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# An ergodic theorem for asymptotically periodic time-inhomogeneous Markov processes, with application to quasi-stationarity with moving boundaries.

William Oçafrain<sup>1</sup>

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## Abstract

This paper deals with ergodic theorems for particular time-inhomogeneous Markov processes, whose time-inhomogeneity is asymptotically periodic. Under a Lyapunov/minorization condition, it is shown that, for any measurable bounded function  $f$ , the time average  $\frac{1}{t} \int_0^t f(X_s) ds$  converges in  $\mathbb{L}^2$  towards a limiting distribution, starting from any initial distribution for the process  $(X_t)_{t \geq 0}$ . This convergence can be improved to an almost sure convergence under an additional assumption on the initial measure. This result will be then applied to show the existence of a quasi-ergodic distribution for processes absorbed by an asymptotically periodic moving boundary, satisfying a conditional Doeblin condition.

*Key words: ergodic theorem; law of large numbers; time-inhomogeneous Markov processes; quasi-stationarity; quasi-ergodic distribution; moving boundaries.*

## Notation

Throughout we shall use the following notation:

- $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ .
- $\mathcal{M}_1(E)$  denotes the space of the probability measures whose support is included in  $E$ .
- $\mathcal{B}(E)$  denotes the set of the measurable bounded functions defined on  $E$ .
- $\mathcal{B}_1(E)$ : denotes the set of the measurable functions  $f$  defined on  $E$  such that  $\|f\|_\infty \leq 1$ .
- For all  $\mu \in \mathcal{M}_1(E)$  and  $p \in \mathbb{N}$ ,  $\mathbb{L}^p(\mu)$  denotes the set of the measurable functions  $f : E \mapsto \mathbb{R}$  such that  $\int_E |f(x)|^p \mu(dx) < +\infty$ .
- For any  $\mu \in \mathcal{M}_1(E)$  and  $f \in \mathbb{L}^1(\mu)$ , denote

$$\mu(f) := \int_E f(x) \mu(dx).$$

- For any positive function  $\psi$ ,

$$\mathcal{M}_1(\psi) := \{\mu \in \mathcal{M}_1(E) : \mu(\psi) < +\infty\}.$$

- Id denotes the Identity operator.

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# 1 Introduction

In general, an ergodic theorem for a Markov process  $(X_t)_{t \geq 0}$  and probability measure  $\pi$  refers to the almost sure convergence

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \pi(f), \quad \forall f \in \mathbb{L}^1(\pi). \quad (1)$$

In the time-homogeneous setting, such an ergodic theorem holds for positive Harris recurrent Markov processes with the limiting distribution  $\pi$  corresponding to an invariant measure for the underlying Markov process. For time-inhomogeneous Markov processes, such a result does not hold in general (in particular the notion of invariant measure is in general not well-defined), except for specific types of time-inhomogeneity such as *periodic time-inhomogeneous Markov processes*, defined as time-inhomogeneous Markov processes for which there exists  $\gamma > 0$  such that, for any  $s \leq t$ ,  $k \in \mathbb{Z}_+$  and  $x$ ,

$$\mathbb{P}[X_t \in \cdot | X_s = x] = \mathbb{P}[X_{t+k\gamma} \in \cdot | X_{s+k\gamma} = x]. \quad (2)$$

In other words, a time-inhomogeneous Markov process is periodic when the transition law between any times  $s$  and  $t$  remains unchanged when the time interval  $[s, t]$  is shifted by a multiple of the period  $\gamma$ . In particular, this implies that, for any  $s \in [0, \gamma)$ , the Markov chain  $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$  is time-homogeneous. This fact allowed Höpfner et al. (in [20, 21, 22]) to show that, if the skeleton Markov chain  $(X_{n\gamma})_{n \in \mathbb{Z}_+}$  is Harris recurrent, then the chains  $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$ , for all  $s \in [0, \gamma)$ , are also Harris recurrent and

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \pi_s(f) ds, \quad \text{almost surely, from any initial measure,}$$

where  $\pi_s$  is the invariant measure for  $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$ .

This paper aims to prove a similar result for time-inhomogeneous Markov processes said to be *asymptotically periodic*. Roughly speaking (a precise definition of which will be explicitly given later), an asymptotically periodic Markov process is such that, given a time interval  $T \geq 0$ , its transition law on the interval  $[s, s+T]$  is asymptotically "close to" the one, on the same interval, of a periodic time-inhomogeneous Markov process called an *auxiliary Markov process*, when  $s \rightarrow \infty$ . This definition is very similar to the notion of *asymptotic homogenization*, defined as follows in [1, Subsection 3.3]: a time-inhomogeneous Markov process  $(X_t)_{t \geq 0}$  is said to be *asymptotically homogeneous* if there exists a time-homogeneous Markovian semigroup  $(Q_t)_{t \geq 0}$  such that, for all  $s \geq 0$ ,

$$\lim_{t \rightarrow \infty} \sup_x \|\mathbb{P}[X_{t+s} \in \cdot | X_t = x] - \delta_x Q_s\|_{TV} = 0, \quad (3)$$

where, for two positive measures with finite mass  $\mu_1$  and  $\mu_2$ ,  $\|\mu_1 - \mu_2\|_{TV}$  is the *total variation distance* between  $\mu_1$  and  $\mu_2$ :

$$\|\mu_1 - \mu_2\|_{TV} := \sup_{f \in \mathcal{B}_1(E)} |\mu_1(f) - \mu_2(f)|. \quad (4)$$

In particular, it is well-known (see [1, Theorem 3.11]) that, under this suitable additional conditions, an asymptotically homogeneous Markov process converges towards a probability measure which is invariant for  $(Q_t)_{t \geq 0}$ . It is similarly expected that an asymptotically periodic process has the same asymptotic properties as a periodic Markov process; in particular an ergodic theorem holds for the asymptotically periodic process.

The main result of this paper provides for an asymptotically periodic Markov process to satisfy

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{\mathbb{L}^2(\mathbb{P}_{0,\mu})} \frac{1}{\gamma} \int_0^\gamma \beta_s(f) ds, \quad \forall f \in \mathcal{B}(E), \forall \mu \in \mathcal{M}_1(E), \quad (5)$$

whereby  $\mathbb{P}_{0,\mu}$  is a probability measure under which  $X_0 \sim \mu$ , and where  $\beta_s$  is the limiting distribution of the skeleton Markov chain  $(X_{s+n\gamma})_{n \in \mathbb{Z}_+}$ , if it satisfies a Lyapunov-type condition, a local Doeblin condition (defined further in Section 2), and is such that its auxiliary process satisfies a Lyapunov/minorization condition.

Furthermore, this convergence result holds almost surely if a Lyapunov function of the process  $(X_t)_{t \geq 0}$ , denoted by  $\psi$ , is integrable with respect to the initial measure:

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{\mathbb{P}_{0,\mu}\text{-almost surely}} \frac{1}{\gamma} \int_0^\gamma \beta_s(f) ds, \quad \forall \mu \in \mathcal{M}_1(\psi).$$

This will be more precisely stated and proved in Section 2.

The main motivation of this paper is then to deal with *quasi-stationarity with moving boundaries*, that is the study of asymptotic properties for the process  $X$ , conditioned not to reach some moving subset of the state space. In particular, such a study is motivated by models such as those presented in [3], studying Brownian particles absorbed by cells whose volume may vary over time.

Quasi-stationarity with moving boundaries has been studied in particular in [24, 25], where a "*conditional ergodic theorem*" (see further the definition of a *quasi-ergodic distribution*) has been shown when the absorbing boundaries move periodically. In this paper, we show that a similar result holds when the boundary is asymptotically periodic, assuming that the process satisfies a conditional Doeblin condition (see Assumption (A')). This will be dealt with in Section 3.

The paper will be concluded by using these results in two examples: an ergodic theorem for an asymptotically periodic Ornstein-Uhlenbeck process, and the existence of a unique quasi-ergodic distribution for a Brownian motion confined between two symmetric asymptotically periodic functions.

## 2 Ergodic theorem for asymptotically periodic time-inhomogeneous semigroup.

### Asymptotic periodicity: the definition.

Let  $(E, \mathcal{E})$  be a measurable space. Consider  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$  a Markovian time-inhomogeneous semigroup, giving a family of measurable subspaces of  $(E, \mathcal{E})$ , denoted by  $(E_t, \mathcal{E}_t)_{t \geq 0}$ , and a family of linear operator  $(P_{s,t})_{s \leq t}$ , with  $P_{s,t} : \mathcal{B}(E_t) \rightarrow \mathcal{B}(E_s)$ , satisfying for any  $r \leq s \leq t$ ,

$$P_{s,s} = \text{Id}, \quad P_{s,t} \mathbb{1}_{E_t} = \mathbb{1}_{E_s}, \quad P_{r,s} P_{s,t} = P_{r,t}.$$

In particular, associated to  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$  is a Markov process  $(X_t)_{t \geq 0}$  and a family of probability measures  $(\mathbb{P}_{s,x})_{s \geq 0, x \in E_s}$  such that, for any  $s \leq t$ ,  $x \in E_s$  and  $A \in \mathcal{E}_t$ ,

$$\mathbb{P}_{s,x}[X_t \in A] = P_{s,t} \mathbb{1}_A(x).$$

We denote  $\mathbb{P}_{s,\mu} := \int_{E_s} \mathbb{P}_{s,x} \mu(dx)$  for any probability measure  $\mu$  supported on  $E_s$ . We also denote by  $\mathbb{E}_{s,x}$  and  $\mathbb{E}_{s,\mu}$  the expectations associated to  $\mathbb{P}_{s,x}$  and  $\mathbb{P}_{s,\mu}$  respectively. Finally, the following notation will be used for  $\mu \in \mathcal{M}_1(E_s)$ ,  $s \leq t$  and  $f \in \mathcal{B}(E_t)$ ,

$$\mu P_{s,t} f := \mathbb{E}_{s,\mu}[f(X_t)], \quad \mu P_{s,t} := \mathbb{P}_{s,\mu}[X_t \in \cdot].$$

The periodicity of a time-inhomogeneous semigroup is defined as follows. We say a semigroup  $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$  is  $\gamma$ -*periodic* (for  $\gamma > 0$ ), if, for any  $s \leq t$ ,

$$(F_t, \mathcal{F}_t) = (F_{t+k\gamma}, \mathcal{F}_{t+k\gamma}), \quad Q_{s,t} = Q_{s+k\gamma, t+k\gamma}, \quad \forall k \in \mathbb{Z}_+.$$

It is now possible to define an *asymptotically periodic semigroup*.

**Definition 1** (Asymptotically periodic semigroups). A time-inhomogeneous semigroup  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$  is said to be *asymptotically periodic* if (for some  $\gamma > 0$ ) there exists a  $\gamma$ -periodic semigroup  $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$  and two families of functions  $(\psi_s)_{s \geq 0}$  and  $(\tilde{\psi}_s)_{s \geq 0}$  such that  $\tilde{\psi}_{s+\gamma} = \psi_s$  for all  $s \geq 0$ , and for any  $s \in [0, \gamma)$ :

1.  $\bigcup_{k=0}^{\infty} \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s \neq \emptyset$ ;
2. there exists  $x_s \in \bigcup_{k=0}^{\infty} \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s$  such that, for any  $n \in \mathbb{Z}_+$ ,

$$\|\delta_{x_s} P_{s+k\gamma, s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \delta_{x_s} Q_{s, s+n\gamma}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0. \quad (6)$$

The semigroup  $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$  is then called the *auxiliary semigroup of  $(P_{s,t})_{s \leq t}$* .

When  $\psi_s = \tilde{\psi}_s = \mathbb{1}$  for all  $s \geq 0$ , we say that the semigroup  $(P_{s,t})_{s \leq t}$  is *asymptotically periodic in total variation*. By extension, we will say that the process  $(X_t)_{t \geq 0}$  is asymptotically periodic (in total variation) if the associated semigroup  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}\}$  is asymptotically periodic (in total variation).

In what follows, the functions  $(\psi_s)_{s \geq 0}$  and  $(\tilde{\psi}_s)_{s \in [0, \gamma)}$  will play the role of Lyapunov functions (that is to say satisfying Assumption 1 (ii) below) for the semigroups  $(P_{s,t})_{s \leq t}$  and  $(Q_{s,t})_{s \leq t}$  respectively. The introduction of these functions in the definition of asymptotically periodic semigroups will allow us to establish an ergodic theorem for processes satisfying the Lyapunov/minorization conditions written below.

### Lyapunov/minorization conditions.

The main assumption of Theorem 1, which will be provided later, will be that the asymptotically periodic Markov process satisfies the following assumption.

**Assumption 1.** *There exist  $t_1 \geq 0$ ,  $n_0 \in \mathbb{N}$ ,  $c > 0$ ,  $\theta \in (0, 1)$ , a family of measurable sets  $(K_t)_{t \geq 0}$  such that  $K_t \subset E_t$  for all  $t \geq 0$ , a family of probability measures  $(\nu_s)_{s \geq 0}$  on  $(K_s)_{s \geq 0}$ , and a family of functions  $(\psi_s)_{s \geq 0}$ , all lower-bounded by 1, such that:*

(i) *For any  $s \geq 0$ ,  $x \in K_s$  and  $n \geq n_0$ ,*

$$\delta_x P_{s, s+nt_1} \geq c\nu_{s+nt_1}.$$

(ii) *For any  $s \geq 0$ ,*

$$P_{s, s+t_1} \psi_{s+t_1} \leq \theta \psi_s + C \mathbb{1}_{K_s}.$$

(iii) *For any  $s \geq 0$  and  $t \in [0, t_1)$ ,*

$$P_{s, s+t} \psi_{s+t} \leq C \psi_s.$$

When a semigroup  $(P_{s,t})_{s \leq t}$  satisfies Assumption 1 as stated above, we will say that the family of functions  $(\psi_s)_{s \geq 0}$  are *Lyapunov functions* for the semigroup  $(P_{s,t})_{s \leq t}$ . In particular, under (ii) and (iii), it is easy to prove that for any  $s \leq t$ ,

$$P_{s,t} \psi_t \leq C \left( 1 + \frac{C}{1-\theta} \right) \psi_s. \quad (7)$$

We remark in particular that Assumption 1 implies an *exponential weak ergodicity in  $\psi_t$ -distance*, that is we have the existence of two constants  $C' > 0$  and  $\kappa > 0$  such that, for all  $s \leq t$  and for all probability measures  $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$ ,

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{\psi_t} \leq C' [\mu_1(\psi_s) + \mu_2(\psi_s)] e^{-\kappa(t-s)}, \quad (8)$$

whereby, for a given function  $\psi$ ,  $\|\mu - \nu\|_{\psi}$  is the  $\psi$ -distance defined to be

$$\|\mu - \nu\|_{\psi} := \sup_{|f| \leq \psi} |\mu(f) - \nu(f)|, \quad \forall \mu, \nu \in \mathcal{M}_1(\psi).$$

In particular, when  $\psi = \mathbb{1}$  for all  $t \geq 0$ , the  $\psi$ -distance is the total variation distance. If we have weak ergodicity (8) in the time-homogeneous setting (see in particular [15]), the proof of [15, Theorem 1.3.] can be adapted to a general time-inhomogeneous framework (see for example [6, Subsection 9.5]).

### The main theorem and proof.

The main result of this paper is the following.

**Theorem 1.** *Let  $\{(E_t, \mathcal{E}_t)_{t \geq 0}, (P_{s,t})_{s \leq t}, (X_t)_{t \geq 0}, (\mathbb{P}_{s,x})_{s \geq 0, x \in E_s}\}$  be an asymptotically  $\gamma$ -periodic time-inhomogeneous Markov process, with  $\gamma > 0$ , and denote by  $\{(F_t, \mathcal{F}_t)_{t \geq 0}, (Q_{s,t})_{s \leq t}\}$  its periodic auxiliary semigroup. Also, denote by  $(\psi_s)_{s \geq 0}$  and  $(\tilde{\psi}_s)_{s \geq 0}$  the two families of functions as defined in Definition 1. Assume moreover that:*

1. *The semigroups  $(P_{s,t})_{s \leq t}$  and  $(Q_{s,t})_{s \leq t}$  satisfy Assumption 1, with  $(\psi_s)_{s \geq 0}$  and  $(\tilde{\psi}_s)_{s \geq 0}$  as Lyapunov functions respectively.*
2. *For any  $s \in [0, \gamma)$ ,  $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$  converges pointwisely to  $\tilde{\psi}_s$ .*

Then, for any  $\mu \in \mathcal{M}_1(E_0)$  such that  $\mu(\psi_0) < +\infty$ ,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s}[\psi_s \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0, \quad (9)$$

whereby  $\beta_\gamma \in \mathcal{M}_1(F_0)$  is the unique invariant probability measure of the skeleton semigroup  $(Q_{0,n\gamma})_{n \in \mathbb{Z}_+}$  satisfying  $\beta_\gamma(\tilde{\psi}_0) < +\infty$ . Moreover, for any  $f \in \mathcal{B}(E)$  we have:

1. For any  $\mu \in \mathcal{M}_1(E_0)$ ,

$$\mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0. \quad (10)$$

2. If moreover  $\mu(\psi_0) < +\infty$ , then

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow{t \rightarrow \infty} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely}. \quad (11)$$

*Remark 1.* When Assumption 1 hold for  $K_s = E_s$  for any  $s$ , then the condition (i) in Assumption 1 implies Doebelin condition.

**Doebelin condition.** There exists  $t_0 \geq 0$ ,  $c > 0$  and a family of probability measure  $(\nu_t)_{t \geq 0}$  on  $(E_t)_{t \geq 0}$  such that, for any  $s \geq 0$  and  $x \in E_s$ ,

$$\delta_x P_{s,s+t_0} \geq c\nu_{s+t_0}. \quad (12)$$

In fact, if we assume that Assumption 1 (i) holds for  $K_s = E_s$ , Doebelin condition holds by setting  $t_0 := n_0 t_1$ . Conversely, the Doebelin condition implies the conditions (i)-(ii)-(iii) with  $K_s = E_s$  and  $\psi_s = \mathbb{1}_{E_s}$  for all  $s \geq 0$ , so that these conditions are equivalent. In fact, (ii) and (iii) straightforwardly hold true for  $(K_s)_{s \geq 0} = (E_s)_{s \geq 0}$ ,  $(\psi_s)_{s \geq 0} = (\mathbb{1}_{E_s})_{s \geq 0}$ ,  $C = 1$ , any  $\theta \in (0, 1)$  and any  $t_1 \geq 0$ . Setting  $t_1 = t_0$  and  $n_0 = 1$ , the Doebelin condition entails that, for any  $s \in [0, t_1)$ ,

$$\delta_x P_{s,s+t_1} \geq c\nu_{s+t_1}, \quad \forall x \in E_s.$$

Integrating this inequality over  $\mu \in \mathcal{M}_1(E_s)$ , one obtains

$$\mu P_{s,s+t_1} \geq c\nu_{s+t_1}, \quad \forall s \in [0, t_1), \forall \mu \in \mathcal{M}_1(E_s).$$

Then, by the Markov property, for all  $s \in [0, t_1)$ ,  $x \in E_s$  and  $n \in \mathbb{N}$ , we have

$$\delta_x P_{s,s+nt_1} = (\delta_x P_{s,s+(n-1)t_1}) P_{s+(n-1)t_1,s+nt_1} \geq c\nu_{s+nt_1},$$

which is (i).

Theorem 1 then entails the following corollary.

**Corollary 1.** Let  $(X_t)_{t \geq 0}$  be asymptotically  $\gamma$ -periodic in total variation distance. If  $(X_t)_{t \geq 0}$  and its auxiliary semigroup satisfy a Doebelin condition, then the convergence (10) is improved to

$$\sup_{\mu \in \mathcal{M}_1(E_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0.$$

Moreover, the almost sure convergence (11) holds for any initial measure  $\mu$ .

*Remark 2.* We also note that, if the convergence (6) holds for all  $x \in \bigcup_{k=0}^\infty \bigcap_{l \geq k} E_{s+l\gamma} \cap F_s$ , then this implies (6) and therefore the pointwise convergence of  $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$  to  $\tilde{\psi}_s$  (by taking  $n = 0$  in (6)).

*Proof of Theorem 1.* The proof is divided into five steps.

*First step.* Since the auxiliary semigroup  $(Q_{s,t})_{s \leq t}$  satisfies Assumption 1 with  $(\tilde{\psi}_s)_{s \geq 0}$  as Lyapunov functions, the time-homogeneous semigroup  $(Q_{0,n\gamma})_{n \in \mathbb{Z}_+}$  satisfies assumptions 1 and 2 of [15], which we now recall (using our notation).

**Assumption 2** (Assumption 1, [15]). *There exists  $V : F_0 \rightarrow [0, +\infty)$ ,  $n_1 \in \mathbb{N}$  and constants  $K \geq 0$  and  $\kappa \in (0, 1)$  such that*

$$Q_{0, n_1 \gamma} V \leq \kappa V + K.$$

**Assumption 3** (Assumption 2, [15]). *There exists a constant  $\alpha \in (0, 1)$  and a probability measure  $\nu$  such that*

$$\inf_{x \in \mathcal{C}_R} \delta_x Q_{0, n_1 \gamma} \geq \alpha \nu(\cdot),$$

with  $\mathcal{C}_R := \{x \in F_0 : V(x) \leq R\}$  for some  $R > 2K/(1 - \kappa)$ , whereby  $n_1$ ,  $K$  and  $\kappa$  are the constants from Assumption 2.

In fact, since  $(Q_{s,t})_{s \leq t}$  satisfies (ii) and (iii) of Assumption 1, there exists  $C > 0$ ,  $\theta \in (0, 1)$ ,  $t_1 \geq 0$  and  $(K_s)_{s \geq 0}$  such that

$$Q_{s, s+t_1} \tilde{\psi}_{s+t_1} \leq \theta \tilde{\psi}_s + C \mathbb{1}_{K_s}, \quad \forall s \geq 0, \quad (13)$$

and

$$Q_{s, s+t} \tilde{\psi}_{s+t} \leq C \tilde{\psi}_s, \quad \forall s \geq 0, \forall t \in [0, t_1].$$

We let  $n_2 \in \mathbb{N}$  be such that  $\theta^{n_2} C (1 + \frac{C}{1-\theta}) < 1$ . By (13) and recalling that  $\tilde{\psi}_t = \tilde{\psi}_{t+\gamma}$  for all  $t \geq 0$ , one has for any  $s \geq 0$  and  $n \in \mathbb{N}$ ,

$$Q_{s, s+nt_1} \tilde{\psi}_{s+nt_1} \leq \theta^n \tilde{\psi}_s + \frac{C}{1-\theta}. \quad (14)$$

Thus, for all  $n_1 \geq \lceil \frac{n_2 t_1}{\gamma} \rceil$ ,

$$Q_{0, n_1 \gamma} \tilde{\psi}_0 = Q_{0, n_1 \gamma - n_2 t_1} Q_{n_1 \gamma - n_2 t_1, n_1 \gamma} \tilde{\psi}_{n_1 \gamma} \leq \theta^{n_2} Q_{0, n_1 \gamma - n_2 t_1} \tilde{\psi}_{n_1 \gamma - n_2 t_1} + \frac{C}{1-\theta} \leq \theta^{n_2} C (1 + \frac{C}{1-\theta}) \tilde{\psi}_0 + \frac{C}{1-\theta},$$

where we successively used the property of semigroup of  $(Q_{s,t})_{s \leq t}$ , (14) and (7) applied to  $(Q_{s,t})_{s \leq t}$ . Hence one has Assumption 2 by setting  $V = \tilde{\psi}_0$ ,  $\kappa := \theta^{n_2} C (1 + \frac{C}{1-\theta})$  and  $K := \frac{C}{1-\theta}$ .

We now prove Assumption 3. To this end, we introduce a Markov process  $(Y_t)_{t \geq 0}$  and a family of probability measures  $(\hat{\mathbb{P}}_{s,x})_{s \geq 0, x \in F_s}$  such that

$$\hat{\mathbb{P}}_{s,x}(Y_t \in A) = Q_{s,t} \mathbb{1}_A(x), \quad \forall s \leq t, x \in F_s, A \in \mathcal{F}_t.$$

In what follows, for all  $s \geq 0$  and  $x \in F_s$ , we will use the notation  $\hat{\mathbb{E}}_{s,x}$  for the expectation associated to  $\hat{\mathbb{P}}_{s,x}$ . Moreover, we define

$$T_K := \inf\{n \in \mathbb{Z}_+ : Y_{nt_1} \in K_{nt_1}\}.$$

Then, using (13) recursively, for all  $k \in \mathbb{N}$ ,  $R > 0$  and  $x \in \mathcal{C}_R$  (recalling that  $\mathcal{C}_R$  is defined in the statement of Assumption 3) we have

$$\begin{aligned} \hat{\mathbb{E}}_{0,x}[\tilde{\psi}_{kt_1}(Y_{kt_1}) \mathbb{1}_{T_K > k}] &= \hat{\mathbb{E}}_{0,x}[\mathbb{1}_{T_K > k-1} \hat{\mathbb{E}}_{(k-1)t_1, Y_{(k-1)t_1}}(\tilde{\psi}_{kt_1}(Y_{kt_1}) \mathbb{1}_{T_K > k})] \\ &\leq \theta \hat{\mathbb{E}}_{0,x}[\tilde{\psi}_{(k-1)t_1}(Y_{(k-1)t_1}) \mathbb{1}_{T_K > k-1}] \leq \theta^k \tilde{\psi}_0(x) \leq R \theta^k. \end{aligned}$$

Since  $\tilde{\psi}_{kt_1} \geq 1$  for all  $k \in \mathbb{Z}_+$ , then for all  $x \in \mathcal{C}_R$ , for all  $k \in \mathbb{Z}_+$ ,

$$\hat{\mathbb{P}}_{0,x}(T_K > k) \leq R \theta^k.$$

In particular, there exists  $k_0 \geq n_0$  such that, for all  $k \geq k_0 - n_0$ ,

$$\hat{\mathbb{P}}_{0,x}(T_K > k) \leq \frac{1}{2}.$$

Hence, for all  $x \in \mathcal{C}_R$ ,

$$\begin{aligned} \delta_x Q_{0,k_0 t_1} &= \hat{\mathbb{P}}_{0,x}(Y_{k_0 t_1} \in \cdot) \geq \sum_{i=0}^{k_0-n_0} \hat{\mathbb{E}}_{0,x}(\mathbb{1}_{T_k=i} \hat{\mathbb{P}}_{it_1, X_{it_1}}(Y_{k_0 t_1} \in \cdot)) \\ &\geq c \sum_{i=0}^{k_0-n_0} \hat{\mathbb{E}}_{0,x}(\mathbb{1}_{T_K=i}) \times \nu_{k_0 t_1} \\ &= c \hat{\mathbb{P}}_{0,x}(T_K \leq k_0 - n_0) \nu_{k_0 t_1} \\ &\geq \frac{c}{2} \nu_{k_0 t_1}. \end{aligned}$$

Hence, for all  $n_1 \geq \lceil \frac{k_0 t_1}{\gamma} \rceil$ , for all  $x \in \mathcal{C}_R$ ,

$$\delta_x Q_{0,k_0 t_1} Q_{k_0 t_1, n_1 \gamma} \geq \frac{c}{2} \nu_{k_0 t_1} Q_{k_0 t_1, n_1 \gamma}.$$

Thus, Assumption 3 is satisfied taking  $n_1 := \lceil \frac{n_2 t_1}{\gamma} \rceil \vee \lceil \frac{k_0 t_1}{\gamma} \rceil$ ,  $\alpha := \frac{c}{2}$  and  $\nu(\cdot) := \nu_{k_0 t_1} Q_{k_0 t_1, n_1 \gamma}$ .

Then, by [15, Theorem 1.2], Assumptions 2 and 3 imply that  $Q_{0, n_1 \gamma}$  admits a unique invariant probability measure  $\beta_\gamma$ . Furthermore, there exists constants  $C > 0$  and  $\delta \in (0, 1)$  such that, for all  $\mu \in \mathcal{M}_1(F_0)$ ,

$$\|\mu Q_{0, n n_1 \gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \leq C \mu(\tilde{\psi}_0) \delta^n. \quad (15)$$

Since  $\beta_\gamma$  is the unique invariant probability measure of  $Q_{0, n_1 \gamma}$  and noting that  $\beta_\gamma Q_{0, \gamma}$  is invariant for  $Q_{0, n_1 \gamma}$ , we deduce that  $\beta_\gamma$  is the unique invariant probability measure for  $Q_{0, \gamma}$  and, by (15), for all  $\mu$  such that  $\mu(\tilde{\psi}_0) < +\infty$ ,

$$\|\mu Q_{0, n \gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \xrightarrow{n \rightarrow \infty} 0.$$

Now, for any  $s \geq 0$ , note that  $\delta_x Q_{s, \lceil \frac{s}{\gamma} \rceil \gamma} \tilde{\psi}_0 < +\infty$  for all  $x \in F_s$  (this is a consequence of (7) applied to the semigroup  $(Q_{s,t})_{s \leq t}$ ), and therefore, taking  $\mu = \delta_x Q_{s, \lceil \frac{s}{\gamma} \rceil \gamma}$  in the above convergence,

$$\|\delta_x Q_{s, n \gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \xrightarrow{n \rightarrow \infty} 0$$

for all  $x \in F_s$ . Hence, since  $Q_{n \gamma, n \gamma + s} \tilde{\psi}_s \leq C(1 + \frac{C}{1-\theta}) \tilde{\psi}_{n \gamma}$  by (7), we conclude from the above convergence that

$$\|\delta_x Q_{s, s+n \gamma} - \beta_\gamma Q_{0,s}\|_{\tilde{\psi}_s} \leq C(1 + \frac{C}{1-\theta}) \|\delta_x Q_{s, n \gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \xrightarrow{n \rightarrow \infty} 0. \quad (16)$$

Moreover,  $\beta_\gamma(\tilde{\psi}_0) < +\infty$ .

*Second step.* The first part of this step (until the equality (20)) is inspired by the proof of [1, Theorem 3.11].

We fix  $s \in [0, \gamma]$ . Without loss of generality, we assume that  $\bigcap_{l \geq 0} E_{s+l \gamma} \cap F_s \neq \emptyset$ . Then, by Definition 1, there exists  $x_s \in \bigcap_{l \geq 0} E_{s+l \gamma} \cap F_s$  such that for any  $n \geq 0$ ,

$$\|\delta_{x_s} P_{s+k \gamma, s+(k+n) \gamma}[\psi_{s+(k+n) \gamma} \times \cdot] - \delta_{x_s} Q_{s, s+n \gamma}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow{k \rightarrow \infty} 0,$$

which implies by (16) that

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \|\delta_{x_s} P_{s+k \gamma, s+(k+n) \gamma}[\psi_{s+(k+n) \gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = 0. \quad (17)$$

Then, by the Markov property, (8), and (7), one obtains that, for any  $k, n \in \mathbb{N}$  and  $x \in \bigcap_{l \geq 0} E_{s+l \gamma}$ ,

$$\begin{aligned} \|\delta_x P_{s, s+(k+n) \gamma} - \delta_x P_{s+k \gamma, s+(k+n) \gamma}\|_{\psi_{s+(k+n) \gamma}} &= \|(\delta_x P_{s, s+k \gamma}) P_{s+k \gamma, s+(k+n) \gamma} - \delta_x P_{s+k \gamma, s+(k+n) \gamma}\|_{\psi_{s+(k+n) \gamma}} \\ &\leq C' [P_{s, s+k \gamma} \psi_{s+k \gamma}(x) + \psi_{s+k \gamma}(x)] e^{-\kappa \gamma n} \leq C'' [\psi_s(x) + \psi_{s+k \gamma}(x)] e^{-\kappa \gamma n}, \end{aligned} \quad (18)$$



whereby  $C'' := C' \left( C \left( 1 + \frac{C}{1-\theta} \right) \vee 1 \right)$ . Then, for any  $k, n \in \mathbb{N}$ ,

$$\begin{aligned} & \|\delta_{x_s} P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} \\ & \leq C''[\psi_s(x) + \psi_{s+k\gamma}(x)]e^{-\kappa\gamma n} + \|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV}, \end{aligned} \quad (19)$$

which by (17) and the pointwise convergence of  $(\psi_{s+k\gamma})_{k \in \mathbb{Z}_+}$  implies that

$$\lim_{n \rightarrow \infty} \|\delta_{x_s} P_{s,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = \lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \|\delta_{x_s} P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} = 0. \quad (20)$$

The weak ergodicity (8) implies therefore that the previous convergence actually holds for any initial distribution  $\mu \in \mathcal{M}_1(E_0)$  satisfying  $\mu(\psi_0) < +\infty$ , so that

$$\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} \xrightarrow{n \rightarrow \infty} 0. \quad (21)$$

Since  $\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} \leq 2$  for all  $\mu \in \mathcal{M}_1(E_0)$ ,  $s \geq 0$  and  $n \in \mathbb{Z}_+$ , (21) and Lebesgue's dominated convergence theorem implies that

$$\frac{1}{\gamma} \int_0^\gamma \|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} ds \xrightarrow{n \rightarrow \infty} 0,$$

which implies that

$$\left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \xrightarrow{n \rightarrow \infty} 0.$$

By Cesaro's lemma, this allows us to conclude that, for any  $\mu \in \mathcal{M}_1(E_0)$  such that  $\mu(\psi_0) < +\infty$ ,

$$\begin{aligned} & \left\| \frac{1}{t} \int_0^t \mu P_{0,s}[\psi_s \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} \\ & \leq \frac{1}{\lfloor \frac{t}{\gamma} \rfloor} \sum_{k=0}^{\lfloor \frac{t}{\gamma} \rfloor} \left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+k\gamma}[\psi_{s+k\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV} + \left\| \frac{1}{t} \int_{\lfloor \frac{t}{\gamma} \rfloor \gamma}^t \mu P_{0,s}[\psi_s \times \cdot] ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0, \end{aligned}$$

which concludes the proof of (9).

*Third step.* In the same manner, we now prove that, for any  $\mu \in \mathcal{M}_1(E_0)$  such that  $\mu(\psi_0) < +\infty$ ,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s} ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0. \quad (22)$$

In fact, for any function  $f$  bounded by 1 and  $\mu \in \mathcal{M}_1(E_0)$  such that  $\mu(\psi_0) < +\infty$ ,

$$\begin{aligned} & \left| \mu P_{0,s+n\gamma} \left[ \psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \\ & \leq \left| \mu P_{0,s+n\gamma} \left[ \psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] \right| + \left| \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \\ & \leq \|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]\|_{TV} + \left| \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right|. \end{aligned}$$

We now remark that, since  $\psi_{s+n\gamma} \geq 1$  for any  $s$  and  $n \in \mathbb{Z}_+$ , one has that

$$\left| \frac{\tilde{\psi}_s}{\psi_{s+n\gamma}} - 1 \right| \leq 1 + \tilde{\psi}_s.$$

Since  $(\psi_{s+n\gamma})_{n \in \mathbb{Z}^+}$  converges pointwisely towards  $\tilde{\psi}_s$  and  $\beta_\gamma Q_{0,s} \tilde{\psi}_s < +\infty$ , Lebesgue's dominated convergence theorem implies

$$\sup_{f \in \mathcal{B}_1(E)} \left| \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[ \tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Then, using (21), one has

$$\|\mu P_{0,s+n\gamma} - \beta_\gamma Q_{0,s}\|_{TV} \xrightarrow{n \rightarrow \infty} 0,$$

which allows us to conclude (22), using the same argument as in the first step.

*Fourth step.* In order to show the  $\mathbb{L}^2$ -ergodic theorem, we let  $f \in \mathcal{B}(E)$ . For any  $x \in E_0$  and  $t \geq 0$ ,

$$\begin{aligned} \mathbb{E}_{0,x} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[ \frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \\ = \frac{2}{t^2} \int_0^t \int_s^t (\mathbb{E}_{0,x}[f(X_s)f(X_u)] - \mathbb{E}_{0,x}[f(X_s)]\mathbb{E}_{0,x}[f(X_u)]) duds \\ = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [f(X_s) (f(X_u) - \mathbb{E}_{0,x}[f(X_u)])] duds \\ = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [f(X_s) (\mathbb{E}_{s,X_s}[f(X_u)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_u)])] duds, \end{aligned}$$

whereby the Markov property was used on the last line. By (8) (weak ergodicity) and (7), one obtains for any  $s \leq t$ ,

$$|\mathbb{E}_{s,X_s}[f(X_t)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_t)]| \leq C'' \|f\|_\infty [\psi_s(X_s) + \psi_0(x)] e^{-\kappa(t-s)}, \quad \mathbb{P}_{0,x} - \text{almost surely}, \quad (23)$$

whereby  $C''$  was defined in the first part. As a result, for any  $x \in E_0$  and  $t \geq 0$ ,

$$\begin{aligned} \mathbb{E}_{0,x} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[ \frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] &\leq \frac{2C'' \|f\|_\infty}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] e^{-\kappa(u-s)} duds \\ &= \frac{2C'' \|f\|_\infty}{t^2} \int_0^t \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] e^{\kappa s} \frac{e^{-\kappa s} - e^{-\kappa t}}{\kappa} ds \\ &= \frac{2C'' \|f\|_\infty}{\kappa t} \times \mathbb{E}_{0,x} \left[ \frac{1}{t} \int_0^t |f(X_s)| (\psi_s(X_s) + \psi_0(x)) ds \right] - \frac{2C'' \|f\|_\infty e^{-\kappa t}}{\kappa t^2} \int_0^t e^{\kappa s} \mathbb{E}_{0,x} [|f(X_s)| (\psi_s(X_s) + \psi_0(x))] ds. \end{aligned}$$

Then, by (9), there exists a constant  $\tilde{C} > 0$  such that, for any  $x \in E_0$ , when  $t \rightarrow \infty$ ,

$$\mathbb{E}_{0,x} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,x} \left[ \frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \leq \frac{\tilde{C} \|f\|_\infty \psi_0(x)}{t} \times \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} [|f| \tilde{\psi}_s] ds + o\left(\frac{1}{t}\right). \quad (24)$$

Since  $f \in \mathcal{B}(E)$  and by definition of the total variation distance, (22) implies that, for all  $x \in E_0$ ,

$$\left| \frac{1}{t} \int_0^t P_{0,s} f(x) - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right| \leq \|f\|_\infty \left\| \frac{1}{t} \int_0^t \delta_x P_{0,s} ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} ds \right\|_{TV} \xrightarrow{t \rightarrow \infty} 0.$$

Then, using (22), one deduces that, for any  $x \in E_0$  and bounded function  $f$ ,

$$\begin{aligned} \mathbb{E}_{0,x} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right] \\ \leq 2 \left( \mathbb{E}_{0,x} \left[ \left( \frac{1}{t} \int_0^t f(X_s) ds - \frac{1}{t} \int_0^t P_{0,s} f(x) \right)^2 \right] + \left| \frac{1}{t} \int_0^t P_{0,s} f(x) - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds \right|^2 \right) \xrightarrow{t \rightarrow \infty} 0. \end{aligned}$$

The convergence for any probability measure  $\mu \in \mathcal{M}_1(E_0)$  comes from Lebesgue's dominated convergence theorem.

*Fifth step.* We now fix non-negative  $f \in \mathcal{B}(E)$ , and  $\mu \in \mathcal{M}_1(E_0)$  satisfying  $\mu(\psi_0) < +\infty$ . The following proof is inspired by the proof of [26, Theorem 12].

Since  $\mu(\psi_0) < +\infty$ , the inequality (24) entails that there exists a finite constant  $C_{f,\mu} \in (0, \infty)$  such that, for  $t$  large enough,

$$\mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \mathbb{E}_{0,\mu} \left[ \frac{1}{t} \int_0^t f(X_s) ds \right] \right|^2 \right] \leq \frac{C_{f,\mu}}{t}.$$

Then, for  $n$  large enough,

$$\mathbb{E}_{0,\mu} \left[ \left| \frac{1}{n^2} \int_0^{n^2} f(X_s) ds - \mathbb{E}_{0,\mu} \left[ \frac{1}{n^2} \int_0^{n^2} f(X_s) ds \right] \right|^2 \right] \leq \frac{C_{f,\mu}}{n^2}.$$

Then, by Chebyshev's inequality and the Borel-Cantelli lemma, this last inequality implies that

$$\left| \frac{1}{n^2} \int_0^{n^2} f(X_s) ds - \mathbb{E}_{0,\mu} \left[ \frac{1}{n^2} \int_0^{n^2} f(X_s) ds \right] \right| \xrightarrow[n \rightarrow \infty]{} 0, \quad \mathbb{P}_{0,\mu} - \text{almost surely.}$$

One thereby obtains by the convergence (22) that

$$\frac{1}{n^2} \int_0^{n^2} f(X_s) ds \xrightarrow[n \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely.} \quad (25)$$

Since the nonnegativity of  $f$  is assumed, this implies that for any  $t > 0$  we have

$$\int_0^{\lfloor \sqrt{t} \rfloor^2} f(X_s) ds \leq \int_0^t f(X_s) ds \leq \int_0^{\lceil \sqrt{t} \rceil^2} f(X_s) ds.$$

These inequalities and (25) then give that

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} f ds, \quad \mathbb{P}_{0,\mu} - \text{almost surely.}$$

In order to conclude that the result holds for any bounded measurable function  $f$ , it is enough to decompose  $f = f_+ - f_-$  with  $f_+ := f \vee 0$  and  $f_- = (-f) \vee 0$  and apply the above convergence to  $f_+$  and  $f_-$ . This concludes the proof of Theorem 1.  $\square$

*Proof of Corollary 1.* We remark as in the previous proof that, if  $\|f\|_\infty \leq 1$  and  $\psi_s = \mathbb{1}$ , an upper-bound for the inequality (24) can be obtained, which does not depend on  $f$  and  $x$ . Likewise, the convergence (21) holds uniformly in the initial measure due to (23).  $\square$

*Remark 3.* The proof of Theorem 1, as written above, does not allow to deal with semigroups satisfying a Doeblin condition with time-dependent constant  $c_s$ , that is such that there exists  $t_0 \geq 0$  and a family of probability measure  $(\nu_t)_{t \geq 0}$  on  $(E_t)_{t \geq 0}$  such that, for all  $s \geq 0$  and  $x \in E_s$ ,

$$\delta_x P_{s,s+t_0} \geq c_{s+t_0} \nu_{s+t_0}.$$

In fact, under the condition written above, we can show (see for example the proof of the formula (2.7) of [9, Theorem 2.1]) that, for all  $s \leq t$  and  $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$ ,

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{TV} \leq 2 \prod_{k=1}^{\lfloor \frac{t-s}{t_0} \rfloor} (1 - c_{t-kt_0}).$$

Hence, by this last inequality with  $\mu_1 = \delta_x P_{s,s+k\gamma}$ ,  $\mu_2 = \delta_x$ , replacing  $s$  by  $s + k\gamma$  and  $t$  by  $s + (k+n)\gamma$ , one obtains

$$\|\delta_x P_{s,s+(k+n)\gamma} - \delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{TV} \leq 2 \prod_{l=1}^{\lfloor \frac{n\gamma}{t_0} \rfloor} (1 - c_{s+(k+n)\gamma-lt_0}),$$

replacing therefore the inequality (18) in the proof of Theorem 1. Plugging therefore this last inequality into the formula (19), one obtains

$$\|\delta_x P_{s,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV} \leq 2 \prod_{l=1}^{\lfloor \frac{n\gamma}{t_0} \rfloor} (1 - c_{s+(k+n)\gamma-lt_0}) + \|\delta_x P_{s+k\gamma,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV}.$$

Hence, we see that we cannot conclude a similar result when  $c_s \rightarrow 0$ , as  $s \rightarrow +\infty$  since, for  $n$  fixed,

$$\limsup_{k \rightarrow \infty} \prod_{l=1}^{\lfloor \frac{n\gamma}{t_0} \rfloor} (1 - c_{s+(k+n)\gamma-lt_0}) = 1.$$

### 3 Application to quasi-stationarity with moving boundaries

In this section,  $(X_t)_{t \geq 0}$  is assumed as being a time-homogeneous Markov process. We consider a family of measurable subsets  $(A_t)_{t \geq 0}$  of  $E$ , and denote the hitting time

$$\tau_A := \inf\{t \geq 0 : X_t \in A_t\}.$$

For all  $s \leq t$ , denote by  $\mathcal{F}_{s,t}$  the  $\sigma$ -field generated by the family  $(X_u)_{s \leq u \leq t}$  and  $\mathcal{F}_t := \mathcal{F}_{0,t}$ . Assume that  $\tau_A$  is a stopping time with respect to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ . Assume also that for any  $x \notin A_0$ ,

$$\mathbb{P}_{0,x}[\tau_A < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{0,x}[\tau_A > t] > 0, \quad \forall t \geq 0.$$

We will be interested in a notion of *quasi-stationarity with moving boundaries*, which studies the asymptotic behavior of the Markov process  $(X_t)_{t \geq 0}$  conditioned not to hit  $(A_t)_{t \geq 0}$  up to the time  $t$ . For non-moving boundaries ( $A_t = A_0$  for any  $t \geq 0$ ), the *quasi-limiting distribution* is defined as a probability measure  $\alpha$  such that, for at least one initial measure  $\mu$  and for all measurable subsets  $\mathcal{A} \subset E$ ,

$$\mathbb{P}_{0,\mu}[X_t \in \mathcal{A} | \tau_A > t] \xrightarrow[t \rightarrow \infty]{} \alpha(\mathcal{A}).$$

Such a definition is equivalent (still in the non-moving framework) to the notion of *quasi-stationary distribution* defined as a probability measure  $\alpha$  such that, for any  $t \geq 0$ ,

$$\mathbb{P}_{0,\alpha}[X_t \in \cdot | \tau_A > t] = \alpha. \tag{26}$$

If quasi-limiting and quasi-stationary distributions are in general well-defined for time-homogeneous Markov processes and non-moving boundaries (see [11, 23] for a general overview on the theory of quasi-stationarity), these notions could be not well-defined for time-inhomogeneous Markov processes or moving boundaries and are not equivalent anymore. In particular, under reasonable assumptions on irreducibility, it was shown in [24] that the notion of quasi-stationary distribution as defined by (26) is not well-defined for time-homogeneous Markov processes absorbed by moving boundaries.

Another asymptotic notion to study is the *quasi-ergodic distribution*, related to a conditional version of the ergodic theorem and usually defined as follows.

**Definition 2.** A probability measure  $\beta$  is a *quasi-ergodic distribution* if, for some initial measure  $\mu \in \mathcal{M}_1(E \setminus A_0)$  and for any bounded continuous function  $f$ ,

$$\mathbb{E}_{0,\mu} \left[ \frac{1}{t} \int_0^t f(X_s) ds \middle| \tau_A > t \right] \xrightarrow[t \rightarrow \infty]{} \beta(f).$$

In the time-homogeneous setting (in particular for non-moving boundaries), this notion has been extensively studied (see for example [2, 8, 10, 12, 13, 16, 17, 18, 24]). In the "moving boundaries" framework, the existence of quasi-ergodic distributions has been dealt with in [24] for Markov chains on finite state spaces absorbed by periodic boundaries, and in [25] for processes satisfying a Champagnat-Villemonais condition (see Assumption (A') set later) absorbed by converging or periodic boundaries. In this last paper, the existence of the quasi-ergodic distribution is dealt with through the following inequality (see [25, Theorem 1]), holding for any initial state  $x$ ,  $s \leq t$  and for some constant  $C, \gamma > 0$  independent of  $x, s$  and  $t$ :

$$\|\mathbb{P}_{0,x}(X_s \in \cdot | \tau_A > t) - \mathbb{Q}_{0,x}(X_s \in \cdot)\|_{TV} \leq Ce^{-\gamma(t-s)},$$

where the family of probability measure  $(\mathbb{Q}_{s,x})_{s \geq 0, x \in E_s}$  is defined by

$$\mathbb{Q}_{s,x}[\Gamma] := \lim_{T \rightarrow \infty} \mathbb{P}_{s,x}[\Gamma | \tau_A > T], \quad \forall s \leq t, x \in E \setminus A_s, \Gamma \in \mathcal{F}_{s,t}.$$

Moreover, [9, Proposition 3.1], there exists a family of positive bounded functions  $(\eta_t)_{t \geq 0}$  defined in such that, for all  $s \leq t$  and  $x \in E_s$ ,

$$\mathbb{E}_{s,x}(\eta_t(X_t) \mathbb{1}_{\tau_A > t}) = \eta_s(x).$$

Then, we can show (this is actually shown in [9]) that

$$\mathbb{Q}_{s,x}(\Gamma) = \mathbb{E}_{s,x}(\mathbb{1}_{\Gamma, \tau_A > t} \frac{\eta_t(X_t)}{\eta_s(x)})$$

and that, for all  $\mu \in \mathcal{M}_1(E_0)$ ,

$$\|\mathbb{P}_{0,\mu}(X_s \in \cdot | \tau_A > t) - \mathbb{Q}_{0,\eta_0 * \mu}(X_s \in \cdot)\|_{TV} \leq Ce^{-\gamma(t-s)}.$$

where  $\eta_0 * \mu(dx) := \frac{\eta_0(x)\mu(dx)}{\mu(\eta_0)}$ . By triangular inequality, one has

$$\left\| \frac{1}{t} \int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t] ds - \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}[X_s \in \cdot] ds \right\|_{TV} \leq \frac{C}{\gamma t}, \quad \forall t > 0, \quad (27)$$

In particular, the inequality (27) implies that there exists a quasi-ergodic distribution  $\beta$  for the process  $(X_t)_{t \geq 0}$  absorbed by  $(A_t)_{t \geq 0}$  if and only if there exist some probability measures  $\mu \in \mathcal{M}_1(E_0)$  such that  $\frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}[X_s \in \cdot] ds$  converges weakly to  $\beta$ , when  $t$  goes to infinity. In other terms, under Assumption (A'), the existence of a quasi-ergodic distribution for the absorbed process is equivalent to the law of large number for its  $Q$ -process.

Assumption (A') is now set.

**Assumption (A').** *There exists a family of probability measures  $(\nu_t)_{t \geq 0}$ , defined on  $E \setminus A_t$  for each  $t$ , such that:*

(A'1) *There exists  $t_0 \geq 0$  and  $c_1 > 0$  such that*

$$\mathbb{P}_{s,x}[X_{s+t_0} \in \cdot | \tau_A > s + t_0] \geq c_1 \nu_{s+t_0}, \quad \forall s \geq 0, \forall x \in E \setminus A_s.$$

(A'2) *There exists  $c_2 > 0$  such that*

$$\mathbb{P}_{s,\nu_s}[\tau_A > t] \geq c_2 \mathbb{P}_{s,x}[\tau_A > t], \quad \forall s \leq t, \forall x \in E \setminus A_s.$$

In what follows, we say that the couple  $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$  satisfies Assumption (A') when it holds for the Markov process  $(X_t)_{t \geq 0}$  considered as absorbed by the moving boundary  $(A_t)_{t \geq 0}$ .

The condition (A'1) is a conditional version of the Doeblin condition (12) and (A'2) is a Harnack-like inequality on the probabilities of surviving, necessary to deal with the conditioning. They are equivalent to the set of conditions presented in [1, Definition 2.2], when the non-conservative semigroup is sub-Markovian. In the time-homogeneous framework, we obtain the Champagnat-Villemonais condition defined in [5] (see Assumption (A)), shown as being equivalent to the exponential uniform convergence to quasi-stationarity in total variation.

In [25], the existence of a unique quasi-ergodic distribution has been only proved for converging or periodic boundaries. However, we can expect such a result of existence (and uniqueness) for other kinds of movement for the boundary. Hence, the aim of this section is to extend the results on the existence of quasi-ergodic distributions obtained in [25] for Markov processes absorbed by asymptotically periodic moving boundaries.

Now, let us state the following theorem.

**Theorem 2.** Assume that there exists a  $\gamma$ -periodic sequence of subsets  $(B_t)_{t \geq 0}$  such that, for any  $s \in [0, \gamma)$ ,

$$E'_s := E \setminus \bigcap_{k \in \mathbb{Z}_+} \bigcup_{l \geq k} A_{s+l\gamma} \cup B_s \neq \emptyset$$

and there exists  $x_s \in E_s$  such that, for any  $n \leq N$ ,

$$\|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0. \quad (28)$$

Assume also that Assumption (A') is satisfied by the couples  $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$  and  $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$ .

Then there exists a probability measure  $\beta \in \mathcal{M}_1(E)$  such that

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0, \mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] \xrightarrow[t \rightarrow \infty]{} 0. \quad (29)$$

*Remark 4.* Remark that the condition (28) implies that, for any  $n \in \mathbb{Z}_+$ ,

$$\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+n)\gamma] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{s, x_s}[\tau_B > s + n\gamma].$$

Under the additional condition  $B_t \subset A_t$  for all  $t \geq 0$ , these two conditions are equivalent, since for all  $n \leq N$ ,

$$\begin{aligned} & \|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \\ &= \|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_B \leq s + (k+N)\gamma < \tau_A]\|_{TV} \\ &\leq \mathbb{P}_{s+k\gamma, x_s}[\tau_B \leq s + (k+N)\gamma < \tau_A] \\ &= |\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[\tau_B > s + N\gamma]|, \end{aligned}$$

where we used the periodicity of  $(B_t)_{t \geq 0}$ , writing that  $\mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma] = \mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_B > s + (k+N)\gamma]$  for all  $k \in \mathbb{Z}_+$ . This implies the following corollary.

**Corollary 2.** Assume that there exists a  $\gamma$ -periodic sequence of subsets  $(B_t)_{t \geq 0}$ , with  $B_t \subset A_t$  for all  $t \geq 0$ , such that, for any  $s \in [0, \gamma)$ , there exists  $x_s \in E'_s$  such that, for any  $n \leq N$ ,

$$\mathbb{P}_{s+k\gamma, x_s}[\tau_A > s + (k+n)\gamma] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{s, x_s}[\tau_B > s + n\gamma].$$

Assume also that Assumption (A') is satisfied by  $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$  and  $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$ .

Then there exists  $\beta \in \mathcal{M}_1(E)$  such that (29) holds.

*Proof of Theorem 2.* Since  $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$  satisfies Assumption (A') and  $(B_t)_{t \geq 0}$  is a periodic boundary, we already know by [25, Theorem 2] that, for any initial distribution  $\mu$ ,  $t \mapsto \frac{1}{t} \int_0^t \mathbb{P}_{0, \mu}[X_s \in \cdot | \tau_B > t] ds$  converges weakly to a quasi-ergodic distribution  $\beta$ .

The main idea of this proof is to apply Corollary 1. Since  $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$  and  $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$  satisfy Assumption (A'), [25, Theorem 1] implies that there exist two families of probability measures  $(\mathbb{Q}_{s, x}^A)_{s \geq 0, x \in E \setminus A_s}$  and  $(\mathbb{Q}_{s, x}^B)_{s \geq 0, x \in E \setminus B_s}$  such that, for any  $s \leq t$ ,  $x \in E \setminus A_s$ ,  $y \in E \setminus B_s$  and  $\Gamma \in \mathcal{F}_{s, t}$ ,

$$\mathbb{Q}_{s, x}^A[\Gamma] = \lim_{T \rightarrow \infty} \mathbb{P}_{s, x}[\Gamma | \tau_A > T], \quad \text{and} \quad \mathbb{Q}_{s, y}^B[\Gamma] = \lim_{T \rightarrow \infty} \mathbb{P}_{s, y}[\Gamma | \tau_B > T].$$

In particular, the quasi-ergodic distribution  $\beta$  is the limit of  $t \mapsto \frac{1}{t} \int_0^t \mathbb{Q}_{0, \mu}^B[X_s \in \cdot] ds$ , when  $t$  goes to infinity (see [25, Theorem 5]). Also, by [25, Theorem 1], there exists a constant  $C > 0$  and  $\kappa > 0$  such that, for any  $s \leq t \leq T$ , for any  $x \in E \setminus A_s$ ,

$$\|\mathbb{Q}_{s, x}^A[X_t \in \cdot] - \mathbb{P}_{s, x}[X_t \in \cdot | \tau_A > T]\|_{TV} \leq Ce^{-\kappa(T-t)},$$

and for any  $x \in E \setminus B_s$ ,

$$\|\mathbb{Q}_{s, x}^B[X_t \in \cdot] - \mathbb{P}_{s, x}[X_t \in \cdot | \tau_B > T]\|_{TV} \leq Ce^{-\kappa(T-t)}.$$

Moreover, for any  $s \leq t \leq T$  and  $x \in E'_s$ ,

$$\begin{aligned}
& \|\mathbb{P}_{s,x}[X_t \in \cdot | \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_B > T]\|_{TV} \\
&= \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_A > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&= \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T) \mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}(\tau_A > T) \mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\leq \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T) \mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}(\tau_A > T) \mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\quad + \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\leq \frac{|\mathbb{P}_{s,x}(\tau_B > T) - \mathbb{P}_{s,x}(\tau_A > T)|}{\mathbb{P}_{s,x}(\tau_B > T)} + \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]} \\
&\leq 2 \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}, \tag{30}
\end{aligned}$$

since  $|\mathbb{P}_{s,x}(\tau_B > T) - \mathbb{P}_{s,x}(\tau_A > T)| \leq \|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}$ . Then, we obtain for any  $s \leq t \leq T$  and  $x \in E'_s$ ,

$$\|\mathbb{Q}_{s,x}^A[X_t \in \cdot] - \mathbb{Q}_{s,x}^B[X_t \in \cdot]\|_{TV} \leq 2Ce^{-\kappa(T-t)} + 2 \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}. \tag{31}$$

The condition (28) implies the existence of  $x_s \in E_s$  such that, for any  $n \leq N$ , for all  $k \in \mathbb{Z}_+$ ,

$$\lim_{k \rightarrow \infty} \|\mathbb{P}_{s+k\gamma, x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s, x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} = 0,$$

which implies by (31) that, for any  $n \leq N$ ,

$$\limsup_{k \rightarrow \infty} \|\mathbb{Q}_{s+k\gamma, x_s}^A[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}_{s+k\gamma, x_s}^B[X_{s+(k+n)\gamma} \in \cdot]\|_{TV} \leq 2Ce^{-\kappa\gamma(N-n)}.$$

Now, letting  $N \rightarrow \infty$ , for any  $n \in \mathbb{Z}_+$

$$\lim_{k \rightarrow \infty} \|\mathbb{Q}_{s+k\gamma, x_s}^A[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}_{s+k\gamma, x_s}^B[X_{s+(k+n)\gamma} \in \cdot]\|_{TV} = \lim_{k \rightarrow \infty} \|\mathbb{Q}_{s+k\gamma, x_s}^A(X_{s+(k+n)\gamma} \in \cdot) - \mathbb{Q}_{s, x_s}^B(X_{s+n\gamma} \in \cdot)\|_{TV} = 0.$$

In other words, the semigroup  $(Q_{s,t}^A)_{s \leq t}$  defined by

$$Q_{s,t}^A f(x) := \mathbb{E}_{s,x}^{\mathbb{Q}^A}(f(X_t)), \quad \forall s \leq t, \forall f \in \mathcal{B}(E \setminus A_t), \forall x \in E \setminus A_s,$$

is asymptotically periodic (according to the Definition 1, with  $\psi_s = \tilde{\psi}_s = 1$  for all  $s \geq 0$ ), associated to the auxiliary semigroup  $(Q_{s,t}^B)_{s \leq t}$  defined by

$$Q_{s,t}^B f(x) := \mathbb{E}_{s,x}^{\mathbb{Q}^B}(f(X_t)), \quad \forall s \leq t, \forall f \in \mathcal{B}(E \setminus B_t), \forall x \in E \setminus B_s.$$

Moreover, assumptions (A') satisfied for  $\{(X_t)_{t \geq 0}, (A_t)_{t \geq 0}\}$  and  $\{(X_t)_{t \geq 0}, (B_t)_{t \geq 0}\}$  imply that Doeblin condition holds for these two  $Q$ -processes. As a matter of fact, by the Markov property, for all  $s \leq t \leq T$  and  $x \in E \setminus A_s$ ,

$$\begin{aligned}
\mathbb{P}_{s,x}(X_t \in \cdot | \tau_A > T) &= \mathbb{E}_{s,x} \left[ \mathbb{1}_{X_t \in \cdot, \tau_A > t} \frac{\mathbb{P}_{t, X_t}(\tau_A > T)}{\mathbb{P}_{s, \mu}(\tau_A > T)} \right] \\
&= \mathbb{E}_{s,x} \left[ \frac{\mathbb{1}_{X_t \in \cdot, \tau_A > t} \mathbb{P}_{t, X_t}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_A > t) \mathbb{P}_{t, \phi_{t,s}(\mu)}(\tau_A > T)} \right] \\
&= \mathbb{E}_{s,x} \left[ \mathbb{1}_{X_t \in \cdot} \frac{\mathbb{P}_{t, X_t}(\tau_A > T)}{\mathbb{P}_{t, \phi_{t,s}(\delta_x)}(\tau_A > T)} \middle| \tau_A > t \right]. \tag{32}
\end{aligned}$$

By  $(A'_1)$ , for all  $s \geq 0$ ,  $T \geq s + t_0$  and  $x \in E \setminus A_s$ ,

$$\mathbb{E}_{s,x} \left[ \mathbb{1}_{X_{s+t_0} \in \cdot} \frac{\mathbb{P}_{s+t_0, X_{s+t_0}}(\tau_A > T)}{\mathbb{P}_{s+t_0, \phi_{s+t_0, s}(\delta_x)}(\tau_A > T)} \middle| \tau_A > s + t_0 \right] \geq c_1 \int \nu_{s+t_0}(dy) \frac{\mathbb{P}_{s+t_0, y}(\tau_A > T)}{\mathbb{P}_{s+t_0, \phi_{s+t_0, s}(\delta_x)}(\tau_A > T)},$$

that is to say by (32) that

$$\mathbb{P}_{s,x}(X_{s+t_0} \in \cdot | \tau_A > T) \geq c_1 \int \nu_{s+t_0}(dy) \frac{\mathbb{P}_{s+t_0, y}(\tau_A > T)}{\mathbb{P}_{s+t_0, \phi_{s+t_0, s}(\delta_x)}(\tau_A > T)}.$$

Letting  $T \rightarrow \infty$  in this last inequality and using [9, Proposition 3.1], for all  $s \geq 0$  and  $x \in E \setminus A_s$ ,

$$\mathbb{Q}_{s,x}^A(X_{s+t_0} \in \cdot) \geq c_1 \int \nu_{s+t_0}(dy) \frac{\eta_{s+t_0}(y)}{\phi_{s+t_0, s}(\delta_x)(\eta_{s+t_0})}.$$

The measure  $\int \nu_{s+t_0}(dy) \frac{\eta_{s+t_0}(y)}{\phi_{s+t_0, s}(\delta_x)(\eta_{s+t_0})}$  is then a positive measure whose the mass is lower-bounded by  $c_2$  by  $(A'_2)$ , since for all  $s \geq 0$  and  $T \geq s + t_0$ ,

$$\int_{E \setminus A_{s+t_0}} \nu_{s+t_0} \frac{\mathbb{P}_{t,x}(\tau_A > T)}{\mathbb{P}_{t, \phi_{t,s}(\delta_x)}(\tau_A > T)} \geq c_2.$$

This proves therefore a Doeblin condition for the semigroup  $(Q_{s,t}^A)_{s \leq t}$ . The same reasoning applies to show also a Doeblin condition for the semigroup  $(Q_{s,t}^B)_{s \leq t}$ . Then, using (27) then Corollary 1,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t] ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}^A(X_s \in \cdot) ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}^B[X_s \in \cdot] ds = \beta,$$

where the limits refer to the convergence in total variation and hold uniformly in the initial measure.

For any  $\mu \in \mathcal{M}_1(E \setminus A_0)$ ,  $f \in \mathcal{B}_1(E)$  and  $t \geq 0$ ,

$$\mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds \right|^2 \middle| \tau_A > t \right] = \frac{2}{t^2} \int_0^t \int_s^t \mathbb{E}_{0,\mu}[f(X_s)f(X_u) | \tau_A > t] duds.$$

Then, by [25, Theorem 1], for any  $s \leq u \leq t$ , for any  $\mu \in \mathcal{M}_1(E \setminus A_0)$  and  $f \in \mathcal{B}(E)$ ,

$$\left| \mathbb{E}_{0,\mu}[f(X_s)f(X_u) | \tau_A > t] - \mathbb{E}_{0,\eta_0 * \mu}^A[f(X_s)f(X_u)] \right| \leq C \|f\|_\infty e^{-\kappa(t-u)},$$

where the expectation  $\mathbb{E}_{0,\eta_0 * \mu}^A$  is associated to the probability measure  $\mathbb{Q}_{0,\eta_0 * \mu}^A$ . Hence, for any  $\mu \in \mathcal{M}_1(E \setminus A_0)$ ,  $f \in \mathcal{B}_1(E)$  and  $t > 0$ ,

$$\begin{aligned} \left| \mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] - \mathbb{E}_{0,\eta_0 * \mu}^A \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \right] \right| &\leq \frac{4C}{t^2} \int_0^t \int_s^t e^{-\kappa(t-u)} duds \\ &\leq \frac{4C}{\kappa t} - \frac{4C(1 - e^{-\kappa t})}{\kappa^2 t^2}. \end{aligned}$$

Moreover, since  $(Q_{s,t}^A)_{s \leq t}$  is asymptotically periodic in total variation and satisfies the Doeblin condition, like  $(Q_{s,t}^B)_{s \leq t}$ , Corollary 1 implies that

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\eta_0 * \mu}^A \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \right] \xrightarrow{t \rightarrow \infty} 0.$$

Then

$$\sup_{\mu \in \mathcal{M}_1(E \setminus A_0)} \sup_{f \in \mathcal{B}_1(E)} \mathbb{E}_{0,\mu} \left[ \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_A > t \right] \xrightarrow{t \rightarrow \infty} 0.$$

□

*Remark 5.* It seems that Assumption (A') can be weakened by a conditional version of Assumption 1. In particular, such conditions can be derived from the Assumption (F) presented in [6], as it will be shown later by the preprint [4] in preparation.



## 4 Examples

### 4.1 Asymptotically periodic Ornstein-Uhlenbeck processes

Let  $(X_t)_{t \geq 0}$  be a time-inhomogeneous diffusion process on  $\mathbb{R}$  satisfying the following stochastic differential equation

$$dX_t = dW_t - \lambda(t)X_t dt,$$

where  $(W_t)_{t \geq 0}$  is a one-dimensional Brownian motion and  $\lambda : [0, \infty) \rightarrow [0, \infty)$  is a function such that

$$\sup_{t \geq 0} |\lambda(t)| < +\infty$$

and such that there exists  $\gamma > 0$  such that

$$\inf_{s \geq 0} \int_s^{s+\gamma} \lambda(u) du > 0.$$

By Itô's lemma, for any  $s \leq t$ ,

$$X_t = e^{-\int_s^t \lambda(u) du} \left[ X_s + \int_s^t e^{\int_s^u \lambda(v) dv} dW_u \right].$$

In particular, denoting  $(P_{s,t})_{s \leq t}$  the semigroup associated to  $(X_t)_{t \geq 0}$ , for any  $f \in \mathcal{B}(\mathbb{R})$ ,  $t \geq 0$  and  $x \in \mathbb{R}$ ,

$$P_{s,t}f(x) = \mathbb{E} \left[ f \left( e^{-\int_s^t \lambda(u) du} x + e^{-\int_s^t \lambda(u) du} \sqrt{\int_s^t e^{2\int_s^u \lambda(v) dv} du} \times \mathcal{N}(0, 1) \right) \right],$$

where  $\mathcal{N}(0, 1)$  denotes a standard Gaussian variable.

**Theorem 3.** *Assume that there exists a  $\gamma$ -periodic function  $g$ , bounded on  $\mathbb{R}$ , such that  $\lambda \sim_{t \rightarrow \infty} g$ . Then the assumptions of Theorem 1 hold.*

*Proof.* In our case, the auxiliary semigroup  $(Q_{s,t})_{s \leq t}$  of Definition 1 will be defined as follows: for any  $f \in \mathcal{B}(\mathbb{R})$ ,  $t \geq 0$  and  $x \in \mathbb{R}$ ,

$$Q_{s,t}f(x) = \mathbb{E} \left[ f \left( e^{-\int_s^t g(u) du} x + e^{-\int_s^t g(u) du} \sqrt{\int_s^t e^{2\int_s^u g(v) dv} du} \times \mathcal{N}(0, 1) \right) \right].$$

In particular, the semigroup  $(Q_{s,t})_{s \leq t}$  is associated to the process  $(Y_t)_{t \geq 0}$  following

$$dY_t = dW_t - g(t)Y_t dt.$$

We first remark that the function  $\psi : x \mapsto 1 + x^2$  is a Lyapunov function for  $(P_{s,t})_{s \leq t}$  and  $(Q_{s,t})_{s \leq t}$ . In fact, for any  $s \geq 0$  and  $x \in \mathbb{R}$ ,

$$\begin{aligned} P_{s,s+\gamma} \psi(x) &= 1 + e^{-2\int_s^{s+\gamma} \lambda(u) du} x^2 + e^{-2\int_s^{s+\gamma} \lambda(u) du} \int_s^{s+\gamma} e^{2\int_s^u \lambda(v) dv} du \\ &= e^{-2\int_s^{s+\gamma} \lambda(u) du} \psi(x) + 1 - e^{-2\int_s^{s+\gamma} \lambda(u) du} + e^{-2\int_s^{s+\gamma} \lambda(u) du} \int_s^{s+\gamma} e^{2\int_s^u \lambda(v) dv} du \\ &\leq e^{-2\gamma c_{\inf}} \psi(x) + C, \end{aligned}$$

where  $C \in (0, +\infty)$  and  $c_{\inf} := \inf_{t \geq 0} \frac{1}{\gamma} \int_t^{t+\gamma} \lambda(u) du > 0$ . Taking  $\theta \in (e^{-2\gamma c_{\inf}}, 1)$ , there exists a compact set  $K$  such that, for any  $s \geq 0$ ,

$$P_{s,s+\gamma} \psi(x) \leq \theta \psi(x) + C \mathbb{1}_K(x).$$

Moreover, for any  $s \geq 0$  and  $t \in [0, \gamma)$ , the function  $P_{s,s+t} \psi / \psi$  is upper-bounded uniformly in  $s$  and  $t$ . It remains therefore to prove Assumption 1 (i) for  $(P_{s,t})_{s \leq t}$ , which is a consequence of the following lemma.

**Lemma 1.** For any  $a, b_-, b_+ > 0$ , define the subset  $\mathcal{C}(a, b_-, b_+) \subset \mathcal{M}_1(\mathbb{R})$  as

$$\mathcal{C}(a, b_-, b_+) := \{\mathcal{N}(m, \sigma) : m \in [-a, a], \sigma \in [b_-, b_+]\}.$$

Then, for any  $a, b_-, b_+ > 0$ , there exists a probability measure  $\nu$  and a constant  $c > 0$  such that, for any  $\mu \in \mathcal{C}(a, b_-, b_+)$ ,

$$\mu \geq c\nu.$$

The proof of this lemma is postponed after the end of this proof.

Since  $\lambda \sim_{t \rightarrow \infty} g$  and these two functions are bounded on  $\mathbb{R}_+$ , Lebesgue's dominated convergence theorem implies that, for all  $s \leq t$ ,

$$\left| \int_{s+k\gamma}^{t+k\gamma} \lambda(u) du - \int_s^t g(u) du \right| \xrightarrow[k \rightarrow \infty]{} 0.$$

In the same way, for all  $s \leq t$ ,

$$\int_{s+k\gamma}^{t+k\gamma} e^{2 \int_{s+k\gamma}^u \lambda(v) dv} du \xrightarrow[k \rightarrow \infty]{} \int_s^t e^{2 \int_s^u g(v) dv} du.$$

Hence, for any  $s \leq t$ ,

$$e^{-\int_{s+k\gamma}^{t+k\gamma} \lambda(u) du} \xrightarrow[k \rightarrow \infty]{} e^{-\int_s^t g(u) du},$$

and

$$e^{-\int_{s+k\gamma}^{t+k\gamma} \lambda(u) du} \sqrt{\int_{s+k\gamma}^{t+k\gamma} e^{2 \int_{s+k\gamma}^u \lambda(v) dv} du} \xrightarrow[k \rightarrow \infty]{} e^{-\int_s^t g(u) du} \sqrt{\int_s^t e^{2 \int_s^u g(v) dv} du}.$$

Using [14, Theorem 1.3.], for any  $x \in \mathbb{R}$ ,

$$\|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s+k\gamma, t+k\gamma}\|_{TV} \xrightarrow[k \rightarrow \infty]{} 0. \quad (33)$$

To deduce the convergence in  $\psi$ -distance, we will inspire from the proof of [19, Lemma 3.1]. Since the variances are uniformly bounded in  $k$  (for  $s \leq t$  fixed), there exists  $H > 0$  such that, for any  $k \in \mathbb{N}$  and  $s \leq t$ ,

$$\delta_x P_{s+k\gamma, t+k\gamma}[\psi^2] \leq H \quad \text{and} \quad \delta_x Q_{s, t}[\psi^2] \leq H. \quad (34)$$

Since  $\lim_{|x| \rightarrow \infty} \frac{\psi(x)}{\psi^2(x)} = 0$ , for any  $\epsilon > 0$ , there exists  $l_\epsilon > 0$  such that, for any function  $f$  such that  $|f| \leq \psi$  and for any  $|x| \geq l_\epsilon$ ,

$$|f(x)| \leq \frac{\epsilon \psi(x)^2}{H}.$$

This implies with (34) that, denoting  $K_\epsilon := [-l_\epsilon, l_\epsilon]$ , for any  $k \in \mathbb{Z}_+$ ,  $f$  such that  $|f| \leq \psi$  and  $x \in \mathbb{R}$ ,

$$\delta_x P_{s+k\gamma, t+k\gamma}[f \mathbb{1}_{K_\epsilon^c}] \leq \epsilon \quad \text{and} \quad \delta_x Q_{s, t}[f \mathbb{1}_{K_\epsilon^c}] \leq \epsilon.$$

Then, for any  $k \in \mathbb{Z}_+$  and  $f$  such that  $|f| \leq \psi$ ,

$$|\delta_x P_{s+k\gamma, t+k\gamma} f - \delta_x Q_{s, t} f| \leq 2\epsilon + |\delta_x P_{s+k\gamma, t+k\gamma}[f \mathbb{1}_{K_\epsilon}] - \delta_x Q_{s, t}[f \mathbb{1}_{K_\epsilon}]| \quad (35)$$

$$\leq 2\epsilon + (1 + l_\epsilon^2) \|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s, t}\|_{TV} \quad (36)$$

Hence, (33) implies that, for  $k$  large enough, for any  $f$  bounded by  $\psi$ ,

$$|\delta_x P_{s+k\gamma, t+k\gamma} f - \delta_x Q_{s, t} f| \leq 3\epsilon, \quad (37)$$

implying that

$$\|\delta_x P_{s+k\gamma, t+k\gamma} - \delta_x Q_{s, t}\|_\psi \xrightarrow[k \rightarrow \infty]{} 0.$$

□

We now prove Lemma 1.

*Proof of Lemma 1.* Defining

$$f_\nu(x) := e^{-\frac{(x-a)^2}{2b_-^2}} \wedge e^{-\frac{(x+a)^2}{2b_-^2}},$$

we conclude easily that, for any  $m \in [-a, a]$  and  $\sigma \geq b_-$ , for any  $x \in \mathbb{R}$ ,

$$e^{-\frac{(x-m)^2}{2\sigma^2}} \geq f_\nu(x).$$

Imposing moreover that  $\sigma \leq b_+$ , one has

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \geq \frac{1}{\sqrt{2\pi}b_+} f_\nu(x),$$

which concludes the proof.  $\square$

## 4.2 Quasi-ergodic distribution for Brownian motion absorbed by an asymptotically periodic moving boundary

Let  $(W_t)_{t \geq 0}$  be a one-dimensional Brownian motion and  $h$  be a  $\mathcal{C}^1$ -function such that

$$h_{\min} := \inf_{t \geq 0} h(t) > 0, \quad \text{and} \quad h_{\max} := \sup_{t \geq 0} h(t) < +\infty.$$

We assume also that

$$-\infty < \inf_{t \geq 0} h'(t) \leq \sup_{t \geq 0} h'(t) < +\infty.$$

Denote by

$$\tau_h := \inf\{t \geq 0 : |W_t| \geq h(t)\}.$$

Since  $h$  is continuous, the hitting time  $\tau_h$  is a stopping time with respect to the natural filtration of  $(W_t)_{t \geq 0}$ . Moreover, since  $\sup_{t \geq 0} h(t) < +\infty$  and  $\inf_{t \geq 0} h(t) > 0$ ,

$$\mathbb{P}_{s,x}[\tau_h < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{s,x}[\tau_h > t] > 0, \quad \forall s \leq t, \forall x \in [-h(s), h(s)].$$

The main assumption on the function  $h$  is the existence of a  $\gamma$ -periodic function  $g$  such that  $h(t) \leq g(t)$ , for any  $t \geq 0$ , and such that

$$h \sim_{t \rightarrow \infty} g, \quad \text{and} \quad h' \sim_{t \rightarrow \infty} g'.$$

Similarly to  $\tau_h$ , denote

$$\tau_g := \inf\{t \geq 0 : |W_t| = g(t)\}.$$

Finally, let us assume that there exists  $n_0 \in \mathbb{N}$  such that, for any  $s \geq 0$ ,

$$\inf\{u \geq s : h(u) = \inf_{t \geq s} h(t)\} - s \leq n_0 \gamma. \tag{38}$$

This condition says that there exists  $n_0 \in \mathbb{N}$  such that, for any time  $s \geq 0$ , the infimum of the function  $h$  on the domain  $[s, +\infty)$  is reached on the subset  $[s, s + n_0 \gamma]$ .

We first show the following proposition.

**Proposition 1.** *The Markov process  $(W_t)_{t \geq 0}$ , considered as absorbed by  $h$  or by  $g$ , satisfies Assumption (A').*

*Proof.* In what follows, Assumption (A') w.r.t. the absorbing function  $h$  will be shown. The following proof could easily be adapted for the function  $g$ .

- *Proof of (A'1).* Denote  $\mathcal{T} := \{s \geq 0 : h(s) = \inf_{t \geq s} h(t)\}$ . The condition (38) implies that this set contains an infinity of times.

In what follows, the following notation is needed: for any  $z \in \mathbb{R}$ , define  $\tau_z$  as

$$\tau_z := \inf\{t \geq 0 : |W_t| = z\}.$$

Also, let us state that, since the Brownian motion absorbed at  $\{-1, 1\}$  satisfies Assumption (A) of [5] at any time (see [7]), it follows that, for a given  $t_0 > 0$ , there exists  $c > 0$  and  $\nu \in \mathcal{M}_1((-1, 1))$  such that, for any  $x \in (-1, 1)$ ,

$$\mathbb{P}_{0,x} \left[ W_{\frac{t_0}{h_{\max}^2} \wedge t_0} \in \cdot \mid \tau_1 > \frac{t_0}{h_{\max}^2} \wedge t_0 \right] \geq c\nu. \quad (39)$$

Moreover, regarding the proof of [7, Section 5.1], the probability measure  $\nu$  can be expressed as

$$\nu = \frac{1}{2} (\mathbb{P}_{0,1-\epsilon}[W_{t_2} \in \cdot \mid \tau_1 > t_2] + \mathbb{P}_{0,-1+\epsilon}[W_{t_2} \in \cdot \mid \tau_1 > t_2]), \quad (40)$$

for some  $0 < t_2 < \frac{t_0}{h_{\max}^2} \wedge t_0$  and  $\epsilon \in (0, 1)$ .

The following lemma is very important for the following.

**Lemma 2.** *For all  $z \in [h_{\min}, h_{\max}]$ ,*

$$\mathbb{P}_{0,x}[W_u \in \cdot \mid \tau_z > u] \geq c\nu_z, \quad \forall x \in (-z, z), \forall u \geq t_0,$$

where  $t_0$  was evoked before,  $c > 0$  is the same constant as in (39) and

$$\nu_z(f) = \int_{(-1,1)} f(zx)\nu(dx),$$

with  $\nu \in \mathcal{M}_1((-1, 1))$  defined in (40).

The proof of this lemma is postponed after the current proof.

Let  $s \in \mathcal{T}$ . Then, for all  $x \in (-h(s), h(s))$  and  $t \geq 0$ ,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot \mid \tau_h > s+t] \geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} \mathbb{P}_{s,x}[W_{s+t} \in \cdot \mid \tau_{h(s)} > s+t],$$

By Lemma 2, for all  $x \in (-h(s), h(s))$  and  $t \geq t_0$ ,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot \mid \tau_{h(s)} > s+t] \geq c\nu_{h(s)},$$

which implies therefore that, for any  $t \in [t_0, t_0 + n_0\gamma]$ ,

$$\begin{aligned} \mathbb{P}_{s,x}[W_{s+t} \in \cdot \mid \tau_h > s+t] &\geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} c\nu_{h(s)} \\ &\geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0 + n_0\gamma]}{\mathbb{P}_{s,x}[\tau_h > s+t_0]} c\nu_{h(s)}. \end{aligned} \quad (41)$$

Let us introduce the process  $X^h$  defined by, for all  $t \geq 0$ ,

$$X_t^h := \frac{W_t}{h(t)}.$$

By Itô's formula, for any  $t \geq 0$ ,

$$X_t^h = X_0^h + \int_0^t \frac{dW_s}{h(s)} - \int_0^t \frac{h'(s)}{h(s)} X_s^h ds.$$

Denote by  $(M_t^h)_{t \geq 0} := \left( \int_0^t \frac{1}{h(s)} dW_s \right)_{t \geq 0}$ . By the Dubins-Schwarz theorem, it is well-known that the process  $M^h$  has the same law as  $\left( W \int_0^t \frac{1}{h^2(s)} ds \right)_{t \geq 0}$ . Then, denoting  $I^h(s) := \int_0^s \frac{1}{h^2(u)} du$  and, for any  $s \leq t$  and for any trajectory  $w$ ,

$$\mathcal{E}_{s,t}^h(w) := \sqrt{\frac{h(t)}{h(s)}} \exp \left( -\frac{1}{2} \left[ h'(t)h(t)w_{I^h(t)}^2 - h'(s)h(s)w_{I^h(s)}^2 + \int_s^t w_{I^h(u)}^2 [(h'(u))^2 - [h(u)h'(u)]'] du \right] \right), \quad (42)$$

Girsanov theorem implies that, for all  $x \in (-h(s), h(s))$ ,

$$\mathbb{P}_{s,x}[\tau_h > s + t_0] = \mathbb{E}_{I^h(s), \frac{x}{h(s)}} \left[ \mathcal{E}_{s,s+t_0}^h(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du} \right]. \quad (43)$$

On the event  $\{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du\}$ , and since  $h$  and  $h'$  are bounded on  $\mathbb{R}_+$ , the random variable  $\mathcal{E}_{s,s+t_0}^h(W)$  is almost surely bounded by a constant  $C > 0$ , uniformly in  $s$ , such that for all  $x \in (-h(s), h(s))$ ,

$$\mathbb{E}_{I^h(s), \frac{x}{h(s)}} \left[ \mathcal{E}_{s,s+t_0}^h(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du} \right] \leq C \mathbb{P}_{0, \frac{x}{h(s)}} \left[ \tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right]. \quad (44)$$

Since  $h(t) \geq h(s)$  for all  $t \geq s$  (since  $s \in \mathcal{T}$ ),  $I^h(s+t_0) - I^h(s) \leq \frac{t_0}{h(s)^2}$ . By the scaling property of the Brownian motion and by Markov property, one has for all  $x \in (-h(s), h(s))$ ,

$$\begin{aligned} \mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0] &= \mathbb{P}_{0,x}[\tau_{h(s)} > t_0] \\ &= \mathbb{P}_{0, \frac{x}{h(s)}} \left[ \tau_1 > \frac{t_0}{h^2(s)} \right] \\ &= \mathbb{E}_{0, \frac{x}{h(s)}} \left[ \mathbb{1}_{\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du} \mathbb{P}_{0,W} \int_s^{s+t_0} \frac{1}{h^2(u)} du \left[ \tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(s)} ds \right] \right] \\ &= \mathbb{P}_{0, \frac{x}{h(s)}} \left[ \tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right] \mathbb{P}_{0, \phi_{I^h(s+t_0) - I^h(s)}(\delta_x)} \left[ \tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right], \end{aligned}$$

where, for any initial distribution  $\mu$  and any  $t \geq 0$ ,

$$\phi_t(\mu) := \mathbb{P}_{0,\mu}[W_t \in \cdot | \tau_1 > t].$$

The family  $(\phi_t)_{t \geq 0}$  satisfies the equality  $\phi_t \circ \phi_s = \phi_{t+s}$  for all  $s, t \geq 0$ . By this property, and using that  $I^h(s+t_0) - I^h(s) \geq \frac{t_0}{h_{\max}^2}$  for any  $s \geq 0$ , the minorization (39) implies that, for all  $s \geq 0$  and  $x \in (-1, 1)$ ,

$$\phi_{I^h(s+t_0) - I^h(s)}(\delta_x) \geq c\nu.$$

Hence, by this minorization, and using that  $h$  is upper-bounded and lower-bounded positively on  $\mathbb{R}_+$ , one has for all  $x \in (-1, 1)$ ,

$$\mathbb{P}_{0, \phi_{I^h(s+t_0) - I^h(s)}(\delta_x)} \left[ \tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right] \geq c \mathbb{P}_{0,\nu} \left[ \tau_1 > \inf_{s \geq 0} \left\{ \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right],$$

that is to say,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]}{\mathbb{P}_{0, \frac{x}{h(s)}} \left[ \tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du \right]} \geq c \mathbb{P}_{0,\nu} \left[ \tau_1 > \inf_{s \geq 0} \left\{ \frac{\gamma}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right].$$

In other words, we just showed that, for all  $x \in (-h(s), h(s))$ ,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]}{\mathbb{P}_{s,x}[\tau_h > s + t_0]} \geq \frac{c}{C} \mathbb{P}_{0,\nu} \left[ \tau_1 > \inf_{s \geq 0} \left\{ \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du \right\} \right] > 0. \quad (45)$$

Moreover, by Lemma 2 and the scaling property of the Brownian motion, for all  $x \in (-h(s), h(s))$ ,

$$\begin{aligned} \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0 + n_0\gamma]}{\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0]} &= \mathbb{P}_{0,\mathbb{P}_{0,x}[W_{t_0} \in \cdot | \tau_{h(s)} > t_0]}[\tau_{h(s)} > n_0\gamma] \\ &\geq c \mathbb{P}_{0,\nu_{h(s)}}[\tau_{h(s)} > n_0\gamma] \\ &= c \int_{(-1,1)} \nu(dy) \mathbb{P}_{h(s)y}[\tau_{h(s)} > n_0\gamma] \\ &\geq c \mathbb{P}_{0,\nu} \left[ \tau_1 > \frac{n_0\gamma}{h_{\min}^2} \right] > 0. \end{aligned} \quad (46)$$

Thus, gathering (41), (45) and (46), for any  $x \in (-h(s), h(s))$  and any  $t \in [t_0, t_0 + n_0\gamma]$ ,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s + t] \geq c_1 \nu_{h(s)}, \quad (47)$$

where

$$c_1 := c \mathbb{P}_{0,\nu} \left[ \tau_1 > \frac{n_0\gamma}{h_{\max}^2} \right] \times \frac{c}{C} \mathbb{P}_{0,\nu} \left[ \tau_1 > \inf_{s \geq 0} \left\{ \frac{\gamma}{h^2(s)} - \int_s^{s+\gamma} \frac{1}{h^2(u)} du \right\} \right] c.$$

We recall that the Doeblin condition (47) is, for now, only obtained for  $s \in \mathcal{T}$ . Consider now  $s \notin \mathcal{T}$ . Then, by the condition (38), there exists  $s_1 \in \mathcal{T}$  such that  $s < s_1 \leq s + n_0\gamma$ . Markov property and (47) implies therefore that, for any  $x \in (-h(s), h(s))$ ,

$$\mathbb{P}_{s,x}[W_{s+t_0+n_0\gamma} \in \cdot | \tau_h > s + t_0 + n_0\gamma] = \mathbb{P}_{s_1,\phi_{s_1,s}}[W_{s+t_0+n_0\gamma} \in \cdot | \tau_h > s + t_0 + n_0\gamma] \geq c_1 \nu_{h(s_1)},$$

where, for all  $s \leq t$  and  $\mu \in \mathcal{M}_1((-h(s), h(s)))$ ,

$$\phi_{t,s}(\mu) := \mathbb{P}_{s,\mu}[W_t \in \cdot | \tau_h > t].$$

This concludes the proof of (A'1).

- *Proof of (A'2).* Since  $(W_t)_{t \geq 0}$  is a Brownian motion, note that, for any  $s \leq t$ ,

$$\sup_{x \in (-1,1)} \mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,0}[\tau_h > t].$$

Also, for any  $a \in (0, h(s))$ ,

$$\inf_{[-a,a]} \mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,a}[\tau_h > t].$$

Thus, by Markov property, and using that the function  $s \mapsto \mathbb{P}_{s,0}[\tau_g > t]$  is non-decreasing on  $[0, t]$  (for all  $t \geq 0$ ), one has for any  $s \leq t$ ,

$$\mathbb{P}_{s,a}[\tau_h > t] \geq \mathbb{E}_{s,a}[\mathbb{1}_{\tau_0 < s + \gamma < \tau_h} \mathbb{P}_{\tau_0,0}[\tau_h > t]] \geq \mathbb{P}_{s,a}[\tau_0 < s + \gamma < \tau_h] \mathbb{P}_{s,0}[\tau_h > t]. \quad (48)$$

Denoting  $a := \frac{h_{\min}}{h_{\max}}$ , by Lemma 2 and taking  $s_1 := \inf\{u \geq s : u \in \mathcal{T}\}$ , one obtains that, for all  $s \leq t$ ,

$$\begin{aligned} \mathbb{P}_{s,\nu_{h(s_1)}}[\tau_h > t] &= \int_{(-1,1)} \nu(dx) \mathbb{P}_{s,h(s_1)x}[\tau_h > t] \\ &\geq \nu([-a, a]) \mathbb{P}_{s,h(s_1)a}[\tau_h > t] \\ &\geq \nu([-a, a]) \mathbb{P}_{0,h_{\min}}[\tau_0 < \gamma < \tau_h] \sup_{x \in (-h(s), h(s))} \mathbb{P}_{s,x}[\tau_h > t]. \end{aligned}$$

This concludes the proof since, using (40), one has  $\nu([-a, a]) > 0$ .

□

We now prove Lemma 2.

*Proof of Lemma 2.* This result comes from the scaling property of a Brownian motion. In fact, for any  $z \in [h_{\min}, h_{\max}]$ ,  $x \in (-z, z)$  and  $t \geq 0$ , and for any measurable bounded function  $f$ ,

$$\begin{aligned} \mathbb{E}_{0,x}[f(W_t)|\tau_z > t] &= \mathbb{E}_{0,x} \left[ f \left( z \times \frac{1}{z} W_{z^2 \frac{t}{z^2}} \right) \middle| \tau_z > t \right] \\ &= \mathbb{E}_{0, \frac{x}{z}} \left[ f \left( z \times W_{\frac{t}{z^2}} \right) \middle| \tau_1 > \frac{t}{z^2} \right]. \end{aligned}$$

Then, the minorization (39) implies that, for any  $x \in (-1, 1)$ ,

$$\mathbb{P}_{0,x} \left[ W_{\frac{t_0}{h_{\max}^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{h_{\max}^2} \right] \geq c\nu.$$

This inequality holds for any time greater than  $\frac{t_0}{h_{\max}^2}$ . In particular, for any  $z \in [h_{\min}, h_{\max}]$  and  $x \in (-1, 1)$ ,

$$\mathbb{P}_{0,x} \left[ W_{\frac{t_0}{z^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{z^2} \right] \geq c\nu.$$

Then, for any  $z \in [a, b]$ ,  $f$  positive and measurable, and  $x \in (-z, z)$ ,

$$\mathbb{E}_{0,x}[f(W_{t_0})|\tau_z > t_0] \geq c\nu_z(f),$$

where  $\nu_z(f) := \int_E f(z \times x)\nu(dx)$ . This completes the proof of Lemma 2. □

The section is now concluded by stating and proving the following result.

**Theorem 4.** *For any  $s \leq t$ ,  $n \in \mathbb{N}$  and any  $x \in \mathbb{R}$ ,*

$$\mathbb{P}_{s+k\gamma,x}[\tau_h \leq t + k\gamma < \tau_g] \xrightarrow[k \rightarrow \infty]{} 0.$$

*In particular, Corollary 2 holds for  $(W_t)_{t \geq 0}$  absorbed by  $h$ .*

*Proof.* Reminding (42), by Markov property for the Brownian motion, one has for any  $k, n \in \mathbb{N}$  and any  $x \in \mathbb{R}$ ,

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t + k\gamma] = \sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \mathbb{E}_{0,x} \left[ \exp \left( -\frac{1}{2} \mathcal{A}_{s,t,k}^h(W) \right) \mathbb{1}_{\tau_1 > I^h(t+k\gamma) - I^h(s+k\gamma)} \right],$$

where, for any trajectory  $w = (w_u)_{u \geq 0}$ ,

$$\begin{aligned} \mathcal{A}_{s,t,k}^h(w) &= h'(t+k\gamma)h(t+k\gamma)w_{I^h(t+k\gamma) - I^h(s+k\gamma)}^2 - h'(s+k\gamma)h(s+k\gamma)w_0^2 \\ &\quad + \int_0^{t-s} w_{I^h(u+s+k\gamma) - I^h(s+k\gamma)}^2 [h'(u+s+k\gamma)]^2 - [h(u+s+k\gamma)h'(u+s+k\gamma)]' du. \end{aligned}$$

Since  $h \sim_{t \rightarrow \infty} g$ , one has for any  $s, t \in [0, \gamma]$

$$\sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \xrightarrow[k \rightarrow \infty]{} \sqrt{\frac{g(t)}{g(s)}}.$$

For the same reasons, and using that the function  $h$  is bounded on  $[s+k\gamma, t+k\gamma]$  for all  $s \leq t$ , Lebesgue's dominated convergence theorem implies that

$$I^h(t+k\gamma) - I^h(s+k\gamma) \xrightarrow[k \rightarrow \infty]{} I^g(t) - I^g(s)$$

for all  $s \leq t \in [0, \gamma]$ . Moreover, since  $h \sim_{t \rightarrow \infty} g$  and  $h' \sim_{t \rightarrow \infty} g'$ , one has for all trajectories  $w = (w_u)_{u \geq 0}$  and  $s \leq t \in [0, \gamma]$ ,

$$\mathcal{A}_{s,t,k}^h(w) \xrightarrow[k \rightarrow \infty]{} g'(t)g(t)w_{I^g(t)-I^g(s)}^2 - g'(s)g(s)w_0^2 + \int_s^t w_{I^g(u)}^2 [(g'(u))^2 - [g(u)g'(u)]'] du.$$

Since the random variable  $\exp\left(-\frac{1}{2}\mathcal{A}_{s,t,k}^h(W)\right) \mathbb{1}_{\tau_1 > I^h(t+k\gamma) - I^h(s+k\gamma)}$  is bounded almost surely, Lebesgue's dominated convergence theorem implies that

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t + k\gamma] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{s,x}[\tau_g > t],$$

which concludes the proof. □

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