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## Breathing modes of a fast rotating Fermi gas

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We derive the frequency spectrum of the lowest compressional oscillations of a three-dimensional harmonically trapped Fermi superfluid in the presence of a vortex lattice, treated in the diffused vorticity approximation within a hydrodynamic approach. We consider the general case of a superfluid at  $T=0$  characterized by a polytropic equation of state ( $\sim n^\gamma$ ), which includes both the Bose-Einstein condensed regime of dimers ( $\gamma=1$ ) and the unitary limit of infinite scattering length ( $\gamma=2/3$ ). Important limiting cases are considered, including the centrifugal limit, the isotropic trapping, and the cigar geometry. The conditions required to enter the lowest Landau level and quantum Hall regimes at unitarity are also discussed.

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The experimental realization of quantized vortices in interacting ultracold Fermi gases [1] has opened new challenging perspectives in the experimental and theoretical study of superfluidity. These perspectives are particularly important because in Fermi gases the measurement of the order parameter is not directly accessible. Actually, only for small and positive values of the scattering length, when dimers built up with pairs of atoms of opposite spin are formed, the fermionic system gives rise to the phenomenon of Bose-Einstein condensation (BEC), whose onset is clearly revealed by the bimodal structure of the density [2]. The observation of vortices for negative values of the scattering length as well as in the unitary limit close to a Feshbach resonance, where the scattering length is larger than the interparticle distance, consequently provides a unique source of information on the superfluid nature of these novel configurations.

The measurements of Ref. [1] have, however, shown that vortices in atomic Fermi gases are not directly observable *in situ* nor after expansion, unless one suddenly ramps the scattering length to small and positive values (corresponding to the BEC regime) just after the release of the trap. Indeed, the visibility of vortices is limited in both the BCS and unitary limits, in the first case due to the reduced contrast and in the latter mainly due to the smallness of their size, fixed by the interparticle distance.

For the above reasons it is interesting to explore more macroscopic signatures of the presence of vortices. A first important source of information comes from the bulge effect associated with the increase of the radial size of the cloud produced by the centrifugal force. More systematic information comes from the study of the collective oscillations. For example the splitting of the quadrupole frequencies with opposite angular momentum [3] provides direct information on the angular momentum carried by the vortical configuration and has been used to measure even the quantization of a single vortex line in Bose-Einstein condensed atomic gases [4]. In this work we focus on the study of the compressional modes whose frequency is affected by the presence of the vortex lines and, at the same time, is sensitive to the nature of the configuration (BEC gas of dimers, unitary limit, etc.).

The study of the compressional modes actually provides a unique information on the equation of state of these systems. In the absence of rotation it has been the object of recent theoretical [5,6] and experimental [7–9] work. Theoretical studies of the collective oscillations in rotating Fermi gases were so far limited to small angular velocities in the absence of vortices [10]. First calculations based on the diffused vorticity approximation were recently carried out in cylindrical geometry [11].

We consider here a two-component Fermi gas with balanced spin population, trapped by a three-dimensional axisymmetric harmonic potential, so that the collective oscillations can be labeled by the axial component  $m$  of angular momentum. The typical wavelength of the lowest modes is of the order of the system size. When the number of vortices in the sample is large, this length scale is much larger than the intervortex distance. In order to evaluate the corresponding oscillation frequencies one can then rely on a coarse grain description of the system, the so-called diffused vorticity approximation [12,13], which does not require to deal with the microscopic details of single vortices. By considering the case of a regular lattice of singly quantized vortices, this long-wavelength description assumes the vorticity to be uniformly spread in the fluid. In practice, if the vortex lattice rotates at angular velocity  $\mathbf{\Omega}$ , the average curl of the velocity field  $\mathbf{v}$  is given by  $\nabla \wedge \mathbf{v} = 2\mathbf{\Omega}$ , characterizing the rigid body rotation  $\mathbf{v} = \mathbf{\Omega} \wedge \mathbf{r}$ . This also corresponds to a uniform vortex density, which, for a Fermi superfluid, is  $n_v = 2M\Omega / \pi\hbar$ , i.e., a factor 2 larger than in the case of a Bose superfluid with the same value of the atomic mass  $M$ .

The main consequence of the diffused vorticity approximation is the introduction of an effective velocity field which does not satisfy anymore the irrotationality constraint of the microscopic superfluid flow, but accounts for the presence of the vortex lattice. Within this framework, we consider the problem of solving the equations of rotational hydrodynamics with a polytropic equation of state, where the chemical potential  $\mu$  is assumed to have a power law dependence on the density  $n$ , namely,  $\mu \propto n^\gamma$ . This parametrization treats exactly several important configurations of interacting Fermi

gases, including the Bose-Einstein condensed regime of dimers ( $\gamma=1$ ), where the scattering length is small and positive, and the unitary regime of infinite scattering length where the equation of state takes a universal density dependence characterized by the value  $\gamma=2/3$ . Moreover, as discussed in the literature (see, e.g., Ref. [6], and references therein), it is possible to provide an accurate, although approximate, description of the entire BEC-BCS crossover by introducing an effective exponent  $\gamma$  for the equation of state. Since the compressional modes are sensitive to the equation of state, the accurate study of their frequency can then provide a useful insight on the various regimes achieved in the experiments.

The equations of rotational hydrodynamics, written, in the laboratory frame, are given by

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0, \quad (1)$$

$$M\partial_t \mathbf{v} + \nabla(Mv^2/2 + V_{\text{ext}} + \mu_{\text{loc}}) = M\mathbf{v} \wedge (\nabla \wedge \mathbf{v}), \quad (2)$$

where  $\mu_{\text{loc}}(\mathbf{r}, t) \propto n^\gamma(\mathbf{r}, t)$  is the local chemical potential fixed by the equation of state of the uniform matter,  $\mathbf{v}(\mathbf{r}, t)$  is the velocity field, and  $V_{\text{ext}}$  is the external potential which is assumed to be the same for both the spin components of the Fermi gas. For an axisymmetric harmonic potential  $V_{\text{ext}} = M[\omega_\perp^2(x^2 + y^2) + \omega_z^2 z^2]/2$  and for a rotation of the trap in the  $x$ - $y$  plane at frequency  $\Omega_0$ , the equilibrium solutions of Eqs. (1) and (2) are given by  $\mathbf{v}_0(\mathbf{r}) = \Omega_0 \wedge \mathbf{r}$  and  $n_0(\mathbf{r}) \propto [\mu_0 - \tilde{V}_{\text{ext}}(\mathbf{r})]^{1/\gamma}$ , where  $\tilde{V}_{\text{ext}} = V_{\text{ext}} - M\Omega_0^2(x^2 + y^2)/2$  is the renormalized trapping potential accounting for the centrifugal effect produced by the rotation and the chemical potential  $\mu_0$  is obtained from the normalization condition for the density. The centrifugal force causes a bulge effect which modifies the aspect ratio of the rotating cloud according to the relationship

$$R_z^2/R_\perp^2 = (\omega_\perp^2 - \Omega_0^2)/\omega_z^2, \quad (3)$$

where  $R_z$  and  $R_\perp$  are, respectively, the axial and radial Thomas-Fermi radii of the cloud. It also fixes a natural limit for the angular velocity  $\Omega_0$  which cannot exceed the radial trapping frequency  $\omega_\perp$ .

By expanding Eqs. (1) and (2) with respect to small perturbations of the density and velocity field,  $n = n_0 + \delta n$  and  $\mathbf{v} = \mathbf{v}_0 + \delta \mathbf{v}$ , one obtains two coupled linearized equations which admit several solutions of relevant physical interest. On the one side, one has surface solutions carrying angular momentum and characterized by irrotational flow. For example, the most relevant  $m = \pm 2$  quadrupole solutions are described by density variations of the form  $\delta n \propto (x \pm iy)^2$  and by the velocity field  $\delta \mathbf{v} \propto \nabla(x \pm iy)^2$  [11, 13–15]. These surface modes exhibit the dispersion  $\omega_\pm = \sqrt{2\omega_\perp^2 - \Omega_0^2} \pm \Omega_0$  and are strongly affected by the rotational effect, as experimentally proven in the case of Bose-Einstein condensed atomic gases [16]. Their frequency is, however, independent of the equation of state and is not expected to exhibit a new behavior in the case of a Fermi superfluid.

In addition to the surface modes the hydrodynamic equations exhibit an important class of  $m=0$  compressional modes. In order to solve the linearized equations of motion we use the ansatz [10, 13]

$$\delta \mathbf{v} = \{\delta \Omega \wedge \mathbf{r} + \nabla[\alpha_\perp(x^2 + y^2) + \alpha_z z^2]\}e^{-i\omega t}, \quad (4)$$

$$\delta n = n_0^{1-\gamma}[a_0 + a_\perp(x^2 + y^2) + a_z z^2]e^{-i\omega t}, \quad (5)$$

where  $\delta \Omega$ , parallel to the axial direction, accounts for the proper variation of the angular velocity during the oscillation. The inclusion of this term is crucial to ensure the conservation of angular momentum. It was ignored in previous works [11, 14] on the collective oscillations of two-dimensional Bose and Fermi gases containing a vortex lattice, where pure irrotational flow was assumed. While the irrotational assumption is valid for the  $m \neq 0$  modes, it turns out to be inadequate for the compressional  $m=0$  oscillations which are associated with variations of the vortex density and hence of the angular velocity.

The ansatz (4) and (5) gives rise to a linear system yielding three solutions for the oscillation frequency  $\omega$ . One of them is the trivial solution  $\omega=0$ , corresponding to a change of the equilibrium configuration due to the adiabatic change of the angular velocity of the system. The other two solutions instead correspond to the radial and axial breathing modes of the gas and their frequency is given by [17]

$$\omega_\pm^2 = (1 + \gamma)\omega_\perp^2 + \frac{2 + \gamma}{2}\omega_z^2 + (1 - \gamma)\Omega_0^2 \pm \sqrt{(1 + \gamma)^2\omega_\perp^4 + \left(\frac{2 + \gamma}{2}\right)^2\omega_z^4 + (1 - \gamma)^2\Omega_0^4 + (\gamma^2 - 3\gamma - 2)\omega_z^2\omega_\perp^2 + 2(1 + \gamma)(1 - \gamma)\omega_\perp^2\Omega_0^2 - (\gamma^2 - \gamma + 2)\omega_z^2\Omega_0^2}. \quad (6)$$

These two solutions arise from the coupling between the radial and axial motion caused by the hydrodynamic forces in the presence of the rigid rotation of the gas. It is easy to see that Eq. (6) reproduces, as limiting cases, the result of Ref. [18] for the nonrotating Bose condensed gas ( $\Omega_0=0$ ,

$\gamma=1$ ), the result of Ref. [13] for the rotational Bose gas ( $\Omega_0 \neq 0$ ,  $\gamma=1$ ), and finally the result of Ref. [10] for the nonrotating superfluid Fermi gas ( $\Omega_0=0$ , for generic  $\gamma$ ). The predicted behavior of the frequencies for a typical cigar-shaped geometry ( $\omega_\perp/\omega_z=10$ ) is shown in Fig. 1, where we

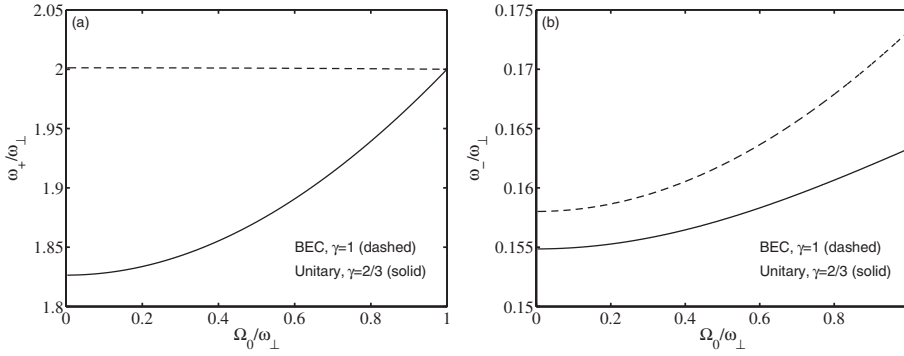


FIG. 1. Frequencies of the breathing modes  $\omega_+/\omega_\perp$  (a) and  $\omega_-/\omega_\perp$  (b) of Eq. (6), as a function of the rotation frequency  $\Omega_0/\omega_\perp$ . We consider a cigar-shape trapping ( $\omega_\perp/\omega_z=10$ ), and show results for the BEC ( $\gamma=1$ ) and the unitary ( $\gamma=2/3$ ) regimes.

explicitly compare the BEC ( $\gamma=1$ ) and the unitary ( $\gamma=2/3$ ) regimes.

Let us now discuss some limiting cases predicted by Eq. (6). A first important case is given by the centrifugal limit  $\Omega_0 \rightarrow \omega_\perp$  where the cloud assumes a disk shape as a consequence of the bulge effect (3) and the two solutions (6) take the simple form

$$\omega_+ = 2\omega_\perp, \quad (7)$$

$$\omega_- = \sqrt{2 + \gamma}\omega_z. \quad (8)$$

In the centrifugal limit the frequency of the radial breathing mode approaches the universal value  $2\omega_\perp$ , independent of the equation of state (see Fig. 1) and of the value of trap deformation  $\omega_\perp/\omega_z$ , while the  $\gamma$  dependent frequency of the axial breathing mode coincides with the value predicted by the hydrodynamic equations at  $\Omega_0=0$ , in the disk-shaped configuration  $\omega_\perp \ll \omega_z$ . In this regard, one should notice that result (8), as well as the more general result (6), has been derived assuming a three-dimensional (3D) configuration, i.e., assuming the validity of the local density approximation along the three directions. When  $\Omega_0$  becomes too close to  $\omega_\perp$  the gas becomes extremely dilute and the Thomas-Fermi condition  $\mu_0 \gg \hbar\omega_z$  is eventually violated with the consequent transition to a 2D configuration. In this case the axial frequency takes the ideal gas value  $2\omega_z$  instead of  $\sqrt{2 + \gamma}\omega_z$ . For a Bose-Einstein condensed atomic gas ( $\gamma=1$ ) this transition has been investigated experimentally [19]. In the case of a Fermi gas at unitarity the conditions required to reach the 2D regime by approaching the centrifugal limit are much more severe (see discussion below).

Another interesting configuration is given by the isotropic trap geometry  $\omega_z = \omega_\perp \equiv \omega_0$  for which Eq. (6) reduces to

$$\omega_\pm^2 = \frac{4 + 3\gamma}{2}\omega_0^2 + (1 - \gamma)\Omega_0^2 \pm \sqrt{\frac{9}{4}\gamma^2\omega_0^4 + (1 - \gamma)^2\Omega_0^4 - \gamma(3\gamma - 1)\omega_0^2\Omega_0^2}. \quad (9)$$

It is worth noticing that, while in the absence of rotation the frequency  $\omega_-$  reduces to the result  $\sqrt{2}\omega_0$  for the surface quadrupole  $m=0$  mode and the frequency  $\omega_+$  approaches the value  $\sqrt{2 + 3\gamma}\omega_0$  of the pure monopole compression mode, the rotation provides a coupling between the two modes even for isotropic trapping, so that they are both sensitive to the

value of the equation of state. A special case is the unitary regime ( $\gamma=2/3$ ) where the two solutions reduce to

$$\omega_+ = 2\omega_0, \quad (10)$$

$$\omega_- = \sqrt{2\omega_0^2 + \frac{2}{3}\Omega_0^2}. \quad (11)$$

The spherical trapping geometry in the unitary regime is actually of particular interest since in this case the Schrödinger equation exhibits important scaling properties [20]. These give rise to universal features for the free expansion as well as for the radial monopole frequency  $\omega_+$ , which turns out to be independent of  $\Omega_0$ . It is worth stressing that, because of the centrifugal effect, the shape of the gas is not spherical in spite of the spherical symmetry of the trap. The dynamics are however isotropic, the solution of the equations of motion being exactly fixed by an isotropic scaling transformation.

Let us now study the experimentally relevant case of a strongly anisotropic trap  $\omega_z \ll \omega_\perp$  (cigar shape). In this case Eq. (6) yields the useful results [21]

$$\omega_+ = \sqrt{2(1 + \gamma)\omega_\perp^2 + 2(1 - \gamma)\Omega_0^2}, \quad (12)$$

$$\omega_- = \sqrt{\frac{2 + 3\gamma + (2 - \gamma)(\Omega_0/\omega_\perp)^2}{1 + \gamma + (1 - \gamma)(\Omega_0/\omega_\perp)^2}}\omega_z, \quad (13)$$

which correspond to the solution for the radial and axial breathing modes, respectively. It is remarkable to see that in the BEC case ( $\gamma=1$ ) the frequency of the radial breathing mode is independent of  $\Omega_0$ , reflecting the peculiar behavior exhibited by the Gross-Pitaevskii equation in 2D [22]. Conversely, in the unitary regime ( $\gamma=2/3$ ), the previous equations reduce to

$$\omega_+ = \sqrt{\frac{10}{3}\omega_\perp^2 + \frac{2}{3}\Omega_0^2}, \quad (14)$$

$$\omega_- = 2\sqrt{\frac{3 + (\Omega_0/\omega_\perp)^2}{5 + (\Omega_0/\omega_\perp)^2}}\omega_z, \quad (15)$$

showing that the radial breathing mode depends on the actual value of the angular velocity and its frequency ranges from the value  $\sqrt{10/3}\omega_\perp$  at  $\Omega_0=0$  (recently measured in Lithium gases [7–9]) to the universal value  $2\omega_\perp$  holding in the centrifugal limit (see Fig. 1).

It is useful to recall that the hydrodynamic description

employed in the present work is based on the applicability of the Thomas-Fermi approximation  $\mu_0 \gg \hbar\omega_\perp$  which, in the presence of vortices implies that the vortex size, fixed by the healing length  $\xi$ , be much smaller than the intervortex distance  $d_v = \sqrt{\hbar/M\Omega}$ :

$$\xi \ll d_v. \quad (16)$$

When  $\xi \simeq d_v$  one enters the lowest Landau level (LLL) regime, a regime already explored in rotating Bose gases both theoretically [24] and experimentally [19]. The condition  $\mu_0 \gg \hbar\omega_z$  instead ensures the 3D nature of the configuration [23]. At unitarity, where the healing length is of the order of the interparticle distance  $d$ , the transition to the LLL regime ( $\xi \simeq d_v$ ) is reached for angular velocities extremely close to the centrifugal limit, satisfying the condition

$$\left[ 1 - \left( \frac{\Omega}{\omega_0} \right)^2 \right] \simeq \frac{1}{N}. \quad (17)$$

Here, for simplicity, we have considered isotropic trapping ( $\omega_\perp = \omega_z \equiv \omega_0$ ), so that the condition (17) also implies the transition to 2D. It is immediate to see that in a Fermi gas at unitarity this condition is equivalent to requiring that the number of vortices be close to the number of particles ( $N_v \simeq N$ ) [25], a regime where one expects to observe quantum-Hall- (QH-) like effects.

It is worth comparing Eq. (17) with the condition required to reach the LLL and QH regimes in a dilute Bose gas. In this case the LLL regime (also corresponding to the 2D regime) is obtained under the less severe requirement  $[1 - (\Omega/\omega_0)^2] \simeq 1/G$ , where  $G = Na/a_{ho}$ . Indeed, since the scattering length  $a$  is much smaller than the oscillator length

$a_{ho} = \sqrt{\hbar/M\omega_0}$  one has  $G \ll N$ . The condition for reaching the quantum Hall regime ( $N_v \simeq N$ ) is instead given by the much more severe requirement  $[1 - (\Omega/\omega_0)^2] \simeq G/N^2$ .

An other interesting case concerns the BCS regime of negative values of the scattering length. As recently pointed out in Ref. [26], in this case the centrifugal limit cannot be reached by keeping the system in the superfluid phase. In fact, since the density of the gas becomes smaller and smaller as  $\Omega \rightarrow \omega_\perp$ , the pairing gap eventually becomes of the order of the trapping frequency and superfluidity is lost, with the emergence of a smooth transition between a superfluid central core containing a vortex lattice and a rotating normal fluid at the periphery. While the equations of rotational hydrodynamics are expected to hold also in the normal phase, the detailed structure of elementary excitations might be influenced by the co-existence of the normal and superfluid components. Let us finally point out that the equations of hydrodynamics are well suited also to study the problem of the expansion and in particular to predict how the presence of vorticity affects the time evolution of the aspect ratio after release of the trap.

In conclusion we have derived the frequency spectrum of the breathing modes of a 3D harmonically trapped Fermi superfluid in the presence of a vortex lattice. Special attention has been devoted to the unitarity regime, where the collective frequencies are found to exhibit a different dependence on the angular velocity with respect to the case of a dilute Bose gas.

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