



HAL
open science

Decays rates for Kelvin-Voigt damped wave equations II: The geometric control condition

Nicolas Burq, Chenmin Sun

► **To cite this version:**

Nicolas Burq, Chenmin Sun. Decays rates for Kelvin-Voigt damped wave equations II: The geometric control condition. Proceedings of the American Mathematical Society, 2021, 150 (3), pp.1021–1039. 10.1090/proc/15493 . hal-02964395v2

HAL Id: hal-02964395

<https://hal.science/hal-02964395v2>

Submitted on 15 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DECAY RATES FOR KELVIN-VOIGT DAMPED WAVE EQUATIONS II: THE GEOMETRIC CONTROL CONDITION

NICOLAS BURQ AND CHENMIN SUN

ABSTRACT. We study in this article decay rates for Kelvin-Voigt damped wave equations under a geometric control condition. When the damping coefficient is sufficiently smooth (C^1 vanishing nicely, see (1.3)) we show that exponential decay follows from geometric control conditions (see [5, 12] for similar results under stronger assumptions on the damping function).

1. INTRODUCTION

In this paper we investigate decay rates for Kelvin-Voigt damped wave equations under geometric control conditions. We work in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ and consider the following equation

$$(1.1) \quad \begin{cases} (\partial_t^2 - \Delta)u - \operatorname{div}(a(x)\nabla_x \partial_t u) = 0 \\ u|_{t=0} = u_0 \in H_0^1(\Omega), \quad \partial_t u|_{t=0} = u_1 \in L^2(\Omega) \\ u|_{\partial\Omega} = 0 \end{cases}$$

with a non negative damping term $a(x)$. The solution can be written as

$$(1.2) \quad U(t) = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{\mathcal{A}t} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

where the generator \mathcal{A} of the semi-group is given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & \operatorname{div} a \nabla \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

with domain

$$D(\mathcal{A}) = \{(u_0, u_1) \in H_0^1 \times L^2; \Delta u_0 + \operatorname{div} a \nabla u_1 \in L^2; u_1 \in H_0^1\}.$$

The energy of solutions

$$E(u)(t) = \int_{\Omega} (|\nabla_x u|^2 + |\partial_t u|^2) dx$$

satisfies

$$E((u_0, u_1))(t) - E((u_0, u_1))(0) = - \int_0^t \int_{\Omega} a(x) |\nabla_x \partial_t u|^2(s, x) ds.$$

Our purpose here is to show that if the damping a is sufficiently smooth, the exponential decay rate holds, dropping some unnecessary assumptions on the behaviour of the damping term where it becomes positive in previous works [5]. Namely we shall assume $a(x) \geq 0$ is $C^1(\bar{\Omega})$ and satisfy the regularity hypothesis

$$(1.3) \quad |\nabla a| \leq C a^{\frac{1}{2}}.$$

Our main result is

Theorem 1. *Assume that Ω is a compact Riemannian manifold with smooth boundary. Let $a \in C^1(\bar{\Omega})$ be a nonnegative function satisfying (1.3), such that the following geometric control condition is satisfied:*

- *There exists $\delta > 0$ such that all rays of geometric optics (straight lines) reflecting on the boundary according to the laws of geometric optics eventually reach the set $\omega_{\delta} = \{x \in \Omega : a(x) > \delta\}$ in finite time.*

Then there exists $\alpha > 0$, such that for all $t \geq 0$ and every $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the energy of solution $u(t)$ of (1.1) with initial data (u_0, u_1) satisfies

$$E[u](t) \leq e^{-\alpha t} E[u](0).$$

To prove this result, we first reduce it very classically in Section 2 to resolvent estimates. Since the low frequency estimates are true, we are reduced to the high frequency regime. The proof relies on resolvent estimates which are proved through a contradiction argument that we establish in Section 2. In Section 3 we prove *a priori* estimates for our sequences. The main task then is to prove a propagation invariance for these measures. A main difficulty to overcome is that it is not possible to put the damping term in the r.h.s. of the equation (1.1) and treat it as a *perturbation*. Instead we have to keep it on the left hand side and revisit the proof of the propagation property from [7]. In Section 4, we introduce the geometric tools necessary to tackle the boundary value problem and define semi-classical measures associated to our sequences. In Section 5 we prove the interior propagation result for our measures. Finally, in Section 6, we finish the proof of the contradiction argument by establishing the invariance of the semi-classical measures we defined up to the boundary. Here the proof uses crucially the main result in [7, Théorème 1].

Remark 1.1. Throughout this note, we shall prove that some operators of the type $P - \lambda \text{Id}$, $\lambda \in \mathbb{R}$ (resp. $\lambda \in i\mathbb{R}$) are invertible with estimates on the inverse. All these operators share the feature that they have compact resolvent, i.e. $\exists z_0 \in \mathbb{C}; (P - z_0)^{-1}$ exists and is compact (or it will be possible to reduce the question to this situation). As a consequence, since

$$(P - \lambda) = (P - z_0)^{-1}(\text{Id} + (z_0 - \lambda)^{-1}),$$

and $(\text{Id} + (z_0 - \lambda)^{-1})$ is Fredholm with index 0, to show that $(P - \lambda)$ is invertible with inverse bounded in norm by A , it is enough to bound the solutions of $(P - \lambda)u = f$ and prove

$$(P - \lambda)u = f \Rightarrow \|u\|_{L^2} \leq A\|f\|_{L^2}.$$

Remark 1.2. Assume that a is the restriction to Ω of a nonnegative $C^2(\mathbb{R}^d)$ function. Then the hypothesis (1.3) is satisfied.

Proof. It is enough to prove (1.3) for $\Omega = \mathbb{R}^d$, $a \in C^2(\Omega)$. Let $x_0 \in \mathbb{R}^d$ and denote by $z_0 = \nabla a(x_0)$. From Taylor's formula, we have for any $s \in \mathbb{R}$, there exists $\theta \in (0, 1)$, such that

$$a(x_0 + sz_0) = a(x_0) + s|z_0|^2 + \frac{s^2}{2} a''(x_0 + \theta sz_0)(z_0, z_0) \geq 0$$

Since this polynomial in s is non negative, we deduce that its discriminant is non positive

$$|z_0|^4 - 2\|a''\|_\infty |z_0|^2 a(z_0) \leq 0 \Rightarrow |\nabla_x a(x_0)|^2 \leq 2\|a''\|_\infty a(z_0).$$

Notice that in the above lemma, the condition *cannot* be relaxed to $a \in C^2(\overline{\Omega})$, $a \geq 0$. Indeed, consider the following example: $\Omega = B(0, 1)$ and $a(x) = 1 - |x|^2$ for $|x| \leq 1$. Then obviously $a \in C^2(\overline{\Omega})$, $a \geq 0$, but on the boundary, $\nabla_x a \neq 0$, while $a = 0$. \square

Acknowledgment. The first author is supported by Institut Universitaire de France and ANR grant ISDEEC, ANR-16-CE40-0013. The second author is supported by the postdoc programme: "Initiative d'Excellence Paris Seine" of CY Cergy-Paris Université and ANR grant ODA (ANR-18-CE40- 0020-01).

2. CONTRADICTION ARGUMENT

It is well known that decay estimates for the evolution semi-group follow from resolvent estimates [1, 2, 4]. Here we shall need the classical (see e.g. [6, Proposition A.1])

Theorem 2. *The exponential decay of the Kelvin Voigt semi-group is equivalent to the following resolvent estimate: There exists C such that for all $\lambda \in \mathbb{R}$, the operator $(\mathcal{A} - i\lambda)$ is invertible from $D(\mathcal{A})$ to \mathcal{H} and its inverse satisfies*

$$(2.1) \quad \|(\mathcal{A} - i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C$$

Let us first recall that

$$(2.2) \quad (\mathcal{A} - i\lambda) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \Leftrightarrow \begin{cases} -i\lambda u + v = f \\ \Delta u + \operatorname{div} a \nabla_x v - i\lambda v = g \end{cases}$$

From [8, Section 2], we have the following low frequencies estimates of the resolvent of the operator \mathcal{A} :

Proposition 2.1. *Assume that $a \in L^\infty$ is non negative $a \geq 0$ and non trivial $\int_\Omega a(x)dx > 0$. Then for any $M > 0$, there exists $C > 0$ such that for all $\lambda \in \mathbb{R}$, $|\lambda| \leq M$, the operator $\mathcal{A} - i\lambda$ is invertible from $D(\mathcal{A})$ to \mathcal{H} with estimate*

$$(2.3) \quad \|(\mathcal{A} - i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C.$$

As a consequence, to prove Theorem 1 it is enough to study the high frequency regime $\lambda \rightarrow +\infty$ and prove

Proposition 2.2. *Assume that $a \in C^1(\overline{\Omega})$ is a nonnegative function satisfying (1.3). Then under the geometric control condition, there exists $\Lambda_0 > 0$ such that for any $|\lambda| > \Lambda_0$ we have*

$$\|(\mathcal{A} - i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C.$$

By standard argument, we can reduce the proof of Proposition 2.2 to a semi-classical estimate. We denote by $0 < h = |\lambda|^{-1} \ll 1$ and

$$P_h = -h^2 \Delta - 1 - ih \operatorname{div} a(x) \nabla.$$

Proposition 2.3. *There exists $C > 0$, such that for all $0 < h \ll 1$,*

$$(2.4) \quad \|P_h^{-1}\|_{\mathcal{L}(L^2)} \leq Ch^{-1}.$$

For the proof of Proposition 2.3, we argue by contradiction. Assume that there exist sequences $(u_n) \subset H^2 \cap H_0^1$, $(f_n) \subset L^2$ and $h_n \rightarrow 0$, such that $P_{h_n} u_n = f_n$, $\|u_n\|_{L^2} = 1$ and $\|f_n\|_{L^2} = o(h_n)$. We will use a semi-classical notation and denote by (u_h, f_h) the sequences with the properties

$$(2.5) \quad \|u_h\|_{L^2} = 1, \quad \|f_h\|_{L^2} = o(h), \quad P_h u_h = f_h.$$

Sometimes we even omit the subindex for u_h, f_h . In the following subsections, we will prove propagation estimates for such sequences.

3. A PRIORI ESTIMATES

In this section we establish a series of *a priori* estimates for the sequence defined in (2.5).

Lemma 3.1. *Assume that $a \in L^\infty(\Omega)$ is a non-negative function, then*

$$(3.1) \quad (1) \quad \int_\Omega (|u_h|^2 - |h \nabla u_h|^2) = \operatorname{Re} \int_\Omega f_h \overline{u_h} = o(h);$$

$$(3.2) \quad (2) \quad \int_\Omega a(x) |h \nabla u_h|^2 = h \operatorname{Im} \int_\Omega f_h \overline{u_h} = o(h^2).;$$

$$(3.3) \quad (3) \quad h^2 \|u_h\|_{H^2} = O(1).$$

Proof. We get (1) and (2) by multiplying the equation $P_h u = f$ by \overline{u} , integrating by part and taking the real and imaginary parts respectively. For (3), from the equation, we have

$$h^2 \Delta u + u + ih \nabla a \cdot \nabla u + iha \Delta u = -f, \text{ i.e. } h^2 \Delta u = -\frac{u + f + i \nabla a \cdot h \nabla u}{1 + ih^{-1} a(x)}.$$

From the global estimate of the Poisson equation

$$\|w\|_{H^2} \leq C \|\Delta w\|_{L^2}, \forall w \in H^2 \cap H_0^1,$$

we obtain that $\|h^2 u\|_{H^2} = O(1)$. □

Corollary 3.2. *Assume that $a \in C^1(\overline{\Omega})$ is a non-negative function satisfying (1.3), then*

$$(3.4) \quad \|a^{\frac{1}{2}} u_h\|_{L^2} + \|a^{\frac{1}{2}} h \nabla u_h\| + \|a^{\frac{1}{2}} h^2 \Delta u_h\|_{L^2} = o(h).$$

Proof. We only need to estimate $\int_{\Omega} a(x)|u|^2$, since $\int_{\Omega} a(x)|h\nabla u|^2 = o(h^2)$ is just (3.2). Multiplying $P_h u = f$ by $a\bar{u}$ and taking the real part, we have

$$\int_{\Omega} a(x)|u|^2 = \operatorname{Re} \int_{\Omega} h\nabla u \cdot h\nabla(a\bar{u}) - \operatorname{Im} h \int_{\Omega} a(x)\nabla u \cdot \nabla(a(x)\bar{u}) + \operatorname{Im} \int_{\Omega} a(x)f\bar{u}.$$

Since $|\nabla a| \leq Ca^{\frac{1}{2}}$, the first term on the r.h.s. can be bounded by

$$\|a^{\frac{1}{2}}h\nabla u\|_{L^2}^2 + h\|\nabla ah\nabla u\|_{L^2}\|u\|_{L^2} = o(h^2).$$

The third term of r.h.s is bounded by $o(h)\|a^{\frac{1}{2}}u\|_{L^2}$, and the second term can be bounded by

$$h\left|\int_{\Omega} a\nabla a \cdot \bar{u}\nabla u\right| \leq h\|a^{\frac{1}{2}}\bar{u}\|_{L^2}\|a^{\frac{1}{2}}\nabla a\nabla u\|_{L^2} \leq C\|a^{\frac{1}{2}}h\nabla u\|\|a^{\frac{1}{2}}u\|_{L^2} = o(h)\|a^{\frac{1}{2}}u\|_{L^2}.$$

For the second derivative, we observe that

$$a(x)^{\frac{1}{2}}h^2\Delta u = -\frac{a^{\frac{1}{2}}u + a^{\frac{1}{2}}f + iha^{\frac{1}{2}}\nabla a \cdot h\nabla u}{1 + ih^{-1}a(x)},$$

thus $\|a^{\frac{1}{2}}h^2\Delta u\|_{L^2} = o(h)$. This completes the proof of Corollary 3.2. \square

Let ν be the out-normal vector field on $\partial\Omega$. We denote by $L^2(\partial) = L^2(\partial\Omega)$. The following hidden regularity holds:

Lemma 3.3. *Assume that $a \in C^1(\bar{\Omega})$ is a nonnegative function satisfying (1.3), then*

$$\|h\partial_{\nu}u\|_{L^2(\partial)} = O(1), \quad \|a^{\frac{1}{2}}h\partial_{\nu}u\|_{L^2(\partial)} = O(h^{\frac{1}{2}}).$$

Proof. We use the standard multiplier method. Let $L = b_j(x)\partial_j$ be an C^2 extension of the out-normal vector field ν , where b_j 's are supported in a neighborhood of $\partial\Omega$. Write $P_h = P_{h,0} + iM_h$, where

$$P_{h,0} = -h^2\Delta - 1, \quad M_h = -h\operatorname{div} a(x)\nabla$$

are self-adjoint operators. Consider the commutator $[P_h, L] = [P_{h,0}, L] + i[M_h, L]$. Note that $[P_{h,0}, L] = \frac{1}{h}[P_{h,0}, hL]$ belongs to $h^2\operatorname{Op}(S^2)^*$, we deduce that $([P_{h,0}, L]u, u)_{L^2} = O(1)$. By direct computation, we have

$$\begin{aligned} -([M_h, L]u, u)_{L^2} &= (h\partial_k[a\partial_k, b_j\partial_j]u, u)_{L^2} + h([\partial_k, b_j\partial_j]a\partial_k u, u)_{L^2} \\ &= (h\partial_k(a(\partial_k b_j)\partial_j u - b_j(\partial_j a)\partial_k u), u)_{L^2} + ((\partial_k b_j)h\partial_j(a\partial_k u), u)_{L^2} \\ &= -((a\partial_k b_j)\partial_j u - b_j(\partial_j a)\partial_k u, h\partial_k u)_{L^2} - (a\partial_k u, h\partial_j((\partial_k b_j)u))_{L^2} \end{aligned}$$

From Corollary 3.2, the absolute value of the r.h.s. can be bounded by constant times

$$\|a\nabla u\|_{L^2} + \|\nabla a\nabla u\|_{L^2} = o(1).$$

Therefore, $([P_h, L]u, u)_{L^2} = O(1)$. On the other hand, by developing the commutator and exploiting the equation, we have

$$\begin{aligned} ([P_h, L]u, u)_{L^2} &= (P_h Lu, u)_{L^2} - (Lf, u)_{L^2} \\ &= (Lu, P_h^* u)_{L^2} - (f, L^* u)_{L^2} + h^2\|\partial_{\nu}u\|_{L^2(\partial)}^2 + ih\|a^{\frac{1}{2}}\partial_{\nu}u\|_{L^2(\partial)}^2. \end{aligned}$$

Observe that

$$(Lu, P_h^* u)_{L^2} = (Lu, f - 2M_h u)_{L^2} = o(1) - 2(Lu, M_h u)_{L^2} = o(1) - 2(Lu, h\nabla a \cdot \nabla u + ha\Delta u)_{L^2}.$$

Since L is a first order differential operator and from Corollary 3.2 that $\|a^{\frac{1}{2}}h\Delta u\|_{L^2} = o(1)$, we have

$$|(Lu, h\nabla a \cdot \nabla u + ha\Delta u)_{L^2}| \leq \|h\nabla u\|_{L^2}\|\nabla a\nabla u\|_{L^2} + \|a^{\frac{1}{2}}\nabla u\|_{L^2}\|a^{\frac{1}{2}}h\Delta u\|_{L^2} = o(1).$$

Therefore,

$$\|h\partial_{\nu}u\|_{L^2(\partial)}^2 + ih\|a^{\frac{1}{2}}\partial_{\nu}u\|_{L^2(\partial)}^2 = O(1).$$

*Strictly speaking, the symbol of L is not C^∞ , but here we only need $\frac{1}{h}[P_{h,0}, hL] = O(h^2)$ on L^2 .

The proof of Lemma 3.3 is then completed by taking real and imaginary parts. \square

Let $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(z) \equiv 1$ for $|z| \leq 1$ and $\chi(z) \equiv 0$ for $|z| > 2$. We decompose

$$(3.5) \quad u_h = v_h + w_h, \quad v_h = \chi(ah^{-1})u_h, \quad w_h = (1 - \chi(ah^{-1}))u_h.$$

In the rest of this note, we always assume that $a \in C^1(\overline{\Omega})$ is a nonnegative function satisfying (1.3).

Lemma 3.4. *We have*

$$\|w_h\|_{L^2} + \|h\nabla w_h\|_{L^2} = o(h^{\frac{1}{2}}), \quad \|a^{\frac{1}{2}}v_h\|_{L^2} + \|a^{\frac{1}{2}}h\nabla v_h\|_{L^2} = o(h),$$

and

$$\|u_h\|_{H_h^1(a \geq ch)} + \|v_h\|_{H_h^1(a \geq ch)} = o(h^{\frac{1}{2}}).$$

Proof. By definition,

$$\int_{\Omega} (|w|^2 + |h\nabla w|^2) \leq \int_{a \geq h} (|u|^2 + |h\nabla u|^2 + |\nabla a|^2 |u|^2).$$

The conclusion then follows from Corollary 3.2 and the fact that $|\nabla a|^2 \leq Ca$. Similarly, for any other cutoff to the region $a \geq ch$, we deduce that $\|u_h\|_{H_h^1(a \geq ch)} = o(h^{\frac{1}{2}})$. For the estimate of v , note that $a^{\frac{1}{2}}h\nabla v = a^{\frac{1}{2}}\chi h\nabla u + a^{\frac{1}{2}}\nabla a \chi' v$, from Corollary 3.2, we have

$$\|a^{\frac{1}{2}}h\nabla v\|_{L^2} \leq \|a^{\frac{1}{2}}h\nabla u\|_{L^2} + \|\chi' a^{\frac{1}{2}}(a^{\frac{1}{2}}v)\|_{L^2} = o(h).$$

This completes the proof of Lemma 3.4. \square

4. GEOMETRY, SEMI-CLASSICAL MEASURES

Having the *a priori* estimates of the previous section at hand, we can now study v_h . For some subsequence of v_h , we will associate it a semi-classical measure and then prove the invariance of the measure under the generalized geodesic flow. First recall some geometric preliminaries from [7].

4.1. Geometry. Denote by ${}^bT\Omega$ the bundle of rank d whose sections are the vector fields tangent to $\partial\Omega$, ${}^bT^*\Omega$ the dual bundle (Melrose's compressed cotangent bundle) and $j : T^*\Omega \rightarrow {}^bT^*\Omega$ the canonical map. In any coordinate system where $\Omega = \{x = (x_d > 0, x')\}$, the bundle ${}^bT\Omega$ is generated by the fields $\frac{\partial}{\partial x'}$, $x_d \frac{\partial}{\partial x_d}$ and j is defined by

$$(4.1) \quad j(x_d, x', \xi_d, \xi') = (x_d, x', v = x_d \xi_d, \xi').$$

Denote by $\text{Car}P_0$ the semi-classical characteristic manifold of $P_0 = -h^2\Delta - 1$ and Z its projection

$$(4.2) \quad \text{Car}P_0 = \{(x, \xi) = (x', x_d, \xi', \xi_d) \in T^*\mathbb{R}^d |_{\overline{\Omega}}; p(x, \xi) = 0\}, \quad Z = j(\text{Car}P_0).$$

The set Z is a locally compact metric space.

Consider, near a point $x_0 \in \partial\Omega$ a geodesic system of coordinates for which $x_0 = (0, 0)$, $\Omega = \{(x_d, x') \in \mathbb{R}^+ \times \mathbb{R}^{d-1}\}$ and the operator P_0 has the form (near x_0)

$$(4.3) \quad P_{h,0} = -h^2\Delta - 1 = h^2 D_{x_d}^2 - R(x_d, x', hD_{x'}) + hQ(x, hD_x),$$

with R a second order tangential operator and Q a first order operator.

We recall now the usual decomposition of $T^*\partial\Omega$ (in this coordinate system). Denote by $r(x', x_d, \xi')$ the semi-classical principal symbol of R and $r_0 = r|_{x_d=0}$. Then $T^*\partial\Omega$ is the disjoint union of $\mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$ with

$$(4.4) \quad \mathcal{E} = \{r_0 < 0\}, \mathcal{G} = \{r_0 = 0\}, \mathcal{H} = \{r_0 > 0\}.$$

Remark that j gives a natural identification between $Z|_{\partial M}$ and $\mathcal{H} \cup \mathcal{G} \subset T^*\partial M$. In \mathcal{G} we distinguish between the *diffractive* points $\mathcal{G}^{2,+} = \{r_0 = 0, r_1 = \partial_{x_d} r|_{x_d=0} > 0\}$ and the *gliding* points $\mathcal{G}^- = \{r_0 = 0, r_1 = \partial_{x_d} r|_{x_d=0} \leq 0\}$. We will make the assumption (Ω has no infinite order contact with its tangents) that for any $\varrho_0 \in T^*\partial M$, there exists $N \in \mathbb{N}$ such that

$$H_{r_0}^N(r_1) \neq 0$$

The definition of the generalized bicharacteristic flow, φ_s associated to the operator P_0 is essentially the definition given in [11]:

Definition 4.1. A generalized bicharacteristic curve $\gamma(s)$ is a continuous curve from an interval $I \subset \mathbb{R}$ to Z such that

- (1) if $s_0 \in I$ and $\gamma(s_0) \in T^*\Omega$ then close to s_0 , γ is an integral curve of the Hamiltonian vector field $H_{\tilde{P}}$
- (2) If $s_0 \in I$ and $\gamma(s_0) \in \mathcal{H} \cup \mathcal{G}^{2,+}$ then there exists $\varepsilon > 0$ such that for $0 < |s - s_0| < \varepsilon$, $x_d(\gamma(s)) > 0$
- (3) If $s_0 \in I$ and $\gamma(s_0) \in \mathcal{G}^-$ then for any function $f \in C^\infty(T^*\mathbb{R}^d |_{\tilde{\Omega}})$ satisfying the symmetry condition

$$(4.5) \quad \forall \varrho_0 \in Z, \forall \widehat{\varrho}_0, \widetilde{\varrho}_0 \in j^{-1}(\varrho_0) \cap \text{Car}(\tilde{P}), f(\widehat{\varrho}_0) = f(\widetilde{\varrho}_0)$$

then

$$\frac{d}{ds} f(j(\gamma(s)) |_{s=s_0} = H_{\tilde{P}} |_{j^{-1}(\gamma(s_0))} f(j^{-1}(\gamma(s_0)))$$

It is proved in [11] that under the assumption of no infinite order contact, through every point $\varrho_o \in {}^bT^*M \setminus \{0\}$ there exists a unique generalized bicharacteristic (which is furthermore a limit of bicharacteristics having only hyperbolic contacts with the boundary). This defines the flow Φ .

4.2. Wigner measures. Consider functions $a = a_i + a_\partial$ with $a_i \in C_0^\infty(T^*M)$, and $a_\partial \in C_0^\infty(\mathbb{R}^{2d-1})$. Such symbols are quantized in the following way: take $\varphi_i \in C_0^\infty(M)$ (resp $\varphi_\partial \in C_0^\infty(\mathbb{R}^d)$) equal to 1 near the x -projection of $\text{supp}(a_i)$ (resp the x -projection of $\text{supp}(a_\partial)$) and define

$$(4.6) \quad \text{Op}_h^{\varphi_i, \varphi_\partial}(a)(x, hD_x)f = \frac{1}{(2\pi h)^d} \int e^{i(x-y)\cdot\xi/h} a_i(x, \xi) \varphi_i(y) f(y) dy d\xi \\ + \frac{1}{(2\pi h)^{d-1}} \int e^{i(x'-y')\cdot\xi'/h} a_\partial(x_d, x', \xi) \varphi_\partial(x_d, y') f(x_d, y') dy' d\xi'.$$

Remark that according to the symbolic semi-classical calculus, the operator $\text{Op}_h^{\varphi_i, \varphi_\partial}(a)$ does not depend on the choice of functions $\varphi_i, \varphi_\partial$, modulo operators on L^2 of norms bounded by $O(h^\infty)$. For conciseness we shall in the sequel drop the index $\varphi_i, \varphi_\partial$.

Denote by \mathcal{A}^h the space of the operators which are a finite sum of operators obtained as above in suitable coordinate systems near the boundary and for $B \in \mathcal{A}$, by $b = \sigma(B)$ the semiclassical symbol of the operator A . For such functions b we can define $\kappa(b) \in C^0(Z)$ by

$$(4.7) \quad \kappa(b)(\rho) = b(j^{-1}(\rho))$$

(the value is independent of the choice of $j^{-1}(\rho)$ since the operator is tangential).

The set

$$(4.8) \quad \{\kappa(b), b = \sigma(B), B \in \mathcal{A}^h\}$$

is a locally dense subset of $C_c^0(Z)$.

4.2.1. Elliptic regularity. The sequence v_h satisfies (with $\chi_h = \chi(a/h)$)

$$P_h v_h = \chi_h P_h u_h - h^2 \text{div}(\nabla \chi_h u_h) - h^2 \nabla \chi_h \nabla u_h - ih \text{div}(a \nabla \chi_h u_h) - iha \nabla \chi_h \cdot \nabla u_h.$$

Since $\nabla \chi_h = h^{-1} \chi'(a/h) \nabla a$ and $|\nabla a| \lesssim a^{\frac{1}{2}}$, by Corollary 3.2, we have

$$P_h v_h = o_{L^2}(h) + o_{H^{-1}}(h^2).$$

Thus

$$(h^2 \Delta + 1)v_h = -ih \text{div}(a \nabla(\chi_h u)) + o_{L^2}(h) + o_{H^{-1}}(h^2).$$

Using Corollary 3.2 again and the fact that $a \lesssim h$ on the support of χ_h , we deduce that $ha \nabla(\chi_h u) = o_{L^2}(h^{\frac{3}{2}})$, hence

$$(h^2 \Delta + 1)v_h = o(h^{\frac{3}{2}})_{H^{-1}(\Omega)} + o_{L^2}(h).$$

We deduce, by standard elliptic regularity results

Proposition 4.2. *If a_i is equal to 0 near $\text{Car}(P_0)$ then*

$$(4.9) \quad \lim_{k \rightarrow +\infty} (\text{Op}_{h_k}(a_i)v_{h_k}, v_{h_k})_{L^2} = 0,$$

while near the boundary (see e.g. [7, Appendice A.1] in a slightly different context) we get

Proposition 4.3. *If a_∂ is equal to 0 near Z (i.e. a_i is supported in the elliptic region) then*

$$(4.10) \quad \lim_{k \rightarrow +\infty} (a_\partial(x', x_d, h_k D_{x'})v_{h_k}, v_{h_k})_{L^2} = 0.$$

Remark 4.4. Note that if we regard the damping term $h \text{div}(a \nabla(\chi_h u)) = o_{H^{-1}}(h^{\frac{3}{2}})$ as a source term, we are not able to use the classical propagation theorem for $h^2 \Delta + 1$ as a black box, as such a strategy would require smaller r.h.s., namely $o_{H^{-1}}(h^2) + o_{L^2}(h)$. On the other hand, an integration by parts shows from Lemma 3.4

$$\left(\text{div}(a \nabla(\chi_h u)), \chi_h u \right)_{L^2} = -\|a^{1/2} \nabla_x \chi_h u\|_{L^2}^2 = o(1),$$

and this will ensure that in the propagation estimates such terms are invisible. The key of our analysis in the sequel will be to systematically use this procedure: testing the damping term on expressions like $Q_h \chi_h u$, doing the integration by part and then balancing $a^{\frac{1}{2}}$ to the other side. It is to perform this analysis that we need the condition $|\nabla a| \leq C a^{\frac{1}{2}}$ to ensure the gain $O(h)$ from the commutator $[a^{\frac{1}{2}}, Q_h]$. More precisely, we shall need the following lemma:

Lemma 4.5. *Assume that $Q_h, B_{0,h}, B_{1,h}$ are tangential h -pseudodifferential operators of order 0 and $B_h = B_{0,h} + B_{1,h} h D_{x_d}$, then*

$$h^{-1} (Q_h M_h u, B_h u)_{L^2} = o(1),$$

where $M_h = -h \text{div} a \nabla$.

Proof. Since $M_h u = -\nabla a h \nabla u - a h \Delta u$, from Corollary 3.2, we have

$$h^{-1} |(Q_h \nabla a h \nabla u, B_h u)_{L^2}| \leq C \|\nabla a \nabla u\|_{L^2} \|u\|_{H_h^1} = o(1).$$

To estimate $(Q_h a \Delta u, B_h u)_{L^2}$, we write $Q_h a \Delta u = a^{\frac{1}{2}} Q_h a^{\frac{1}{2}} \Delta u + [Q_h, a^{\frac{1}{2}}] a^{\frac{1}{2}} \Delta u$. From Corollary 3.2, $a^{\frac{1}{2}} \Delta u = o_{L^2}(h^{-1})$. By Corollary A.2, $[Q_h, a^{\frac{1}{2}}], [B_h, a^{\frac{1}{2}}] = O_{\mathcal{L}(L^2)}(h)$. Therefore,

$$(Q_h a \Delta u, B_h u)_{L^2} = (Q_h a^{\frac{1}{2}} \Delta u, B_h a^{\frac{1}{2}} u)_{L^2} + o(1).$$

Again from Corollary 3.2, we have $B_h a^{\frac{1}{2}} u = O_{L^2}(h)$, hence $(Q_h a \Delta u, B_h u)_{L^2} = o(1)$. The proof of Lemma 4.5 is complete. \square

4.2.2. *Definition of the measure.* The following results gives the existence of semi-classical measures.

Proposition 4.6. *Let $(v_{h_{k_p}})$ be a sequence bounded in $L^2(\Omega)$. There exists a subsequence (k_p) and a Radon positive measure μ on Z such that*

$$(4.11) \quad \forall Q \in \mathcal{A}^{h_{k_p}} \quad \lim_{p \rightarrow \infty} (Q v_{h_{k_p}}, v_{h_{k_p}})_{L^2} = \langle \mu, \kappa(\sigma(Q)) \rangle.$$

The proof of this result relies on the Gårding inequality for tangential operators (see G. Lebeau [10] for a proof in the classical context and [3, 9] for the semi-classical construction). As before, we drop the indexes k_p and denote by (v_h) the extracted sequence.

Proposition 4.7 (First properties of the measure μ). *We have*

$$(4.12) \quad \mu(\mathcal{H}) = 0.$$

Moreover, for any tangential symbol b ,

$$(4.13) \quad \limsup_{k \rightarrow +\infty} |(\text{Op}_h(b) h_k D_{x_d} v_{h_k}, v_{h_k})_{L^2}| \leq C \sup_{\varrho \in \text{supp}(b)} |r|^{1/2} |b|.$$

Proof. The first property (4.12) follows from the fact that the trajectories near a hyperbolic point is transversal to the boundary. It follows from [7], with additional attention to the damping term $ih\operatorname{div}a\nabla v_h$. We factorize $P_{h,0} = -h^2\Delta - 1$ as $(hD_{x_d} - L_h^+)(hD_{x_d} - L_h^-) + \mathcal{O}_{H^\infty}(h^\infty)$ (see [7, Lemme 6.1]) near $\rho_0 \in \mathcal{H}$ and choose L_h^\pm with principal symbols $\pm l(x', x_d, \xi') = \pm\sqrt{r(x', x_d, \xi')}$. we denote by $q_0(x', \xi') \in C_c^\infty(\mathcal{H})$ and $q^\pm(y, x', \xi')$ solutions of

$$\partial_{x_d} q^\pm \mp \{l, q^\pm\} = 0, \quad q^\pm|_{x_d=0} = q_0.$$

Denote by

$$u^\pm := \psi(x_d)Q_h^\pm(hD_{x_d} - L_h^\mp)u,$$

where $\psi(x_d) \equiv 1$ if $0 \leq x_d \leq \epsilon_0$. We have

$$\begin{aligned} (hD_{x_d} - L_h^\pm)u^\pm &= \psi(x_d)[hD_{x_d} - L_h^\pm, Q_h^\pm](hD_{x_d} - L_h^\mp)u + Q_h^\pm f_h + \frac{h}{i}\psi'(x_d)Q_h^\pm(hD_{x_d} - L_h^\mp)u \\ &\quad + iQ_h^\pm M_h u + \mathcal{O}_{L^2}(h^\infty) \end{aligned}$$

where $M_h = h\operatorname{div}a\nabla$ and $f_h = o_{L^2}(h)$. By definition of q^\pm , the first term of r.h.s. is $\mathcal{O}(h^2)$, hence

$$(4.14) \quad (hD_{x_d} - L_h^\pm)u^\pm = g_h^\pm - ih\psi'(x_d)Q_h^\pm(hD_{x_d} - L_h^\mp)u + iQ_h^\pm M_h u, \quad g_h^\pm = o_{L^2}(h).$$

We have

$$h \frac{d}{dx_d} (u^\pm, u^\pm)_{L^2(\partial)} = -2 \operatorname{Im} (g_h^\pm - ih\psi'(x_d)Q_h^\pm(hD_{x_d} - L_h^\mp)u + iQ_h^\pm M_h u, u^\pm)_{L^2(\partial)} + i((L_h^\pm - L_h^{\pm,*})u^\pm, u^\pm)_{L^2(\partial)}.$$

For $y_0 \leq \epsilon_0$, we have

$$\begin{aligned} \|u^\pm(y_0)\|_{L^2(\partial)}^2 &\leq \|u^\pm(0)\|_{L^2(x_d=0)}^2 + Ch^{-1}\|g_h^\pm\|_{L^2(x_d \leq y_0)}\|u^\pm\|_{L^2(x_d \leq y_0)} + C\|u^\pm\|_{L^2(x_d \leq y_0)}^2 \\ &\quad + Ch^{-1} |(Q_h^\pm M_h u, u^\pm)_{L^2(x_d \leq y_0)}| \end{aligned}$$

The second line of r.h.s. is $o(1)$, due to Lemma 4.5, and the first line of r.h.s. can be bounded by

$$\|u^\pm(0)\|_{L^2(x_d=0)}^2 + C\|u^\pm\|_{L^2(x_d \leq y_0)}^2 + o(1).$$

Integrating both sides over $y_0 \leq \epsilon_0$, letting $h \rightarrow 0$ and then $\epsilon_0 \rightarrow 0$, we deduce that $\langle \mu \mathbf{1}_{y=0}, q_0 \rangle = 0$. This proves (4.12).

For (4.13), it suffices to prove the inequality for u instead of v . By Cauchy-Schwarz,

$$|(\operatorname{Op}_h(b)h\partial_{x_d}u, u)_{L^2}| \leq |(\operatorname{Op}_h(b)^*\operatorname{Op}_h(b)h\partial_{x_d}u, h\partial_{x_d}u)_{L^2}|^{\frac{1}{2}} \|u\|_{L^2}.$$

Doing integration by part, we have

$$(\operatorname{Op}_h(b)^*\operatorname{Op}_h(b)h\partial_{x_d}u, h\partial_{x_d}u)_{L^2} = -(\operatorname{Op}_h(b)^*\operatorname{Op}_h(b)h^2\partial_{x_d}^2u, u)_{L^2} + \mathcal{O}(h).$$

Replacing $h^2\partial_{x_d}^2u$ by equation $h^2\partial_{x_d}^2u = -R_h u - iM_h u + \mathcal{O}_{L^2}(h)$, we deduce that

$$|(\operatorname{Op}_h(b)^*\operatorname{Op}_h(b)h^2\partial_{x_d}^2u, u)_{L^2}| \leq |(\operatorname{Op}_h(|b|^2)R_h u, u)_{L^2}| + \mathcal{O}(h) + |(\operatorname{Op}_h(b)^*\operatorname{Op}_h(b)M_h u, u)_{L^2}|.$$

From Lemma 4.5, the third term of r.h.s. is $o(h)$. Passing $h \rightarrow 0$, we complete the proof of Proposition 4.7. \square

4.3. Invariance of the measure. The key to prove the invariance of the measure will be to apply the propagation theorem in [7, Théorème 1].

Theorem. *The two following properties are equivalent*

- (1) *The measure μ is invariant along the generalised flow.*
- (2) *The measure μ satisfies $\dot{\mu} = 0$ and $\mu(\mathcal{G}_2^+) = 0$ in the sense that $\langle \mu, \{p, q\} \rangle = 0$ holds for any even symbol $q \in C_c^\infty(\operatorname{Car}(P_0))$, i.e. $q(x', x_d = 0, \xi', \xi_d) = q(x', x_d = 0, \xi', -\xi_d)$.*

Remark 4.8. Technically, Theorem 4.3 is proved in [7] for time dependent measures, i.e. measures depending in addition on two additional variables $(t, \tau) \in T^*\mathbb{R}$, and p is replaced by $p - \tau^2$. However, it is easy to apply the results from [7] by considering the measure

$$(4.15) \quad \tilde{\mu} = \mu_{x, \xi} \otimes dt \otimes \delta_{\tau=1},$$

which is supported in $\text{Car}(-\Delta + \partial_t^2)$ and satisfies $\tilde{\mu} = 0$ in the sense that $\langle \mu, \{p - \tau^2, q\} \rangle = 0$ holds for any even symbol $q \in C_c^\infty(\text{Car}(P_0 - \tau^2))$, i.e. $q(x', x_d = 0, t, \xi', \xi_d, \tau) = q(x', x_d = 0, t, \xi', -\xi_d, \tau)$. Remark that though we shall not use it, the measure $\tilde{\mu}$ is, in the sense of [7, Section 2], the microlocal defect measure on the sequence $v_n(t, x) = h_n e^{it h_n^{-1}} u_n(x)$ (the pre-factor h_n comes from the H^1 normalisation of the sequence v_n in [7]). Now, the generalised bicharacteristic flow for $p_0 - \tau^2$, Ψ_s is given in terms of the generalised bicharacteristic flow for p_0 , ψ_s by

$$\Psi_s(t, x, \tau = 1, \xi) = (t - 2s, \tau = 1, \psi_s(x, \xi)),$$

The set of diffractive points $\tilde{\mathcal{G}}^{2,+}$ in the time dependent frame-work is given by

$$\tilde{\mathcal{G}}^{2,+} = \mathcal{G}^{2,+} \times \mathbb{R} \times \{\tau = \pm 1\}$$

and consequently,

$$\mu(\mathcal{G}^{2,+}) = 0 \Leftrightarrow \tilde{\mu}(\tilde{\mathcal{G}}^{2,+}) = 0,$$

and in view of the particular form (4.15), the invariance of $\tilde{\mu}$ by Ψ_s is equivalent to the invariance of μ by ψ_s .

Let us now briefly explain the procedure we are going to follow.

- First from Proposition 4.6 and the ellipticity (Proposition 4.9, Proposition 4.10), the measure μ is defined on $Z = j(\text{Car}(P_0))$ by testing on symbols of the form $q = q_i + q_\partial$, $q_i \in C_c^\infty(T^*\Omega)$ and q_∂ tangential (which is dense in $C_0(Z)$).
- Using the fact $\mu(\mathcal{H}) = 0$ (Proposition 4.7), the measure μ can be extended to test on functions of $\text{Car}(P_0)$ which admits a representation (thanks to Malgrange's theorem)

$$q(x', x_d, \xi', \xi_d) = q_0(x', x_d, \xi') + \xi_d q_1(x', x_d, \xi'), \quad \text{on } \xi_d^2 = r(x', x_d, \xi').$$

Then, we will show in Proposition 4.9 that for tangential h -pseudodifferential operators $B_{0,h}, B_{1,h}$, the quadratic form

$$((B_{0,h_k} + B_{1,h_k} \frac{1}{i} h_k \partial_{x_d})) v_{h_k}, v_{h_k}$$

converges to $\langle \mu, b_0 + b_1 \xi_d \mathbf{1}_{\rho \notin \mathcal{H}} \rangle$, by a suitable limit procedure for symbols in \mathcal{A}^h . Consequently, for any $q \in C_c^\infty(\text{Car}(P_0))$, we can make sense of the expression

$$\langle \mu, \{p, q\} \rangle$$

μ -a.e., by viewing $\{p, q\} = 2\xi_d \partial_{x_d} q \mathbf{1}_{\rho \notin \mathcal{H}} - \{r, q\}$. We remark that to calculate $\{p, q\}$, it is enough to choose one representation $q = q_0 + q_1 \xi_d$ on $\text{Car}(P_0)$, since $\{p, p\} = 0$ and $p = 0$ on $\text{supp}(\mu)$.

- Finally, to prove that the measure μ is invariant along the Melrose-Sjöstrand flow, we apply Theorem 4.3, for which we need to verify the following conditions:

(a) $\mu(\mathcal{G}^{2,+}) = 0$

(b) $\dot{\mu} = 0$, in the sense that $\langle \mu, \{p, q\} \rangle = 0$ holds for any even symbol $q \in C_c^\infty(\text{Car}(P_0))$, i.e. $q(x', x_d = 0, \xi', \xi_d) = q(x', x_d = 0, \xi', -\xi_d)$.

The verification of (a),(b) in our context is based on the propagation formula: Proposition 5.1 and Proposition 6.1. Especially, starting from Proposition 6.1, by choosing suitable test symbols of the form $q_0 + q_1 \xi_d$, we are able to verify the conditions (a) and (b).

Proposition 4.9. *If $B_{0,h}, B_{1,h}$ are two tangential h -pseudodifferential operators of with principal symbols b_0, b_1 of order 0, then we have*

$$\lim_{k \rightarrow \infty} ((B_{0,h_k} + B_{1,h_k} \frac{1}{i} h_k \partial_{x_d})) v_{h_k}, v_{h_k} L^2 = \langle \mu, b_0 + b_1 \xi_d \mathbf{1}_{\rho \notin \mathcal{H}} \rangle.$$

Proof. Since $B_{0,h}$ and $B_{1,h}$ are all tangential, by the definition of the measure, the first term $(B_{0,h_k} v_{h_k}, v_{h_k})_{L^2}$ converges to $\langle \mu, b_0 \rangle$. It remains to prove the convergence of the second term $(B_{1,h_k} \frac{1}{i} h_k \partial_{x_d} v_{h_k}, v_{h_k})_{L^2}$. For this, we pick $\epsilon > 0, \delta > 0$ and define

$$\begin{aligned} B_{1,h_k,\epsilon} &= (1 - \psi(\frac{x_d}{\epsilon})) B_{1,h_k} (1 - \psi(\frac{x_d}{2\epsilon})), & B_{1,h_k}^\epsilon &= B_{1,h_k} - B_{1,h_k,\epsilon}, \\ B_{1,h_k}^{\epsilon,\delta} &= \text{Op}_{h_k}(\psi(\frac{r}{\delta})) B_{1,h_k}^\epsilon, & B_{1,h_k,\delta}^\epsilon &= B_{1,h_k}^\epsilon - B_{1,h_k}^{\epsilon,\delta}, \end{aligned}$$

where ψ is a cutoff function which is 1 near 0. Now by the definition of μ and the dominating convergence,

$$\lim_{\epsilon \rightarrow 0} \lim_{k \rightarrow \infty} (B_{1,h_k,\epsilon} \frac{1}{i} h_k \partial_{x_d} v_{h_k}, v_{h_k})_{L^2} = \langle \mu, b_1 \xi_d \mathbf{1}_{x_d > 0} \rangle = \langle \mu, b_1 \xi_d \mathbf{1}_{\rho \notin \mathcal{H}} \rangle,$$

since $\mu(\mathcal{E}) = \mu(\mathcal{H}) = 0$. Now from Proposition 4.7, the contribution of

$$\lim_{\epsilon \rightarrow 0} \limsup_{k \rightarrow \infty} |(B_{1,h_k}^{\epsilon,\delta} h_k \partial_{x_d} v_{h_k}, v_{h_k})_{L^2}| \leq C \delta^{\frac{1}{2}},$$

which converges to 0 if we let $\delta \rightarrow 0$. Finally, by Cauchy-Schwarz,

$$|(B_{1,h_k,\delta}^\epsilon h_k \partial_{x_d} v_{h_k}, v_{h_k})_{L^2}| \leq \|h_k \partial_{x_d} v_{h_k}\|_{L^2} \|(B_{1,h_k}^{\epsilon,\delta})^* v_{h_k}\|_{L^2}.$$

Notice that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|(B_{1,h_k}^{\epsilon,\delta})^* v_{h_k}\|_{L^2}^2 &= \lim_{k \rightarrow \infty} (B_{1,h_k}^{\epsilon,\delta} (B_{1,h_k}^{\epsilon,\delta})^* v_{h_k}, v_{h_k})_{L^2} \\ &= \langle \mu, (1 - \psi(\frac{r}{\delta})) [\psi(\frac{x_d}{\epsilon}) (1 - \psi(\frac{x_d}{2\epsilon})) b_1 + (1 - \psi(\frac{x_d}{\epsilon}) \psi(\frac{x_d}{2\epsilon})) b_1] \rangle, \end{aligned}$$

taking the double limit $\limsup_{\delta \rightarrow 0} \limsup_{\epsilon \rightarrow 0}$, we obtain that

$$\limsup_{\delta \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \lim_{k \rightarrow \infty} \|(B_{1,h_k}^{\epsilon,\delta})^* v_{h_k}\|_{L^2}^2 \leq \langle \mu, b_1^2 \mathbf{1}_{x_d=0} \mathbf{1}_{r \neq 0} \rangle = 0,$$

since $\mu \mathbf{1}_{\mathcal{E} \cup \mathcal{H}} = 0$. This completes the proof of Proposition 4.9. \square

5. INTERIOR PROPAGATION ESTIMATE

Proposition 5.1 (Interior propagation). *Let $Q_h = \tilde{\chi} Q_h \tilde{\chi}$ be a h -pseudodifferential operator of order 0, where $\tilde{\chi} \in C_c^\infty(\Omega)$, then we have*

$$\frac{1}{i\hbar} ([h^2 \Delta + 1, Q_h] v_h, v_h)_{L^2} = o(1).$$

Proof. Denote by $P_h = P_{h,0} + iM_h$ with $M_h = -h \text{div } a \nabla$ and $P_{h,0} = -h^2 \Delta - 1$, we have

$$\begin{aligned} \frac{1}{i\hbar} ([P_{h,0}, Q_h] v, v)_{L^2} &= \frac{1}{i\hbar} (Q_h v, P_{h,0} v)_{L^2} - \frac{1}{i\hbar} (P_{h,0} v, Q_h^* v)_{L^2} \\ &= \frac{1}{i\hbar} (Q_h v, \chi P_{h,0} u)_{L^2} - \frac{1}{i\hbar} (\chi P_{h,0} u, Q_h^* v)_{L^2} + \mathcal{R}_1 \\ \text{with } \mathcal{R}_1 &= \frac{1}{i\hbar} (Q_h v, [P_{h,0}, \chi] u)_{L^2} - \frac{1}{i\hbar} ([P_{h,0}, \chi] u, Q_h^* v)_{L^2}. \end{aligned}$$

By using the equation $P_{h,0} u = f - iM_h u$, we have

$$(5.1) \quad \frac{1}{i\hbar} ([P_{h,0}, Q_h] v, v)_{L^2} = \mathcal{R}_1 + \mathcal{R}_2 + o(1),$$

where

$$\mathcal{R}_2 = \frac{1}{h} (Q_h v, \chi M_h u)_{L^2} + \frac{1}{h} (\chi M_h u, Q_h^* v)_{L^2}.$$

Note that

$$-[P_{h,0}, \chi] = h \nabla (\nabla a \chi'(a/h)) + 2 \nabla a \chi'(a/h) h \nabla,$$

since $h \nabla (\chi(a/h)) = \nabla a \chi'(a/h)$.

• **Claim 1:** $\mathcal{R}_1 = o(1)$

It suffices to show that $ih^{-1}(B_h v, [P_{h,0}, \chi]u)_{L^2} = o(1)$ for any compact supported h -pseudo B_h of degree 0. By integration by part,

$$-\frac{1}{ih}(B_h v, [P_{h,0}, \chi]u)_{L^2} = -\frac{1}{ih}(B_h v, h \operatorname{div}(\chi'(\frac{a}{h})\nabla a u) + \chi'(\frac{a}{h})\nabla a h \nabla u)_{L^2}$$

and we simply apply Corollary 3.2, to get for each term $o(1)$.

• **Claim 2:** $\mathcal{R}_2 = o(1)$

It suffices to prove that $(Q_h v, \chi \operatorname{div}(a \nabla u))_{L^2} = o(1)$. We write

$$(Q_h v, \chi \operatorname{div} a \nabla u)_{L^2} = -((\nabla \chi) Q_h v, a \nabla u)_{L^2} - (\chi [\nabla, Q_h] v, a \nabla u)_{L^2} - (\chi Q_h \nabla v, a \nabla u)_{L^2}.$$

Since $|a^{\frac{1}{2}} \nabla \chi| = h^{-1} |a^{\frac{1}{2}} \chi' \nabla a| \leq C$, from Corollary 3.2, the first term of r.h.s. can be bounded by

$$\|Q_h v\|_{L^2} \|a^{\frac{1}{2}} \nabla u\|_{L^2} = o(1).$$

The second term of r.h.s. can be bounded by $o(h)$. Observe that $\nabla(a^{\frac{1}{2}}) = \frac{1}{2} a^{-\frac{1}{2}} \nabla a$ is bounded, thus from Corollary A.2,

$$[a^{\frac{1}{2}}, Q_h] = O_{\mathcal{L}(L^2)}(h).$$

Therefore,

$$|(\chi Q_h \nabla v, a \nabla u)_{L^2}| \leq |(\chi Q_h a^{\frac{1}{2}} \nabla v, a^{\frac{1}{2}} \nabla u)_{L^2}| + |(\chi [a^{\frac{1}{2}}, Q_h] \nabla v, a^{\frac{1}{2}} \nabla u)_{L^2}|.$$

The second term is bounded by $Ch \|\nabla v\|_{L^2} \|a^{\frac{1}{2}} \nabla u\|_{L^2} = o(1)$, and the first term can be bounded by $o(1)$, due to Lemma 3.4. This completes the proof of Proposition 5.1. \square

6. PROPAGATION NEAR THE BOUNDARY

Recall that $v_h = \chi(a/h)u_h$. Consider the operator

$$B_h = B_{0,h} + B_{1,h} \frac{h}{i} \partial_{x_d}$$

where $B_{j,h} = \tilde{\chi}_1 \operatorname{Op}_h(b_j) \tilde{\chi}_1$, $j = 0, 1$ are two tangential operators and $\tilde{\chi}_1$ has compact support near a point $z_0 \in \partial\Omega$. Note that in the local coordinate system,

$$P_{h,0} = -h^2 \Delta - 1 = -\frac{1}{\sqrt{|g|}} h \partial_{x_d} \sqrt{|g|} h \partial_{x_d} - R_h,$$

where R_h is a self-adjoint tangential operator of order 2. The operator involving the damping can be written as

$$M_h = -\frac{h}{\sqrt{|g|}} \partial_{x_d} \sqrt{|g|} a \partial_{x_d} - \frac{h}{\sqrt{|g|}} \partial_{x'_k} \sqrt{|g|} a g^{jk} \partial_{x'_j}$$

Proposition 6.1 (Boundary propagation).

$$\frac{1}{ih} ([P_{h,0}, B_h] v, v)_{L^2} = (B_{1,h}|_{x_d=0} (h \partial_{x_d} v)|_{x_d=0}, (h \partial_{x_d} v)|_{x_d=0})_{L^2(\partial)} + o(1).$$

Proof. We give the proof in the case $B_{0,h} = 0$. The $B_{0,h}$ terms are handled by slightly simpler versions of the same computations. By developing the commutator, we have

$$\frac{1}{ih} ([P_{h,0}, B_h] v, v)_{L^2} = \frac{1}{ih} (B_h v, P_{h,0} v)_{L^2} - \frac{1}{ih} (B_h P_{h,0} v, v)_{L^2} + (B_{1,h}|_{x_d=0} (h \partial_{x_d} v)|_{y=0}, h \partial_{x_d} v|_{x_d=0})_{L^2(\partial)},$$

where the boundary term (the third) comes from the integration by part of the term

$$\frac{1}{ih} \left(\frac{1}{\sqrt{|g|}} h \partial_{x_d} \sqrt{|g|} h \partial_{x_d} v, v \right)_{L^2},$$

since R_h is self-adjoint tangential operator. It suffices to show that

$$(6.1) \quad I_h := \frac{1}{ih} (B_{1,h} h \partial_{x_d} v, P_{h,0} v)_{L^2} - \frac{1}{ih} (B_{1,h} h \partial_{x_d} P_{h,0} v, v)_{L^2} = o(1).$$

Since $v = \chi u$ and $P_{h,0}u = P_h u - iM_h u = f_h - iM_h u$, we have

$$P_{h,0}v = \chi P_{h,0}u + [P_{h,0}, \chi]u = \chi f_h - i\chi M_h u + [P_{h,0}, \chi]u.$$

Therefore,

$$\mathbf{I}_h = o(1) + \mathbf{I}_{h,1} + \mathbf{I}_{h,2},$$

where

$$\mathbf{I}_{h,1} = \frac{1}{ih} (B_{1,h} h \partial_{x_d} v, [P_{h,0}, \chi]u)_{L^2} - \frac{1}{ih} (B_{1,h} h \partial_{x_d} [P_{h,0}, \chi]u, v)_{L^2}$$

and

$$\mathbf{I}_{h,2} = \frac{1}{h} (B_{1,h} h \partial_{x_d} v, \chi M_h u)_{L^2} - \frac{1}{h} (B_{1,h} h \partial_{x_d} \chi M_h u, v)_{L^2}$$

• **Claim 1:** $\mathbf{I}_{h,1} = o(1)$.

Indeed, from integration by part, the second term

$$ih^{-1} (B_{1,h} h \partial_{x_d} [P_{h,0}, \chi]u, v)_{L^2} = ih^{-1} ([P_{h,0}, \chi]u, h \partial_{x_d} A_h v)_{L^2}$$

for some tangential operator A_h , hence it has the same structure as the first term. It suffices to show that

$$h^{-1} (B_{1,h} h \partial_{x_d} v, [P_{h,0}, \chi]u)_{L^2} = o(1).$$

Since

$$-[P_{h,0}, \chi]u = h \nabla \cdot (\nabla a \chi'(a/h))u + 2 \nabla a \chi'(a/h) \cdot h \nabla u,$$

doing integration by part, we obtain that

$$\begin{aligned} -h^{-1} (B_{1,h} h \partial_{x_d} v, [P_{h,0}, \chi]u)_{L^2} &= -(\nabla(\bar{u} B_{1,h} h \partial_{x_d} v), \nabla a \chi')_{L^2} + 2h^{-1} (B_{1,h} h \partial_{x_d} v, \nabla a \chi' h \nabla u)_{L^2} \\ &= -(\nabla B_{1,h} h \partial_{x_d} v, \nabla a \chi' u)_{L^2} + h^{-1} (B_{1,h} h \partial_{x_d} v, \nabla a \chi' h \nabla u)_{L^2}. \end{aligned}$$

Note that $v = \chi u$, if one of the derivatives $h \partial_{x_d}$, $h \nabla$ fall on $\chi(a/h)$ we can bound them from Corollary 3.2 by $o(h)$. If all the derivatives fall on u in anyone of the two terms, by Lemma 3.1 and Corollary 3.2, these terms can be bounded by

$$\|h \nabla \partial_{x_d} u\|_{L^2} \|\nabla a u\|_{L^2} + h^{-1} \|h \partial_{x_d} u\|_{L^2} \|\nabla a h \nabla u\|_{L^2} = o(1).$$

• **Claim 2:** $\mathbf{I}_{h,2} = o(1)$.

It suffices to prove that

$$h^{-1} (B_{1,h} h \partial_{x_d} (\chi u), \chi M_h u)_{L^2} = o(1).$$

Note that $-M_h u = \nabla a h \nabla u + a h \Delta u = o_{L^2}(1)$ and $h \partial_{x_d} (\chi u) = \partial_{x_d} a \chi' u + \chi h \partial_{x_d} u$. We have

$$h^{-1} |(B_{1,h} \partial_{x_d} a \chi' u, \chi M_h u)_{L^2}| \leq h^{-1} \|\nabla a u\|_{L^2} \|\chi M_h u\|_{L^2} = o(1),$$

since $\|\nabla a u\|_{L^2} = o(h)$ from Corollary 3.2. It remains to show that

$$h^{-1} (B_{1,h} \chi h \partial_{x_d} u, \chi (\nabla a \cdot h \nabla u + a h \Delta u))_{L^2} = o(1).$$

Since $\|\nabla a h \nabla u\|_{L^2} = o(h)$, we have $h^{-1} (B_{1,h} \chi h \partial_{x_d} u, \chi (\nabla a \cdot h \nabla u))_{L^2} = o(1)$. Finally, we show that

$$h^{-1} (B_{1,h} \chi h \partial_{x_d} u, \chi a h \Delta u)_{L^2} = o(1).$$

Recall that from $|\nabla(a^{\frac{1}{2}})| \leq C$ and Corollary A.2,

$$[B_{1,h}, a^{\frac{1}{2}}] = O_{\mathcal{L}(L^2)}(h),$$

we have

$$\begin{aligned} h^{-1} |(B_{1,h} \chi h \partial_{x_d} u, \chi a h \Delta u)_{L^2}| &\leq h^{-1} |(B_{1,h} a^{\frac{1}{2}} \chi h \partial_{x_d} u, \chi a^{\frac{1}{2}} h \Delta u)_{L^2}| + h^{-1} |([B_{1,h}, a^{\frac{1}{2}}] \chi h \partial_{x_d} u, \chi a^{\frac{1}{2}} h \Delta u)_{L^2}| \\ &\leq C h^{-1} \|a^{\frac{1}{2}} h \nabla u\|_{L^2} \|a^{\frac{1}{2}} h \Delta u\|_{L^2} + C \|h \nabla u\|_{L^2} \|a^{\frac{1}{2}} h \Delta u\|_{L^2} = o(1). \end{aligned}$$

This completes the proof of Proposition 6.1. \square

To show that the semi-classical measure μ of (v_{h_k}) is invariant along the Melrose-Sjöstrand flow (to complete the proof of Proposition 2.3), we need to verify the condition (2) in Theorem 4.3. We will make use of the propagation formula, i.e. Proposition 6.1. Formally, for $B_h = B_{0,h} + B_{1,h} \frac{1}{i} h \partial_{x_d}$, the principal symbol of $\frac{i}{h} [P_{h,0}, B_h]$ is given by

$$\{\eta^2 - r, b_0 + b_1 \xi_d\} = a_0 + a_1 \xi_d + a_2 \xi_d^2,$$

where

$$(6.2) \quad a_0 = b_1 \partial_{x_d} r - \{r, b_0\}', \quad a_1 = 2 \partial_{x_d} b_0 - \{r, b_1\}', \quad a_2 = 2 \partial_{x_d} b_1,$$

and $\{\cdot, \cdot\}'$ is the Poisson bracket for (x', ξ') variables. On the other hand, by calculating the commutator, we find

$$(6.3) \quad \frac{i}{h} [P_{h,0}, B_h] = A_0 + A_1 h D_{x_d} + A_2 h^2 D_{x_d}^2 + h \text{Op}_h(S_\partial^0 + S_\partial^0 \xi_d),$$

where A_0, A_1, A_2 are tangential operators with symbols a_0, a_1, a_2 , with respectively. We will prove the following propagation formula:

Corollary 6.2. *Assume that $B_h = B_{h,0} + B_{h,1} h D_{x_d}$, where $B_{h,0}, B_{h,1}$ are tangential operators of order 0 with symbols b_0, b_1 , with respectively. Assume that $b = b_0 + b_1 \xi_d$. Define the formal Poisson bracket*

$$\{p, b\} = (a_0 + a_2 r) + a_1 \xi_d \mathbf{1}_{\rho \notin \mathcal{H}},$$

where a_0, a_1, a_2 are given by (6.2). Then the defect measure μ satisfies the equation

$$\langle \mu, \{p, b\} \rangle = -\langle \nu_\partial, b_1 \rangle,$$

where ν_∂ is the semiclassical measure of $(h \partial_{x_d} v_h|_{x_d=0})$. Moreover, if b is an even symbol (i.e. $b(x', x_d = 0, \xi', \xi_d) = b(x', x_d = 0, \xi', -\xi_d)$), then we have

$$\langle \mu, \{p, b\} \rangle = 0.$$

In particular, by combining Proposition 5.1, we have $\dot{\mu} = 0$.

Proof. From Proposition 6.1 and the decomposition (6.3), we have

$$(6.4) \quad ((A_0 + A_1 h_k D_{x_d} + A_2 h_k^2 D_{x_d}^2) v_{h_k}, v_{h_k})_{L^2} = -\langle \nu_\partial, b_1 \rangle + o(1).$$

From Lemma 3.4, we can also replace the function v_{h_k} on the l.h.s. by u_{h_k} . Using the equation of u_{h_k} :

$$(h^2 D_{x_d}^2 - R_{h_k}) u_{h_k} = i M_{h_k} u_{h_k} - f_{h_k} + O_{L^2}(h_k),$$

we deduce that

$$(A_2 h_k^2 D_{x_d}^2 u_{h_k}, u_{h_k}) = (A_2 R_{h_k} u_{h_k}, u_{h_k})_{L^2} + o(1),$$

thanks to Lemma 4.5. Therefore, from Proposition 4.9,

$$\lim_{kh \rightarrow \infty} ((A_0 + A_1 h_k D_{x_d} + A_2 R_{h_k}) u_{h_k}, u_{h_k})_{L^2} = \langle \mu, a_0 + a_1 \xi_d \mathbf{1}_{\rho \notin \mathcal{H}} + a_2 r \rangle = \langle \mu, \{p, b\} \rangle.$$

Now if $b = b_0 + b_1 \xi_d$ is an even symbol, we must have $b_1|_{x_d=0} = 0$, therefore, $\langle \mu, \{p, b\} \rangle = -\langle \nu_\partial, b_1 \rangle = 0$. The proof of Lemma 6.2 is complete. \square

Corollary 6.3. *We have $\mu(\mathcal{G}^{2,+}) = 0$.*

Proof. We will make use of the formula $\langle \mu, \{p, b\} \rangle = -\langle \nu_\partial, b_1 \rangle$ by choosing $b = b_{1,\epsilon} \eta$ with

$$b_{1,\epsilon}(x', x_d, \xi') = \psi\left(\frac{x_d}{\epsilon^{\frac{1}{2}}}\right) \psi\left(\frac{r(x_d, x', \xi')}{\epsilon}\right) \kappa(x_d, x', \xi'),$$

where $\psi \in C_c^\infty(\mathbb{R})$ equals to 1 near the origin and $\kappa(y, x', \xi') \geq 0$ near a point $\rho_0 \in \mathcal{G}^{2,+}$. Since $\{p, b_\epsilon\} = (a_0 + a_2 r) + a_1 \xi_d \mathbf{1}_{\rho \notin \mathcal{H}}$, and a_0, a_1, a_2 are given by the relation (6.2). In particular for our choice, by direct calculation we have

$$a_0 = b_{1,\epsilon} \partial_{x_d} r, \quad a_1 = -\{r, \kappa\}' \psi\left(\frac{x_d}{\epsilon^{\frac{1}{2}}}\right) \psi\left(\frac{r}{\epsilon}\right),$$

and

$$a_2 = 2\partial_{x_d} b_{1,\epsilon} = 2\epsilon^{-\frac{1}{2}} \psi' \left(\frac{x_d}{\epsilon} \right) \psi \left(\frac{r}{\epsilon} \right) \kappa + 2 \frac{\partial_{x_d} r}{\epsilon} \psi \left(\frac{x_d}{\epsilon} \right) \psi' \left(\frac{r}{\epsilon} \right) \kappa + 2\psi \left(\frac{x_d}{\epsilon} \right) \psi \left(\frac{r}{\epsilon} \right) \partial_{x_d} \kappa.$$

Note that a_2 is uniformly bounded in ϵ and for any fixed (y, x', ξ') , $ra_2 \rightarrow 0$ as $\epsilon \rightarrow 0$. Thus by dominating convergence, we have

$$\lim_{\epsilon \rightarrow 0} \langle \mu, \{p, b_\epsilon\} \rangle = \langle \mu, \kappa|_{x_d=0} \partial_{x_d} r \mathbf{1}_{r=0} \rangle \geq 0$$

since $\partial_{x_d} r > 0$ on $\mathcal{G}^{2,+}$. However, $-\langle \nu_\partial, b_\epsilon \rangle \leq 0$, we must have $\mu \mathbf{1}_{\mathcal{G}^{2,+}} = 0$. This completes the proof of Lemma 6.3. \square

From Lemma 6.2 and Lemma 6.3, we have verified that $\dot{\mu} = 0$ and $\mu(\mathcal{G}^{2,+}) = 0$, thus from Theorem 4.3, the semi-classical μ is invariant along the Melrose-Sjöstrand flow. Thanks to the geometric control condition and the fact that $a^{\frac{1}{2}} v_{h_k} = o_{L^2}(1)$, we deduce that $\mu = 0$. This contradicts to the assumption that $\|v_{h_k}\|_{L^2} = \|u_{h_k}\|_{L^2} = 1 + o(1)$, as $k \rightarrow \infty$. The proof of Proposition 2.3 is now complete.

APPENDIX A. SOME COMMUTATOR ESTIMATES

Lemma A.1. *Assume that $b(x, y, \xi) \in L^\infty(\mathbb{R}^{3d}_{x,y,\xi})$ such that*

$$|\partial_\xi^\alpha b(x, y, \xi)| \lesssim_\alpha \langle \xi \rangle^{-(|\alpha|+1)}$$

for all multi-index $\alpha \in \mathbb{N}^d$, $|\alpha| \leq d+1$. Then the operator T_h associated with the Schwartz kernel

$$K_h(x, y) := \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} b(x, y, \xi) e^{\frac{i(x-y)\cdot\xi}{h}} d\xi$$

is bounded on $L^2(\mathbb{R}^d)$, uniformly in $0 < h \leq 1$.

Proof. Using the Littlewood-Paley decomposition, we can decompose the operator $T_h = \sum_{j \geq 0} T_{h,j}$ where each $T_{h,j}$ has the Schwartz kernel

$$K_{h,j}(x, y) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} b_j(x, y, \xi) e^{\frac{i(x-y)\cdot\xi}{h}} d\xi,$$

with $b_j(x, y, \xi) = b(x, y, \xi) \psi_j(\xi)$ and $\psi_j(\xi) = \psi(2^{-j}\xi)$ for some $\psi \in C_c^\infty(\frac{1}{2} \leq |\xi| \leq 2)$, if $j \geq 1$ and $\psi_0(\xi)$ is supported on $|\xi| \leq 1$. Note that

$$(x-y)^\alpha K_{h,j}(x, y) = \frac{i^{-|\alpha|} h^{|\alpha|}}{(2\pi h)^d} \int_{\mathbb{R}^d} D_\xi^\alpha b_j(x, y, \xi) \cdot e^{\frac{i(x-y)\cdot\xi}{h}} d\xi,$$

we have

$$|K_{h,j}(x, y)| \lesssim_\alpha \frac{h^{|\alpha|-d}}{|x-y|^{|\alpha|}} \cdot 2^{-j(|\alpha|+1-d)}.$$

We have another trivial bound $|K_{h,j}(x, y)| \lesssim_\alpha 2^{jd} h^{-d}$. Therefore, for fixed $x \in \mathbb{R}^d$, by choosing $|\alpha| = d+1$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |K_{h,j}(x, y)| dy &\lesssim \int_{\mathbb{R}^d} \min \left\{ 2^{-j} \frac{2^{-j} h}{|x-y|^{d+1}}, 2^{jd} h^{-d} \right\} dy \\ &\leq \int_{|z| \leq 2^{-\frac{j}{d+1}} \cdot 2^{-j} h} 2^{jd} h^{-d} dz + \int_{|z| > 2^{-\frac{j}{d+1}} \cdot 2^{-j} h} \frac{2^{-j} 2^{-j} h}{|z|^{d+1}} dz \lesssim 2^{-\frac{jd}{d+1}}. \end{aligned}$$

Similarly, for fixed $y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |K_{h,j}(x, y)| dx \lesssim 2^{-\frac{jd}{d+1}}.$$

By Schur's test, we have $\|T_{h,j}\|_{\mathcal{L}(L^2)} \lesssim 2^{-\frac{jd}{d+1}}$. Using the triangle inequality, we obtain that T_h is bounded on $L^2(\mathbb{R}^d)$, uniformly in $0 < h \leq 1$. The proof of Lemma A.1 is now complete. \square

Corollary A.2. *Assume that $\kappa \in W^{1,\infty}(\mathbb{R}^d)$ and $b \in S^0(\mathbb{R}^{2d})$ is a symbol of order zero, then we have*

$$\|[\text{Op}_h(b), \kappa]\|_{\mathcal{L}(L^2)} = O(h).$$

Proof. The kernel of $[\text{Op}_h(b), \kappa]$ is given by

$$K(x, y) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} b(x, \xi) (\kappa(y) - \kappa(x)) e^{\frac{i(x-y)\cdot\xi}{h}} d\xi.$$

Since $\kappa \in W^{1,\infty}$, there exists $\Psi \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$\kappa(y) - \kappa(x) = (y - x) \cdot \Psi(x, y).$$

Thus

$$K(x, y) = - \sum_{j=1}^d \frac{h}{i(2\pi h)^d} \int_{\mathbb{R}^d} \partial_{\xi_j} b(x, \xi) \Psi_j(x, y) e^{\frac{i(x-y)\cdot\xi}{h}} d\xi$$

Applying Lemma A.1 to each $\partial_{\xi_j} b(x, \xi) \Psi_j(x, y)$, the proof of Corollary A.2 is complete. \square

REFERENCES

- [1] Charles J. K. Batty and Thomas Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *J. Evol. Equ.*, 8(4):765–780, 2008.
- [2] A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. *Math. Ann.*, 347(2):455–478, 2010.
- [3] N. Burq. Mesures semi-classiques et mesures de défaut. *Séminaire Bourbaki*, Mars 1997.
- [4] N. Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Mathematica*, 180:1–29, 1998.
- [5] N. Burq and H. Christianson. Imperfect geometric control and overdamping for the damped wave equation. *Comm. Math. Phys.*, 336(1):101–130, 2015.
- [6] N. Burq and P. Gérard. Stabilization of wave equations on the torus with rough dampings. To appear *Pure and Applied Analysis*. preprint arxiv <https://arxiv.org/abs/1801.00983>, 2020.
- [7] N. Burq and G. Lebeau. Mesures de défaut de compacité, application au système de Lamé. *Ann. Sci. École Norm. Sup. (4)*, 34(6):817–870, 2001.
- [8] Nicolas Burq. Decays for Kelvin-Voigt damped wave equations I: the black box perturbative method. *SIAM J. Control Optim.*, 58(4):1893–1905, 2020.
- [9] P. Gérard and E. Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Mathematical Journal*, 71:559–607, 1993.
- [10] G. Lebeau. Equation des ondes amorties. In A. Boutet de Monvel and V. Marchenko, editors, *Algebraic and Geometric Methods in Mathematical Physics*, pages 73–109. Kluwer Academic, The Netherlands, 1996.
- [11] R.B. Melrose and J. Sjöstrand. Singularities of boundary value problems II. *Communications in Pure Applied Mathematics*, 35:129–168, 1982.
- [12] Louis Tebou. A constructive method for the stabilization of the wave equation with localized Kelvin-Voigt damping. *C. R. Math. Acad. Sci. Paris*, 350(11-12):603–608, 2012.

UNIVERSITÉ PARIS-SACLAY, LABORATOIRE DE MATHÉMATIQUES D'ORSAY, UMR 8628 DU CNRS, BÂTIMENT 307, 91405 ORSAY CEDEX, FRANCE AND INSTITUT UNIVERSITAIRE DE FRANCE

Email address: nicolas.burq@math.u-psud.fr

UNIVERSITÉ DE CERGY-PONTOISE, LABORATOIRE DE MATHÉMATIQUES AGM, UMR 8088 DU CNRS, 2 AV. ADOLPHE CHAUVIN 95302 CERGY-PONTOISE CEDEX, FRANCE

Email address: chenmin.sun@u-cergy.fr