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STRUCTURE OF JULIA SETS FOR POST-CRITICALLY FINITE ENDOMORPHISMS ON \mathbb{P}^2

ZHUCHAO JI

ABSTRACT. Let f be a post-critically finite endomorphism (PCF map for short) on \mathbb{P}^2 , let J_1 denote the Julia set and let J_2 denote the support of the measure of maximal entropy. In this paper we show that: 1. $J_1 \setminus J_2$ is contained in the union of the (finitely many) basins of critical component cycles and stable manifolds of sporadic super-saddle cycles. 2. For every $x \in J_2$ which is not contained in the stable manifold of a sporadic super-saddle cycle, there is no Fatou disk containing x . Here sporadic means that the super-saddle cycle is not contained in a critical component cycle. Under the additional assumption that all branches of $PC(f)$ are smooth and intersect transversally, we show that there is no sporadic super-saddle cycle. Thus in this case $J_1 \setminus J_2$ is contained in the union of the basins of critical component cycles, and for every $x \in J_2$ there is no Fatou disk containing x .

As consequences of our result: 1. We answer some questions of Fornæss-Sibony about the non-wandering set for PCF maps on \mathbb{P}^2 with no sporadic super-saddle cycles. 2. We give a new proof of de Thélin's laminarity of the Green current in $J_1 \setminus J_2$ for PCF maps on \mathbb{P}^2 . 3. We show that for PCF maps on \mathbb{P}^2 an invariant compact set is expanding if and only if it does not contain critical points, and we obtain characterizations of PCF maps on \mathbb{P}^2 which are expanding on J_2 or satisfy Axiom A.

1. INTRODUCTION

1.1. Background. Let $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a holomorphic endomorphism of degree ≥ 2 , where \mathbb{P}^2 is the complex projective plane. The *first Julia set* J_1 is defined as the locus where the iterates $(f^n)_{n \geq 0}$ do not locally form a normal family, i.e. the complement of the *Fatou set*. Let T be the *dynamical Green current* of f , defined by $T = \lim_{n \rightarrow +\infty} d^{-n} (f^n)^* \omega_{FS}$, where ω_{FS} is the Fubini-Study (1,1) form on \mathbb{P}^2 . The Julia set J_1 coincides with $\text{Supp}(T)$, and the self intersection measure $\mu = T \wedge T$ is the unique measure of maximal entropy of f . See Dinh-Sibony [10] for background on holomorphic dynamics on projective spaces.

We define the *second Julia set* to be $J_2 = \text{Supp} \mu$. From the definitions we know that $J_2 \subset J_1$. By Briend-Duval [5], J_2 is contained in the closure of the set of repelling periodic points. However contrary to the situation in dimension one there may exist repelling periodic point outside J_2 . A major problem in holomorphic dynamics is to investigate the structure of $J_1 \setminus J_2$. A promising picture is that $J_1 \setminus J_2$ is foliated (in some appropriate sense) by holomorphic disks D along which $(f^n|_D)_{n \geq 0}$ is a normal family. Such disks are called *Fatou disks*. The dynamical Green current T is called *laminar* in some open set Ω if it expresses as an integral of integration currents over a measurable family of compatible holomorphic disks (which means that these disks have no isolated intersections) in Ω . These disks are automatically Fatou disks. Let $\sigma_T = T \wedge \omega$ be the trace measure of T , which is a natural

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reference measure on J_1 . If T is laminar in Ω , then for σ_T a.e. $x \in \Omega$ there exists a germ of holomorphic Fatou disk D containing x . De Thélin proved in [8] and [9] that T is laminar in $J_1 \setminus J_2$ for *post-critically finite endomorphisms* on \mathbb{P}^2 . We note that by the works of Dujardin in general T is not necessarily laminar in $J_1 \setminus J_2$ [13], but a related weaker result holds [12].

A holomorphic endomorphism f on \mathbb{P}^k , $k \geq 1$ is called post-critically finite (PCF for short) if the post-critical set

$$PC(f) := \bigcup_{n \geq 1} f^n(C(f))$$

is an algebraic subset of \mathbb{P}^k , where

$$C(f) := \left\{ x \in \mathbb{P}^k : Df(x) \text{ is not invertible} \right\}$$

is the critical set.

In dimension 1, this coincides with the usual definition of PCF maps on the Riemann sphere \mathbb{P}^1 . PCF maps play an important role in one-dimensional complex dynamics, mainly because the remarkable topological classification theorem of Thurston [11]. PCF maps are still of interest in higher dimension, and their dynamics have been investigated by many authors. Here are some results which are useful for our purpose:

1. The Fatou set of a PCF maps on \mathbb{P}^2 is the union of the basins of super-attracting cycles (Fornaess-Sibony [15], Ueda [31] and Rong [26]).
2. $J_k = \mathbb{P}^k$ for strictly PCF maps on \mathbb{P}^k ($k = 2$ by Jonsson [20], general k by Ueda [30]), our main result can be seen as a generalization of Jonsson's result.
3. The eigenvalues of periodic points of PCF maps on \mathbb{P}^2 are either 0 or larger than 1 (Le [22]).

The dynamics of PCF maps on \mathbb{P}^k , $k \geq 2$ were further studied by Ueda [30] and Astorg [1]. Moreover, interesting examples of PCF maps were constructed by Crass [6], Fornæss-Sibony [14] and Koch [21].

1.2. Basins of critical component cycles. In this paper we investigate the dynamics on the Julia sets J_1 and J_2 for PCF maps on \mathbb{P}^2 . Let f be a PCF map on \mathbb{P}^2 . Recall that the critical set $C(f)$ is an algebraic curve. We call an irreducible component Λ of $C(f)$ periodic if there exists an integer $n \geq 1$ such that $f^n(\Lambda) = \Lambda$. Such an irreducible component is called a *periodic critical component*. There are finitely many periodic critical components. The set $\{\Lambda, f(\Lambda), \dots, f^{n-1}(\Lambda)\}$ is called a *critical component cycle*. Similarly, a critical point x satisfying $f^n(x) = x$ for some $n \geq 1$ is called a *periodic critical point*. The set $\{x, f(x), \dots, f^{n-1}(x)\}$ is called a *critical point cycle*. Since f^n and f have the same Julia sets, to investigate the structure of $J_1 \setminus J_2$ we may assume that all periodic critical components are invariant.

An important observation is that in \mathbb{P}^2 , any invariant critical component Λ is an *attracting set*. By definition an attracting set Λ in \mathbb{P}^2 is an invariant compact subset such that there is a neighborhood U of Λ satisfying $f(U) \subset\subset U$, and $\Lambda = \bigcap_{n \geq 1} f^n(U)$. The open set U is called a *trapping region* of Λ . The attracting basin $\mathcal{B}(\Lambda)$ of an attracting set Λ is by definition the

set $\bigcup_{n \geq 0} f^{-n}(U)$, where U is a trapping region of Λ . Equivalently $\mathcal{B}(\Lambda)$ is the set of points attracted by Λ , i.e.

$$B(\Lambda) = \{x \in \mathbb{P}^2 : \text{dist}(f^n(x), \Lambda) \rightarrow 0, \text{ as } n \rightarrow +\infty\}.$$

An attracting basin in \mathbb{P}^2 is always disjoint from J_2 , except when $\Lambda = \mathbb{P}^2$, see [28] Proposition 1.1 for a proof.

Fornaess and Sibony [15] proved that for any fixed Riemannian metric on \mathbb{P}^2 , if Λ is an invariant critical component, then for $x \in \mathbb{P}^2$, when $\text{dist}(x, \Lambda) \rightarrow 0$, we have

$$\text{dist}(f(x), \Lambda) = o(\text{dist}(x, \Lambda))$$

It follows that for $\epsilon > 0$ sufficiently small, the ϵ -neighborhood

$$U_\epsilon = \{x \in \mathbb{P}^2 : \text{dist}(x, \Lambda) < \epsilon\}$$

satisfies $f(U_\epsilon) \subset\subset U_\epsilon$, and $\Lambda = \bigcap_{n \geq 1} f^n(U_\epsilon)$. Then Λ is an attracting set with trapping region U_ϵ . Fornaess and Sibony [15] also showed that an invariant irreducible curve in \mathbb{P}^2 has genus 0 or 1. Bonifant, Dabija and Milnor showed that an elliptic curve can not be an attracting set [3], so the invariant critical component Λ must be a (possibly singular) rational curve.

Now let f be a PCF map on \mathbb{P}^2 with an invariant critical component Λ . Let $\pi : \widehat{\Lambda} \rightarrow \Lambda$ be the normalization of Λ , then f restricted to Λ lifts to a map \hat{f} from $\widehat{\Lambda}$ to itself. By [20], \hat{f} is a PCF map on \mathbb{P}^1 of degree ≥ 2 . Let $J(f)$ be the Julia set of f , and let $\hat{\nu}$ be the unique measure of maximal entropy of \hat{f} . Then $\nu = \pi_*(\hat{\nu})$ is an invariant measure on \mathbb{P}^2 . It follows that $\text{Supp } \nu = \pi(J(f))$, and ν is a hyperbolic measure of saddle type. Daurat showed that the Green current T is laminar in the basin $\mathcal{B}(\Lambda)$ and subordinate to the stable manifolds $\bigcup_{x \in \text{Supp } \nu} W^s(x)$. See [7] for the proof and for more details (it is easy to verify in our case the trapping region U_ϵ satisfies conditions (Tub) and (SJ) in Daurat's paper when ϵ sufficiently small). See also Bedford-Jonsson [2] for the case when Λ is totally invariant.

1.3. Main results. At this stage we get a nice description of the dynamics in the attracting basins of critical component cycles for PCF maps on \mathbb{P}^2 . In this paper we will show that except maybe for a possibly "small" set (the stable manifolds of *sporadic* super-saddle cycles), every point in $J_1 \setminus J_2$ is indeed contained in the basin of a critical component cycle, and for every point in J_2 , there is no Fatou disk passing through it. Recall that a fixed point is called super-saddle if it has one 0 eigenvalue and one eigenvalue with modulus larger than 1. A sporadic super-saddle fixed point is by definition a super-saddle fixed point which is not contained in a critical component cycle. We will see in Lemma 2.9 that there are at most finitely many sporadic super-saddle cycles. The following is our main result.

Theorem 1.1. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 , then:*

(1) $J_1 \setminus J_2$ is contained in the union of the basins of critical component cycles and stable manifolds of sporadic super-saddle cycles.

(2) For every $x \in J_2$ which is not contained in the stable manifold of a sporadic super-saddle cycle, there is no Fatou disk containing x .

Here are some remarks of Theorem 1.1. First, for general holomorphic endomorphisms on \mathbb{P}^2 , there can exist an invariant compact set $E \subset J_2$ with positive entropy such that for every $x \in E$ there exists a Fatou disk containing x , see Taffin [29] Theorem 1.6. Second, to the author's knowledge, there are no sporadic super-saddle cycles for all known examples of PCF maps on \mathbb{P}^2 . It is interesting to know whether there exists a PCF map on \mathbb{P}^2 carrying a sporadic super-saddle cycle. If the answer is yes, it is also interesting to know whether the stable manifold of this sporadic super-saddle point is contained in J_2 or $J_1 \setminus J_2$. However, under the additional assumption that all branches of $PC(f)$ are smooth and intersect transversally, the answer of this question is no.

Theorem 1.2. *Let f be a PCF maps on \mathbb{P}^2 of degree ≥ 2 such that all branches of $PC(f)$ are smooth and intersect transversally, then every super-saddle cycle is contained in a critical component cycle, that is, sporadic super-saddle cycles do not exist.*

We note that the assumption that all branches of $PC(f)$ are smooth and intersect transversally are satisfied by examples constructed by Crass [6], Fornaess-Sibony [14] and Koch [21], in fact in these examples $PC(f)$ are union of projective lines. A direct corollary of Theorem 1.1 and 1.2 is the following.

Corollary 1.3. *Let f be a PCF maps on \mathbb{P}^2 of degree ≥ 2 such that all branches of $PC(f)$ are smooth and intersect transversally, then*

- (1) $J_1 \setminus J_2$ is contained in the union of the basins of critical component cycles
- (2) There is no Fatou disk containing x for every $x \in J_2$.

1.4. Some corollaries of the main results. In [18], Fornaess and Sibony asked several questions about the non-wandering set $\Omega(f)$ of a holomorphic endomorphism on \mathbb{P}^2 . The following are the questions:

Q1: Is $\Omega(f)$ the closure of periodic points and Siegel varieties? (Here a Siegel variety is an irreducible analytic set X of positive dimension such that there exists a subsequence $\{n_j\}$ such that $f^{n_j}|_X \rightarrow Id$.)

Q2: Is the closure of the repelling periodic points open in $\Omega(f)$?

Q3: Is J_2 open in $\Omega(f)$?

Q4: Describe $\Omega(f) \setminus J_2$.

As a corollary of Theorem 1.1, we can give answers to the above four questions for PCF maps on \mathbb{P}^2 with no sporadic super-saddle cycles.

Theorem 1.4. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then J_2 is the closure of repelling periodic points. If we further assume that there is no sporadic super-saddle cycle, then $\Omega(f)$ is the closure of periodic points, and $\Omega(f) \setminus J_2$ is the union of super-attracting cycles together with $\cup(C \cap J_1)$, where C ranges over the set of critical component cycles, in particular J_2 is open in $\Omega(f)$.*

Next, as a direct corollary of Theorem 1.1, we obtain a new proof of the laminarity of the Green current in $J_1 \setminus J_2$ for PCF maps on \mathbb{P}^2 , which was first proved by de Thélin [9].

Corollary 1.5 (de Thélin). *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then the Green current T is laminar in $J_1 \setminus J_2$.*

We note that Fornæss and Sibony have studied the non-wandering set and the problem of laminarity of the Green current for s -hyperbolic holomorphic endomorphisms on \mathbb{P}^2 (s -hyperbolic means Axiom A plus some technical conditions), see [16].

Finally, we obtain a characterization of expanding invariant compact sets for PCF maps on \mathbb{P}^2 . By using both Theorem 1.1 and the methods in the proof of Theorem 1.1, we obtain the following result.

Theorem 1.6. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 and let K be an invariant compact set. Then K is expanding if and only if K does not contain critical points.*

We note that for rational maps on \mathbb{P}^1 , an invariant compact set in the Julia set is expanding if it does not intersect the closure of the post-critical set. It is an open problem that whether the same is true for holomorphic endomorphisms on \mathbb{P}^k , i.e. whether an invariant compact set in J_k is expanding if it does not intersect the closure of the post-critical set. See Maegawa [23] for a discussion of this problem. Theorem 1.6 answer this question for PCF maps on \mathbb{P}^2 .

As corollaries of Theorem 1.6 and Theorem 1.1, we obtain characterizations of PCF maps on \mathbb{P}^2 which are expanding on J_2 or satisfy Axiom A.

Corollary 1.7. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then f is expanding on J_2 if and only if every critical component of f is preperiodic to a critical component cycle, and f is Axiom A if and only if f is expanding on J_2 , and for every critical component cycle C , $C \cap J_1$ is a hyperbolic (saddle) set.*

1.5. Outline of the paper. The structure of this paper is as follows. Section 2 is devoted to some preliminaries. In particular we recall Ueda's results about Fatou maps and the normality of backward iterates of holomorphic endomorphisms on \mathbb{P}^k , $k \geq 1$.

In section 3 we prove Theorem 1.2 and the following theorem

Theorem 1.8. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 , then J_2 is the closure of the set of repelling periodic points.*

Here is a comment of Theorem 1.8. We note that for a general holomorphic endomorphism on \mathbb{P}^2 , repelling periodic point may not be contained in J_2 . Indeed there exist examples possessing isolated repelling points outside J_2 , see [17] and [19]. The proof of Theorem 1.8 is a rather quick consequence of a result of Ueda [31].

In section 4 we prove Theorem 1.1, by using the previous results. The proof is divided into two steps. First, we prove that there exists a Fatou disk containing x for every $x \in J_1 \setminus J_2$ which is not contained in the basin of critical component cycle. Second, we prove that if there is a Fatou disk containing x for some $x \in J_1$, then x must be contained in the basin of a critical component cycle or in the stable manifold of a sporadic super-saddle cycle. Theorem 1.1 is a combination of these two steps.

In section 5 we prove Theorem 1.4, Corollary 1.5, Theorem 1.6 and Corollary 1.7.

In section 6 we discuss some open problems about the existence of sporadic super-saddle periodic points (for PCF maps on \mathbb{P}^2) and the possible generalization of Theorem 1.1 to higher dimension.

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2. PRELIMINAIRES

In this section we recall some results of Ueda that we will use later. We start with the following definitions of Ueda [31] Definition 4.5 and [30] Section 1.

Definition 2.1. *Let f be a holomorphic endomorphism on \mathbb{P}^k of degree ≥ 2 . Let Z be a complex analytic space. A holomorphic map $h : Z \rightarrow \mathbb{P}^k$ is called a Fatou map if $\{f^n \circ h\}_{n \geq 1}$ is a normal family. A Fatou disk $D \subset \mathbb{P}^k$ is an image of a non-constant Fatou map $\phi : \mathbb{D} \rightarrow \mathbb{P}^k$, where \mathbb{D} is the unit disk.*

Note that with this definition, a Fatou disk may be singular.

The following fundamental result about Fatou maps can be found in Ueda [30] Proposition 2.1.

Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ be the canonical projection. Given a holomorphic map $\phi : X \rightarrow \mathbb{P}^k$, we say a holomorphic map $\Phi : X \rightarrow \mathbb{C}^{k+1} \setminus \{0\}$ is a lift of ϕ if $\pi \circ \Phi = \phi$ holds.

For a homogeneous regular polynomial endomorphism F on \mathbb{C}^k , we define the dynamical Green function as

$$G(z) := \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \|F^n(z)\|, \text{ for } z \in \mathbb{C}^k.$$

Theorem 2.2. *Let X be a complex analytic space. Let $\phi : X \rightarrow \mathbb{P}^k$ be a holomorphic map. Then the following are equivalent.*

- (1) ϕ is a Fatou map.
- (2) $\{f^j \circ \phi\}$ contains a locally uniformly convergent subsequence.
- (3) If V is an open set of X and $\Phi_V : V \rightarrow \mathbb{C}^{k+1} \setminus \{0\}$ is a holomorphic lift of $\phi|_V$, then $G \circ \Phi_V$ is pluriharmonic on V .
- (4) For any $x \in X$, there exists a neighborhood V of x and $\Phi_V : V \rightarrow \mathbb{C}^{k+1} \setminus \{0\}$ a holomorphic lift of $\phi|_V$, such that $G \circ \Phi_V$ is identically 0 on V .

Next we introduce the conception of points with bounded ramification, first introduced by Ueda in [31] Definition 4.5.

Definition 2.3. *Let f be a holomorphic endomorphism on \mathbb{P}^k of degree ≥ 2 . A point q is said to be a point of bounded ramification if the following conditions are satisfied:*

- (1) there exists a neighborhood W of q such that $PC(f) \cap W$ is an analytic subset of W .
- (2) there exists an integer m such that for every $j > 0$ and every $p \in f^{-j}(q)$, we have that $\text{ord}(f^j, p) \leq m$.

In the case $k = 2$, we have the following characterization of points of bounded ramification for PCF maps on \mathbb{P}^2 . (See Ueda [31] Lemma 5.7.)

Lemma 2.4. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then the points with unbounded ramification are the union of critical component cycles and critical point cycles.*

Next we introduce the following abstract result of Ueda. (See [31] Lemma 3.7 and Lemma 3.8.)

Lemma 2.5. *Let X be a complex manifold and D an analytic subset of X of codimension 1. Let $x \in X$ and let X_0 be a compact neighborhood of x such that the pair $(X_0, X_0 \cap D)$ can be triangulated, i.e. there exist simplicial complex K and its subcomplex L whose underlying spaces are X_0 and $X_0 \cap D$. Let W be the open star of x with respect to the complex K . Then for every integer $m \geq 0$, there exists an irreducible normal complex space Z and $\eta : Z \rightarrow W$ holomorphic such that:*

(1) η is m -universal, in the sense that for every $D \cap W$ -branched holomorphic covering $h : Y \rightarrow W$ (i.e. the ramification locus of h is contained in $D \cap W$) with sheet number $\leq m$, there exists a holomorphic map $\gamma : Z \rightarrow Y$ such that $h \circ \gamma = \eta$.

(2) If $h : Y \rightarrow W$ is a $D \cap W$ branched holomorphic covering, then $h^{-1}(x)$ is a single point.

Specializing to holomorphic endomorphisms on \mathbb{P}^k , we get the following corollary:

Corollary 2.6. *Let f be a holomorphic endomorphism on \mathbb{P}^k of degree ≥ 2 . Let $x \in \mathbb{P}^k$ be a point with bounded ramification. Then for every neighborhood W of x satisfying the assumption in Lemma 2.5, there exists an irreducible normal complex space Z , and $\eta : Z \rightarrow W$ a $PC(f) \cap W$ branched holomorphic covering map such that if W_n denotes a connected component of $f^{-n}(W)$, then there exists a holomorphic map $g_n : Z \rightarrow W_n$ such that $f^n \circ g_n = \eta$, i.e. the following diagram is commute.*

$$\begin{array}{ccc} Z & \xrightarrow{\eta} & W \\ g_n \downarrow & \nearrow f^n & \\ W_n & & \end{array}$$

The map g_n constructed above can be seen as a kind of inverse branch of f^n .

Proof. Since x has bounded ramification, there exists $m \geq 0$ such that $\text{ord}(f^n, x_n) \leq m$ for every $x_n \in f^{-n}(x)$ and $n \geq 0$. Take $X = \mathbb{P}^k$ and $D = PC(f)$ in Lemma 2.5. Then for W satisfying the assumption in Lemma 2.5, there exists an irreducible normal complex space Z , and $\eta : Z \rightarrow W$ a $PC(f) \cap W$ branched holomorphic covering map, satisfy the two conclusion of Lemma 2.5. Let W_n denote a connected component of $f^{-n}(W)$, then by Lemma 2.5 (2), $W_n \cap f^{-n}(x)$ contains a single point, thus $f^n : W_n \rightarrow W$ has sheet number $\leq m$. By Lemma 2.5 (1), there exists a holomorphic map $g_n : Z \rightarrow W_n$ such that $f^n \circ g_n = \eta$. Thus the proof is complete. \square

The sequence $\{g_n\}$ defined in Corollary 2.6 is in fact normal, and any limit map of $\{g_n\}$ is also a Fatou map, by the following result of Ueda ([30] Theorem 2.4).

Lemma 2.7. *Let f be a holomorphic endomorphism on \mathbb{P}^k of degree ≥ 2 . Let Z be a complex analytic space and $h : Z \rightarrow \mathbb{P}^k$ a holomorphic map, for every integer n let $g_n : Z \rightarrow \mathbb{P}^k$ be the holomorphic map such that $f^n \circ g_n = h$. Then $\{g_n\}$ is a normal family. Further more if ϕ is a limit map of a subsequence of $\{g_n\}$, then ϕ is a Fatou map.*

The following lemma is implicitly contained in [30] Theorem 4.2, which will be important in our proof of the main results.

Lemma 2.8. *Let f be a holomorphic endomorphism on \mathbb{P}^k of degree ≥ 2 . Let K be compact connected subset of \mathbb{P}^k such that $\{f^n|_K\}$ contains a uniformly convergent subsequence and K is not a single point. Let ϕ be the limit map of the uniformly convergent subsequence of $f^n|_K$. Then*

(1) *If ϕ is a constant map, then $x := \phi(K)$ is a point of unbounded ramification.*

(2) *If ϕ is not a constant map and there exists $x \in \phi(K)$ which is a point of bounded ramification. Then there exists a Siegel variety Y containing x , i.e. an irreducible analytic set of positive dimension such that there is a subsequence $\{n_j\}$ such that $f^{n_j}|_Y \rightarrow Id$.*

Proof. The proof is contained in the proof of [30] Theorem 4.2, except at one point one should replace the branched covering map $\xi : X \rightarrow \mathbb{P}^k$ in [30] by the branched covering $\eta : Z \rightarrow W$, where W is a neighborhood of x , and $\eta : Z \rightarrow W$ is the branched covering constructed in Corollary 2.6. The idea is to construct locally uniformly convergent subsequences in the following style: $\{g_j : Z \rightarrow \mathbb{P}^k\}$ such that $f^{n_j} \circ g_j = \eta$. The property of the limit map of well-chosen $\{g_j\}$ will either give a contradiction of (1), or give the desired Fatou map of (2). \square

We end this section by showing the following finiteness result.

Lemma 2.9. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then the set of critical point cycles is a finite set.*

Proof. For every critical point cycle, there is a periodic point x in this cycle such that $x \in C(f)$. Let V be the union of periodic components in $PC(f)$, then $x \in V \cap C(f)$. It is clear that every irreducible curve in $V \cap C(f)$ is contained in a critical component cycle. So if x is not contained in a critical component cycle, then x is contained in the dimension 0 subvariety of $V \cap C(f)$. Since this dimension 0 part is just a union of finite number of points, there are at most finitely many critical point cycles which are not contained in a critical component cycle. \square

3. LOCATION OF PERIODIC POINTS.

Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . In this section we prove Theorem 1.2 and Theorem 1.8. By Briend-Duval [5], J_2 is contained in the closure of the set of repelling periodic points. Thus to prove J_2 is the closure of the set of repelling periodic points, we only need to prove that repelling periodic points are contained in J_2 . We start with a definition.

Definition 3.1. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . A fixed point x_0 is called repelling if all eigenvalues of Df at x_0 have modulus larger than 1. A fixed point x_0 is called super-saddle if Df at x_0 has one 0 eigenvalue and one eigenvalue with modulus larger than 1. A fixed point x_0 is called super-attracting if Df at x_0 has only 0 eigenvalues.*

Note that by the result of Le [22], for PCF maps on \mathbb{P}^2 every periodic point is either repelling, super-saddle or super-attracting.

3.1. Repelling points. We first recall the following fundamental result, for the proof see Sibony [27] Corollaire 3.6.5.

Proposition 3.2. *Let f be a holomorphic endomorphism on \mathbb{P}^k of degree ≥ 2 . Then $x \in J_k$ if and only if for every neighborhood U of x , $\mathbb{P}^k \setminus \bigcup_{n=0}^{\infty} f^n(U)$ is a pluri-polar set.*

Now we can prove Theorem 1.8.

Theorem 3.3. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 , then every repelling periodic point belongs to J_2 .*

Proof. Our argument concerns the backward iterates around a fixed point. This kind of argument has already appeared in Fornæss-Sibony [17]. Without loss of generality we may assume x_0 is fixed. Since $x_0 \notin C(f)$, by Lemma 2.4, x_0 is a point of bounded ramification. We are going to show that for every neighborhood U of x_0 , we have $\mathbb{P}^2 \setminus PC(f) \subset \bigcup_{n=0}^{\infty} f^n(U)$. Thus since $PC(f)$ is algebraic, by Proposition 3.2 we will get $x_0 \in J_2$.

Let $y \in \mathbb{P}^2 \setminus PC(f)$ be an arbitrary point. Let W be a neighborhood of x_0 such that $y \in W$ and W satisfies the condition in Corollary 2.6. It can be achieved, by first joining x_0 and y by a smooth embedded (real) curve, and let W be a sufficiently thin tubular neighborhood of this curve such that ∂W is smooth. Let $m \geq 0$ such that $\text{ord}(f^n, x_n) \leq m$ for every $x_n \in f^{-n}(x_0)$ and $n \geq 0$. Let $\eta : Z \rightarrow W$ a $PC(f) \cap W$ branched holomorphic covering map as in Corollary 2.6. For $n \geq 0$, let W_n denote the connected component of $f^{-n}(W)$ containing x_0 . Then by Corollary 2.6 we can define holomorphic map $g_n : Z \rightarrow W_n$ such that $f^n \circ g_n = \eta$. By Lemma 2.7, $\{g_n\}$ is a normal family. We are going to show that actually g_n converges to the fixed point x_0 . Since x_0 is repelling, there exists a small neighborhood $\Omega \subset W$ such that g_n converges to x_0 uniformly on $\eta^{-1}(\Omega)$. Now let ϕ be any limit map of some subsequence of $\{g_n\}$. Since ϕ is constant on an open set $\eta^{-1}(\Omega)$, ϕ is constant on Z . Thus any limit map of some subsequence of $\{g_n\}$ is the constant map $z \mapsto x_0$. This implies that g_n converges to the fixed point x_0 . In particular if z_0 satisfies $\eta(z_0) = y$, we have $g_n(z_0) \rightarrow x_0$ when $n \rightarrow +\infty$.

Now let U be an arbitrary neighborhood of x_0 . Since $g_n(z_0) \rightarrow x_0$, there exists $N > 0$ such that $g_N(z_0) \in U$. Since $f^N \circ g_N = \eta$ we get $y \in f^N(U)$. Since $y \in \mathbb{P}^2 \setminus PC(f)$ is arbitrary we get $\mathbb{P}^2 \setminus PC(f) \subset \bigcup_{n=0}^{\infty} f^n(U)$. By the arbitrariness of U , we have $x_0 \in J_2$, which completes the proof. □

3.2. Super-saddle points. Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . We investigate the location of super-saddle cycles of f . The following is an equivalent formulation of Theorem 1.2.

Theorem 3.4. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 , Let x_0 be a super-saddle fixed point in \mathbb{P}^2 . If the branches of $PC(f)$ are smooth and intersect transversally at x_0 , then x_0 is contained in a critical component cycle.*

Proof. In the following we let $C_{x_0}(f)$ denote the union of critical components containing x_0 and let $PC_{x_0}(f) := \bigcup_{n=1}^{\infty} f^n(C_{x_0}(f))$. It is easy to observe that $PC_{x_0}(f) = PC_{x_0}(f^n)$ for every $n \geq 1$. Then up to an iterate of f , we may assume that all periodic components of $PC_{x_0}(f)$ are invariant, and every component of $PC_{x_0}(f)$ is mapped to an invariant component by at most one iterate. It remains to show that x_0 is contained in an invariant component of $C_{x_0}(f)$. We argue by contradiction. Assume x_0 is not contained in an invariant component

of $C_{x_0}(f)$, then there exists V an invariant component of $PC_{x_0}(f)$ such that $V \not\subset C_{x_0}(f)$. We first show that $V \neq PC_{x_0}(f)$. We argue by contradiction, assuming $V = PC_{x_0}(f)$, then there exists a neighborhood U of x_0 such that $f : U \rightarrow f(U)$ is a V -branched covering. Then, since $x_0 \in C_{x_0}(f)$, $f^{-1}(V)$ contains a component of $C_{x_0}(f)$. Since $f^{-1}(V)$ also contains V , we deduce that $f^{-1}(V)$ is singular at x_0 . We recall the following result of Ueda [31] Lemma 3.5.

Lemma 3.5. *Let $f : U_1 \rightarrow U_2$ be a V -branched holomorphic covering, where U_1, U_2 are complex manifolds and V is a codimension 1 analytic subset of U_2 . Suppose that $x_0 \in U_1$ is a singular point of $f^{-1}(V)$, then $f(x_0)$ is a singular point of V .*

Coming back to our situation, letting $U = U_1$ and $f(U) = U_2$ in above lemma, we know that x_0 is a singular point of V . This is impossible, since by our assumption V is smooth at x_0 . Thus there must exist a component V_1 of $PC_{x_0}(f)$ such that $V_1 \neq V$.

Let $V' = f(V)$, then by our assumption V' is invariant. We recall the following result of Le [22] Proposition 5.5.

Lemma 3.6. *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be a proper holomorphic germ and let Σ_1, Σ_2 be two irreducible germs at 0 such that $\Sigma_1 \neq \Sigma_2$, $f(\Sigma_1) = \Sigma_2$ and $f(\Sigma_2) = \Sigma_2$. If Σ_2 is smooth at 0 then the eigenvalue of Df at 0 are 0 and λ where λ is the eigenvalue of $D_0f|_{T_0\Sigma_2}$.*

Coming back to our situation, take $V = \Sigma_2$ (resp. $V' = \Sigma_2$) in above lemma, since x_0 is super-saddle, we get that the eigenvalue of f restricted to V (resp. V') at x_0 must be repelling. By our assumption V' and V intersect transversally, the only possible case that x_0 being super-saddle is when $V' = V$. Thus we must have $f(V_1) = V$ by our assumption that every component of $PC_{x_0}(f)$ is mapped to an invariant component by one iterate. To summarize, there exists $V_1 \subset PC_{x_0}(f)$ such that $V_1 \neq V$, and every component of $PC_{x_0}(f)$ is mapped to V by one iterate (in particular, $f(V_1) = V$).

We do a local coordinate change such that $x_0 = (0, 0)$, $V = \{x = 0\} \cap U$, where U is a small neighborhood of $(0, 0)$. By [22] Proposition 5.5, 0 is a repelling point of $f|_V$. Let λ be the eigenvalue of $f|_V$ at 0, $|\lambda| > 1$. Thus after a linear coordinate change fixing V , the expression of f in this coordinate has the following form

$$(3.1) \quad f(x, y) = (G(x, y), \lambda y + ax + H(x, y)),$$

Where $G(x, y) = O(|x^2|, |y^2|)$, x is a factor of G and $H(x, y) = O(|x|^2, |y|^2)$.

Since V_1 is a smooth curve that intersects $\{x = 0\}$ transversally, we may assume $V_1 = \{y = \phi(x)\}$ for some holomorphic function ϕ . We do a local coordinate change $x' \rightarrow x$, $y' \rightarrow y - \phi(x)$. In this coordinate the expression of f has the same form as in (3.1), and $V_1 = \{y = 0\}$. In the following we work in this coordinate.

By our assumption that every component of $PC_{x_0}(f)$ is mapped to an invariant component by one iterate, there exists a critical component C such that $f(C) = V_1$. Thus C satisfies the equation

$$(3.2) \quad \lambda y + ax + H(x, y) = 0.$$

By the implicit function theorem, C is a smooth curve that intersects with $\{x = 0\}$ transversally. We let $C = \{y = \psi(x)\}$ for some holomorphic function ψ .

By direct calculation, the Jacobian of f is

$$\text{Jac}(f) = \frac{\partial G}{\partial x} \left(\lambda + \frac{\partial H}{\partial y} \right) - \frac{\partial G}{\partial y} \left(a + \frac{\partial H}{\partial x} \right).$$

Since C is in the critical set of f , ψ satisfies the following equation

$$(3.3) \quad \frac{\partial G}{\partial x}(x, \psi(x)) \left(\lambda + \frac{\partial H}{\partial y}(x, \psi(x)) \right) - \frac{\partial G}{\partial y}(x, \psi(x)) \left(a + \frac{\partial H}{\partial x}(x, \psi(x)) \right) = 0.$$

Take differential of x in the both sides of (3.2) we get

$$(3.4) \quad \lambda \psi'(x) + a + \frac{\partial H}{\partial x}(x, \psi(x)) + \frac{\partial H}{\partial y}(x, \psi(x)) \psi'(x) = 0.$$

Combining (3.3) and (3.4) we get

$$\left(\lambda + \frac{\partial H}{\partial y}(x, \psi(x)) \right) \left(\frac{\partial G}{\partial x}(x, \psi(x)) + \frac{\partial G}{\partial y}(x, \psi(x)) \psi'(x) \right) = 0.$$

Since $\lambda + \frac{\partial H}{\partial y}(x, \psi(x)) \neq 0$ in a neighborhood of 0, ψ satisfies

$$\frac{\partial G}{\partial x}(x, \psi(x)) + \frac{\partial G}{\partial y}(x, \psi(x)) \psi'(x) = 0.$$

This implies that ψ satisfies $G(x, \psi(x)) = 0$ for every x . Then by the expression (3.1), we must have $f(C) \subset V$. Since $f(C) = V_1$, we have $f(C) = (0, 0)$, which is a contradiction since f is a locally finite to one map. Thus x_0 is contained in an invariant critical component, and the proof is complete. \square

Corollary 3.7. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 such that all branches of $PC(f)$ are smooth and intersect transversally, then every periodic point in J_2 is repelling.*

Proof. Let $x_0 \in J_2$ be a periodic point. By Le [22], x_0 is either repelling, super-saddle or super-attracting. Since super-attracting periodic points belong to the Fatou set, x_0 is not super-attracting. By Theorem 3.4, super-saddle periodic points are contained in critical component cycles, in particular they are not in J_2 . So x_0 must be repelling. \square

4. STRUCTURE OF THE JULIA SETS.

In this section we prove Theorem 1.1. We will first prove that there exists a Fatou disk containing x for every $x \in J_1 \setminus J_2$ which is not contained in a basin of critical component cycle (Theorem 4.5). Second we prove that if there is a Fatou disk containing x for $x \in J_1$, then x must be contained in the basin of a critical component cycle or in the stable manifold of a sporadic super-saddle cycle (Theorem 4.8). Then Theorem 1.1 is a combination of these two theorems. We start with several lemmas.

Lemma 4.1. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Let $x_0 \in \mathbb{P}^k$ and let v be a subsequence of integers such that $x_v \rightarrow y$ and y is of bounded ramification, where $x_v = f^v(x_0)$. Let W be a neighborhood of y and $\eta : Z \rightarrow W$ a holomorphic covering as in Corollary 2.6. Let W_v denote the connected component of $f^{-v}(W)$ containing x_0 . Let $g_v : Z \rightarrow W_v$ such that $f^v \circ g_v = \eta$. Assume g_v converges to a constant map, then $x_0 \in J_2$ and $y \in J_2$*

Proof. We first prove $y \in J_2$. We take $W = B(y, r)$ for sufficiently small r . Let $Z' = \eta^{-1}(B(y, r/2))$. Let N large enough such that $x_v \in B(y, r/2)$ when $v \geq N$. Since g_v converges to a constant map and $Z' \subset\subset Z$, we have $\text{diam } g_v(Z') \rightarrow 0$. Let v large enough such that $W' := f^N(g_v(Z')) \subset\subset W$. Then $f^{v-N} : W' \rightarrow W$ is a polynomial-like map, By [10] Theorem 2.22, there exists a fixed point of f^{v-N} in W . Letting $r \rightarrow 0$, we get that y is approximated by periodic points. By the result of Le [22], every periodic point of a PCF map f on \mathbb{P}^2 is repelling, super-saddle or super-attracting. Since y is not in a critical component cycle, and since there are only finitely many super-attracting periodic points and super-saddle periodic points outside critical component cycles (see Lemma 2.9), y is approximated by repelling periodic points. By Theorem 3.3, $y \in J_2$. Let $z = \eta^{-1}(y)$. The convergence of g_v to x_0 implies $g_v(y) \rightarrow x_0$, thus there is a sequence of preimage $\{y_v\}$ of y such that y_v converges to x_0 . By the backward invariance of J_2 , we conclude that $x_0 \in J_2$. \square

Lemma 4.2. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Let $x_0 \in \mathbb{P}^2$ and let v be a subsequence of integers such that $x_v \rightarrow y$ and y is of bounded ramification, where $x_v = f^v(x_0)$. Let W be a neighborhood of y and $\eta : Z \rightarrow W$ a holomorphic covering as in Corollary 2.6. Let W_v denotes the connected component of $f^{-v}(W)$ containing x_0 . Let $g_v : Z \rightarrow W_v$ such that $f^v \circ g_v = \eta$. Assume g_v converges to a non-constant map ϕ , then there exists a Fatou disk containing x_0 .*

Proof. Let $W = B(y, r)$ for small r . Let $M = \phi(Z)$, we will show that M contains a Fatou disk containing x_0 . Let N large enough such that $x_v \in W$ when $v \geq N$. Then for $v \geq N$, there exist $z_v \in Z'$ such that $g_v(z_v) = x_0$. Let $z = \eta^{-1}(y)$, it is clear that $z_v \rightarrow z$ when $v \rightarrow +\infty$. Let $v \rightarrow +\infty$ in the equation $g_v(z_v) = x_0$ we get $\phi(z) = x_0$, then $x_0 \in M$. By Lemma 2.7, $\phi : Z \rightarrow \mathbb{P}^2$ is a Fatou map, by definition, this implies that $\{f^n|_M\}_{n \geq 1}$ is a normal family. Let $D \subset Z$ be a holomorphic disk containing z such that ϕ is not a constant map when restricted to D , then $\phi(D) \subset M$ is a Fatou disk containing x_0 . \square

Lemma 4.3. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Let $x_0 \in J_1$ such that x_0 is not contained in the attracting basin of a critical component cycle nor contained in the stable manifold of a super-saddle cycle, then there exists a subsequence v of positive integers such that $x_v = f^v(x_0) \rightarrow y$, where y is a point of bounded ramification.*

Proof. There are at most finitely many critical point cycles, which are not contained in the critical component cycles (see Lemma 2.9). We denote this finite set by E . We show that if $x_0 \in J_1$ satisfying that x_0 is not contained in the attracting basins of critical component cycles and $\omega(x_0)$ contains only points of unbounded ramification, then x_0 is contained in the stable manifold of a sporadic super-saddle cycle. If x_0 satisfies the above assumption, by Lemma 2.4 we know that $\omega(x_0) \subset E$. We recall the following basic property of ω -limit set:

Lemma 4.4. *Let X be a compact metric space, let $f : X \rightarrow X$ be a continuous map, and let $g : \omega(x_0) \rightarrow \omega(x_0)$ be the restriction of f on the ω -limit set of x_0 , then there is no non-trivial open subset U of $\omega(x_0)$ such that $g(\overline{U}) \subset U$.*

Proof. Bowen ([4] Theorem 1) proved the above lemma for homeomorphisms, but the proof also holds for non-invertible maps. For the completion we give a proof here. Assume by contradiction that such open subset U exists, let $Y := \omega(x_0)$. By our assumption $2\epsilon := \text{dist}(Y \setminus U, g(\overline{U})) > 0$. Choose $0 < \delta < \epsilon$ such that $\text{dist}(x_1, x_2) < \delta$ implies $\text{dist}(f(x_1), f(x_2)) < \epsilon$ for every $x_1, x_2 \in X$. Now it is clear that there is $N > 0$ such that $\text{dist}(f^N(x_0), Y) < \delta$

when $n > N$ (otherwise $\omega(x_0)$ will be strictly larger than Y). Pick $M \geq N$ such that $\text{dist}(f^M(x_0), g(\overline{U})) < \epsilon$ and $\text{dist}(f^M(x_0), y) < \delta$ for some $y \in Y$. Then $\text{dist}(g(\overline{U}), y) < 2\epsilon$, which implies $y \in U$. Then

$$\text{dist}(f^{M+1}(x_0), g(\overline{U})) \leq \text{dist}(f^{M+1}(x_0), g(y)) < \epsilon.$$

Inductively for all $m \geq M$ we have $\text{dist}(f^m(x_0), g(\overline{U})) < \epsilon$. This implies $Y \cap (Y \setminus U) = \emptyset$, which is a contradiction. \square

Coming back to our situation, since $\omega(x_0)$ is a finite set, the only possibility is that $\omega(x_0)$ contains a single periodic cycle. Since $x_0 \in J_1$ and $\omega(x_0)$ contains no points of bounded ramification, this cycle must be a super-saddle cycle. Thus x_0 is contained in the stable manifold of this super-saddle cycle. It follows that if x_0 is not contained in the attracting basin of a critical component cycle nor contained in the stable manifold of a super-saddle cycle, $\omega(x_0)$ has non-empty intersection with the set of points of bounded ramification. This completes the proof. \square

Now we can prove the following theorem.

Theorem 4.5. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . If $x \in J_1 \setminus J_2$ which is not contained in the basin of a critical component cycle, then there is a Fatou disk D containing x .*

Proof. Suppose first x_0 is contained in the stable manifold of a super-saddle cycle. Then there exists an embedded holomorphic disc D containing x_0 such that D coincides with the local stable manifold at x_0 , and it is clear that $\{f^n|_D\}_{n \geq 1}$ is normal. Now suppose x_0 satisfies the assumptions of the theorem, and x_0 is not contained in the stable manifold of a super-saddle cycle. By Lemma 4.3, we can choose a subsequence v of positive integers such that $x_v = f^v(x_0) \rightarrow y$, and y is of bounded ramification. Consider a neighborhood W of y and $\eta : Z \rightarrow W$ the holomorphic covering as in Corollary 2.6. Let W_v denotes the connected component of $f^{-v}(W)$ containing x_0 . Let $g_v : Z \rightarrow W_v$ such that $f^v \circ g_v = \eta$. By Lemma 2.7 $\{g_v\}$ is a normal family. By passing to some subsequence, we may assume g_v converges to a holomorphic map ϕ . ϕ can not be a constant map, since otherwise by Lemma 4.1, $x_0 \in J_2$, which is a contradiction. Thus ϕ is a non-constant map. By Lemma 4.2, there is a Fatou disk containing x_0 , which completes the proof. \square

Before proving Theorem 4.8, we need some lemmas.

Lemma 4.6. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Let $x \in J_1$ which is not contained in the basin of a critical component cycle, nor in the stable manifold of a super-saddle cycle. Assume moreover that there is a Fatou disk X containing x . Then there is a point $y \in \omega(x)$ such that there is a Siegel variety containing y .*

Proof. We choose $y \in \omega(x)$ such that y is not a super-saddle periodic point. Then by Lemma 2.4, y is a point of bounded ramification. Let ϕ be a limit map of $\{f^n|_X\}$ such that $\phi(x) = y$. Then by Lemma 2.8, there is a Siegel variety X containing y . \square

Lemma 4.7. *let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Let X be a Siegel variety, then $X \not\subset PC(f)$.*

Proof. Assume by contradiction that $X \subset PC(f)$. By passing to an iterate of f we may assume X is contained in an invariant component V of $PC(f)$. Let $\pi : \widehat{V} \rightarrow V$ be the normalization of V , and let \hat{f} be the lift of f on \widehat{V} . Then by Jonsson [20] $\widehat{V} = \mathbb{P}^1$ or a torus, and \hat{f} is a PCF map on \mathbb{P}^1 or an expanding torus map. We have the following commutative diagram

$$\begin{array}{ccc} \widehat{V} & \xrightarrow{\hat{f}} & \widehat{V} \\ \pi \downarrow & & \downarrow \pi \\ V & \xrightarrow{f} & V \end{array}$$

Let $U \subset \pi^{-1}(X)$ be an open set which is contained in the unramification locus of π . Then U is a Siegel variety in \widehat{V} . It is well-known that a PCF map on \mathbb{P}^1 or a expanding torus map can not have such a Siegel variety, a contradiction. \square

Next we can prove Theorem 4.8, thus completes the proof of Theorem 1.1.

Theorem 4.8. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . If $x \in J_1$ such that there is a Fatou disk containing x . Then x must be contained in the basin of a critical component cycle, or in the stable manifold of a super-saddle cycle.*

Proof. Our argument borrows some ingredients in Ueda's paper [30] and Le's paper [22]. We argue by contradiction. Suppose x is not contained in the basin of a critical component cycle, nor in the stable manifold of a super-saddle cycle. Then by Lemma 4.6, there is a point $y \in \omega(x)$ such that there is a Siegel variety X containing y . By Lemma 4.7 we may assume $X \cap PC(f) = \emptyset$ (shrink X if necessary).

Let Z be the universal covering space of $\mathbb{P}^2 \setminus PC(f)$, and let $\eta : Z \rightarrow \mathbb{P}^2 \setminus PC(f)$ be the universal covering map. Let $z_0 \in Z$ satisfying $\eta(z_0) \in \phi(X)$. Recall that $\phi = \lim_{j \rightarrow \infty} f^{n_j}|_X$. By passing to a subsequence we may assume $k_j := n_{j+1} - n_j \rightarrow +\infty$. Let W be a neighborhood of $\eta(z_0)$ such that $W \cap PC(f) = \emptyset$, and let Z_0 be a neighborhood of z_0 such that $\eta : Z_0 \rightarrow W$ is a biholomorphism. Let $x_0 \in X$ satisfying $f^{n_j}(x_0) \rightarrow \eta(z_0)$. Let W_j be the connected component of $f^{-k_j}(W)$ containing $f^{n_j}(x_0)$. Let $g_j : Z_0 \rightarrow W_j$ be the holomorphic map such that $f^{k_j} \circ g_j = \eta$ as in Corollary 2.6. Since Z is simply connected, there is no obstruction to extend g_j as a globally defined map on Z . In the following g_j will denote this globally defined map. By Lemma 2.7 $\{g_j\}$ is a normal family. By shifting to a subsequence we may assume $\{g_j\}$ converges locally uniformly to a holomorphic map g .

Let $V := \eta^{-1}(X) \cap Z_0$. We claim that $g = \eta$ on V . To show this, let $z \in V$ be an arbitrary point, we are going to show $g(z) = \eta(z)$. Let $x \in X$ satisfying $f^{n_j}(x) \rightarrow \eta(z)$. Since for j large enough we have $f^{n_{j+1}}(x) \in W$, we get $f^{n_j}(x) \in W_j$. Let $z_{j+1} \in Z_0$ such that $g_j(z_{j+1}) = f^{n_j}(x)$. By the equation $f^{k_j} \circ g_j = \eta$ we have that $\eta(z_{j+1}) = f^{n_{j+1}}(x)$. Since $f^{n_{j+1}}(x) \rightarrow \eta(z)$, we get that $z_{j+1} \rightarrow z$. Now let $j \rightarrow +\infty$ in the equation $g_j(z_{j+1}) = f^{n_j}(x)$ we get that $g(z) = \eta(z)$. Thus $g = \eta$ on V .

The equation $g = \eta$ defines a closed analytic set $Y \subset Z$. Since $V \subset Y$, Y has dimension one. Let Y_0 be an irreducible component of Y_0 of pure dimension 1. By Lemma 2.7 g is a Fatou map, thus $\eta|_{Y_0}$ is a Fatou map since $g = \eta$ on Y_0 . Since Y_0 is a closed analytic set in Z and η is a covering map, the following property holds: for every $x \in \mathbb{P}^2 \setminus PC(f)$, there is a neighborhood U of x such that either $\eta^{-1}(U) \cap Y_0 = \emptyset$ or every connected component of $\eta^{-1}(U) \cap Y_0$ is relatively compact in Y_0 .

Let $\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ be the canonical projection. Let F be a homogeneous polynomial endomorphism on \mathbb{C}^3 which induce f . It is clear that F is a PCF map on \mathbb{C}^3 . Let G_F be the Green function of F . Recall that $\eta : Y_0 \rightarrow \mathbb{P}^2$ is a Fatou map. By Theorem 2.2, there exists a holomorphic lift $\Phi : Y_0 \rightarrow \mathbb{C}^3 \setminus \{0\}$ of $\eta|_{Y_0}$ such that $\pi \circ \Phi = \eta$ and $G_F \circ \Phi$ is identically zero on Y_0 . Since $\eta(Y_0) \cap PC(f) = \emptyset$ we get that $\Phi(Y_0) \cap PC(F) = \emptyset$. By the construction of Φ and by the fact that π is an open map, the following property holds: for every $x \in \mathbb{C}^3 \setminus PC(F)$, there is a neighborhood U of x such that either $\Phi^{-1}(U) \cap Y_0 = \emptyset$ or every connected component of $\Phi^{-1}(U) \cap Y_0$ is relatively compact in Y_0 .

Let Y_1 be the smooth part of Y_0 , then Y_1 is a Riemann surface. Let $S := \Phi(Y_0 \setminus Y_1)$, then S is a discrete set in $\mathbb{C}^3 \setminus PC(F)$. Since $G_F \circ \Phi$ is identically zero on Y_1 , $\Phi(Y_1)$ is contained in a bounded domain in \mathbb{C}^3 . This implies that Y_1 is a hyperbolic Riemann surface, i.e. there exists a universal covering map $\kappa : \mathbb{D} \rightarrow Y_1$, where \mathbb{D} is the unit disk. Since κ is a covering map, the following property holds:

for every $x \in \mathbb{C}^3 \setminus (PC(F) \cup S)$, there is a neighborhood U of x such that either $(\Phi \circ \kappa)^{-1}(U) \cap \mathbb{D} = \emptyset$ or every connected component of $(\Phi \circ \kappa)^{-1}(U) \cap \mathbb{D}$ is relatively compact in \mathbb{D} (\star).

We consider the map $\Phi \circ \kappa : \mathbb{D} \rightarrow \mathbb{C}^3$. The following idea of using Fatou-Riesz's theorem was first appeared in Le [22]. As we mentioned before $\Phi \circ \kappa(\mathbb{D})$ is contained in a bounded domain in \mathbb{C}^3 . By the well known Fatou-Riesz's theorem (see Milnor [24] Theorem A.3), for Lebesgue a.e. $\theta \in [0, 2\pi)$, the radial limit

$$\tau_\theta = \lim_{r \rightarrow 1^-} \Phi \circ \kappa(re^{i\theta})$$

exists for some $\tau_\theta \in \mathbb{C}^3$.

We next show that for every limit τ_θ , $\tau_\theta \in PC(F) \cup S$. Assume by contradiction that $\tau_\theta \notin PC(F) \cup S$. Let U be a small neighborhood of τ_θ , then by property (\star), $(\Phi \circ \kappa)^{-1}(U)$ is a disjoint union of relatively compact open set in \mathbb{D} . Then for every $R \in (0, 1)$ there exists a r satisfying $R < r < 1$ such that $re^{i\theta} \notin (\Phi \circ \kappa)^{-1}(U)$ (by the connectedness of the interval $[R, 1]$). This contradicts the fact that $\tau_\theta = \lim_{r \rightarrow 1^-} \Phi \circ \kappa(re^{i\theta})$. Thus $\tau_\theta \in PC(F) \cup S$.

Since S is a countable set, there must exist a positive Lebesgue measure set $E \subset [0, 2\pi)$ such that for every $\theta \in E$ either $\lim_{r \rightarrow 1^-} \Phi \circ \kappa(re^{i\theta}) \in PC(F)$ or $\lim_{r \rightarrow 1^-} \Phi \circ \kappa(re^{i\theta}) = s$ for an element $s \in S$. We will show that these two cases are both impossible, then we will get a contradiction.

We first treat the case that for every $\theta \in E$, $\lim_{r \rightarrow 1^-} \Phi \circ \kappa(re^{i\theta}) \in PC(F)$. Let P be a homogeneous polynomial that defines $PC(F)$, i.e.

$$PC(F) = \{x \in \mathbb{C}^3 : P(x) = 0\}.$$

Then $P \circ \Phi \circ \kappa$ is a bounded holomorphic function on \mathbb{D} such that the radial limit

$$\lim_{r \rightarrow 1^-} P \circ \Phi \circ \kappa(re^{i\theta}) = 0$$

for every $\theta \in E$, where E has positive Lebesgue measure. By Fatou-Riesz's theorem $P \circ \Phi \circ \kappa$ is identically zero on \mathbb{D} . This implies $\Phi \circ \kappa(\mathbb{D}) \subset PC(F)$ which implies $\Phi(Y_1) \subset PC(F)$, a contradiction.

Second we treat the case that for every $\theta \in E$, $\lim_{r \rightarrow 1^-} \Phi \circ \kappa(re^{i\theta}) = s$, where $s \in S$. Let P_1, P_2, P_3 be three linear polynomial in \mathbb{C}^3 that define s , i.e.

$$\{s\} = \{x \in \mathbb{C}^3 : P_i(x) = 0, i = 1, 2, 3\}.$$

Then for every i we get that $P_i \circ \Phi \circ \kappa$ is a bounded holomorphic function on \mathbb{D} such that the radial limit

$$\lim_{r \rightarrow 1^-} P_i \circ \Phi \circ \kappa(re^{i\theta}) = 0$$

for every $\theta \in E$, where E has positive Lebesgue measure. By Fatou-Riesz's theorem, for every i we get that $P_i \circ \Phi \circ \kappa$ is identically zero on \mathbb{D} . This implies $\Phi \circ \kappa(\mathbb{D}) = \{s\}$ which implies $\Phi(Y_1) = \{s\}$, contradicts the fact that Y_1 has dimension 1. This completes the proof. \square

We end this section by showing the sporadic super-saddle cycle is contained in J_2 .

Lemma 4.9. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 , then every sporadic super-saddle cycle is contained in J_2 .*

Proof. Let x be a super-saddle cycle which is not contained in a critical component cycle. Passing to an iterate of f we may assume every x is fixed and every periodic component of $PC(f)$ is invariant. Let V be an invariant component of $PC(f)$ containing x , by our assumption $V \not\subset C(f)$. By Lemma 3.6, the eigenvalue λ of $Df|_{T_x V}$ satisfies $|\lambda| > 1$. Let $\pi : \widehat{V} \rightarrow V$ be the normalization of V , and let \widehat{f} be the lift of f on \widehat{V} . Let $\widehat{x} \in \widehat{V}$ such that $\pi(\widehat{x}) = x$. Since V is smooth at x , such a \widehat{x} is unique. Then \widehat{x} is a repelling fixed point of \widehat{f} . Since \widehat{V} is either \mathbb{P}^1 or a torus and \widehat{f} is either a rational function of degree ≥ 2 or an expanding torus map, \widehat{x} can be approximated by a sequence of periodic points $\{\widehat{y}_n\} \subset \widehat{V}$ such that $y_n = \pi(\widehat{y}_n)$ is in the smooth part of $PC(f)$. Then by the result of Le [22], y_n is a repelling periodic point of f for every n . By Theorem 1.8, $y_n \in J_2$. Since $y_n \rightarrow x$ we get $x \in J_2$ as desired. \square

5. SOME COROLLARIES OF THE MAIN RESULTS

In this section we prove Theorem 1.4, Corollary 1.5, Theorem 1.6 and Corollary 1.7.

5.1. Non-wandering set for PCF maps on \mathbb{P}^2 . Before proving Theorem 1.4, we prove the following lemma.

Lemma 5.1. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Let C be an invariant critical component. Then $\Omega(f) \cap C$ is the union of super-attracting cycles and $J_1 \cap C$, moreover periodic points are dense in $\Omega(f) \cap C$.*

Proof. Let $\pi : \widehat{C} \rightarrow C$ be the normalization of V , and let \widehat{f} be the lift of f on \widehat{C} . Then $\widehat{C} = \mathbb{P}^1$, and \widehat{f} is a rational function of degree ≥ 2 , see [15]. Moreover by Jonsson [20], \widehat{f} is also a PCF map. We have the following commutative diagram.

$$\begin{array}{ccc} \widehat{C} & \xrightarrow{\widehat{f}} & \widehat{C} \\ \pi \downarrow & & \downarrow \pi \\ C & \xrightarrow{f} & C \end{array}$$

It is clear that if p is a periodic point of \hat{f} , then $\pi(p)$ is a periodic point of f . Let $J(\hat{f})$ be the Julia set of \hat{f} . Since periodic points of \hat{f} are dense in $J(\hat{f})$, periodic points of f are dense in $\pi(J(\hat{f}))$. We claim that $J_1 \cap C = \pi(J(\hat{f}))$. By the above discussion $\pi(J(\hat{f}))$ is contained in the closure of periodic points. These periodic points must be super-saddle. Thus $\pi(J(\hat{f})) \subset J_1 \cap C$. To show $J_1 \cap C \subset \pi(J(\hat{f}))$, it is equivalent to show that for every y in the Fatou set of \hat{f} , $\pi(y)$ is in the Fatou set of f . Since \hat{f} is a PCF map, y is contained in the basin of a super-attracting cycle of \hat{f} . For every super-attracting fixed point z of \hat{f} , $\pi(z)$ is a fixed point of f . It is clear that $\pi(z)$ is not repelling, since $\pi(z) \in C(f)$. It is also clear that $\pi(z)$ is not super-saddle, since otherwise $\pi(z)$ is a smooth point of C (see Le [22] Proposition 5.4), and z must be a repelling point of \hat{f} . Thus $\pi(z)$ must be a super-attracting fixed point of f . Thus $\pi(y)$ is in the basin of $\pi(z)$, this implies $\pi(y)$ is in the Fatou set of f . Thus $J_1 \cap C = \pi(J(\hat{f}))$ as desired. Since $\pi(J(\hat{f}))$ is contained in the closure of periodic points, $J_1 \cap C$ is contained in $\Omega(f)$. Since the Fatou set of f is the union of basins of super-attracting cycles, we get that $\Omega(f) \cap C$ is exactly the union of super-attracting cycles and $J_1 \cap C$. This implies periodic points are dense in $\Omega(f) \cap C$. \square

Now we can prove Theorem 1.4. Recall the statement.

Theorem 5.2. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then J_2 is the closure of repelling periodic points. If further assumed there is no sporadic super-saddle cycle, then $\Omega(f)$ is the closure of periodic points and $\Omega(f) \setminus J_2$ is the union of super-attracting cycles together with $\cup(C \cap J_1)$, where C ranges over the set of critical component cycles, in particular J_2 is open in $\Omega(f)$.*

Proof. By Theorem 1.8, J_2 is the closure of the set of repelling periodic points. In the following we assume there is no sporadic super-saddle cycle. First we show that $\Omega(f)$ is the closure of periodic points. Let $x \in \Omega(f)$ such that $x \notin J_2$. By Theorem 1.1 and the fact that Fatou set are only super-attracting basins, either x is contained in the basin of a super-attracting cycles, or x is contained in the basin of a critical component cycle. In the first case it is clear that x must be a super-attracting periodic point. In the second case, x must be contained in a critical component cycle, thus x is contained in the closure of periodic points by Lemma 5.1. Again by Lemma 5.1, for every critical component C we have $\Omega(f) \cap C$ is the union of super-attracting cycles and $J_1 \cap C$. Thus $\Omega(f) \setminus J_2$ is the union of super-attracting cycles together with $\cup(C \cap J_1)$, where C ranges over the set of critical component cycles. \square

5.2. Laminarity of the Green current for PCF maps on \mathbb{P}^2 . Recall the statement of Corollary 1.5.

Corollary 5.3 (de Thélin). *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then the Green current T is laminar in $J_1 \setminus J_2$.*

Proof. Let E be the union of sporadic super-saddle cycles. Let W be the union of stable manifolds of super-saddle cycles in E . Then W is a pluri-polar set, and T put no mass on W (since the local potential of T is Hölder continuous, see [10]). By Theorem 1.1, $T|_{J_1 \setminus J_2}$ put full mass on the union of the basins of critical component cycles. By Daurat [7], T is laminar in every basin of critical component cycle. Thus T is laminar in $J_1 \setminus J_2$. \square

5.3. Expanding sets for PCF maps on \mathbb{P}^2 . Recall the statement of Theorem 1.6. The following proof is an adaption of the argument in Maegawa [23].

Theorem 5.4. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 and let K be an invariant compact set. Then K is expanding if and only if K does not contain critical points.*

Proof. The only if part is obvious. In the following we prove K is expanding if K does not contain critical points. To prove that K is an expanding set, it is equivalent to prove the following limit tends to infinity:

$$\min_{x \in K} \min_{v \in T_x \mathbb{P}^2, |v|=1} |D(f^n)(v)| \rightarrow +\infty, \quad n \rightarrow +\infty.$$

Assume by contradiction that the limit does not go to infinity. Then there exists a constant $C > 0$ and a sequence of points $\{x_m\} \subset K$ and vectors $\{v_m\} \subset T_K \mathbb{P}^2$ such that $|v_m| = 1$ and $|D(f^{n_m})_{x_m}(v_m)| < C$, where $\{n_m\}$ is a sequence of integers. By shifting to a subsequence we may assume $f^{n_m}(x_m) \rightarrow x \in K$. Since K is invariant and $K \cap C(f) = \emptyset$, by Lemma 2.4 we know that x is a point of bounded ramification. Let W be a neighborhood of x and let $\eta : Z \rightarrow W$ be the branched covering constructed in Corollary 2.6. Let W_m be the connected component of $f^{-n_m}(W)$ containing x_m . Let $g_m : Z \rightarrow W_m$ be the holomorphic map such that $f^{n_m} \circ g_m = \eta$. Then by Lemma 2.7 by shifting to a subsequence we may assume $\{g_m\}$ locally uniformly converges to a Fatou map g . Since K is an invariant compact set and $K \cap C(f) = \emptyset$, by Theorem 1.1 we get $K \subset J_2$. Again by Theorem 1.1 and the fact that $K \cap C(f) = \emptyset$, there is no Fatou disk containing points in K . Thus g must be a constant map. Thus we have $\text{diam } W_m \rightarrow 0$. Since $K \cap C(f) = \emptyset$, $W_m \cap C(f) = \emptyset$ for m large enough. Thus $f^{n_m} : W_m \rightarrow W$ is a biholomorphism when m large. We can then define $g'_m : W \rightarrow W_m$ be the inverse map of $f^{n_m}|_{W_m}$, and it easily follows that g'_m converges locally uniformly to a constant map. This implies

$$(5.1) \quad \max_{y \in U} \max_{v \in T_y \mathbb{P}^2, |v|=1} |Dg'_m(v)| \rightarrow 0, \quad m \rightarrow +\infty,$$

where U is a small neighborhood of x .

By the definition of g'_m (5.1) is equivalent to the following

$$(5.2) \quad \min_{z \in U_m} \min_{v \in T_z \mathbb{P}^2, |v|=1} |D(f^{n_m})(v)| \rightarrow +\infty, \quad m \rightarrow +\infty,$$

where $U_m := f^{-n_m}(U) \cap W_m$.

By our construction of $\{x_m\}$ and $\{v_m\}$, it follows that $x_m \in U_m$ for m large and

$$|D(f^{n_m})_{x_m}(v_m)| < C.$$

This contradicts (5.2). Thus the proof is complete. \square

5.4. PCF maps on \mathbb{P}^2 that are expanding on J_2 or Axiom A. We first discuss the PCF maps on \mathbb{P}^2 that are expanding on J_2 . Recall the first part of the statement of Corollary 1.7.

Corollary 5.5. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then f is expanding on J_2 if and only if every critical component of f is preperiodic to a critical component cycle.*

Proof. We first show the if part. If every critical component of f is preperiodic to a critical component cycle, then $C(f)$ is contained in the basins of critical component cycles. Thus $J_2 \cap C(f) = \emptyset$ and J_2 is expanding follows from Theorem 1.6.

Next we show the only if part. Assume by contradiction that there is a critical component C which is not preperiodic to a critical component cycle. Let V be a periodic component of $PC(f)$ such that $V \not\subset C(f)$. Then by the argument in the proof of Lemma 5.1, there is a repelling periodic point $x \in V$. By Theorem 1.8 we know that $x \in J_2$. Let $y \in C(f)$ such that $f^n(y) = x$ for some $n > 0$. Then $y \in J_2$ since J_2 is totally invariant. Thus $J_2 \cap C(f) \neq \emptyset$. Thus J_2 is not expanding, which is a contradiction. This completes the proof. \square

Next we introduce the notion of Axiom A for holomorphic endomorphisms on \mathbb{P}^k . (For the general definition for smooth endomorphisms on Riemannian manifold, see for instance [25].)

Definition 5.6. *Let f be a holomorphic endomorphism on \mathbb{P}^2 of degree ≥ 2 . Then f satisfies the Axiom A if the following two conditions hold:*

- (1) *The non-wandering set $\Omega(f)$ is hyperbolic.*
- (2) *$\Omega(f)$ is the closure of periodic points.*

We give the following characterization of Axiom A for PCF maps on \mathbb{P}^2 . Some interesting examples of Axiom A PCF maps can be found in Ueno [32].

Corollary 5.7. *Let f be a PCF map on \mathbb{P}^2 of degree ≥ 2 . Then f is Axiom A if and only if f is expanding on J_2 , and for every critical component cycle C , $C \cap J_1$ is a hyperbolic (saddle) set.*

Proof. By Theorem 1.4 we know that $\Omega(f)$ is the closure of periodic points. Thus f is Axiom A if and only if f is hyperbolic on $\Omega(f)$. By Theorem 1.4, $\Omega(f) \setminus J_2$ is the union of super-attracting cycles and $C \cap J_1$, where C is a critical component cycle. Thus f is Axiom A if and only if f is expanding on J_2 , and for every critical component cycle C , $C \cap J_1$ is a hyperbolic (saddle) set (since super-attracting cycle is automatically a unstable dimension 0 hyperbolic set). \square

6. FURTHER DISCUSSION

In Lemma 2.9 we proved that if f is a PCF map on \mathbb{P}^2 , then the set of sporadic super-saddle cycles is a finite set. However we do not know any examples of PCF maps on \mathbb{P}^2 carrying such sporadic super-saddle fixed points. One may conjecture that such sporadic super-saddle fixed points actually can not exist for PCF maps on \mathbb{P}^2 . Theorem 3.4 may be seen as an evidence supporting this conjecture.

It is natural to ask whether our main result Theorem 1.1 can be generalized to higher dimension. Let f be a PCF map on \mathbb{P}^k , one may conjecture that in nice enough cases, the picture might be the following:

- (1) Let J_m be the Julia set of order m . Then $J_m \setminus J_{m+1}$ is contained in the union of the attracting basins of m -dimensional periodic submanifolds, where these submanifolds are contained in the critical set.
- (2) There is no Fatou disk containing x for every $x \in J_k$.

At least two difficulties need to be overcome for proving this conjectural picture. The first is that the classification of points of bounded ramification is not known for higher dimensional PCF maps. In dimension 2 this classification is known by Lemma 2.4. A more serious problem is that, even if the classification of the points of bounded ramification is known, one can possibly prove that for $x \in \mathbb{P}^k \setminus J_k$, $\omega(x)$ is contained in the set of points of unbounded ramification. But it is not possible to show (1), the structure of the Julia sets filtration, by the argument in Theorem 4.5 and 4.8, since in our argument the points of bounded ramification play an important role.

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