta_0 q_0$, $\beta = B_{\delta,h}^0 Q_h^+ h \mathcal{N} q_0$, hence

$$w_{\delta,0,\perp}^\pm = i\alpha \mp \beta + O_{L_x^2}(h).$$

From the assumption $\|w_{\delta,0,\perp}^\pm\|_{L^2} = o(1)$, we have that $\|i\alpha - \beta\|_{L^2}^2 = o(1)$, and this implies that $\|\alpha\|^2 + \|\beta\|^2 - 2\text{Im}(\alpha|\beta) = o(1)$. We claim that $\text{Im}(\alpha|\beta) = o(1)$.

Indeed, from Proposition 6.5 and the ellipticity of the Dirichlet-Neumann operator \mathcal{N} , we have that $q_0 = O_{L^2(\mathbb{R}_x^{d-1})}(1)$, micro-locally away from $\xi = 0$. Now from the trace theorem and Proposition 3.8, we have

$$\beta = A_{\delta,h} q_0 + O_{L^2(\mathbb{R}_x^{d-1})}(h^{1/3})$$

for some PdO with real-valued principal symbol a_δ , compactly supported and vanishing when $\lambda(y, x, \xi) \leq \delta/4$. Similarly,

$$\alpha = A'_{\delta,h} q_0 + o_{L^2(\mathbb{R}_x^{d-1})}(1)$$

for some PdO with real-valued principal symbol a'_δ . Thus $\text{Im}(\alpha|\beta)_{L^2} = o(1)$, since all the principal symbols involved in the inner product are real-valued. Now from $\|\alpha\|_{L^2} = o(1)$, $\|\beta\|_{L^2} = o(1)$, one deduce that the terms on the righthand side of $w_{\delta,0,\parallel}^\pm$ involving pressure are also $o_{L_x^2}(1)$, and $v = o_{L_x^2}(1)$ follows since $w_{\delta,0,\parallel}^- = o_{L^2(\mathbb{R}^{d-1})}(1)$. Therefore $\mu_{\delta,0}^- = 0$ and consequently $\mu_\delta^- = 0$ from Lemma 6.8. This implies that $\mu_0^- \mathbf{1}_{\lambda > \delta} = \mu^- \mathbf{1}_{\lambda > \delta} = 0$. Since $\delta > 0$ is arbitrary, we have that $b^- \mu^- \mathbf{1}_{\xi \neq 0} = 0$. Moreover, Corollary 6.6 implies that $\mu \mathbf{1}_{\xi \neq 0} = 0$. This completes the proof of Lemma 6.10. \blacksquare

Now we finish the proof of Proposition 6.7 by showing the following lemma.

Lemma 6.11. $\mu^+ = 0$ implies that $\mu^- = 0$.

Proof. We only need deal with $\xi = 0$. Take $\tilde{\psi}$ to be a cut-off function which equals to 1 near the origin. Pick any $\epsilon > 0$, we define the operator

$$B_h^{\epsilon, \pm} = \text{Op}_h(\tilde{\psi}(\lambda(y, x, \xi)/\epsilon))B_h^{\pm}.$$

Applying divergence equation for w^{\pm}

$$B_h^{\epsilon, \pm} h \text{div}_{\parallel} w_{\parallel}^{\pm} + B_h^{\epsilon, \pm} h \partial_y w_{\perp}^{\pm} = O_{L^2(\mathbb{R}_+^d)}(h),$$

we have

$$\|B_h^{\epsilon, \pm} h \partial_y w_{\perp}^{\pm}\|_{L^2(\mathbb{R}_+^d)} \leq \|B_h^{\epsilon, \pm} h \text{div}_{\parallel} w_{\parallel}^{\pm}\|_{L^2(\mathbb{R}_+^d)} + R_{\epsilon}(h)$$

with $R_{\epsilon}(h) \rightarrow 0$, as $h \rightarrow 0$ for each fixed $\epsilon > 0$. By estimating the operator norm from its symbol, we have

$$\|B_h^{\epsilon, \pm} h \partial_y w_{\perp}^{\pm}\|_{L^2(\mathbb{R}_+^d)} \leq C\epsilon + R_{\epsilon}(h),$$

and

$$\limsup_{h \rightarrow 0^+} \|B_h^{\epsilon, \pm} h \partial_y w_{\perp}^{\pm}\|_{L^2(\mathbb{R}_+^d)} \leq C\epsilon.$$

Using the equation $hD_y w_{\perp}^{\pm} = Q_h^{\pm} w_{\perp}^{\pm} + O_{L^2(\mathbb{R}_+^d)}(h)$, we have

$$\limsup_{h \rightarrow 0^+} \|B_h^{\epsilon, \pm} Q_h^{\pm} w_{\perp}^{\pm}\|_{L^2(\mathbb{R}_+^d)} \leq C\epsilon.$$

Finally let $\epsilon \rightarrow 0$, we have $\mu_{\perp}^{\pm} \mathbf{1}_{\xi=0} = 0$. Therefore $\mu_{\perp}^- = 0$. As a consequence of the proof of Lemma 6.8, $\mu_{0, \perp}^{\pm} \mathbf{1}_{\xi=0} = 0$. Now let $\mu_{\alpha}, \mu_{\beta}$ be the defect measures of $\alpha = -B_h^0 h^2 \Delta_0 q_0, \beta = B_h^0 Q_h h \mathcal{N} q_0$, and let $\mu_{i\alpha \pm \beta}$ be the defect measure of $i\alpha \pm \beta$. Denote also by $\mu_{\alpha\beta}$ the limit corresponding to the quadratic form $(A_h \alpha | \beta)$. Similarly for $\mu_{\beta\alpha}$. Note that $\mu_{\alpha\beta} = \overline{\mu_{\beta\alpha}}$. From

$$\begin{aligned} \langle \mu_{i\alpha + \beta}, \mathbf{1}_{\xi=0} \rangle &= \langle \mu_{\alpha}, \mathbf{1}_{\xi=0} \rangle + \langle \mu_{\beta}, \mathbf{1}_{\xi=0} \rangle - \langle 2 \text{Im} \mu_{\alpha\beta}, \mathbf{1}_{\xi=0} \rangle = 0 \\ \langle \mu_{i\alpha + \beta}, \mathbf{1}_{\xi=0} \rangle &= \langle \mu_{\alpha}, \mathbf{1}_{\xi=0} \rangle + \langle \mu_{\beta}, \mathbf{1}_{\xi=0} \rangle + \langle 2 \text{Im} \mu_{\alpha\beta}, \mathbf{1}_{\xi=0} \rangle = 0, \end{aligned}$$

we have that $\mu_{\alpha} \mathbf{1}_{\xi=0} = \mu_{\beta} \mathbf{1}_{\xi=0} = 0$.

Next we consider parallel components. The key claim is that the measure corresponding to $B_h^0 Q_h^{\pm} h \nabla_{\parallel} q_0$ vanishes on the set $\{\xi = 0\}$. Indeed, from Lemma 3.3 and the trace theorem, $h q_0 \rightarrow 0$ strongly in $L^2(\partial\Omega)$. From the ellipticity of \mathcal{N} , there exists a classical pseudo-differential operator E of order -1 such that $EN = I + R$, where R is a classical smoothing operator. Our goal is to show that

$$\lim_{\epsilon \rightarrow 0} \limsup_{h \rightarrow 0} \|B_h^0 B_h^{\epsilon, 0} Q_h^{\pm} h \nabla_{\parallel} q_0\|_{L^2(\mathbb{R}^{d-1})} = 0.$$

From symbolic calculus and the strong convergence of $h q_0$ in $L^2(\mathbb{R}_x^{d-1})$, it suffices to prove

$$\lim_{\epsilon \rightarrow 0} \limsup_{h \rightarrow 0} \|h \nabla_{\parallel} B_h^0 B_h^{\epsilon, 0} Q_h^{\pm} q_0\|_{L^2(\mathbb{R}^{d-1})} = 0. \quad (6.2)$$

We write

$$\begin{aligned} h \nabla_{\parallel} B_h^0 B_h^{\epsilon, 0} Q_h^{\pm} q_0 &= h \nabla_{\parallel} B_h^0 B_h^{\epsilon, 0} Q_h^{\pm} EN q_0 - h \nabla_{\parallel} B_h^0 B_h^{\epsilon, 0} Q_h^{\pm} R q_0 \\ &= \nabla_{\parallel} E B_h^0 B_h^{\epsilon, 0} Q_h^{\pm} h \mathcal{N} q_0 + h \nabla_{\parallel} [B_h^0 B_h^{\epsilon, 0} Q_h^{\pm}, E] \mathcal{N} q_0 \\ &\quad - h \nabla_{\parallel} B_h^0 B_h^{\epsilon, 0} Q_h^{\pm} R q_0. \end{aligned} \quad (6.3)$$

Here we are taking the commutator between a semi-classical PdO and a classical PdO, hence the semi-classical symbolic calculus is not applicable. Yet, it is not difficult to check that for any $a \in C_c^\infty(T^*\partial\Omega)$, $E \in S_{x,\xi}^{-1}$,

$$[a(x, hD_x), E(x, D_x)] = h\text{Op}(S^{-1}) + \text{Op}(S^{-2}),$$

where the implicit constants only depend on the semi-norms of the symbols $a(x, \xi)$ and $E(x, \xi)$. Notice that $h\nabla_{\parallel} B_h^0, B_h^{\epsilon, 0}, Q_h^{\pm}$ are uniformly bounded operators in L_x^2 with respect to h , thus $\nabla_{\parallel} B_h^0 B_h^{\epsilon, 0} Q_h^{\pm} R$, $\nabla_{\parallel} B_h^0 \text{Op}(S^{-2}) \mathcal{N}$, $h\nabla_{\parallel} B_h^0 \text{Op}(S^{-1}) \mathcal{N}$ are uniformly bounded operators in L_x^2 with respect to h . Thus from the strong convergence of hq_0 , the last two terms on the right hand side of (6.3) are killed when we let $h \rightarrow 0$ first. Thus (6.2) follows from the vanishing of the measure of $\pm\beta = B_h^0 Q_h^{\pm} h\mathcal{N}q_0$ on the set $\{\xi = 0\}$. Combining the assumption that $\mu_{0,\parallel}^+ \mathbf{1}_{\xi=0} = 0$, we deduce that $\mu_{0,\parallel}^- \mathbf{1}_{\xi=0} = 0$. The proof of Proposition 6.7 is now complete. \blacksquare

7. NEAR $\mathcal{G}^{2,+}$

In this section, we follow the strategy of V. Ivrii (see [9] or [8]) to prove Proposition 4.4. Denote by $G = \det(\bar{g})$ and $P_h = h^2 \Delta_H - 1$, we have

Lemma 7.1. *In local coordinate Y_+ , we have*

$$P_h = -h^2 \frac{\bar{g}}{\sqrt{G}} \partial_y (\sqrt{G} \bar{g}^{-1} \partial_y) + R_h = h^2 D_y^2 + \text{Op}_h(r) + O_{L^2 \rightarrow L^2}(h),$$

where R_h is a matrix-valued second order differential operator in x with scalar principal symbol $r(y, x, \xi) = 1 - \lambda(y, x, \xi)^2$, which is self-ajoint with respect to the $(\cdot, \cdot)_{L^2(Y_+)}$.

The proof will be given in the appendix.

To simplify the notations, in a fix local coordinate in Y_+ , we will identify $u = \varphi_1 u$, $q = \varphi_1 q$ and all the operators B by $\varphi B \varphi_1$.

Proposition 7.2. *For any tangential operator B with scalar principal symbol $b(y, x, \xi)$ vanishing near $\xi = 0$, we have*

$$\limsup_{h \rightarrow 0} \|BhD_y u\|_{L^2(Y_+)} \leq \sup_{\rho \in \text{supp}(b)} |r(\rho)|^{1/2} |b(\rho)|.$$

Proof. We calculate

$$\begin{aligned} (BhD_y u | BhD_y u)_{Y_+} &= ([B, hD_y]u | BhD_y u)_{Y_+} + (hD_y B u | BhD_y u)_{Y_+} \\ &= O(h) + (B u | Bh^2 D_y^2 u)_{L^2(Y_+)} \\ &= O(h) - (B u | BRu)_{L^2(Y_+)} + (B u | BP_h u)_{Y_+} \\ &= O(h) - (B u | BRu)_{Y_+} - (B u | Bh d q)_{Y_+}. \end{aligned}$$

Integrating by part and using symbolic calculus, we have

$$\begin{aligned} (B u | Bh d q)_{Y_+} &= (B u | h d B q)_{Y_+} + (B u | [B, h d] q)_{Y_+} \\ &= -([h d^*, B] u | B q)_{Y_+} + (B u | [B, h d] q)_{Y_+} \\ &= O(h), \end{aligned}$$

thanks to the fact that B has scalar-valued principal symbol. \blacksquare

The proof of Proposition 4.4 is based on the following integration by part result.

Proposition 7.3. *Given real scalar-valued tangential symbols a_0, a_1 , there exist tangential operators A_0, A_1 (constructed in the local coordinate) with real, scalar-valued principal symbol a_0, a_1 and $A = A_1 hD_y + A_0$, such that for any 1-form w with compact support in Y_+ , we have*

$$\frac{2}{h} \operatorname{Im}(P_h w | Aw)_{Y_+} = (A_1 hD_y w | hD_y w)_{\partial Y_+} + \operatorname{Re} \sum_{j=0}^2 (C_j (hD_y)^j w | w)_{Y_+} + O(h),$$

where the tangential operators C_j have scalar-valued principal symbol $c_j(y, x, \xi)$ and

$$\sum_{j=0}^2 c_j(y, x, \xi) \eta^j = \{p, a\}.$$

Proof. We first calculate

$$\begin{aligned} I &= \frac{1}{ih} \left(-\frac{h^2}{\sqrt{G}} \bar{g} h \partial_y (\sqrt{G} \bar{g}^{-1} h \partial_y w) | Aw \right)_{Y_+} - \frac{1}{ih} \left(Aw | -\frac{h^2}{\sqrt{G}} \bar{g} h \partial_y (\sqrt{G} \bar{g}^{-1} h \partial_y w) \right)_{Y_+} \\ &= (hD_y w | A_1 hD_y w)_{\partial Y_+} + (A_1 hD_y w | hD_y w)_{\partial Y_+} \\ &\quad + \frac{1}{ih} (hD_y w | hD_y A w)_{Y_+} - \frac{1}{ih} (hD_y A w | hD_y w)_{Y_+} \\ &= (hD_y w | A_1 hD_y w)_{\partial Y_+} + (A_1 hD_y w | hD_y w)_{\partial Y_+} \\ &\quad + \frac{1}{ih} (hD_y w | [hD_y, A] w)_{Y_+} - \frac{1}{ih} ([hD_y, A] w | hD_y w)_{Y_+} \\ &\quad + \frac{1}{ih} (hD_y w | A hD_y w)_{Y_+} - \frac{1}{ih} (A hD_y w | hD_y w)_{Y_+}, \end{aligned}$$

and the last two terms on the right hand side equal to

$$\frac{1}{ih} (A^* hD_y w | hD_y w) - (A_1^* hD_y w | hD_y w)_{\partial Y_+} - \frac{1}{ih} (A hD_y w | hD_y w)_{Y_+}.$$

We want to construct operators A_0, A_1 such that $A_1^* = A_1 + O(h^2)$ and $A^* = A + O(h^2)$. Assume that

$$\tilde{a}_1 \simeq a_1^{(0)} + \frac{h}{i} a_1^{(1)}$$

with real-valued $a_1^{(j)}$ (not necessarily scalar-valued). From

$$\int_{Y_+} \langle A_1 u | v \rangle_{\mathbb{R}^{d-1}} \sqrt{G} dy dx = \int_{Y_+} \langle \bar{g}^{-1} A_1 u, v \rangle_{\mathbb{R}^{d-1}} \sqrt{G} dy dx,$$

the symbol of A_1^* is equal to the symbol of $\frac{\bar{g}}{\sqrt{G}} \operatorname{Op}_h(\tilde{a}_1^*) \sqrt{G} \bar{g}^{-1}$, which can be expressed by

$$b_1(y, x, \xi) \simeq \sum_{k \geq 0} \left(\frac{h}{i} \right)^k b_1^{(k)}(y, x, \xi),$$

with

$$b_1^{(k)}(y, x, \xi) = \sum_{j=0}^1 \sum_{|\alpha|+|\beta|+j=k} (-1)^j \partial_\xi^\beta \left(\frac{\bar{g}}{\sqrt{G}} \partial_\xi^\alpha \partial_x^\alpha a_1^{(j)} \right) \cdot \partial_x^\beta (\sqrt{G} \bar{g}^{-1}), \quad k \geq 1.$$

We have that

$$b_1^{(0)} = a_1^{(0)}, \quad b_1^{(1)} = -a_1^{(1)} + \sum_{|\alpha|+|\beta|=1} \partial_\xi^\beta \left(\frac{\bar{g}}{\sqrt{G}} \partial_\xi^\alpha \partial_x^\alpha a_1^{(0)} \right) \cdot \partial_x^\beta (\sqrt{G} \bar{g}^{-1})$$

We set

$$a_1^{(0)} = a_1, \quad a_1^{(1)} = \frac{1}{2} \sum_{|\alpha|+|\beta|=1} \partial_\xi^\beta \left(\frac{\bar{g}}{\sqrt{G}} \partial_\xi^\alpha \partial_x^\alpha a_1^{(0)} \right) \cdot \partial_x^\beta (\sqrt{G\bar{g}^{-1}}),$$

thus $A_1^* = A_1 + O(h^2)$. Note that $a_1^{(1)}$ is matrix-valued while $a_1^{(0)}$ is real and scalar-valued.

The construction of A_0 is similar. We observe that $(hD_y)^* = hD_y + h\frac{g}{\sqrt{G}}D_y(\sqrt{G\bar{g}^{-1}})$ and set

$$\tilde{a}_0 = a_0^{(0)} + \frac{h}{i}a_0^{(1)}.$$

A_0^* has symbol which can be expanded as

$$b_0 \simeq \sum_{k \geq 0} \left(\frac{h}{i}\right)^k b_0^{(k)}$$

with $b_0^{(0)} = a_0^{(0)}$ and

$$b_0^{(k)}(y, x, \xi) = \sum_{j=0}^1 \sum_{|\alpha|+|\beta|+j=k} (-1)^j \partial_\xi^\beta \left(\frac{\bar{g}}{\sqrt{G}} \partial_\xi^\alpha \partial_x^\alpha a_0^{(j)} \right) \cdot \partial_x^\beta (\sqrt{G\bar{g}^{-1}}), \quad k \geq 1. \quad (7.1)$$

Note that

$$\begin{aligned} (hD_y)^* A_1^* - A_1^* hD_y &= [(hD_y)^*, A_1^*] + A_1^* (hD_y)^* - A_1^* hD_y \\ &= \frac{h}{i} (\partial_y A_1^*) + \frac{h}{i} \left[\frac{\bar{g}}{\sqrt{G}} \partial_y (\sqrt{G\bar{g}^{-1}}), A_1^* \right] + \frac{h}{i} A_1^* \frac{\bar{g}}{\sqrt{G}} \partial_y (\sqrt{G\bar{g}^{-1}}), \end{aligned}$$

and its symbol can be expanded as

$$\sum_{k \geq 0} \left(\frac{h}{i}\right)^k \kappa_k(y, x, \xi)$$

with $\kappa_0 = 0$ and

$$\begin{aligned} \kappa_1 &= \partial_y b_1 + b_1 \frac{\bar{g}}{\sqrt{G}} \partial_y (\sqrt{G\bar{g}^{-1}}), \\ \kappa_k &= \sum_{|\alpha|=k-1} \frac{1}{i^{|\alpha|+1}} \{\partial_\xi^\alpha, \partial_x^\alpha\} \left(\frac{\bar{g}}{\sqrt{G}} \partial_y (\sqrt{G\bar{g}^{-1}}), b_1 \right) \\ &\quad + \frac{h}{i} (\partial_y b_1 + b_1 \frac{\bar{g}}{\sqrt{G}} \partial_y (\sqrt{G\bar{g}^{-1}})) + \frac{h}{i} \sum_{|\alpha| \geq 1} \frac{h^{|\alpha|}}{i^{|\alpha|}} \{\partial_\xi^\alpha, \partial_x^\alpha\} \left(\frac{\bar{g}}{\sqrt{G}} \partial_y (\sqrt{G\bar{g}^{-1}}), b_1 \right), \quad k \geq 2, \end{aligned}$$

where

$$\{\partial_\xi^\alpha, \partial_x^\alpha\}(f_1, f_2) = \partial_\xi^\alpha f_1 \partial_x^\alpha f_2 - \partial_\xi^\alpha f_2 \partial_x^\alpha f_1.$$

We set $b_0^{(0)} = a_0$ and $a_0^{(1)}$ such that $a_0^{(1)} = b_0^{(1)} + \kappa_1$ (it has a solution thanks to (7.1)). Finally, we construct A_j by $\varphi_1 \text{Op}_h(\tilde{a}_j) \varphi_1$ in local coordinate and it can be easily verified that

$$A_1^* = A_1 + O_{L^2 \rightarrow L^2}(h^2), \quad A^* = A + O_{L^2 \rightarrow L^2}(h^2).$$

Therefore

$$I = (hD_y w | A_1 hD_y w)_{\underline{\partial}Y_+} + \frac{1}{ih} (hD_y w | [hD_y, A] w)_{Y_+} - \frac{1}{ih} ([hD_y, A] w | hD_y w)_{Y_+} + O(h).$$

We next calculate

$$\begin{aligned} \frac{1}{i\hbar}(R_h w|Aw)_{Y_+} - \frac{1}{i\hbar}(Aw|R_h w)_{Y_+} &= \frac{1}{i\hbar}((A^* R_h - R_h^* A)w|w)_{Y_+} \\ &= \frac{1}{i\hbar}([A, R_h]w|w)_{Y_+} + O(\hbar), \end{aligned}$$

since R_h is self-adjoint and $A^* - A = O_{L^2 \rightarrow L^2}(\hbar^2)$. Moreover, the principal symbol of $\frac{1}{i\hbar}[A, R_h]$ is $\{r, a\}$. This completes the proof of Proposition 7.3. \blacksquare

Now assume that we are working near a diffractive point $\rho \in \mathcal{G}^{2,+}$ in Y_+ where

$$\partial_y r \geq c_0 > 0$$

The following lemma is a semi-classical version of Lemma 24.4.5 in [8]. The proof is slightly more complicated, due to the different equation that we are considering.

Lemma 7.4. *Let $B_j = \varphi B_j \varphi_1$, with real, scalar-valued tangential principal symbols $b_j, j = 0, 1, 2$, compactly supported and*

$$\sum_{j=0}^2 b_j(y, x, \xi) \eta^j = -\psi(y, x, \eta, \xi)^2 \text{ when } \eta^2 = r(y, x, \xi),$$

with some smooth function $\psi \in C_c^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$. Assume that

$$dr \neq 0, \partial_y r > 0, \text{ on } \{y = r = 0\} \cap \bigcup_{j=1}^2 \text{supp}(b_j).$$

Then one can choose compactly supported, tangential operators $\Psi_j, j = 0, 1$ with real, scalar-valued principal symbols $\psi_j, j = 0, 1$, satisfying

$$\psi_0(y, x, \xi) = \psi(y, x, 0, \xi), \psi_1(y, x, \xi) = \partial_\eta \psi(y, x, 0, \xi) \text{ when } \eta = r(y, x, \xi) = 0,$$

so that for any solution u of $P_h u = f - h\nabla q, h\text{div} u = 0$ with $u|_{y=0} = 0$, we have

$$\begin{aligned} &\text{Re} \sum_{j=0}^2 (B_j (hD_y)^j v|v)_{Y_+} + \|\Psi_0 v + \Psi_1 hD_y v\|_{L^2(Y_+)}^2 + (\Theta P_h v|v)_{Y_+} \\ &= o(1) \end{aligned} \quad (7.2)$$

as $h \rightarrow 0$, where $v = \varphi \text{Op}_h(\chi) \varphi_1 u$ and $\chi \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$ has support near $\rho \in \mathcal{G}^{2,+}$. Θ is a tangential operator, depending on ψ_j, b_j whose principal symbols are scalar-valued.

The proof is based on the following elementary lemma, for which the proof can be found as Lemma 24.4.3 in [8],

Lemma 7.5. *Let X be an open subset of $\overline{\mathbb{R}_+^n} = \{x \in \mathbb{R}^n : x_1 \geq 0\}$, and let $r \in C^\infty(X)$. Assume that r is real-valued, that $dr \neq 0$ when $r = 0$ and that $\frac{\partial r}{\partial x_1} > 0$ when $r = x_1 = \frac{\partial r}{\partial x_j} = 0$ for $j \neq 1$. Let*

$$F(t, x) = \sum_{j=0}^2 f_j(x) t^j$$

be a quadratic polynomial in t with coefficients in $C^\infty(X)$ such that

$$F(t, x) = -\psi(t, x)^2 \text{ when } t^2 = r(x),$$

where $\psi \in C^\infty(\mathbb{R} \times X)$. Then one can find $\psi_0, \psi_1, \theta \in C^\infty(X)$ such that $\psi_0(x) = \psi(0, x)$, $\psi_1(x) = \frac{\partial \psi}{\partial t}(0, x)$ when $r(x) = 0$, and

$$F(t, x) + (\psi_0(x) + t\psi_1(x))^2 \leq \theta(x)(t^2 - r(x)), \quad \forall t \in \mathbb{R}, x \in X.$$

Proof of Lemma 7.4. Choose C^∞ functions $\psi_0(y, x, \xi)$ and $\psi_1(y, x, \xi)$ as in Lemma 7.5, such that $\psi_j(y, x, \xi) = \partial_\eta^j \psi|_{y=0}$, $j = 0, 1$ when $\eta = r(y, x, \xi) = 0$ and

$$\sum_{j=0}^2 b_j \eta^j + (\psi_0 + \eta \psi_1)^2 \leq \theta(y, x, \xi)(\eta^2 - r).$$

Since ψ_0, ψ_1 and each b_j are compactly supported in variables (y, x, ξ) , we may assume that θ is smooth and with compact support. Define $\Theta = \varphi \text{Op}_h(\theta) \varphi_1$, $\Psi_j = \varphi \text{Op}_h(\psi_j) \varphi_1$, $j = 0, 1$ and consider the quantity

$$\text{Re} \sum_{j=0}^2 (B_j (hD_y)^j v|v)_{Y_+} + ((\Psi_0 + \Psi_1 hD_y)^2 v|v)_{Y_+} - (\Theta hD_y v|hD_y v)_{Y_+} + (\Theta R_h v|v)_{Y_+}.$$

The expression above can be written under the form below

$$\sum_{j=0}^2 (C_j (hD_y)^j v|v)_{Y_+},$$

where the tangential operators C_j have real, scalar-valued principle symbol. Moreover,

$$\sum_{j=0}^2 c_j(y, x, \xi) \eta^j \leq 0.$$

However, since the symbol is not bounded in η and we can not apply sharp Gårding inequality directly. To resolve this issue, we extend each c_j to $\tilde{c}_j \in C_c^m(\mathbb{R} \times \mathbb{R}^{2d-2})$ who agrees with c_j on $y \geq 0$ up to order m , any given order, of derivatives. This is possible since any order of y derivative of all the symbols has continuous limit as $y \rightarrow 0$. We still use the notation c_j in what follows. Let $\underline{v} = v \mathbf{1}_{y \geq 0}$ and we use the boundary condition $v|_{y=0} = 0$ and calculate

$$\begin{aligned} \left(\sum_{j=0}^2 C_j (hD_y)^j v|v \right)_{Y_+} &= \left(\sum_{j=0}^2 C_j (hD_y)^j \underline{v}|\underline{v} \right)_{Y_+} \\ &= \left(\psi \left(\frac{hD_y}{A} \right) \sum_{j=0}^2 C_j (hD_y)^j \underline{v}|\underline{v} \right)_{Y_+} \\ &\quad + \left(\left(1 - \psi \left(\frac{hD_y}{A} \right) \right) \sum_{j=0}^2 C_j (hD_y)^j \underline{v}|\underline{v} \right)_{Y_+} \\ &=: \text{I} + \text{II}, \end{aligned}$$

for any big number $A > 0$. Now we apply sharp Gårding inequality to the first term to get

$$I \leq C_A h,$$

with some constant C_A depending on A . For the second term, the principle symbol is supported in the elliptic region and we define

$$\Xi(y, x, \eta, \xi) := \left(1 - \psi\left(\frac{\eta}{A}\right)\right) \sum_{j=0}^2 \frac{c_j(y, x, \xi) \eta^j}{\eta^2 - r(y, x, \xi)} \in S^0(\mathbb{R}^{2d}),$$

hence we can bound

$$\begin{aligned} |\text{II}| &\leq O(h) + C(\Xi(y, x, hD_y, hD_x) \chi(y, x, hD_x) P_h \underline{v}) \left(1 - \psi\left(\frac{2hD_y}{A}\right) \underline{v}\right)_{Y_+} \\ &= O(h) + C(\Xi(y, x, hD_y, hD_x) \chi(y, x, hD_x) (hw \otimes \delta_{y=0})) \left(1 - \psi\left(\frac{2hD_y}{A}\right) \underline{v}\right)_{Y_+} \\ &\quad + C(\Xi(y, x, hD_y, hD_x) \chi(y, x, hD_x) (\mathbf{1}_{y \geq 0} h \nabla q)) \left(1 - \psi\left(\frac{2hD_y}{A}\right) \underline{v}\right)_{Y_+}, \end{aligned}$$

with $w = hD_y u|_{y=0}$. Note that to obtain the expression above, one can not use symbolic calculus to deal with commutator between semi-classical tangential symbol and the classical symbol. However, since P_h is a differential operator, we can compute its commutator with $\chi(y, x, hD_x)$ explicitly.

Now from Proposition 8.17, the limsup of the third term on the right hand side when $h \rightarrow 0$ can be bounded by $\epsilon(A)$ with $\lim_{A \rightarrow \infty} \epsilon(A) = 0$. Here we can use the flat metric to estimate the L^2 norm. The second term on the right hand side can be bounded by

$$Ch \|(1 - h^2 \Delta_{y,x})^{-\frac{s}{2}} (w \otimes \delta_{y=0})\|_{L^2(\mathbb{R}^d)} \|(1 - h^2 \Delta_{y,x})^{\frac{s}{2}} \underline{v}\|_{L^2(\mathbb{R}^d)} \leq Ch^{1-s},$$

for any $s \in (\frac{1}{2}, 1)$. Here we have used the fact that $\delta_{y=0} \in H^{-s}(\mathbb{R}_y)$ for any $s > \frac{1}{2}$ and $h^s \underline{v}$ is bounded in $H^s(\mathbb{R}^d)$ since $\underline{v}|_{y=0} = 0$ and $h \nabla_{y,x'} v$ is bounded in $L^2(\mathbb{R}^d)$. Therefore, for any $A > 0$, we have showed that

$$\limsup_{h \rightarrow 0} |\text{II}| \leq \epsilon(A),$$

and this completes the proof of Lemma 7.4. \blacksquare

Adapting to the notations in this section, Proposition 4.4 can be rephrased as follows

Proposition 7.6. *Suppose that $\rho \in \mathcal{G}^{2,+}$, and $\rho_0 \in T^*\Omega$ approaching to ρ such that $\partial_y r(\rho_0) \geq \frac{1}{2} \partial_y r(\rho) \geq c_0$. Let $\gamma_- = [\rho_0, \rho]$ be a segment of the generalized ray issued from ρ_0 to ρ (the trajectory under the canonical projection is tangent to the boundary at ρ). Then if $\rho_0 \notin \text{supp}(\mu)$, we have $\rho \notin \text{supp}(\mu)$.*

Proof. Take a small neighborhood Γ_0 of ρ_0 such that $\Gamma_0 \cap \text{supp}(\mu) = \emptyset$. Take a small neighborhood $W_0 \subset \overline{\Omega} \times \mathbb{R}^{d-1}$ such that $\frac{\partial r}{\partial y}(y, x, \xi) \geq c_0/4 > 0$. Shrinking W_0 if necessary, we assume that each point $(y, x, \xi) \in W_0$ with $r(y, x, \xi) \geq 0$ can be connected by a (possibly broken) ray issued from Γ_0 with at most one reflection or tangency at $\partial\Omega$. It suffices to prove the following statement:

For any $\chi \in C_c^\infty(\overline{\Omega} \times \mathbb{R}^{d-1})$ with $\text{supp}(\chi) \subset W_0$, small enough, we have

$$\varphi \text{Op}_h(\chi) \varphi_1 u = o_{L^2}(1), h \rightarrow 0.$$

As in [8], we construct test functions which satisfy the following properties:

Lemma 7.7. *There exists*

$$a(y, x, \eta, \xi) = a_0(y, x, \xi) + a_1(y, x, \xi)\eta, \quad a_j \in C_c^\infty(W_0)$$

with the following properties:

- (1) $a_1(0, x, \xi) = -t(x, \xi)^2$, for some $t \in C_c^\infty(T^*\partial Y_+)$,
- (2) For some large $M \geq 0$, when $p = \eta^2 - r(y, x, \xi) = 0$, we have

$$\{p, a\} + aM = -\psi(y, x, \eta, \xi)^2 + \omega(y, x, \xi)(\eta - r^{1/2}(y, x, \xi)), \quad a = s^2,$$

where $s \in C^\infty(Y_+ \times (\mathbb{R}^d \setminus \{0\}))$, $\psi \in C_c^\infty(Y_+ \times \mathbb{R}^d \setminus \{0\})$ and $\omega \in C_c^\infty(W_0)$.

Moreover, $r|_{\text{supp}(\omega)} > 0$.

The construction is exactly the same as in [8] and will be given in the appendix D for the sake of completeness.

Now we take $\chi \in C_c^\infty(W_0)$ with $\chi \equiv 1$, in a neighborhood of $\text{supp}(a_0) \cup \text{supp}(a_1)$. Let $v = \varphi \text{Op}_h(\chi) \varphi_1 u$, and we calculate

$$\begin{aligned} (P_h v | Av)_{Y_+} &= (\varphi \text{Op}_h(\chi) \varphi_1 P_h u | Av)_{Y_+} + ([P_h, \varphi \text{Op}_h(\chi) \varphi_1] u | Av)_{Y_+} \\ &= (\varphi \text{Op}_h(\chi) \varphi_1 f | Av)_{Y_+} - (\varphi \text{Op}_h(\chi) \varphi_1 h d q | Av)_{Y_+} \\ &\quad + ([P_h, \varphi \text{Op}_h(\chi) \varphi_1] u | Av)_{Y_+}. \end{aligned}$$

Here we have used the differential form to calculate the inner product. Notice that $\{p, \chi\} = 0$ on $\text{supp}(a_j)$ and $f = o_{L^2}(h)$, $h D_y u_\perp|_{y=0} = 0$, thus

$$\begin{aligned} \frac{2}{h} \text{Im}(P_h v | Av)_\Omega &= o(1) - \frac{2}{h} \text{Im}([\varphi \text{Op}_h(\chi) \varphi_1, h d] q | A \varphi \text{Op}_h(\chi) \varphi_1 u)_{Y_+} \\ &\quad + \frac{2}{h} \text{Im}(\text{Op}_h(\chi) q | h d^*(A \varphi \text{Op}_h(\chi) \varphi_1 u))_{Y_+}. \end{aligned} \quad (7.3)$$

From Proposition 7.3,

$$\begin{aligned} \sum_{j=0}^2 (C_j (h D_y)^j v | v)_{Y_+} &= -(A_1 h D_y v | h D_y v)_{\partial Y_+} - \frac{2}{h} \text{Im}([\varphi \text{Op}_h(\chi) \varphi_1, h d] q | Av)_{Y_+} \\ &\quad + \frac{2}{h} \text{Im}(\varphi \text{Op}_h(\chi) \varphi_1 q | h d^*(A \varphi \text{Op}_h(\chi) \varphi_1 u))_{Y_+} + o(1). \end{aligned} \quad (7.4)$$

Since the principal symbol of A is scalar-valued, by using $d^*u = 0$, we have

$$\frac{2}{h} \text{Im}(\varphi \text{Op}_h(\chi) \varphi_1 q | h d^*(A \varphi \text{Op}_h(\chi) \varphi_1 u))_{Y_+} = (\varphi \text{Op}_h(\chi) \varphi q | \Upsilon u)_{Y_+} + O(h)$$

and

$$-\frac{2}{h} \text{Im}([\varphi \text{Op}_h(\chi) \varphi_1, h d] q | Av)_{Y_+} = (\Upsilon_2 q | Av)_{Y_+} + O(h),$$

where $\Upsilon = \Upsilon_0 + \Upsilon_1 h D_y$, and Υ_j are matrix-valued tangential pseudo-differential operators with principal symbols supported in $\text{supp}(\chi)$. Applying Lemma 7.4 to the function

$$\sum_{j=0}^2 c_j \eta^j + aM - \omega(\eta - r^{1/2}) = -\psi^2,$$

we have

$$\begin{aligned}
& \operatorname{Re} \left(\sum_{j=0}^2 C_j (hD_y)^j u |u \right)_{Y_+} - \operatorname{Re} (\Phi (hD_y - \tilde{Q}_+) v | \varphi(y, x, hD_x) v)_{Y_+} \\
& + \operatorname{Re} (Mv | Av)_{Y_+} + (\Theta P_h v | v)_{Y_+} + \|\Psi_0 v + \Psi_1 hD_y v\|_{L^2(Y_+)}^2 \\
& \leq o(1) + Ch \|v\|_{L^2(Y_+)}^2,
\end{aligned} \tag{7.5}$$

where the compact supported tangential operator Φ has scalar-valued principal symbol $\phi \in C_c^\infty(W_0)$ and $r|_{\operatorname{supp}(\phi)} > 0$, $\phi = 1$ in a neighborhood of $\operatorname{supp} \omega$. \tilde{Q}_+ is the operator constructed in the hyperbolic region with principal symbol $r^{1/2}$. This is possible since in the proof of Lemma 7.7, we indeed have $r \geq \delta^2 |\xi|^2$ on the support of ω . Note that the principal symbol of A is positive on $\eta^2 - r = 0$, we can apply Lemma 7.4 again to the term $(Mv | Av)_\Omega$ and bound it from below by

$$o(1) - |(\Theta_1 P_h v | v)_{Y_+}|.$$

Thus we have

$$\begin{aligned}
& - (A_1 hD_y v | hD_y v)_{\underline{Y}_+} + \|\Psi_0 v + \Psi_1 hD_y v\|_{L^2(Y_+)}^2 \\
& \leq o(1) + Ch \|v\|_{L^2(Y_+)}^2 + C |(\Theta P_h v | v)_{Y_+}| + C |(\Theta_1 P_h v | v)_{Y_+}| \\
& + \left| \operatorname{Re} (\varphi \operatorname{Op}_h(\phi) \varphi_1 (hD_y - \tilde{Q}_+) v | \varphi \operatorname{Op}_h(\omega) \varphi_1 v)_{Y_+} \right| \\
& + |(\Upsilon_2 q | Av)_{Y_+}| + |(\varphi \operatorname{Op}_h(\chi) \varphi_1 q | \Upsilon u)_{Y_+}|.
\end{aligned} \tag{7.6}$$

The terms on the left hand side are essentially positive from the sharp Gårding (semi-classical, see [18]) inequality, hence we only need to control the terms on the right hand side. The term $|(\Theta P_h v | v)_\Omega| + |(\Theta_1 P_h v | v)_\Omega| = o(1)$ follows from the equation and symbolic calculus since the principal symbols of Θ and Θ_1 are scalar-valued. Next we claim that

$$\left| \operatorname{Re} (\varphi \operatorname{Op}_h(\phi) \varphi_1 (hD_y - \tilde{Q}_+) v | \varphi \operatorname{Op}_h(\omega) \varphi_1 v)_{Y_+} \right| = o(1), \quad h \rightarrow 0. \tag{7.7}$$

Indeed, micro-locally on $\operatorname{supp}(\phi)$, $r \gtrsim \delta^2 > 0$, hence in the region where $\lambda^2(y, x, \xi) < 1$, we could construct \tilde{Q}_+, \tilde{Q}_- micro-locally such that

$$P_h = (hD_y - \tilde{Q}_-)(hD_y - \tilde{Q}_+) + O(h^\infty)$$

as we have done in the hyperbolic case. From symbolic calculus and Corollary 6.6, we have

$$(hD_y - \tilde{Q}_-)(hD_y - \tilde{Q}_+)u = O_{L^2_{y,x}}(h) + h\nabla q = o_{L^2_{y,x}}(1), \quad \text{micro-locally on } \operatorname{supp}(\phi).$$

Therefore the measure μ concentrates on $\{\eta = -\sqrt{r}\} \cup \{\eta = \sqrt{r}\}$. For any point $\rho_1 \in \operatorname{supp}(\phi) \cap \operatorname{supp}(\mu)$, with $\eta(\rho_1) = -\sqrt{r(\rho_1)} < 0$, the backward generalized ray issued from ρ_1 must enter Γ_0 without meeting any point in $\mathcal{G}^{2,+}$, since along the backward flow, η is decreasing. Consequently, away from the boundary, $u = o_{L^2(Y_+)}(1)$ and hence $(hD_y - \tilde{Q}_+)u = o_{L^2(Y_+)}(1)$, micro-localized near $\eta = -\sqrt{r}$, due to the fact that $hD_y - \tilde{Q}_-$ is micro-locally elliptic near $\eta = \sqrt{r}$. Near the boundary and some point $\rho_1 \in \mathcal{H} \cap \operatorname{supp}(\phi)$, any point can be connected backwardly to Γ_0 by at most transversal reflection. Thus (7.7) holds true.

It remains to control the last two terms involving pressure. We just treat one of them, and the other can be treated in the same way. Pick $\varphi_0 \in C_c^\infty((-2, 2))$ which

is equal to 1 on $(-1, 1)$. Define

$$\chi_\epsilon(y, x, \xi) = \chi(y, x, \xi) \varphi_0(r(y, x, \xi) \epsilon^{-1}).$$

We fix any $\epsilon > 0$, small enough, and write

$$\begin{aligned} (\Upsilon_2 q | Av)_{Y_+} &= (\Upsilon_2 q | A \varphi \text{Op}_h(\chi_\epsilon) \varphi_1 u)_{Y_+} \\ &\quad + (\Upsilon_2 q | A \varphi \text{Op}_h(\chi - \chi_\epsilon) \varphi_1 u)_{Y_+} \\ &=: \mathbb{I}_{h, \epsilon} + \mathbb{II}_{h, \epsilon}. \end{aligned} \quad (7.8)$$

We first deal with $\mathbb{I}_{h, \epsilon}$. Notice that from Proposition 7.2, we have

$$\limsup_{h \rightarrow 0} \|h D_y \varphi \text{Op}_h(\chi_\epsilon) \varphi_1 u\|_{L^2(Y_+)} \leq C \epsilon^{1/2}.$$

Applying Cauchy Schwartz, we have

$$\begin{aligned} \int_0^{y_0} \|\varphi \text{Op}_h(\chi_\epsilon) \varphi_1 u\|_{L^2(\partial Y_+, \sqrt{G} dx)}^2 dy &\leq C h^{-2} \int_0^{y_0} \int \left| \int_0^y h D_y \varphi \text{Op}_h(\chi_\epsilon) \varphi_1 u(s, x) ds \right|^2 dx dy \\ &\leq \frac{C y_0^2}{h^2} \|h D_y \varphi \text{Op}_h(\chi_\epsilon \varphi_1 u)\|_{L^2_{x,y}}^2. \end{aligned}$$

By choosing $\theta \in (0, 1/2)$ and $y_0 = h \epsilon^{-\theta}$, we estimate

$$\begin{aligned} |\mathbb{I}_{h, \epsilon}| &\leq \left(\int_0^{y_0} + \int_{y_0}^{\epsilon_0} \right) |(\Upsilon_2 q | A \varphi \text{Op}_h(\chi_\epsilon) \varphi_1 u)_{L^2(\partial Y_+, \sqrt{G} dx)}| dy \\ &\leq C \frac{1}{\epsilon^{2\theta}} (\|h D_y \text{Op}_h(\chi_\epsilon) u\|_{L^2_{x,y}}^2 + O(h)) + C e^{-\frac{c}{\epsilon^\theta}}, \end{aligned}$$

where we have used Lemma 3.9. Note that Lemma 3.9 is applicable even when the micro-local cut-off χ_{δ_0} is matrix-valued. In summary we have

$$\limsup_{h \rightarrow 0} |\mathbb{I}_{h, \epsilon}| \leq C(\epsilon^{1-2\theta} + e^{-\frac{c}{\epsilon^\theta}}).$$

We now turn to the estimates of $\mathbb{II}_{h, \epsilon}$. This can be done from geometric argument. Let

$$S_\epsilon := \{(y, x, \xi) : r(y, x, \xi) \geq \epsilon, y \leq 4\epsilon/c_0\} \cap W_0.$$

We claim that for any ray γ with $\gamma(0) \in \Gamma_0$ and $\Gamma(s_0) \in S_\epsilon$, $\gamma|_{[0, s_0]} \cap \mathcal{G}^{2,+} = \emptyset$.

Indeed, by contradiction, assume that for some γ and $s_1 \in [0, s_0]$, we have $\rho_1 = \gamma(s_1) \in \mathcal{G}^{2,+}$. After time s_1 , along γ we have

$$\dot{y} = 2\eta, \dot{\eta} = \partial_y r \geq c_0/4,$$

with $y(s_1) = \eta(s_1) = 0, \eta(s_0) \geq \sqrt{\epsilon}$. This implies that $s_0 - s_1 \geq 4\sqrt{\epsilon}/c_0$ and $y(s_0) \geq c_0 T^2/4 \geq \frac{4\epsilon\delta_0}{c_0}$. The claim then follows.

Now we write

$$\begin{aligned} \mathbb{II}_\epsilon &= (\varphi_0(c_0 y/\epsilon) \Upsilon_2 q | A \varphi \text{Op}_h(\chi - \chi_\epsilon) \varphi_1 u)_{Y_+} \\ &\quad + ((1 - \varphi_0(c_0 y/\epsilon)) \Upsilon_2 q | A \varphi \text{Op}_h(\chi - \chi_\epsilon) \varphi_1 u)_{Y_+}. \end{aligned}$$

From the discussion above, the first term on the right hand side above tends to 0 as $h \rightarrow 0$ for any fixed $\epsilon > 0$, while the second term is controlled from above by

$$\int_{\frac{\epsilon\delta_0}{4C}}^{\epsilon_0} \int |C(y, x, h D_x) q|^2 dx dy$$

for some zero order semi-classical tangential operator with principal symbol $c(y, x, \xi)$ such that $\text{supp } c \cap \{\xi = 0\} = \emptyset$. Applying Lemma 3.9, we have $\limsup_{h \rightarrow 0} |\mathbb{II}_\epsilon| = 0$ is

true for any $\epsilon > 0$. Notice that the left hand side of (7.5) is independent of ϵ , we have

$$\limsup_{h \rightarrow 0} ((-A_1 h D_y v | h D_y v)_{\partial Y_+} + \|\Psi_0 v + \Psi_1 h D_y v\|_{L^2(Y_+)}^2) = 0.$$

From the construction of a_0, a_1 and the corresponding expression of ψ_0, ψ_1 , we can choose another different \tilde{a}_0, \tilde{a}_1 , such that the function $\tilde{\psi}_0 + \tilde{\psi}_1 \eta$ is independent of $\psi_0 + \psi_1 \eta$ on $\text{supp}(\chi)$ (see appendix D). It follows then

$$\|v\|_{L^2(Y_+)} + \|h D_y v\|_{L^2(Y_+)} = o(1), \quad h \rightarrow 0,$$

and this completes the proof of Proposition 7.6. \blacksquare

8. NEAR $\mathcal{G}^{2,-}$ AND \mathcal{G}^k FOR $k \geq 3$

This section is devoted to the proof of Proposition 4.6. Before proving it, we need some preparation. In what follows, we take tangential operators

$$A = \varphi \text{Op}_h(a) \varphi_1, \quad A^* = A + O_{L^2(\partial Y_+)}(h^2).$$

Proposition 8.1.

$$\frac{1}{h} (([P, A]u|u)_{Y_+} = \frac{1}{h} (Au|Pu)_{Y_+} - \frac{1}{h} (APu|u)_{Y_+} + O(h).$$

Proof. The proof goes in exactly the same way and much simpler than the diffractive case, and we omit it here. \blacksquare

Recall that $r_0 = r|_{y=0}$ and $r_1 = \partial_y r|_{y=0}$. Direct calculation gives

$$H_p a = 2\eta \frac{\partial a}{\partial y} + \frac{\partial r}{\partial y} \frac{\partial a}{\partial \eta} + H_{-r} a.$$

Pick $\rho_0 \in \mathcal{G}^{2,-} \subset T^* \partial \Omega \setminus \{0\}$ and a small neighborhood $U \subset T^* \partial \Omega \setminus \{0\}$ of ρ_0 . Let $L \subset U$ be a co-dimension 1 hypersurface containing ρ_0 in $T^* \partial \Omega$ and transversal to the vector field H_{-r_0} . For small positive numbers $\delta, \tau > 0$, define

$$L^\pm(\delta, \tau; \rho_0) := \{\exp(tH_{-r_0})(\rho) \in U : \rho \in L, \text{dist}(\rho, \rho_0) \leq \delta^2, 0 \leq \pm t \leq \tau\}.$$

When there is no risk of confusion, we write it simply as $L^\pm(\delta, \tau)$. Define also

$$F^\pm(\delta, \tau) := \{(y, x, \xi) : 0 \leq y \leq \delta^2, (x, \xi) \in L^\pm(\delta, \tau)\},$$

$$F(\delta, \tau) = F^+(\delta, \tau) \cup F^-(\delta, \tau).$$

Let $C_1 > 0$ sufficiently large and $\delta_0 > 0, \tau_0 > 0$ sufficiently small so that $\delta < \delta_0, \tau < \tau_0$

$$|r(y, x, \xi)| \leq \frac{1}{2} C_1^2 \delta^2 \tag{8.1}$$

in $F(\delta, \tau)$ for all $0 < \delta \leq \delta_0, 0 < \tau \leq \tau_0$. With the same constant C_1 , we further define the sets

$$\begin{aligned} V^\pm(\delta, \tau) &:= \{(y, x, \eta, \xi) : 0 \leq y \leq \delta^2/2, (x, \xi) \in L^\pm(\delta, \tau)\} \\ &\quad \cup \{(y, x, \eta, \xi) : \delta^2/2 \leq y \leq \delta^2, (x, \xi) \in L^\pm(\delta, \tau), |\eta| \leq C_1 \delta\}, \\ W^\pm(\delta, \tau) &:= \{(y, x, \eta, \xi) : 0 \leq y \leq \delta^2/2, (x, \xi) \in L^\pm(\delta, \tau)\} \\ &\quad \cup \{(y, x, \eta, \xi) : \delta^2/2 \leq y \leq \delta^2, (x, \xi) \in L^\pm(\delta, \tau), |\eta| \leq 2C_1 \delta\}, \end{aligned}$$

$$V(\delta, \tau) := V^+(\delta, \tau) \cup V^-(\delta, \tau), \quad W(\delta, \tau) = W^+(\delta, \tau) \cup W^-(\delta, \tau).$$

We need test functions constructed in [12]:

Lemma 8.2 ([12]). *Let $I = [0, \epsilon_0)_y$. There exist $\sigma > 0, \delta_0 > 0, \tau_0 > 0$, small enough with the hierarchy $\delta_0 \ll \sigma \ll 1$, and families of smooth functions $a_\delta \in C_c^\infty(I \times U), g_\delta, h_\delta \in C^\infty(Y_+ \times \mathbb{R}_\eta \times \mathbb{R}_\xi^{d-1} \setminus \{0\})$, where $0 < \delta \leq \delta_0$, satisfying the following properties:*

- (1) $a_\delta \geq 0$, $\text{supp}(a_\delta) \subset F^+(\delta, \sigma\delta) \cup F^-(\delta, \delta^2)$.
- (2) $a_\delta(0, \exp(tH_{-r_0}(\rho_0))) \neq 0$, $\forall 0 \leq t < \delta\sigma$.
- (3) $a_\delta > 0$ on $\text{supp}(a_{\delta'})$ if $0 < \delta' < \delta \leq \delta_0$ and a_δ is independent of y for $0 \leq y < \delta^2/2$.
- (4) $g_\delta + h_\delta = -H_p a_\delta$.
- (5) in $W(\delta, \tau)$, $g_\delta \geq 0$ and $g_\delta > 0$ when $a_\delta \neq 0$.
- (6) For any $m > 1$ and any multiple index $\alpha \in \mathbb{N}^d$, $|g_\delta^{-\frac{1}{m}} \partial^\alpha g_\delta| = O_\delta(1)$, locally uniformly on $W(\delta, \tau_0)$, where the implicit constant inside $O_\delta(1)$ depends on α, m and δ .
- (7) $\text{supp}(h_\delta) \subset I \times L^-(\delta, \delta^2) \times \mathbb{R}_\eta$, and $\text{supp}(g_\delta) \cup \text{supp}(h_\delta) \subset \text{supp}(a_\delta)$, g_δ, h_δ are independent of η for $0 \leq y \leq \delta^2/2$.

For the convenience of the reader, we will recall the proof in the appendix D.

According to the lemma, we have $\partial(g_\delta^{1/2}) = 2g_\delta^{-1/2}\partial g_\delta = O(1)$, this implies that $g_\delta^{1/2} \in C^\infty(W(\delta, \tau))$. Set $b_\delta := g_\delta^{1/2} \in C^\infty(W(\delta, \tau))$. Note b_δ may not be smooth with compact support. We need split it into two parts as follows: Let $\phi_1 \in C^\infty(\mathbb{R})$ such that $\phi_1 \equiv 1$ if $0 \leq y \leq \frac{\delta^2}{4}$ and $\phi_1 \equiv 0$ if $y > \frac{3\delta^2}{8}$. Let $\phi_2 \in C^\infty(\Omega \times \mathbb{R}^d \setminus \{0\})$ with compact support in x, ξ, η variables, such that $\phi_2 \geq 0$ and $\phi_2 \equiv 0$ whenever $y \leq \frac{\delta^2}{4}$ or $|\eta| > 2C_1\delta$. Indeed, we can choose $\kappa \in C_c^\infty(\mathbb{R})$, non-negative, smooth and with compact support, such that $\kappa(z) \equiv 0$ when $|z| > 2C_1\delta$ and $\kappa(z) \equiv 1$ when $|z| \leq \frac{3}{2}C_1\delta$. Now let $\phi_2(y, x, \eta, \xi)^2 = (1 - \phi_1(y)^2)\kappa(\delta^{-1}\eta)\chi_\delta(y, x, \xi)$ with $\chi_\delta|_{\text{supp}(a_\delta)} \equiv 1$, $\text{supp}(\chi_\delta) \subset F^+(\delta, \sigma\delta) \cup F^-(\delta, \sigma\delta)$. We observe that

$$W^\pm(\delta, \tau) \cap \text{supp}(1 - \phi_1^2 - \phi_2^2) \subset \{(y, x, \eta, \xi) : \frac{\delta^2}{4} \leq y \leq \delta^2, |\eta| > \frac{3}{2}C_1\delta, \}.$$

We finally put $b_{\delta,j} := \phi_j b_\delta, j = 1, 2$. Note that $b_{\delta,1} \in C_c^\infty(F(\delta, \tau))$ is a tangential symbol (since for $y \geq \frac{\delta^2}{2}$, $\text{supp}(g_\delta) \subset \text{supp}(a_\delta)$ is compact) while $b_{\delta,2} \in C_c^\infty(W(\delta, \tau))$ is a usual interior symbol with compact support in $T^*\Omega$.

8.1. Gliding case. The propagation of support of μ near a gliding point in $\mathcal{G}^{2,-}$ can be stated as follows:

Proposition 8.3. *Suppose $\rho_0 \in \mathcal{G}^{2,-}$ and $L^+(\delta_0, \tau_0) \cup L^-(\delta_0, \tau_0) \subset \mathcal{G}^{2,-}$ for some $\delta_0, \tau_0 > 0$. Then for any $\sigma > 0$ with $\sigma\delta_0 < \tau_0$, such that if*

$$\{(y, x, \eta, \xi) : 0 \leq y \leq \delta^2, (x, \xi) \in L^-(\delta, \delta^2; \rho_0)\} \cap \text{supp}(\mu) = \emptyset$$

for some $0 < \delta \leq \delta_0$, then $\exp(tH_{-r_0})(\rho_0) \notin \text{supp}(\mu)$ for any $t \in [0, \sigma\delta)$.

We need several lemmas.

Lemma 8.4. *Suppose $a \in C_c^\infty(\mathbb{R}^{2d}), b \in C^\infty([0, 1], C_c^\infty(\mathbb{R}^{2(d-1)}))$ with the following support property:*

$$a(y, x, \eta, \xi) \equiv 0 \text{ if } y \leq c_0 < 1 \text{ or } |\eta| > C_0|\xi|.$$

Then the usual symbolic calculus for $a(y, x, hD_y, hD_x)b(y, x, hD_x)$ still valid. In particular,

$$a(y, x, hD_y, hD_x)b(y, x, hD_x) = c(y, x, hD_y, hD_x) + O_{L^2 \rightarrow L^2}(h),$$

with

$$c(y, x, \eta, \xi) = a(y, x, \eta, \xi)b(y, x, \xi)$$

We postpone the proof in the appendix F.

Lemma 8.5. *Given any $\rho_1 \in \mathcal{G}$, there exist $\delta_1 > 0, \tau_1 > 0, \sigma_1 > 0$ with $\delta_1 \ll \sigma_1$ and $\sigma_1 \delta_1 < \tau_1$ such that if $\rho \in T^*\partial\Omega$ and $\text{dist}(\rho, \rho_1) \leq \delta^2$ for some $0 < \delta \leq \delta_1$, then $\text{dist}(\gamma(s, \rho), \gamma(s, \rho_1)) \leq C\delta^2$ for $|s| \leq \sigma_1\delta$. In particular, $\gamma(s, \rho) \in W(\delta, \tau_1)$ for all $|s| \leq \sigma_1\delta$.*

Proof. Write $\gamma(s, \rho)$ and $\exp(sH_{-r_0})(\rho)$ in coordinate as

$$\gamma_1(s) = (y(s), \eta(s), x(s), \xi(s)) \text{ and } \gamma_2(s) = (\tilde{y}(s), \tilde{\eta}(s), \tilde{x}(s), \tilde{\xi}(s)).$$

From $\dot{y} = 2\eta, \dot{\eta} = O(1)$, we have $y(s) \leq Cs^2$ and the same estimate holds for $\tilde{y}(s)$. Let

$$d(s) = |x(s) - \tilde{x}(s)|^2 + |\xi(s) - \tilde{\xi}(s)|^2,$$

and then $\dot{d}(s) \leq C\sqrt{d(s)}$. This implies $d(s) \leq C_1\delta^2$ for all $|s| \leq \sigma_1\delta$. By the same argument, we have $\text{dist}(\exp(sH_{-r_0})(\rho), \exp(sH_{-r_0})(\rho_1)) \leq C\delta^2$. The conclusion then follows from the triangle inequality. \blacksquare

We will see the crucial role of $\rho_0 \in \mathcal{G}^{2,-}$ in the following lemma:

Lemma 8.6. *Assume that δ_1, τ_1 are parameters given in the previous lemma. Suppose that $-C_0 \leq \partial_y r(\rho) \leq -c_0 < 0$ for all $\rho \in W(\delta_1, \tau_1)$. Define $S_\epsilon = W(\delta_1, \tau_1) \cap \{r \geq \epsilon, y \leq \epsilon\}$ for sufficiently small $\epsilon > 0$. Then along any ray $\gamma(s, \rho_1)$ in $W(\delta_1, \tau_1)$ with $\rho_1 \in S_\epsilon$, if $y(\gamma(-t, \rho_1)) = 0$ for some $0 \leq t \leq \tau_1$, we have $r(y(\gamma(-t, \rho_1))) \geq c_1\epsilon$, where c_1 depends only on $W(\delta_1, \tau_1)$.*

Proof. Assume $\rho_1 = (y_1, x_1, \eta_1, \xi_1) \in S_\epsilon$ and $\gamma(s, \rho_1) = (y(s), x(s); \eta(s), \xi(s))$. Let $s_3 = \inf\{0 \leq s \leq \tau_1 : y(-s) = 0\}$. For $s \in [-s_3, 0]$, $\dot{y} = 2\eta, -C_0 \leq \dot{\eta} = \partial_y r \leq -c_0$. There are two possibilities. If $\eta_1 \geq \sqrt{\epsilon}$, then $\eta(-s_3^+) \geq \eta_1 > 0$ since $\dot{\eta} < 0$. Otherwise, $\eta_1 \leq -\sqrt{\epsilon}$, and we denote by $s_2 = \inf\{s \in [0, s_1] : \eta(-s) = 0\}$. From $\eta_1 = \int_{-s_2}^0 \dot{\eta} ds \geq -C_0 s_2$, we have $s_2 \geq \frac{|\eta_1|}{C_0}$. Moreover,

$$y_1 - y(-s_2) = 2\eta_1 s_2 - \int_{-s_2}^0 ds \int_s^0 \dot{\eta} ds' \leq 2\eta_1 s_2 + C_0 s_2^2 \leq -\frac{|\eta_1|^2}{C_0}.$$

Now from

$$y(-s_2) = y(-s_2) - y(-s_3) = -\int_{-s_3}^{-s_2} ds \int_s^{-s_2} \dot{\eta} ds' \leq C_0 |s_3 - s_2|^2,$$

we have $|s_3 - s_2|^2 \geq \frac{y(-s_2)}{C_0} \geq \frac{(y(-s_2) - y_1)}{C_0} \geq \frac{|\eta_1|^2}{C_0^2}$ and finally

$$\eta(-s_3^+) = -\int_{-s_3}^{-s_2} \dot{\eta} ds \geq c_0 |s_3 - s_2| \geq \frac{c_0 \sqrt{\epsilon}}{C_0}.$$

The proof of Lemma 8.6 is then complete by applying the argument above between any two adjacent zeros of $s \mapsto y(\gamma(-s, \rho_1))$. \blacksquare

Proof of Proposition 8.3. For any $\delta' > 0$, we define the operator

$$N_{\delta'} = \frac{1}{i\hbar} [P, A_{\delta'}]$$

with principal symbol $n_{\delta'} = -H_p a_{\delta'} = g_{\delta'} + h_{\delta'}$. Define operators

$$B_{\delta',j} := \text{Op}_h(b_{\delta',j}), j = 1, 2, \quad N_{\delta,3} = \text{Op}_h((1 - \phi_1^2 - \phi_2^2)n_{\delta'}).$$

Write $h_{\delta',j} = \phi_j^2 h_{\delta'}$, $H_{\delta',j} = \text{Op}_h(h_{\delta',j})$, $j = 1, 2$. The proposition will follow if we can show that for any $\delta' < \delta$,

$$\lim_{h \rightarrow 0} \sum_{j=1}^2 \|B_{\delta',j} u\|_{L^2(Y_+)}^2 = 0 \quad (8.2)$$

We remark that $h_{\delta',1}, b_{\delta',1}$ are both tangential symbols while $h_{\delta',2}, b_{\delta',2}$ are interior symbols vanishing near the boundary. Observe also that $N_{\delta',3}$ is interior pseudo-differential operator with symbol vanishing near the boundary as well as on $p^{-1}(0)$, thanks to the fact that in $W(\delta', \tau)$, $|r(y, x, \xi)| \leq \frac{1}{2} C_1^2 \delta'^2$. Thus $N_{\delta',3} u = o_{L^2(Y_+)}(1)$ as $h \rightarrow 0$ for $\delta' > 0$ small enough. Moreover, from the assumption on the support of μ near the original point ρ_0 we have $H_{\delta',j} u = o_{L^2_{y,x}}(1)$. Now set

$$M_{\delta',j} = \phi_j^2 N_{\delta',j} - B_{\delta',j}^* B_{\delta',j} - H_{\delta',j}, j = 1, 2.$$

From symbolic calculus, we deduce that $M_{\delta',1} = O_{L^2 \rightarrow L^2}(h)$ and it has the principal symbol depending only on y, x, ξ . Note that by definition of $M_{\delta',2}$, we will encounter the composition of tangential operator with interior operator $\text{Op}_h(\phi_2^2)$. Since ϕ_2 has support far away from $y = 0$ and $\eta = 0$, the symbolic calculus still valid thanks to Lemma 8.4. Therefore $M_{\delta',2} = O_{L^2 \rightarrow L^2}(h)$ is an interior operator. Finally, we obtain that

$$N_{\delta'} = N_{\delta',3} + \sum_{j=1}^2 (B_{\delta',j}^* B_{\delta',j} + H_{\delta',j}) + O_{L^2(Y_+) \rightarrow L^2(Y_+)}(h).$$

Combining all the analysis above and applying Proposition 8.1, we have

$$\begin{aligned} \sum_{j=1}^2 \|B_{\delta',j} u\|_{L^2(Y_+)}^2 &\leq o(1) + \frac{2}{h} |\text{Im}([A_{\delta'}, hd]q|u)_{Y_+}| + \frac{1}{h} |\text{Im}(q|hd^*(A_{\delta'} u))_{Y_+}| \\ &= o(1) + |(q|\Upsilon_1 u)_{Y_+}| + |(\Upsilon_2 q|u)_{Y_+}| \end{aligned} \quad (8.3)$$

where Υ_1, Υ_2 are compactly supported matrix-valued tangential operators with principal symbols vanishing outside $\text{supp}(a_{\delta'})$.

To finish the proof, we need show that the right hand side of (8.3) is $o(1)$ as $h \rightarrow 0$. Pick $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(s) \equiv 1$ if $0 \leq s \leq \frac{1}{2}$ and $\chi(s) \equiv 0$ if $s \geq 1$. Let $\chi_\epsilon(y, x, \xi) = \chi(\epsilon^{-1}r(y, x, \xi))$. Denote by

$$I_{h,\epsilon} = |(\Upsilon_1 \varphi \text{Op}_h(\chi_\epsilon) \varphi_1 u | q)_{Y_+}|, \quad \text{II}_{h,\epsilon} = |(\Upsilon_1 (1 - \varphi \text{Op}_h(\chi_\epsilon) \varphi_1) u | q)_{Y_+}|.$$

The treatment of $I_{h,\epsilon}$ is exactly the same as in the diffractive case, so we have

$$\lim_{\epsilon \rightarrow 0} \limsup_{h \rightarrow 0} I_{h,\epsilon} = 0.$$

For $\text{II}_{h,\epsilon}$, we may assume that the interval of the integration over y variable is $[0, \epsilon]$, since for $y \geq \epsilon$ we can use the rapid decreasing of q as in the treatment of $I_{h,\epsilon}$. According to Lemma 8.5 and Lemma 8.6, the measure of $\Upsilon_1(1 - \varphi \text{Op}_h(\chi_\epsilon) \varphi_1) u$ vanishes, since all the backward generalized rays starting from each point in S_ϵ

will enter the small neighborhood of $\rho_0 \in \mathcal{G}^{2,-}$ by at most reflection at boundary. From the propagation theorem in the hyperbolic case (Proposition 4.3), the proof of Proposition 8.3 is complete. \blacksquare

Remark 8.7. *We remark that as a consequence of Proposition 8.3, the measure of q (or $h\nabla q$) also vanishes along $\exp(tH_{-r_0})$ for $t \in [0, \sigma\delta]$.*

8.2. high order contact. In this subsection we will use a new coordinate system in a neighborhood \widetilde{W}_k of $\rho_k \in \mathcal{G}^k$ in $[0, \epsilon_0] \times T^*\partial\Omega$:

$$(y, \eta, x, \xi) \mapsto (y, \eta, z, \zeta), \quad z = (z_1, z'), \zeta = (\zeta_1, \zeta')$$

with $p = \eta^2 - r$, $r = \zeta_1 + yr_1(z, \zeta) + O(y^2)$, $\zeta_1 = r_0$, where $r_0 = r|_{y=0}$, $r_1 = \partial_y r|_{y=0}$. This is possible since $d_{x,\xi}r_0 \neq 0$, if $\xi \neq 0$. Along the generalized bicharacteristic curve $\gamma(s)$, (z, ζ) satisfies

$$\dot{z} = -\partial_\zeta r(y(s), z(s), \zeta(s)), \quad \dot{\zeta} = \partial_z r(y(s), z(s), \zeta(s)).$$

This implies that in \widetilde{W}_k , $-z_1 \sim 1 > 0$, as $y \rightarrow 0$, and thus $s \mapsto z_1(s)$ is strictly decreasing. Moreover, $\dot{\zeta}_1 \sim y\partial_{z_1}r_1$, as $y \rightarrow 0$.

Suppose now $k \geq 3$, we have locally that

$$\mathcal{G}^k := \{(z, \zeta) : \zeta_1 = 0, \partial_{z_1}^l r_1(z, \zeta) = 0, \forall l \leq k-3, \partial_{z_1}^{k-2} r_1(z, \zeta) \neq 0\}.$$

Define $\Sigma_k := \{(z, \zeta) : \partial_{z_1}^{k-3} r_1(z, \zeta) = 0, \partial_{z_1}^{k-2} r_1(z, \zeta) \neq 0\}$. From implicit function theorem, Σ_k is locally a hypersurface and we can write it as

$$\Sigma_k = \{(z, \zeta) : z_1 = \Theta_k(z', \zeta)\}.$$

\mathcal{G}^k can be viewed locally as a closed subset of Σ_k . Since the map $s \mapsto z_1(s)$ is bijective, we may assume that along each ray, $z_1(0) = \Theta_k(z'(0), \zeta(0))$, and

$$z_1(s) < \Theta_k(z'(s), \zeta(s)), s > 0; \quad z_1(s) > \Theta_k(z'(s), \zeta(s)), s < 0.$$

We see that all the generalized rays are transversal to the codimension 2 manifold (locally) Σ_k . Moreover, a ray passes Σ_k if and only if $y(0) = 0$ and $\zeta_1(0) = 0$. Now we define the set near ρ_k :

$$\Sigma_k^\pm := \{(y, \eta, z, \zeta) \in \text{Car}(P) \cap \widetilde{W}_k : z_1 \mp \Theta_k(z', \zeta) > 0\}.$$

Note that the gliding rays $\exp(sH_{-r_0})$ intersect transversally to Σ_k and $H_{-r_0} = -\partial_{z_1}$ inside $T^*\partial\Omega$. Thus we can re-parametrize the gliding flows by z_1 . Moreover, $\Sigma_k^\pm \cap \mathcal{G}^j = \emptyset$, $\forall j \geq k$, provided that we choose \widetilde{W}_k small enough. In other word, z_1 gives a foliation of $T^*\partial\Omega$ near Σ_k for small $|z_1 - \Theta_k(z', \zeta)|$.

The following proposition is a long time extension of Proposition 8.3, adapted to the notations introduced above.

Proposition 8.8. *Suppose $\rho_0 \in \mathcal{G}^{2,-}$ near $\rho_k \in \Sigma_k$ with coordinate (z, ζ) , $z_1 > \Theta_k(z', \zeta)$. Then there exists $\delta_0 > 0$, sufficiently small such that if*

$$\{(y, x, \eta, \xi) : 0 \leq y \leq \delta^2, (x, \xi) \in L^-(\delta, \delta^2; \rho_0)\} \cap \text{supp}(\mu) = \emptyset$$

for $0 < \delta \leq \delta_0$, then $\exp(sH_{-r_0})(\rho_0) \notin \text{supp}(\mu)$ for any $s < z_1 - \Theta_k(z', \zeta)$.

In other words, each generalized ray, issued from gliding set outside $\text{supp}(\mu)$ does not carry any singularity until it reaches some point in \mathcal{G}^k for $k \geq 3$.

Proof. The proof is purely topological. For each $\rho_0 = (z, \zeta) \notin \text{supp}(\mu)$ and $z_1 > 0$, let $s_1 := \sup\{s : s \leq z_1 - \Theta_k(z', \zeta), \exp(s'H_{-r_0}) \notin \text{supp}(\mu), \forall s' \in [0, s]\}$. The existence of s_1 is guaranteed by Proposition 8.3. It remains to show that $s_1 = z_1 - \Theta_k(z', \zeta)$. By contradiction, suppose $s_1 < z_1 - \Theta_k(z', \zeta)$, then the point $\rho_1 = (z_1 - s_1, z', \zeta)$ is in $\mathcal{G}^{2,-}$. We can apply Proposition 8.3 again to obtain that for some small $\sigma_1 > 0$, $\exp(\theta\sigma H_{-r_0})(\rho_1) \notin \text{supp}(\mu)$ for any $\theta \in [0, 1]$. This is a contradiction of the choice of s_1 . \blacksquare

As a consequence, we have

Corollary 8.9. *Suppose $\rho_0 \in \mathcal{G}^{2,-}$ and $\rho_0 \notin \text{supp}(\mu)$. Let $\gamma(s)$ be the generalized ray passing ρ_0 with $\gamma(0) = \rho_0$. Then $\gamma(s) \notin \text{supp}(\mu)$ for any $s \in [-s_0, s_0]$, provided that $\gamma|_{[-s_0, s_0]} \subset \mathcal{G}^{2,-}$.*

Combining the analysis near a diffractive point and a gliding point, we have already established the k -propagation property for $k = 2$. We will argue by induction to prove k -propagation property for all $k \geq 3$. To this end, we need an intermediate step. Let us first introduce a notation

$$\Gamma(\rho_0; \delta) := \{(y, x; z, \zeta) : 0 \leq y \leq \delta^2, (z, \zeta) \in L^-(\delta, \delta^2; \rho_0)\}$$

and a definition

Definition 8.10 (k -pre-propagation property). *For $k \geq 2$, we say that k -pre-propagation property holds, if the following statement is true:*

For any $\rho_k \in \mathcal{G}^k$, there exists a neighborhood V_k of ρ_k in $T^\partial\Omega$, and $0 < \delta_k \ll \sigma_k \ll 1$, depending on V_k , such that for any $\rho_0 \in (\mathcal{G}^{2,-} \cup \bigcup_{3 \leq j \leq k} \mathcal{G}^j) \cap V_k$,*

if $\Gamma(\rho_0; \delta) \cap \text{supp}(\mu) = \emptyset$ for some $0 < \delta < \delta_k$, then $\exp(sH_{-r_0})(\rho_0) \notin \text{supp}(\mu)$ for all $s \in [0, \sigma_k \delta]$.

The key step is the following inductive proposition.

Proposition 8.11. *Suppose $k \geq 3$ and $(k-1)$ -propagation property holds true, then k -pre-propagation property also holds true.*

We do some preparation before proving this proposition. Select a neighborhood W_k of $\rho_k \in \mathcal{G}^k$ in $T^*\partial\Omega$ (and contained in \widetilde{W}_k) with compact closure such that $|\partial_{z_1}^{k-2} r_1(\rho)| \geq c_0 > 0$ for all $\rho \in \overline{W}_k$. By abusing the notation, we will refer \mathcal{G}^k to be $\mathcal{G}^k \cap W_k$ in the sequel. According to the asymptotic behaviour of the flow $\exp(sH_{-r_0})$ as $s \rightarrow 0$, we have for any given $(z_1 = \Theta_k(z'_0, \zeta_0), z'_0, \zeta_0) \in \mathcal{G}^k$,

$$r_1 \circ \exp(sH_{-r_0})(z'_0, \zeta_0) = b_k(z'_0, \zeta_0)s^{k-2} + O(s^{k-1}),$$

where $b_k \neq 0$ can be viewed as a function of points in \mathcal{G}^k . From compactness, we can choose $\sigma > 0, \theta > 0$ depending only on \overline{W}_k such that for all $\rho \in \mathcal{G}^k$,

$$|b_k(\rho)| \geq \theta > 0, |r_1 \circ \exp(sH_{-r_0})(\rho)| \geq \frac{1}{2}|b_k s^{k-2}|, \quad \forall s \in [-\sigma, 0) \cup (0, \sigma].$$

Now we define a smaller neighborhood V_k of ρ_k such that for any $\rho_0 \in V_k$, and $\delta_k > 0, \sigma_k > 0$, $\exp(sH_{-r_0})(L^\pm(\delta_k, \delta_k^2; \rho_0)) \subset W_k$ for all $|s| \leq \sigma_k \delta_k$. Moreover, $|r_1 \circ \exp(sH_{-r_0})(\rho_0)| \leq \delta_k$. We also put $\widetilde{W}_k = [0, \delta_k^2] \times W_k$, $\widetilde{V}_k = [0, \delta_k^2] \times V_k$.

Choosing a cut-off $\tilde{a}_\delta \in C_c^\infty$ with $\tilde{a}_\delta \equiv 1$ near ρ_k , we define

$$S_{\delta, \epsilon} := \text{supp}(\tilde{a}_\delta) \cap \{y \leq \epsilon, r \geq \epsilon\}$$

for any $0 < \epsilon \ll \delta$. Note that near $S_{\delta, \epsilon}$ (thus near $\rho_k \in \mathcal{G}^k, k \geq 3$) we have $|r_1| \leq \delta_k$, and this implies that $\zeta_1 \gtrsim \epsilon$, near $S_{\delta, \epsilon}$. We divide the proof of Proposition 8.11 into several lemmas.

Lemma 8.12. *Given any generalized ray $\gamma(s) = (y(s), \eta(s), z(s), \zeta(s))$ with $\gamma(s_0) \in \Gamma(\rho_0; \delta) \cap \mathcal{G}^{2,-}$ and $\gamma(s_1) \in S_{\delta, \epsilon}$. Assume that $\gamma|_{[s_0, s_1]} \subset \text{Car}(P) \cap \widetilde{W}_k$, then $\gamma(s) \notin \mathcal{G}^k$ for all $s \in [s_0, s_1]$.*

Proof. Take $\Gamma^+(\rho_0; \delta) := \Gamma(\rho_0; \delta) \cap \Sigma_k^+$ and identify points in Σ_k^\pm as their projection to (y, x, ξ) . Let F_k (may be empty) be the union of generalized rays issued from $\Gamma^+(\rho_0; \delta)$ which meet \mathcal{G}^k . Note that along both real trajectories $\gamma(s)$ and $\exp(sH_{-r_0})$, $s \mapsto z_1$ is strictly decreasing, it suffices to show that $F_k \cap S_{\delta, \epsilon} \subset \Sigma_k^+$ since generalized rays intersect with Σ_k transversally.

We argue by contradiction. Assume that some ray in F_k satisfies $\gamma(s_0) \in \Gamma^+(\rho_0; \delta)$, $\gamma(0) \in \mathcal{G}^k$, and $\gamma(s_1) \in S_{\delta, \epsilon}$ for $s_0 < 0 < s_1$. Write $\exp(sH_{-r_0})(\gamma(0)) = (\tilde{z}(s), \tilde{\zeta}(s))$, and

$$r_1 \circ \exp(sH_{-r_0})(z'(0), \zeta(0)) = r_1(\tilde{z}(s), \tilde{\zeta}(s)) = b_k s^{k-2} + O(s^{k-1}), s \rightarrow 0,$$

More precisely, we have

$$|b_k(z'(0), \zeta(0))| \geq \theta > 0, |r_1(\tilde{z}(s), \tilde{\zeta}(s))| \geq \frac{1}{2}|b_k s^{k-2}|, \forall s \in [-\sigma, 0) \cup (0, \sigma].$$

After shrinking support of a_δ if necessary, we may assume that $s_1 < \sigma$. According to the parity of k and the sign of b_k , there are several situations.

If $b_k < 0$, then for any k we have $\gamma(s) \in \mathcal{G}^{2,-}$ for all $s \in (0, \sigma)$. This is impossible since $r \circ \gamma(s_1) \geq \epsilon$. Otherwise $b_k > 0$, in this case we have $r_1(\tilde{z}(s), \tilde{\zeta}(s)) \geq b_k s^{k-2}/2$, for all $s \in (0, \sigma)$, and

$$\begin{aligned} & (\partial_{z_1} r_1)(\tilde{z}(s), \tilde{\zeta}(s)) \\ &= (\partial_{z_1} r_1) \circ \exp(sH_{-r_0})(z'(0), \zeta(0)) \\ &= -\partial_s (r_1 \circ \exp(sH_{-r_0})(z'(0), \zeta(0))) \\ &= -(k-2)b_k s^{k-3} + O(s^{k-2}) \leq 0, \forall s \in [0, \sigma), \end{aligned} \tag{8.4}$$

thanks to $\partial_{z'} r|_{y=0} = \partial_z r|_{y=0} = 0$. Taking the difference with real trajectory $\gamma(s) = (y(s), \eta(s); z(s), \zeta(s))$, we have

$$\begin{aligned} & \partial_{z_1} r_1(y(s), z(s), \zeta(s)) - \partial_{z_1} r_1(\tilde{z}(s), \tilde{\zeta}(s)) \\ &= (\partial_{z_1} r_1(0, z(s), \zeta(s)) - \partial_{z_1} r_1(\tilde{z}(s), \zeta(s))) + (\partial_{z_1} r_1(\tilde{z}(s), \zeta(s)) - \partial_{z_1} r_1(\tilde{z}(s), \tilde{\zeta}(s))) \\ &+ (\partial_{z_1} r_1(y(s), z(s), \zeta(s)) - \partial_{z_1} r_1(0, z(s), \zeta(s))). \end{aligned}$$

Using the fact that $(z(0), \zeta(0)) = (\tilde{z}(0), \tilde{\zeta}(0))$ and $y(s) = O(s^2)$, we have

$$\partial_{z_1} r_1(y(s), z(s), \zeta(s)) - \partial_{z_1} r_1(\tilde{z}(s), \tilde{\zeta}(s)) = O(s).$$

This together with (8.4) yield

$$\begin{aligned} \dot{\zeta}_1 &\leq y \partial_{z_1} r_1(y(s), z(s), \zeta(s)) + C_0 y^2 \leq C_0 (y^2 + ys), \quad \dot{y} = 2\eta, \\ \eta^2 &= \zeta_1 + y r_1(z, \zeta) + O(y^2), \quad (\zeta_1(0), y(0)) = (0, 0), \end{aligned} \tag{8.5}$$

where the constant C_0 and the implicit constant inside the big O only depends on $\text{supp}(a_\delta)$.

Applying the formula $H_p^k y(0) = 2(H_{-r_0})^{k-2} r_1 = 2b_k(k-2)! > 0$ and Taylor expansion, we have

$$\begin{aligned} y(s) &= \frac{2b_k}{k(k-1)} s^k + O(s^{k+1}) \geq \frac{b_k}{k(k-1)} s^k, s \in (0, \sigma), \\ \dot{y}(s) &= \frac{2b_k}{k-1} s^{k-1} + O(s^k) > \frac{b_k}{k-1} s^{k-1} > 0, s \in (0, \sigma). \end{aligned} \quad (8.6)$$

Injecting in (8.5), we have $\dot{\zeta}_1(s) \leq C_0(\epsilon^2 + \epsilon s)$ for all $s > 0$ small as long as $y(s) \leq \epsilon$ and $\gamma(s) \notin S_{\delta, \epsilon}$. For these s ,

$$\zeta_1(s) \leq C_0(\epsilon^2 s + \epsilon s^2/2).$$

Setting $s_2 = \inf\{0 \leq s \leq s_1 : \gamma(s) \in S_{\delta, \epsilon}\}$, we know that along the flow, $2\sqrt{\epsilon} = 2\eta(s_2) = \dot{y}(s_2)$, and this implies that $s_2 \sim \epsilon^{\frac{1}{2(k-1)}}$ since $y(s) > \epsilon$ if $s > \epsilon^{\frac{1}{2(k-1)}}$.

In summary, we have

$$\epsilon \leq r \circ \phi_{s_2} \leq 2C_0\epsilon^{1+\frac{1}{2(k-1)}} + \delta_k\epsilon + C_1\epsilon^2.$$

However, this contradicts to $r = \zeta_1 + yr_1 + O(y^2)$, provided that $\delta_k < 1$, $\epsilon \ll \delta_k < 1$. \blacksquare

Lemma 8.13. *The conclusion of Proposition 8.11 holds if $\rho_0 \in \mathcal{G}^{2,-}$*

Proof. Adapting the notations and argument in the proof of Proposition 8.3, we have

$$\sum_{j=1}^2 \|B_{\delta, j} u\|_{L^2(Y_+)}^2 \leq o(1) + \frac{2}{h} |\operatorname{Im}([A_\delta, \operatorname{hd}]q|u)_{Y_+}| + \frac{1}{h} |\operatorname{Im}(q|\operatorname{hd}^*(A_\delta u))_{Y_+}|. \quad (8.7)$$

The goal is to show that the last two terms on the right hand side tend to 0 as $h \rightarrow 0$.

We denote by $\tilde{\gamma}(s)$ the gliding ray $\exp(sH_{-r_0})$ such that $\tilde{\gamma}(s_0) = \rho_0$ for some $s_0 < 0$. Suppose $\tilde{\gamma}(0) = \rho \in \mathcal{G}^k$ for some $k \geq 3$ and $\tilde{\gamma}(s) \in \mathcal{G}^{2,-}$ for $s \in (s_0, 0)$. In view of Corollary 8.9, we may assume that ρ_0 is close enough to ρ , and $|s_0|$ is small. Pick $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(s) \equiv 1$ if $0 \leq s \leq \frac{1}{2}$ and $\chi(s) \equiv 0$ if $s \geq 1$. For any $\epsilon > 0$, let $\chi_\epsilon(y, x, \xi) = \chi(\epsilon^{-1}r(y, x, \xi))$. Let

$$I_{h, \epsilon} = \frac{2}{h} |(hd^*(\varphi \operatorname{Op}_h(\chi_\epsilon) \varphi_1 u)|q)_{Y_+}|, \quad \text{II}_{h, \epsilon} = \frac{2}{h} |(hd^*(1 - \varphi \operatorname{Op}_h(\chi_\epsilon) \varphi_1) u|q)_{Y_+}|.$$

The treatment of $I_{h, \epsilon}$ is exactly the same as in the diffractive case, we have

$$\lim_{\epsilon \rightarrow 0} \limsup_{h \rightarrow 0} I_{h, \epsilon} = 0.$$

For $\text{II}_{h, \epsilon}$, again, we only concern about the integration over $[0, \epsilon]$ in y variable. From Lemma 8.12, any ray entering $S_{\delta, \epsilon}$ can at most pass \mathcal{G}^j for $j < k$. Applying $(k-1)$ -propagation property, we deduce that for any cut-off φ_ϵ with $\operatorname{supp}(\varphi_\epsilon) \subset S_{\delta, \epsilon}$, $\operatorname{supp}(\varphi_\epsilon) \cap \operatorname{supp}(\mu) = \emptyset$. Therefore

$$\lim_{h \rightarrow 0} \text{II}_{h, \epsilon} = 0$$

for any $\epsilon > 0$. This completes the proof of Lemma 8.13. \blacksquare

Lemma 8.14. *The conclusion of Proposition 8.11 holds if $\rho_0 \in \mathcal{G}^j$ for some $3 \leq j \leq k$.*

Proof. Taking a micro-local cut-off $\psi(y, x, \xi)$ with support near ρ_0 , we have

$$\|\varphi \text{Op}_h(\psi) \varphi_1 u\|_{L^2(Y_+)} = o(1)$$

from the assumption that $\rho_0 \notin \text{supp}(\mu)$. Note that along the flow of H_{-r_0} and on $\text{supp}(1 - \psi) \cap V_k$ we have $|r_1(0, x, \xi)| \geq c(\psi, \delta) > 0$. Hence from Corollary 8.9, if $\exp(tH_{-r_0})(\rho_0) \in \mathcal{G}^{2,-}$ for all $t \in (0, \sigma\delta)$, and then $\exp(tH_{-r_0})(\rho_0) \notin \text{supp}(\mu)$. Otherwise $\exp(tH_{-r_0})(\rho_0) \in \mathcal{G}^{2,+}$ for all $t \in (0, \sigma\delta)$, we claim that we still have $\exp(tH_{-r_0})(\rho_0) \notin \text{supp}(\mu)$ from geometric consideration.

Indeed, by considering the backward generalized ray, we conclude that for any $s_0 \in (0, \sigma_k\delta)$, there exists $\rho \in \widetilde{W}_k$, so that $\gamma(s_0, \rho) = \exp(s_0 H_{-r_0})(\rho_0)$ where $\gamma(s, \rho)$ is the generalized ray issued from ρ . From this fact we must have $\gamma(s, \rho) \notin \mathcal{G}^k$ for $s \in (0, s_0)$, since any ray intersecting with \mathcal{G}^k will enter $T^*\Omega$ or $\mathcal{G}^{2,-}$ immediately, provided that the neighborhood W_k is chosen to be small enough. By $(k-1)$ -propagation property, it suffices to show that $\rho \notin \text{supp}(\mu)$.

Therefore, by definition of $\Gamma(\rho_0; \delta)$, we only need to show that

$$\rho \in \{(y, z, \zeta) : 0 \leq y \leq \delta^2, |(z, \zeta) - \rho_0| \leq \delta^2\}.$$

We will prove this by comparing two flows $\exp(sH_{-r_0})(\rho_0) = (\tilde{z}(s), \tilde{\zeta}(s))$ and $\gamma(s, \rho) = (y(s), \eta(s), z(s), \zeta(s))$. Taking the difference of the two, we have

$$\frac{d}{ds}(z_1(s) - \tilde{z}_1(s)) = -\partial_{\zeta_1} r(y(s), z(s), \zeta(s)) + \partial_{\zeta_1} r(0, \tilde{z}(s), \tilde{\zeta}(s)) = O(y(s)),$$

$$\frac{d}{ds}(z'(s) - \tilde{z}'(s)) = O(y(s)), \quad \frac{d}{ds}(\zeta(s) - \tilde{\zeta}(s)) = O(y(s)), \quad \frac{dy}{ds} = 2\eta(s).$$

Note that $|\eta|^2 = |r| = O(1)$ and $y(s_0) = 0, \tilde{z}(s_0) = z(s_0), \tilde{\zeta}(s_0) = \zeta(s_0)$, we have

$$y(s) \leq C(s - s_0)^2 \text{ for all } s \in [0, s_0].$$

Hence $y(0) \leq C\sigma_k^2\delta^2 < \delta^2$, provided that $\sigma_k^2 < 1/C$. Moreover,

$$|(z(0), \zeta(0)) - \rho_0| \leq Cs_0^3 \leq C\sigma_k^3\delta^3 \leq \delta^2.$$

This completes the proof of Lemma 8.14 as well as Proposition 8.11. \blacksquare

Proposition 8.15. *Suppose that $(k-1)$ -propagation property holds. Then k -pre-propagation property implies k -propagation property.*

Proof. Up to re-parameter the flow, we may assume that $\rho_0 \in \mathcal{G}^k$ and $\gamma(s)$ is the generalized ray such that $\gamma(0) = \rho_0$. We also denote $\gamma(s)$ by $\gamma(s, \rho_0)$ in view of flow map. Suppose $\gamma(s_0) \notin \text{supp}(\mu)$ for some $s_0 < 0$ and $\gamma|_{[s_0, 0]} \cap \text{supp}(\mu) = \emptyset$. Our goal is to show that $\rho_0 \notin \text{supp}(\mu)$. Let $\sigma_{k-1} > 0$ be the required length in the definition of $(k-1)$ -propagation property.

Let $\delta_k > 0, \sigma_k > 0$ and V_k , neighborhood of $\rho_0 \in \mathcal{G}^k$ in $T^*\partial\Omega$ and \widetilde{V}_k , neighborhood of ρ_0 in $[0, \epsilon_0] \times T^*\partial\Omega$, as in the definition of k -pre-propagation property which satisfy the conditions in the paragraph in front of Lemma 8.12. Note in particular that we have $V_k \cap \mathcal{G}_j = \emptyset$ for all $j > k$. Without loss of generality, we may assume that $|s_0| < \min\{\sigma_{k-1}, \sigma_k\}$ and $\gamma(s_0) \in \widetilde{V}_k$, since otherwise we can choose $s'_0 < 0, |s'_0|$ small enough and replace $\gamma(s_0)$ by $\gamma(s'_0)$.

Let $\Gamma_0 \subset \widetilde{V}_k$ be a neighborhood of $\gamma(s_0)$ so that $\Gamma_0 \cap \text{supp}(\mu) = \emptyset$. For $\delta_1 > 0$ small with $\delta_1 \ll \sigma_k$, we set $\rho_1 = \exp(-\frac{\sigma_k \delta_1}{2} H_{-r_0})(\rho_0)$ and define

$$U_{\delta_1} := \{\rho = (y, \eta, z, \zeta) \in \text{Car}(P) : 0 \leq y \leq \delta_1^2, |(z, \zeta) - \rho_1| \leq \delta_1^2\}.$$

From continuous dependence of the generalized bicharacteristic flow, we have

$$U_{\delta_1} \subset \gamma(s_0, \Gamma_0), \quad \text{provided that } \delta_1 \text{ small enough.}$$

Now we claim that for possibly smaller $\delta_1 > 0$, we have

$$\gamma(s_1, U_{\delta_1}) \cap \bigcup_{j \geq k} \mathcal{G}^j = \emptyset, \quad \forall s_1 \in (s_0, 0).$$

Indeed, it suffices to prove that $\gamma(s_1, U_{\delta_1}) \cap \mathcal{G}^k = \emptyset$ since there are no point of \mathcal{G}^j in \widetilde{V}_k for $j > k$. Firstly, from the transversality of the flow $\exp(sH_{-r_0})$ and Σ_k , we deduce that at ρ_1 , $z_1 > \Theta_k(z', \zeta)$. By choosing δ_1 smaller, there exists $\epsilon_1 > 0$, such that for all $\rho \in U_{\delta_1}$, $z_1 > \Theta_k(z', \zeta) + \epsilon_1$ holds. In particular, $U_{\delta_1} \subset \Sigma_k^+$. We calculate

$$\begin{aligned} \frac{d}{ds} \Theta_k(z'(s), \zeta(s)) &= \frac{\partial \Theta_k}{\partial z'} \frac{dz'}{ds} + \frac{\partial \Theta_k}{\partial \zeta} \frac{d\zeta}{ds} \\ &= - \frac{\partial \Theta_k}{\partial z'} \frac{\partial r}{\partial \zeta'} + \frac{\partial \Theta_k}{\partial \zeta} \frac{\partial r}{\partial z}. \end{aligned}$$

Note that in \widetilde{V}_k , we can write $r = \zeta_1 + yr_1(z, \zeta) + O(y^2)$, hence

$$\frac{d}{ds} \Theta_k(z'(s), \zeta(s)) = O(y(s)).$$

Next we argue by contradiction, assume that for some $s_1 \in (s_0, 0)$ and $\rho \in U_{\delta_1}$ we have $\gamma(s_1, \rho) \in \mathcal{G}^k$ and $\gamma(s, \rho) \notin \mathcal{G}^k$ for any $0 > s > s_1$. In this case we have $|y(s)| \leq C|s - s_1|$ for all $s \in [s_1, 0]$. Therefore we must have

$$\left| \frac{d}{ds} \Theta_k(z'(s), \zeta(s)) \right| \leq C|s - s_1|.$$

Combining with $z_1 \sim -1$, we have

$$\begin{aligned} \Theta_k(z'(s_1), \zeta(s_1)) &\leq \Theta_k(z'(0), \zeta(0)) + C \int_{s_1}^0 |s - s_1| ds \\ &< z_1(0) + Cs_1^2 \\ &= z_1(s_1) + \int_{s_1}^0 \frac{dz_1}{ds} ds + Cs_1^2 \\ &\leq z_1(s_1) - C_1|s_1| + Cs_1^2 \\ &\leq z_1(s_1), \end{aligned}$$

provided that $|s_0|$ is chosen to be small enough. This implies that $\gamma(s_1, \rho) \in \Sigma_k^+$, which is a contradiction.

From $(k-1)$ -propagation property, we know that $U_{\delta_1} \cap \text{supp}(\mu) = \emptyset$. Therefore, applying k -pre-propagation property with respect to ρ_1 and U_{δ_1} , we deduce that $\rho_0 \notin \text{supp}(\mu)$, and this completes the proof of Proposition (8.15). \blacksquare

A. PROOF OF LEMMA 3.2

Proof of Lemma 3.2. The first assertion follows from $h\text{div}u = 0$ and Dirichlet boundary condition, while we apply a multiplier method to prove the second. From the geometric assumption on Ω , we can find a vector field $L \in C^1(\overline{\Omega})$ such that

$L|_{\partial\Omega} = \nu$ (see [14], page 36). In global coordinate system, we write $L = L_j(x)\partial_{x_j}$. By using the equation, we have

$$\begin{aligned} \int_{\Omega} Lu \cdot f dx &= \int_{\Omega} Lu \cdot (-h^2 \Delta u - u + h \nabla q) dx, \\ - \int_{\Omega} Lu \cdot u dx &= - \int_{\Omega} L_j(x) \partial_{x_j} u^i u^i dx \\ &= - \int_{\Omega} \partial_{x_j} (L_j(x) u^i) u^i dx + \int_{\Omega} \operatorname{div} L(x) |u|^2 dx \\ &= \int_{\Omega} L_j(x) u^i(x) \partial_{x_j} u^i dx + \int_{\Omega} \operatorname{div} L(x) |u|^2 dx \\ &= \int_{\Omega} Lu \cdot u dx + \int_{\Omega} \operatorname{div} L(x) |u|^2 dx, \end{aligned}$$

and thus $\int_{\Omega} Lu \cdot u dx = -\frac{1}{2} \int_{\Omega} \operatorname{div} L(x) |u|^2 dx = O(1)$. Next we calculate

$$\begin{aligned} h \int_{\Omega} Lu \cdot \nabla q dx &= -h \int_{\Omega} u^i \partial_{x_j} (L_j \partial_{x_i} q) dx \\ &= -h \int_{\Omega} u \cdot L(\nabla q) dx - h \int_{\Omega} (\operatorname{div} L(x)) u \cdot \nabla q dx \\ &= -h \int_{\Omega} u \cdot [L, \nabla] q dx - h \int_{\Omega} \operatorname{div} L(x) u \cdot \nabla q dx \\ &= O(1), \end{aligned}$$

$$\begin{aligned} -h^2 \int_{\Omega} Lu^i \Delta u^i dx &= -h^2 \int_{\partial\Omega} |\partial_{\nu} u^i|^2 d\sigma + h^2 \int_{\Omega} \nabla L(\nabla u^i, \nabla u^i) dx \\ &\quad + h^2 \int_{\Omega} L_j(x) \partial_{x_j x_k}^2 u^i \partial_{x_k} u^i \\ &= -h^2 \int_{\partial\Omega} |\partial_{\nu} u^i|^2 d\sigma + h^2 \int_{\Omega} \nabla L(x) (\nabla u^i, \nabla u^i) dx \\ &\quad + h^2 \int_{\Omega} \partial_{x_j} (L_j \partial_{x_k} u^i) \partial_{x_k} u^i dx - h^2 \int_{\Omega} \operatorname{div} L(x) \nabla u^i \cdot \nabla u^i(x) dx, \end{aligned}$$

$$h^2 \int_{\Omega} \partial_{x_j} (L_j \partial_{x_k} u^i) \partial_{x_k} u^i dx = h^2 \int_{\partial\Omega} L \cdot \nu |\partial_{\nu} u^i|^2 d\sigma - h^2 \int_{\Omega} L_j(x) \partial_{x_k} u^i \partial_{x_j x_k}^2 u^i dx,$$

$$-h^2 \int_{\Omega} Lu^i \Delta u^i dx = -\frac{h^2}{2} \int_{\partial\Omega} |\partial_{\nu} u^i|^2 d\sigma + \int_{\Omega} \nabla L(x) (h \nabla u^i, h \nabla u^i) dx - \frac{h^2}{2} \int_{\Omega} \operatorname{div} L(x) |\nabla u^i|^2 dx.$$

Observing that $\int_{\Omega} Lu \cdot f dx = o(1)$, we have

$$\int_{\partial\Omega} |h \partial_{\nu} u|^2 d\sigma = O(1).$$

■

B. STANDARD ELLIPTIC THEORY

The differential operator is given by

$$P_h = \operatorname{Op}_h(\eta^2 + \lambda(y, x', \xi')^2 - 1 + hm(y, x', \eta, \xi')),$$

where the principal symbol $p = \eta^2 + \lambda^2 - 1$ is scalar while m is matrix valued. When micro-locally near the region $p > 0$, we want to construct the parametrix of the inverse of P . Denote by U the tubular neighborhood (two sided) of $\partial\Omega$. Take $\varphi \in C_c^\infty(U)$, $\chi_0 \in C^\infty(\mathbb{R}^{d-1})$ and the support of φ is contained in a coordinate patch near the boundary. We put

$$E^0 := \text{Op}_h\left(\frac{\chi_0(\xi')\varphi(y, x')}{\eta^2 + \lambda(y, x', \xi')^2 - 1}\right),$$

and we define matrix valued PdO E^l , $l \geq 1$ inductively via

$$\begin{aligned} E^1 \times p &= - \sum_{|\alpha|=1} \frac{1}{i} \partial_{\xi', \eta}^\alpha E^0 \times \partial_{x', y}^\alpha p - E^0 \times m, \\ E^n \times p &= - \sum_{|\alpha|+k=n, k \neq n} \frac{1}{i^{|\alpha|}} \partial_{\xi', \eta}^\alpha E^k \times \partial_{x', y}^\alpha p - \sum_{|\alpha|+k=n-1} \frac{1}{i^{|\alpha|}} \partial_{\xi', \eta}^\alpha E^k \times \partial_{x', y}^\alpha m. \end{aligned} \quad (8.8)$$

For any $N \in \mathbb{N}$, we set

$$E_N = \sum_{j=0}^N h^j E^j,$$

and then

$$E_N \circ P = \chi_0(hD_{x'})\varphi(y, x')\text{Id} + R_N, \quad \|R_N\|_{L^2 \rightarrow L^2} = O(h^{N+1}).$$

Proposition 8.16. *The sequence of solutions (u_k) is h_k -oscillating in the following sense:*

$$\begin{aligned} \lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_{|\xi| \geq Rh_k^{-1}} |\widehat{\varphi u_k}(\xi)|^2 d\xi &= 0 \quad \forall \psi \in C_c^\infty(\Omega), \\ \lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_0^{\varepsilon_0} dy \int_{|\xi'| \geq Rh_k^{-1}} |\widehat{\varphi u_k}(y, \xi')|^2 d\xi' &= 0, \quad \forall \psi \in C_c^\infty(\overline{\Omega}), \end{aligned}$$

where in the second formula, the support of φ is contained in some local coordinate patch near the boundary, and the Fourier transform is only taken for the x' direction.

Proof. We drop the subindex k in the proof. For the first formula, one can use the equation of u to obtain

$$(-h^2 \Delta - 1)(\varphi u) = g = O_{L^2}(1),$$

and

$$\int_{|\xi| \geq Rh^{-1}} |\widehat{\varphi u}(\xi)|^2 d\xi \leq \int_{h|\xi| \geq R} \frac{|\widehat{g}(\xi)|^2}{|h^2|\xi|^2 - 1|^2} d\xi \leq \frac{C}{(R^2 - 1)^2}.$$

For the second formula, it will be sufficient to show that

$$\lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \left\| \left(1 - \chi\left(\frac{hD_{x'}}{R}\right)\right)(\varphi u) \right\|_{L^2} = 0$$

for some $\chi \in C_c^\infty(-1, 1)$. We write $w = u\mathbf{1}_{y \geq 0}$, $\tilde{g} = g\mathbf{1}_{y \geq 0}$, $v = h\partial_y u|_{y=0} = O_{L^2(y=0)}(1)$. We apply the parametrix construction above with $\chi_0(\xi') = 1 - \chi\left(\frac{\xi'}{R}\right)$. Let $e_N(y, x', \eta, \xi')$ be the symbol of the operator E_N , which is meromorphic in η with poles $\eta_0^\pm = \pm i\sqrt{\lambda^2(y, x', \xi')^2 - 1}$. Moreover,

$$|\partial^\alpha e_N(y, x', \eta, \xi')| \leq \frac{C_{N, \alpha}}{R}. \quad (8.9)$$

We take $\tilde{\varphi}$ to be a slight enlargement of φ such that $\tilde{\varphi}\varphi = \varphi$. Then

$$E_N \circ \tilde{\varphi} P w = (1 - \chi(\frac{hD_{x'}}{R}))(\varphi w) + R_N w.$$

From the jump formula, $P w = \tilde{g} + h v(x') \otimes \delta_{y=0}$. We have

$$(1 - \chi(\frac{hD_{x'}}{R}))(\varphi w) = E_N(\tilde{\varphi}\tilde{g} + \tilde{\varphi}h v \otimes \delta_{y=0}) + O_{L^2_{y,x'}}(h^{N+1}).$$

From symbolic calculus,

$$\|E_N(\tilde{\varphi}\tilde{g})\|_{L^2_{y,x'}} \leq \sum_{|\alpha| \leq C_d} \sup_{(y,x',\eta,\xi') \in \mathbb{R}^{2d}} |\partial^\alpha e_N(y, x', \eta, \xi')| + Ch,$$

and it vanishes after the taking the limit $h \rightarrow 0$ and then $R \rightarrow \infty$, thanks to (8.9).

We next write

$$E_N(\tilde{\varphi}h v \otimes \delta_{y=0}) = \frac{\pi}{(2\pi h)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{\frac{i(x'-z') \cdot \xi'}{h}} \omega_N(y, x', \xi') \tilde{\varphi}(0, z') v(z') dz' d\xi',$$

with

$$\omega_N(y, x', \xi') = \int_{-\infty}^{\infty} e_N(y, x', \eta, \xi') e^{\frac{iy\eta}{h}} d\eta.$$

To calculate ω_N for $y > 0$, we deform the contour of integral in η in the half plane $\text{Im } \eta > 0$. From the Residue formula, we have

$$\omega_N(y, x', \xi') = 2\pi i \text{Res}(e_N(y, x', \eta, \xi'); i\eta_0^+) e^{\frac{iy\eta_0^+}{h}}.$$

The principal symbol of ω_N is given by

$$\pi \exp\left(-\frac{yQ(y, x', \xi')}{h}\right) \frac{\varphi(y, x')(1 - \chi(\frac{\xi'}{R}))}{2Q(y, x', \xi')}, \quad Q(y, x', \xi') = \sqrt{\lambda(y, x', \xi')^2 - 1}.$$

Therefore

$$\begin{aligned} & \limsup_{h \rightarrow 0} \|E_N(\tilde{\varphi}2h v \otimes \delta_{y=0})\|_{L^2_{y,x'}} \\ & \leq C_{N,d} \int_0^\infty \sum_{|\alpha| \leq C_d} \sup_{(x', \xi')} |\partial_{x', \xi'}^\alpha \omega_N(y, x', \xi')| \|v\|_{L^2_{x'}} dy \leq \frac{C}{R}, \end{aligned}$$

where we have used the point-wise estimate

$$|\partial_{x', \xi'}^\alpha \omega_N(y, x', \xi')| \leq \frac{C_\alpha e^{-\frac{y\sqrt{R^2-1}}{2Rh}}}{\sqrt{R^2-1}} \left(1 + \left(\frac{y}{h}\right)^{|\alpha|}\right).$$

■

Given $\chi(y, x', \xi') \in C_c^\infty([0, \epsilon_0) \times \mathbb{R}^{2d-2})$, the following proposition can be deduced in the same manner.

Proposition 8.17. *Let $w_k = \chi(y, x', hD_{x'}) (\varphi u_k)$, $\underline{w}_k = \mathbf{1}_{y \geq 0} u_k$. Then for $\chi_1 \in C_c^\infty(\mathbb{R})$, $0 \leq \chi_1 \leq 1$, $\chi_1 = 1$ near 0, we have*

$$\lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} \left\| \left(1 - \chi_1\left(\frac{h_k D_y}{R}\right)\right) \underline{w}_k \right\|_{L^2(\mathbb{R}^d)} = 0.$$

Proof. We have $P\underline{w} = \underline{g} + hv(x') \otimes \delta_{y=0}$ with $\|\underline{g}\|_{L^2_{y,x'}} = O(1)$, $\|v\|_{L^2_{x'}} = O(1)$. Note that the functions $\underline{g}, \underline{v}$ here may not coincide with the functions in the proof of Proposition 2.7. We define

$$E_N = \sum_{j=0}^N h^j E^j + R_N, \quad \|R_N\|_{L^2 \rightarrow L^2} = O(h^{N+1}),$$

with $E^0 = \text{Op}_h\left(\frac{\tilde{\varphi}(y, x')(1 - \chi_1(\frac{\eta}{R}))}{\eta^2 + \lambda(y, x', \xi')^2 - 1}\right)\text{Id}$ and $E^l, l \geq 1$ as in (8.8). This implies that

$$(1 - \chi_1(\frac{hD_y}{R}))(\varphi\underline{w}) = E_N(\tilde{\varphi}\underline{g} + 2h\tilde{\varphi}v \otimes \delta_{y=0}) - R_N\underline{w}.$$

Consequently, we have

$$\begin{aligned} & \lim_{R \rightarrow \infty} \limsup_{h \rightarrow 0} \|E_N(\tilde{\varphi}\underline{g})\|_{L^2_{y,x'}} = 0. \\ & E_N(h\tilde{\varphi}v \otimes \delta_{y=0})(y, x') \\ &= \frac{\pi}{(2\pi h)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{\frac{i(x'-z')\xi'}{h}} a(y, x', \xi') \tilde{\varphi}(0, z') v(z') dz' d\xi', \\ & \text{with } a(y, x', \xi') = \int_{-\infty}^{\infty} e^{\frac{i y \eta}{h}} e_N(y, x', \eta, \xi') d\eta. \end{aligned}$$

Observe that

$$\sup_{(x', \xi')} |\partial_{x', \xi'}^\alpha e_N(y, x', \eta, \xi')| \leq \frac{C_\alpha (1 - \chi_1(\frac{\eta}{R}))}{(1 + \eta^2)(1 + y^2)},$$

and this implies that

$$\lim_{R \rightarrow \infty} \limsup_{h \rightarrow 0} \|E_N(h\tilde{\varphi}v \otimes \delta_{y=0})\|_{L^2_{y,x'}} = 0. \quad \blacksquare$$

C. PROOF OF TECHNICAL RESULTS IN SECTION 3

Proof of Lemma 3.7. The proof can be reduced to the point-wise estimate of the solution $F(y)$ of the ODE:

$$-h^2 \frac{d^2 F}{dy^2} + \lambda(y)^2 F(y) = G(y), \quad F(0) = F'(0) = 0,$$

with $0 < c_1 \leq \lambda(y)^2 \leq c_2$, $G \in C^\infty([0, \infty))$, and $|G(y)| \leq C e^{-\frac{c_1 y}{h}}$ for all $y \geq 0$. By rescaling $z = \frac{y}{h}$, it reduces to prove the exponential decay of the solution W of the ODE:

$$-\frac{d^2 W}{dz^2} + V(z)W(z) = g(z), \quad W(0) = W'(0) = 0,$$

with $0 < c_1 \leq V(z) \leq c_2$, $g \in C^\infty([0, \infty))$ and $|g(z)| \leq C e^{-c_1 z}$ for all $z \geq 0$.

For this, we first notice that W is smooth and in $H^s(\mathbb{R}_+)$ for all $s \geq 0$. To prove the exponential decay, we pick $\theta_\epsilon(z) = e^{\frac{2\delta_0 z}{1+\epsilon z}}$ with $\delta_0 > 0$ to be chosen later. One observe easily that $0 < \theta'_\epsilon(z) \leq 2\delta_0 \theta_\epsilon(z)$ for $z \geq 0$. We multiply by $\theta_\epsilon W$ to the both sides of the equation and integrate it, then

$$\int_0^\infty ((\theta_\epsilon W)' W' + \theta_\epsilon V W^2) dz = \int_0^\infty \theta_\epsilon W G dz.$$

Notice that $\theta'_\epsilon W'W \geq -2\delta_0\theta_\epsilon|W||W'| \geq -\delta_0\theta_\epsilon(|W|^2 + |W'|^2)$, by choosing

$$\delta_0 < \min \left\{ \frac{c_1}{4}, \frac{1}{4}, \frac{c}{2} \right\},$$

we have that

$$\int_0^\infty \theta_\epsilon (|W'|^2 + |W|^2) dz \leq 2 \|Ge^{2\delta_0 z}\|_{L^2(\mathbb{R}_+)} \|W\|_{L^2(\mathbb{R}_+)},$$

thanks to Cauchy-Schwartz and the fact that $\theta_\epsilon \leq e^{2\delta_0 z}$, uniformly in ϵ . From the dominating convergence theorem, we have $We^{\delta_0 z} \in L^2(\mathbb{R}_+)$ and $W'e^{\delta_0 z} \in L^2(\mathbb{R}_+)$. Finally, by elliptic regularity, we have that $We^{\delta_0 z} \in L^\infty(\mathbb{R}_+)$, $W'e^{\delta_0 z} \in L^\infty(\mathbb{R}_+)$. \blacksquare

Proof of Proposition 3.8. We choose $\varphi_2 \in C_c^\infty(Y_+)$ such that

$$\varphi_1|_{\text{supp}(\varphi_2)} = 1, \varphi_2|_{\text{supp}(\varphi)} = 1.$$

We first claim that

$$\varphi_2 \text{Op}_h(\chi_{\delta_0} A_j)(\varphi_{1,0} q_0) = O_{L^2(\mathbb{R}_+^d)}(1). \quad (8.10)$$

Indeed, we can write

$$\varphi_2 \text{Op}_h(A_j \chi_{\delta_0})(\varphi_{1,0} q_0) = \varphi_2 \text{Op}_h(A_j) \varphi_1 \text{Op}_h(\chi_{\delta_0})(\varphi_{1,0} q_0) + h \varphi_2 \text{Op}_h(B_j \tilde{\chi}_{\delta_0})(\varphi_{1,0} q_0)$$

with $B_j \in \mathcal{E}_\partial^{-j}$ and $\tilde{\chi}_{\delta_0}$ has similar support property as χ_{δ_0} . Thus from symbolic calculus, we have for each fixed $y > 0$,

$$\|\varphi_2 \text{Op}_h(A_j(y) \chi_{\delta_0}(y))(\varphi_{1,0} q_0)\|_{L^2(\mathbb{R}^{d-1})}^2 \leq C_j e^{-\frac{c_j y}{h}} h^{-1} \left(1 + \left(\frac{y}{h}\right)^{n_j}\right),$$

thanks to Lemma 3.5. Integrating over $y > 0$ yields (8.10).

By taking $\text{supp}(\chi_{\delta_0})$ small such that $\varphi_2 \chi_{\delta_0} = \chi_{\delta_0}$, we have that

$$\varphi_2 (\text{Op}_h(\chi_{\delta_0})(\varphi_1 q) - \text{Op}_h(\chi_{\delta_0} A)(\varphi_{1,0} q_0)) = w + O_{L^2(\mathbb{R}_+^d)}(h)$$

with

$$w = \varphi_2 (\text{Op}_h(\chi_{\delta_0})(\varphi_1 q) - \text{Op}_h(\chi_{\delta_0}) \varphi_1 \varphi_2 \text{Op}_h(A)(\varphi_{1,0} q_0)).$$

From Lemma 3.5, we have $w = O_{L^2(\mathbb{R}_+^d)}(1)$ and $hD_y w = O_{L^2(\mathbb{R}_+^d)}(1)$. The trace of w satisfies

$$w|_{y=0} = \varphi_{2,0} \text{Op}_h(\chi_{\delta_0}(1 - \psi_{\delta_0}(\lambda_0)))(\varphi_{1,0} q_0) = O_{H^\infty(\mathbb{R}^{d-1})}(h^\infty),$$

and w satisfies the equation (we use $\varphi_2 = \varphi_1 \varphi_2$ here)

$$P_0 w = \varphi_1 [P_0, \varphi_2 \text{Op}_h(\chi_{\delta_0})] (\varphi_1 q - \varphi_2 \text{Op}_h(A)(\varphi_{1,0} q_0)) + O_{H^\infty(\mathbb{R}^d)}(h^\infty). \quad (8.11)$$

Notice that $\varphi_1 q - \varphi_2 \text{Op}_h(A)(\varphi_{1,0} q_0) = O_{L^2(\mathbb{R}_+^d)}(1)$, micro-locally for $\lambda \geq \frac{\delta_0}{2}$, the right hand side of (8.11) is equal to $O_{L^2(\mathbb{R}_+^d)}(h)$ as well as $O_{H^1(\mathbb{R}_+^d)}(1)$. Multiply by $w = \varphi_1 w$ to the both sides of (8.11) and integrate it, we have

$$\int_0^\infty \int_{\mathbb{R}^{d-1}} w(y, x') P_0 w dx' dy = O(h).$$

Doing integration by part for the left hand side, we have

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^{d-1}} w P_0 w dx' dy &= - \int_0^\infty \int_{\mathbb{R}^{d-1}} |h \partial_y w|^2 dx' dy - \int_{\mathbb{R}^{d-1}} h^2 (w \partial_y w)|_{y=0} dx' \\ &\quad - \int_0^\infty \int_{\mathbb{R}^{d-1}} \sum_{1 \leq j, k \leq d-1} g^{jk} h \partial_j w h \partial_k w dx' dy + O(h). \end{aligned}$$

This implies that

$$\|hD_{y,x'}w\|_{L^2(\mathbb{R}_+^d)} + \|w\|_{L^2(\mathbb{R}_+^d)} = O(h^{1/2}).$$

Using this smallness and redoing the integration by part, we can improve each bound in the procedure above and obtain that

$$\|hD_{y,x'}w\|_{L^2(\mathbb{R}_+^d)} + \|w\|_{L^2(\mathbb{R}_+^d)} = O(h^{3/4}).$$

To conclude, we observe that

$$\begin{aligned} \varphi h \partial_y w &= \varphi h \partial_y \text{Op}_h(\chi_{\delta_0})(\varphi_1 q) - \varphi h \partial_y \text{Op}_h(\chi_{\delta_0}) \varphi_2 \text{Op}_h(A)(\varphi_{1,0} q_0) \\ &= \varphi \text{Op}_h(\chi_{\delta_0}) h \partial_y(\varphi_1 q) - \varphi \text{Op}_h(\chi_{\delta_0}) \varphi_2 \text{Op}_h(h \partial_y A)(\varphi_{1,0} q_0) \\ &\quad + O_{L^2(\mathbb{R}_+^d)}(h) \\ &= \varphi \text{Op}_h(\chi_{\delta_0}) h \partial_y(\varphi_1 q) - \varphi \text{Op}_h(\chi_{\delta_0} \lambda A)(\varphi_{1,0} q_0) \\ &\quad + O_{L^2(\mathbb{R}_+^d)}(h), \end{aligned} \tag{8.12}$$

where we have used symbolic calculus and Lemma 3.5 several times. Plugging into (8.11), we have that $P_0 w = O_{L^2(\mathbb{R}_+^d)}(h)$, $w|_{y=0} = O_{H^\infty(\mathbb{R}^{d-1})}(h^\infty)$. We decompose $w = w_1 + w_2$ with $P_0 w_1 = P_0 w$, $w_1|_{y=0} = 0$ and $P_0 w_2 = 0$, $w_2|_{y=0} = w|_{y=0}$. From elliptic regularity of boundary value problem, we have $h^2 w_1 = O_{H^2(\mathbb{R}_+^d)}(h)$ and $h^2 w_2 = O_{H^2(\mathbb{R}_+^d)}(h^\infty)$ and thus $h \partial_y w = O_{H^1(\mathbb{R}_+^d)}(1)$. From interpolation, we deduce that $h \partial_y w = O_{H^{\frac{2}{3}}(\mathbb{R}_+^d)}(h^{\frac{1}{4}})$. Observe that the error terms on the right hand side of (8.12) can be also bounded by $O_{H^1(\mathbb{R}_+^d)}(1)$ and thus $O_{H^{\frac{2}{3}}(\mathbb{R}_+^d)}(h^{\frac{1}{4}})$ by interpolation. This completes the proof. \blacksquare

D. CONSTRUCTION OF TEST FUNCTIONS

We first give the detailed construction of $a = a_0 + a_1 \eta$ used in the first step of the proof of Proposition 7.6, which follows closely to [8].

Proof of Lemma 7.7. Given $\chi_1 \in C_c^\infty(-2, 2)$ with $\chi_1|_{(-1,1)} = 1$ and $\chi_2 \in C_c^\infty(-3, 3)$ such that $\chi_2|_{(-2,2)} = 1$. Consider the smooth function $\chi_0(t) = e^{-1/t} \mathbf{1}_{t>0}$. We work in the local coordinate (y, x, ξ) , and assume that $(0, x_0, \xi_0) \in \mathcal{G}^{2,+}$ with $|\xi_0| \sim 1$. Set $\phi = \phi_0 + \phi_1 \eta$ with

$$\phi_1(y, x, \xi) = \frac{1}{|\xi|}, \phi_0(y, x, \xi) = y^2 + |x - x_0|^2 + |\xi - \xi_0|^2.$$

We calculate

$$H_p \phi = \eta(2\partial_y \phi_0 - \{r, \phi_1\}) + \phi_1 \partial_y r - \{r, \phi_0\} \geq 2c > 0,$$

provided that $|\eta| \leq c_0$ for some $c_0 > 0$ and W_0 is chosen small enough such that $\frac{\partial r}{\partial y} \geq 4c$ on it. The positivity then follows from the direct calculation:

$$\begin{aligned} \{r, \phi_0\} &= 2\partial_\xi r \cdot (x - x_0) - 2\partial_x r \cdot (\xi - \xi_0), \\ \partial_y \phi_0 &= 2y, \{r, \phi_1\} = \partial_x r \cdot \frac{\xi}{|\xi|^3}. \end{aligned}$$

We next take

$$f(y, x, \eta, \xi) := \chi_2\left(\frac{\phi_0}{\delta}\right)^2 \chi_0\left(1 - \frac{\phi}{\delta}\right).$$

The desired functions a_0, a_1 are chosen to be the remainders when f is divided by $p = \eta^2 - r(y, x, \xi)$, thanks to the Malgrange preparation theorem:

$$f(y, x, \eta, \xi) = (\eta^2 - r(y, x, \xi))g(y, x, \eta, \xi) + a_1(y, x, \xi)\eta + a_0(y, x, \xi).$$

On the support of f , we observe that

$$\phi_0(y, x, \xi) = |(y, x, \xi) - (0, x_0, \xi_0)|^2 \leq 3\delta, \quad \frac{\eta}{|\xi|} + \phi_0 \leq \delta,$$

which implies $\eta \leq \delta|\xi|$. Moreover, on $\text{supp}(f) \cap \text{supp}(\partial\chi_2(\delta^{-1}\phi_0))$, we have $\phi_0 \geq 2\delta, \phi_0 + \phi_1\eta \leq \delta$, and these imply $\eta \leq -\delta|\xi|$, hence $r(y, x, \xi) = \eta^2 \geq \delta^2|\xi|^2$, when $p = \eta^2 - r = 0$.

Direct calculation yields

$$H_p f + fM|\xi| + \psi^2 = \chi_0(1 - \frac{\phi}{\delta})H_p(\chi_2(\frac{\phi_0}{\delta})^2) - (1 - \chi_1(\frac{\eta}{\delta|\xi|})^2)N,$$

with

$$N = \chi_2(\frac{\phi_0}{\delta})^2(\chi_0'(1 - \frac{\phi}{\delta})\frac{H_p\phi}{\delta} - \chi_0(1 - \frac{\phi}{\delta})M|\xi|) \in C^\infty, \quad \psi = \chi_1(\frac{\eta}{\delta|\xi|})N^{1/2}.$$

Here $N \geq 0$ on $\text{supp}(\psi)$ if we choose $\delta > 0$ small enough. Observe that when $\eta = r^{1/2} \geq 0$, we have $\chi_0(1 - \delta^{-1}\phi)H_p(\chi_2(\delta^{-1}\phi_0)^2) = 0, (1 - \chi_1(\frac{\eta}{\delta|\xi|})^2)N = 0$. We then define a function

$$\varphi(y, x, \xi) = -\frac{\chi_0(1 - \frac{\phi}{\delta})H_p(\chi_2(\frac{\phi_0}{\delta})^2) - (1 - \chi_1(\frac{\eta}{\delta|\xi|})^2)N}{2r^{1/2}}|_{\eta = -r^{1/2}} \mathbf{1}_{r(y, x, \xi) > 0}$$

and then

$$H_p f + fM|\xi| + \psi^2 = \varphi(\eta - r^{1/2}), \quad \text{when } p = \eta^2 - r = 0.$$

Therefore, on $p = 0$, we have

$$H_p a + aM|\xi| + \psi^2 = \varphi(\eta - r^{1/2}).$$

It is left to check the smoothness of functions φ, ψ and ρ . Indeed, on the support of $\psi, |\phi_1\eta| \leq 2\delta, \phi_0 \leq 3\delta$, and then $1 - \frac{\phi}{\delta} \leq 3$. Notice that $\frac{\chi_0(t)}{\chi_0'(t)} = t^2$, we have

$$N^{1/2} = \chi_2(\frac{\phi_0}{\delta})\sqrt{\chi_0'(1 - \frac{\phi}{\delta})\frac{H_p\phi}{\delta}}G(\frac{M|\xi|\delta}{H_p\phi}, 1 - \frac{\phi}{\delta}) \in C^\infty,$$

since the function $G(a, t) = \sqrt{1 - at^2} \in C^\infty$ for $t \leq 3, |a| \ll 1$. This implies that $\psi \in C^\infty$, provided that δ is chosen small enough.

For φ , the smoothness comes from the fact that on the support of

$$\chi_0(1 - \frac{\phi}{\delta})H_p(\chi_2(\frac{\phi_0}{\delta})^2)|_{\eta^2=r} - (1 - \chi_1(\frac{\eta}{\delta|\xi|})^2)N|_{\eta^2=r},$$

we have $r \geq \delta^2|\xi|^2$. Moreover, φ has compact support.

Finally, from the definition of a , we have

$$a_1(y, x, \xi) = \frac{f(y, x, \eta, \xi) - f(y, x, -\eta, \xi)}{2\eta}|_{\eta = \sqrt{r(y, x, \xi)}} \in C_c^\infty,$$

$$a_0(y, x, \xi) = \frac{f(y, x, \eta, \xi) + f(y, x, -\eta, \xi)}{2}|_{\eta = \pm\sqrt{r(y, x, \xi)}} \in C_c^\infty.$$

we deduce that $\frac{\partial f}{\partial \eta} = -\frac{\phi_1}{\delta} \chi_2 \left(\frac{\phi_0}{\delta}\right)^2 \chi'_0 \left(1 - \frac{\phi}{\delta}\right) < 0$, hence

$$\frac{f(y, x, \eta, \xi) - f(y, x, -\eta, \xi)}{2\eta} = \frac{1}{2} \int_{-1}^1 \frac{\partial f}{\partial \eta}(y, x, s\eta, \xi) ds < 0.$$

Define

$$t(y, x, \xi) = \left(-\frac{1}{2} \int_{-1}^1 \frac{\partial f}{\partial \eta}(y, x, s\eta, \xi) ds \Big|_{\eta=\sqrt{r(y, x, \xi)}} \right)^{1/2},$$

one can show that t is a smooth function with compact support.

The last observation is that $a = f > 0$ on $p = 0$, hence $s = f^{1/2}|_{p=0} \in C_c^\infty$.

We give some more calculations: Let $\psi_0 = \psi|_{y=0}$, $\psi_1 = \frac{\partial \psi}{\partial \eta}|_{y=0}$, when $\eta = r = 0$. Thus at (x_0, ξ_0) ,

$$\begin{aligned} t(x_0, \xi_0) &= \sqrt{\frac{\chi'_0(1)\phi_1(x_0, \xi_0)}{\delta}}, \\ \psi_0(x_0, \xi_0)^2 &= \chi'_0(1) \frac{\partial r}{\partial y}(0, x_0, \xi_0) \frac{\phi_1(0, x_0, \xi_0)}{\delta} - \chi_0(1)M|\xi| > 0, \\ 2\psi_0\psi_1(x_0, \xi_0) &= -\chi''_0(1) \frac{\partial r}{\partial y}(x_0, \xi_0) \frac{\phi_1(x_0, \xi_0)^2}{\delta^2} - \chi'_0(1)\{r, \phi_1\}(0, x_0, \xi_0) \frac{1}{\delta} \\ &\quad + \frac{\chi'_0(1)M\phi_1(0, x_0, \xi_0)}{\delta} > 0, \text{ for } \delta \text{ small enough since } \chi''_0(1) < 0. \end{aligned}$$

Observe that near (x_0, ξ_0) , we have $\frac{\psi_1}{\psi_0} \sim -\frac{\phi_1(x_0, \xi_0)\chi''_0(1)}{2\chi'_0(1)\delta}$, provided that δ is small enough. Now if we make a different choice of $\tilde{\delta} > 0$, the difference between two ratios $\frac{\psi_1}{\psi_0}$ and $\frac{\tilde{\psi}_1}{\tilde{\psi}_0}$ is non-zero. This implies that we can choose a further cut-off χ near $(0, x_0, \xi_0)$ such that $\|\varphi \text{Op}_h(\chi)\varphi_1 u\|_{L^2(Y_+)} = o(1)$ and $\|\varphi \text{Op}_h(\chi)hD_y \varphi_1 u\|_{L^2(Y_+)} = o(1)$ from $\|\varphi \text{Op}_h(\psi_0 + \psi_1 \eta)\varphi_1 u\|_{L^2(Y_+)} = o(1)$ and $\|\varphi \text{Op}_h(\tilde{\psi}_0 + \tilde{\psi}_1 \eta)\varphi_1 u\|_{L^2(Y_+)} = o(1)$. \blacksquare

Next we recall the proof of Lemma 8.2, which is essentially given in [12].

Proof of Lemma 8.2. : From the transversality, we can choose a new coordinate (s, t) in U such that $\rho_0 = (0, 0)$ and $H_{-r_0} = \partial_t$ in this coordinate.

Step 1. Consider the function $\chi(u) = e^{\frac{1}{u-3/4}} \mathbf{1}_{u < 3/4}$. It is easy to check that χ is smooth and non-increasing with the property:

$$\partial^N \chi(u) = O((-\chi')^{1/m}), \quad \forall N \in \mathbb{N}, m > 1, \text{ locally uniformly.}$$

Step 2. Next we choose $\beta \in C^\infty(\mathbb{R})$ such that $\beta \geq 0$ vanishing on $(-\infty, -1)$ and strictly increasing on $(-1, -\frac{1}{2})$, equaling to 1 on $(-\frac{1}{2}, \infty)$. We modify β such that

$$\partial^N \beta = O(\beta^{1/m}), \quad \forall N \in \mathbb{N}, m > 1, \text{ locally uniformly.}$$

Step 3. Choose $f \in C^\infty(\mathbb{R})$ so that f vanishes on $(-\infty, 1/2)$ and is strictly increasing and convex on $(1/2, \infty)$ with $f(1) > 1$.

Now we set

$$a_\delta = \beta\left(\frac{3t}{4\delta^2}\right) \chi\left(\frac{t}{\sigma\delta} + \frac{|s|^2}{\delta^4} + f\left(\frac{y}{\delta^2}\right)\right),$$

and

$$g_\delta = -\beta\left(\frac{3t}{4\delta^2}\right) H_p(\chi(u))$$

with $u = \frac{3t}{4\sigma\delta} + \frac{|s|^2}{\delta^4} + f\left(\frac{y}{\delta^2}\right)$. Finally we define $h_\delta = -H_p a_\delta - g_\delta$. Note that

$$\text{supp}(a_\delta) = \left\{ (y, s, t) : -\delta^2 \leq t, \frac{t}{\sigma\delta} + \frac{|s|^2}{\delta^4} + f\left(\frac{y}{\delta^2}\right) \leq \frac{3}{4} \right\},$$

hence it is clear that (1)(2)(3)(4) in Lemma 8.2 are satisfied.

Since $r = r_0 + O(y)$, when H_{-r} acts on functions independent of η we have

$$H_p = \partial_t + O(y)\partial_s + O(y)\partial_t + O(\delta)\partial_y,$$

due to the bound $|\eta| = O(\delta)$. Therefore, we have

$$\begin{aligned} -g_\delta &= \beta\left(\frac{t}{\delta^2}\right)\left(\left(\frac{1}{\delta\sigma} + O(\delta^2)\frac{|s|}{\delta^4} + O(\delta^2)\frac{1}{\delta\sigma} + O(\delta)\frac{1}{\delta^2}\right)\chi'(u) + O(1)\chi(u)\right) \\ &= \beta\left(\frac{t}{\delta^2}\right)\left(\frac{3}{4\delta\sigma} + O\left(\frac{\delta}{\sigma}\right) + O\left(\frac{1}{\delta}\right) + O(1)\right)\chi'(u) \\ &\sim \beta\left(\frac{t}{\delta^2}\right)\frac{1}{\delta\sigma}\chi'(u), \end{aligned}$$

provided that $\delta, \sigma \ll 1$. In the calculation above, we have used the fact that $\chi(u) = O(\chi'(u))$ on the support of g_δ . Thus (5) in Lemma 8.2 follows.

(6) follows from the construction of χ .

To check (7), we observe that $\text{supp}(g_\delta) \cup \text{supp}(h_\delta) \subset \text{supp}(a_\delta)$. Moreover, from the construction, g_δ, h_δ are independent of η whenever $0 \leq y < \frac{\delta^2}{2}$. Finally, to check the support of h_δ , we write $h_\delta = -H_p\left(\beta\left(\frac{t}{\delta^2}\right)\right)\chi(u)$. Since β is independent of y, η , we have $H_p\left(\beta\left(\frac{t}{\delta^2}\right)\right) = H_{-r}\left(\beta\left(\frac{t}{\delta^2}\right)\right)$, which is supported on $I \times L^-(\delta, \delta^2) \times \mathbb{R}_\eta$, thanks to $\text{supp}\beta' \subset [-1, -\frac{1}{2}]$. \blacksquare

E. PROOF OF LEMMA 7.1

Lemma 8.18. *In local coordinate Y_+ , we have*

$$P_h = -h^2 \frac{\bar{g}}{\sqrt{G}} \frac{\partial}{\partial y} (\sqrt{G} \bar{g}^{-1} \partial_y) + R_h = h^2 D_y^2 + \text{Op}_h(r) + O_{L^2 \rightarrow L^2}(h).$$

Moreover, R_h is a matrix-valued second order differential operator in x with scalar principal symbol $r(y, x, \xi) = 1 - \lambda(y, x, \xi)^2$, which is self-adjoint with respect to the $(\cdot|\cdot)_{L^2(Y_+)}$.

Proof. Denote by $y = x^0$, $\partial_0 = \partial_y$, $\partial_j = \partial_{x_j}$, $j = 1, 2, \dots, d-1$. Let $u \in \Lambda^1(Y_+)$ and $w \in \Lambda^2(Y_+)$ written in the form

$$u = u_0 dx^0 + u_j dx^j, \quad w = w_{0j} dx^0 \wedge dx^j + w_{jk} dx^j \wedge dx^k.$$

We have from direct calculation that

$$\begin{aligned} du &= (\partial_0 u_j - \partial_j u_0) dx^0 \wedge dx^j + \partial_k u_j dx^j \wedge dx^k, \\ d^* u &= -\frac{1}{\sqrt{G}} \partial_0 (u_0 \sqrt{G}) - \frac{1}{\sqrt{G}} \partial_j (g^{jk} u_k \sqrt{G}), \\ d^* w &= \frac{1}{\sqrt{G}} \partial_k (w_{0j} g^{jk} \sqrt{G}) dx^0 \\ &\quad - \frac{1}{\sqrt{G}} g_{jl} \left(\partial_0 (w_{0k} g^{kl} \sqrt{G}) + \partial_m (\sqrt{G} w_{pk} (g^{pl} g^{km} - g^{pm} g^{kl})) \right) dx^j \end{aligned}$$

From direct calculation, $h^2 \Delta_H u = h^2 (dd^* + d^*d)u = v_0 dx^0 + v_j dx^j + R_h u$ with

$$\begin{aligned} v_0 &= -h^2 \partial_0^2 u_0 - h^2 \frac{\partial_0(\sqrt{G})}{\sqrt{G}} \partial_0 u_0, \\ v_j &= -h^2 \partial_0^2 u_l - \frac{h^2}{\sqrt{G}} g_{jl} \partial_0 (g^{kl} \sqrt{G}) \partial_0 u_k \end{aligned}$$

and the $R_h u$ consists only the tangential derivatives ∂_j . Hence in the matrix form,

$$v = L_h u := -h^2 \frac{\bar{g}}{\sqrt{G}} \frac{\partial}{\partial y} (\sqrt{G} \bar{g}^{-1} \frac{\partial u}{\partial y}).$$

Moreover, one easily verified that $L_h^* = L_h$, thus $R_h^* = R_h$. \blacksquare

F. PROOF OF LEMMA 8.4

Proof of Lemma 8.4. For our need, it suffices to prove the last assertion. We first let $A_h = a(y, x, hD_y, hD_x)$ and $B_h = b(y, x, hD_x)$, then

$$A_h B_h u(y, x) = \frac{1}{(2\pi h)^d} \iint e^{\frac{i(x-x')\xi + i(y-y')\eta}{h}} \varphi(y', y, x, \eta, \xi) u(y', x') dy' dx' d\xi d\eta,$$

where

$$\varphi(y', y, x, \eta, \xi) = \frac{1}{(2\pi h)^{d-1}} \iint e^{\frac{i(x-z)(\xi' - \xi)}{h}} a(y, x, \eta, \xi') b(y', z, \xi) d\xi' dz.$$

Talor expansion gives

$$\varphi(y', y, x, \eta, \xi) = \varphi(y, y, x, \eta, \xi) + (y' - y) \int_0^1 \partial_{y'} \varphi(ty' + (1-t)y, y, x, \eta, \xi) dt.$$

Denote by $c(y, x, \eta, \xi) = \varphi(y, y, x, \eta, \xi)$, it is obvious that c is an interior symbol, since it can be viewed as a tangential symbol for fixed η , and we have $(1 + |\xi|)^m \lesssim (1 + |\xi| + |\eta|)^m$ for all $m \in \mathbb{R}$ on the support of c , thanks to the support property of a . Now we note $C_h = c(y, x, hD_y, hD_x)$, and write $A_h B_h u = C_h u + R'_h u$, where

$$\begin{aligned} R'_h u(y, x) &= \int_0^1 dt \frac{1}{(2\pi h)^d} \iint e^{\frac{i(x-x')\xi + i(y-y')\eta}{h}} (y - y') \partial_{y'} c_t(y', y, x, \eta, \xi) u(y', x') dy' dx' d\xi d\eta \\ &= ih \int_0^1 dt \frac{1}{(2\pi h)^d} \iint e^{\frac{i(x-x')\xi + i(y-y')\eta}{h}} \partial_\eta \partial_{y'} c_t(y', y, x, \eta, \xi) u(y', x') dy' dx' d\xi d\eta \\ &=: ih \int_0^1 C_t u(y, x) dt, \end{aligned}$$

with $c_t(y', y, x, \eta, \xi) = \varphi(ty' + (1-t)y, y, x, \eta, \xi)$. Notice that

$$\partial_\eta \partial_{y'} c_t(y', y, x, \eta, \xi) = \frac{1}{(2\pi h)^{d-1}} \iint e^{\frac{i(x-z)(\xi' - \xi)}{h}} \partial_\eta a(y, x, \eta, \xi') (\partial_{y'} b)(ty' + (1-t)y, z, \xi) d\xi' dz.$$

We need to be careful here since $\partial_y b$ only exists for $y > 0$ and at the point $y = 0$, the right derivative $(\partial_y^m)^+ b(0) := \lim_{y \rightarrow 0^+} \partial_y^m b(y)$ exists for any order m . Since we are dealing with Dirichlet boundary condition, we always apply a tangential operator $B(y, x, hD_x)$ to functions $u(y, x)$ with $u|_{y=0} = 0$ in the trace sense. We could thus extend $u(y', x')$ by $u(y', x') \mathbf{1}_{y' \geq 0}$ in y' in the expression of the form

$$\frac{1}{(2\pi h)^d} \iint e^{\frac{i(x-x')\xi' + i(y-y')\eta}{h}} \varphi(y', y, x, \eta, \xi) u(y', x') dy' dx' d\xi d\eta.$$

Therefore, we have

$$\sup_{y, y' \geq 0, 0 < t < 1, z, \xi} |\partial_z^\alpha \partial_\xi^\beta b(ty' + (1-t)y, z, \xi)| \leq C_{m, \alpha, \beta}, \forall m \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^{d-1}.$$

Now it is reduced to prove the uniform L^2 boundness of the operator

$$T_h u = \int_{\mathbb{R}^d} K_h(y', x', y, x) u(y', x') dy' dx',$$

with kernel

$$K_h(y', x', y, x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} e^{\frac{i(x-x')\xi + i(y-y')\eta}{h}} H_t(y', y, x, \eta, \xi) d\eta d\xi,$$

where

$$H_t(y', y, x, \eta, \xi) = \mathbf{1}_{y', y \geq 0} \frac{1}{(2\pi h)^{d-1}} \int e^{\frac{iz\zeta}{h}} a_1(y, x, \eta, \xi + \zeta) b_1(ty' + (1-t)y, x - z, \zeta) dz d\zeta.$$

From Schur's test, we need to show

$$\begin{aligned} \sup_{(y, x) \in \mathbb{R}_+^d} \int_{\mathbb{R}_+^d} |K_h(y', x', y, x)| dy' dx' &\leq C_1 < \infty, \\ \sup_{(y', x') \in \mathbb{R}_+^d} \int_{\mathbb{R}_+^d} |K_h(y', x', y, x)| dy dx &\leq C_2 < \infty, \end{aligned}$$

with C_1, C_2 independent of h and t .

To this end, we define

$$k_h(y, x, w, v) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iv\xi + iw\eta} H_t(y - hw, y, x, \eta, \xi) d\eta d\xi,$$

hence,

$$T_h u(y, x) = \frac{1}{h^d} \int_{\mathbb{R}_+^d} k_h(y, x, \frac{y-y'}{h}, \frac{x-x'}{h}) u(y', x') dy' dx'.$$

Notice that $H_t(y', y, x, \eta, \xi)$ is a tangential symbol, parametrized by (y', y, η) . Moreover, it is compactly supported in (y, x, η, ξ) variables, uniformly in the first variable y' . Thus, $\partial_\eta^m \partial_\xi^\alpha H_t(y - hw, y, x, \eta, \xi)$ has compact support in (η, ξ) and

$$|\partial_\eta^m \partial_\xi^\alpha H_t(y - hw, y, x, \eta, \xi)| \leq C_{m, \alpha}$$

for any $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{d-1}$. Thus, doing integration by part in the expression of k_h , we have

$$\sup_{(y, x)} |k_h(y, x, w, v)| \leq C(1 + |w| + |v|)^{-(d+1)}.$$

Therefore, we obtain

$$\begin{aligned} \int_{\mathbb{R}_+^d} |K_h(y', x', y, x)| dy' dx' &= \frac{1}{h^d} \int_{\mathbb{R}_+^d} |k_h(y, x, \frac{y-y'}{h}, \frac{x-x'}{h})| dy' dx' \\ &= \int_{\mathbb{R}^d} |k_h(y, x, w, v)| dw dv \\ &\leq C_1, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}_+^d} |K_h(y, x, y', x')| dy dx &= \frac{1}{h^d} \int_{\mathbb{R}_+^d} |k_h(y, x, \frac{y-y'}{h}, \frac{x-x'}{h})| dy dx \\ &\leq \int_{\mathbb{R}^d} \sup_{(y,x)} |k_h(y, x, w, v)| dw dv \\ &\leq C_2. \end{aligned}$$

■

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