Ergodic theorem for asymptotically periodic time-inhomogeneous Markov processes, with application to quasi-stationarity with moving boundaries

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Abstract

This paper deals with ergodic theorems for particular time-inhomogeneous Markov processes, whose the time-inhomogeneity is asymptotically periodic. Under a Lyapunov/minorization condition, it is shown that, for any measurable bounded function f, the time average $\frac{1}{t}\int_0^t f(X_s)ds$ converges in $\mathbb{L}^2$ towards a limiting distribution, starting from any initial distribution for the process $(X_t)_{t\geq 0}$. This convergence can be improved to an almost sure convergence under an additional assumption on the initial measure. This result will be then applied to show the existence of a quasi-ergodic distribution for processes absorbed by an asymptotically periodic moving boundary, satisfying a conditional Doeblin's condition.

Key words: ergodic theorem; law of large numbers; time-inhomogeneous Markov processes; quasi-stationarity; quasi-ergodic distribution; moving boundaries.

Notation

- $\mathbb{N} = \{1, 2, \dots, \}$ and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.
- $\mathcal{M}_1(E)$: Space of the probability measures whose the support is included in E.
- $\mathcal{B}(E)$: Set of the measurable bounded functions defined on E.
- $\mathcal{B}_1(E)$: Set of the measurable functions f defined on E such that $\|f\|_\infty \leq 1$.
- $\mathbb{L}^p(\mu)$: Set of the measurable functions f such that $\int |f|^p d\mu < +\infty$ ($p \in \mathbb{N}$).
- For any $\mu \in \mathcal{M}_1(E)$ and $f \in \mathbb{L}^1(\mu)$, denote

$$\mu(f) := \int_E f(x)\mu(dx).$$

- Id: Identity operator.

1 Introduction

In a general way, the ergodic theorem (for Markov processes) refers to the almost sure convergence

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \pi(f), \quad \forall f \in \mathbb{L}^1(\pi), \tag{1}$$

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$(X_t)_{t\geq 0}$ being a Markov process and π being a probability measure. In the time-homogeneous setting, the ergodic theorem holds for positive Harris recurrent Markov processes and the limiting distribution π corresponds to an invariant measure for the underlying Markov process. For time-inhomogeneous Markov processes, such a result does not hold in general (in particular the notion of invariant measure is not generally well-defined), except for specific time-inhomogeneity such as for *periodic time-inhomogeneous Markov processes*, defined as time-inhomogeneous Markov processes for which there exists $\gamma > 0$ such that, for any $s \leq t$, $k \in \mathbb{Z}_+$ and x,

$$\mathbb{P}[X_t \in \cdot | X_s = x] = \mathbb{P}[X_{t+k\gamma} \in \cdot | X_{s+k\gamma} = x]. \tag{2}$$

In other words, a time-inhomogeneous Markov process is periodic when the transition law between any times s and t remains unchanged when the time interval $[s,t]$ is shifted by a multiple of the period γ. In particular, this implies that, for any $s \in [0,\gamma)$, the Markov chain $(X_{s+n\gamma})_{n\in\mathbb{Z}_+}$ is time-homogeneous. This fact allowed to Höpfner et al. (in [20, 21, 22]) to show that, if the skeleton Markov chain $(X_{n\gamma})_{n\in\mathbb{Z}_+}$ is Harris recurrent, then the chains $(X_{s+n\gamma})_{n\in\mathbb{Z}_+}$, for all $s \in [0,\gamma)$, are also Harris recurrent and

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \frac{1}{\gamma}\int_0^\gamma \pi_s(f)ds, \quad \text{almost surely, from any initial measure,}$$

where π_s is the invariant measure for $(X_{s+n\gamma})_{n\in\mathbb{Z}_+}$.

This paper aims to prove a similar result for time-inhomogeneous Markov processes said to be *asymptotically periodic*. Roughly speaking (the definition will be explicitly given further), an asymptotically periodic Markov process is such that, given a duration $T \geq 0$, its transition law on the interval $[s, s+T]$ is asymptotically *"close to"* the one, on the same interval, of a periodic time-inhomogeneous Markov process called *auxiliary Markov process*, when $s \to \infty$. This definition is very similar to the notion of *asymptotic homogeneization* (when the approximating semi-group is time-homogeneous), or the notion of *asymptotic pseudotrajectories* as introduced in [2]. In particular, it is known that these rapprochements entail that, under suitable additional conditions, an asymptotically homogeneous Markov process converges towards a probability measure, which is invariant for the approximating semi-group. In the same way, it is expected that an asymptotically periodic process gets the same asymptotic properties as a periodic Markov process; in particular that an ergodic theorem holds for the asymptotically periodic process.

Main result. The main result of this paper states that an asymptotically periodic Markov process satisfying a Lyapunov-type condition and a local Doeblin's condition (defined further in Section 2), and such that its auxiliary process also satisfies a Lyapunov/minorization condition, satisfies

$$\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow[t\to\infty]{\mathbb{L}^2(\mathbb{P}_{0,\mu})} \frac{1}{\gamma}\int_0^\gamma \beta_s(f)ds, \quad \forall f \text{ measurable bounded}, \forall \mu \text{ initial distribution}, \tag{3}$$

where $\mathbb{P}_{0,\mu}$ is a probability measure under which $X_0 \sim \mu$, and where β_s is the limiting distribution of the skeleton Markov chain $(X_{s+n\gamma})_{n\in\mathbb{Z}_+}$. Furthermore, the convergence 3 holds almost surely if the initial measure integrates the Lyapunov function of the process $(X_t)_{t\geq 0}$:

$$\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow[t\to\infty]{\mathbb{P}_{0,\mu}-\text{almost surely}} \frac{1}{\gamma}\int_0^\gamma \beta_s(f)ds, \quad \forall \mu \text{ integrating the Lyapunov function.}$$

This will be more precisely stated and proved in Section 2.

Quasi-stationarity with moving boundaries. One point of interest in this paper is to apply the main result (3) to the theory of *quasi-stationarity with moving boundaries*, which refers to the study of asymptotical behaviors for Markov processes conditioned not to reach a moving subset of the state space. See in particular [24, 25], where a *"conditional ergodic theorem"* (see further the definition of a *quasi-ergodic distribution*) has been shown when the absorbing boundaries move periodically. In this paper, one shows that a similar result holds when the boundary is asymptotically periodic, assuming that the process satisfies a conditional Doeblin's condition (see Assumption (A')). This part will be dealt with in Section 3.

The paper will be concluded by using these results in two examples: an ergodic theorem for an asymptotically periodic Ornstein-Uhlenbeck process, and the existence of a unique quasi-ergodic distribution for a Brownian motion confined between two symmetric asymptotically periodic functions.

2 Ergodic theorem for asymptotically periodic time-inhomogeneous semi-group.

Asymptotic periodicity: the definition.

Let $(E,\mathcal{E})$ be a measurable space. Consider $\{(E_t,\mathcal{E}_t)_{t\geq 0},(P_{s,t})_{s\leq t}\}$ a Markovian time-inhomogeneous semi-group, giving a family of measurable subspaces of $(E,\mathcal{E})$, denoted by $(E_t,\mathcal{E}_t)_{t\geq 0}$, and a family of linear operator $(P_{s,t})_{s\leq t}$, with $P_{s,t}:\mathcal{B}(E_t)\to\mathcal{B}(E_s)$, satisfying for any $r\leq s\leq t$,

$$P_{s,s}=\mathrm{Id},\qquad P_{s,t}\mathbb{1}_{E_t}=\mathbb{1}_{E_s},\qquad P_{r,s}P_{s,t}=P_{r,t}.$$

In particular, it is associated to $\{(E_t,\mathcal{E}_t)_{t\geq 0},(P_{s,t})_{s\leq t}\}$ a Markov process $(X_t)_{t\geq 0}$ and a family of probability measure $(\mathbb{P}_{s,x})_{s\geq 0,x\in E_s}$ such that, for any $s\leq t$, $x\in E_s$ and $A\in\mathcal{E}_t$,

$$\mathbb{P}_{s,x}[X_t\in A]=P_{s,t}\mathbb{1}_A(x).$$

For any probability measure μ supported on E_s, denote $\mathbb{P}_{s,\mu}:=\int_{E_s}\mathbb{P}_{s,x}\mu(dx)$. Also, denote by $\mathbb{E}_{s,x}$ and $\mathbb{E}_{s,\mu}$ the expectations associated to $\mathbb{P}_{s,x}$ and $\mathbb{P}_{s,\mu}$, respectively. Finally, the following notation will be used for $\mu\in\mathcal{M}_1(E_s)$, $s\leq t$ and $f\in\mathcal{B}(E_t)$:

$$\mu P_{s,t}f:=\mathbb{E}_{s,\mu}[f(X_t)],\qquad \mu P_{s,t}:=\mathbb{P}_{s,\mu}[X_t\in\cdot].$$

Respectively to the definition (2), the periodicity of a time-inhomogeneous semi-group can be defined as follows: a semi-group $\{(F_t,\mathcal{F}_t)_{t\geq 0},(Q_{s,t})_{s\leq t}\}$ is said to be γ-*periodic*, with $\gamma>0$, if, for any $s\leq t$,

$$(F_t,\mathcal{F}_t)=(F_{t+k\gamma},\mathcal{F}_{t+k\gamma}),\qquad Q_{s,t}=Q_{s+k\gamma,t+k\gamma},\qquad \forall k\in\mathbb{Z}_+.$$

Following to this last definition, it is now possible to define an *asymptotically periodic semi-group*:

Definition 1 (Asymptotical periodicity for semi-groups)**.** A time-inhomogeneous semi-group $\{(E_t,\mathcal{E}_t)_{t\geq 0},(P_{s,t})_{s\leq t}\}$ is said to be *asymptotically periodic* if there exists a γ-periodic semi-group $\{(F_t,\mathcal{F}_t)_{t\geq 0},(Q_{s,t})_{s\leq t}\}$ and two families of functions $(\psi_s)_{s\geq 0}$ and $(\tilde{\psi}_s)_{s\in[0,\gamma)}$ such that, for any $s\in[0,\gamma)$,

- $\bigcup_{k=0}^{\infty}\bigcap_{l\geq k}E_{s+l\gamma}\cap F_s\neq\emptyset$
- and there exists $x_s\in\bigcup_{k=0}^{\infty}\bigcap_{l\geq k}E_{s+l\gamma}\cap F_s$ such that, for any $n\in\mathbb{Z}_+$,

$$\|\delta_{x_s}P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot]-\delta_{x_s}Q_{s,s+n\gamma}[\tilde{\psi}_s\times\cdot]\|_{TV}\underset{k\to\infty}{\longrightarrow}0, \tag{4}$$

where, for two positive measures with finite mass μ_1 and μ_2, $\|\mu_1-\mu_2\|_{TV}$ is the *total variation distance* between μ_1 and μ_2:

$$\|\mu_1-\mu_2\|_{TV}:=\sup_{f\in\mathcal{B}_1(E)}|\mu_1(f)-\mu_2(f)|.$$

Lyapunov/minorization conditions.

The main assumption of the theorem will be the satisfaction of the following assumptions by the asymptotically periodic Markov process:

Lyapunov/minorization conditions. *There exists $t_1\geq 0$, $n_0\in\mathbb{N}$, $c>0$, $\theta\in(0,1)$, a family of subsets $(K_t)_{t\geq 0}$, a family of probability measure $(\nu_{s,n})_{s\in[0,t_1),n\in\mathbb{N}}$ on $(K_{s+nt_1})_{s\in[0,t_1),n\in\mathbb{N}}$, and a family of Lyapunov functions $(\psi_s)_{s\geq 0}$, all lower-bounded by 1, such that*

(i) For any $s\in[0,t_1)$, $x\in K_s$ and $n\geq n_0$,

$$\delta_x P_{s,s+nt_1}\geq c\nu_{s,n},$$

(ii) For any $s\geq 0$,

$$P_{s,s+t_1}\psi_{s+t_1}\leq\theta\psi_s+C\mathbb{1}_{K_s},$$

(iii) For any $s \geq 0$, for any $t \in [0, t_1)$,

$$P_{s,s+t}\psi_{s+t} \leq C\psi_s.$$

In particular, under (ii) and (iii), it is easy to prove that, for any $s \leq t$,

$$P_{s,t}\psi_t \leq C\left(1 + \frac{C}{1-\theta}\right)\psi_s. \tag{5}$$

An important remark is that the Lyapunov/minorisation condition implies an *exponential uniform-in-law weak ergodicity in ψ_t-distance*, i.e. the existence of two constants $C' > 0$ and $\kappa > 0$ such that, for all $s \leq t$ and for all probability measures $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$,

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{\psi_t} \leq C'[\mu_1(\psi_s) + \mu_2(\psi_s)]e^{-\kappa(t-s)}, \tag{6}$$

where, for a given function ψ, $\|\mu - \nu\|_\psi$ is the ψ-distance and is defined as

$$\|\mu - \nu\|_\psi := \sup_{|f| \leq \psi} |\mu(f) - \nu(f)|, \qquad \forall \mu, \nu \text{ such that } \mu(\psi) < +\infty \text{ and } \nu(\psi) < +\infty.$$

In particular, when $\psi_t = \mathbb{1}$ for all $t \geq 0$, the ψ_t-distance is *total variation distance*. If the weak ergodicity (6) is well-known in the time-homogeneous setting (see in particular [15]), the proof of [15, Theorem 1.3.] can be adapted in a general time-inhomogeneous framework (see for example [6, Subsection 9.5]).

The main theorem and proof.

The main result is then the following:

Theorem 1. *Let $\{(E_t, \mathcal{E}_t)_{t\geq 0}, (P_{s,t})_{s\leq t}, (X_t)_{t\geq 0}, (\mathbb{P}_{s,x})_{s\geq 0, x\in E_s}\}$ be an asymptotically γ-periodic time-inhomogeneous Markov process, with $\gamma > 0$, and denote by $\{(F_t, \mathcal{F}_t)_{t\geq 0}, (Q_{s,t})_{s\leq t}\}$ its periodic auxiliary semi-group. Also, denote by $(\psi_s)_{s\geq 0}$ and $(\tilde{\psi}_s)_{s\in[0,\gamma)}$ the two families of functions as defined in Definition 1. Assume moreover that*

- *the semi-groups $(P_{s,t})_{s\leq t}$ and $(Q_{s,t})_{s\leq t}$ satisfy a Lyapunov/minorisation condition, with $(\psi_s)_{s\geq 0}$ and $(\tilde{\psi}_s)_{s\in[0,\gamma)}$ as Lyapunov functions respectively,*
- *and, for any $s \in [0, \gamma)$, $(\psi_{s+n\gamma})_{n\in\mathbb{Z}_+}$ converges pointwisely to $\tilde{\psi}_s$.*

Then, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\left\| \frac{1}{t}\int_0^t \mu P_{0,s}[\psi_s \times \cdot]ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]ds \right\|_{TV} \underset{t\to\infty}{\longrightarrow} 0, \tag{7}$$

where $\beta_\gamma \in \mathcal{M}_1(F_0)$ is the unique invariant of the skeleton semi-group $(Q_{0,n\gamma})_{n\in\mathbb{Z}_+}$ satisfying $\beta_\gamma(\tilde{\psi}_0) < +\infty$. Moreover, for any $f \in \mathcal{B}(E)$,

- *for any $\mu \in \mathcal{M}_1(E_0)$,*

$$\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}f ds\right|^2\right] \underset{t\to\infty}{\longrightarrow} 0. \tag{8}$$

- *If moreover $\mu(\psi_0) < +\infty$, then*

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}f ds, \qquad \mathbb{P}_{0,\mu} - \textit{almost surely.} \tag{9}$$

Remark 1. Also note that, if the Lyapunov/minorization conditions hold for $\psi_s = \mathbb{1}$ and $K_s = E_s$ for any s, then these conditions are boiled down to the *Doeblin's condition*:

Doeblin's condition. *there exists $t_0 \geq 0$, $c > 0$ and a family of probability measure $(\nu_t)_{t\geq 0}$ on $(E_t)_{t\geq 0}$ such that, for any $s \geq 0$ and $x \in E_s$,*

$$\delta_x P_{s,s+t_0} \geq c\nu_{s+t_0}. \tag{10}$$

Then, Theorem 1 entails the following corollary:

Corollary 1. *Let $(X_t)_{t\geq 0}$ be asymptotically γ-periodic in total variation distance (i.e. $\psi_s = \tilde{\psi}_s = \mathbb{1}$ in the Definition 1, for all $s \geq 0$). If $(X_t)_{t\geq 0}$ and its auxiliary semi-group satisfy a Doeblin's condition, then the convergence* (8) *is improved to*

$$\sup_{\mu\in\mathcal{M}_1(E_0)} \sup_{f\in\mathcal{B}_1(E)} \mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s} f ds\right|^2\right] \underset{t\to\infty}{\longrightarrow} 0.$$

Moreover, the almost sure convergence (9) *holds for all initial measure μ.*

Remark 2. Note also that, if the convergence (4) holds actually for all $x \in \bigcup_{k=0}^{\infty}\bigcap_{l\geq k} E_{s+l\gamma}\cap F_s$, then this implies (4), obviously, and the pointwise convergence of $(\psi_{s+n\gamma})_{n\in\mathbb{Z}_+}$ to $\tilde{\psi}_s$ (taking $n = 0$ in (4)).

Proof of Theorem 1. The proof is divided into four steps.

First step. For the first part of this step (until the equality (14)), we will inspire from the proof of [1, Theorem 3.11]. Fix $s \in [0, \gamma]$. Since the auxiliary semi-group $(Q_{s,t})_{s\leq t}$ satisfies a Lyapunov/minorization condition with $(\tilde{\psi})_{s\geq 0}$ as Lyapunov functions, this entails that, for any $x \in F_s$,

$$\|\delta_x Q_{s,s+n\gamma} - \beta_\gamma Q_{0,s}\|_{\tilde{\psi}_s} \underset{n\to\infty}{\longrightarrow} 0, \tag{11}$$

recalling that β_γ is the unique invariant measure of $(Q_{0,n\gamma})_{n\in\mathbb{Z}_+}$ satisfying $\beta_\gamma(\tilde{\psi}_0) < +\infty$.

Without loss of generality, let us assume that $\bigcap_{l\geq 0} E_{s+l\gamma}\cap F_s \neq \emptyset$. Then, by Definition 1, for any $n \geq 0$,

$$\|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot] - \delta_{x_s} Q_{s,s+n\gamma}[\tilde{\psi}_s\times\cdot]\|_{TV} \underset{k\to\infty}{\longrightarrow} 0.,$$

which entails using (11) that

$$\lim_{n\to\infty}\limsup_{k\to\infty} \|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s\times\cdot]\|_{TV} = 0. \tag{12}$$

Now, by Markov property, (6), and using (5), one obtains that, for any $k, n \in \mathbb{N}$ and $x \in \bigcap_{l\geq 0} E_{s+l\gamma}$,

$$\begin{aligned}\|\delta_x P_{s,s+(k+n)\gamma} - \delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{\psi_{s+(k+n)\gamma}} &= \|\left(\delta_x P_{s,s+k\gamma}\right) P_{s+k\gamma,s+(k+n)\gamma} - \delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{\psi_{s+(k+n)\gamma}} \\ &\leq C'[P_{s,s+k\gamma}\psi_{s+k\gamma}(x) + \psi_{s+k\gamma}(x)]e^{-\kappa\gamma n} \\ &\leq C''[\psi_s(x) + \psi_{s+k\gamma}(x)]e^{-\kappa\gamma n},\end{aligned}$$

where $C'' := C'\left(C\left(1 + \frac{C}{1-\theta}\right)\wedge 1\right)$. Then, for any $k, n \in \mathbb{N}$,

$$\begin{aligned}&\|\delta_{x_s} P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s\times\cdot]\|_{TV} \\ &\qquad\leq C''[\psi_s(x) + \psi_{s+k\gamma}(x)]e^{-\kappa\gamma n} + \|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s\times\cdot]\|_{TV},\end{aligned} \tag{13}$$

which entails, by (12) and the pointwise convergence of $(\psi_{s+k\gamma})_{k\in\mathbb{Z}_+}$, that

$$\lim_{n\to\infty}\|\delta_{x_s} P_{s,s+n\gamma}[\psi_{s+n\gamma}\times\cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s\times\cdot]\|_{TV} = \lim_{n\to\infty}\limsup_{k\to\infty}\|\delta_{x_s} P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s\times\cdot]\|_{TV} = 0. \tag{14}$$

The weak ergodicity (6) implies therefore that the following convergence actually holds for any initial distribution $\mu \in \mathcal{M}_1(E_0)$ satisfying $\mu(\psi_0) < +\infty$, so that

$$\left\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma}\times\cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s\times\cdot]\right\|_{TV} \underset{n\to\infty}{\longrightarrow} 0. \tag{15}$$

In particular, by Lebesgue's dominated convergence theorem, for any μ satisfying $\mu(\psi_0) < +\infty$,

$$\left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \underset{n\to\infty}{\longrightarrow} 0.$$

By Cesaro's lemma, this allows to conclude that, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\begin{aligned}
&\left\| \frac{1}{t} \int_0^t \mu P_{0,s}[\psi_s \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \\
&\leq \frac{1}{\lfloor \frac{t}{\gamma} \rfloor} \sum_{k=0}^{\lfloor \frac{t}{\gamma} \rfloor} \left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} + \left\| \frac{1}{t} \int_{\lfloor \frac{t}{\gamma} \rfloor \gamma}^t \mu P_{0,s}[\psi_s \times \cdot] ds \right\|_{TV} \underset{t\to\infty}{\longrightarrow} 0,
\end{aligned}$$

which concludes the proof of (7).

Second step. In the same way, one can show that, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s} ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} ds \right\|_{TV} \underset{t\to\infty}{\longrightarrow} 0. \tag{16}$$

As a matter of fact, for any function f bounded by 1 and $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\begin{aligned}
&\left| \mu P_{0,s+n\gamma} \left[\psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \\
&\quad \leq \left| \mu P_{0,s+n\gamma} \left[\psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] \right| + \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \\
&\quad \leq \left\| \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] \right\|_{TV} + \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right|.
\end{aligned}$$

Now, remark that, since $\psi_{s+n\gamma} \geq 1$ for any s and $n \in \mathbb{Z}_+$, one obtains that

$$\left| \frac{\tilde{\psi}_s}{\psi_{s+n\gamma}} - 1 \right| \leq 1 + \tilde{\psi}_s.$$

Then, since $(\psi_{s+n\gamma})_{n\in\mathbb{Z}_+}$ converges pointwisely towards $\tilde{\psi}_s$ and $\beta_\gamma Q_{0,s}\tilde{\psi}_s < +\infty$, Lebesgue's dominated convergence theorem entails that

$$\sup_{f\in\mathcal{B}_1(E)} \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \underset{n\to\infty}{\longrightarrow} 0.$$

Then, using now (15), one has

$$\|\mu P_{0,s+n\gamma} - \beta_\gamma Q_{0,s}\|_{TV} \underset{n\to\infty}{\longrightarrow} 0,$$

which allows to conclude in (16), using the same argument as in the first step.

Third step. In order to show the $\mathbb{L}^2$-ergodic theorem, let $f \in \mathcal{B}(E)$. For any $x \in E_0$ and $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}_{0,x} \left[\left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[\frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \\
&\qquad = \frac{2}{t^2} \int_0^t \int_s^t \left(\mathbb{E}_{0,x}[f(X_s) f(X_u)] - \mathbb{E}_{0,x}[f(X_s)] \mathbb{E}_{0,x}[f(X_u)] \right) du ds \\
&\qquad\quad = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} \left[f(X_s) \left(f(X_u) - \mathbb{E}_{0,x}[f(X_u)] \right) \right] du ds \\
&\qquad\qquad = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} \left[f(X_s) \left(\mathbb{E}_{s,X_s}[f(X_u)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_u)] \right) \right] du ds,
\end{aligned}$$

where Markov property was used at the last line. By the weak ergodicity (6) and (5), one obtains for any $s \leq t$,

$$\left|\mathbb{E}_{s,X_s}[f(X_t)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_t)]\right| \leq C''\|f\|_\infty[\psi_s(X_s)+\psi_0(x)]e^{-\kappa(t-s)}, \quad \mathbb{P}_{0,x}-\text{almost surely,} \tag{17}$$

where C'' was defined in the first part. As a result, for any $x \in E_0$ and $t \geq 0$,

$$\begin{aligned}\mathbb{E}_{0,x}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \mathbb{E}_{0,x}\left[\frac{1}{t}\int_0^t f(X_s)ds\right]\right|^2\right] &\leq \frac{2C''\|f\|_\infty}{t^2}\int_0^t\int_s^t \mathbb{E}_{0,x}[|f(X_s)|(\psi_s(X_s)+\psi_0(x))]e^{-\kappa(u-s)}duds \\ &= \frac{2C''\|f\|_\infty}{t^2}\int_0^t \mathbb{E}_{0,x}[|f(X_s)|(\psi_s(X_s)+\psi_0(x))]e^{\kappa s}\frac{e^{-\kappa s}-e^{-\kappa t}}{\kappa}ds \\ = \frac{2C''\|f\|_\infty}{\kappa t}\times\mathbb{E}_{0,x}\left[\frac{1}{t}\int_0^t |f(X_s)|(\psi_s(X_s)+\psi_0(x))ds\right] &- \frac{2C''\|f\|_\infty e^{-\kappa t}}{\kappa t^2}\int_0^t e^{\kappa s}\mathbb{E}_{0,x}[|f(X_s)|(\psi_s(X_s)+\psi_0(x))]ds.\end{aligned}$$

Then, by (7) and (16), there exists a constant $\tilde{C} > 0$ such that, for any $x \in E_0$, when $t \to \infty$,

$$\begin{aligned}&\mathbb{E}_{0,x}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \mathbb{E}_{0,x}\left[\frac{1}{t}\int_0^t f(X_s)ds\right]\right|^2\right] \\ &\qquad \leq \frac{2\tilde{C}\|f\|_\infty}{\kappa t}\times\left(\frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}[|f|\psi_s]ds + \frac{\psi_0(x)}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}[|f|]ds\right) + o\left(\frac{1}{t}\right). \end{aligned}\tag{18}$$

Then, using (16), one deduces that, for any $x \in E_0$ and bounded function f,

$$\mathbb{E}_{0,x}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}fds\right|^2\right] \underset{t\to\infty}{\longrightarrow} 0.$$

The convergence for any probability measure $\mu \in \mathcal{M}_1(E_0)$ comes from Lebesgue's dominated convergence theorem.

Fourth step. Now, let us fix $f \in \mathcal{B}(E)$, assumed nonnegative on E, and $\mu \in \mathcal{M}_1(E_0)$ satisfying $\mu(\psi_0) < +\infty$. The following proof is inspired from the proof of [26, Theorem 12].

Since $\mu(\psi_0) < +\infty$, the inequality (18) entails that there exists a finite constant $C_{f,\mu} \in (0,\infty)$ such that, for any $t > 0$,

$$\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \mathbb{E}_{0,\mu}\left[\frac{1}{t}\int_0^t f(X_s)ds\right]\right|^2\right] \leq \frac{C_{f,\mu}}{t}.$$

Then, for any $n \in \mathbb{N}$,

$$\mathbb{E}_{0,\mu}\left[\left|\frac{1}{n^2}\int_0^{n^2} f(X_s)ds - \mathbb{E}_{0,\mu}\left[\frac{1}{n^2}\int_0^{n^2} f(X_s)ds\right]\right|^2\right] \leq \frac{C_{f,\mu}}{n^2}.$$

Then, by Bienayme-Tchebychev's inequality and Borel-Cantelli's lemma, this last inequality entails that

$$\left|\frac{1}{n^2}\int_0^{n^2} f(X_s)ds - \mathbb{E}_{0,\mu}\left[\frac{1}{n^2}\int_0^{n^2} f(X_s)ds\right]\right| \underset{n\to\infty}{\longrightarrow} 0,, \quad \mathbb{P}_{0,\mu}-\text{almost surely,}$$

and, by the convergence (16), one obtains

$$\frac{1}{n^2}\int_0^{n^2} f(X_s)ds \underset{n\to\infty}{\longrightarrow} \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}fds, \quad \mathbb{P}_{0,\mu}-\text{almost surely.} \tag{19}$$

Since the nonnegativity of f is assumed, this entails that, for any $t > 0$,

$$\int_0^{\lfloor\sqrt{t}\rfloor^2} f(X_s)ds \leq \int_0^t f(X_s)ds \leq \int_0^{\lceil\sqrt{t}\rceil^2} f(X_s)ds.$$

Then, these inequalities and (19) show that

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s} f ds, \qquad \mathbb{P}_{0,\mu} - \text{almost surely.}$$

Then, in order to conclude that the following result holds actually for any bounded measurable function f, it is enough to say that, for such a function, one has the decomposition $f = f_+ - f_-$, with $f_+ := f \vee 0$ and $f_- = (-f)\vee 0$ which are positive bounded, and to apply the previous convergence to f_+ and f_-. This concludes the proof of Theorem 1. □

Proof of Corollary 1. Just remark in the previous proof that, if $\|f\|_\infty \leq 1$ and $\psi_s = \mathbb{1}$, an upper-bound for the inequality (18), which does not depend on f and x, can be obtained. Likewise, the convergence (15) holds uniformly in the initial measure due to weak ergodicity (17), which becomes uniform-in-law when $\psi = \mathbb{1}$. □

Remark 3. It seems that the theorem does not hold if the Doeblin's constant c depends on the starting time s. As proof, considering a Markov process satisfying the Doeblin's condition with time-dependant constants c_s, the weak ergodicity holds as soon as, for any $s \geq 0$,

$$\prod_{k=1}^{\left\lfloor \frac{t-s}{t_0}\right\rfloor} (1 - c_{t-kt_0}) \underset{t\to\infty}{\longrightarrow} 0,$$

since one has

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{TV} \leq 2 \prod_{k=1}^{\left\lfloor \frac{t-s}{t_0}\right\rfloor} (1 - c_{t-kt_0}).$$

Then the line (13) has to be replaced by

$$\|\delta_x P_{s,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV} \leq 2 \prod_{l=1}^{\left\lfloor \frac{n\gamma}{t_0}\right\rfloor} (1 - c_{s+(k+n)\gamma - lt_0}) + \|\delta_x P_{s+k\gamma, s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV}.$$

This does not allows to conclude in the interesting cases ($c_s \underset{s\to\infty}{\longrightarrow} 0$), since

$$\lim_{n\to\infty} \limsup_{k\to\infty} \prod_{l=1}^{\left\lfloor \frac{n\gamma}{t_0}\right\rfloor} (1 - c_{s+(k+n)\gamma - lt_0}) = 1.$$

In the same way, it seems that the constants θ and C defined in the Lyapunov/minorization condition have also to be independent on the starting times.

3 Application to quasi-stationarity with moving boundaries

In this section, consider a family of measurable subsets $(A_t)_{t\geq 0}$ of E, and denote the hitting time

$$\tau_A := \inf\{t \geq 0 : X_t \in A_t\}.$$

Assume that τ_A is a stopping time with respect to the natural filtration of the Markov process, denoted by $(\mathcal{F}_{s,t})_{s\leq t}$. Assume also that for any $x \notin A_0$,

$$\mathbb{P}_{0,x}[\tau_A < +\infty] = 1 \qquad \text{and} \qquad \mathbb{P}_{0,x}[\tau_A > t] > 0, \quad \forall t \geq 0.$$

We will be interested in a notion of *quasi-stationarity with moving boundaries*, which studies the asymptotic behavior of the Markov process $(X_t)_{t\geq 0}$ conditioned not to hit $(A_t)_{t\geq 0}$ up to the time t. For non-moving boundaries

($A_t = A_0$ for any $t \geq 0$) and time-homogeneous processes, the *quasi-limiting distribution* is defined as a probability measure α such that, for at least one initial measure μ,

$$\mathbb{P}_{0,\mu}[X_t \in \cdot | \tau_A > t] \xrightarrow[t\to\infty]{\mathcal{L}} \alpha.$$

Such a definition is actually equivalent (still in the non-moving framework) to the notion of the *quasi-stationary distribution* defined as a probability measure α such that, for any $t \geq 0$,

$$\mathbb{P}_{0,\alpha}[X_t \in \cdot | \tau_A > t] = \alpha. \tag{20}$$

If quasi-limiting and quasi-stationary distributions are generally well-defined for time-homogeneous Markov process and non-moving boundaries (see [11, 23] for a general overview on the topic), these notions could be not well-defined for time-inhomogeneous Markov processes or moving boundaries and are not equivalent anymore. In particular, under reasonable assumptions on irreducibility, it was shown in [24] that the notion of quasi-stationary distribution as defined by (20) is not well-defined for time-homogeneous Markov processes absorbed by moving boundaries.

In the theory of quasi-stationarity with moving boundaries, another asymptotic notion to study is the *quasi-ergodic distribution*, related to a conditional version of the ergodic theorem and usually defined as follows:

Definition 2. A probability measure β is a *quasi-ergodic distribution* if, for some initial measure $\mu \in \mathcal{M}_1(E \setminus A_0)$ and for any bounded continuous function f,

$$\mathbb{E}_{0,\mu}\left[\frac{1}{t}\int_0^t f(X_s)ds \middle| \tau_A > t\right] \xrightarrow[t\to\infty]{} \beta(f).$$

In the time-homogeneous setting (in particular for non-moving boundaries), this notion has been extensively studied (see for example [3, 8, 10, 12, 13, 16, 17, 18, 24]). In the *"moving boundaries"* framework, the existence of quasi-ergodic distributions has been dealt with in [24] for Markov chains on finite state spaces, and in [25] for processes satisfying a Champagnat-Villemonais condition (see Assumption (A') set further). In this last paper, the existence of the quasi-ergodic distribution is actually obtained through the following inequality holding for any initial distribution μ and for some constant $C > 0$ independant on μ:

$$\left\| \frac{1}{t}\int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t]ds - \frac{1}{t}\int_0^t \mathbb{Q}_{0,\eta_0 * \mu}[X_s \in \cdot]ds \right\|_{TV} \leq \frac{C}{t}, \quad \forall t > 0, \tag{21}$$

where

- the family of probability measure $(\mathbb{Q}_{s,x})_{s\geq 0, x\in E_s}$ is defined as follows:

$$\mathbb{Q}_{s,x}[\Gamma] := \lim_{T\to\infty} \mathbb{P}_{s,x}[\Gamma | \tau_A > T], \quad \forall s \geq 0, x \notin A_s, \Gamma \in \mathcal{F}_{s,t},$$

- η_0 is a function on E_0
- and $\eta_0 * \mu(dx) := \frac{\eta_0(x)\mu(dx)}{\mu(\eta_0)}$.

In particular, the law of $(X_t)_{t\geq 0}$ under $(\mathbb{Q}_{s,x})_{s\geq 0, x\in E_s}$, called *Q-process*, is a time-inhomogeneous Markov process and there exists a unique quasi-ergodic distribution for the process $(X_t)_{t\geq 0}$ absorbed by $(A_t)_{t\geq 0}$ if and only if the probability measures $\frac{1}{t}\int_0^t \mathbb{Q}_{0,\eta_0*\mu}[X_s \in \cdot]ds$ converge weakly, when t goes to infinity. Hence, if one shows that the Q-process is asymptotically periodic, according to Definition 1, then Theorem 1 entails the existence of a quasi-ergodic distribution.

For the following statement, the following assumption is needed:

Assumption (A'). *there exists a family of probability measures $(\nu_t)_{t\geq 0}$, defined on $E \setminus A_t$ for each t, such that*

(A'1) there exists $t_0 \geq 0$ and $c_1 > 0$ such that

$$\mathbb{P}_{s,x}[X_{s+t_0} \in \cdot | \tau_A > s + t_0] \geq c_1 \nu_{s+t_0}, \quad \forall s \geq 0, \forall x \notin A_s,$$

(A'2) and there exists $c_2 > 0$ such that

$$\mathbb{P}_{s,\nu_s}[\tau_A > t] \geq c_2 \mathbb{P}_{s,x}[\tau_A > t], \quad \forall s \leq t, \forall x \notin A_s.$$

These assumptions are the main assumptions on which relies the results of [25] and are an analogous version of the Doeblin's condition (10). When these assumptions are written in the time-homogeneous framework, one obtains the Champagnat-Villemonais condition defined in [5] (see Assumption (A)), shown as being equivalent to the exponential uniform-in-law convergence to quasi-stationarity in total variation.

Now, let us state the following theorem:

Theorem 2. *Assume that there exists a γ-periodic sequence of subsets $(B_t)_{t\geq 0}$ such that, for any $s \in [0, \gamma)$,*

$$E'_s := E \setminus \bigcap_{k\in\mathbb{Z}_+} \bigcup_{l\geq k} A_{s+l\gamma} \cup B_s \neq \emptyset$$

and there exists $x_s \in E_s$ such that, for any $n \leq N$,

$$\|\mathbb{P}_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s,x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \underset{k\to\infty}{\longrightarrow} 0. \tag{22}$$

Assume also that Assumption (A') are satisfied for the Markov process $(X_t)_{t\geq 0}$ considered as absorbed either by $(A_t)_{t\geq 0}$ or by $(B_t)_{t\geq 0}$.

Then there exists a probability measure $\beta \in \mathcal{M}_1(E)$ such that

$$\sup_{\mu\in\mathcal{M}_1(E\setminus A_0)} \sup_{f\in\mathcal{B}_1(E)} \mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \beta(f)\right|^2 \middle| \tau_A > t\right] \underset{t\to\infty}{\longrightarrow} 0. \tag{23}$$

Remark 4. Remark that the condition (22) implies that, for any $n \in \mathbb{Z}_+$,

$$\mathbb{P}_{s+k\gamma,x_s}[\tau_A > s + (k+n)\gamma] \underset{k\to\infty}{\longrightarrow} \mathbb{P}_{s,x_s}[\tau_B > s + n\gamma].$$

However, under the additional condition $B_t \subset A_t$ for all $t \geq 0$, these two conditions are actually equivalent, since forall $n \leq N$,

$$\begin{aligned}\|\mathbb{P}_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s,x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \\ \leq |\mathbb{P}_{s+k\gamma,x_s}[\tau_A > s + (k+N)\gamma] - \mathbb{P}_{s,x_s}[\tau_B > s + N\gamma]|.\end{aligned}$$

This entails the following corollary:

Corollary 2. *Assume that there exists a γ-periodic sequence of subsets $(B_t)_{t\geq 0}$, with $B_t \subset A_t$ for all $t \geq 0$, such that, for any $s \in [0, \gamma)$, there exists $x_s \in E'_s$ such that, for any $n \leq N$,*

$$\mathbb{P}_{s+k\gamma,x_s}[\tau_A > s + (k+n)\gamma] \underset{k\to\infty}{\longrightarrow} \mathbb{P}_{s,x_s}[\tau_B > s + n\gamma].$$

Assume also that Assumption (A') are satisfied for the Markov process $(X_t)_{t\geq 0}$ considered as absorbed either by $(A_t)_{t\geq 0}$ or by $(B_t)_{t\geq 0}$.

Then there exists $\beta \in \mathcal{M}_1(E)$ such that (23) holds.

Proof of Theorem 2. Since $(X_t)_{t\geq 0}$ absorbed by $(B_t)_{t\geq 0}$ satisfies Assumption (A'), we already know by [25] that, for any initial distribution μ, $t \mapsto \frac{1}{t}\int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_B > t]ds$ converges in total variation to a quasi-ergodic distribution β.

The main idea of this proof is to apply Corollary 1. Since $(X_t)_{t\geq 0}$, absorbed by A or B, satisfies Assumption (A'), then [9, Theorem 3.3.] entails that there exist two families of probability measures $(\mathbb{Q}^A_{s,x})_{s\geq 0, x\notin A_s}$ and $(\mathbb{Q}^B_{s,x})_{s\geq 0, x\notin B_s}$ such that, for any $s \leq t$, $x \notin A_s$, $y \notin B_s$ and $\Gamma \in \mathcal{F}_{s,t}$,

$$\mathbb{Q}^A_{s,x}[\Gamma] = \lim_{T\to\infty} \mathbb{P}_{s,x}[\Gamma | \tau_A > T], \quad \text{and} \quad \mathbb{Q}^B_{s,y}[\Gamma] = \lim_{T\to\infty} \mathbb{P}_{s,y}[\Gamma | \tau_B > T].$$

In particular, the quasi-ergodic distribution β is actually the limit of $t \mapsto \frac{1}{t}\int_0^t \mathbb{Q}^B_{0,\mu}[X_s \in \cdot]ds$, when t goes to infinity. Also, by [25, Theorem 1], there exists a constant $C > 0$ and $\kappa > 0$ such that, for any $s \leq t \leq T$, for any $x \notin A_s$,

$$\|\mathbb{Q}^A_{s,x}[X_t \in \cdot] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_A > T]\|_{TV} \leq Ce^{-\kappa(T-t)},$$

and for any $x \notin B_s$,

$$\|\mathbb{Q}^B_{s,x}[X_t \in \cdot] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_B > T]\|_{TV} \leq Ce^{-\kappa(T-t)}.$$

Moreover, for any $s \leq t \leq T$ and $x \in E'_s$,

$$\begin{aligned}
&\|\mathbb{P}_{s,x}[X_t \in \cdot | \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_B > T]\|_{TV} \\
&\qquad = \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_A > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\qquad = \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T)}{\mathbb{P}_{s,x}(\tau_A > T)} \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\qquad \leq \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T)}{\mathbb{P}_{s,x}(\tau_A > T)} \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\qquad\quad + \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\qquad \leq \left| \frac{\mathbb{P}_{s,x}(\tau_B > T)}{\mathbb{P}_{s,x}(\tau_A > T)} - 1 \right| \frac{\mathbb{P}_{s,x}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_B > T)} + \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}.
\end{aligned} \tag{24}$$

Then, one obtains, for any $s \leq t \leq T$ and $x \in E'_s$,

$$\begin{aligned}
&\|\mathbb{Q}^A_{s,x}[X_t \in \cdot] - \mathbb{Q}^B_{s,x}[X_t \in \cdot]\|_{TV} \\
&\qquad \leq 2Ce^{-\kappa(T-t)} + \left| \frac{\mathbb{P}_{s,x}(\tau_B > T)}{\mathbb{P}_{s,x}(\tau_A > T)} - 1 \right| \frac{\mathbb{P}_{s,x}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_B > T)} + \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}.
\end{aligned}$$

The condition (22) implies that, for any $n \leq N$,

$$\begin{aligned}
&\left| \frac{\mathbb{P}_{s+k\gamma,x_s}(\tau_B > s + (k+N)\gamma)}{\mathbb{P}_{s+k\gamma,x_s}(\tau_A > s + (k+N)\gamma)} - 1 \right| \frac{\mathbb{P}_{s+k\gamma,x_s}(\tau_A > s + (k+N)\gamma)}{\mathbb{P}_{s+k\gamma,x_s}(\tau_B > s + (k+N)\gamma)} \\
&\qquad + \frac{\|\mathbb{P}_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s,x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV}}{\mathbb{P}_{s,x_s}[\tau_B > s + N\gamma]} \underset{k\to\infty}{\longrightarrow} 0,
\end{aligned}$$

which implies that, for any $n \leq N$,

$$\limsup_{k\to\infty} \|\mathbb{Q}^A_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}^B_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot]\|_{TV} \leq 2Ce^{-\kappa\gamma(N-n)}.$$

Now, letting $N \to \infty$, one has for any $n \in \mathbb{Z}_+$

$$\limsup_{k\to\infty} \|\mathbb{Q}^A_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}^B_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot]\|_{TV} = 0.$$

In other words, the Markov process $\{(X_t)_{t\geq 0}, (\mathbb{Q}^A_{s,x})_{s\geq 0, x\notin A_s}\}$ is asymptotically periodic in total variation distance, with $\{(X_t)_{t\geq 0}, (\mathbb{Q}^B_{s,x})_{s\geq 0, x\notin B_s}\}$ as auxiliary Markov process. Moreover, Assumptions (A') satisfied for $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ entail that Doeblin's conditions hold for these two Q-processes. Then, by Corollary 1 and using (21), one obtains that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t]ds = \lim_{t\to\infty} \int_0^t \mathbb{Q}^B_{0,\eta_0 * \mu}[X_s \in \cdot]ds = \beta,$$

where the limits refer to the convergence in total variation and hold uniformly in the initial measure. Now, for any $\mu \in \mathcal{M}_1(E \setminus A_0)$, $f \in \mathcal{B}_1(E)$ and $t \geq 0$,

$$\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds\right|^2 \middle| \tau_A > t\right] = \frac{2}{t^2}\int_0^t \int_s^t \mathbb{E}_{0,\mu}[f(X_s)f(X_u)|\tau_A > t]duds.$$

Then, by [25, Theorem 1], for any $s \leq u \leq t$, for any $\mu \in \mathcal{M}_1(E \setminus A_0)$ and $f \in \mathcal{B}(E)$,

$$\left|\mathbb{E}_{0,\mu}[f(X_s)f(X_u)|\tau_A > t] - \mathbb{E}^{\mathbb{Q}^A}_{0,\eta_0 * \mu}[f(X_s)f(X_u)]\right| \leq C\|f\|_\infty e^{-\kappa(t-u)},$$

where the expectation $\mathbb{E}^{\mathbb{Q}^A}_{0,\eta_0*\mu}$ is associated to the probability measure $\mathbb{Q}^A_{0,\eta_0*\mu}$. Hence, for any $\mu \in \mathcal{M}_1(E \setminus A_0)$, $f \in \mathcal{B}_1(E)$ and $t > 0$,

$$\begin{aligned}\left|\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \beta(f)\right|^2 \middle| \tau_A > t\right] - \mathbb{E}^{\mathbb{Q}^A}_{0,\eta_0*\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \beta(f)\right|^2\right]\right| &\leq \frac{4C}{t^2}\int_0^t\int_s^t e^{-\kappa(t-u)}duds \\ &\leq \frac{4C}{\kappa t} - \frac{4C(1-e^{-\kappa t})}{\kappa^2 t^2}.\end{aligned}$$

Moreover, Corollary 1 entails that, for any $\mu \in \mathcal{M}_1(E \setminus A_0)$,

$$\sup_{\mu \in \mathcal{M}_1(E\setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}^{\mathbb{Q}^A}_{0,\eta_0*\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \beta(f)\right|^2\right] \underset{t\to\infty}{\longrightarrow} 0.$$

Then, this implies that

$$\sup_{\mu \in \mathcal{M}_1(E\setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \beta(f)\right|^2 \middle| \tau_A > t\right] \underset{t\to\infty}{\longrightarrow} 0.$$

□

Remark 5. It seems that Assumption (A') can be weaken by a conditional version of the Lyapunov/minorization conditions. In particular, such conditions can be derived from the Assumption (F) presented in [6], as it will be shown later by the preprint [4] in preparation.

4 Examples

4.1 Asymptotical periodic Ornstein-Uhlenbeck processes

Let $(X_t)_{t\geq 0}$ be a time-inhomogeneous diffusion process on $\mathbb{R}$ satisfying the following stochastic differential equation

$$dX_t = dW_t - \lambda(t)X_t dt,$$

where $(W_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $\lambda : [0,\infty) \to [0,\infty)$ is a function such that

$$0 < \inf_{t\geq 0}\lambda(t) \leq \sup_{t\geq 0}\lambda(t) < +\infty.$$

By Itô's formula, one obtains for any $s \leq t$,

$$X_t = e^{-\int_s^t \lambda(u)du}\left[X_s + \int_s^t e^{\int_s^u \lambda(v)dv}dW_u\right].$$

In particular, denoting $(P_{s,t})_{s\leq t}$ the semi-group associated to $(X_t)_{t\geq 0}$, for any $f \in \mathcal{B}(\mathbb{R})$, $t \geq 0$ and $x \in \mathbb{R}$,

$$P_{s,t}f(x) = \mathbb{E}\left[f\left(e^{-\int_s^t \lambda(u)du}x + e^{-\int_s^t \lambda(u)du}\sqrt{\int_s^t e^{2\int_s^u \lambda(v)dv}du} \times \mathcal{N}(0,1)\right)\right],$$

where $\mathcal{N}(0,1)$ denotes a standard Gaussian variable.

Theorem 3. *Assume that there exists a γ-periodic function g, positive and bounded on $\mathbb{R}$, such that $\lambda \sim_{t\to\infty} g$. Then the assumptions of Theorem 1 hold.*

Proof. In our case, the auxiliary semi-group $(Q_{s,t})_{s\le t}$ of Definition 1 will be defined as follows: for any $f \in \mathcal{B}(\mathbb{R})$, $t \ge 0$ and $x \in \mathbb{R}$,

$$Q_{s,t}f(x) = \mathbb{E}\left[f\left(e^{-\int_s^t g(u)du} x + e^{-\int_s^t g(u)du} \sqrt{\int_s^t e^{2\int_s^u g(v)dv} du} \times \mathcal{N}(0,1) \right) \right].$$

In particular, the semi-group $(Q_{s,t})_{s\le t}$ is associated to the process $(Y_t)_{t\ge 0}$ defined as following

$$dY_t = dW_t - g(t)Y_t dt.$$

First of all, one remarks that the function $\psi : x \mapsto 1 + x^2$ is a Lyapunov function for $(P_{s,t})_{s\le t}$ and $(Q_{s,t})_{s\le t}$. As a matter of fact, for any $s \ge 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} P_{s,s+\gamma}\psi(x) &= 1 + e^{-2\int_s^{s+\gamma}\lambda(u)du} x^2 + e^{-2\int_s^{s+\gamma}\lambda(u)du} \int_s^{s+\gamma} e^{2\int_s^u \lambda(v)dv} du \\ &= e^{-2\int_s^{s+\gamma}\lambda(u)du}\psi(x) + 1 - e^{-2\int_s^{s+\gamma}\lambda(u)du} + e^{-2\int_s^{s+\gamma}\lambda(u)du} \int_s^{s+\gamma} e^{2\int_s^u \lambda(v)dv} du \\ &\le e^{-2\gamma c_{\inf}}\psi(x) + C, \end{aligned}$$

for some constant $C > 0$, and where $c_{\inf} := \inf_{t\ge 0}\lambda(t) > 0$. Taking $\theta \in (e^{-2\gamma c_{\inf}}, 1)$, there exists a compact set K such that, for any $s \ge 0$,

$$P_{s,s+\gamma}\psi(x) \le \theta\psi(x) + C\mathbb{1}_K.$$

Also, it is simple to check that, for any $s \ge 0$ and $t \in [0,\gamma)$, the function $P_{s,s+t}\psi/\psi$ is upper-bounded uniformly in s and t. Now, the local Doeblin's condition for $(P_{s,t})_{s\le t}$ is actually a consequence of the following lemma:

Lemma 1. *For any $a, b_-, b_+ > 0$, define the subset $\mathcal{C}(a, b_-, b_+) \subset \mathcal{M}_1(\mathbb{R})$ as*

$$\mathcal{C}(a, b_-, b_+) := \{\mathcal{N}(m, \sigma) : m \in [-a, a], \sigma \in [b_-, b_+]\}.$$

Then, for any $a, b_-, b_+ > 0$, there exists a probability measure ν and a constant $c > 0$ such that, for any $\mu \in \mathcal{C}(a, b_-, b_+)$,

$$\mu \ge c\nu.$$

The proof of this lemma is postponed after the end of this proof. Note that the condition $\lambda \sim_{t\to\infty} g$ and the fact that λ is bounded on $\mathbb{R}$ entails by Lebesgue's dominated convergence theorem that, for all $s \le t$,

$$\left| \int_{s+k\gamma}^{t+k\gamma} \lambda(u)du - \int_s^t g(u)du \right| \underset{k\to\infty}{\longrightarrow} 0,$$

and, in the same way,

$$\int_{s+k\gamma}^{t+k\gamma} e^{2\int_{s+k\gamma}^u \lambda(v)dv} du \underset{k\to\infty}{\longrightarrow} \int_s^t e^{2\int_s^u g(v)dv} du.$$

Then, for any $s \le t$,

$$e^{-\int_{s+k\gamma}^{t+k\gamma}\lambda(u)du} \underset{k\to\infty}{\longrightarrow} e^{-\int_s^t g(u)du},$$

and

$$e^{-\int_{s+k\gamma}^{t+k\gamma}\lambda(u)du} \sqrt{\int_{s+k\gamma}^{t+k\gamma} e^{2\int_{s+k\gamma}^u \lambda(v)dv} du} \underset{k\to\infty}{\longrightarrow} e^{-\int_s^t g(u)du} \sqrt{\int_s^t e^{2\int_s^u g(v)dv} du}.$$

Using [14, Theorem 1.3.], one obtains, for any $x \in \mathbb{R}$,

$$\|\delta_x P_{s+k\gamma,t+k\gamma} - \delta_x Q_{s+k\gamma,t+k\gamma}\|_{TV} \underset{k\to\infty}{\longrightarrow} 0. \tag{25}$$

To deduce the convergence in ψ-distance, we will inspire from the proof of [19, Lemma 3.1]. The variances being bounded uniformly in k (for $s \leq t$ fixed), then there exists $H > 0$ such that, for any $k \in \mathbb{N}$,

$$\delta_x P_{s+k\gamma,t+k\gamma}[\psi^2] \leq H, \quad \delta_x Q_{s,t}[\psi^2] \leq H. \tag{26}$$

Then, since $\lim_{|x|\to\infty} \frac{\psi(x)}{\psi^2(x)} = 0$, for any $\epsilon > 0$, there exists $l_\epsilon > 0$ such that, for any function f such that $|f| \leq \psi$ and for any $|x| \geq l_\epsilon$,

$$|f(x)| \leq \frac{\epsilon\psi(x)^2}{H}.$$

This implies with (26) that, denoting $K_\epsilon := [-l_\epsilon, l_\epsilon]$, for any $k \in \mathbb{Z}_+$, f such that $|f| \leq \psi$ and $x \in \mathbb{R}$,

$$\delta_x P_{s+k\gamma,t+k\gamma}[f\mathbb{1}_{K_\epsilon^c}] \leq \epsilon, \quad \delta_x Q_{s,t}[f\mathbb{1}_{K_\epsilon^c}] \leq \epsilon.$$

Then, for any $k \in \mathbb{Z}_+$ and f such that $|f| \leq \psi$,

$$\begin{aligned} |\delta_x P_{s+k\gamma,t+k\gamma} f - \delta_x Q_{s,t} f| &\leq 2\epsilon + |\delta_x P_{s+k\gamma,t+k\gamma}[f\mathbb{1}_{K_\epsilon}] - \delta_x Q_{s,t}[f\mathbb{1}_{K_\epsilon}]| &(27)\\ &\leq 2\epsilon + (1+l_\epsilon^2)\|\delta_x P_{s+k\gamma,t+k\gamma} - \delta_x Q_{s,t}\|_{TV} &(28)\end{aligned}$$

Hence, (25) entails that, for k large enough, for any f bounded by ψ,

$$|\delta_x P_{s+k\gamma,t+k\gamma} f - \delta_x Q_{s,t} f| \leq 3\epsilon, \tag{29}$$

which entails that

$$\|\delta_x P_{s+k\gamma,t+k\gamma} - \delta_x Q_{s,t}\|_\psi \underset{k\to\infty}{\longrightarrow} 0.$$

□

Now, let us prove Lemma 1.

Proof of Lemma 1. Defining

$$f_\nu(x) := e^{-\frac{(x-a)^2}{2b_-^2}} \wedge e^{-\frac{(x+a)^2}{2b_-^2}},$$

one concludes easily that, for any $m \in [-a, a]$ and $\sigma \geq b_-$, for any $x \in \mathbb{R}$,

$$e^{-\frac{(x-m)^2}{2\sigma^2}} \geq f_\nu(x).$$

Imposing moreover that $\sigma \leq b_+$, one has

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \geq \frac{1}{\sqrt{2\pi}b_+} f_\nu(x),$$

which concludes the proof. □

4.2 Quasi-ergodic distribution for Brownian motion absorbed by an asymptotically periodic moving boundary

Let $(W_t)_{t\geq 0}$ be a one-dimensional Brownian motion and h be a $\mathcal{C}^1$-function such that

$$h_{\min} := \inf_{t\geq 0} h(t) > 0, \quad \text{and} \quad \sup_{t\geq 0} h(t) < +\infty.$$

Assume also that

$$0 < \inf_{t\geq 0} h'(t) \leq \sup_{t\geq 0} h'(t) < +\infty.$$

Denote by

$$\tau_h := \inf\{t \geq 0 : |W_t| \geq h(t)\}.$$

Since h is continuous, the hitting time τ_h is a stopping time with respect to the natural filtration of $(W_t)_{t\geq 0}$. Moreover, since $\sup_{t\geq 0} h(t) < +\infty$ and $\inf_{t\geq 0} h(t) > 0$,

$$\mathbb{P}_{s,x}[\tau_h < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{s,x}[\tau_h > t] > 0, \quad \forall s \leq t, \forall x \in [-h(s), h(s)].$$

The main assumption on the function h is the existence of a γ-periodic function g such that $h(t) \leq g(t)$, for any $t \geq 0$, and such that

$$h \sim_{t\to\infty} g, \quad \text{and} \quad h' \sim_{t\to\infty} g'.$$

Similarly to τ_h, denote

$$\tau_g := \inf\{t \geq 0 : |W_t| = g(t)\}.$$

Finally, let us assume that there exists $n_0 \in \mathbb{N}$ such that, for any $s \geq 0$,

$$\inf\{u \geq s : h(u) = \inf_{t\geq s} h(t)\} - h(s) \leq n_0\gamma. \tag{30}$$

Then, first of all, let us show the following proposition:

Proposition 1. *The Markov process $(W_t)_{t\geq 0}$, considered as absorbed by h or by g, follows the condition (A'1)-(A'2).*

Proof. In what follows, Assumption (A') w.r.t. the absorbing function h will be shown. The following proof could be easily adapt for the function g.

- *Proof of (A'1).* Denote $\mathcal{T} := \{s \geq 0 : h(s) = \inf_{t\geq s} h(t)\}$. The condition (30) entails that this set contains an infinity of times.

  In what follows, the following notation is needed: for any $z \in \mathbb{R}$, define τ_z as

  $$\tau_z := \inf\{t \geq 0 : |W_t| = z\}.$$

  Also, let us state that, since the Brownian motion absorbed at $\{-1, 1\}$ satisfies the Champagnat-Villemonais condition at any time (see [7]), it follows that, for a given $t_0 > 0$, there exists $c > 0$ and $\nu \in \mathcal{M}_1((-1, 1))$ such that, for any $x \in (-1, 1)$,

  $$\mathbb{P}_{0,x}\left[W_{\frac{t_0}{h^2_{\max}}\wedge t_0} \in \cdot \middle| \tau_1 > \frac{t_0}{h^2_{\max}} \wedge t_0\right] \geq c\nu. \tag{31}$$

  Moreover, regarding the proof of [7, Thereom 3.1.], the probability measure ν can be expressed as

  $$\nu = \frac{1}{2}\left(\mathbb{P}_{0,1-\epsilon}[W_{t_2} \in \cdot | \tau_1 > t_2] + \mathbb{P}_{0,-1+\epsilon}[W_{t_2} \in \cdot | \tau_1 > t_2]\right), \tag{32}$$

  for some $0 < t_2 < \frac{t_0}{h^2_{\max}} \wedge t_0$ and $\epsilon \in (0, 1)$.

  The following lemma is very important for the following:

  Lemma 2. *For all $z \in [h_{\min}, h_{max}]$,*

  $$\mathbb{P}_{0,x}[W_u \in \cdot | \tau_z > u] \geq c\nu_z, \quad \forall x \in (-z, z), \forall u \geq t_0,$$

  where t_0 was evoked before, $c > 0$ is the same constant as in (31) and

  $$\nu_z(f) = \int_{(-1,1)} f(zx)\nu(dx),$$

  with $\nu \in \mathcal{M}_1((-1, 1))$ defined in (32).

The proof of this lemma is postponed after the current proof.
At first, let $s \in \mathcal{T}$. Then, for any $x \in (-h(s), h(s))$, for any $t \geq 0$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s+t] \geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} \mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_{h(s)} > s+t],$$

By Lemma 2, for any $x \in (-h(s), h(s))$ and $t \geq t_0$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_{h(s)} > s+t] \geq c\nu_{h(s)},$$

which entails that, for any $t \in [t_0, t_0 + n_0\gamma]$,

$$\begin{aligned}\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s+t] &\geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} c\nu_{h(s)} \\ &\geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0+n_0\gamma]}{\mathbb{P}_{s,x}[\tau_h > s+t_0]} c\nu_{h(s)}. \end{aligned} \tag{33}$$

Let us introduce the process X^h defined by, for any $t \geq 0$,

$$X^h_t := \frac{W_t}{h(t)}.$$

By Itô's formula, one has for any t,

$$X^h_t = X^h_0 + \int_0^t \frac{dW_s}{h(s)} - \int_0^t \frac{h'(s)}{h(s)} X^h_s ds.$$

Denote by $(M^h_t)_{t\geq 0} := \left(\int_0^t \frac{1}{h(s)} dW_s\right)_{t\geq 0}$. By Dubin-Schwartz's theorem, it is well known that the process M^h has the same law as $\left(W_{\int_0^t \frac{1}{h^2(s)} ds}\right)_{t\geq 0}$. Then, denoting $I^h(s) := \int_0^s \frac{1}{h^2(u)} du$ and, for any $s \leq t$ and for any trajectory w,

$$\mathcal{E}^h_{s,t}(w) := \sqrt{\frac{h(t)}{h(s)}} \exp\left(-\frac{1}{2}\left[h'(t)h(t)w^2_{I^h(t)} - h'(s)h(s)w^2_{I^h(s)} + \int_s^t w^2_{I^h(u)}[(h'(u))^2 - [h(u)h'(u)]']du\right]\right), \tag{34}$$

Girsanov's theorem entails therefore that, for any $x \in (-h(s), h(s))$,

$$\mathbb{P}_{s,x}[\tau_h > s+t_0] = \mathbb{E}_{I^h(s), \frac{x}{h(s)}}\left[\mathcal{E}^h_{s,s+t_0}(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du}\right].$$

On the event $\{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du\}$, and since h and h' are bounded on $\mathbb{R}_+$, the random variable $\mathcal{E}^h_{s,s+t_0}(W)$ is almost surely bounded by a constant $C > 0$, uniformly in s, so that for any $x \in (-h(s), h(s))$,

$$\mathbb{E}_{s, \frac{x}{h(s)}}\left[\mathcal{E}^h_{s,s+t_0}(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du}\right] \leq C\mathbb{P}_{0, \frac{x}{h(s)}}\left[\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du\right].$$

Since $h(t) \geq h(s)$ for any $t \geq s$ (recalling that $s \in \mathcal{T}$), $I^h(s+t_0) - I^h(s) \leq \frac{t_0}{h(s)^2}$. Then, by the scaling property of the Brownian motion and by Markov property, for any $x \in (-h(s), h(s))$,

$$\begin{aligned}\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0] &= \mathbb{P}_{0,x}[\tau_{h(s)} > t_0] \\ &= \mathbb{P}_{0, \frac{x}{h(s)}}\left[\tau_1 > \frac{t_0}{h^2(s)}\right] \\ &= \mathbb{E}_{0, \frac{x}{h(s)}}\left[\mathbb{1}_{\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du} \mathbb{P}_{0, W_{\int_s^{s+t_0} \frac{1}{h^2(u)} du}}\left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(s)} ds\right]\right] \\ &= \mathbb{P}_{0, \frac{x}{h(s)}}\left[\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du\right] \mathbb{P}_{0, \phi_{I^h(s+t_0) - I^h(s)}(\delta_x)}\left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du\right], \end{aligned}$$

where, for any initial distribution μ and any $t \geq 0$,

$$\phi_t(\mu) := \mathbb{P}_{0,\mu}[W_t \in \cdot | \tau_1 > t].$$

In particular, the family $(\phi_t)_{t\geq 0}$ is a semi-flow, i.e. $\phi_t \circ \phi_s = \phi_{t+s}$ for all s, t. Hence, by this property of semi-flow and using that $I^h(s+t_0) - I^h(s) \geq \frac{t_0}{h^2_{\max}}$ for any $s \geq 0$, the minorization (31) entails that, for all $s \geq 0$ and $x \in (-1, 1)$,

$$\phi_{I^h(s+t_0)-I^h(s)}(\delta_x) \geq c\nu.$$

Then, by this minorization and using that h is upper-bounded and lower-bounded positively on $\mathbb{R}_+$, one has, for any $x \in (-1, 1)$,

$$\mathbb{P}_{0,\phi_{I^h(s+t_0)-I^h(s)}(\delta_x)}\left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)}du\right] \geq c\mathbb{P}_{0,\nu}\left[\tau_1 > \inf_{s\geq 0}\left\{\frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)}du\right\}\right],$$

that is to say,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]}{\mathbb{P}_{0,\frac{x}{h(s)}}\left[\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)}du\right]} \geq c\mathbb{P}_{0,\nu}\left[\tau_1 > \inf_{s\geq 0}\left\{\frac{\gamma}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)}du\right\}\right].$$

In other words, we just showed that, for any $x \in (-h(s), h(s))$,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]}{\mathbb{P}_{s,x}[\tau_h > s + t_0]} \geq \frac{c}{C}\mathbb{P}_{0,\nu}\left[\tau_1 > \inf_{s\geq 0}\left\{\frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)}du\right\}\right] > 0. \tag{35}$$

Moreover, by Lemma 2 and the scaling property of the Brownian motion, for any $x \in (-h(s), h(s))$,

$$\begin{aligned}
\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0 + n_0\gamma]}{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]} &= \mathbb{P}_{0,\mathbb{P}_{0,x}[W_{t_0}\in\cdot|\tau_{h(s)}>t_0]}[\tau_{h(s)} > n_0\gamma] \\
&\geq c\mathbb{P}_{0,\nu_{h(s)}}[\tau_{h(s)} > n_0\gamma] \\
&= c\int_{(-1,1)} \nu(dy)\mathbb{P}_{h(s)y}[\tau_{h(s)} > n_0\gamma] \\
&\geq c\mathbb{P}_{0,\nu}\left[\tau_1 > \frac{n_0\gamma}{h^2_{\min}}\right] > 0.
\end{aligned} \tag{36}$$

To sum up, gathering (33), (35) and (36), for any $x \in (-h(s), h(s))$, for any $t \in [t_0, t_0 + n_0\gamma]$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s + t] \geq c_1\nu_{h(s)},$$

where $c_1 := c\mathbb{P}_{0,\nu}\left[\tau_1 > \frac{n_0\gamma}{h^2_{\max}}\right] \times \frac{c}{C}\mathbb{P}_{0,\nu}\left[\tau_1 > \inf_{s\geq 0}\left\{\frac{\gamma}{h^2(s)} - \int_s^{s+\gamma} \frac{1}{h^2(u)}du\right\}\right] c$.

Now, consider $s \geq 0$. If $s \notin \mathcal{T}$, there exists $s_1 \in \mathcal{T}$ such that $s < s_1 \leq s + n_0\gamma$. Then, Markov property entails that, for any $x \in (-h(s), h(s))$,

$$\mathbb{P}_{s,x}[W_{s+t_0+n_0\gamma} \in \cdot | \tau_h > s + t_0 + n_0\gamma] = \mathbb{P}_{s_1,\phi_{s_1,s}}[W_{s+t_0+n_0\gamma} \in \cdot | \tau_h > s + t_0 + n_0\gamma] \geq c_1\nu_{h(s_1)},$$

where, for any $s \leq t$ and $\mu \in \mathcal{M}_1((-h(s), h(s)))$, $\phi_{t,s}(\mu) := \mathbb{P}_{s,\mu}[W_t \in \cdot | \tau_h > t]$. This concludes the proof of (A'1).

- *Proof of (A'2).* Since $(W_t)_{t\geq 0}$ is a Brownian motion, note that, for any $s \leq t$,

$$\sup_{x\in(-1,1)} \mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,0}[\tau_h > t],$$

and that, for any $a \in (0, h(s))$,

$$\inf_{[-a,a]} \mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,a}[\tau_h > t].$$

Then, by Markov property, and using that the function $s \mapsto \mathbb{P}_{s,0}[\tau_g > t]$ is non-decreasing on $[0,t]$ (for all $t \geq 0$), then one has for any $s \leq t$,

$$\mathbb{P}_{s,a}[\tau_h > t] \geq \mathbb{E}_{s,a}[\mathbb{1}_{\tau_0 < s+\gamma < \tau_h} \mathbb{P}_{\tau_0,0}[\tau_h > t]] \geq \mathbb{P}_{s,a}[\tau_0 < s+\gamma < \tau_h]\mathbb{P}_{s,0}[\tau_h > t]. \tag{37}$$

Then, denoting $a := \frac{h_{\min}}{h_{\max}}$, by Lemma 2 and taking $s_1 := \inf\{u \geq s : u \in \mathcal{T}\}$, one obtains for any $s \leq t$,

$$\begin{aligned}
\mathbb{P}_{s,\nu_{h(s_1)}}[\tau_h > t] &= \int_{(-1,1)} \nu(dx) \mathbb{P}_{s,h(s_1)x}[\tau_h > t] \\
&\geq \nu([-a,a]) \mathbb{P}_{s,h(s_1)a}[\tau_h > t] \\
&\geq \nu([-a,a]) \mathbb{P}_{0,h_{\min}}[\tau_0 < \gamma < \tau_h] \sup_{x \in (-h(s),h(s))} \mathbb{P}_{s,x}[\tau_h > t].
\end{aligned}$$

This concludes the proof since, using (32), one has $\nu([-a,a]) > 0$.

□

Proof of Lemma 2. This result comes from the scaling property of a Brownian motion: as a matter of fact, for any $z \in [h_{\min}, h_{\max}]$, $x \in (-z,z)$ and $t \geq 0$, and for any measurable bounded function f,

$$\begin{aligned}
\mathbb{E}_{0,x}[f(W_t)|\tau_z > t] &= \mathbb{E}_{0,x}\left[f\left(z \times \frac{1}{z} W_{z^2 \frac{t}{z^2}} \right) \middle| \tau_z > t \right] \\
&= \mathbb{E}_{0,\frac{x}{z}}\left[f\left(z \times W_{\frac{t}{z^2}} \right) \middle| \tau_1 > \frac{t}{z^2} \right].
\end{aligned}$$

Then, the minorization (31) entails that, for any $x \in (-1,1)$,

$$\mathbb{P}_{0,x}\left[W_{\frac{t_0}{h_{\max}^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{h_{\max}^2} \right] \geq c\nu.$$

This inequality actually hold for any time greater than $\frac{t_0}{h_{\max}^2}$. In particular, for any $z \in [h_{\min}, h_{\max}]$, $x \in (-1,1)$,

$$\mathbb{P}_{0,x}\left[W_{\frac{t_0}{z^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{z^2} \right] \geq c\nu.$$

Then, for any $z \in [a,b]$, f positive measurable and $x \in (-z,z)$,

$$\mathbb{E}_{0,x}[f(W_{t_0})|\tau_z > t_0] \geq c\nu_z(f),$$

where $\nu_z(f) = \int_E f(z \times x)\nu(dx)$.

□

Now, the section is concluded by stating and proving the following result:

Theorem 4. *For any $s \leq t$, $n \in \mathbb{N}$ and any $x \in \mathbb{R}$,*

$$\mathbb{P}_{s+k\gamma,x}[\tau_h \leq t + k\gamma < \tau_g] \underset{k\to\infty}{\longrightarrow} 0.$$

In particular, Corollary 2 holds for $(W_t)_{t\geq 0}$ absorbed by h.

Proof. Reminding (34) and by Markov property for the Brownian motion, one obtains that, for any $k, n \in \mathbb{N}$, for any $x \in \mathbb{R}$,

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t + k\gamma] = \sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \mathbb{E}_{0,x}\left[\exp\left(-\frac{1}{2} \mathcal{A}^h_{s,t,k}(W) \right) \mathbb{1}_{\tau_1 > I^h(t+k\gamma) - I^h(s+k\gamma)} \right],$$

where, for any trajectory w,

$$\mathcal{A}^h_{s,t,k}(w) = h'(t+k\gamma)h(t+k\gamma)w^2_{I^h(t+k\gamma)-I^h(s+k\gamma)} - h'(s+k\gamma)h(s+k\gamma)w_0^2 + \int_0^{t-s} w^2_{I^h(u+s+k\gamma)-I^h(s+k\gamma)}[(h'(u+s+k\gamma))^2 - [h(u+s+k\gamma)h'(u+s+k\gamma)]']du.$$

First, since $h \sim_{t\to\infty} g$, for any $s \in [0,\gamma]$ and $n \in \mathbb{N}$,

$$\sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \underset{k\to\infty}{\longrightarrow} \sqrt{\frac{g(t)}{g(s)}}.$$

Then, for the same reasons and using that the function h is bounded on $[s+k\gamma, t+k\gamma]$ for any $s \le t$, Lebesgue's theorem entails that

$$I^h(t+k\gamma) - I^h(s+k\gamma) \underset{k\to\infty}{\longrightarrow} I^g(t) - I^g(s).$$

Moreover, since $h \sim_{t\to\infty} g$ and $h' \sim_{t\to\infty} g'$, one has the following convergence for any trajectory w:

$$\mathcal{A}^h_{s,t,k}(w) \underset{k\to\infty}{\longrightarrow} g'(t)g(t)w^2_{I^g(t)-I^g(s)} - g'(s)g(s)w_0^2 + \int_s^t w^2_{I^g(u)}[(g'(u))^2 - [g(u)g'(u)]']du.$$

Then, since the random variable $\exp\left(-\frac{1}{2}\mathcal{A}^h_{s,t,k}(W)\right)\mathbb{1}_{\tau_1 > I^h(t+k\gamma)-I^h(s+k\gamma)}$ is bounded almost surely, by Lebesgue's theorem,

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t+k\gamma] \underset{k\to\infty}{\longrightarrow} \mathbb{P}_{s,x}[\tau_g > t],$$

which concludes the proof. □

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