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# A splitting double sweep method for the Helmholtz equation

N. Bouziani\*, H. Calandra †, F. Nataf ‡

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Statement of the problem and some algorithms</b>	<b>2</b>
<b>3</b>	<b>Splitting double sweep</b>	<b>3</b>
3.1	Substructuring . . . . .	3
3.2	Splitting double sweep preconditioner . . . . .	4
<b>4</b>	<b>Convergence rates</b>	<b>5</b>
<b>5</b>	<b>Numerical results</b>	<b>6</b>
5.1	Homogeneous waveguide . . . . .	6
5.2	Influence of the overlap . . . . .	8
5.3	Open cavity test . . . . .	8
5.4	Wedge test . . . . .	10
<b>6</b>	<b>Prospects</b>	<b>10</b>

## Abstract

We consider the domain decomposition method approach to solve the Helmholtz equation. A new double sweep based approach is presented valid for any type of interface boundary conditions and that benefits from the overlap. It makes use of a splitting of the local problems in the subdomain. Despite of the fact that a first order interface boundary conditions is used, the splitting double sweep method demonstrates good stability properties with respect to the number of subdomains and the frequency even for heterogeneous media. Convergence is improved when compared to the double sweep method for all of our test cases: waveguide, open cavity and wedge problems.

## 1 Introduction

Solving the Helmholtz equation numerically is a difficult task, especially when dealing with high-frequency regimes, heterogeneous media or reflecting boundary conditions. Over the last decades a lot of effort and progress has been made in developing efficient algorithms to solve the ill-conditioned linear system resulting from the Helmholtz operator's discretization. Domain decomposition methods (DDM) try to overcome these difficulties. Domain decomposition methods are hybrid methods that combine direct solvers in subdomains and iterative matching of the solutions across the subdomains. The original domain decomposition method introduced by Schwarz [15] only works for overlapping domain decomposition. P. L. Lions [12] introduced a new variant of this algorithm where the Dirichlet interface conditions are replaced by Robin interface conditions, his method can be applied to both overlapping and nonoverlapping subdomains. He showed convergence for the elliptic case and the proof was extended by Després [4] to the Helmholtz equation and later on to the time-harmonic Maxwell equations [5]. It is known that the presence of overlaps helps to speed up the convergence, however nonoverlapping based methods are often used to avoid to deal with the construction of the normal derivative of the solution. More recently, sweeping-type domain decomposition methods have been made popular due to their capability to achieve nearly-linear asymptotic complexity. The sweeping algorithm was first proposed and analyzed in [13] for convection-diffusion operators. Sweeping approach for Helmholtz problems has recently seen its interest renewed as a preconditioner to speed up the convergence

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of the solver: the double sweep preconditionner of Vion and Geuzaine for non overlapping decomposition with high order interface conditions [16, 17], the PML-based sweep method of Stolk [2], and the polarized traces method of Zepeda-Núñez and Demanet [18]. There also exists sweeping-type methods that are not domain decomposition based methods, such as the sweeping PML preconditionner of Engquist and Ying [8, 7], the source transfer method [3], see [10] for a complete panorama and relations between these methods.

For a decomposition of the domain into layers and when equipping the local subproblems with exact absorbing boundary conditions (ABC), the total number of iterations is equal to the number of subdomains, this is due to the nilpotency of the iteration operator valued matrix. In practice, the exact ABC (which are also the optimal interface conditions, see [14]) procedure is tedious to implement and computationally expensive. On the other hand, for the non-exact ABC such as Robin condition, the nilpotency property is lost and convergence deteriorates. More precisely, these boundary conditions at the interfaces produce spurious reflected waves that significantly increase the number of iterations to converge, in particular for heterogeneous media and high frequency regimes.

We propose to precondition the discrete Helmholtz system by a double-sweep type algorithm modified in two ways compared to previous works:

- non-overlapping or overlapping subdomains can be used
- a novel splitting of the local subproblems prevents spurious interface reflections from hindering the convergence

Enabling overlapping subdomains enables to leverage its beneficial effect on the damping of high frequency modes (see e.g. [6], § 2.2.) of the error whereas the novel splitting prevents its adversary effect on the convergence of propagative modes. This is useful since in the non-overlapping approach, the quality of the ABC is nearly the only way of impacting the convergence of the algorithm, and when dealing with more complex problem such as Maxwell equations high order ABCs are harder to handle.

Note that in double sweep algorithms only one or two subdomain solves are done concurrently. In order to avoid having idle processes, in [17] it is proposed to pipeline the algorithm with respect to multiple right hand sides. This is useful in seismic inversion problems or in far-field pattern computations.

We will first state the problem in § 2. Then we explain in § 3 how we substructure the Helmholtz problem and formulate a modified version of the double sweep algorithm introduced in [13], then how we use this modified double sweep algorithm to build a preconditionner that efficiently speed up the convergence of the solver. In § 4, the convergence of the algorithm is studied when it is applied to a convection-diffusion equation in place of the Helmholtz equation. In § 5, we present numerical results for the original problem, i.e. the Helmholtz problem.

## 2 Statement of the problem and some algorithms

We consider the Helmholtz equation in a bounded domain  $\Omega \subset \mathbb{R}^2$  with frequency  $\omega$ , velocity  $c$  and wavenumber  $k$  defined by  $k^2 = \omega^2/c^2$ :

$$\begin{aligned} (-k^2 - \Delta) u &= f \text{ in } \Omega \\ &+ \text{appropriate boundary conditions on } \partial\Omega \end{aligned} \quad (1)$$

We consider a layered decomposition of  $\Omega$  into  $N$  slices  $(\Omega_i)_{1 \leq i \leq N}$  with or without overlap, see Figure 1. More precisely, for each  $1 \leq i \leq N$ ,  $\Omega \setminus \Omega_i$  is written as the disjoint union of two open subsets  $\Omega_{i,l}$  and  $\Omega_{i,r}$  where  $\Omega_{i,l}$  is on the left of  $\Omega_i$  and  $\Omega_{i,r}$  on its right. The boundary  $\partial\Omega_i \setminus \partial\Omega$  is written as the disjoint union of  $\Gamma_{i,l}$  and  $\Gamma_{i,r}$  where  $\Gamma_{i,l}$  is on the left of  $\Omega_i$  and  $\Gamma_{i,r}$  is on its right ( $\Omega_{1,l} = \emptyset$  and  $\Omega_{N,r} = \emptyset$ ) (see Fig. 2). The outward normal from  $\Omega_i$  on  $\Gamma_{i,l}$  (resp.  $\Gamma_{i,r}$ ) is denoted by  $\vec{n}_{i,l}$  (resp.  $\vec{n}_{i,r}$ ). The problem (1) can be solved iteratively using a domain decomposition method where we solve locally on each subdomain  $\Omega_i$  the equation (1) with appropriate boundary conditions on the physical boundaries and interfaces [4]. The method writes:

Solve in parallel:

$$\left\{ \begin{aligned} (-k^2 - \Delta) u_i^{n+1} &= f \text{ in } \Omega_i, 1 \leq i \leq N \\ \mathcal{B}_{i,l}(u_i^{n+1}) &= \mathcal{B}_{i,l}(u_{i-1}^n) \text{ on } \Gamma_{i,l}, 2 \leq i \leq N \\ \mathcal{B}_{i,r}(u_i^{n+1}) &= \mathcal{B}_{i,r}(u_{i+1}^n) \text{ on } \Gamma_{i,r}, 1 \leq i \leq N - 1 \\ &+ \text{appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i. \end{aligned} \right. \quad (2)$$

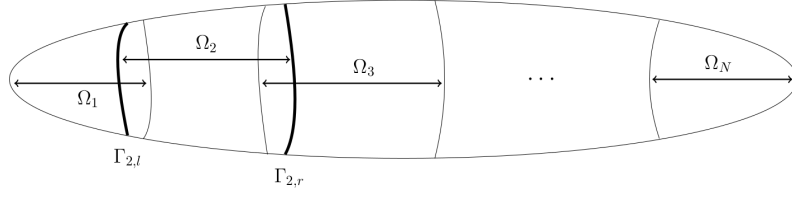


Figure 1: Decomposition into vertical strips

where  $\mathcal{B}_{i,l}$  and  $\mathcal{B}_{i,r}$  are the interface conditions. For sake of simplicity, we consider first-order ABC as interface conditions:

$$\begin{cases} \mathcal{B}_{i,l} = \partial_{\vec{n}_{i,l}} + Ik \\ \mathcal{B}_{i,r} = \partial_{\vec{n}_{i,r}} + Ik \end{cases} \quad (3)$$

where  $I^2 = -1$  and  $\vec{n}_{i,r}$  (resp.  $\vec{n}_{i,l}$ ) is the outward normal to domain  $\Omega_i$  on  $\Gamma_{i,r}$  (resp.  $\Gamma_{i,l}$ ). It is known that higher-order ABC lead to significant improve of the convergence speed, see e.g. [9, 1].

A more efficient variant of algorithm 2 was introduced in [13] which enables pipelining techniques when solving for multiple right hand sides [17]. It consists in double sweeps over the subdomains:

**Left to right sweep:**

$$\begin{cases} (-k^2 - \Delta) u_i^{n+1/2} = f \text{ in } \Omega_i, 1 \leq i \leq N \\ \mathcal{B}_{i,l}(u_i^{n+1/2}) = \mathcal{B}_{i,l}(u_{i-1}^{n+1/2}) \text{ on } \Gamma_{i,l}, 2 \leq i \leq N \\ \mathcal{B}_{i,r}(u_i^{n+1/2}) = \mathcal{B}_{i,r}(u_{i+1}^n) \text{ on } \Gamma_{i,r}, 1 \leq i \leq N-1 \\ + \text{ appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i. \end{cases} \quad (4)$$

**Right to left sweep:**

$$\begin{cases} (-k^2 - \Delta) u_i^{n+1} = f \text{ in } \Omega_i, 1 \leq i \leq N \\ \mathcal{B}_{i,l}(u_i^{n+1}) = \mathcal{B}_{i,l}(u_{i-1}^{n+1/2}) \text{ on } \Gamma_{i,l}, 2 \leq i \leq N \\ \mathcal{B}_{i,r}(u_i^{n+1}) = \mathcal{B}_{i,r}(u_{i+1}^{n+1}) \text{ on } \Gamma_{i,r}, 1 \leq i \leq N-1 \\ + \text{ appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i. \end{cases} \quad (5)$$

### 3 Splitting double sweep

In this section, we define a variant of algorithm (4)-(5) which has a superior convergence. Numerical results will show that it benefits better from the overlap and have a better parallelism. This algorithm is written in terms of the substructured problem that we define first.

#### 3.1 Substructuring

Substructuring the algorithm (2), the iterative method can be reformulated considering only surfacic unknowns on the interfaces:

$$\begin{cases} h_{i,l}^n := \mathcal{B}_{i,l}(u_i^n), \text{ on } \Gamma_{i,l} \text{ for } 2 \leq i \leq N \\ h_{i,r}^n := \mathcal{B}_{i,r}(u_i^n), \text{ on } \Gamma_{i,r} \text{ for } 1 \leq i \leq N-1. \end{cases} \quad (6)$$

Considering the global vector  $h^n$  containing the local unknowns  $(h_{i,l}^n)_{2 \leq i \leq N}$  and  $(h_{i,r}^n)_{1 \leq i \leq N-1}$ , we can reformulate the additive Schwarz method (2) as a Jacobi algorithm on  $h^n$ :

$$h^{n+1} := \mathcal{T}(h^n) + G \quad (7)$$

where the iteration operator  $\mathcal{T}$  can be written in the form of an operator valued matrix and  $G$  refers to the contribution of the right-hand side  $f$ , see [14]. Therefore, we look for a vector  $h$  such that,

$$(Id - \mathcal{T})(h) = G. \quad (8)$$

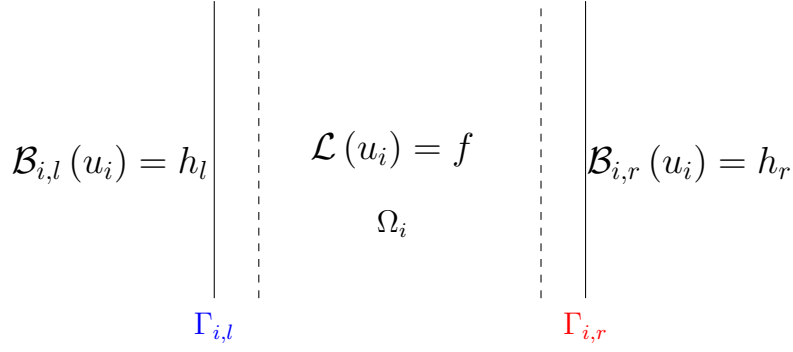


Figure 2: Local problem on the subdomain  $\Omega_i$

In order to define more precisely the operator  $\mathcal{T}$ , we introduce for each subdomain an operator  $S_i$  which takes three arguments, two surfacic functions  $h_l$  and  $h_r$  and a volume function  $f$ :

$$S_i(h_l, h_r, f) := v \quad (9)$$

where  $v : \Omega_i \mapsto \mathbb{C}$  satisfies:

$$\begin{cases} (-k^2 - \Delta)v = f & \text{in } \Omega_i \\ \mathcal{B}_{i,l}(v) = h_l & \text{on } \Gamma_{i,l} \quad (2 \leq i \leq N) \\ \mathcal{B}_{i,r}(v) = h_r & \text{on } \Gamma_{i,r} \quad (1 \leq i \leq N-1) \\ + \text{appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i, \end{cases} \quad (10)$$

for  $1 < i < N$ . For  $i = 1$ , the definition of  $S_1$  is similar except that it takes only the two arguments  $(h_r, f)$  since domain  $\Omega_1$  has no left interface and similarly operator  $S_N$  takes only the two arguments  $(h_l, f)$  since domain  $\Omega_N$  has no right interface. **As of now, for sake of simplicity and by abuse of notation,  $S_1(h_l, h_r, f)$  (resp.  $S_N(h_l, h_r, f)$ ) will refer to  $S_1(h_r, f)$  (resp.  $S_N(h_l, f)$ ).**

Next, we introduce the surfacic right hand-side  $G$  by

$$\begin{aligned} G_{i+1,l} &:= \mathcal{B}_{i+1,l}(S_i(0, 0, f)), \quad 1 \leq i \leq N-1 \\ G_{i-1,r} &:= \mathcal{B}_{i-1,r}(S_i(0, 0, f)), \quad 2 \leq i \leq N. \end{aligned} \quad (11)$$

and the substructured operator  $\mathcal{T}$  by:

$$\begin{aligned} \mathcal{T}(h)_{i+1,l} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l}, h_{i,r}, 0)), \quad 1 \leq i \leq N-1 \\ \mathcal{T}(h)_{i-1,r} &:= \mathcal{B}_{i-1,r}(S_i(h_{i,l}, h_{i,r}, 0)), \quad 2 \leq i \leq N. \end{aligned} \quad (12)$$

Using this framework, we write the substructured form of the **double sweep algorithm** as follows:

**Forward sweep**

$$\begin{aligned} h_{i+1,l}^{n+1/2} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l}^{n+1/2}, h_{i,r}^n, f)), \\ h_{i-1,r}^{n+1/2} &:= \mathcal{B}_{i-1,r}(S_i(h_{i,l}^{n+1/2}, h_{i,r}^n, f)), \end{aligned} \quad (13)$$

**followed by a Backward sweep**

$$\begin{aligned} h_{i+1,l}^{n+1} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l}^{n+1/2}, h_{i,r}^{n+1}, f)), \\ h_{i-1,r}^{n+1} &:= \mathcal{B}_{i-1,r}(S_i(h_{i,l}^{n+1/2}, h_{i,r}^{n+1}, f)). \end{aligned} \quad (14)$$

### 3.2 Splitted double sweep preconditioner

We explain now the rationale behind the splitting double sweep preconditioner that we define in this section. Note first that by linearity of the operators  $(S_i)_{1 \leq i \leq N}$ , the contribution of each subdomain can be split into two contributions, one for each of its two interfaces:

$$\begin{aligned} \mathcal{T}(h)_{i+1,l} &= \mathcal{B}_{i+1,l}(S_i(h_{i,l}, 0, 0)) + \mathcal{B}_{i+1,l}(S_i(0, h_{i,r}, 0)), \quad 1 \leq i \leq N-1 \\ \mathcal{T}(h)_{i-1,r} &= \mathcal{B}_{i-1,r}(S_i(0, h_{i,r}, 0)) + \mathcal{B}_{i-1,r}(S_i(h_{i,l}, 0, 0)), \quad 2 \leq i \leq N \end{aligned} \quad (15)$$

Had we used exact absorbing boundary conditions (EABC)  $\mathcal{B}^{EABC}$  instead of the zero-th order ones (3) in equations (9)-(10), we would have with obvious abuse of notation:

$$\begin{aligned}\mathcal{B}_{i+1,l}^{EABC}(S_i^{EABC}(0, h_{i,r}, 0)) &= 0, \quad 1 \leq i \leq N-1 \\ \mathcal{B}_{i-1,r}^{EABC}(S_i^{EABC}(h_{i,l}, 0, 0)) &= 0, \quad 2 \leq i \leq N.\end{aligned}\quad (16)$$

Then, the operator would have the following form

$$\begin{aligned}\mathcal{T}^{EABC}(h)_{i+1,l} &= \mathcal{B}_{i+1,l}^{EABC}(S_i^{EABC}(h_{i,l}, 0, 0)), \quad 1 \leq i \leq N-1, \\ \mathcal{T}^{EABC}(h)_{i-1,r} &= \mathcal{B}_{i-1,r}^{EABC}(S_i^{EABC}(0, h_{i,r}, 0)), \quad 2 \leq i \leq N.\end{aligned}\quad (17)$$

The operator  $\mathcal{T}^{EABC}$  is a nilpotent operator of order  $N-1$  so that the inverse of  $I - \mathcal{T}^{EABC}$  is easy to compute as a finite Neumann series:

$$(I - \mathcal{T}^{EABC})^{-1} = \sum_{i=0}^{N-2} (\mathcal{T}^{EABC})^i.$$

This formula induces a convergence in  $N-1$  steps of a domain decomposition method with exact absorbing boundary conditions. In practice, the absorbing boundary conditions are non exact, therefore we have

$$\begin{aligned}\mathcal{B}_{i+1,l}(S_i(0, h_{i,r}, 0)) &\neq 0, \quad 1 \leq i \leq N-1, \\ \mathcal{B}_{i-1,r}(S_i(h_{i,l}, 0, 0)) &\neq 0, \quad 2 \leq i \leq N,\end{aligned}\quad (18)$$

and we loose the nilpotency property of  $\mathcal{T}$ . This led us to define a new operator  $\mathcal{T}_{SDS}$

$$\begin{aligned}\mathcal{T}_{SDS}(h)_{i+1,l} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l}, 0, 0)), \quad 1 \leq i \leq N-1 \\ \mathcal{T}_{SDS}(h)_{i-1,r} &:= \mathcal{B}_{i-1,r}(S_i(0, h_{i,r}, 0)), \quad 2 \leq i \leq N\end{aligned}\quad (19)$$

which by definition is a nilpotent operator of order  $N-1$ . We propose to use this newly defined operator to build a preconditioner for (8). The left-preconditioned system reads:

$$(Id - \mathcal{T}_{SDS})^{-1} (Id - \mathcal{T})(h) = (Id - \mathcal{T}_{SDS})^{-1} G \quad (20)$$

More intuitively, the key idea is to cancel out the reverse contribution at the interfaces that should not happen for the exact ABC case in order to prevent spurious interface reflections from hindering the convergence. In fact, these boundary conditions at the interfaces produce spurious reflected waves that significantly increase the number of iterations to converge, in particular for heterogeneous media and high frequency regimes.

## 4 Convergence rates

Unless mentioned explicitly, convergence rates estimates in this section are only valid for domain decomposition methods applied to the convection-diffusion equation with constant coefficients in the plane  $\mathbb{R}^2$  decomposed into  $N$  overlapping vertical strips. We first recall the convergence rates proved in [13] for the fixed point method related to the Jacobi and double sweep methods. Then, we state a similar estimate for the fixed point method related to the splitting double sweep algorithm.

It was proved in Theorem 2.4 of [13] that under some geometric assumptions that the fixed point algorithms achieve geometric convergence with the estimates:

$$\|u_i^n - u\|_{H_{st}^2(\Omega_i)} \leq C \rho^{\lfloor \frac{n}{2(N-1)} \rfloor} \sup_j \|u_j^0 - u\|_{H_{st}^2(\Omega_j)} \quad n \geq 2N+1, \quad (21)$$

for the Jacobi method (6),

$$\|u_i^n - u\|_{H_{st}^2(\Omega_i)} \leq C \rho^n \sup_j \|u_j^0 - u\|_{H_{st}^2(\Omega_j)} \quad n \geq 3, \quad (22)$$

for the double sweep algorithm (13)-(14) (writing  $[\cdot]$  for the integer part) where  $C$  and  $\rho$  are constants,  $C > 0$ ,  $0 < \rho < 1$ ,  $\rho$  is independent of  $N$  and  $\|\cdot\|_{H_{st}^2}$  denotes some Sobolev norm.

Under the same assumptions, it can be checked that for  $n$  even, we have

$$\|u_i^n - u\|_{H_{st}^2(\Omega_i)} \leq 2 \sqrt{\rho}^n \sup_j \|u_j^0 - u\|_{H_{st}^2(\Omega_j)} \quad n \geq 2, \quad (23)$$

for the fixed point method related to the splitting double sweep defined as follows:

$$(Id - \mathcal{T}_{SDS})(h^{n+1}) = (\mathcal{T} - \mathcal{T}_{SDS})(h^n) + G. \quad (24)$$

Note that the expression for  $\rho$  (not shown here) is the same for all these estimates. The differences lie in the exponent of  $\rho$  and in the constant. The constant  $C$  in eq. (22) may be large whereas in eq. (23) its value is simply 2. A seemingly advantage of (22) with respect to (23) is the square root of  $\rho$  ratio in the convergence rate. In fact this difference is nullified when you realize that an iteration of DS takes twice as much time as an iteration of SDS. Indeed, during one iteration of the DS algorithm, only one subdomain is active at a time in the order  $1 \rightarrow 2 \rightarrow \dots \rightarrow N$  and then in the reverse order  $N \rightarrow N-1 \rightarrow \dots \rightarrow 1$ . Whereas during one iteration of SDS, two subdomains are active at a time in the order  $1 \rightarrow 2 \rightarrow \dots \rightarrow N$  for one subdomain and concurrently in the reverse order  $N \rightarrow N-1 \rightarrow \dots \rightarrow 1$  for the other one. Thus, the elapsed time of one iteration of DS is thus twice as much time as one iteration of SDS using two cores. Note that using two cores for the DS algorithm brings no advantage in terms of elapsed time. So overall, these estimates give an edge to SDS over DS in terms of the constants in formulas (22) and (23).

As a side remark for the interested reader, we give a few hints on the proof of (23). The convergence of SDS depends on the spectral norm of

$$T_{SDS} := (Id - \mathcal{T}_{SDS})^{-1}(\mathcal{T} - \mathcal{T}_{SDS}). \quad (25)$$

whose expression, using notations from [13] without redefining them, is:

$$T_{SDS} = \sum_{i=0}^{N-2} M_r^i A_r + \sum_{i=0}^{N-2} M_l^i A_l.$$

Using relations (23) page 369 in [13], it can be checked that for  $n$  even:

$$T_{SDS}^n = \left( \sum_{i=0}^{N-2} M_r^i A_r \sum_{i=0}^{N-2} M_l^i A_l \right)^{n/2} + \left( \sum_{i=0}^{N-2} M_l^i A_l \sum_{i=0}^{N-2} M_r^i A_r \right)^{n/2}.$$

Then, the estimate follows from the formula for  $\rho$  given in Theorem 5.1. of [13].

## 5 Numerical results

In this section, we present numerical results when solving the substructured equation (8) with the GMRES algorithm right preconditioned by  $Id$  (Jacobi method),  $(Id - \mathcal{T}_{DS})^{-1}$  (Double sweep algorithm) and  $(Id - \mathcal{T}_{SDS})^{-1}$  (Splitting Double sweep algorithm). The Helmholtz equation is discretized with a P1 finite element using the FreeFem++ domain specific language [11]. Note that we use a careful variational discretisation of the normal derivative ensuring that the solution obtained converges to the solution of the problem without domain decompositions.

### 5.1 Homogeneous waveguide

First, we consider the homogeneous waveguide test case with a layered decomposition into  $N$  subdomains. More specifically, we consider a rectangular geometry ( $\Omega = [0, N] \times [0, 1]$ ) made of a homogeneous medium. On the upper and lower sides of the waveguide, we impose homogeneous Dirichlet conditions. In addition, we perform a multimode excitation on the left side and we impose an absorbing boundary conditions on the right side. The global problem is written as

$$\begin{cases} (-k^2 - \Delta) u = f & \text{in } \Omega \\ (\partial_{\bar{n}} + Ik) u = 0 & \text{on } \{x = N\} \times [0, 1] \\ (\partial_{\bar{n}} + Ik) u = u_g & \text{on } \{x = 0\} \times [0, 1] \\ u = 0 & \text{on } [0, N] \times \{y = 0, y = 1\} \end{cases} \quad (26)$$

where  $u_g = e^{-120(y-0.5)^2} \sin(\pi y)$ .

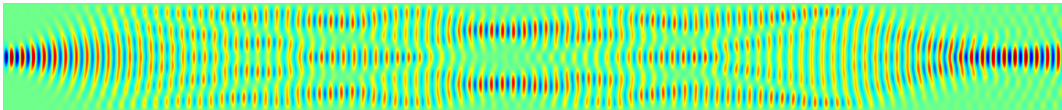


Figure 3: Homogeneous waveguide ( $k = 20\pi$ )

We considered two values for the wave number:  $k = 20$  and  $k = 20\pi$ . The width of the overlap is 4 mesh size. In Tables 1 and 2 we show iteration counts. We use two stopping criteria for the relative residual:  $10^{-6}$  and  $10^{-3}$  the latter is written in brackets. As expected, the Jacobi method needs many iterations to converge. For both Jacobi and double sweep algorithms, the iteration counts vary linearly w.r.t. the number of subdomains. The iteration counts for the Splitting Double Sweep algorithm are significantly lower and display a sublinear behavior w.r.t. the number of subdomains.

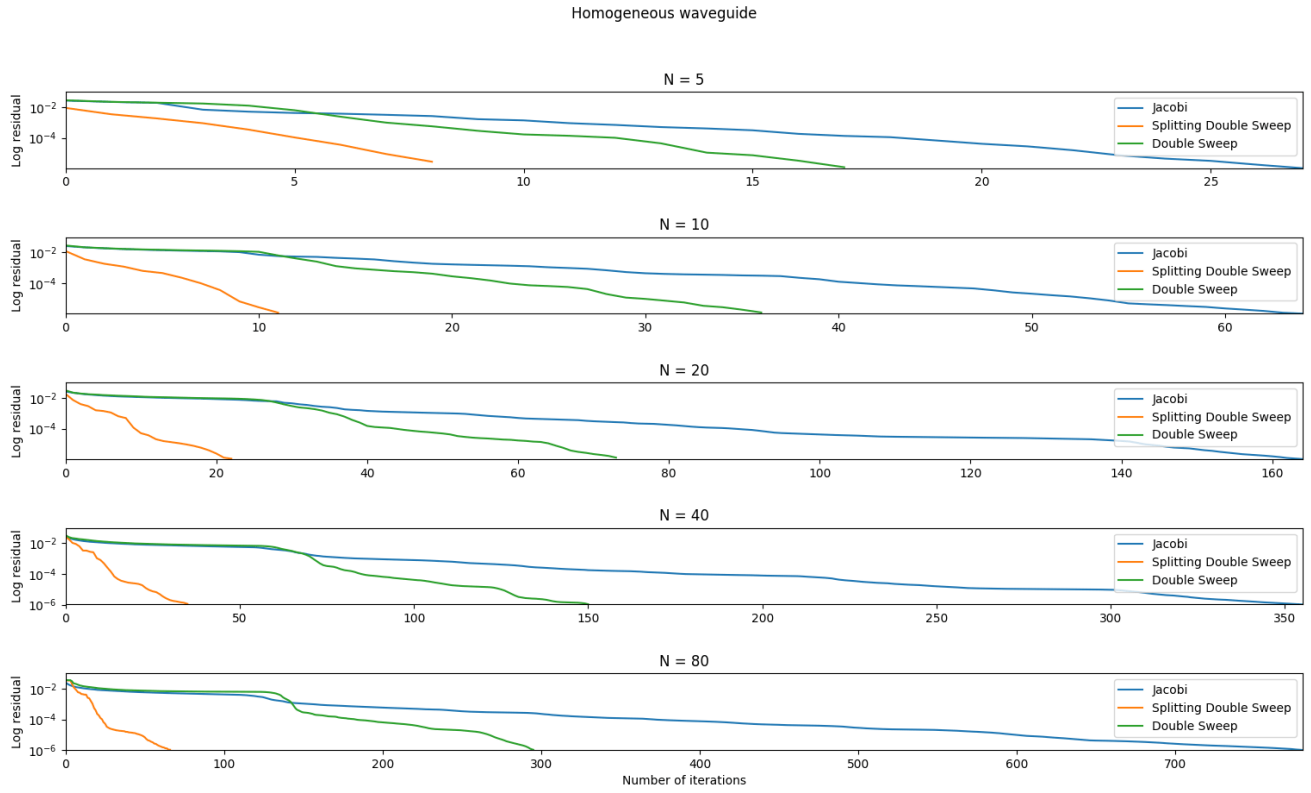


Figure 4: Homogeneous waveguide residual curves (tol=  $10^{-7}$ )

N	Jacobi	DS	Splitting DS
5	28 (11)	18 (7)	9 (3)
10	65 (27)	37 (15)	12 (4)
20	165 (51)	74 (36)	23 (7)
40	356 (86)	151 (72)	36 (10)
80	781 (155)	296 (144)	67 (18)

Table 1: Homogeneous waveguide,  $k = 20$ ,  $\delta = 4h$ , TOL= $10^{-6}(10^{-3})$ , nppw1 = 24 , P1

N	Jacobi	DS	Splitting DS
5	25	16	9
10	62	33	20
20	130	76	33
40	280	170	60

Table 2: Homogeneous waveguide,  $k = 20\pi$ ,  $\delta = 4h$ , TOL= $10^{-6}$ , nppw1 = 24 , P1



## 5.2 Influence of the overlap

We have also tested the effect of the width of the overlap on the convergence. We considered two test cases: the homogeneous waveguide (see Table 3) and the wedge (see Table 4) that is more precisely defined in § 5.4. We observe that for the waveguide solved by the Splitting Double Sweep method, the iteration count decreases significantly with the overlap. This monotonical decrease in the iteration count contrasts with the behaviour of the other two methods. We see that for the Jacobi and double sweep (DS) methods, the overlap has a very little effect. For the Jacobi method it improves slightly the iteration counts whereas for the DS method, it might deteriorate the iteration count. For the wedge test case, all methods benefit monotonically from the size of the overlap but once again the reduction in the iteration count is more pronounced for the Splitting Double Sweep method where the iteration count is reduced by a factor 1.83 when the overlap is increased from  $2h$  to  $16h$ .

$\delta$	Jacobi	DS	Splitting DS
2	159	69	27
4	165	74	23
8	160	76	20
16	143	73	18

Table 3: Homogeneous waveguide,  $k = 20$ ,  $\delta$  varies, TOL= $10^{-6}$ , nppwl = 24 , P1

$\delta$	Jacobi	DS	Splitting DS
2	259	127	97
4	245	117	83
8	221	105	69
16	202	91	53

Table 4: Wedge,  $\omega = 40\pi$ ,  $\delta$  varies, TOL= $10^{-6}$ , nppwl = 24 , P1

## 5.3 Open cavity test

Same as before, the domain is rectangular with an homogeneous medium and its length increases with the number of subdomains. The open cavity test is challenging due to the homogeneous Dirichlet conditions imposed on three sides. In addition, we perform an excitation on the left side. The Dirichlet conditions create rebounds leading to an increase in the number of reflections, this phenomenon is exacerbated for high-frequency regimes. The global problem can be written as

$$\begin{cases} (-k^2 - \Delta) u = f & \text{in } \Omega \\ (\partial_{\vec{n}} + Ik) u = g & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \setminus \Gamma \end{cases} \quad (27)$$

where  $\Gamma := \{x = 0\} \times [0, 1]$  and  $g = \exp^{-ik(\sin(\theta)*y+x*\cos(\theta))}$ ,  $\theta = \frac{\pi}{8}$ . It corresponds to an incident plane wave propagating at an angle  $\frac{\pi}{8}$  w.r.t. to the horizontal direction, see Fig. 5. This creates numerous reflections on the lateral boundaries of the open cavity.

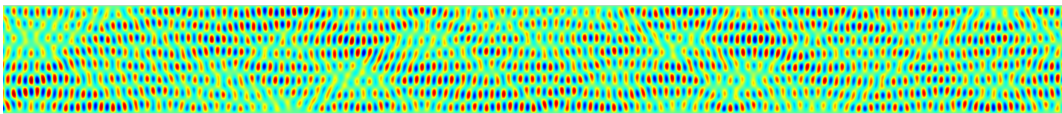


Figure 5: Open cavity solution ( $k = 20\pi$ )

Iteration counts are given in Tables 5 and 6 for two wave numbers. One can notice that we took an overlap of  $8h$  instead of  $4h$  for the waveguide because we want to take advantage of the fact that the overlap contributes to accelerate convergence. Once again, for a given number of subdomains, the Splitting DS method is systematically better than the other methods. As for the scalability, we observe that in terms of the number of subdomains, the iteration count varies surlinearly for the Jacobi method, linearly for the DS method and sublinearly for the Splitting DS method.

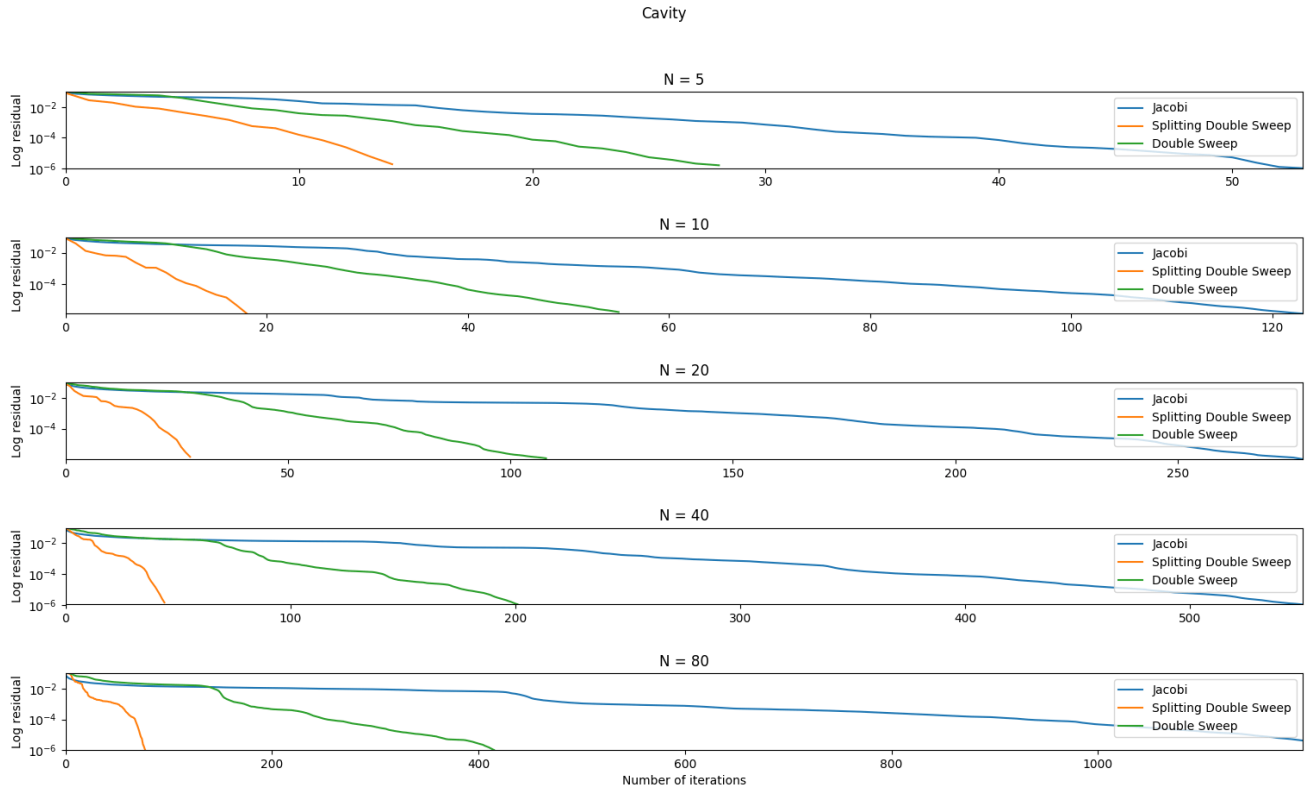


Figure 6: Open cavity residual curves (tol=  $10^{-7}$ )

N	Jacobi	DS	Splitting DS
5	54	29	15
10	124	56	19
20	279	109	29
40	551	202	45
80	+1200	416	78

Table 5: Open cavity,  $k = 20$ ,  $\delta = 8h$ , TOL= $10^{-6}$ ,  $nppwl = 24$ , P1

N	Jacobi	DS	Splitting DS
5	72	47	31
10	154	94	47
20	331	223	82
40	754	453	149

Table 6: Open cavity,  $k = 20\pi$ ,  $\delta = 8h$ , TOL= $10^{-6}$ ,  $nppwl = 24$ , P1

## 5.4 Wedge test

We consider the classical test case of the wedge, see e.g. [16], a rectangular domain  $[0, 600] \times [0, 1000]$  with three different velocities in regions separated by non-parallel boundaries (Fig. 7a). Starting from the top, the velocities are  $c = 2000$ ,  $c = 1500$  and  $c = 3000$ . Sommerfeld conditions are imposed on the bottom, right and left boundaries. The abrupt variations of the wavenumber produce internal reflections in different directions. A typical solution is shown on Figure 7b. Convergence curves are plotted in Fig. 8 and iteration counts are given in Tables 7 and 8. The SDS method is clearly superior to the Jacobi and DS methods. When increasing the number of subdomains, the ratio in favor of the SDS method compared to the DS method increases up to reaching a value of nearly 4 for a domain decomposition into 40 strips. Interestingly, we see that for a low tolerance on the residual ( $\text{TOL}=10^{-3}$ ), the SDS iteration counts are almost independent of the number of subdomains. Note also that compared to the DS method, the convergence curves of the SDS method does not suffer of any stagnation.

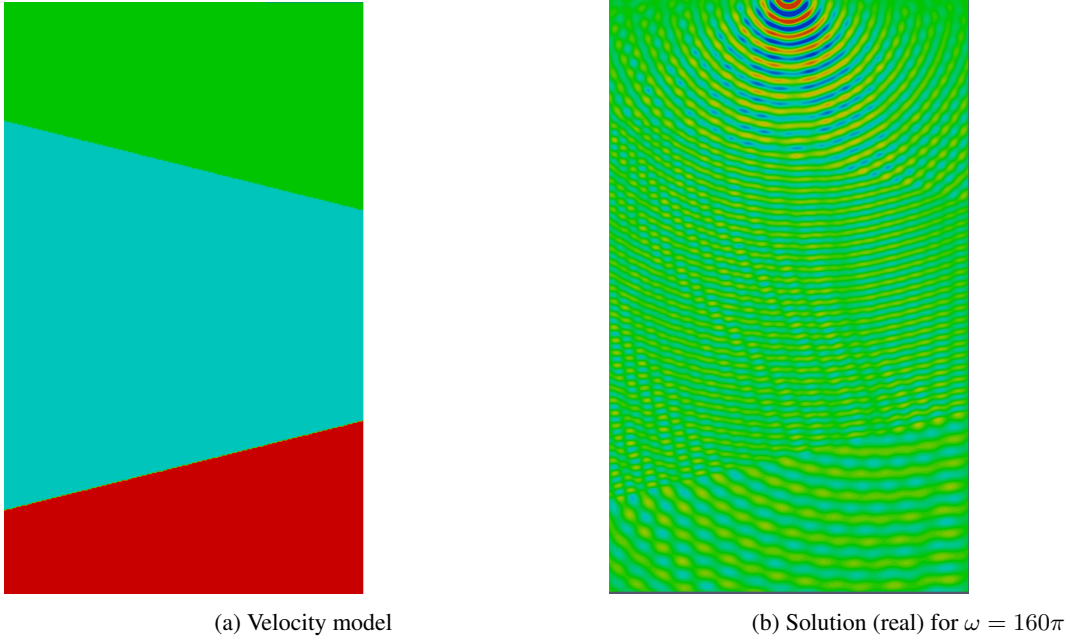


Figure 7: Heterogeneous media: Wedge ( $\omega = 160\pi$ )

N	Jacobi	DS	Splitting DS
5	28 (17)	19 (11)	13 (6)
10	55 (31)	31 (16)	14 (7)
20	110 (55)	58 (29)	18 (8)
40	203 (88)	103 (47)	27 (9)

Table 7: Wedge,  $\omega = 40\pi$ ,  $\delta = 16h$ ,  $\text{TOL}=10^{-6}(10^{-3})$ ,  $nppwl = 24$ , P1

N	Jacobi	DS	Splitting DS
5	28 (15)	18 (10)	12 (5)
10	56 (30)	31 (15)	14 (6)
20	111 (53)	57 (28)	18 (7)
40	206 (85)	111 (55)	30 (9)

Table 8: Wedge,  $\omega = 60\pi$ ,  $\delta = 16h$ ,  $\text{TOL}=10^{-6}(10^{-3})$ ,  $nppwl = 24$ , P1

## 6 Prospects

We have introduced a variant of the double algorithm which yields improved convergence over the double sweep method for various problems: the waveguide, the open cavity and wedge test cases. Many aspects deserve further investigations. We

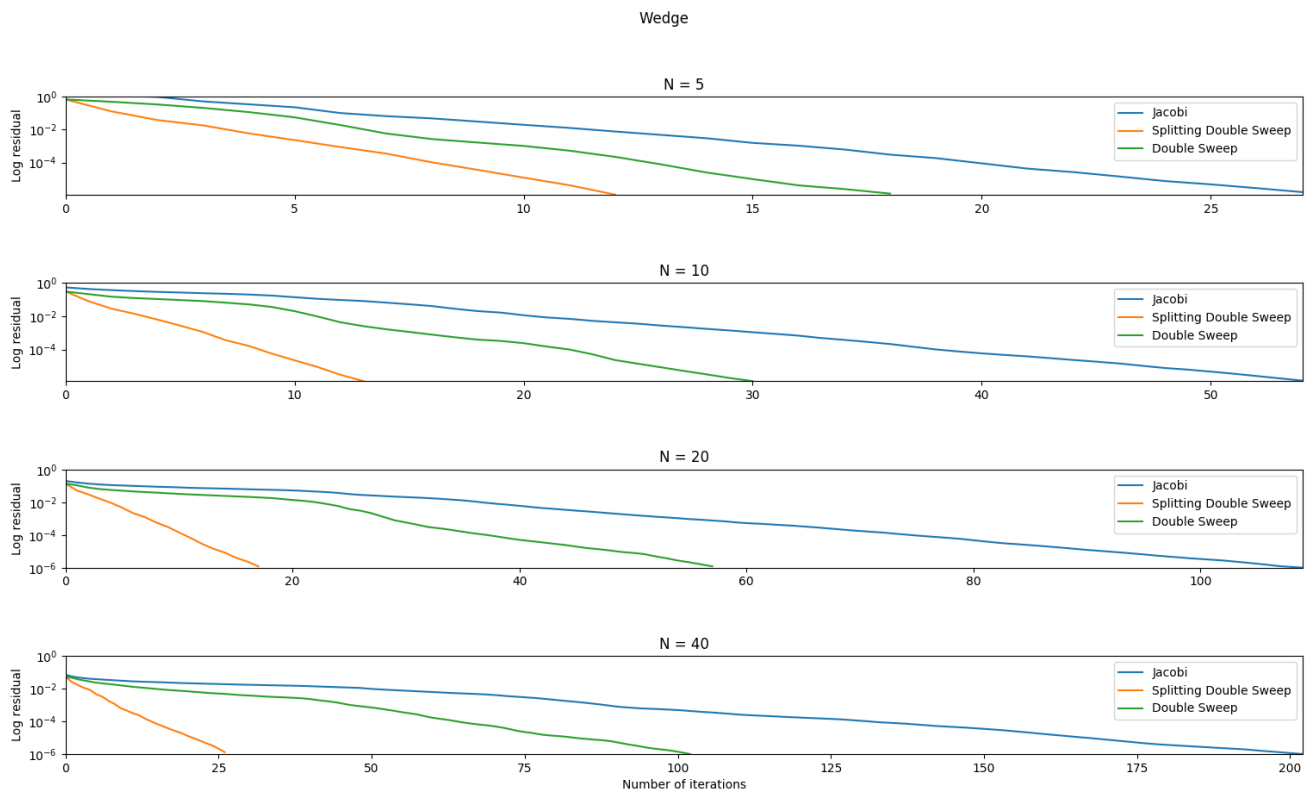


Figure 8: Wedge residual curves ( $\text{tol} = 10^{-7}$ )

list some of them. As previously mentioned, for a first study of the new algorithm we only use first-order absorbing boundary conditions at the interfaces. It is known that higher-order ABC lead to drastically better stability properties and to a quicker convergence. This should reflect in better iteration counts for all methods. Also, an intrinsic problem with both double sweep and splitting double sweep methods is that due to the sequentiality of the algorithm, subdomains are idle most of the time. To be more precise, in a double sweep, only one process is active at a time whereas in a splitting double sweep, two processes are active at a time (one for the left sweep and one for the right sweep performed concurrently). A fix to this is to introduce a pipelining technique that can be applied to multiple right handside problems to improve parallelism and achieve significant speed-ups, see [17]. As a last remark, note that the local solve induced by the definition of the splitting double sweep operator  $\mathcal{T}_{SDS}$ , eq. (19) can be replaced by a cheaper approximate solve such as incomplete factorizations [8].

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