



HAL
open science

An ergodic theorem for asymptotically periodic time-inhomogeneous Markov processes, with application to quasi-stationarity with moving boundaries

William Oçafrain

► **To cite this version:**

William Oçafrain. An ergodic theorem for asymptotically periodic time-inhomogeneous Markov processes, with application to quasi-stationarity with moving boundaries. 2022. hal-02963683v2

HAL Id: hal-02963683

<https://hal.science/hal-02963683v2>

Preprint submitted on 5 Apr 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An ergodic theorem for asymptotically periodic time-inhomogeneous Markov processes, with application to quasi-stationarity with moving boundaries.

William Oçafrain¹

5th April 2022

Abstract

This paper deals with ergodic theorems for particular time-inhomogeneous Markov processes, whose time-inhomogeneity is asymptotically periodic. Under a Lyapunov/minorization condition, it is shown that, for any measurable bounded function f , the time average $\frac{1}{t} \int_0^t f(X_s) ds$ converges in \mathbb{L}^2 towards a limiting distribution, starting from any initial distribution for the process $(X_t)_{t \geq 0}$. This convergence can be improved to an almost sure convergence under an additional assumption on the initial measure. This result will be then applied to show the existence of a quasi-ergodic distribution for processes absorbed by an asymptotically periodic moving boundary, satisfying a conditional Doeblin condition.

Key words: ergodic theorem; law of large numbers; time-inhomogeneous Markov processes; quasi-stationarity; quasi-ergodic distribution; moving boundaries.

Notation

Throughout we shall use the following notation:

- $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.
- $\mathcal{M}_1(E)$ denotes the space of the probability measures whose support is included in E .
- $\mathcal{B}(E)$ denotes the set of the measurable bounded functions defined on E .
- $\mathcal{B}_1(E)$: denotes the set of the measurable functions f defined on E such that $\|f\|_\infty \leq 1$.
- For all $\mu \in \mathcal{M}_1(E)$ and $p \in \mathbb{N}$, $\mathbb{L}^p(\mu)$ denotes the set of the measurable functions $f : E \mapsto \mathbb{R}$ such that $\int_E |f(x)|^p \mu(dx) < +\infty$.
- For any $\mu \in \mathcal{M}_1(E)$ and $f \in \mathbb{L}^1(\mu)$, denote

$$\mu(f) := \int_E f(x) \mu(dx).$$

- For any positive function ψ ,

$$\mathcal{M}_1(\psi) := \{\mu \in \mathcal{M}_1(E) : \mu(\psi) < +\infty\}.$$

- Id denotes the Identity operator.

¹Université de Lorraine, CNRS, Inria, IECL, UMR 7502, F-54000, Nancy, France.
E-mail: w.oacafrain@hotmail.fr

1 Introduction

In general, an ergodic theorem for a Markov process $(X_t)_{t \geq 0}$ and probability measure π refers to the almost sure convergence

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \pi(f), \quad \forall f \in \mathbb{L}^1(\pi). \quad (1)$$

In the time-homogeneous setting, such an ergodic theorem holds for positive Harris recurrent Markov processes with the limiting distribution π corresponding to an invariant measure for the underlying Markov process. For time-inhomogeneous Markov processes, such a result does not hold in general (in particular the notion of invariant measure is in general not well-defined), except for specific types of time-inhomogeneity such as *periodic time-inhomogeneous Markov processes*, defined as time-inhomogeneous Markov processes for which there exists $\gamma > 0$ such that, for any $s \leq t$, $k \in \mathbb{Z}_+$ and x ,

$$\mathbb{P}[X_t \in \cdot | X_s = x] = \mathbb{P}[X_{t+k\gamma} \in \cdot | X_{s+k\gamma} = x]. \quad (2)$$

In other words, a time-inhomogeneous Markov process is periodic when the transition law between any times s and t remains unchanged when the time interval $[s, t]$ is shifted by a multiple of the period γ . In particular, this implies that, for any $s \in [0, \gamma)$, the Markov chain $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$ is time-homogeneous. This fact allowed Höpfner et al. (in [20, 21, 22]) to show that, if the skeleton Markov chain $(X_{n\gamma})_{n \in \mathbb{Z}_+}$ is Harris recurrent, then the chains $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$, for all $s \in [0, \gamma)$, are also Harris recurrent and

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \pi_s(f) ds, \quad \text{almost surely, from any initial measure,}$$

where π_s is the invariant measure for $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$.

This paper aims to prove a similar result for time-inhomogeneous Markov processes said to be *asymptotically periodic*. Roughly speaking (a precise definition of which will be explicitly given later), an asymptotically periodic Markov process is such that, given a time interval $T \geq 0$, its transition law on the interval $[s, s+T]$ is asymptotically "close to" the one, on the same interval, of a periodic time-inhomogeneous Markov process called an *auxiliary Markov process*, when $s \rightarrow \infty$. This definition is very similar to the notion of *asymptotic homogenization*, defined as follows in [1, Subsection 3.3]: a time-inhomogeneous Markov process $(X_t)_{t \geq 0}$ is said to be *asymptotically homogeneous* if there exists a time-homogeneous Markovian semigroup $(Q_t)_{t \geq 0}$ such that, for all $s \geq 0$,

$$\lim_{t \rightarrow \infty} \sup_x \|\mathbb{P}[X_{t+s} \in \cdot | X_t = x] - \delta_x Q_s\|_{TV} = 0, \quad (3)$$

where, for two positive measures with finite mass μ_1 and μ_2 , $\|\mu_1 - \mu_2\|_{TV}$ is the *total variation distance* between μ_1 and μ_2 :

$$\|\mu_1 - \mu_2\|_{TV} := \sup_{f \in \mathcal{B}_1(E)} |\mu_1(f) - \mu_2(f)|. \quad (4)$$

In particular, it is well-known (see [1, Theorem 3.11]) that, under this suitable additional conditions, an asymptotically homogeneous Markov process converges towards a probability measure which is invariant for $(Q_t)_{t \geq 0}$. It is similarly expected that an asymptotically periodic process has the same asymptotic properties as a periodic Markov process; in particular an ergodic theorem holds for the asymptotically periodic process.

The main result of this paper provides for an asymptotically periodic Markov process to satisfy

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{\mathbb{L}^2(\mathbb{P}_{0,\mu})} \frac{1}{\gamma} \int_0^\gamma \beta_s(f) ds, \quad \forall f \in \mathcal{B}(E), \forall \mu \in \mathcal{M}_1(E), \quad (5)$$

whereby $\mathbb{P}_{0,\mu}$ is a probability measure under which $X_0 \sim \mu$, and where β_s is the limiting distribution of the skeleton Markov chain $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$, if it satisfies a Lyapunov-type condition, a local Doeblin condition (defined further in Section 2), and is such that its auxiliary process satisfies a Lyapunov/minorization condition.

Furthermore, this convergence result holds almost surely if a Lyapunov function of the process $(X_t)_{t \geq 0}$, denoted by ψ , is integrable with respect to the initial measure:

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{\mathbb{P}_{0,\mu}\text{-almost surely}} \frac{1}{\gamma} \int_0^\gamma \beta_s(f) ds, \quad \forall \mu \in \mathcal{M}_1(\psi).$$

This will be more precisely stated and proved in Section 2.

The main motivation of this paper is then to deal with *quasi-stationarity with moving boundaries*, that is the study of asymptotic properties for the process X , conditioned not to reach some moving subset of the state space. In particular, such a study is motivated by models such as those presented in [3], studying Brownian particles absorbed by cells whose volume may vary over time.

Quasi-stationarity with moving boundaries has been studied in particular in [24, 25], where a "*conditional ergodic theorem*" (see further the definition of a *quasi-ergodic distribution*) has been shown when the absorbing boundaries move periodically. In this paper, we show that a similar result holds when the boundary is asymptotically periodic, assuming that the process satisfies a conditional Doeblin condition (see Assumption (A')). This will be dealt with in Section 3.

The paper will be concluded by using these results in two examples: an ergodic theorem for an asymptotically periodic Ornstein-Uhlenbeck process, and the existence of a unique quasi-ergodic distribution for a Brownian motion confined between two symmetric asymptotically periodic functions.

2 Ergodic theorem for asymptotically periodic time-inhomogeneous semigroup.

Asymptotic periodicity: the definition.

Let (E, \mathcal{E}) be a measurable space. Consider $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$ a Markovian time-inhomogeneous semigroup, giving a family of measurable subspaces of (E, \mathcal{E}) , denoted by $(E_t, \mathcal{E}_t)_{t \geq 0}$, and a family of linear operator $(P_{s,t})_{s \leq t}$, with $P_{s,t} : \mathcal{B}(E_t) \rightarrow \mathcal{B}(E_s)$, satisfying for any $r \leq s \leq t$,

$$P_{s,s} = \text{Id}, \quad P_{s,t} \mathbb{1}_{E_t} = \mathbb{1}_{E_s}, \quad P_{r,s} P_{s,t} = P_{r,t}.$$

In particular, associated to $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$ is a Markov process $(X_t)_{t \geq 0}$ and a family of probability measures $(\mathbb{P}_{s,x})_{s \geq 0, x \in E_s}$ such that, for any $s \leq t$, $x \in E_s$ and $A \in \mathcal{E}_t$,

$$\mathbb{P}_{s,x}[X_t \in A] = P_{s,t} \mathbb{1}_A(x).$$

We denote $\mathbb{P}_{s,\mu} := \int_{E_s} \mathbb{P}_{s,x} \mu(dx)$ for any probability measure μ supported on E_s . We also denote by $\mathbb{E}_{s,x}$ and $\mathbb{E}_{s,\mu}$ the expectations associated to $\mathbb{P}_{s,x}$ and $\mathbb{P}_{s,\mu}$ respectively. Finally, the following notation will be used for $\mu \in \mathcal{M}_1(E_s)$, $s \leq t$ and $f \in \mathcal{B}(E_t)$,

$$\mu P_{s,t} f := \mathbb{E}_{s,\mu}[f(X_t)], \quad \mu P_{s,t} := \mathbb{P}_{s,\mu}[X_t \in \cdot].$$

The periodicity of a time-inhomogeneous semigroup is defined as follows. We say a semigroup $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$ is γ -*periodic* (for $\gamma > 0$), if, for any $s \leq t$,

$$(F_t, \mathcal{F}_t) = (F_{t+k\gamma}, \mathcal{F}_{t+k\gamma}), \quad Q_{s,t} = Q_{s+k\gamma, t+k\gamma}, \quad \forall k \in \mathbb{Z}_+.$$

It is now possible to define an *asymptotically periodic semigroup*.

Definition 1 (Asymptotically periodic semigroups). A time-inhomogeneous semigroup $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$ is said to be *asymptotically periodic* if (for some $\gamma > 0$) there exists a γ -periodic semigroup $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$ and two families of functions $(\psi_s)_{s \geq 0}$ and $(\tilde{\psi}_s)_{s \geq 0}$ such that $\tilde{\psi}_{s+\gamma} = \psi_s$ for all $s \geq 0$, and for any $s \in [0, \gamma)$:

1. $\bigcup_{k=0}^{\infty} \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s \neq \emptyset$;
2. there exists $x_s \in \bigcup_{k=0}^{\infty} \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s$ such that, for any $n \in \mathbb{Z}_+$,

$$\|\delta_{x_s} P_{s+k\gamma, s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \delta_{x_s} Q_{s, s+n\gamma}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0. \quad (6)$$

The semigroup $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$ is then called the *auxiliary semigroup of* $(P_{s,t})_{s \leq t}$.

When $\psi_s = \tilde{\psi}_s = \mathbb{1}$ for all $s \geq 0$, we say that the semigroup $(P_{s,t})_{s \leq t}$ is *asymptotically periodic in total variation*. By extension, we will say that the process $(X_t)_{t \geq 0}$ is asymptotically periodic (in total variation) if the associated semigroup $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$ is asymptotically periodic (in total variation).

In what follows, the functions $(\psi_s)_{s \geq 0}$ and $(\tilde{\psi}_s)_{s \in [0, \gamma)}$ will play the role of Lyapunov functions (that is to say satisfying Assumption 1 (ii) below) for the semigroups $(P_{s,t})_{s \leq t}$ and $(Q_{s,t})_{s \leq t}$ respectively. The introduction of these functions in the definition of asymptotically periodic semigroups will allow us to establish an ergodic theorem for processes satisfying the Lyapunov/minorization conditions written below.

Lyapunov/minorization conditions.

The main assumption of Theorem 1, which will be provided later, will be that the asymptotically periodic Markov process satisfies the following assumption.

Assumption 1. *There exist $t_1 \geq 0$, $n_0 \in \mathbb{N}$, $c > 0$, $\theta \in (0, 1)$, a family of measurable sets $(K_t)_{t \geq 0}$ such that $K_t \subset E_t$ for all $t \geq 0$, a family of probability measures $(\nu_s)_{s \geq 0}$ on $(K_s)_{s \geq 0}$, and a family of functions $(\psi_s)_{s \geq 0}$, all lower-bounded by 1, such that:*

(i) *For any $s \geq 0$, $x \in K_s$ and $n \geq n_0$,*

$$\delta_x P_{s, s+nt_1} \geq c\nu_{s+nt_1}.$$

(ii) *For any $s \geq 0$,*

$$P_{s, s+t_1} \psi_{s+t_1} \leq \theta \psi_s + C \mathbb{1}_{K_s}.$$

(iii) *For any $s \geq 0$ and $t \in [0, t_1)$,*

$$P_{s, s+t} \psi_{s+t} \leq C \psi_s.$$

When a semigroup $(P_{s,t})_{s \leq t}$ satisfies Assumption 1 as stated above, we will say that the family of functions $(\psi_s)_{s \geq 0}$ are *Lyapunov functions* for the semigroup $(P_{s,t})_{s \leq t}$. In particular, under (ii) and (iii), it is easy to prove that for any $s \leq t$,

$$P_{s,t} \psi_t \leq C \left(1 + \frac{C}{1-\theta} \right) \psi_s. \quad (7)$$

We remark in particular that Assumption 1 implies an *exponential weak ergodicity in ψ_t -distance*, that is we have the existence of two constants $C' > 0$ and $\kappa > 0$ such that, for all $s \leq t$ and for all probability measures $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$,

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{\psi_t} \leq C' [\mu_1(\psi_s) + \mu_2(\psi_s)] e^{-\kappa(t-s)}, \quad (8)$$

whereby, for a given function ψ , $\|\mu - \nu\|_{\psi}$ is the ψ -distance defined to be

$$\|\mu - \nu\|_{\psi} := \sup_{|f| \leq \psi} |\mu(f) - \nu(f)|, \quad \forall \mu, \nu \in \mathcal{M}_1(\psi).$$

In particular, when $\psi = \mathbb{1}$ for all $t \geq 0$, the ψ -distance is the total variation distance. If we have weak ergodicity (8) in the time-homogeneous setting (see in particular [15]), the proof of [15, Theorem 1.3.] can be adapted to a general time-inhomogeneous framework (see for example [6, Subsection 9.5]).

The main theorem and proof.

The main result of this paper is the following.

Theorem 1. *Let $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}, (X_t)_{t \geq 0}, (\mathbb{P}_{s,x})_{s \geq 0, x \in E_s}\}$ be an asymptotically γ -periodic time-inhomogeneous Markov process, with $\gamma > 0$, and denote by $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$ its periodic auxiliary semigroup. Also, denote by $(\psi_s)_{s \geq 0}$ and $(\tilde{\psi}_s)_{s \geq 0}$ the two families of functions as defined in Definition 1. Assume moreover that:*

1. *The semigroups $(P_{s,t})_{s \leq t}$ and $(Q_{s,t})_{s \leq t}$ satisfy Assumption 1, with $(\psi_s)_{s \geq 0}$ and $(\tilde{\psi}_s)_{s \geq 0}$ as Lyapunov functions respectively.*
2. *For any $s \in [0, \gamma)$, $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$ converges pointwisely to $\tilde{\psi}_s$.*

Then, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s}[\psi_s \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0, \quad (9)$$

whereby $\beta_\gamma \in \mathcal{M}_1(F_0)$ is the unique invariant probability measure of the skeleton semigroup $(Q_{0,n\gamma})_{n \in \mathbb{Z}_+}$ satisfying $\beta_\gamma(\tilde{\psi}_0) < +\infty$. Moreover, for any $f \in \mathcal{B}(E)$ we have:

1. For any $\mu \in \mathcal{M}_1(E_0)$,

$$\mathbb{E}_{0,\mu} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0. \quad (10)$$

2. If moreover $\mu(\psi_0) < +\infty$, then

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow{t \rightarrow \infty} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely}. \quad (11)$$

Remark 1. When Assumption 1 hold for $K_s = E_s$ for any s , then the condition (i) in Assumption 1 implies Doebelin condition.

Doebelin condition. There exists $t_0 \geq 0$, $c > 0$ and a family of probability measure $(\nu_t)_{t \geq 0}$ on $(E_t)_{t \geq 0}$ such that, for any $s \geq 0$ and $x \in E_s$,

$$\delta_x P_{s,s+t_0} \geq c\nu_{s+t_0}. \quad (12)$$

In fact, if we assume that Assumption 1 (i) holds for $K_s = E_s$, Doebelin condition holds by setting $t_0 := n_0 t_1$. Conversely, the Doebelin condition implies the conditions (i)-(ii)-(iii) with $K_s = E_s$ and $\psi_s = \mathbb{1}_{E_s}$ for all $s \geq 0$, so that these conditions are equivalent. In fact, (ii) and (iii) straightforwardly hold true for $(K_s)_{s \geq 0} = (E_s)_{s \geq 0}$, $(\psi_s)_{s \geq 0} = (\mathbb{1}_{E_s})_{s \geq 0}$, $C = 1$, any $\theta \in (0, 1)$ and any $t_1 \geq 0$. Setting $t_1 = t_0$ and $n_0 = 1$, the Doebelin condition entails that, for any $s \in [0, t_1)$,

$$\delta_x P_{s,s+t_1} \geq c\nu_{s+t_1}, \quad \forall x \in E_s.$$

Integrating this inequality over $\mu \in \mathcal{M}_1(E_s)$, one obtains

$$\mu P_{s,s+t_1} \geq c\nu_{s+t_1}, \quad \forall s \in [0, t_1), \forall \mu \in \mathcal{M}_1(E_s).$$

Then, by the Markov property, for all $s \in [0, t_1)$, $x \in E_s$ and $n \in \mathbb{N}$, we have

$$\delta_x P_{s,s+nt_1} = (\delta_x P_{s,s+(n-1)t_1}) P_{s+(n-1)t_1,s+nt_1} \geq c\nu_{s+nt_1},$$

which is (i).

Theorem 1 then entails the following corollary.

Corollary 1. Let $(X_t)_{t \geq 0}$ be asymptotically γ -periodic in total variation distance. If $(X_t)_{t \geq 0}$ and its auxiliary semigroup satisfy a Doebelin condition, then the convergence (10) is improved to

$$\sup_{\mu \in \mathcal{M}_1(E_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\mu} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0.$$

Moreover, the almost sure convergence (11) holds for any initial measure μ .

Remark 2. We also note that, if the convergence (6) holds for all $x \in \bigcup_{k=0}^\infty \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s$, then this implies (6) and therefore the pointwise convergence of $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$ to $\tilde{\psi}_s$ (by taking $n = 0$ in (6)).

Proof of Theorem 1. The proof is divided into five steps.

First step. Since the auxiliary semigroup $(Q_{s,t})_{s \leq t}$ satisfies Assumption 1 with $(\tilde{\psi}_s)_{s \geq 0}$ as Lyapunov functions, the time-homogeneous semigroup $(Q_{0,n\gamma})_{n \in \mathbb{Z}_+}$ satisfies assumptions 1 and 2 of [15], which we now recall (using our notation).

Assumption 2 (Assumption 1, [15]). *There exists $V : F_0 \rightarrow [0, +\infty)$, $n_1 \in \mathbb{N}$ and constants $K \geq 0$ and $\kappa \in (0, 1)$ such that*

$$Q_{0, n_1 \gamma} V \leq \kappa V + K.$$

Assumption 3 (Assumption 2, [15]). *There exists a constant $\alpha \in (0, 1)$ and a probability measure ν such that*

$$\inf_{x \in \mathcal{C}_R} \delta_x Q_{0, n_1 \gamma} \geq \alpha \nu(\cdot),$$

with $\mathcal{C}_R := \{x \in F_0 : V(x) \leq R\}$ for some $R > 2K/(1 - \kappa)$, whereby n_1 , K and κ are the constants from Assumption 2.

In fact, since $(Q_{s,t})_{s \leq t}$ satisfies (ii) and (iii) of Assumption 1, there exists $C > 0$, $\theta \in (0, 1)$, $t_1 \geq 0$ and $(K_s)_{s \geq 0}$ such that

$$Q_{s, s+t_1} \tilde{\psi}_{s+t_1} \leq \theta \tilde{\psi}_s + C \mathbb{1}_{K_s}, \quad \forall s \geq 0, \quad (13)$$

and

$$Q_{s, s+t} \tilde{\psi}_{s+t} \leq C \tilde{\psi}_s, \quad \forall s \geq 0, \forall t \in [0, t_1].$$

We let $n_2 \in \mathbb{N}$ be such that $\theta^{n_2} C (1 + \frac{C}{1-\theta}) < 1$. By (13) and recalling that $\tilde{\psi}_t = \tilde{\psi}_{t+\gamma}$ for all $t \geq 0$, one has for any $s \geq 0$ and $n \in \mathbb{N}$,

$$Q_{s, s+nt_1} \tilde{\psi}_{s+nt_1} \leq \theta^n \tilde{\psi}_s + \frac{C}{1-\theta}. \quad (14)$$

Thus, for all $n_1 \geq \lceil \frac{n_2 t_1}{\gamma} \rceil$,

$$Q_{0, n_1 \gamma} \tilde{\psi}_0 = Q_{0, n_1 \gamma - n_2 t_1} Q_{n_1 \gamma - n_2 t_1, n_1 \gamma} \tilde{\psi}_{n_1 \gamma} \leq \theta^{n_2} Q_{0, n_1 \gamma - n_2 t_1} \tilde{\psi}_{n_1 \gamma - n_2 t_1} + \frac{C}{1-\theta} \leq \theta^{n_2} C (1 + \frac{C}{1-\theta}) \tilde{\psi}_0 + \frac{C}{1-\theta},$$

where we successively used the property of semigroup of $(Q_{s,t})_{s \leq t}$, (14) and (7) applied to $(Q_{s,t})_{s \leq t}$. Hence one has Assumption 2 by setting $V = \tilde{\psi}_0$, $\kappa := \theta^{n_2} C (1 + \frac{C}{1-\theta})$ and $K := \frac{C}{1-\theta}$.

We now prove Assumption 3. To this end, we introduce a Markov process $(Y_t)_{t \geq 0}$ and a family of probability measures $(\hat{\mathbb{P}}_{s,x})_{s \geq 0, x \in F_s}$ such that

$$\hat{\mathbb{P}}_{s,x}(Y_t \in A) = Q_{s,t} \mathbb{1}_A(x), \quad \forall s \leq t, x \in F_s, A \in \mathcal{F}_t.$$

In what follows, for all $s \geq 0$ and $x \in F_s$, we will use the notation $\hat{\mathbb{E}}_{s,x}$ for the expectation associated to $\hat{\mathbb{P}}_{s,x}$. Moreover, we define

$$T_K := \inf\{n \in \mathbb{Z}_+ : Y_{nt_1} \in K_{nt_1}\}.$$

Then, using (13) recursively, for all $k \in \mathbb{N}$, $R > 0$ and $x \in \mathcal{C}_R$ (recalling that \mathcal{C}_R is defined in the statement of Assumption 3) we have

$$\begin{aligned} \hat{\mathbb{E}}_{0,x}[\tilde{\psi}_{kt_1}(Y_{kt_1}) \mathbb{1}_{T_K > k}] &= \hat{\mathbb{E}}_{0,x}[\mathbb{1}_{T_K > k-1} \hat{\mathbb{E}}_{(k-1)t_1, Y_{(k-1)t_1}}(\tilde{\psi}_{kt_1}(Y_{kt_1}) \mathbb{1}_{T_K > k})] \\ &\leq \theta \hat{\mathbb{E}}_{0,x}[\tilde{\psi}_{(k-1)t_1}(Y_{(k-1)t_1}) \mathbb{1}_{T_K > k-1}] \leq \theta^k \tilde{\psi}_0(x) \leq R \theta^k. \end{aligned}$$

Since $\tilde{\psi}_{kt_1} \geq 1$ for all $k \in \mathbb{Z}_+$, then for all $x \in \mathcal{C}_R$, for all $k \in \mathbb{Z}_+$,

$$\hat{\mathbb{P}}_{0,x}(T_K > k) \leq R \theta^k.$$

In particular, there exists $k_0 \geq n_0$ such that, for all $k \geq k_0 - n_0$,

$$\hat{\mathbb{P}}_{0,x}(T_K > k) \leq \frac{1}{2}.$$

Hence, for all $x \in \mathcal{C}_R$,

$$\begin{aligned} \delta_x Q_{0,k_0 t_1} &= \hat{\mathbb{P}}_{0,x}(Y_{k_0 t_1} \in \cdot) \geq \sum_{i=0}^{k_0-n_0} \hat{\mathbb{E}}_{0,x}(\mathbb{1}_{T_k=i} \hat{\mathbb{P}}_{it_1, X_{it_1}}(Y_{k_0 t_1} \in \cdot)) \\ &\geq c \sum_{i=0}^{k_0-n_0} \hat{\mathbb{E}}_{0,x}(\mathbb{1}_{T_K=i}) \times \nu_{k_0 t_1} \\ &= c \hat{\mathbb{P}}_{0,x}(T_K \leq k_0 - n_0) \nu_{k_0 t_1} \\ &\geq \frac{c}{2} \nu_{k_0 t_1}. \end{aligned}$$

Hence, for all $n_1 \geq \lceil \frac{k_0 t_1}{\gamma} \rceil$, for all $x \in \mathcal{C}_R$,

$$\delta_x Q_{0,k_0 t_1} Q_{k_0 t_1, n_1 \gamma} \geq \frac{c}{2} \nu_{k_0 t_1} Q_{k_0 t_1, n_1 \gamma}.$$

Thus, Assumption 3 is satisfied taking $n_1 := \lceil \frac{n_2 t_1}{\gamma} \rceil \vee \lceil \frac{k_0 t_1}{\gamma} \rceil$, $\alpha := \frac{c}{2}$ and $\nu(\cdot) := \nu_{k_0 t_1} Q_{k_0 t_1, n_1 \gamma}$.

Then, by [15, Theorem 1.2], Assumptions 2 and 3 imply that $Q_{0, n_1 \gamma}$ admits a unique invariant probability measure β_γ . Furthermore, there exists constants $C > 0$ and $\delta \in (0, 1)$ such that, for all $\mu \in \mathcal{M}_1(F_0)$,

$$\|\mu Q_{0, n n_1 \gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \leq C \mu(\tilde{\psi}_0) \delta^n. \quad (15)$$

Since β_γ is the unique invariant probability measure of $Q_{0, n_1 \gamma}$ and noting that $\beta_\gamma Q_{0, \gamma}$ is invariant for $Q_{0, n_1 \gamma}$, we deduce that β_γ is the unique invariant probability measure for $Q_{0, \gamma}$ and, by (15), for all μ such that $\mu(\tilde{\psi}_0) < +\infty$,

$$\|\mu Q_{0, n \gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \xrightarrow{n \rightarrow \infty} 0.$$

Now, for any $s \geq 0$, note that $\delta_x Q_{s, \lceil \frac{s}{\gamma} \rceil \gamma} \tilde{\psi}_0 < +\infty$ for all $x \in F_s$ (this is a consequence of (7) applied to the semigroup $(Q_{s,t})_{s \leq t}$), and therefore, taking $\mu = \delta_x Q_{s, \lceil \frac{s}{\gamma} \rceil \gamma}$ in the above convergence,

$$\|\delta_x Q_{s, n \gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \xrightarrow{n \rightarrow \infty} 0$$

for all $x \in F_s$. Hence, since $Q_{n \gamma, n \gamma + s} \tilde{\psi}_s \leq C(1 + \frac{C}{1-\theta}) \tilde{\psi}_{n \gamma}$ by (7), we conclude from the above convergence that

$$\|\delta_x Q_{s, s+n \gamma} - \beta_\gamma Q_{0,s}\|_{\tilde{\psi}_s} \leq C(1 + \frac{C}{1-\theta}) \|\delta_x Q_{s, n \gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \xrightarrow{n \rightarrow \infty} 0. \quad (16)$$

Moreover, $\beta_\gamma(\tilde{\psi}_0) < +\infty$.

Second step. The first part of this step (until the equality (20)) is inspired by the proof of [1, Theorem 3.11].

We fix $s \in [0, \gamma]$. Without loss of generality, we assume that $\bigcap_{l \geq 0} E_{s+l \gamma} \cap F_s \neq \emptyset$. Then, by Definition 1, there exists $x_s \in \bigcap_{l \geq 0} E_{s+l \gamma} \cap F_s$ such that for any $n \geq 0$,

$$\|\delta_{x_s} P_{s+k \gamma, s+(k+n) \gamma}[\psi_{s+(k+n) \gamma} \times \cdot] - \delta_{x_s} Q_{s, s+n \gamma}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow{k \rightarrow \infty} 0,$$

which implies by (16) that

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \|\delta_{x_s} P_{s+k \gamma, s+(k+n) \gamma}[\psi_{s+(k+n) \gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = 0. \quad (17)$$

Then, by the Markov property, (8), and (7), one obtains that, for any $k, n \in \mathbb{N}$ and $x \in \bigcap_{l \geq 0} E_{s+l \gamma}$,

$$\begin{aligned} \|\delta_x P_{s, s+(k+n) \gamma} - \delta_x P_{s+k \gamma, s+(k+n) \gamma}\|_{\psi_{s+(k+n) \gamma}} &= \|(\delta_x P_{s, s+k \gamma}) P_{s+k \gamma, s+(k+n) \gamma} - \delta_x P_{s+k \gamma, s+(k+n) \gamma}\|_{\psi_{s+(k+n) \gamma}} \\ &\leq C' [P_{s, s+k \gamma} \psi_{s+k \gamma}(x) + \psi_{s+k \gamma}(x)] e^{-\kappa \gamma n} \leq C'' [\psi_s(x) + \psi_{s+k \gamma}(x)] e^{-\kappa \gamma n}, \end{aligned} \quad (18)$$

whereby $C'' := C' \left(C \left(1 + \frac{C}{1-\theta} \right) \vee 1 \right)$. Then, for any $k, n \in \mathbb{N}$,

$$\begin{aligned} & \|\delta_{x_s} P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} \\ & \leq C''[\psi_s(x) + \psi_{s+k\gamma}(x)]e^{-\kappa\gamma n} + \|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV}, \end{aligned} \quad (19)$$

which by (17) and the pointwise convergence of $(\psi_{s+k\gamma})_{k \in \mathbb{Z}_+}$ implies that

$$\lim_{n \rightarrow \infty} \|\delta_{x_s} P_{s,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = \lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \|\delta_{x_s} P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = 0. \quad (20)$$

The weak ergodicity (8) implies therefore that the previous convergence actually holds for any initial distribution $\mu \in \mathcal{M}_1(E_0)$ satisfying $\mu(\psi_0) < +\infty$, so that

$$\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow{n \rightarrow \infty} 0. \quad (21)$$

Since $\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} \leq 2$ for all $\mu \in \mathcal{M}_1(E_0)$, $s \geq 0$ and $n \in \mathbb{Z}_+$, (21) and Lebesgue's dominated convergence theorem implies that

$$\frac{1}{\gamma} \int_0^\gamma \|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} ds \xrightarrow{n \rightarrow \infty} 0,$$

which implies that

$$\left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \xrightarrow{n \rightarrow \infty} 0.$$

By Cesaro's lemma, this allows us to conclude that, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\begin{aligned} & \left\| \frac{1}{t} \int_0^t \mu P_{0,s}[\psi_s \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \\ & \leq \frac{1}{\lfloor \frac{t}{\gamma} \rfloor} \sum_{k=0}^{\lfloor \frac{t}{\gamma} \rfloor} \left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+k\gamma}[\psi_{s+k\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} + \left\| \frac{1}{t} \int_{\lfloor \frac{t}{\gamma} \rfloor \gamma}^t \mu P_{0,s}[\psi_s \times \cdot] ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0, \end{aligned}$$

which concludes the proof of (9).

Third step. In the same manner, we now prove that, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s} ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0. \quad (22)$$

In fact, for any function f bounded by 1 and $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\begin{aligned} & \left| \mu P_{0,s+n\gamma} \left[\psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \\ & \leq \left| \mu P_{0,s+n\gamma} \left[\psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] \right| + \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \\ & \leq \|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} + \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right|. \end{aligned}$$

We now remark that, since $\psi_{s+n\gamma} \geq 1$ for any s and $n \in \mathbb{Z}_+$, one has that

$$\left| \frac{\tilde{\psi}_s}{\psi_{s+n\gamma}} - 1 \right| \leq 1 + \tilde{\psi}_s.$$

Since $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$ converges pointwisely towards $\tilde{\psi}_s$ and $\beta_\gamma Q_{0,s} \tilde{\psi}_s < +\infty$, Lebesgue's dominated convergence theorem implies

$$\sup_{f \in \mathcal{B}_1(E)} \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Then, using (21), one has

$$\|\mu P_{0,s+n\gamma} - \beta_\gamma Q_{0,s}\|_{TV} \xrightarrow{n \rightarrow \infty} 0,$$

which allows us to conclude (22), using the same argument as in the first step.

Fourth step. In order to show the \mathbb{L}^2 -ergodic theorem, we let $f \in \mathcal{B}(E)$. For any $x \in E_0$ and $t \geq 0$,

$$\begin{aligned} \mathbb{E}_{0,x} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[\frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] &= \frac{2}{t^2} \int_0^t \int_s^t (\mathbb{E}_{0,x}[f(X_s)f(X_u)] - \mathbb{E}_{0,x}[f(X_s)]\mathbb{E}_{0,x}[f(X_u)]) duds \\ &= \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [f(X_s) (f(X_u) - \mathbb{E}_{0,x}[f(X_u)])] duds \\ &= \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [f(X_s) (\mathbb{E}_{s,X_s}[f(X_u)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_u)])] duds, \end{aligned}$$

whereby the Markov property was used on the last line. By (8) (weak ergodicity) and (7), one obtains for any $s \leq t$,

$$|\mathbb{E}_{s,X_s}[f(X_t)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_t)]| \leq C'' \|f\|_\infty [\psi_s(X_s) + \psi_0(x)] e^{-\kappa(t-s)}, \quad \mathbb{P}_{0,x} - \text{almost surely}, \quad (23)$$

whereby C'' was defined in the first part. As a result, for any $x \in E_0$ and $t \geq 0$,

$$\begin{aligned} \mathbb{E}_{0,x} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[\frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] &\leq \frac{2C'' \|f\|_\infty}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] e^{-\kappa(u-s)} duds \\ &= \frac{2C'' \|f\|_\infty}{t^2} \int_0^t \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] e^{\kappa s} \frac{e^{-\kappa s} - e^{-\kappa t}}{\kappa} ds \\ &= \frac{2C'' \|f\|_\infty}{\kappa t} \times \mathbb{E}_{0,x} \left[\frac{1}{t} \int_0^t |f(X_s)| (\psi_s(X_s) + \psi_0(x)) ds \right] - \frac{2C'' \|f\|_\infty e^{-\kappa t}}{\kappa t^2} \int_0^t e^{\kappa s} \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] ds. \end{aligned}$$

Then, by (9), there exists a constant $\tilde{C} > 0$ such that, for any $x \in E_0$, when $t \rightarrow \infty$,

$$\mathbb{E}_{0,x} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[\frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \leq \frac{\tilde{C} \|f\|_\infty \psi_0(x)}{t} \times \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} [|f| \tilde{\psi}_s] ds + o\left(\frac{1}{t}\right). \quad (24)$$

Since $f \in \mathcal{B}(E)$ and by definition of the total variation distance, (22) implies that, for all $x \in E_0$,

$$\left| \frac{1}{t} \int_0^t P_{0,s} f(x) - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right| \leq \|f\|_\infty \left\| \frac{1}{t} \int_0^t \delta_x P_{0,s} ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0.$$

Then, using (22), one deduces that, for any $x \in E_0$ and bounded function f ,

$$\begin{aligned} \mathbb{E}_{0,x} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] &\leq 2 \left(\mathbb{E}_{0,x} \left[\left(\frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{t} \int_0^t P_{0,s} f(x) \right)^2 \right] + \left| \frac{1}{t} \int_0^t P_{0,s} f(x) - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right) \xrightarrow{t \rightarrow \infty} 0. \end{aligned}$$

The convergence for any probability measure $\mu \in \mathcal{M}_1(E_0)$ comes from Lebesgue's dominated convergence theorem.

Fifth step. We now fix non-negative $f \in \mathcal{B}(E)$, and $\mu \in \mathcal{M}_1(E_0)$ satisfying $\mu(\psi_0) < +\infty$. The following proof is inspired by the proof of [26, Theorem 12].

Since $\mu(\psi_0) < +\infty$, the inequality (24) entails that there exists a finite constant $C_{f,\mu} \in (0, \infty)$ such that, for t large enough,

$$\mathbb{E}_{0,\mu} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,\mu} \left[\frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \leq \frac{C_{f,\mu}}{t}.$$

Then, for n large enough,

$$\mathbb{E}_{0,\mu} \left[\left| \frac{1}{n^2} \int_0^{n^2} f(X_s) ds - \mathbb{E}_{0,\mu} \left[\frac{1}{n^2} \int_0^{n^2} f(X_s) ds \right] \right|^2 \right] \leq \frac{C_{f,\mu}}{n^2}.$$

Then, by Chebyshev's inequality and the Borel-Cantelli lemma, this last inequality implies that

$$\left| \frac{1}{n^2} \int_0^{n^2} f(X_s) ds - \mathbb{E}_{0,\mu} \left[\frac{1}{n^2} \int_0^{n^2} f(X_s) ds \right] \right| \xrightarrow[n \rightarrow \infty]{} 0, \quad \mathbb{P}_{0,\mu} - \text{almost surely.}$$

One thereby obtains by the convergence (22) that

$$\frac{1}{n^2} \int_0^{n^2} f(X_s) ds \xrightarrow[n \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely.} \quad (25)$$

Since the nonnegativity of f is assumed, this implies that for any $t > 0$ we have

$$\int_0^{\lfloor \sqrt{t} \rfloor^2} f(X_s) ds \leq \int_0^t f(X_s) ds \leq \int_0^{\lceil \sqrt{t} \rceil^2} f(X_s) ds.$$

These inequalities and (25) then give that

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely.}$$

In order to conclude that the result holds for any bounded measurable function f , it is enough to decompose $f = f_+ - f_-$ with $f_+ := f \vee 0$ and $f_- = (-f) \vee 0$ and apply the above convergence to f_+ and f_- . This concludes the proof of Theorem 1. \square

Proof of Corollary 1. We remark as in the previous proof that, if $\|f\|_\infty \leq 1$ and $\psi_s = \mathbb{1}$, an upper-bound for the inequality (24) can be obtained, which does not depend on f and x . Likewise, the convergence (21) holds uniformly in the initial measure due to (23). \square

Remark 3. The proof of Theorem 1, as written above, does not allow to deal with semigroups satisfying a Doeblin condition with time-dependent constant c_s , that is such that there exists $t_0 \geq 0$ and a family of probability measure $(\nu_t)_{t \geq 0}$ on $(E_t)_{t \geq 0}$ such that, for all $s \geq 0$ and $x \in E_s$,

$$\delta_x P_{s,s+t_0} \geq c_{s+t_0} \nu_{s+t_0}.$$

In fact, under the condition written above, we can show (see for example the proof of the formula (2.7) of [9, Theorem 2.1]) that, for all $s \leq t$ and $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$,

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{TV} \leq 2 \prod_{k=1}^{\lfloor \frac{t-s}{t_0} \rfloor} (1 - c_{t-kt_0}).$$

Hence, by this last inequality with $\mu_1 = \delta_x P_{s,s+k\gamma}$, $\mu_2 = \delta_x$, replacing s by $s + k\gamma$ and t by $s + (k+n)\gamma$, one obtains

$$\|\delta_x P_{s,s+(k+n)\gamma} - \delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{TV} \leq 2 \prod_{l=1}^{\lfloor \frac{n\gamma}{t_0} \rfloor} (1 - c_{s+(k+n)\gamma-lt_0}),$$

replacing therefore the inequality (18) in the proof of Theorem 1. Plugging therefore this last inequality into the formula (19), one obtains

$$\|\delta_x P_{s,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV} \leq 2 \prod_{l=1}^{\lfloor \frac{n\gamma}{t_0} \rfloor} (1 - c_{s+(k+n)\gamma-lt_0}) + \|\delta_x P_{s+k\gamma,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV}.$$

Hence, we see that we cannot conclude a similar result when $c_s \rightarrow 0$, as $s \rightarrow +\infty$ since, for n fixed,

$$\limsup_{k \rightarrow \infty} \prod_{l=1}^{\lfloor \frac{n\gamma}{t_0} \rfloor} (1 - c_{s+(k+n)\gamma-lt_0}) = 1.$$

3 Application to quasi-stationarity with moving boundaries

In this section, $(X_t)_{t \geq 0}$ is assumed as being a time-homogeneous Markov process. We consider a family of measurable subsets $(A_t)_{t \geq 0}$ of E , and denote the hitting time

$$\tau_A := \inf\{t \geq 0 : X_t \in A_t\}.$$

For all $s \leq t$, denote by $\mathcal{F}_{s,t}$ the σ -field generated by the family $(X_u)_{s \leq u \leq t}$ and $\mathcal{F}_t := \mathcal{F}_{0,t}$. Assume that τ_A is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Assume also that for any $x \notin A_0$,

$$\mathbb{P}_{0,x}[\tau_A < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{0,x}[\tau_A > t] > 0, \quad \forall t \geq 0.$$

We will be interested in a notion of *quasi-stationarity with moving boundaries*, which studies the asymptotic behavior of the Markov process $(X_t)_{t \geq 0}$ conditioned not to hit $(A_t)_{t \geq 0}$ up to the time t . For non-moving boundaries ($A_t = A_0$ for any $t \geq 0$), the *quasi-limiting distribution* is defined as a probability measure α such that, for at least one initial measure μ and for all measurable subsets $\mathcal{A} \subset E$,

$$\mathbb{P}_{0,\mu}[X_t \in \mathcal{A} | \tau_A > t] \xrightarrow[t \rightarrow \infty]{} \alpha(\mathcal{A}).$$

Such a definition is equivalent (still in the non-moving framework) to the notion of *quasi-stationary distribution* defined as a probability measure α such that, for any $t \geq 0$,

$$\mathbb{P}_{0,\alpha}[X_t \in \cdot | \tau_A > t] = \alpha. \tag{26}$$

If quasi-limiting and quasi-stationary distributions are in general well-defined for time-homogeneous Markov processes and non-moving boundaries (see [11, 23] for a general overview on the theory of quasi-stationarity), these notions could be not well-defined for time-inhomogeneous Markov processes or moving boundaries and are not equivalent anymore. In particular, under reasonable assumptions on irreducibility, it was shown in [24] that the notion of quasi-stationary distribution as defined by (26) is not well-defined for time-homogeneous Markov processes absorbed by moving boundaries.

Another asymptotic notion to study is the *quasi-ergodic distribution*, related to a conditional version of the ergodic theorem and usually defined as follows.

Definition 2. A probability measure β is a *quasi-ergodic distribution* if, for some initial measure $\mu \in \mathcal{M}_1(E \setminus A_0)$ and for any bounded continuous function f ,

$$\mathbb{E}_{0,\mu} \left[\frac{1}{t} \int_0^t f(X_s) ds \middle| \tau_A > t \right] \xrightarrow[t \rightarrow \infty]{} \beta(f).$$

In the time-homogeneous setting (in particular for non-moving boundaries), this notion has been extensively studied (see for example [2, 8, 10, 12, 13, 16, 17, 18, 24]). In the "moving boundaries" framework, the existence of quasi-ergodic distributions has been dealt with in [24] for Markov chains on finite state spaces absorbed by periodic boundaries, and in [25] for processes satisfying a Champagnat-Villemonais condition (see Assumption (A') set later) absorbed by converging or periodic boundaries. In this last paper, the existence of the quasi-ergodic distribution is dealt with through the following inequality (see [25, Theorem 1]), holding for any initial state x , $s \leq t$ and for some constant $C, \gamma > 0$ independent of x, s and t :

$$\|\mathbb{P}_{0,x}(X_s \in \cdot | \tau_A > t) - \mathbb{Q}_{0,x}(X_s \in \cdot)\|_{TV} \leq Ce^{-\gamma(t-s)},$$

where the family of probability measure $(\mathbb{Q}_{s,x})_{s \geq 0, x \in E_s}$ is defined by

$$\mathbb{Q}_{s,x}[\Gamma] := \lim_{T \rightarrow \infty} \mathbb{P}_{s,x}[\Gamma | \tau_A > T], \quad \forall s \leq t, x \in E \setminus A_s, \Gamma \in \mathcal{F}_{s,t}.$$

Moreover, [9, Proposition 3.1], there exists a family of positive bounded functions $(\eta_t)_{t \geq 0}$ defined in such that, for all $s \leq t$ and $x \in E_s$,

$$\mathbb{E}_{s,x}(\eta_t(X_t) \mathbb{1}_{\tau_A > t}) = \eta_s(x).$$

Then, we can show (this is actually shown in [9]) that

$$\mathbb{Q}_{s,x}(\Gamma) = \mathbb{E}_{s,x}(\mathbb{1}_{\Gamma, \tau_A > t} \frac{\eta_t(X_t)}{\eta_s(x)})$$

and that, for all $\mu \in \mathcal{M}_1(E_0)$,

$$\|\mathbb{P}_{0,\mu}(X_s \in \cdot | \tau_A > t) - \mathbb{Q}_{0,\eta_0 * \mu}(X_s \in \cdot)\|_{TV} \leq Ce^{-\gamma(t-s)}.$$

where $\eta_0 * \mu(dx) := \frac{\eta_0(x)\mu(dx)}{\mu(\eta_0)}$. By triangular inequality, one has

$$\left\| \frac{1}{t} \int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t] ds - \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}[X_s \in \cdot] ds \right\|_{TV} \leq \frac{C}{\gamma t}, \quad \forall t > 0, \quad (27)$$

In particular, the inequality (27) implies that there exists a quasi-ergodic distribution β for the process $(X_t)_{t \geq 0}$ absorbed by $(A_t)_{t \geq 0}$ if and only if there exist some probability measures $\mu \in \mathcal{M}_1(E_0)$ such that $\frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}[X_s \in \cdot] ds$ converges weakly to β , when t goes to infinity. In other terms, under Assumption (A'), the existence of a quasi-ergodic distribution for the absorbed process is equivalent to the law of large number for its Q -process.

Assumption (A') is now set.

Assumption (A'). *There exists a family of probability measures $(\nu_t)_{t \geq 0}$, defined on $E \setminus A_t$ for each t , such that:*

(A'1) *There exists $t_0 \geq 0$ and $c_1 > 0$ such that*

$$\mathbb{P}_{s,x}[X_{s+t_0} \in \cdot | \tau_A > s + t_0] \geq c_1 \nu_{s+t_0}, \quad \forall s \geq 0, \forall x \in E \setminus A_s.$$

(A'2) *There exists $c_2 > 0$ such that*

$$\mathbb{P}_{s,\nu_s}[\tau_A > t] \geq c_2 \mathbb{P}_{s,x}[\tau_A > t], \quad \forall s \leq t, \forall x \in E \setminus A_s.$$

In what follows, we say that the couple $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$ satisfies Assumption (A') when it holds for the Markov process $(X_t)_{t \geq 0}$ considered as absorbed by the moving boundary $(A_t)_{t \geq 0}$.

The condition (A'1) is a conditional version of the Doeblin condition (12) and (A'2) is a Harnack-like inequality on the probabilities of surviving, necessary to deal with the conditioning. They are equivalent to the set of conditions presented in [1, Definition 2.2], when the non-conservative semigroup is sub-Markovian. In the time-homogeneous framework, we obtain the Champagnat-Villemonais condition defined in [5] (see Assumption (A)), shown as being equivalent to the exponential uniform convergence to quasi-stationarity in total variation.

In [25], the existence of a unique quasi-ergodic distribution has been only proved for converging or periodic boundaries. However, we can expect such a result of existence (and uniqueness) for other kinds of movement for the boundary. Hence, the aim of this section is to extend the results on the existence of quasi-ergodic distributions obtained in [25] for Markov processes absorbed by asymptotically periodic moving boundaries.

Now, let us state the following theorem.

Theorem 2. Assume that there exists a γ -periodic sequence of subsets $(B_t)_{t \geq 0}$ such that, for any $s \in [0, \gamma)$,

$$E'_s := E \setminus \bigcap_{k \in \mathbb{Z}_+} \bigcup_{l \geq k} A_{s+l\gamma} \cup B_s \neq \emptyset$$

and there exists $x_s \in E_s$ such that, for any $n \leq N$,

$$\|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0. \quad (28)$$

Assume also that Assumption (A') is satisfied by the couples $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$ and $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$.

Then there exists a probability measure $\beta \in \mathcal{M}_1(E)$ such that

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0, \mu} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] \xrightarrow[t \rightarrow \infty]{} 0. \quad (29)$$

Remark 4. Remark that the condition (28) implies that, for any $n \in \mathbb{Z}_+$,

$$\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+n)\gamma] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{s, x_s}[\tau_B > s + n\gamma].$$

Under the additional condition $B_t \subset A_t$ for all $t \geq 0$, these two conditions are equivalent, since for all $n \leq N$,

$$\begin{aligned} & \|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \\ &= \|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_B \leq s + (k+N)\gamma < \tau_A]\|_{TV} \\ &\leq \mathbb{P}_{s+k\gamma, x_s}[\tau_B \leq s + (k+N)\gamma < \tau_A] \\ &= |\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[\tau_B > s + N\gamma]|, \end{aligned}$$

where we used the periodicity of $(B_t)_{t \geq 0}$, writing that $\mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma] = \mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_B > s + (k+N)\gamma]$ for all $k \in \mathbb{Z}_+$. This implies the following corollary.

Corollary 2. Assume that there exists a γ -periodic sequence of subsets $(B_t)_{t \geq 0}$, with $B_t \subset A_t$ for all $t \geq 0$, such that, for any $s \in [0, \gamma)$, there exists $x_s \in E'_s$ such that, for any $n \leq N$,

$$\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+n)\gamma] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{s, x_s}[\tau_B > s + n\gamma].$$

Assume also that Assumption (A') is satisfied by $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$ and $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$.

Then there exists $\beta \in \mathcal{M}_1(E)$ such that (29) holds.

Proof of Theorem 2. Since $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$ satisfies Assumption (A') and $(B_t)_{t \geq 0}$ is a periodic boundary, we already know by [25, Theorem 2] that, for any initial distribution μ , $t \mapsto \frac{1}{t} \int_0^t \mathbb{P}_{0, \mu}[X_s \in \cdot | \tau_B > t] ds$ converges weakly to a quasi-ergodic distribution β .

The main idea of this proof is to apply Corollary 1. Since $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$ and $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$ satisfy Assumption (A'), [25, Theorem 1] implies that there exist two families of probability measures $(\mathbb{Q}_{s, x}^A)_{s \geq 0, x \in E \setminus A_s}$ and $(\mathbb{Q}_{s, x}^B)_{s \geq 0, x \in E \setminus B_s}$ such that, for any $s \leq t$, $x \in E \setminus A_s$, $y \in E \setminus B_s$ and $\Gamma \in \mathcal{F}_{s, t}$,

$$\mathbb{Q}_{s, x}^A[\Gamma] = \lim_{T \rightarrow \infty} \mathbb{P}_{s, x}[\Gamma | \tau_A > T], \quad \text{and} \quad \mathbb{Q}_{s, y}^B[\Gamma] = \lim_{T \rightarrow \infty} \mathbb{P}_{s, y}[\Gamma | \tau_B > T].$$

In particular, the quasi-ergodic distribution β is the limit of $t \mapsto \frac{1}{t} \int_0^t \mathbb{Q}_{0, \mu}^B[X_s \in \cdot] ds$, when t goes to infinity (see [25, Theorem 5]). Also, by [25, Theorem 1], there exists a constant $C > 0$ and $\kappa > 0$ such that, for any $s \leq t \leq T$, for any $x \in E \setminus A_s$,

$$\|\mathbb{Q}_{s, x}^A[X_t \in \cdot] - \mathbb{P}_{s, x}[X_t \in \cdot | \tau_A > T]\|_{TV} \leq Ce^{-\kappa(T-t)},$$

and for any $x \in E \setminus B_s$,

$$\|\mathbb{Q}_{s, x}^B[X_t \in \cdot] - \mathbb{P}_{s, x}[X_t \in \cdot | \tau_B > T]\|_{TV} \leq Ce^{-\kappa(T-t)}.$$

Moreover, for any $s \leq t \leq T$ and $x \in E'_s$,

$$\begin{aligned}
& \|\mathbb{P}_{s,x}[X_t \in \cdot | \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_B > T]\|_{TV} \\
&= \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_A > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&= \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T) \mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}(\tau_A > T) \mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\leq \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T) \mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}(\tau_A > T) \mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\quad + \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\leq \frac{|\mathbb{P}_{s,x}(\tau_B > T) - \mathbb{P}_{s,x}(\tau_A > T)|}{\mathbb{P}_{s,x}(\tau_B > T)} + \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]} \\
&\leq 2 \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}, \tag{30}
\end{aligned}$$

since $|\mathbb{P}_{s,x}(\tau_B > T) - \mathbb{P}_{s,x}(\tau_A > T)| \leq \|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}$. Then, we obtain for any $s \leq t \leq T$ and $x \in E'_s$,

$$\|\mathbb{Q}_{s,x}^A[X_t \in \cdot] - \mathbb{Q}_{s,x}^B[X_t \in \cdot]\|_{TV} \leq 2Ce^{-\kappa(T-t)} + 2 \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}. \tag{31}$$

The condition (28) implies the existence of $x_s \in E_s$ such that, for any $n \leq N$, for all $k \in \mathbb{Z}_+$,

$$\lim_{k \rightarrow \infty} \|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} = 0,$$

which implies by (31) that, for any $n \leq N$,

$$\limsup_{k \rightarrow \infty} \|\mathbb{Q}_{s+k\gamma, x_s}^A[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}_{s+k\gamma, x_s}^B[X_{s+(k+n)\gamma} \in \cdot]\|_{TV} \leq 2Ce^{-\kappa\gamma(N-n)}.$$

Now, letting $N \rightarrow \infty$, for any $n \in \mathbb{Z}_+$

$$\lim_{k \rightarrow \infty} \|\mathbb{Q}_{s+k\gamma, x_s}^A[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}_{s+k\gamma, x_s}^B[X_{s+(k+n)\gamma} \in \cdot]\|_{TV} = \lim_{k \rightarrow \infty} \|\mathbb{Q}_{s+k\gamma, x_s}^A(X_{s+(k+n)\gamma} \in \cdot) - \mathbb{Q}_{s, x_s}^B(X_{s+n\gamma} \in \cdot)\|_{TV} = 0.$$

In other words, the semigroup $(Q_{s,t}^A)_{s \leq t}$ defined by

$$Q_{s,t}^A f(x) := \mathbb{E}_{s,x}^{\mathbb{Q}^A}(f(X_t)), \quad \forall s \leq t, \forall f \in \mathcal{B}(E \setminus A_t), \forall x \in E \setminus A_s,$$

is asymptotically periodic (according to the Definition 1, with $\psi_s = \tilde{\psi}_s = 1$ for all $s \geq 0$), associated to the auxiliary semigroup $(Q_{s,t}^B)_{s \leq t}$ defined by

$$Q_{s,t}^B f(x) := \mathbb{E}_{s,x}^{\mathbb{Q}^B}(f(X_t)), \quad \forall s \leq t, \forall f \in \mathcal{B}(E \setminus B_t), \forall x \in E \setminus B_s.$$

Moreover, assumptions (A') satisfied for $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$ and $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$ imply that Doeblin condition holds for these two Q -processes. As a matter of fact, by the Markov property, for all $s \leq t \leq T$ and $x \in E \setminus A_s$,

$$\begin{aligned}
\mathbb{P}_{s,x}(X_t \in \cdot | \tau_A > T) &= \mathbb{E}_{s,x} \left[\mathbb{1}_{X_t \in \cdot, \tau_A > t} \frac{\mathbb{P}_{t, X_t}(\tau_A > T)}{\mathbb{P}_{s, \mu}(\tau_A > T)} \right] \\
&= \mathbb{E}_{s,x} \left[\frac{\mathbb{1}_{X_t \in \cdot, \tau_A > t} \mathbb{P}_{t, X_t}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_A > t) \mathbb{P}_{t, \phi_{t,s}(\mu)}(\tau_A > T)} \right] \\
&= \mathbb{E}_{s,x} \left[\mathbb{1}_{X_t \in \cdot} \frac{\mathbb{P}_{t, X_t}(\tau_A > T)}{\mathbb{P}_{t, \phi_{t,s}(\delta_x)}(\tau_A > T)} \middle| \tau_A > t \right]. \tag{32}
\end{aligned}$$

By (A'_1) , for all $s \geq 0$, $T \geq s + t_0$ and $x \in E \setminus A_s$,

$$\mathbb{E}_{s,x} \left[\mathbb{1}_{X_{s+t_0} \in \cdot} \frac{\mathbb{P}_{s+t_0, X_{s+t_0}}(\tau_A > T)}{\mathbb{P}_{s+t_0, \phi_{s+t_0, s}(\delta_x)}(\tau_A > T)} \middle| \tau_A > s + t_0 \right] \geq c_1 \int \nu_{s+t_0}(dy) \frac{\mathbb{P}_{s+t_0, y}(\tau_A > T)}{\mathbb{P}_{s+t_0, \phi_{s+t_0, s}(\delta_x)}(\tau_A > T)},$$

that is to say by (32) that

$$\mathbb{P}_{s,x}(X_{s+t_0} \in \cdot | \tau_A > T) \geq c_1 \int \nu_{s+t_0}(dy) \frac{\mathbb{P}_{s+t_0, y}(\tau_A > T)}{\mathbb{P}_{s+t_0, \phi_{s+t_0, s}(\delta_x)}(\tau_A > T)}.$$

Letting $T \rightarrow \infty$ in this last inequality and using [9, Proposition 3.1], for all $s \geq 0$ and $x \in E \setminus A_s$,

$$\mathbb{Q}_{s,x}^A(X_{s+t_0} \in \cdot) \geq c_1 \int \nu_{s+t_0}(dy) \frac{\eta_{s+t_0}(y)}{\phi_{s+t_0, s}(\delta_x)(\eta_{s+t_0})}.$$

The measure $\int \nu_{s+t_0}(dy) \frac{\eta_{s+t_0}(y)}{\phi_{s+t_0, s}(\delta_x)(\eta_{s+t_0})}$ is then a positive measure whose the mass is lower-bounded by c_2 by (A'_2) , since for all $s \geq 0$ and $T \geq s + t_0$,

$$\int_{E \setminus A_{s+t_0}} \nu_{s+t_0} \frac{\mathbb{P}_{t,x}(\tau_A > T)}{\mathbb{P}_{t, \phi_{t,s}(\delta_x)}(\tau_A > T)} \geq c_2.$$

This proves therefore a Doeblin condition for the semigroup $(Q_{s,t}^A)_{s \leq t}$. The same reasoning applies to show also a Doeblin condition for the semigroup $(Q_{s,t}^B)_{s \leq t}$. Then, using (27) then Corollary 1,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t] ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}^A(X_s \in \cdot) ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}^B[X_s \in \cdot] ds = \beta,$$

where the limits refer to the convergence in total variation and hold uniformly in the initial measure.

For any $\mu \in \mathcal{M}_1(E \setminus A_0)$, $f \in \mathcal{B}_1(E)$ and $t \geq 0$,

$$\mathbb{E}_{0,\mu} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds \right|^2 \middle| \tau_A > t \right] = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,\mu}[f(X_s)f(X_u) | \tau_A > t] duds.$$

Then, by [25, Theorem 1], for any $s \leq u \leq t$, for any $\mu \in \mathcal{M}_1(E \setminus A_0)$ and $f \in \mathcal{B}(E)$,

$$\left| \mathbb{E}_{0,\mu}[f(X_s)f(X_u) | \tau_A > t] - \mathbb{E}_{0,\eta_0 * \mu}^A[f(X_s)f(X_u)] \right| \leq C \|f\|_\infty e^{-\kappa(t-u)},$$

where the expectation $\mathbb{E}_{0,\eta_0 * \mu}^A$ is associated to the probability measure $\mathbb{Q}_{0,\eta_0 * \mu}^A$. Hence, for any $\mu \in \mathcal{M}_1(E \setminus A_0)$, $f \in \mathcal{B}_1(E)$ and $t > 0$,

$$\begin{aligned} \left| \mathbb{E}_{0,\mu} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] - \mathbb{E}_{0,\eta_0 * \mu}^A \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \right] \right| &\leq \frac{4C}{t^2} \int_0^t \int_s^t e^{-\kappa(t-u)} duds \\ &\leq \frac{4C}{\kappa t} - \frac{4C(1 - e^{-\kappa t})}{\kappa^2 t^2}. \end{aligned}$$

Moreover, since $(Q_{s,t}^A)_{s \leq t}$ is asymptotically periodic in total variation and satisfies the Doeblin condition, like $(Q_{s,t}^B)_{s \leq t}$, Corollary 1 implies that

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\eta_0 * \mu}^A \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0.$$

Then

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\mu} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] \xrightarrow{t \rightarrow \infty} 0.$$

□

Remark 5. It seems that Assumption (A') can be weakened by a conditional version of Assumption 1. In particular, such conditions can be derived from the Assumption (F) presented in [6], as it will be shown later by the preprint [4] in preparation.

4 Examples

4.1 Asymptotically periodic Ornstein-Uhlenbeck processes

Let $(X_t)_{t \geq 0}$ be a time-inhomogeneous diffusion process on \mathbb{R} satisfying the following stochastic differential equation

$$dX_t = dW_t - \lambda(t)X_t dt,$$

where $(W_t)_{t \geq 0}$ is a one-dimensional Brownian motion and $\lambda : [0, \infty) \rightarrow [0, \infty)$ is a function such that

$$\sup_{t \geq 0} |\lambda(t)| < +\infty$$

and such that there exists $\gamma > 0$ such that

$$\inf_{s \geq 0} \int_s^{s+\gamma} \lambda(u) du > 0.$$

By Itô's lemma, for any $s \leq t$,

$$X_t = e^{-\int_s^t \lambda(u) du} \left[X_s + \int_s^t e^{\int_s^u \lambda(v) dv} dW_u \right].$$

In particular, denoting $(P_{s,t})_{s \leq t}$ the semigroup associated to $(X_t)_{t \geq 0}$, for any $f \in \mathcal{B}(\mathbb{R})$, $t \geq 0$ and $x \in \mathbb{R}$,

$$P_{s,t}f(x) = \mathbb{E} \left[f \left(e^{-\int_s^t \lambda(u) du} x + e^{-\int_s^t \lambda(u) du} \sqrt{\int_s^t e^{2\int_s^u \lambda(v) dv} du} \times \mathcal{N}(0, 1) \right) \right],$$

where $\mathcal{N}(0, 1)$ denotes a standard Gaussian variable.

Theorem 3. *Assume that there exists a γ -periodic function g , bounded on \mathbb{R} , such that $\lambda \sim_{t \rightarrow \infty} g$. Then the assumptions of Theorem 1 hold.*

Proof. In our case, the auxiliary semigroup $(Q_{s,t})_{s \leq t}$ of Definition 1 will be defined as follows: for any $f \in \mathcal{B}(\mathbb{R})$, $t \geq 0$ and $x \in \mathbb{R}$,

$$Q_{s,t}f(x) = \mathbb{E} \left[f \left(e^{-\int_s^t g(u) du} x + e^{-\int_s^t g(u) du} \sqrt{\int_s^t e^{2\int_s^u g(v) dv} du} \times \mathcal{N}(0, 1) \right) \right].$$

In particular, the semigroup $(Q_{s,t})_{s \leq t}$ is associated to the process $(Y_t)_{t \geq 0}$ following

$$dY_t = dW_t - g(t)Y_t dt.$$

We first remark that the function $\psi : x \mapsto 1 + x^2$ is a Lyapunov function for $(P_{s,t})_{s \leq t}$ and $(Q_{s,t})_{s \leq t}$. In fact, for any $s \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} P_{s,s+\gamma} \psi(x) &= 1 + e^{-2\int_s^{s+\gamma} \lambda(u) du} x^2 + e^{-2\int_s^{s+\gamma} \lambda(u) du} \int_s^{s+\gamma} e^{2\int_s^u \lambda(v) dv} du \\ &= e^{-2\int_s^{s+\gamma} \lambda(u) du} \psi(x) + 1 - e^{-2\int_s^{s+\gamma} \lambda(u) du} + e^{-2\int_s^{s+\gamma} \lambda(u) du} \int_s^{s+\gamma} e^{2\int_s^u \lambda(v) dv} du \\ &\leq e^{-2\gamma c_{\inf}} \psi(x) + C, \end{aligned}$$

where $C \in (0, +\infty)$ and $c_{\inf} := \inf_{t \geq 0} \frac{1}{\gamma} \int_t^{t+\gamma} \lambda(u) du > 0$. Taking $\theta \in (e^{-2\gamma c_{\inf}}, 1)$, there exists a compact set K such that, for any $s \geq 0$,

$$P_{s,s+\gamma} \psi(x) \leq \theta \psi(x) + C \mathbb{1}_K(x).$$

Moreover, for any $s \geq 0$ and $t \in [0, \gamma)$, the function $P_{s,s+t} \psi / \psi$ is upper-bounded uniformly in s and t . It remains therefore to prove Assumption 1 (i) for $(P_{s,t})_{s \leq t}$, which is a consequence of the following lemma.

Lemma 1. For any $a, b_-, b_+ > 0$, define the subset $\mathcal{C}(a, b_-, b_+) \subset \mathcal{M}_1(\mathbb{R})$ as

$$\mathcal{C}(a, b_-, b_+) := \{\mathcal{N}(m, \sigma) : m \in [-a, a], \sigma \in [b_-, b_+]\}.$$

Then, for any $a, b_-, b_+ > 0$, there exists a probability measure ν and a constant $c > 0$ such that, for any $\mu \in \mathcal{C}(a, b_-, b_+)$,

$$\mu \geq c\nu.$$

The proof of this lemma is postponed after the end of this proof.

Since $\lambda \sim_{t \rightarrow \infty} g$ and these two functions are bounded on \mathbb{R}_+ , Lebesgue's dominated convergence theorem implies that, for all $s \leq t$,

$$\left| \int_{s+k\gamma}^{t+k\gamma} \lambda(u) du - \int_s^t g(u) du \right| \xrightarrow[k \rightarrow \infty]{} 0.$$

In the same way, for all $s \leq t$,

$$\int_{s+k\gamma}^{t+k\gamma} e^{2 \int_{s+k\gamma}^u \lambda(v) dv} du \xrightarrow[k \rightarrow \infty]{} \int_s^t e^{2 \int_s^u g(v) dv} du.$$

Hence, for any $s \leq t$,

$$e^{-\int_{s+k\gamma}^{t+k\gamma} \lambda(u) du} \xrightarrow[k \rightarrow \infty]{} e^{-\int_s^t g(u) du},$$

and

$$e^{-\int_{s+k\gamma}^{t+k\gamma} \lambda(u) du} \sqrt{\int_{s+k\gamma}^{t+k\gamma} e^{2 \int_{s+k\gamma}^u \lambda(v) dv} du} \xrightarrow[k \rightarrow \infty]{} e^{-\int_s^t g(u) du} \sqrt{\int_s^t e^{2 \int_s^u g(v) dv} du}.$$

Using [14, Theorem 1.3.], for any $x \in \mathbb{R}$,

$$\|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s+k\gamma, t+k\gamma}\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0. \quad (33)$$

To deduce the convergence in ψ -distance, we will inspire from the proof of [19, Lemma 3.1]. Since the variances are uniformly bounded in k (for $s \leq t$ fixed), there exists $H > 0$ such that, for any $k \in \mathbb{N}$ and $s \leq t$,

$$\delta_x P_{s+k\gamma, t+k\gamma}[\psi^2] \leq H \quad \text{and} \quad \delta_x Q_{s, t}[\psi^2] \leq H. \quad (34)$$

Since $\lim_{|x| \rightarrow \infty} \frac{\psi(x)}{\psi^2(x)} = 0$, for any $\epsilon > 0$, there exists $l_\epsilon > 0$ such that, for any function f such that $|f| \leq \psi$ and for any $|x| \geq l_\epsilon$,

$$|f(x)| \leq \frac{\epsilon \psi(x)^2}{H}.$$

This implies with (34) that, denoting $K_\epsilon := [-l_\epsilon, l_\epsilon]$, for any $k \in \mathbb{Z}_+$, f such that $|f| \leq \psi$ and $x \in \mathbb{R}$,

$$\delta_x P_{s+k\gamma, t+k\gamma}[f \mathbb{1}_{K_\epsilon^c}] \leq \epsilon \quad \text{and} \quad \delta_x Q_{s, t}[f \mathbb{1}_{K_\epsilon^c}] \leq \epsilon.$$

Then, for any $k \in \mathbb{Z}_+$ and f such that $|f| \leq \psi$,

$$|\delta_x P_{s+k\gamma, t+k\gamma} f - \delta_x Q_{s, t} f| \leq 2\epsilon + |\delta_x P_{s+k\gamma, t+k\gamma}[f \mathbb{1}_{K_\epsilon}] - \delta_x Q_{s, t}[f \mathbb{1}_{K_\epsilon}]| \quad (35)$$

$$\leq 2\epsilon + (1 + l_\epsilon^2) \|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s, t}\|_{TV} \quad (36)$$

Hence, (33) implies that, for k large enough, for any f bounded by ψ ,

$$|\delta_x P_{s+k\gamma, t+k\gamma} f - \delta_x Q_{s, t} f| \leq 3\epsilon, \quad (37)$$

implying that

$$\|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s, t}\|_\psi \xrightarrow[k \rightarrow \infty]{} 0.$$

□

We now prove Lemma 1.

Proof of Lemma 1. Defining

$$f_\nu(x) := e^{-\frac{(x-a)^2}{2b_-^2}} \wedge e^{-\frac{(x+a)^2}{2b_-^2}},$$

we conclude easily that, for any $m \in [-a, a]$ and $\sigma \geq b_-$, for any $x \in \mathbb{R}$,

$$e^{-\frac{(x-m)^2}{2\sigma^2}} \geq f_\nu(x).$$

Imposing moreover that $\sigma \leq b_+$, one has

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \geq \frac{1}{\sqrt{2\pi}b_+} f_\nu(x),$$

which concludes the proof. \square

4.2 Quasi-ergodic distribution for Brownian motion absorbed by an asymptotically periodic moving boundary

Let $(W_t)_{t \geq 0}$ be a one-dimensional Brownian motion and h be a \mathcal{C}^1 -function such that

$$h_{\min} := \inf_{t \geq 0} h(t) > 0, \quad \text{and} \quad h_{\max} := \sup_{t \geq 0} h(t) < +\infty.$$

We assume also that

$$-\infty < \inf_{t \geq 0} h'(t) \leq \sup_{t \geq 0} h'(t) < +\infty.$$

Denote by

$$\tau_h := \inf\{t \geq 0 : |W_t| \geq h(t)\}.$$

Since h is continuous, the hitting time τ_h is a stopping time with respect to the natural filtration of $(W_t)_{t \geq 0}$. Moreover, since $\sup_{t \geq 0} h(t) < +\infty$ and $\inf_{t \geq 0} h(t) > 0$,

$$\mathbb{P}_{s,x}[\tau_h < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{s,x}[\tau_h > t] > 0, \quad \forall s \leq t, \forall x \in [-h(s), h(s)].$$

The main assumption on the function h is the existence of a γ -periodic function g such that $h(t) \leq g(t)$, for any $t \geq 0$, and such that

$$h \sim_{t \rightarrow \infty} g, \quad \text{and} \quad h' \sim_{t \rightarrow \infty} g'.$$

Similarly to τ_h , denote

$$\tau_g := \inf\{t \geq 0 : |W_t| = g(t)\}.$$

Finally, let us assume that there exists $n_0 \in \mathbb{N}$ such that, for any $s \geq 0$,

$$\inf\{u \geq s : h(u) = \inf_{t \geq s} h(t)\} - s \leq n_0 \gamma. \tag{38}$$

This condition says that there exists $n_0 \in \mathbb{N}$ such that, for any time $s \geq 0$, the infimum of the function h on the domain $[s, +\infty)$ is reached on the subset $[s, s + n_0 \gamma]$.

We first show the following proposition.

Proposition 1. *The Markov process $(W_t)_{t \geq 0}$, considered as absorbed by h or by g , satisfies Assumption (A').*

Proof. In what follows, Assumption (A') w.r.t. the absorbing function h will be shown. The following proof could easily be adapted for the function g .

- *Proof of (A'1).* Denote $\mathcal{T} := \{s \geq 0 : h(s) = \inf_{t \geq s} h(t)\}$. The condition (38) implies that this set contains an infinity of times.

In what follows, the following notation is needed: for any $z \in \mathbb{R}$, define τ_z as

$$\tau_z := \inf\{t \geq 0 : |W_t| = z\}.$$

Also, let us state that, since the Brownian motion absorbed at $\{-1, 1\}$ satisfies Assumption (A) of [5] at any time (see [7]), it follows that, for a given $t_0 > 0$, there exists $c > 0$ and $\nu \in \mathcal{M}_1((-1, 1))$ such that, for any $x \in (-1, 1)$,

$$\mathbb{P}_{0,x} \left[W_{\frac{t_0}{h_{\max}^2} \wedge t_0} \in \cdot \mid \tau_1 > \frac{t_0}{h_{\max}^2} \wedge t_0 \right] \geq c\nu. \quad (39)$$

Moreover, regarding the proof of [7, Section 5.1], the probability measure ν can be expressed as

$$\nu = \frac{1}{2} (\mathbb{P}_{0,1-\epsilon}[W_{t_2} \in \cdot \mid \tau_1 > t_2] + \mathbb{P}_{0,-1+\epsilon}[W_{t_2} \in \cdot \mid \tau_1 > t_2]), \quad (40)$$

for some $0 < t_2 < \frac{t_0}{h_{\max}^2} \wedge t_0$ and $\epsilon \in (0, 1)$.

The following lemma is very important for the following.

Lemma 2. *For all $z \in [h_{\min}, h_{\max}]$,*

$$\mathbb{P}_{0,x}[W_u \in \cdot \mid \tau_z > u] \geq c\nu_z, \quad \forall x \in (-z, z), \forall u \geq t_0,$$

where t_0 was evoked before, $c > 0$ is the same constant as in (39) and

$$\nu_z(f) = \int_{(-1,1)} f(zx)\nu(dx),$$

with $\nu \in \mathcal{M}_1((-1, 1))$ defined in (40).

The proof of this lemma is postponed after the current proof.

Let $s \in \mathcal{T}$. Then, for all $x \in (-h(s), h(s))$ and $t \geq 0$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot \mid \tau_h > s+t] \geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} \mathbb{P}_{s,x}[W_{s+t} \in \cdot \mid \tau_{h(s)} > s+t],$$

By Lemma 2, for all $x \in (-h(s), h(s))$ and $t \geq t_0$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot \mid \tau_{h(s)} > s+t] \geq c\nu_{h(s)},$$

which implies therefore that, for any $t \in [t_0, t_0 + n_0\gamma]$,

$$\begin{aligned} \mathbb{P}_{s,x}[W_{s+t} \in \cdot \mid \tau_h > s+t] &\geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} c\nu_{h(s)} \\ &\geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0 + n_0\gamma]}{\mathbb{P}_{s,x}[\tau_h > s+t_0]} c\nu_{h(s)}. \end{aligned} \quad (41)$$

Let us introduce the process X^h defined by, for all $t \geq 0$,

$$X_t^h := \frac{W_t}{h(t)}.$$

By Itô's formula, for any $t \geq 0$,

$$X_t^h = X_0^h + \int_0^t \frac{dW_s}{h(s)} - \int_0^t \frac{h'(s)}{h(s)} X_s^h ds.$$

Denote by $(M_t^h)_{t \geq 0} := \left(\int_0^t \frac{1}{h(s)} dW_s \right)_{t \geq 0}$. By the Dubins-Schwarz theorem, it is well-known that the process M^h has the same law as $\left(W \int_0^t \frac{1}{h^2(s)} ds \right)_{t \geq 0}$. Then, denoting $I^h(s) := \int_0^s \frac{1}{h^2(u)} du$ and, for any $s \leq t$ and for any trajectory w ,

$$\mathcal{E}_{s,t}^h(w) := \sqrt{\frac{h(t)}{h(s)}} \exp \left(-\frac{1}{2} \left[h'(t)h(t)w_{I^h(t)}^2 - h'(s)h(s)w_{I^h(s)}^2 + \int_s^t w_{I^h(u)}^2 [(h'(u))^2 - [h(u)h'(u)]'] du \right] \right), \quad (42)$$

Girsanov theorem implies that, for all $x \in (-h(s), h(s))$,

$$\mathbb{P}_{s,x}[\tau_h > s + t_0] = \mathbb{E}_{I^h(s), \frac{x}{h(s)}} \left[\mathcal{E}_{s,s+t_0}^h(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du} \right]. \quad (43)$$

On the event $\{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du\}$, and since h and h' are bounded on \mathbb{R}_+ , the random variable $\mathcal{E}_{s,s+t_0}^h(W)$ is almost surely bounded by a constant $C > 0$, uniformly in s , such that for all $x \in (-h(s), h(s))$,

$$\mathbb{E}_{I^h(s), \frac{x}{h(s)}} \left[\mathcal{E}_{s,s+t_0}^h(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du} \right] \leq C \mathbb{P}_{0, \frac{x}{h(s)}} \left[\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right]. \quad (44)$$

Since $h(t) \geq h(s)$ for all $t \geq s$ (since $s \in \mathcal{T}$), $I^h(s+t_0) - I^h(s) \leq \frac{t_0}{h(s)^2}$. By the scaling property of the Brownian motion and by Markov property, one has for all $x \in (-h(s), h(s))$,

$$\begin{aligned} \mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0] &= \mathbb{P}_{0,x}[\tau_{h(s)} > t_0] \\ &= \mathbb{P}_{0, \frac{x}{h(s)}} \left[\tau_1 > \frac{t_0}{h^2(s)} \right] \\ &= \mathbb{E}_{0, \frac{x}{h(s)}} \left[\mathbb{1}_{\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du} \mathbb{P}_{0,W} \int_s^{s+t_0} \frac{1}{h^2(u)} du \left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(s)} ds \right] \right] \\ &= \mathbb{P}_{0, \frac{x}{h(s)}} \left[\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right] \mathbb{P}_{0, \phi_{I^h(s+t_0) - I^h(s)}(\delta_x)} \left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right], \end{aligned}$$

where, for any initial distribution μ and any $t \geq 0$,

$$\phi_t(\mu) := \mathbb{P}_{0,\mu}[W_t \in \cdot | \tau_1 > t].$$

The family $(\phi_t)_{t \geq 0}$ satisfies the equality $\phi_t \circ \phi_s = \phi_{t+s}$ for all $s, t \geq 0$. By this property, and using that $I^h(s+t_0) - I^h(s) \geq \frac{t_0}{h_{\max}^2}$ for any $s \geq 0$, the minorization (39) implies that, for all $s \geq 0$ and $x \in (-1, 1)$,

$$\phi_{I^h(s+t_0) - I^h(s)}(\delta_x) \geq c\nu.$$

Hence, by this minorization, and using that h is upper-bounded and lower-bounded positively on \mathbb{R}_+ , one has for all $x \in (-1, 1)$,

$$\mathbb{P}_{0, \phi_{I^h(s+t_0) - I^h(s)}(\delta_x)} \left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right] \geq c \mathbb{P}_{0,\nu} \left[\tau_1 > \inf_{s \geq 0} \left\{ \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right],$$

that is to say,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]}{\mathbb{P}_{0, \frac{x}{h(s)}} \left[\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right]} \geq c \mathbb{P}_{0,\nu} \left[\tau_1 > \inf_{s \geq 0} \left\{ \frac{\gamma}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right].$$

In other words, we just showed that, for all $x \in (-h(s), h(s))$,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]}{\mathbb{P}_{s,x}[\tau_h > s + t_0]} \geq \frac{c}{C} \mathbb{P}_{0,\nu} \left[\tau_1 > \inf_{s \geq 0} \left\{ \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right] > 0. \quad (45)$$

Moreover, by Lemma 2 and the scaling property of the Brownian motion, for all $x \in (-h(s), h(s))$,

$$\begin{aligned} \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0 + n_0\gamma]}{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]} &= \mathbb{P}_{0,\mathbb{P}_{0,x}[W_{t_0} \in \cdot | \tau_{h(s)} > t_0]}[\tau_{h(s)} > n_0\gamma] \\ &\geq c \mathbb{P}_{0,\nu_{h(s)}}[\tau_{h(s)} > n_0\gamma] \\ &= c \int_{(-1,1)} \nu(dy) \mathbb{P}_{h(s)y}[\tau_{h(s)} > n_0\gamma] \\ &\geq c \mathbb{P}_{0,\nu} \left[\tau_1 > \frac{n_0\gamma}{h_{\min}^2} \right] > 0. \end{aligned} \quad (46)$$

Thus, gathering (41), (45) and (46), for any $x \in (-h(s), h(s))$ and any $t \in [t_0, t_0 + n_0\gamma]$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s + t] \geq c_1 \nu_{h(s)}, \quad (47)$$

where

$$c_1 := c \mathbb{P}_{0,\nu} \left[\tau_1 > \frac{n_0\gamma}{h_{\max}^2} \right] \times \frac{c}{C} \mathbb{P}_{0,\nu} \left[\tau_1 > \inf_{s \geq 0} \left\{ \frac{\gamma}{h^2(s)} - \int_s^{s+\gamma} \frac{1}{h^2(u)} du \right\} \right] c.$$

We recall that the Doeblin condition (47) is, for now, only obtained for $s \in \mathcal{T}$. Consider now $s \notin \mathcal{T}$. Then, by the condition (38), there exists $s_1 \in \mathcal{T}$ such that $s < s_1 \leq s + n_0\gamma$. Markov property and (47) implies therefore that, for any $x \in (-h(s), h(s))$,

$$\mathbb{P}_{s,x}[W_{s+t_0+n_0\gamma} \in \cdot | \tau_h > s + t_0 + n_0\gamma] = \mathbb{P}_{s_1,\phi_{s_1,s}}[W_{s+t_0+n_0\gamma} \in \cdot | \tau_h > s + t_0 + n_0\gamma] \geq c_1 \nu_{h(s_1)},$$

where, for all $s \leq t$ and $\mu \in \mathcal{M}_1((-h(s), h(s)))$,

$$\phi_{t,s}(\mu) := \mathbb{P}_{s,\mu}[W_t \in \cdot | \tau_h > t].$$

This concludes the proof of (A'1).

- *Proof of (A'2).* Since $(W_t)_{t \geq 0}$ is a Brownian motion, note that, for any $s \leq t$,

$$\sup_{x \in (-1,1)} \mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,0}[\tau_h > t].$$

Also, for any $a \in (0, h(s))$,

$$\inf_{[-a,a]} \mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,a}[\tau_h > t].$$

Thus, by Markov property, and using that the function $s \mapsto \mathbb{P}_{s,0}[\tau_g > t]$ is non-decreasing on $[0, t]$ (for all $t \geq 0$), one has for any $s \leq t$,

$$\mathbb{P}_{s,a}[\tau_h > t] \geq \mathbb{E}_{s,a}[\mathbb{1}_{\tau_0 < s + \gamma < \tau_h} \mathbb{P}_{\tau_0,0}[\tau_h > t]] \geq \mathbb{P}_{s,a}[\tau_0 < s + \gamma < \tau_h] \mathbb{P}_{s,0}[\tau_h > t]. \quad (48)$$

Denoting $a := \frac{h_{\min}}{h_{\max}}$, by Lemma 2 and taking $s_1 := \inf\{u \geq s : u \in \mathcal{T}\}$, one obtains that, for all $s \leq t$,

$$\begin{aligned} \mathbb{P}_{s,\nu_{h(s_1)}}[\tau_h > t] &= \int_{(-1,1)} \nu(dx) \mathbb{P}_{s,h(s_1)x}[\tau_h > t] \\ &\geq \nu([-a, a]) \mathbb{P}_{s,h(s_1)a}[\tau_h > t] \\ &\geq \nu([-a, a]) \mathbb{P}_{0,h_{\min}}[\tau_0 < \gamma < \tau_h] \sup_{x \in (-h(s), h(s))} \mathbb{P}_{s,x}[\tau_h > t]. \end{aligned}$$

This concludes the proof since, using (40), one has $\nu([-a, a]) > 0$.

□

We now prove Lemma 2.

Proof of Lemma 2. This result comes from the scaling property of a Brownian motion. In fact, for any $z \in [h_{\min}, h_{\max}]$, $x \in (-z, z)$ and $t \geq 0$, and for any measurable bounded function f ,

$$\begin{aligned} \mathbb{E}_{0,x}[f(W_t)|\tau_z > t] &= \mathbb{E}_{0,x} \left[f \left(z \times \frac{1}{z} W_{z^2 \frac{t}{z^2}} \right) \middle| \tau_z > t \right] \\ &= \mathbb{E}_{0, \frac{x}{z}} \left[f \left(z \times W_{\frac{t}{z^2}} \right) \middle| \tau_1 > \frac{t}{z^2} \right]. \end{aligned}$$

Then, the minorization (39) implies that, for any $x \in (-1, 1)$,

$$\mathbb{P}_{0,x} \left[W_{\frac{t_0}{h_{\max}^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{h_{\max}^2} \right] \geq c\nu.$$

This inequality holds for any time greater than $\frac{t_0}{h_{\max}^2}$. In particular, for any $z \in [h_{\min}, h_{\max}]$ and $x \in (-1, 1)$,

$$\mathbb{P}_{0,x} \left[W_{\frac{t_0}{z^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{z^2} \right] \geq c\nu.$$

Then, for any $z \in [a, b]$, f positive and measurable, and $x \in (-z, z)$,

$$\mathbb{E}_{0,x}[f(W_{t_0})|\tau_z > t_0] \geq c\nu_z(f),$$

where $\nu_z(f) := \int_E f(z \times x)\nu(dx)$. This completes the proof of Lemma 2. □

The section is now concluded by stating and proving the following result.

Theorem 4. For any $s \leq t$, $n \in \mathbb{N}$ and any $x \in \mathbb{R}$,

$$\mathbb{P}_{s+k\gamma, x}[\tau_h \leq t + k\gamma < \tau_g] \xrightarrow[k \rightarrow \infty]{} 0.$$

In particular, Corollary 2 holds for $(W_t)_{t \geq 0}$ absorbed by h .

Proof. Reminding (42), by Markov property for the Brownian motion, one has for any $k, n \in \mathbb{N}$ and any $x \in \mathbb{R}$,

$$\mathbb{P}_{s+k\gamma, x}[\tau_h > t + k\gamma] = \sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \mathbb{E}_{0,x} \left[\exp \left(-\frac{1}{2} \mathcal{A}_{s,t,k}^h(W) \right) \mathbb{1}_{\tau_1 > I^h(t+k\gamma) - I^h(s+k\gamma)} \right],$$

where, for any trajectory $w = (w_u)_{u \geq 0}$,

$$\begin{aligned} \mathcal{A}_{s,t,k}^h(w) &= h'(t+k\gamma)h(t+k\gamma)w_{I^h(t+k\gamma) - I^h(s+k\gamma)}^2 - h'(s+k\gamma)h(s+k\gamma)w_0^2 \\ &\quad + \int_0^{t-s} w_{I^h(u+s+k\gamma) - I^h(s+k\gamma)}^2 [h'(u+s+k\gamma)]^2 - [h(u+s+k\gamma)h'(u+s+k\gamma)]' du. \end{aligned}$$

Since $h \sim_{t \rightarrow \infty} g$, one has for any $s, t \in [0, \gamma]$

$$\sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \xrightarrow[k \rightarrow \infty]{} \sqrt{\frac{g(t)}{g(s)}}.$$

For the same reasons, and using that the function h is bounded on $[s+k\gamma, t+k\gamma]$ for all $s \leq t$, Lebesgue's dominated convergence theorem implies that

$$I^h(t+k\gamma) - I^h(s+k\gamma) \xrightarrow[k \rightarrow \infty]{} I^g(t) - I^g(s)$$

for all $s \leq t \in [0, \gamma]$. Moreover, since $h \sim_{t \rightarrow \infty} g$ and $h' \sim_{t \rightarrow \infty} g'$, one has for all trajectories $w = (w_u)_{u \geq 0}$ and $s \leq t \in [0, \gamma]$,

$$\mathcal{A}_{s,t,k}^h(w) \xrightarrow[k \rightarrow \infty]{} g'(t)g(t)w_{I^g(t)-I^g(s)}^2 - g'(s)g(s)w_0^2 + \int_s^t w_{I^g(u)}^2 [(g'(u))^2 - [g(u)g'(u)]'] du.$$

Since the random variable $\exp\left(-\frac{1}{2}\mathcal{A}_{s,t,k}^h(W)\right) \mathbb{1}_{\tau_1 > I^h(t+k\gamma) - I^h(s+k\gamma)}$ is bounded almost surely, Lebesgue's dominated convergence theorem implies that

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t + k\gamma] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{s,x}[\tau_g > t],$$

which concludes the proof. □

Acknowledgement

I would like to thank the anonymous reviewers for their valuable and relevant comments and suggestions, as well as Oliver Kelsey Tough for reviewing a part of this paper.

References

- [1] V. Bansaye, B. Cloez, and P. Gabriel. Ergodic behavior of non-conservative semigroups via generalized Doeblin's conditions. *Acta Applicandae Mathematicae*, 166:29–72, 2020.
- [2] L. Breyer and G. Roberts. A quasi-ergodic theorem for evanescent processes. *Stochastic Processes and their Applications*, 84:177–186, 1999.
- [3] P. Cattiaux, C. Christophe, and S. Gadat. A stochastic model for cytotoxic T lymphocyte interaction with tumor nodules. *preprint*, 2016.
- [4] N. Champagnat, W. Oçafrain, and D. Villemonais. Quasi-stationarity for time-(in)homogeneous Markov processes absorbed by moving boundaries through Lyapunov criteria. *in preparation*, 2021.
- [5] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q -process. *Probability Theory and Related Fields*, 164(1-2):243–283, 2016.
- [6] N. Champagnat and D. Villemonais. General criteria for the study of quasi-stationarity. *arXiv preprint arXiv:1712.08092*, 2017.
- [7] N. Champagnat and D. Villemonais. Uniform convergence of conditional distributions for absorbed one-dimensional diffusions. *Advances in Applied Probability*, 50(1):178–203, 2017.
- [8] N. Champagnat and D. Villemonais. Uniform convergence to the Q -process. *Electron. Commun. Probab.*, 22:Paper No. 33, 7, 2017.
- [9] N. Champagnat and D. Villemonais. Uniform convergence of penalized time-inhomogeneous Markov processes. *ESAIM: Probability and Statistics*, 22:129–162, 2018.
- [10] J. Chen and S. Jian. A deviation inequality and quasi-ergodicity for absorbing Markov processes. *Ann. Mat. Pura Appl. (4)*, 197(2):641–650, 2018.
- [11] P. Collet, S. Martínez, and J. San Martín. *Quasi-stationary distributions*. Probability and its Applications (New York). Springer, Heidelberg, 2013. Markov chains, diffusions and dynamical systems.
- [12] F. Colonus and M. Rasmussen. Quasi-ergodic limits for finite absorbing Markov chains. *Linear Algebra and its Applications*, 609:253–288, 2021.

- [13] J. N. Darroch and E. Seneta. On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *J. Appl. Probability*, 2:88–100, 1965.
- [14] L. Devroye, A. Mehrabian, and T. Reddad. The total variation distance between high-dimensional gaussians. *arXiv preprint arXiv:1810.08693*, 2018.
- [15] M. Hairer and J. C. Mattingly. Yet another look at Harris’ ergodic theorem for markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, pages 109–117. Springer, 2011.
- [16] G. He. A note on the quasi-ergodic distribution of one-dimensional diffusions. *C. R. Math. Acad. Sci. Paris*, 356(9):967–972, 2018.
- [17] G. He, G. Yang, and Y. Zhu. Some conditional limiting theorems for symmetric Markov processes with tightness property. *Electron. Commun. Probab.*, 24:Paper No. 60, 11, 2019.
- [18] G. He, H. Zhang, and Y. Zhu. On the quasi-ergodic distribution of absorbing Markov processes. *Statistics & Probability Letters*, 149:116–123, 2019.
- [19] A. Hening and D. H. Nguyen. Stochastic Lotka-Volterra food chains. *J. Math. Biol.*, 77(1):135–163, 2018.
- [20] R. Höpfner and Y. Kutoyants. Estimating discontinuous periodic signals in a time inhomogeneous diffusion. *Stat. Inference Stoch. Process.*, 13(3):193–230, 2010.
- [21] R. Höpfner, E. Löcherbach, and M. Thieullen. Ergodicity and limit theorems for degenerate diffusions with time periodic drift. application to a stochastic hodgkin- huxley model. *ESAIM: Probability and Statistics*, 20:527–554, 2016.
- [22] R. Höpfner, E. Löcherbach, and M. Thieullen. Ergodicity for a stochastic Hodgkin–Huxley model driven by Ornstein–Uhlenbeck type input. In *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, volume 52, pages 483–501. Institut Henri Poincaré, 2016.
- [23] S. Méléard and D. Villemonais. Quasi-stationary distributions and population processes. *Probab. Surv.*, 9:340–410, 2012.
- [24] W. Oçafrain. Quasi-stationarity and quasi-ergodicity for discrete-time Markov chains with absorbing boundaries moving periodically. *ALEA*, 15:429–451, 2018.
- [25] W. Oçafrain. Q -processes and asymptotic properties of Markov processes conditioned not to hit moving boundaries. *Stochastic Process. Appl.*, 130(6):3445–3476, 2020.
- [26] P.-C. Vassiliou. Laws of large numbers for non-homogeneous Markov systems. *Methodology and Computing in Applied Probability*, pages 1–28, 2018.