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# Ergodic theorem for asymptotically periodic time-inhomogeneous Markov processes, with application to quasi-stationarity with moving boundaries.

William Oçafrain<sup>1</sup>

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## Abstract

This paper deals with ergodic theorems for particular time-inhomogeneous Markov processes, whose the time-inhomogeneity is asymptotically periodic. Under a Lyapunov/minorization condition, it is shown that, for any measurable bounded function  $f$ , the time average  $\frac{1}{t} \int_0^t f(X_s) ds$  converges in  $\mathbb{L}^2$  towards a limiting distribution, starting from any initial distribution for the process  $(X_t)_{t \geq 0}$ . This convergence can be improved to an almost sure convergence under an additional assumption on the initial measure. This result will be then applied to show the existence of a quasi-ergodic distribution for processes absorbed by an asymptotically periodic moving boundary, satisfying a conditional Doeblin's condition.

*Key words: ergodic theorem; law of large numbers; time-inhomogeneous Markov processes; quasi-stationarity; quasi-ergodic distribution; moving boundaries.*

## Notation

- $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ .
- $\mathcal{M}_1(E)$ : Space of the probability measures whose the support is included in  $E$ .
- $\mathcal{B}(E)$ : Set of the measurable bounded functions defined on  $E$ .
- $\mathcal{B}_1(E)$ : Set of the measurable functions  $f$  defined on  $E$  such that  $\|f\|_\infty \leq 1$ .
- $\mathbb{L}^p(\mu)$ : Set of the measurable functions  $f$  such that  $\int |f|^p d\mu < +\infty$  ( $p \in \mathbb{N}$ ).
- For any  $\mu \in \mathcal{M}_1(E)$  and  $f \in \mathbb{L}^1(\mu)$ , denote

$$\mu(f) := \int_E f(x) \mu(dx).$$

- Id: Identity operator.

## 1 Introduction

In a general way, the ergodic theorem (for Markov processes) refers to the almost sure convergence

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \pi(f), \quad \forall f \in \mathbb{L}^1(\pi), \quad (1)$$

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$(X_t)_{t \geq 0}$  being a Markov process and  $\pi$  being a probability measure. In the time-homogeneous setting, the ergodic theorem holds for positive Harris recurrent Markov processes and the limiting distribution  $\pi$  corresponds to an invariant measure for the underlying Markov process. For time-inhomogeneous Markov processes, such a result does not hold in general (in particular the notion of invariant measure is not generally well-defined), except for specific time-inhomogeneity such as for *periodic time-inhomogeneous Markov processes*, defined as time-inhomogeneous Markov processes for which there exists  $\gamma > 0$  such that, for any  $s \leq t$ ,  $k \in \mathbb{Z}_+$  and  $x$ ,

$$\mathbb{P}[X_t \in \cdot | X_s = x] = \mathbb{P}[X_{t+k\gamma} \in \cdot | X_{s+k\gamma} = x]. \quad (2)$$

In other words, a time-inhomogeneous Markov process is periodic when the transition law between any times  $s$  and  $t$  remains unchanged when the time interval  $[s, t]$  is shifted by a multiple of the period  $\gamma$ . In particular, this implies that, for any  $s \in [0, \gamma)$ , the Markov chain  $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$  is time-homogeneous. This fact allowed to Höpfner et al. (in [20, 21, 22]) to show that, if the skeleton Markov chain  $(X_{n\gamma})_{n \in \mathbb{Z}_+}$  is Harris recurrent, then the chains  $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$ , for all  $s \in [0, \gamma)$ , are also Harris recurrent and

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \pi_s(f) ds, \quad \text{almost surely, from any initial measure,}$$

where  $\pi_s$  is the invariant measure for  $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$ .

This paper aims to prove a similar result for time-inhomogeneous Markov processes said to be *asymptotically periodic*. Roughly speaking (the definition will be explicitly given further), an asymptotically periodic Markov process is such that, given a duration  $T \geq 0$ , its transition law on the interval  $[s, s+T]$  is asymptotically "close to" the one, on the same interval, of a periodic time-inhomogeneous Markov process called *auxiliary Markov process*, when  $s \rightarrow \infty$ . This definition is very similar to the notion of *asymptotic homogenization* (when the approximating semi-group is time-homogeneous), or the notion of *asymptotic pseudotrajectories* as introduced in [2]. In particular, it is known that these rapprochements entail that, under suitable additional conditions, an asymptotically homogeneous Markov process converges towards a probability measure, which is invariant for the approximating semi-group. In the same way, it is expected that an asymptotically periodic process gets the same asymptotic properties as a periodic Markov process; in particular that an ergodic theorem holds for the asymptotically periodic process.

**Main result.** The main result of this paper states that an asymptotically periodic Markov process satisfying a Lyapunov-type condition and a local Doeblin's condition (defined further in Section 2), and such that its auxiliary process also satisfies a Lyapunov/minorization condition, satisfies

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{L^2(\mathbb{P}_{0,\mu})} \frac{1}{\gamma} \int_0^\gamma \beta_s(f) ds, \quad \forall f \text{ measurable bounded, } \forall \mu \text{ initial distribution,} \quad (3)$$

where  $\mathbb{P}_{0,\mu}$  is a probability measure under which  $X_0 \sim \mu$ , and where  $\beta_s$  is the limiting distribution of the skeleton Markov chain  $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$ . Furthermore, the convergence 3 holds almost surely if the initial measure integrates the Lyapunov function of the process  $(X_t)_{t \geq 0}$ :

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{\mathbb{P}_{0,\mu}\text{-almost surely}} \frac{1}{\gamma} \int_0^\gamma \beta_s(f) ds, \quad \forall \mu \text{ integrating the Lyapunov function.}$$

This will be more precisely stated and proved in Section 2.

**Quasi-stationarity with moving boundaries.** One point of interest in this paper is to apply the main result (3) to the theory of *quasi-stationarity with moving boundaries*, which refers to the study of asymptotical behaviors for Markov processes conditioned not to reach a moving subset of the state space. See in particular [24, 25], where a "conditional ergodic theorem" (see further the definition of a *quasi-ergodic distribution*) has been shown when the absorbing boundaries move periodically. In this paper, one shows that a similar result holds when the boundary is asymptotically periodic, assuming that the process satisfies a conditional Doeblin's condition (see Assumption (A')). This part will be dealt with in Section 3.

The paper will be concluded by using these results in two examples: an ergodic theorem for an asymptotically periodic Ornstein-Uhlenbeck process, and the existence of a unique quasi-ergodic distribution for a Brownian motion confined between two symmetric asymptotically periodic functions.

## 2 Ergodic theorem for asymptotically periodic time-inhomogeneous semi-group.

**Asymptotic periodicity: the definition.**

Let  $(E, \mathcal{E})$  be a measurable space. Consider  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$  a Markovian time-inhomogeneous semi-group, giving a family of measurable subspaces of  $(E, \mathcal{E})$ , denoted by  $(E_t, \mathcal{E}_t)_{t \geq 0}$ , and a family of linear operator  $(P_{s,t})_{s \leq t}$ , with  $P_{s,t} : \mathcal{B}(E_t) \rightarrow \mathcal{B}(E_s)$ , satisfying for any  $r \leq s \leq t$ ,

$$P_{s,s} = \text{Id}, \quad P_{s,t} \mathbb{1}_{E_t} = \mathbb{1}_{E_s}, \quad P_{r,s} P_{s,t} = P_{r,t}.$$

In particular, it is associated to  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$  a Markov process  $(X_t)_{t \geq 0}$  and a family of probability measure  $(\mathbb{P}_{s,x})_{s \geq 0, x \in E_s}$  such that, for any  $s \leq t$ ,  $x \in E_s$  and  $A \in \mathcal{E}_t$ ,

$$\mathbb{P}_{s,x}[X_t \in A] = P_{s,t} \mathbb{1}_A(x).$$

For any probability measure  $\mu$  supported on  $E_s$ , denote  $\mathbb{P}_{s,\mu} := \int_{E_s} \mathbb{P}_{s,x} \mu(dx)$ . Also, denote by  $\mathbb{E}_{s,x}$  and  $\mathbb{E}_{s,\mu}$  the expectations associated to  $\mathbb{P}_{s,x}$  and  $\mathbb{P}_{s,\mu}$ , respectively. Finally, the following notation will be used for  $\mu \in \mathcal{M}_1(E_s)$ ,  $s \leq t$  and  $f \in \mathcal{B}(E_t)$ :

$$\mu P_{s,t} f := \mathbb{E}_{s,\mu}[f(X_t)], \quad \mu P_{s,t} := \mathbb{P}_{s,\mu}[X_t \in \cdot].$$

Respectively to the definition (2), the periodicity of a time-inhomogeneous semi-group can be defined as follows: a semi-group  $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$  is said to be  $\gamma$ -periodic, with  $\gamma > 0$ , if, for any  $s \leq t$ ,

$$(F_t, \mathcal{F}_t) = (F_{t+k\gamma}, \mathcal{F}_{t+k\gamma}), \quad Q_{s,t} = Q_{s+k\gamma, t+k\gamma}, \quad \forall k \in \mathbb{Z}_+.$$

Following to this last definition, it is now possible to define an *asymptotically periodic semi-group*:

**Definition 1** (Asymptotical periodicity for semi-groups). A time-inhomogeneous semi-group  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$  is said to be *asymptotically periodic* if there exists a  $\gamma$ -periodic semi-group  $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$  and two families of functions  $(\psi_s)_{s \geq 0}$  and  $(\tilde{\psi}_s)_{s \in [0, \gamma]}$  such that, for any  $s \in [0, \gamma)$ ,

- $\bigcup_{k=0}^{\infty} \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s \neq \emptyset$
- and there exists  $x_s \in \bigcup_{k=0}^{\infty} \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s$  such that, for any  $n \in \mathbb{Z}_+$ ,

$$\|\delta_{x_s} P_{s+k\gamma, s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \delta_{x_s} Q_{s, s+n\gamma}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0, \quad (4)$$

where, for two positive measures with finite mass  $\mu_1$  and  $\mu_2$ ,  $\|\mu_1 - \mu_2\|_{TV}$  is the *total variation distance* between  $\mu_1$  and  $\mu_2$ :

$$\|\mu_1 - \mu_2\|_{TV} := \sup_{f \in \mathcal{B}_1(E)} |\mu_1(f) - \mu_2(f)|.$$

**Lyapunov/minorization conditions.**

The main assumption of the theorem will be the satisfaction of the following assumptions by the asymptotically periodic Markov process:

**Lyapunov/minorization conditions.** *There exists  $t_1 \geq 0$ ,  $n_0 \in \mathbb{N}$ ,  $c > 0$ ,  $\theta \in (0, 1)$ , a family of subsets  $(K_t)_{t \geq 0}$ , a family of probability measure  $(\nu_{s,n})_{s \in [0, t_1], n \in \mathbb{N}}$  on  $(K_{s+nt_1})_{s \in [0, t_1], n \in \mathbb{N}}$ , and a family of Lyapunov functions  $(\psi_s)_{s \geq 0}$ , all lower-bounded by 1, such that*

(i) *For any  $s \in [0, t_1)$ ,  $x \in K_s$  and  $n \geq n_0$ ,*

$$\delta_x P_{s, s+nt_1} \geq c \nu_{s,n},$$

(ii) *For any  $s \geq 0$ ,*

$$P_{s, s+t_1} \psi_{s+t_1} \leq \theta \psi_s + C \mathbb{1}_{K_s},$$

(iii) For any  $s \geq 0$ , for any  $t \in [0, t_1]$ ,

$$P_{s,s+t}\psi_{s+t} \leq C\psi_s.$$

In particular, under (ii) and (iii), it is easy to prove that, for any  $s \leq t$ ,

$$P_{s,t}\psi_t \leq C \left(1 + \frac{C}{1-\theta}\right) \psi_s. \quad (5)$$

An important remark is that the Lyapunov/minorisation condition implies an *exponential uniform-in-law weak ergodicity in  $\psi_t$ -distance*, i.e. the existence of two constants  $C' > 0$  and  $\kappa > 0$  such that, for all  $s \leq t$  and for all probability measures  $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$ ,

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{\psi_t} \leq C' [\mu_1(\psi_s) + \mu_2(\psi_s)] e^{-\kappa(t-s)}, \quad (6)$$

where, for a given function  $\psi$ ,  $\|\mu - \nu\|_{\psi}$  is the  $\psi$ -distance and is defined as

$$\|\mu - \nu\|_{\psi} := \sup_{|f| \leq \psi} |\mu(f) - \nu(f)|, \quad \forall \mu, \nu \text{ such that } \mu(\psi) < +\infty \text{ and } \nu(\psi) < +\infty.$$

In particular, when  $\psi_t = \mathbb{1}$  for all  $t \geq 0$ , the  $\psi_t$ -distance is *total variation distance*. If the weak ergodicity (6) is well-known in the time-homogeneous setting (see in particular [15]), the proof of [15, Theorem 1.3.] can be adapted in a general time-inhomogeneous framework (see for example [6, Subsection 9.5]).

### The main theorem and proof.

The main result is then the following:

**Theorem 1.** *Let  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}, (X_t)_{t \geq 0}, (\mathbb{P}_{s,x})_{s \geq 0, x \in E_s}\}$  be an asymptotically  $\gamma$ -periodic time-inhomogeneous Markov process, with  $\gamma > 0$ , and denote by  $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$  its periodic auxiliary semi-group. Also, denote by  $(\psi_s)_{s \geq 0}$  and  $(\tilde{\psi}_s)_{s \in [0, \gamma]}$  the two families of functions as defined in Definition 1. Assume moreover that*

- the semi-groups  $(P_{s,t})_{s \leq t}$  and  $(Q_{s,t})_{s \leq t}$  satisfy a Lyapunov/minorisation condition, with  $(\psi_s)_{s \geq 0}$  and  $(\tilde{\psi}_s)_{s \in [0, \gamma]}$  as Lyapunov functions respectively,
- and, for any  $s \in [0, \gamma]$ ,  $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$  converges pointwisely to  $\tilde{\psi}_s$ .

Then, for any  $\mu \in \mathcal{M}_1(E_0)$  such that  $\mu(\psi_0) < +\infty$ ,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s}[\psi_s \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0, \quad (7)$$

where  $\beta_\gamma \in \mathcal{M}_1(F_0)$  is the unique invariant of the skeleton semi-group  $(Q_{0,n\gamma})_{n \in \mathbb{Z}_+}$  satisfying  $\beta_\gamma(\tilde{\psi}_0) < +\infty$ . Moreover, for any  $f \in \mathcal{B}(E)$ ,

- for any  $\mu \in \mathcal{M}_1(E_0)$ ,

$$\mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0. \quad (8)$$

- If moreover  $\mu(\psi_0) < +\infty$ , then

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow{t \rightarrow \infty} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely}. \quad (9)$$

*Remark 1.* Also note that, if the Lyapunov/minorization conditions hold for  $\psi_s = \mathbb{1}$  and  $K_s = E_s$  for any  $s$ , then these conditions are boiled down to the *Doebelin's condition*:

**Doebelin's condition.** *there exists  $t_0 \geq 0$ ,  $c > 0$  and a family of probability measure  $(\nu_t)_{t \geq 0}$  on  $(E_t)_{t \geq 0}$  such that, for any  $s \geq 0$  and  $x \in E_s$ ,*

$$\delta_x P_{s,s+t_0} \geq c \nu_{s+t_0}. \quad (10)$$

Then, Theorem 1 entails the following corollary:

**Corollary 1.** *Let  $(X_t)_{t \geq 0}$  be asymptotically  $\gamma$ -periodic in total variation distance (i.e.  $\psi_s = \tilde{\psi}_s = \mathbb{1}$  in the Definition 1, for all  $s \geq 0$ ). If  $(X_t)_{t \geq 0}$  and its auxiliary semi-group satisfy a Doebelin's condition, then the convergence (8) is improved to*

$$\sup_{\mu \in \mathcal{M}_1(E_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0.$$

Moreover, the almost sure convergence (9) holds for all initial measure  $\mu$ .

*Remark 2.* Note also that, if the convergence (4) holds actually for all  $x \in \bigcup_{k=0}^\infty \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s$ , then this implies (4), obviously, and the pointwise convergence of  $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$  to  $\tilde{\psi}_s$  (taking  $n = 0$  in (4)).

*Proof of Theorem 1.* The proof is divided into four steps.

*First step.* For the first part of this step (until the equality (14)), we will inspire from the proof of [1, Theorem 3.11]. Fix  $s \in [0, \gamma]$ . Since the auxiliary semi-group  $(Q_{s,t})_{s \leq t}$  satisfies a Lyapunov/minorization condition with  $(\tilde{\psi})_{s \geq 0}$  as Lyapunov functions, this entails that, for any  $x \in F_s$ ,

$$\|\delta_x Q_{s,s+n\gamma} - \beta_\gamma Q_{0,s}\|_{\tilde{\psi}_s} \xrightarrow{n \rightarrow \infty} 0, \quad (11)$$

recalling that  $\beta_\gamma$  is the unique invariant measure of  $(Q_{0,n\gamma})_{n \in \mathbb{Z}_+}$  satisfying  $\beta_\gamma(\tilde{\psi}_0) < +\infty$ .

Without loss of generality, let us assume that  $\bigcap_{l \geq 0} E_{s+l\gamma} \cap F_s \neq \emptyset$ . Then, by Definition 1, for any  $n \geq 0$ ,

$$\|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \delta_{x_s} Q_{s,s+n\gamma}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow{k \rightarrow \infty} 0.,$$

which entails using (11) that

$$\lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = 0. \quad (12)$$

Now, by Markov property, (6), and using (5), one obtains that, for any  $k, n \in \mathbb{N}$  and  $x \in \bigcap_{l \geq 0} E_{s+l\gamma}$ ,

$$\begin{aligned} \|\delta_x P_{s,s+(k+n)\gamma} - \delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{\psi_{s+(k+n)\gamma}} &= \|(\delta_x P_{s,s+k\gamma}) P_{s+k\gamma,s+(k+n)\gamma} - \delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{\psi_{s+(k+n)\gamma}} \\ &\leq C' [P_{s,s+k\gamma} \psi_{s+k\gamma}(x) + \psi_{s+k\gamma}(x)] e^{-\kappa\gamma n} \\ &\leq C'' [\psi_s(x) + \psi_{s+k\gamma}(x)] e^{-\kappa\gamma n}, \end{aligned}$$

where  $C'' := C' \left( C \left( 1 + \frac{C}{1-\theta} \right) \wedge 1 \right)$ . Then, for any  $k, n \in \mathbb{N}$ ,

$$\begin{aligned} \|\delta_{x_s} P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} \\ \leq C'' [\psi_s(x) + \psi_{s+k\gamma}(x)] e^{-\kappa\gamma n} + \|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV}, \end{aligned} \quad (13)$$

which entails, by (12) and the pointwise convergence of  $(\psi_{s+k\gamma})_{k \in \mathbb{Z}_+}$ , that

$$\lim_{n \rightarrow \infty} \|\delta_{x_s} P_{s,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = \lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \|\delta_{x_s} P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = 0. \quad (14)$$

The weak ergodicity (6) implies therefore that the following convergence actually holds for any initial distribution  $\mu \in \mathcal{M}_1(E_0)$  satisfying  $\mu(\psi_0) < +\infty$ , so that

$$\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow{n \rightarrow \infty} 0. \quad (15)$$

In particular, by Lebesgue's dominated convergence theorem, for any  $\mu$  satisfying  $\mu(\psi_0) < +\infty$ ,

$$\left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \xrightarrow{n \rightarrow \infty} 0.$$

By Cesaro's lemma, this allows to conclude that, for any  $\mu \in \mathcal{M}_1(E_0)$  such that  $\mu(\psi_0) < +\infty$ ,

$$\begin{aligned} & \left\| \frac{1}{t} \int_0^t \mu P_{0,s}[\psi_s \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \\ & \leq \frac{1}{\lfloor \frac{t}{\gamma} \rfloor} \sum_{k=0}^{\lfloor \frac{t}{\gamma} \rfloor} \left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} + \left\| \frac{1}{t} \int_{\lfloor \frac{t}{\gamma} \rfloor \gamma}^t \mu P_{0,s}[\psi_s \times \cdot] ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0, \end{aligned}$$

which concludes the proof of (7).

*Second step.* In the same way, one can show that, for any  $\mu \in \mathcal{M}_1(E_0)$  such that  $\mu(\psi_0) < +\infty$ ,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s} ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0. \quad (16)$$

As a matter of fact, for any function  $f$  bounded by 1 and  $\mu \in \mathcal{M}_1(E_0)$  such that  $\mu(\psi_0) < +\infty$ ,

$$\begin{aligned} & \left| \mu P_{0,s+n\gamma} \left[ \psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \\ & \leq \left| \mu P_{0,s+n\gamma} \left[ \psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] \right| + \left| \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \\ & \leq \left\| \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] \right\|_{TV} + \left| \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right|. \end{aligned}$$

Now, remark that, since  $\psi_{s+n\gamma} \geq 1$  for any  $s$  and  $n \in \mathbb{Z}_+$ , one obtains that

$$\left| \frac{\tilde{\psi}_s}{\psi_{s+n\gamma}} - 1 \right| \leq 1 + \tilde{\psi}_s.$$

Then, since  $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$  converges pointwisely towards  $\tilde{\psi}_s$  and  $\beta_\gamma Q_{0,s} \tilde{\psi}_s < +\infty$ , Lebesgue's dominated convergence theorem entails that

$$\sup_{f \in \mathcal{B}_1(E)} \left| \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Then, using now (15), one has

$$\| \mu P_{0,s+n\gamma} - \beta_\gamma Q_{0,s} \|_{TV} \xrightarrow{n \rightarrow \infty} 0,$$

which allows to conclude in (16), using the same argument as in the first step.

*Third step.* In order to show the  $\mathbb{L}^2$ -ergodic theorem, let  $f \in \mathcal{B}(E)$ . For any  $x \in E_0$  and  $t \geq 0$ ,

$$\begin{aligned} & \mathbb{E}_{0,x} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[ \frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \\ & = \frac{2}{t^2} \int_0^t \int_s^t (\mathbb{E}_{0,x}[f(X_s)f(X_u)] - \mathbb{E}_{0,x}[f(X_s)]\mathbb{E}_{0,x}[f(X_u)]) duds \\ & = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [f(X_s) (f(X_u) - \mathbb{E}_{0,x}[f(X_u)])] duds \\ & = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [f(X_s) (\mathbb{E}_{s,X_s}[f(X_u)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_u)])] duds, \end{aligned}$$

where Markov property was used at the last line. By the weak ergodicity (6) and (5), one obtains for any  $s \leq t$ ,

$$|\mathbb{E}_{s, X_s}[f(X_t)] - \mathbb{E}_{s, \delta_x P_{0,s}}[f(X_t)]| \leq C'' \|f\|_\infty [\psi_s(X_s) + \psi_0(x)] e^{-\kappa(t-s)}, \quad \mathbb{P}_{0,x} - \text{almost surely}, \quad (17)$$

where  $C''$  was defined in the first part. As a result, for any  $x \in E_0$  and  $t \geq 0$ ,

$$\begin{aligned} \mathbb{E}_{0,x} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[ \frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] &\leq \frac{2C'' \|f\|_\infty}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] e^{-\kappa(u-s)} du ds \\ &= \frac{2C'' \|f\|_\infty}{t^2} \int_0^t \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] e^{\kappa s} \frac{e^{-\kappa s} - e^{-\kappa t}}{\kappa} ds \\ &= \frac{2C'' \|f\|_\infty}{\kappa t} \times \mathbb{E}_{0,x} \left[ \frac{1}{t} \int_0^t |f(X_s)| (\psi_s(X_s) + \psi_0(x)) ds \right] - \frac{2C'' \|f\|_\infty e^{-\kappa t}}{\kappa t^2} \int_0^t e^{\kappa s} \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] ds. \end{aligned}$$

Then, by (7) and (16), there exists a constant  $\tilde{C} > 0$  such that, for any  $x \in E_0$ , when  $t \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{E}_{0,x} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[ \frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \\ \leq \frac{2\tilde{C} \|f\|_\infty}{\kappa t} \times \left( \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} [|f| \psi_s] ds + \frac{\psi_0(x)}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} [|f|] ds \right) + o\left(\frac{1}{t}\right). \quad (18) \end{aligned}$$

Then, using (16), one deduces that, for any  $x \in E_0$  and bounded function  $f$ ,

$$\mathbb{E}_{0,x} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] \xrightarrow[t \rightarrow \infty]{} 0.$$

The convergence for any probability measure  $\mu \in \mathcal{M}_1(E_0)$  comes from Lebesgue's dominated convergence theorem.

*Fourth step.* Now, let us fix  $f \in \mathcal{B}(E)$ , assumed nonnegative on  $E$ , and  $\mu \in \mathcal{M}_1(E_0)$  satisfying  $\mu(\psi_0) < +\infty$ . The following proof is inspired from the proof of [26, Theorem 12].

Since  $\mu(\psi_0) < +\infty$ , the inequality (18) entails that there exists a finite constant  $C_{f,\mu} \in (0, \infty)$  such that, for any  $t > 0$ ,

$$\mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,\mu} \left[ \frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \leq \frac{C_{f,\mu}}{t}.$$

Then, for any  $n \in \mathbb{N}$ ,

$$\mathbb{E}_{0,\mu} \left[ \left| \frac{1}{n^2} \int_0^{n^2} f(X_s) ds - \mathbb{E}_{0,\mu} \left[ \frac{1}{n^2} \int_0^{n^2} f(X_s) ds \right] \right|^2 \right] \leq \frac{C_{f,\mu}}{n^2}.$$

Then, by Bienayme-Tchebychev's inequality and Borel-Cantelli's lemma, this last inequality entails that

$$\left| \frac{1}{n^2} \int_0^{n^2} f(X_s) ds - \mathbb{E}_{0,\mu} \left[ \frac{1}{n^2} \int_0^{n^2} f(X_s) ds \right] \right| \xrightarrow[n \rightarrow \infty]{} 0, \quad \mathbb{P}_{0,\mu} - \text{almost surely},$$

and, by the convergence (16), one obtains

$$\frac{1}{n^2} \int_0^{n^2} f(X_s) ds \xrightarrow[n \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely}. \quad (19)$$

Since the nonnegativity of  $f$  is assumed, this entails that, for any  $t > 0$ ,

$$\int_0^{\lfloor \sqrt{t} \rfloor^2} f(X_s) ds \leq \int_0^t f(X_s) ds \leq \int_0^{\lceil \sqrt{t} \rceil^2} f(X_s) ds.$$



Then, these inequalities and (19) show that

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely.}$$

Then, in order to conclude that the following result holds actually for any bounded measurable function  $f$ , it is enough to say that, for such a function, one has the decomposition  $f = f_+ - f_-$ , with  $f_+ := f \vee 0$  and  $f_- = (-f) \vee 0$  which are positive bounded, and to apply the previous convergence to  $f_+$  and  $f_-$ . This concludes the proof of Theorem 1.  $\square$

*Proof of Corollary 1.* Just remark in the previous proof that, if  $\|f\|_\infty \leq 1$  and  $\psi_s = \mathbb{1}$ , an upper-bound for the inequality (18), which does not depend on  $f$  and  $x$ , can be obtained. Likewise, the convergence (15) holds uniformly in the initial measure due to weak ergodicity (17), which becomes uniform-in-law when  $\psi = \mathbb{1}$ .  $\square$

*Remark 3.* It seems that the theorem does not hold if the Doeblin's constant  $c$  depends on the starting time  $s$ . As proof, considering a Markov process satisfying the Doeblin's condition with time-dependant constants  $c_s$ , the weak ergodicity holds as soon as, for any  $s \geq 0$ ,

$$\prod_{k=1}^{\lfloor \frac{t-s}{t_0} \rfloor} (1 - c_{t-kt_0}) \xrightarrow[t \rightarrow \infty]{} 0,$$

since one has

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{TV} \leq 2 \prod_{k=1}^{\lfloor \frac{t-s}{t_0} \rfloor} (1 - c_{t-kt_0}).$$

Then the line (13) has to be replaced by

$$\|\delta_x P_{s,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV} \leq 2 \prod_{l=1}^{\lfloor \frac{n\gamma}{t_0} \rfloor} (1 - c_{s+(k+n)\gamma-lt_0}) + \|\delta_x P_{s+k\gamma,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV}.$$

This does not allow to conclude in the interesting cases ( $c_s \xrightarrow[s \rightarrow \infty]{} 0$ ), since

$$\lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \prod_{l=1}^{\lfloor \frac{n\gamma}{t_0} \rfloor} (1 - c_{s+(k+n)\gamma-lt_0}) = 1.$$

In the same way, it seems that the constants  $\theta$  and  $C$  defined in the Lyapunov/minorization condition have also to be independent on the starting times.

### 3 Application to quasi-stationarity with moving boundaries

In this section, consider a family of measurable subsets  $(A_t)_{t \geq 0}$  of  $E$ , and denote the hitting time

$$\tau_A := \inf\{t \geq 0 : X_t \in A_t\}.$$

Assume that  $\tau_A$  is a stopping time with respect to the natural filtration of the Markov process, denoted by  $(\mathcal{F}_{s,t})_{s \leq t}$ . Assume also that for any  $x \notin A_0$ ,

$$\mathbb{P}_{0,x}[\tau_A < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{0,x}[\tau_A > t] > 0, \quad \forall t \geq 0.$$

We will be interested in a notion of *quasi-stationarity with moving boundaries*, which studies the asymptotic behavior of the Markov process  $(X_t)_{t \geq 0}$  conditioned not to hit  $(A_t)_{t \geq 0}$  up to the time  $t$ . For non-moving boundaries

( $A_t = A_0$  for any  $t \geq 0$ ) and time-homogeneous processes, the *quasi-limiting distribution* is defined as a probability measure  $\alpha$  such that, for at least one initial measure  $\mu$ ,

$$\mathbb{P}_{0,\mu}[X_t \in \cdot | \tau_A > t] \xrightarrow[t \rightarrow \infty]{\mathcal{L}} \alpha.$$

Such a definition is actually equivalent (still in the non-moving framework) to the notion of the *quasi-stationary distribution* defined as a probability measure  $\alpha$  such that, for any  $t \geq 0$ ,

$$\mathbb{P}_{0,\alpha}[X_t \in \cdot | \tau_A > t] = \alpha. \quad (20)$$

If quasi-limiting and quasi-stationary distributions are generally well-defined for time-homogeneous Markov process and non-moving boundaries (see [11, 23] for a general overview on the topic), these notions could be not well-defined for time-inhomogeneous Markov processes or moving boundaries and are not equivalent anymore. In particular, under reasonable assumptions on irreducibility, it was shown in [24] that the notion of quasi-stationary distribution as defined by (20) is not well-defined for time-homogeneous Markov processes absorbed by moving boundaries.

In the theory of quasi-stationarity with moving boundaries, another asymptotic notion to study is the *quasi-ergodic distribution*, related to a conditional version of the ergodic theorem and usually defined as follows:

**Definition 2.** A probability measure  $\beta$  is a *quasi-ergodic distribution* if, for some initial measure  $\mu \in \mathcal{M}_1(E \setminus A_0)$  and for any bounded continuous function  $f$ ,

$$\mathbb{E}_{0,\mu} \left[ \frac{1}{t} \int_0^t f(X_s) ds \middle| \tau_A > t \right] \xrightarrow[t \rightarrow \infty]{} \beta(f).$$

In the time-homogeneous setting (in particular for non-moving boundaries), this notion has been extensively studied (see for example [3, 8, 10, 12, 13, 16, 17, 18, 24]). In the "*moving boundaries*" framework, the existence of quasi-ergodic distributions has been dealt with in [24] for Markov chains on finite state spaces, and in [25] for processes satisfying a Champagnat-Villemonais condition (see Assumption (A') set further). In this last paper, the existence of the quasi-ergodic distribution is actually obtained through the following inequality holding for any initial distribution  $\mu$  and for some constant  $C > 0$  independant on  $\mu$ :

$$\left\| \frac{1}{t} \int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t] ds - \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}[X_s \in \cdot] ds \right\|_{TV} \leq \frac{C}{t}, \quad \forall t > 0, \quad (21)$$

where

- the family of probability measure  $(\mathbb{Q}_{s,x})_{s \geq 0, x \in E_s}$  is defined as follows:

$$\mathbb{Q}_{s,x}[\Gamma] := \lim_{T \rightarrow \infty} \mathbb{P}_{s,x}[\Gamma | \tau_A > T], \quad \forall s \geq 0, x \notin A_s, \Gamma \in \mathcal{F}_{s,t},$$

- $\eta_0$  is a function on  $E_0$
- and  $\eta_0 * \mu(dx) := \frac{\eta_0(x)\mu(dx)}{\mu(\eta_0)}$ .

In particular, the law of  $(X_t)_{t \geq 0}$  under  $(\mathbb{Q}_{s,x})_{s \geq 0, x \in E_s}$ , called *Q-process*, is a time-inhomogeneous Markov process and there exists a unique quasi-ergodic distribution for the process  $(X_t)_{t \geq 0}$  absorbed by  $(A_t)_{t \geq 0}$  if and only if the probability measures  $\frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}[X_s \in \cdot] ds$  converge weakly, when  $t$  goes to infinity. Hence, if one shows that the *Q-process* is asymptotically periodic, according to Definition 1, then Theorem 1 entails the existence of a quasi-ergodic distribution.

For the following statement, the following assumption is needed:

**Assumption (A').** *there exists a family of probability measures  $(\nu_t)_{t \geq 0}$ , defined on  $E \setminus A_t$  for each  $t$ , such that (A'1) there exists  $t_0 \geq 0$  and  $c_1 > 0$  such that*

$$\mathbb{P}_{s,x}[X_{s+t_0} \in \cdot | \tau_A > s + t_0] \geq c_1 \nu_{s+t_0}, \quad \forall s \geq 0, \forall x \notin A_s,$$

(A'2) and there exists  $c_2 > 0$  such that

$$\mathbb{P}_{s,\nu_s}[\tau_A > t] \geq c_2 \mathbb{P}_{s,x}[\tau_A > t], \quad \forall s \leq t, \forall x \notin A_s.$$

These assumptions are the main assumptions on which relies the results of [25] and are an analogous version of the Doeblin's condition (10). When these assumptions are written in the time-homogeneous framework, one obtains the Champagnat-Villemonais condition defined in [5] (see Assumption (A)), shown as being equivalent to the exponential uniform-in-law convergence to quasi-stationarity in total variation.

Now, let us state the following theorem:

**Theorem 2.** *Assume that there exists a  $\gamma$ -periodic sequence of subsets  $(B_t)_{t \geq 0}$  such that, for any  $s \in [0, \gamma)$ ,*

$$E'_s := E \setminus \bigcap_{k \in \mathbb{Z}_+} \bigcup_{l \geq k} A_{s+l\gamma} \cup B_s \neq \emptyset$$

and there exists  $x_s \in E_s$  such that, for any  $n \leq N$ ,

$$\|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0. \quad (22)$$

Assume also that Assumption (A') are satisfied for the Markov process  $(X_t)_{t \geq 0}$  considered as absorbed either by  $(A_t)_{t \geq 0}$  or by  $(B_t)_{t \geq 0}$ .

Then there exists a probability measure  $\beta \in \mathcal{M}_1(E)$  such that

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0, \mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] \xrightarrow[t \rightarrow \infty]{} 0. \quad (23)$$

*Remark 4.* Remark that the condition (22) implies that, for any  $n \in \mathbb{Z}_+$ ,

$$\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+n)\gamma] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{s, x_s}[\tau_B > s + n\gamma].$$

However, under the additional condition  $B_t \subset A_t$  for all  $t \geq 0$ , these two conditions are actually equivalent, since for all  $n \leq N$ ,

$$\begin{aligned} \|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \\ \leq |\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[\tau_B > s + N\gamma]|. \end{aligned}$$

This entails the following corollary:

**Corollary 2.** *Assume that there exists a  $\gamma$ -periodic sequence of subsets  $(B_t)_{t \geq 0}$ , with  $B_t \subset A_t$  for all  $t \geq 0$ , such that, for any  $s \in [0, \gamma)$ , there exists  $x_s \in E'_s$  such that, for any  $n \leq N$ ,*

$$\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+n)\gamma] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{s, x_s}[\tau_B > s + n\gamma].$$

Assume also that Assumption (A') are satisfied for the Markov process  $(X_t)_{t \geq 0}$  considered as absorbed either by  $(A_t)_{t \geq 0}$  or by  $(B_t)_{t \geq 0}$ .

Then there exists  $\beta \in \mathcal{M}_1(E)$  such that (23) holds.

*Proof of Theorem 2.* Since  $(X_t)_{t \geq 0}$  absorbed by  $(B_t)_{t \geq 0}$  satisfies Assumption (A'), we already know by [25] that, for any initial distribution  $\mu$ ,  $t \mapsto \frac{1}{t} \int_0^t \mathbb{P}_{0, \mu}[X_s \in \cdot | \tau_B > t] ds$  converges in total variation to a quasi-ergodic distribution  $\beta$ .

The main idea of this proof is to apply Corollary 1. Since  $(X_t)_{t \geq 0}$ , absorbed by  $A$  or  $B$ , satisfies Assumption (A'), then [9, Theorem 3.3.] entails that there exist two families of probability measures  $(\mathbb{Q}_{s,x}^A)_{s \geq 0, x \notin A_s}$  and  $(\mathbb{Q}_{s,x}^B)_{s \geq 0, x \notin B_s}$  such that, for any  $s \leq t$ ,  $x \notin A_s$ ,  $y \notin B_s$  and  $\Gamma \in \mathcal{F}_{s,t}$ ,

$$\mathbb{Q}_{s,x}^A[\Gamma] = \lim_{T \rightarrow \infty} \mathbb{P}_{s,x}[\Gamma | \tau_A > T], \quad \text{and} \quad \mathbb{Q}_{s,y}^B[\Gamma] = \lim_{T \rightarrow \infty} \mathbb{P}_{s,y}[\Gamma | \tau_B > T].$$

In particular, the quasi-ergodic distribution  $\beta$  is actually the limit of  $t \mapsto \frac{1}{t} \int_0^t \mathbb{Q}_{0,\mu}^B[X_s \in \cdot] ds$ , when  $t$  goes to infinity. Also, by [25, Theorem 1], there exists a constant  $C > 0$  and  $\kappa > 0$  such that, for any  $s \leq t \leq T$ , for any  $x \notin A_s$ ,

$$\|\mathbb{Q}_{s,x}^A[X_t \in \cdot] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_A > T]\|_{TV} \leq Ce^{-\kappa(T-t)},$$

and for any  $x \notin B_s$ ,

$$\|\mathbb{Q}_{s,x}^B[X_t \in \cdot] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_B > T]\|_{TV} \leq Ce^{-\kappa(T-t)}.$$

Moreover, for any  $s \leq t \leq T$  and  $x \in E'_s$ ,

$$\begin{aligned} & \|\mathbb{P}_{s,x}[X_t \in \cdot | \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_B > T]\|_{TV} \\ &= \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_A > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\ &= \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T) \mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}(\tau_A > T) \mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\ &\leq \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T) \mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}(\tau_A > T) \mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\ &\quad + \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\ &\leq \left| \frac{\mathbb{P}_{s,x}(\tau_B > T)}{\mathbb{P}_{s,x}(\tau_A > T)} - 1 \right| \frac{\mathbb{P}_{s,x}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_B > T)} + \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}. \end{aligned} \tag{24}$$

Then, one obtains, for any  $s \leq t \leq T$  and  $x \in E'_s$ ,

$$\begin{aligned} & \|\mathbb{Q}_{s,x}^A[X_t \in \cdot] - \mathbb{Q}_{s,x}^B[X_t \in \cdot]\|_{TV} \\ & \leq 2Ce^{-\kappa(T-t)} + \left| \frac{\mathbb{P}_{s,x}(\tau_B > T)}{\mathbb{P}_{s,x}(\tau_A > T)} - 1 \right| \frac{\mathbb{P}_{s,x}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_B > T)} + \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}. \end{aligned}$$

The condition (22) implies that, for any  $n \leq N$ ,

$$\begin{aligned} & \left| \frac{\mathbb{P}_{s+k\gamma, x_s}(\tau_B > s + (k+N)\gamma)}{\mathbb{P}_{s+k\gamma, x_s}(\tau_A > s + (k+N)\gamma)} - 1 \right| \frac{\mathbb{P}_{s+k\gamma, x_s}(\tau_A > s + (k+N)\gamma)}{\mathbb{P}_{s+k\gamma, x_s}(\tau_B > s + (k+N)\gamma)} \\ & \quad + \frac{\|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV}}{\mathbb{P}_{s, x_s}[\tau_B > s + N\gamma]} \xrightarrow[k \rightarrow \infty]{} 0, \end{aligned}$$

which implies that, for any  $n \leq N$ ,

$$\limsup_{k \rightarrow \infty} \|\mathbb{Q}_{s+k\gamma, x_s}^A[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}_{s+k\gamma, x_s}^B[X_{s+(k+n)\gamma} \in \cdot]\|_{TV} \leq 2Ce^{-\kappa\gamma(N-n)}.$$

Now, letting  $N \rightarrow \infty$ , one has for any  $n \in \mathbb{Z}_+$

$$\limsup_{k \rightarrow \infty} \|\mathbb{Q}_{s+k\gamma, x_s}^A[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}_{s+k\gamma, x_s}^B[X_{s+(k+n)\gamma} \in \cdot]\|_{TV} = 0.$$

In other words, the Markov process  $\{(X_t)_{t \geq 0}, (\mathbb{Q}_{s,x}^A)_{s \geq 0, x \notin A_s}\}$  is asymptotically periodic in total variation distance, with  $\{(X_t)_{t \geq 0}, (\mathbb{Q}_{s,x}^B)_{s \geq 0, x \notin B_s}\}$  as auxiliary Markov process. Moreover, Assumptions (A') satisfied for  $(A_t)_{t \geq 0}$  and  $(B_t)_{t \geq 0}$  entail that Doeblin's conditions hold for these two  $Q$ -processes. Then, by Corollary 1 and using (21), one obtains that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t] ds = \lim_{t \rightarrow \infty} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}^B[X_s \in \cdot] ds = \beta,$$

where the limits refer to the convergence in total variation and hold uniformly in the initial measure. Now, for any  $\mu \in \mathcal{M}_1(E \setminus A_0)$ ,  $f \in \mathcal{B}_1(E)$  and  $t \geq 0$ ,

$$\mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds \right|^2 \middle| \tau_A > t \right] = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,\mu} [f(X_s)f(X_u) | \tau_A > t] duds.$$

Then, by [25, Theorem 1], for any  $s \leq u \leq t$ , for any  $\mu \in \mathcal{M}_1(E \setminus A_0)$  and  $f \in \mathcal{B}(E)$ ,

$$\left| \mathbb{E}_{0,\mu} [f(X_s)f(X_u) | \tau_A > t] - \mathbb{E}_{0,\eta_0 * \mu}^{\mathbb{Q}^A} [f(X_s)f(X_u)] \right| \leq C \|f\|_\infty e^{-\kappa(t-u)},$$

where the expectation  $\mathbb{E}_{0,\eta_0 * \mu}^{\mathbb{Q}^A}$  is associated to the probability measure  $\mathbb{Q}_{0,\eta_0 * \mu}^A$ . Hence, for any  $\mu \in \mathcal{M}_1(E \setminus A_0)$ ,  $f \in \mathcal{B}_1(E)$  and  $t > 0$ ,

$$\begin{aligned} \left| \mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] - \mathbb{E}_{0,\eta_0 * \mu}^{\mathbb{Q}^A} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \right] \right| &\leq \frac{4C}{t^2} \int_0^t \int_s^t e^{-\kappa(t-u)} duds \\ &\leq \frac{4C}{\kappa t} - \frac{4C(1 - e^{-\kappa t})}{\kappa^2 t^2}. \end{aligned}$$

Moreover, Corollary 1 entails that, for any  $\mu \in \mathcal{M}_1(E \setminus A_0)$ ,

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\eta_0 * \mu}^{\mathbb{Q}^A} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0.$$

Then, this implies that

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] \xrightarrow{t \rightarrow \infty} 0.$$

□

*Remark 5.* It seems that Assumption (A') can be weakened by a conditional version of the Lyapunov/minorization conditions. In particular, such conditions can be derived from the Assumption (F) presented in [6], as it will be shown later by the preprint [4] in preparation.

## 4 Examples

### 4.1 Asymptotical periodic Ornstein-Uhlenbeck processes

Let  $(X_t)_{t \geq 0}$  be a time-inhomogeneous diffusion process on  $\mathbb{R}$  satisfying the following stochastic differential equation

$$dX_t = dW_t - \lambda(t)X_t dt,$$

where  $(W_t)_{t \geq 0}$  is a one-dimensional Brownian motion and  $\lambda : [0, \infty) \rightarrow [0, \infty)$  is a function such that

$$0 < \inf_{t \geq 0} \lambda(t) \leq \sup_{t \geq 0} \lambda(t) < +\infty.$$

By Itô's formula, one obtains for any  $s \leq t$ ,

$$X_t = e^{-\int_s^t \lambda(u) du} \left[ X_s + \int_s^t e^{\int_s^u \lambda(v) dv} dW_u \right].$$

In particular, denoting  $(P_{s,t})_{s \leq t}$  the semi-group associated to  $(X_t)_{t \geq 0}$ , for any  $f \in \mathcal{B}(\mathbb{R})$ ,  $t \geq 0$  and  $x \in \mathbb{R}$ ,

$$P_{s,t} f(x) = \mathbb{E} \left[ f \left( e^{-\int_s^t \lambda(u) du} x + e^{-\int_s^t \lambda(u) du} \sqrt{\int_s^t e^{2 \int_s^u \lambda(v) dv} du} \times \mathcal{N}(0, 1) \right) \right],$$

where  $\mathcal{N}(0, 1)$  denotes a standard Gaussian variable.

**Theorem 3.** Assume that there exists a  $\gamma$ -periodic function  $g$ , positive and bounded on  $\mathbb{R}$ , such that  $\lambda \sim_{t \rightarrow \infty} g$ . Then the assumptions of Theorem 1 hold.

*Proof.* In our case, the auxiliary semi-group  $(Q_{s,t})_{s \leq t}$  of Definition 1 will be defined as follows: for any  $f \in \mathcal{B}(\mathbb{R})$ ,  $t \geq 0$  and  $x \in \mathbb{R}$ ,

$$Q_{s,t}f(x) = \mathbb{E} \left[ f \left( e^{-\int_s^t g(u)du} x + e^{-\int_s^t g(u)du} \sqrt{\int_s^t e^{2\int_s^u g(v)dv} du} \times \mathcal{N}(0, 1) \right) \right].$$

In particular, the semi-group  $(Q_{s,t})_{s \leq t}$  is associated to the process  $(Y_t)_{t \geq 0}$  defined as following

$$dY_t = dW_t - g(t)Y_t dt.$$

First of all, one remarks that the function  $\psi : x \mapsto 1 + x^2$  is a Lyapunov function for  $(P_{s,t})_{s \leq t}$  and  $(Q_{s,t})_{s \leq t}$ . As a matter of fact, for any  $s \geq 0$  and  $x \in \mathbb{R}$ ,

$$\begin{aligned} P_{s,s+\gamma}\psi(x) &= 1 + e^{-2\int_s^{s+\gamma} \lambda(u)du} x^2 + e^{-2\int_s^{s+\gamma} \lambda(u)du} \int_s^{s+\gamma} e^{2\int_s^u \lambda(v)dv} du \\ &= e^{-2\int_s^{s+\gamma} \lambda(u)du} \psi(x) + 1 - e^{-2\int_s^{s+\gamma} \lambda(u)du} + e^{-2\int_s^{s+\gamma} \lambda(u)du} \int_s^{s+\gamma} e^{2\int_s^u \lambda(v)dv} du \\ &\leq e^{-2\gamma c_{\inf}} \psi(x) + C, \end{aligned}$$

for some constant  $C > 0$ , and where  $c_{\inf} := \inf_{t \geq 0} \lambda(t) > 0$ . Taking  $\theta \in (e^{-2\gamma c_{\inf}}, 1)$ , there exists a compact set  $K$  such that, for any  $s \geq 0$ ,

$$P_{s,s+\gamma}\psi(x) \leq \theta\psi(x) + C\mathbb{1}_K.$$

Also, it is simple to check that, for any  $s \geq 0$  and  $t \in [0, \gamma)$ , the function  $P_{s,s+t}\psi/\psi$  is upper-bounded uniformly in  $s$  and  $t$ . Now, the local Doeblin's condition for  $(P_{s,t})_{s \leq t}$  is actually a consequence of the following lemma:

**Lemma 1.** For any  $a, b_-, b_+ > 0$ , define the subset  $\mathcal{C}(a, b_-, b_+) \subset \mathcal{M}_1(\mathbb{R})$  as

$$\mathcal{C}(a, b_-, b_+) := \{\mathcal{N}(m, \sigma) : m \in [-a, a], \sigma \in [b_-, b_+]\}.$$

Then, for any  $a, b_-, b_+ > 0$ , there exists a probability measure  $\nu$  and a constant  $c > 0$  such that, for any  $\mu \in \mathcal{C}(a, b_-, b_+)$ ,

$$\mu \geq c\nu.$$

The proof of this lemma is postponed after the end of this proof. Note that the condition  $\lambda \sim_{t \rightarrow \infty} g$  and the fact that  $\lambda$  is bounded on  $\mathbb{R}$  entails by Lebesgue's dominated convergence theorem that, for all  $s \leq t$ ,

$$\left| \int_{s+k\gamma}^{t+k\gamma} \lambda(u)du - \int_s^t g(u)du \right| \xrightarrow[k \rightarrow \infty]{} 0,$$

and, in the same way,

$$\int_{s+k\gamma}^{t+k\gamma} e^{2\int_{s+k\gamma}^u \lambda(v)dv} du \xrightarrow[k \rightarrow \infty]{} \int_s^t e^{2\int_s^u g(v)dv} du.$$

Then, for any  $s \leq t$ ,

$$e^{-\int_{s+k\gamma}^{t+k\gamma} \lambda(u)du} \xrightarrow[k \rightarrow \infty]{} e^{-\int_s^t g(u)du},$$

and

$$e^{-\int_{s+k\gamma}^{t+k\gamma} \lambda(u)du} \sqrt{\int_{s+k\gamma}^{t+k\gamma} e^{2\int_{s+k\gamma}^u \lambda(v)dv} du} \xrightarrow[k \rightarrow \infty]{} e^{-\int_s^t g(u)du} \sqrt{\int_s^t e^{2\int_s^u g(v)dv} du}.$$

Using [14, Theorem 1.3.], one obtains, for any  $x \in \mathbb{R}$ ,

$$\|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s+k\gamma, t+k\gamma}\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0. \quad (25)$$

To deduce the convergence in  $\psi$ -distance, we will inspire from the proof of [19, Lemma 3.1]. The variances being bounded uniformly in  $k$  (for  $s \leq t$  fixed), then there exists  $H > 0$  such that, for any  $k \in \mathbb{N}$ ,

$$\delta_x P_{s+k\gamma, t+k\gamma}[\psi^2] \leq H, \quad \delta_x Q_{s, t}[\psi^2] \leq H. \quad (26)$$

Then, since  $\lim_{|x| \rightarrow \infty} \frac{\psi(x)}{\psi^2(x)} = 0$ , for any  $\epsilon > 0$ , there exists  $l_\epsilon > 0$  such that, for any function  $f$  such that  $|f| \leq \psi$  and for any  $|x| \geq l_\epsilon$ ,

$$|f(x)| \leq \frac{\epsilon \psi(x)^2}{H}.$$

This implies with (26) that, denoting  $K_\epsilon := [-l_\epsilon, l_\epsilon]$ , for any  $k \in \mathbb{Z}_+$ ,  $f$  such that  $|f| \leq \psi$  and  $x \in \mathbb{R}$ ,

$$\delta_x P_{s+k\gamma, t+k\gamma}[f \mathbb{1}_{K_\epsilon}] \leq \epsilon, \quad \delta_x Q_{s, t}[f \mathbb{1}_{K_\epsilon}] \leq \epsilon.$$

Then, for any  $k \in \mathbb{Z}_+$  and  $f$  such that  $|f| \leq \psi$ ,

$$|\delta_x P_{s+k\gamma, t+k\gamma} f - \delta_x Q_{s, t} f| \leq 2\epsilon + |\delta_x P_{s+k\gamma, t+k\gamma}[f \mathbb{1}_{K_\epsilon}] - \delta_x Q_{s, t}[f \mathbb{1}_{K_\epsilon}]| \quad (27)$$

$$\leq 2\epsilon + (1 + l_\epsilon^2) \|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s, t}\|_{TV} \quad (28)$$

Hence, (25) entails that, for  $k$  large enough, for any  $f$  bounded by  $\psi$ ,

$$|\delta_x P_{s+k\gamma, t+k\gamma} f - \delta_x Q_{s, t} f| \leq 3\epsilon, \quad (29)$$

which entails that

$$\|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s, t}\|_\psi \xrightarrow[k \rightarrow \infty]{} 0.$$

□

Now, let us prove Lemma 1.

*Proof of Lemma 1.* Defining

$$f_\nu(x) := e^{-\frac{(x-a)^2}{2b_-^2}} \wedge e^{-\frac{(x+a)^2}{2b_-^2}},$$

one concludes easily that, for any  $m \in [-a, a]$  and  $\sigma \geq b_-$ , for any  $x \in \mathbb{R}$ ,

$$e^{-\frac{(x-m)^2}{2\sigma^2}} \geq f_\nu(x).$$

Imposing moreover that  $\sigma \leq b_+$ , one has

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \geq \frac{1}{\sqrt{2\pi}b_+} f_\nu(x),$$

which concludes the proof. □

## 4.2 Quasi-ergodic distribution for Brownian motion absorbed by an asymptotically periodic moving boundary

Let  $(W_t)_{t \geq 0}$  be a one-dimensional Brownian motion and  $h$  be a  $\mathcal{C}^1$ -function such that

$$h_{\min} := \inf_{t \geq 0} h(t) > 0, \quad \text{and} \quad \sup_{t \geq 0} h(t) < +\infty.$$

Assume also that

$$0 < \inf_{t \geq 0} h'(t) \leq \sup_{t \geq 0} h'(t) < +\infty.$$

Denote by

$$\tau_h := \inf\{t \geq 0 : |W_t| \geq h(t)\}.$$

Since  $h$  is continuous, the hitting time  $\tau_h$  is a stopping time with respect to the natural filtration of  $(W_t)_{t \geq 0}$ . Moreover, since  $\sup_{t \geq 0} h(t) < +\infty$  and  $\inf_{t \geq 0} h(t) > 0$ ,

$$\mathbb{P}_{s,x}[\tau_h < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{s,x}[\tau_h > t] > 0, \quad \forall s \leq t, \forall x \in [-h(s), h(s)].$$

The main assumption on the function  $h$  is the existence of a  $\gamma$ -periodic function  $g$  such that  $h(t) \leq g(t)$ , for any  $t \geq 0$ , and such that

$$h \sim_{t \rightarrow \infty} g, \quad \text{and} \quad h' \sim_{t \rightarrow \infty} g'.$$

Similarly to  $\tau_h$ , denote

$$\tau_g := \inf\{t \geq 0 : |W_t| = g(t)\}.$$

Finally, let us assume that there exists  $n_0 \in \mathbb{N}$  such that, for any  $s \geq 0$ ,

$$\inf\{u \geq s : h(u) = \inf_{t \geq s} h(t)\} - h(s) \leq n_0 \gamma. \quad (30)$$

Then, first of all, let us show the following proposition:

**Proposition 1.** *The Markov process  $(W_t)_{t \geq 0}$ , considered as absorbed by  $h$  or by  $g$ , follows the condition (A'1)-(A'2).*

*Proof.* In what follows, Assumption (A') w.r.t. the absorbing function  $h$  will be shown. The following proof could be easily adapt for the function  $g$ .

- *Proof of (A'1).* Denote  $\mathcal{T} := \{s \geq 0 : h(s) = \inf_{t \geq s} h(t)\}$ . The condition (30) entails that this set contains an infinity of times.

In what follows, the following notation is needed: for any  $z \in \mathbb{R}$ , define  $\tau_z$  as

$$\tau_z := \inf\{t \geq 0 : |W_t| = z\}.$$

Also, let us state that, since the Brownian motion absorbed at  $\{-1, 1\}$  satisfies the Champagnat-Villemonais condition at any time (see [7]), it follows that, for a given  $t_0 > 0$ , there exists  $c > 0$  and  $\nu \in \mathcal{M}_1((-1, 1))$  such that, for any  $x \in (-1, 1)$ ,

$$\mathbb{P}_{0,x} \left[ W_{\frac{t_0}{h_{\max}^2} \wedge t_0} \in \cdot \mid \tau_1 > \frac{t_0}{h_{\max}^2} \wedge t_0 \right] \geq c\nu. \quad (31)$$

Moreover, regarding the proof of [7, Theorem 3.1.], the probability measure  $\nu$  can be expressed as

$$\nu = \frac{1}{2} (\mathbb{P}_{0,1-\epsilon}[W_{t_2} \in \cdot \mid \tau_1 > t_2] + \mathbb{P}_{0,-1+\epsilon}[W_{t_2} \in \cdot \mid \tau_1 > t_2]), \quad (32)$$

for some  $0 < t_2 < \frac{t_0}{h_{\max}^2} \wedge t_0$  and  $\epsilon \in (0, 1)$ .

The following lemma is very important for the following:

**Lemma 2.** *For all  $z \in [h_{\min}, h_{\max}]$ ,*

$$\mathbb{P}_{0,x}[W_u \in \cdot \mid \tau_z > u] \geq c\nu_z, \quad \forall x \in (-z, z), \forall u \geq t_0,$$

where  $t_0$  was evoked before,  $c > 0$  is the same constant as in (31) and

$$\nu_z(f) = \int_{(-1,1)} f(zx)\nu(dx),$$

with  $\nu \in \mathcal{M}_1((-1, 1))$  defined in (32).



The proof of this lemma is postponed after the current proof.

At first, let  $s \in \mathcal{T}$ . Then, for any  $x \in (-h(s), h(s))$ , for any  $t \geq 0$ ,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s+t] \geq \frac{\mathbb{P}_{s,x}[\tau_h(s) > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} \mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h(s) > s+t],$$

By Lemma 2, for any  $x \in (-h(s), h(s))$  and  $t \geq t_0$ ,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h(s) > s+t] \geq c\nu_{h(s)},$$

which entails that, for any  $t \in [t_0, t_0 + n_0\gamma]$ ,

$$\begin{aligned} \mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s+t] &\geq \frac{\mathbb{P}_{s,x}[\tau_h(s) > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} c\nu_{h(s)} \\ &\geq \frac{\mathbb{P}_{s,x}[\tau_h(s) > s+t_0 + n_0\gamma]}{\mathbb{P}_{s,x}[\tau_h > s+t_0]} c\nu_{h(s)}. \end{aligned} \quad (33)$$

Let us introduce the process  $X^h$  defined by, for any  $t \geq 0$ ,

$$X_t^h := \frac{W_t}{h(t)}.$$

By Itô's formula, one has for any  $t$ ,

$$X_t^h = X_0^h + \int_0^t \frac{dW_s}{h(s)} - \int_0^t \frac{h'(s)}{h(s)} X_s^h ds.$$

Denote by  $(M_t^h)_{t \geq 0} := \left( \int_0^t \frac{1}{h(s)} dW_s \right)_{t \geq 0}$ . By Dubin-Schwartz's theorem, it is well known that the process  $M^h$  has the same law as  $\left( W \int_0^t \frac{1}{h^2(s)} ds \right)_{t \geq 0}$ . Then, denoting  $I^h(s) := \int_0^s \frac{1}{h^2(u)} du$  and, for any  $s \leq t$  and for any trajectory  $w$ ,

$$\mathcal{E}_{s,t}^h(w) := \sqrt{\frac{h(t)}{h(s)}} \exp \left( -\frac{1}{2} \left[ h'(t)h(t)w_{I^h(t)}^2 - h'(s)h(s)w_{I^h(s)}^2 + \int_s^t w_{I^h(u)}^2 [(h'(u))^2 - [h(u)h'(u)]'] du \right] \right), \quad (34)$$

Girsanov's theorem entails therefore that, for any  $x \in (-h(s), h(s))$ ,

$$\mathbb{P}_{s,x}[\tau_h > s+t_0] = \mathbb{E}_{I^h(s), \frac{x}{h(s)}} \left[ \mathcal{E}_{s,s+t_0}^h(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du} \right].$$

On the event  $\{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du\}$ , and since  $h$  and  $h'$  are bounded on  $\mathbb{R}_+$ , the random variable  $\mathcal{E}_{s,s+t_0}^h(W)$  is almost surely bounded by a constant  $C > 0$ , uniformly in  $s$ , so that for any  $x \in (-h(s), h(s))$ ,

$$\mathbb{E}_{s, \frac{x}{h(s)}} \left[ \mathcal{E}_{s,s+t_0}^h(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du} \right] \leq C \mathbb{P}_{0, \frac{x}{h(s)}} \left[ \tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right].$$

Since  $h(t) \geq h(s)$  for any  $t \geq s$  (recalling that  $s \in \mathcal{T}$ ),  $I^h(s+t_0) - I^h(s) \leq \frac{t_0}{h(s)^2}$ . Then, by the scaling property of the Brownian motion and by Markov property, for any  $x \in (-h(s), h(s))$ ,

$$\begin{aligned} \mathbb{P}_{s,x}[\tau_h(s) > s+t_0] &= \mathbb{P}_{0,x}[\tau_{h(s)} > t_0] \\ &= \mathbb{P}_{0, \frac{x}{h(s)}} \left[ \tau_1 > \frac{t_0}{h^2(s)} \right] \\ &= \mathbb{E}_{0, \frac{x}{h(s)}} \left[ \mathbb{1}_{\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du} \mathbb{P}_{0,W} \int_s^{s+t_0} \frac{1}{h^2(u)} du \left[ \tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(s)} ds \right] \right] \\ &= \mathbb{P}_{0, \frac{x}{h(s)}} \left[ \tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right] \mathbb{P}_{0, \phi_{I^h(s+t_0) - I^h(s)}(\delta_x)} \left[ \tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right], \end{aligned}$$

where, for any initial distribution  $\mu$  and any  $t \geq 0$ ,

$$\phi_t(\mu) := \mathbb{P}_{0,\mu}[W_t \in \cdot | \tau_1 > t].$$

In particular, the family  $(\phi_t)_{t \geq 0}$  is a semi-flow, i.e.  $\phi_t \circ \phi_s = \phi_{t+s}$  for all  $s, t$ . Hence, by this property of semi-flow and using that  $I^h(s+t_0) - I^h(s) \geq \frac{t_0}{h_{\max}^2}$  for any  $s \geq 0$ , the minorization (31) entails that, for all  $s \geq 0$  and  $x \in (-1, 1)$ ,

$$\phi_{I^h(s+t_0)-I^h(s)}(\delta_x) \geq c\nu.$$

Then, by this minorization and using that  $h$  is upper-bounded and lower-bounded positively on  $\mathbb{R}_+$ , one has, for any  $x \in (-1, 1)$ ,

$$\mathbb{P}_{0,\phi_{I^h(s+t_0)-I^h(s)}(\delta_x)} \left[ \tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right] \geq c \mathbb{P}_{0,\nu} \left[ \tau_1 > \inf_{s \geq 0} \left\{ \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right],$$

that is to say,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0]}{\mathbb{P}_{0,\frac{x}{h(s)}} \left[ \tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right]} \geq c \mathbb{P}_{0,\nu} \left[ \tau_1 > \inf_{s \geq 0} \left\{ \frac{\gamma}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right].$$

In other words, we just showed that, for any  $x \in (-h(s), h(s))$ ,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0]}{\mathbb{P}_{s,x}[\tau_h > s+t_0]} \geq \frac{c}{C} \mathbb{P}_{0,\nu} \left[ \tau_1 > \inf_{s \geq 0} \left\{ \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right] > 0. \quad (35)$$

Moreover, by Lemma 2 and the scaling property of the Brownian motion, for any  $x \in (-h(s), h(s))$ ,

$$\begin{aligned} \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0+n_0\gamma]}{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0]} &= \mathbb{P}_{0,\mathbb{P}_{0,x}[W_{t_0} \in \cdot | \tau_{h(s)} > t_0]}[\tau_{h(s)} > n_0\gamma] \\ &\geq c \mathbb{P}_{0,\nu_{h(s)}}[\tau_{h(s)} > n_0\gamma] \\ &= c \int_{(-1,1)} \nu(dy) \mathbb{P}_{h(s)y}[\tau_{h(s)} > n_0\gamma] \\ &\geq c \mathbb{P}_{0,\nu} \left[ \tau_1 > \frac{n_0\gamma}{h_{\min}^2} \right] > 0. \end{aligned} \quad (36)$$

To sum up, gathering (33), (35) and (36), for any  $x \in (-h(s), h(s))$ , for any  $t \in [t_0, t_0 + n_0\gamma]$ ,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s+t] \geq c_1 \nu_{h(s)},$$

where  $c_1 := c \mathbb{P}_{0,\nu} \left[ \tau_1 > \frac{n_0\gamma}{h_{\max}^2} \right] \times \frac{c}{C} \mathbb{P}_{0,\nu} \left[ \tau_1 > \inf_{s \geq 0} \left\{ \frac{\gamma}{h^2(s)} - \int_s^{s+\gamma} \frac{1}{h^2(u)} du \right\} \right] c$ .

Now, consider  $s \geq 0$ . If  $s \notin \mathcal{T}$ , there exists  $s_1 \in \mathcal{T}$  such that  $s < s_1 \leq s + n_0\gamma$ . Then, Markov property entails that, for any  $x \in (-h(s), h(s))$ ,

$$\mathbb{P}_{s,x}[W_{s+t_0+n_0\gamma} \in \cdot | \tau_h > s+t_0+n_0\gamma] = \mathbb{P}_{s_1,\phi_{s_1,s}}[W_{s+t_0+n_0\gamma} \in \cdot | \tau_h > s+t_0+n_0\gamma] \geq c_1 \nu_{h(s_1)},$$

where, for any  $s \leq t$  and  $\mu \in \mathcal{M}_1((-h(s), h(s)))$ ,  $\phi_{t,s}(\mu) := \mathbb{P}_{s,\mu}[W_t \in \cdot | \tau_h > t]$ . This concludes the proof of (A'1).

- *Proof of (A'2).* Since  $(W_t)_{t \geq 0}$  is a Brownian motion, note that, for any  $s \leq t$ ,

$$\sup_{x \in (-1,1)} \mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,0}[\tau_h > t],$$

and that, for any  $a \in (0, h(s))$ ,

$$\inf_{[-a,a]} \mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,a}[\tau_h > t].$$

Then, by Markov property, and using that the function  $s \mapsto \mathbb{P}_{s,0}[\tau_g > t]$  is non-decreasing on  $[0, t]$  (for all  $t \geq 0$ ), then one has for any  $s \leq t$ ,

$$\mathbb{P}_{s,a}[\tau_h > t] \geq \mathbb{E}_{s,a}[\mathbb{1}_{\tau_0 < s + \gamma < \tau_h} \mathbb{P}_{\tau_0,0}[\tau_h > t]] \geq \mathbb{P}_{s,a}[\tau_0 < s + \gamma < \tau_h] \mathbb{P}_{s,0}[\tau_h > t]. \quad (37)$$

Then, denoting  $a := \frac{h_{\min}}{h_{\max}}$ , by Lemma 2 and taking  $s_1 := \inf\{u \geq s : u \in \mathcal{T}\}$ , one obtains for any  $s \leq t$ ,

$$\begin{aligned} \mathbb{P}_{s,\nu_{h(s_1)}}[\tau_h > t] &= \int_{(-1,1)} \nu(dx) \mathbb{P}_{s,h(s_1)x}[\tau_h > t] \\ &\geq \nu([-a, a]) \mathbb{P}_{s,h(s_1)a}[\tau_h > t] \\ &\geq \nu([-a, a]) \mathbb{P}_{0,h_{\min}}[\tau_0 < \gamma < \tau_h] \sup_{x \in (-h(s), h(s))} \mathbb{P}_{s,x}[\tau_h > t]. \end{aligned}$$

This concludes the proof since, using (32), one has  $\nu([-a, a]) > 0$ . □

*Proof of Lemma 2.* This result comes from the scaling property of a Brownian motion: as a matter of fact, for any  $z \in [h_{\min}, h_{\max}]$ ,  $x \in (-z, z)$  and  $t \geq 0$ , and for any measurable bounded function  $f$ ,

$$\begin{aligned} \mathbb{E}_{0,x}[f(W_t) | \tau_z > t] &= \mathbb{E}_{0,x} \left[ f \left( z \times \frac{1}{z} W_{z^2 \frac{t}{z^2}} \right) \middle| \tau_z > t \right] \\ &= \mathbb{E}_{0, \frac{x}{z}} \left[ f \left( z \times W_{\frac{t}{z^2}} \right) \middle| \tau_1 > \frac{t}{z^2} \right]. \end{aligned}$$

Then, the minorization (31) entails that, for any  $x \in (-1, 1)$ ,

$$\mathbb{P}_{0,x} \left[ W_{\frac{t_0}{h_{\max}^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{h_{\max}^2} \right] \geq c\nu.$$

This inequality actually hold for any time greater than  $\frac{t_0}{h_{\max}^2}$ . In particular, for any  $z \in [h_{\min}, h_{\max}]$ ,  $x \in (-1, 1)$ ,

$$\mathbb{P}_{0,x} \left[ W_{\frac{t_0}{z^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{z^2} \right] \geq c\nu.$$

Then, for any  $z \in [a, b]$ ,  $f$  positive measurable and  $x \in (-z, z)$ ,

$$\mathbb{E}_{0,x}[f(W_{t_0}) | \tau_z > t_0] \geq c\nu_z(f),$$

where  $\nu_z(f) = \int_E f(z \times x) \nu(dx)$ . □

Now, the section is concluded by stating and proving the following result:

**Theorem 4.** *For any  $s \leq t$ ,  $n \in \mathbb{N}$  and any  $x \in \mathbb{R}$ ,*

$$\mathbb{P}_{s+k\gamma,x}[\tau_h \leq t + k\gamma < \tau_g] \xrightarrow[k \rightarrow \infty]{} 0.$$

*In particular, Corollary 2 holds for  $(W_t)_{t \geq 0}$  absorbed by  $h$ .*

*Proof.* Reminding (34) and by Markov property for the Brownian motion, one obtains that, for any  $k, n \in \mathbb{N}$ , for any  $x \in \mathbb{R}$ ,

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t + k\gamma] = \sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \mathbb{E}_{0,x} \left[ \exp \left( -\frac{1}{2} \mathcal{A}_{s,t,k}^h(W) \right) \mathbb{1}_{\tau_1 > I^h(t+k\gamma) - I^h(s+k\gamma)} \right],$$

where, for any trajectory  $w$ ,

$$\begin{aligned} \mathcal{A}_{s,t,k}^h(w) &= h'(t+k\gamma)h(t+k\gamma)w_{I^h(t+k\gamma)-I^h(s+k\gamma)}^2 - h'(s+k\gamma)h(s+k\gamma)w_0^2 \\ &\quad + \int_0^{t-s} w_{I^h(u+s+k\gamma)-I^h(s+k\gamma)}^2 [(h'(u+s+k\gamma))^2 - [h(u+s+k\gamma)h'(u+s+k\gamma)]'] du. \end{aligned}$$

First, since  $h \sim_{t \rightarrow \infty} g$ , for any  $s \in [0, \gamma]$  and  $n \in \mathbb{N}$ ,

$$\sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \xrightarrow{k \rightarrow \infty} \sqrt{\frac{g(t)}{g(s)}}.$$

Then, for the same reasons and using that the function  $h$  is bounded on  $[s+k\gamma, t+k\gamma]$  for any  $s \leq t$ , Lebesgue's theorem entails that

$$I^h(t+k\gamma) - I^h(s+k\gamma) \xrightarrow{k \rightarrow \infty} I^g(t) - I^g(s).$$

Moreover, since  $h \sim_{t \rightarrow \infty} g$  and  $h' \sim_{t \rightarrow \infty} g'$ , one has the following convergence for any trajectory  $w$ :

$$\mathcal{A}_{s,t,k}^h(w) \xrightarrow{k \rightarrow \infty} g'(t)g(t)w_{I^g(t)-I^g(s)}^2 - g'(s)g(s)w_0^2 + \int_s^t w_{I^g(u)}^2 [(g'(u))^2 - [g(u)g'(u)]'] du.$$

Then, since the random variable  $\exp\left(-\frac{1}{2}\mathcal{A}_{s,t,k}^h(W)\right) \mathbb{1}_{\tau_1 > I^h(t+k\gamma)-I^h(s+k\gamma)}$  is bounded almost surely, by Lebesgue's theorem,

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t+k\gamma] \xrightarrow{k \rightarrow \infty} \mathbb{P}_{s,x}[\tau_g > t],$$

which concludes the proof.  $\square$

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