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# Finding geodesics on graphs using reinforcement learning

Daniel Kious\*      Cécile Mailler\*<sup>†</sup>      Bruno Schapira<sup>‡</sup>

October 9, 2020

## Abstract

It is well-known in biology that ants are able to find shortest paths between their nest and the food by successive random explorations, without any mean of communication other than the pheromones they leave behind them. This striking phenomenon has been observed experimentally and modelled by different mean-field reinforcement-learning models in the biology literature.

In this paper, we introduce the first probabilistic reinforcement-learning model for this phenomenon. In this model, the ants explore a finite graph in which two nodes are distinguished as the nest and the source of food. The ants perform successive random walks on this graph, starting from the nest and stopped when first reaching the food, and the transition probabilities of each random walk depend on the realizations of all previous walks through some dynamic weighting of the graph. We discuss different variants of this model based on different reinforcement rules and show that slight changes in this reinforcement rule can lead to drastically different outcomes.

We prove that, in two variants of this model and when the underlying graph is, respectively, any series-parallel graph and a 5-edge non-series-parallel *losange* graph, the ants indeed *eventually find the shortest path(s)* between their nest and the food. Both proofs rely on the electrical network method for random walks on weighted graphs and on Rubin’s embedding in continuous time. The proof in the series-parallel cases uses the recursive nature of this family of graphs, while the proof in the seemingly-simpler losange case turns out to be quite intricate: it relies on a fine analysis of some stochastic approximation, and on various couplings with standard and generalised Pólya urns.

## 1 Introduction and main results

### 1.1 Context and motivation

In this paper, we introduce and analyse two variants of a stochastic, unsupervised, reinforcement-learning algorithm, which, given as an input a graph in which two nodes are marked, gives as output the shortest path(s) between the two marked nodes. This algorithm is inspired by mean-field models introduced in the biology literature as models for the behavior of foraging ants (see, e.g. [DS04, MJT<sup>+</sup>13]): it has been widely empirically observed (see, e.g., [GADP89, MJT<sup>+</sup>13] for experiments) that a colony of ants is able to find shortest paths between their nest and the food. Unsupervised reinforcement learning is widely proposed as a model for this phenomenon in the biology literature. Our contribution is to introduce a new probabilistic reinforcement-learning model for this phenomenon and prove that, in this model, the ants indeed find the shortest path between their nest and the food.

We consider a sequence of random walkers on a finite graph  $\mathcal{G} = (V, \mathcal{E})$  with two distinguished nodes  $N$  and  $F$  (for “nest” and “food” when the walkers are interpreted as ants). At the beginning of time, all edges of  $\mathcal{G}$  are given weight 1. The idea is that the walkers explore the graph from  $N$  to  $F$  one after

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\*Department of Mathematical Sciences, University of Bath, Claverton Down, BA2 7AY Bath, UK.

Email: [d.kious/c.mailler@bath.ac.uk](mailto:d.kious@c.mailler@bath.ac.uk)

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<sup>‡</sup>Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France.

Email: [bruno.schapira@univ-amu.fr](mailto:bruno.schapira@univ-amu.fr)

each other, and the weights of the edges are updated after each walker reaches  $F$ . More precisely, for all  $n \geq 1$ , the  $n$ -th walker starts a random walk from  $N$  and walks randomly on the graph until it reaches  $F$ . At every step, the walker chooses one of the neighboring edges with probability proportional to their weights and crosses the chosen edge to the next vertex. Once the  $n$ -th walker has reached  $F$ , we update the weights of the edges by adding 1 to a subset of the trace of this walker. In this paper, we look at two possible rules for the choice of this subset of edges to reinforce:

- In the *loop-erased* version of the model, we reinforce the loop-erased time-reversed trace of walker  $n$ . This corresponds to how a hiker without a map would go back from  $F$  to  $N$  by walking backwards on their own trace, but avoiding unnecessary loops: when facing a choice between several edges they crossed on their way to  $F$ , they choose the edge that they crossed the earliest on their way forward.
- In the *geodesic* version of the model, we reinforce the shortest path from  $N$  to  $F$  inside the trace of the walker (i.e. we only look at the subgraph of all edges that were crossed by this specific walker). The case when there are several shortest paths presents some subtleties, on which we will come back when we will define more formally the model in Subsection 1.2 and when discussing our main results (see Subsection 1.3).

We call this stochastic process the loop-erased or geodesic ant process.

The interpretation of the model in terms of ants is as follows: (1) the ants only lay pheromones behind them on their way back from the food to the nest, (2) each ant goes back to the nest either following the loop-erasure of their forward trajectory reversed in time (for the loop-erased ant process), or following the shortest path in the subgraph that they have explored on the way forward (for the geodesic ant process), and (3) each ant can sense from the amount of pheromones how many of its predecessors have crossed an edge on their way back to the nest, and crosses each neighboring edge with probability proportional to this number. We conjecture that, following this simple unsupervised reinforcement-learning algorithm, the colony of ants *eventually finds the shortest path(s) between the nest and the food*, more precisely, asymptotically when time goes to infinity, a proportion 1 of all ants go from the nest to the food following a geodesic.

The difficulty of our analysis comes from different factors: (i) This is a linear reinforcement model: indeed, each ant chooses the next edge to cross with probability proportional to the number of previous ants that laid pheromones on it on their way back to the nest. Interestingly, the assumption that ants react linearly to pheromones is supported in the biology literature (see, e.g. [PGG<sup>+</sup>12, VPFV13]). In fact, one can easily find counter-examples that show that the same algorithm with super- or sub-linear reinforcement would not find the shortest path (see Subsection 1.3). (ii) The algorithm is a sequence of interacting reinforced random walks, and the reinforcement of the  $n$ -th random walk depends from the realisations of all previous ones.

Our main contribution is to prove that, as conjectured, the ants indeed find the shortest path if we assume that the underlying graph is either a series-parallel graph (as in [HJ04]) whose “source” is the nest and whose “sink” is the source of food, or the 5-edge losange graph of Figure 4. Surprisingly, the proof for the 5-edge losange graph is more intricate than the proof for the whole class of series-parallel graphs; we therefore expect that finding a proof that would hold for any underlying graph is a very challenging and interesting problem. Both our proofs rely heavily on the electric network method for random walks on graphs (see, e.g., [LP05] for an introduction to this method), and Rubin’s embedding in continuous time (first introduced in [Dav90]). The proof for series-parallel graphs also uses the inductive nature of this family of graphs; the proof for the losange graph relies on the fine analysis of different stochastic approximations (see, e.g., [Duf97, Pem07]). Interestingly, we show that the losange case can be seen as an intricate coupling between two types of Pólya urns (see, e.g., [Pem07] for a survey); a fact that is reminiscent of the proof of Pemantle and Volkov [PV99] of the localisation on five sites with positive probability of the vertex-reinforced random walk (see also [Tar11, Tar04]).

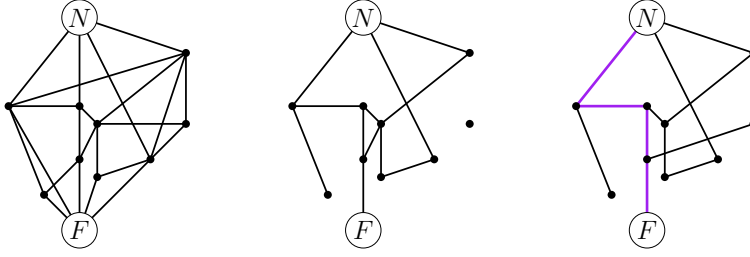


Figure 1: First visual aid for the definition of the uniform-geodesic ant process. On the left is pictured a graph  $\mathcal{G}$ . In the middle is a possible realization of a graph  $\mathcal{G}^{(n)}$ , the trace of the  $n$ -th random walk, and on the right is  $\gamma^{(n)}$  the unique geodesic from  $N$  to  $F$  in  $\mathcal{G}^{(n)}$ .

## 84 1.2 Mathematical description of the model and statement of the main results

85 Let  $\mathcal{G} = (V, E)$  be a finite graph with vertex set  $V$  and edge set  $E$ . Let  $N$  (the nest) and  $F$  (the food)  
 86 be two distinct vertices in  $V$ . In this paper we consider two versions of the same model, which differ by  
 87 their reinforcement rules.

88 We define the sequence  $(\mathbf{W}(n) = (W_e(n) : e \in E))_{n \geq 0}$  recursively as follows:  $W_e(0) = 1$  for all  $e \in E$ ,  
 89 and, for all  $n \geq 1$ :

- 90 • We sample a random walk  $X^{(n)} = (X_i^{(n)})_{i \geq 0}$  on  $\mathcal{G}$  that starts at  $N$ , is killed when first reaching  $F$ ,  
 91 and whose transition probabilities are: for all  $i \geq 1$ , for all  $u, v \in V$ ,

$$92 \quad \mathbb{P}(X_i^{(n)} = v \mid X_{i-1}^{(n)} = u, \mathbf{W}(n-1)) = \frac{W_{\{u,v\}}(n-1) \mathbf{1}_{u \sim v}}{\sum_{w \sim u} W_{\{u,w\}}(n-1)},$$

93 where  $\{u, v\}$  is the (unoriented) edge between  $u$  and  $v$ , and  $u \sim v$  if and only if the edge  $\{u, v\}$  is  
 94 in  $E$ .

- 95 • Let  $\mathcal{G}^{(n)}$  be the trace of  $X^{(n)}$ , that is the subgraph of  $\mathcal{G}$  obtained when removing from  $\mathcal{G}$  all edges  
 96 that the random walk  $X^{(n)}$  did not cross, and choose a path of edges  $\gamma_n$  as follows:
  - 97 – For the loop-erased ant process, we imagine that the walker goes back from  $F$  to  $N$  by following  
 98 its trajectory  $X^{(n)}$  backwards and avoiding loops as follows: when the walker is at a vertex that  
 99 was visited several times on the way forward, possibly coming from different edges at different  
 100 times, it chooses to cross the edge that was crossed the earliest on the way forward. We define  
 101  $\gamma^{(n)}$  as the set of edges crossed by the walker on its way back to the nest.

102 **Remark.** Note that this construction selects a self-avoiding path between  $F$  and  $N$ , which is  
 103 in fact the loop-erased version of the backward trajectory. Indeed, if we assume that  $X^{(n)} =$   
 104  $(X_0^{(n)} = N, X_1^{(n)}, \dots, X_{K_n}^{(n)} = F)$ , for some  $K_n \geq 1$ , and define the time-reversed trajectory  $\overline{X}^{(n)} =$   
 105  $(X_{K_n-i}^{(n)}, 0 \leq i \leq K_n)$ , then, by definition, we have that  $\gamma_i^{(n)} = \overline{X}_{j_i}^{(n)}$  for  $0 \leq i \leq k_n$  for some  
 106  $1 \leq k_n \leq K_n$ , where  $j_0 = 0$  and  $\gamma_{k_n}^{(n)} = F$ , for  $0 \leq i \leq k_n - 1$ ,  $j_{i+1} = \max\{j+1 : \overline{X}_j^{(n)} = \overline{X}_{j_i}^{(n)}\}$ . This  
 107 corresponds to the loop-erasure of  $\overline{X}^{(n)}$ , as defined in [LL10].

- 108 – In the uniform-geodesic version of the model, we define  $\gamma^{(n)}$  as the shortest path from  $N$  to  $F$   
 109 in  $\mathcal{G}^{(n)}$ ; if there are several shortest path, we choose one of them uniformly at random.

- 110 • For all  $e \in E$ , set  $W_e(n+1) = W_e(n) + \mathbf{1}_{e \in \gamma_n}$ .

111 We conjecture that

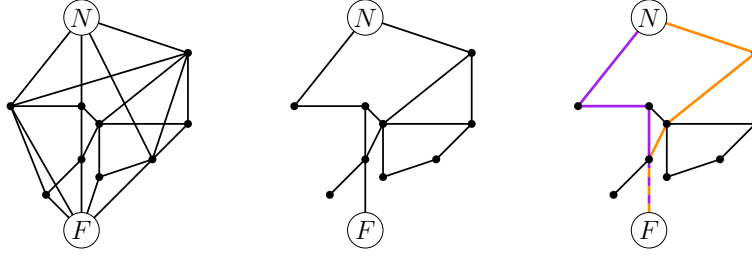


Figure 2: Second visual aid for the definition of the uniform-geodesic ant process. On the left is pictured a graph  $\mathcal{G}$ . In the middle is a possible realization of a graph  $\mathcal{G}^{(n)}$ , the trace of the  $n$ -th random walk. On the right, in orange and purple, are the two geodesics from  $N$  to  $F$  in  $\mathcal{G}^{(n)}$ , and thus the two possible choices for  $\gamma^{(n)}$ .

112 **Conjecture 1.1.** *Let  $\mathcal{G} = (V, E)$  be any finite graph in which two distinct nodes have been marked as  $N$*   
 113 *and  $F$ . Almost surely when  $n \rightarrow +\infty$ , for all  $e \in E$ ,*

114 
$$\frac{W_e(n)}{n} \rightarrow \chi_e,$$

115 *where  $(\chi_e)_{e \in E}$  is a random vector such that:*

116 (1) *For the loop-erased ant process,  $\chi_e \neq 0$  almost surely **if and only if** the edge  $e$  belongs to at least*  
 117 *one of the geodesics from  $N$  to  $F$ .*

118 (2) *For the uniform-geodesic ant process,  $\chi_e \neq 0$  almost surely **only if** the edge  $e$  belongs to at least one*  
 119 *of the geodesics from  $N$  to  $F$ .*

120 *Thus, if there is a unique geodesic  $\gamma$  from  $N$  to  $F$  in  $\mathcal{G}$ , then almost surely  $\chi_e = \mathbf{1}_{e \in \gamma}$ , for all  $e \in E$ , in*  
 121 *the two versions of the model.*

122 This indeed means that the ants *eventually find the shortest paths* between their nest and the source  
 123 of food, because it implies that the probability that the  $n$ -th ant goes from the nest to the food through  
 124 a geodesic path converges to 1 when  $n \rightarrow +\infty$ .

125 The difference between (1) and (2) is that, in the uniform-geodesic ant process, edges that belong to a  
 126 geodesic may have limiting normalised weight  $\chi_e$  that equal zero with positive probability: *the ants find*  
 127 *at least one of the geodesics, but maybe not all of them.* In Proposition 1.5, we provide an example of a  
 128 series-parallel graph where  $\chi_e = 0$  with positive probability for some edge  $e$  on a geodesic path.

129 Our first main contribution is to prove that this conjecture is true for all series-parallel graphs for  
 130 the loop-erased ant process. As their name suggests, series-parallel graphs are classical in electricity;  
 131 in probability theory, they are the object of a famous and still-open conjecture of Hambly and  
 132 Jordan [HJ04]. They have two distinguished nodes called the “source” and the “sink”, which we can  
 133 naturally see as the the nest  $N$  and the source of food  $F$  in our context.

134 **Definition 1.2** (See Figure 3). We define series-parallel (SP) graphs recursively as follows: a series-parallel  
 135 graph is

- 136 • either the single-edge graph (graph made of two vertices joined by one edge) with one node marked  
 137 as the source and the other as the sink,
- 138 • or two series-parallel graphs in series (i.e. we merge the sink of the first graph and the source of the  
 139 second),
- 140 • or two series-parallel graphs in parallel (i.e. we merge the two sources and the two sinks).

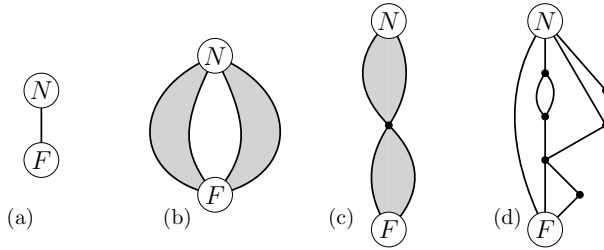


Figure 3: The definition of series-parallel graphs: a SP graph is either (a) the base case, or (b) two SP graphs in parallel, or (c) two SP graphs in series. (d) is an example.

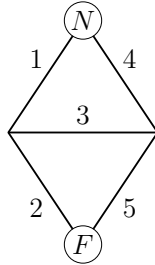


Figure 4: The losange graph

141 **Theorem 1.3.** For any SP graph whose source and sink are respectively marked as  $N$  and  $F$ , and for the  
 142 loop-erased ant process, Conjecture 1.1 is true, i.e. almost surely when  $n \rightarrow +\infty$ , for all  $e \in E$ ,

143 
$$\frac{W_e(n)}{n} \rightarrow \chi_e,$$

144 where  $(\chi_e)_{e \in E}$  is a random vector, such that  $\chi_e \neq 0$  almost surely if and only if the edge  $e$  belongs to at  
 145 least one of the geodesics from  $N$  to  $F$ .

146 Interestingly, the analysis of the loop-erased ant process outside the family of series-parallel graphs  
 147 turns out to be very challenging. To illustrate this, we consider one of the simplest non-series-parallel  
 148 graph one could think of, which is the 5-edge losange of Figure 4, which we call “the losange graph”: even  
 149 on this simple graph, we are not able to prove convergence of the loop-erased ant process. However, we  
 150 are able to prove convergence of the uniform-geodesic ant process, which turns out to be simpler in this  
 151 setting (see the remark before Lemma 3.6).

152 We number the edges of the losange graph from 1 to 5 as in Figure 4. Our second main result is the  
 153 following.

154 **Theorem 1.4.** For all  $1 \leq i \leq 5$  and  $n \geq 0$ , we denote by  $W_i(n)$  ( $\forall 1 \leq i \leq 5$ ) the weight of edge  
 155 number  $i$  after the  $n$ -th walker has reached the food in the uniform-geodesic ant process on the losange  
 156 graph. (Recall that  $W_i(0) = 1$ , by definition.) Almost surely as  $n \rightarrow +\infty$ ,

157 
$$\frac{W_i(n)}{n} \rightarrow \chi_i, \quad \text{for all } 1 \leq i \leq 5,$$

158 where  $(\chi_i)_{1 \leq i \leq 5}$  is a random vector such that almost surely  $\chi_1 = \chi_2 = 1 - \chi_4 = 1 - \chi_5 \in (0, 1)$  and  $\chi_3 = 0$ .

### 159 1.3 Discussion

160 **Discussion on the loop-erased vs. uniform-geodesic reinforcement rules:** While we believe the  
 161 result on the losange graph is also true for the loop-erased ant process, we think the proof would be more  
 162 involved than with the uniform-geodesic ant process.

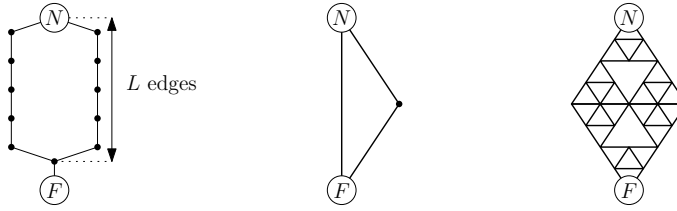


Figure 5: Graphs used in the discussions of Subsection 1.3.

163 The first of three steps in the proof in the uniform-geodesic case is to show that the normalised weight  
 164 of the middle edge (edge number 3) converges to zero, and then use this convergence to zero to prove that  
 165 the speed of convergence to zero is polynomial. Although proving convergence of the normalised weight  
 166 of edge 3 would be similar (and in fact almost identical) in the loop-erased case, proving that the speed of  
 167 convergence is polynomial is, we believe, much harder, and could in fact be wrong. Intuitively, it should  
 168 not be surprising that the weight of edge 3 could be bigger in the loop-erased than in the uniform-geodesic  
 169 version of the model: this comes from the fact that reinforcing the edge 3 is more likely at every step  
 170 in the loop-erased version of the model. Since the proof in the uniform-geodesic case is already quite  
 171 involved, we leave the case of the loop-erased ant process on the losange open.

172 Conversely, the analysis of the uniform-geodesic ant process (and all its variants - see discussion  
 173 below) on series-parallel graphs seems to be a challenging problem, which we also leave for further work.  
 174 In summary, it seems that neither of the two versions of the process is easier to analyse than the other in  
 175 general, but that this depends on the underlying (family of) graph(s).

176 **Discussion on the (uniform-)geodesic version of the model:** First note that on the losange  
 177 graph, the trace of a walker can contain at most one geodesic, and thus the rule of choosing the subset of  
 178 edges to reinforce uniformly among all geodesics in the trace is irrelevant in this case. In fact, we believe  
 179 that the way we choose which shortest path to reinforce when there are several in the trace can have a  
 180 significant impact on the behaviour of the system.

181 Indeed, we first observe that, in the uniform-geodesic version of the model, there could exist an edge  
 182 that belongs to a geodesic between  $N$  and  $F$  whose normalised weight converges to zero:

183 **Proposition 1.5.** *If  $\mathcal{G}$  is the graph on the left-hand side of Figure 5, then the uniform-geodesic version*  
 184 *of the model satisfies: there exists  $e \in E$  such that  $e$  lies on a geodesic between  $N$  and  $F$  (in fact, all edges*  
 185 *lie on such a geodesic in this graph) and, for all  $L$  large enough (see Figure 5 for the definition of  $L$ ),*

$$186 \quad \mathbb{P}(W_e(n)/n \rightarrow 0) > 0.$$

187 This proposition also holds (with an almost identical proof) when the choice of the geodesic is not  
 188 uniform as long as any geodesic within the trace is chosen with a probability bounded away from 0.

189 Another rule for the choice of  $\gamma^{(n)}$  when there are several shortest paths in  $\mathcal{G}^{(n)}$  is the following:  
 190 Consider  $\mathcal{G}_0^{(n)}$  the subgraph of  $\mathcal{G}^{(n)}$  obtained by removing all the edges and vertices that do not belong to  
 191 any of the shortest paths from  $N$  to  $F$  in  $\mathcal{G}^{(n)}$ . As in the loop-erased version of the model, imagine that  
 192 the walker walks back from  $F$  to  $N$ , by only crossing edges from  $\mathcal{G}_0^{(n)}$ , and, when faced with a choice,  
 193 choosing the edge it crossed the earliest on the way forward. Define  $\gamma^{(n)}$  as the set of edges crossed by the  
 194 walker on its way back to the nest. We believe that the same conjecture as for the loop-erased version of  
 195 the model should be true for this version of geodesic ant process.

196 **Other possible reinforcement rules:** An alternative reinforcement rule could be to reinforce all  
 197 edges that the  $n$ -th walker crossed, i.e. all edges in  $\mathcal{G}^{(n)}$ , instead of only reinforce the edges of  $\gamma^{(n)}$ .  
 198 Intuitively, this would mean that ants lay pheromones on their way to the food instead of laying them on  
 199 their way back to the nest. A mean-field version of this alternative model is also considered in the biology

200 literature (see, e.g., [MJT<sup>+</sup>13]). Preliminary work on this alternative reinforcement rule suggests that it  
 201 could lead to surprisingly different results and that the ants may not always find the shortest path, we  
 202 leave this for further work.

203 In this alternative reinforcement rule where ants lay pheromones on their way to the food, one could  
 204 consider that ants cannot sense from the pheromones laid on an edge how many different ants have crossed  
 205 this edge, but rather how many times this edge has been crossed by an ant. This would mean that if  
 206 the  $n$ -th ant crossed an edge  $k$  times the weight of this edge is increased by  $k$  when updating the weights  
 207 after the  $n$ -th ant has reached the food. Finally, one could wonder how the results are impacted if the ants  
 208 are sensitive to their own pheromones, i.e. if the weights are updated during the random walks after every  
 209 steps of the ants, and not after each ant reaches the food. Each ant would then perform a (self-)reinforced  
 210 random walk that starts on an already-weighted graph. We believe that these variants could lead to  
 211 different asymptotic behaviours and raise various interesting mathematical challenges.

212 **Discussion on linear vs. sub- or super-linear reinforcement:** As mentioned in the introduction,  
 213 Conjecture 1.1 would no longer be true if we considered super- or sub-linear reinforcement instead of linear  
 214 reinforcement. Indeed, consider the graph in the middle of Figure 5, and imagine that all the ants perform  
 215 weighted random walks on the graph  $\mathcal{G}$ , but according to the weights  $W_e(n)^\alpha$  ( $\forall e \in E$ ), for some  $\alpha > 0$ .  
 216 One can check that if  $\alpha > 1$  (i.e. in the super-linear case), then, almost surely, the subset of edges from  $E$   
 217 such that  $\liminf_n W_e(n)/n \neq 0$  is either  $\{N, F\}$  or  $E \setminus \{\{N, F\}\}$ , each with positive probability. Also, if  
 218  $\alpha < 1$  (i.e. in the sub-linear case), the subset of all edges from  $E$  such that  $\liminf_n W_e(n)/n \neq 0$  is almost  
 219 surely equal to  $E$  itself.

220 **Discussion on the underlying graph:** Theorems 1.3 and 1.4 confirm Conjecture 1.1 in the cases  
 221 when  $\mathcal{G}$  is a series-parallel graph or when  $\mathcal{G}$  is the losange graph, which is the simplest non-series parallel  
 222 graph. In the proof for series-parallel graph the iterative nature of this family of graph allows us to  
 223 reason by induction. An iterative family of graphs that builds on the losange example is the “double  
 224 Sierpiński gasket” graph, which consists of two Sierpiński gaskets of the same fractal depth whose bases  
 225 have been merged (see the right-hand side of Figure 5 where a double Sierpiński gasket graph of depth 3  
 226 is represented). Interestingly, a version of this graph has been considered in the biology literature under  
 227 the name “tower of Hanoi” (see [MJT<sup>+</sup>13, RSB11]).

228 **Other models of path and network formation by reinforcement:** Our model can be seen as a  
 229 reinforcement path formation model. The idea is that we start from a weighted graph  $\mathcal{G}$  where all edges  
 230 have the same weight 1, and we look at the graph  $\mathcal{G}^{(\infty)}$  of all edges whose normalised weight does not  
 231 tend to zero when time goes to infinity. In the language of Conjecture 1.1,  $\mathcal{G}^\infty = (V, E^\infty)$  where  $e \in E^\infty$  if  
 232 and only if  $e \in E$  and  $\chi_e > 0$ . The fact that  $\mathcal{G}^\infty \neq \mathcal{G}$  means that some path or some network has been  
 233 selected by the dynamics: in our case, we conjecture (and prove for series parallel graphs or the losange  
 234 graph) that the dynamics selects the shortest paths between the nest and the food.

235 Other related models of path formation by reinforcement exist in the literature: for example, Le  
 236 Goff and Raimond [LGR18] look at a model of non-backtracking vertex-reinforced random walk, also  
 237 inspired from ant behaviour. They show that, in this model, with positive probability, the ant eventually  
 238 walks along a cycle of finite edges. This model is very different from ours: the reinforcement is super-  
 239 linear instead of linear, there is one ant as opposed to several ants walking successively in the graph, the  
 240 underlying graph is infinite (although locally finite), and there is no nest or food and thus no geodesics  
 241 involved.

242 Another related model of network formation is the WARM model of van der Hofstad, Holmes, Kuznetsov  
 243 and Ruszel [VDHKK<sup>+</sup>16], where, at every time step, an edge is chosen at random and its weight increased  
 244 by one (see also [HK17]). The choice of the edge to reinforce at each step is done according to a two-step  
 245 procedure that involves super-linear reinforcement. Van der Hofstad et al. prove that the limiting graph  
 246 (i.e. the graph consisting of all edges whose normalised weight does not go to zero) is a linearly-stable  
 247 equilibrium with positive probability. They conjecture that, if the reinforcement is strong enough, all



linearly-stable configuration is a union of trees of diameter at most 3. They prove this conjecture in the simple case of a triangle graph, i.e. the complete graph on three vertices.

A model of network formation with linear reinforcement is the “signaling game” of [HST11, KT16], where at every time step, “Nature” decides which pairs of neighbours are allowed to communicate for this round, and each vertex chooses a neighbour with probability proportional to the number of times they have communicated in the past, and they communicate if they both choose each other and if Nature allows it. In [KT16], the authors show that the limiting graph (consisting of edges between two vertices that communicate asymptotically a positive proportion of rounds) is star-shaped with positive probability.

**Plan of the paper:** Section 2 contains the proof of Theorem 1.3 (i.e. the series-parallel case), and Section 3 the proof of Theorem 1.4 (i.e. the losange case). These two sections can be read independently. Finally we prove Proposition 1.5 in Section 4.

## 2 The loop-erased ant process on series-parallel graphs

In this section, we only consider the loop-erased ant process. We define the size of a graph as its number of edges. For a series-parallel graph  $G$ , we define its height, which we denote by  $h_{\min}(G)$ , as the length of a shortest path from the source to the sink.

### 2.1 Preliminary lemmas

We start with two simple observations. The first one is a direct consequence of the definition of series-parallel graphs:

**Lemma 2.1.** *Let  $G$  be a nonempty series-parallel graph. Then, either  $G$  is reduced to a single edge (it has size one), or one can find two non-empty series-parallel subgraphs  $G_1$  and  $G_2$ , such that  $G$  is obtained by merging  $G_1$  and  $G_2$ , either in series or in parallel.*

The second observation is the following:

**Lemma 2.2.** *Let  $\varphi : (0, +\infty)^2 \rightarrow (0, \infty)$  be the function defined by*

$$\varphi(x, y) = \frac{1}{\frac{1}{x} + \frac{1}{y}} \quad \text{for all } (x, y) \in (0, +\infty)^2.$$

*Then, for all  $(x, y), (x', y') \in (0, +\infty)^2$ , one has*

**(a)**  $\varphi(x + x', y + y') \geq \varphi(x, y) + \varphi(x', y')$ , and

**(b)**  $\varphi(x + 1, y + 1) \leq \varphi(x, y) + 1$ .

*Proof.* Since

$$\varphi\left(\frac{x + x'}{2}, \frac{y + y'}{2}\right) = \frac{1}{2} \cdot \varphi(x + x', y + y'),$$

proving that  $\varphi$  is concave is enough to prove **(a)**. A simple calculation shows that

$$\frac{\partial^2 \varphi}{\partial x^2}(x, y) = \frac{-2y^2}{(x + y)^3}, \quad \frac{\partial^2 \varphi}{\partial y^2}(x, y) = \frac{-2x^2}{(x + y)^3}, \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial x \partial y}(x, y) = \frac{2xy}{(x + y)^3}.$$

This implies that the Hessian of  $\varphi$  is everywhere non-positive, and thus that  $\varphi$  is concave as claimed, which concludes the proof of **(a)**.

To prove **(b)**, fix  $x > 0$  and set  $H(y) := 1 + \varphi(x, y) - \varphi(x + 1, y + 1)$  for all  $y > 0$ . Using the definition of  $\varphi$ , we can calculate

$$H'(y) = \frac{1}{\left(\frac{y}{x} + 1\right)^2} - \frac{1}{\left(\frac{y+1}{x+1} + 1\right)^2},$$

284 implying that the function  $H$  is increasing on  $[0, x)$  and decreasing on  $(x, +\infty)$ . Since  $H(0) > 0$ , and  
 285  $\lim_{y \rightarrow \infty} H(y) = 0$ , it follows that  $H(y) > 0$  for all positive  $y$ , which concludes the proof of **(b)**.  $\square$

286 **Remark.** Item (a) has another simple proof in terms of conductances. Indeed one could note that by  
 287 Rayleigh’s monotonicity’s principle, putting first  $(x, x')$  in parallel and  $(y, y')$  in parallel, and then putting  
 288 the two of them in series has a better conductance than first putting  $(x, y)$  in series and  $(x', y')$  in series,  
 289 and then putting them in parallel (to go to the first one, we need to add an edge with infinite conductance).

## 290 2.2 Our main result in terms of effective conductances

291 The main idea to prove Theorem 1.3 is to reason in terms of the “effective conductance” of the graph.  
 292 We interpret the weight of an edge as its “conductance” and let  $\mathcal{C}_G(n)$  be the effective conductance (from  
 293 the source to the sink) after the  $n$ -th walk has reached the sink, and simply write  $\mathcal{C}_G$  for the initial  
 294 effective conductance. In order to compute the effective conductance of a series parallel graph, one can  
 295 use Lemma 2.1 and the two following rules:

- 296 • If  $G$  is composed of two graphs  $G_1$  and  $G_2$  merged in parallel, then  $\mathcal{C}_G = \mathcal{C}_{G_1} + \mathcal{C}_{G_2}$ .
- 297 • If  $G$  is composed of two graphs  $G_1$  and  $G_2$  merged in series, then  $\mathcal{C}_G = \varphi(\mathcal{C}_{G_1}, \mathcal{C}_{G_2})$ .

298 Our main result in terms of effective conductances reads as follows.

299 **Theorem 2.3.** *If  $G$  is a series-parallel graph, and  $\mathcal{C}_G(n)$  is its conductance after the  $n$ -th walker has*  
 300 *reached the sink, then, almost surely when  $n \rightarrow +\infty$ ,*

$$301 \frac{\mathcal{C}_G(n)}{n} \rightarrow \frac{1}{h_{\min}(G)},$$

302 where  $h_{\min}(G)$  is the graph distance between the source and the sink in  $G$ .

## 303 2.3 Deterministic bounds for the effective conductance of a series-parallel graph after 304 $n$ walks

305 The first step towards proving Theorems 1.3 and 2.3 is the following (deterministic) lemma.

306 **Lemma 2.4. (a)** *Let  $G$  be a series-parallel graph with weighted edges and let  $\mathcal{C}_G$  be its effective con-*  
 307 *ductance from the source to the sink. Consider a self-avoiding path from the source to the sink of length*  
 308  *$L$ , and denote by  $\mathcal{C}'_G$  the effective conductance of  $G$  after the weights of all edges on this path have been*  
 309 *increased by one. Then,*

$$310 \frac{1}{L} \leq \mathcal{C}'_G - \mathcal{C}_G \leq 1.$$

311 **(b)** *Let  $G$  be a series-parallel graph and consider the loop-erased ant process on  $G$ . There exists a constant*  
 312  *$C > 0$  depending only on  $G$ , such that, almost surely,*

$$313 \mathcal{C}_G(n) \leq \frac{n + C}{h_{\min}(G)}, \quad \text{for all } n \geq 0.$$

314 *Proof.* We first prove **(a)** by induction on the size of the graph. If  $G$  has size one, then the result is  
 315 immediate since  $\mathcal{C}'_G = \mathcal{C}_G + 1$ . Now assume that the result holds for all series-parallel graphs with size at  
 316 most  $N$  (for some integer  $N \geq 1$ ) and consider a graph  $G$  of size  $N + 1$ . By Lemma 2.1 we know that  $G$   
 317 is the merging of two non-empty subgraphs  $G_1$  and  $G_2$ , either in parallel or in series. Note that  $G_1$  and  
 318  $G_2$  both have size at most  $N$  and thus that the induction hypothesis applies to them.

319 If  $G_1$  and  $G_2$  are in parallel, then  $\mathcal{C}_G = \mathcal{C}_{G_1} + \mathcal{C}_{G_2}$ . Now since the chosen path is self-avoiding, it either  
 320 lies entirely in  $G_1$  or in  $G_2$ . Assume for instance that it lies in  $G_1$ : using the induction hypothesis, we  
 321 get that  $1 \geq \mathcal{C}'_{G_1} - \mathcal{C}_{G_1} \geq 1/L$ , which concludes the proof since  $\mathcal{C}'_G = \mathcal{C}'_{G_1} + \mathcal{C}_{G_2}$ .

322 If  $G_1$  and  $G_2$  are in series, then first observe that one can write  $L = L_1 + L_2$ , with  $L_i$  the length of  
 323 the restriction of the path to  $G_i$ , for  $i = 1, 2$ . Then,

$$324 \quad \mathcal{C}'_G = \frac{1}{\frac{1}{\mathcal{C}'_{G_1}} + \frac{1}{\mathcal{C}'_{G_2}}} \geq \frac{1}{\frac{1}{\mathcal{C}_{G_1 + L_1}} + \frac{1}{\mathcal{C}_{G_2 + L_2}}} \geq \frac{1}{\frac{1}{\mathcal{C}_{G_1}} + \frac{1}{\mathcal{C}_{G_2}}} + \frac{1}{L_1 + L_2} = \mathcal{C}_G + \frac{1}{L},$$

325 using the induction hypothesis for the first inequality and Lemma 2.2(a) for the second one. This  
 326 concludes the proof of the lower bound of (a). The proof of the upper bound is entirely similar, using  
 327 this time Lemma 2.2(b) instead of Lemma 2.2(a).

328 Let us now prove (b) by induction on the size of the graph again. If  $G$  has only one edge (which  
 329 connects the source and the sink), then  $\mathcal{C}_G(n) = 1 + n$ , which proves the result in this case. Assume  
 330 by induction that the upper bound holds for all graphs with at most  $N$  edges, and assume that  $G$  has  
 331  $N + 1$  edges. By Lemma 2.1,  $G$  consists of two nonempty graphs  $G_1$  and  $G_2$  which are merged either in  
 332 parallel or in series, and such that both  $G_1$  and  $G_2$  have at most  $N$  edges. By hypothesis, there exist two  
 333 constants  $C_1$  and  $C_2$ , such that for all  $n \geq 0$ ,

$$334 \quad \mathcal{C}_{G_1}(n) \leq \frac{n + C_1}{h_{\min}(G_1)}, \quad \text{and} \quad \mathcal{C}_{G_2}(n) \leq \frac{n + C_2}{h_{\min}(G_2)}. \quad (1)$$

335 If  $G_1$  and  $G_2$  are in parallel, then  $\mathcal{C}_G(n) = \mathcal{C}_{G_1}(n_1) + \mathcal{C}_{G_2}(n - n_1)$ , for some (random) integer  $0 \leq n_1 \leq n$ ,  
 336 and the result follows immediately from (1), with the constant  $C := C_1 + C_2$ , using that  $h_{\min}(G) =$   
 337  $\min(h_{\min}(G_1), h_{\min}(G_2))$ . If  $G_1$  and  $G_2$  are in series, then noting that  $h_{\min}(G) = h_{\min}(G_1) + h_{\min}(G_2)$ ,  
 338 we get

$$339 \quad \mathcal{C}_G(n) = \frac{1}{\frac{1}{\mathcal{C}_{G_1}(n)} + \frac{1}{\mathcal{C}_{G_2}(n)}} \stackrel{(1)}{\leq} \frac{1}{\frac{h_{\min}(G_1)}{n + C_1} + \frac{h_{\min}(G_2)}{n + C_2}} \leq \frac{n + \max(C_1, C_2)}{h_{\min}(G_1) + h_{\min}(G_2)} = \frac{n + \max(C_1, C_2)}{h_{\min}(G)}.$$

340 This proves the induction step when  $G_1$  and  $G_2$  are in series, and concludes the proof of the lemma.  $\square$

341 A consequence of this lemma is that one has the deterministic bounds

$$342 \quad \frac{n}{h_{\max}(G)} \leq \mathcal{C}_G(n) - \mathcal{C}_G(0) \leq \frac{n + C}{h_{\min}(G)}, \quad \text{for all } n \geq 0, \quad (2)$$

343 for some constant  $C > 0$  and where  $h_{\max}(G)$  is the length of the longest self-avoiding path from the source  
 344 to the sink of  $G$ . In particular, almost surely  $\mathcal{C}_G(n) \rightarrow \infty$ , as  $n \rightarrow \infty$ . Note also that if the ants were  
 345 always choosing the shortest path, then we would have

$$346 \quad \frac{n}{h_{\min}(G)} \leq \mathcal{C}_G(n) \leq \frac{n + C}{h_{\min}(G)},$$

347 for some constant  $C > 0$ , for all  $n \geq 0$ . While the ants usually do not make this optimal choice, we will  
 348 see that almost surely the asymptotic behavior of the effective conductance of the graph is still of this  
 349 order (with a weaker control on the error term for the lower bound).

## 350 2.4 Bounds for a generalised version of the model

351 In the following, for any series-parallel graph  $G$ , any (series-parallel) subgraph  $H \subseteq G$ , and any  $n \geq 0$ ,  
 352 we let  $\mathbf{W}_H^G(n)$  denote the set of weights on the edges of  $G$  after the  $n$ -th time a path in  $H$  has been  
 353 reinforced. We also simply write  $\mathbf{W}_G(n)$ , when  $H = G$ .

354 In order to implement an induction argument, we need to consider a generalisation of the loop-erased  
 355 ant process. The reason for this is that we want the law of the process to be stable under restriction to a  
 356 subgraph. Unfortunately, the loop-erased ant process does not fulfil this: for instance if  $G$  is the merging

357 of two subgraphs  $G_1$  and  $G_2$  in parallel, then when reinforcing a path in  $G_1$ , an ant on  $G$  tends to visit  
 358 the source less often than an ant restricted to  $G_1$ . We now explain how we go around this problem.

359 In the original model on a graph  $G$ , when the  $n$ -th ant starts its random walk from  $N$ , it comes back  
 360 to  $N$  a random geometric number of times, say  $B_n$ , and then goes from  $N$  to  $F$  without returning to  $N$ .  
 361 We say that the  $n$ -th ant did  $B_n$  *unsuccessful excursions* in  $G$  (i.e. going from  $N$  to  $N$  without hitting  
 362  $F$ ), and one *successful excursion* (i.e. going from  $N$  to  $F$  without returning to  $N$ ).

363 In the original model, for all  $n \geq 1$ ,  $B_n$  is measurable with respect to  $\mathcal{F}_{n-1}(G) := \sigma(\mathbf{W}_G(0), \dots, \mathbf{W}_G(n-1))$ .  
 364 In the generalised model, we allow  $B_n$  and its law to be different and to depend on a larger sigma-  
 365 field. More precisely, given  $\mathcal{F}_{n-1}(G)$  and given some additional integer-valued random variable  $B_n$ , we  
 366 condition the  $n$ -th ant on performing  $B_n$  unsuccessful excursions before hitting  $F$ , and then reinforce a  
 367 path in its range according to the same rule as for the loop-erased ant process, i.e. we increase by one the  
 368 weights of the edges along the loop-erasure of the backwards trajectory of the  $n$ -th ant. The only case of  
 369 interest is when  $B_n$  is measurable with respect to some sigma-field of the type  $\sigma(\mathbf{W}_{G'}^G(0), \dots, \mathbf{W}_{G'}^G(n-1))$ ,  
 370 where  $G'$  is some series-parallel graph containing  $G$ ; however the proofs of the next results work in full  
 371 generality, without assuming anything on the random variables  $B_n$ .

372 For a series-parallel graph  $G$ , we still let  $\mathcal{C}_G(n)$  denote the effective conductance of graph  $G$  after  $n$   
 373 walkers have performed their walks and updated the weights in the generalised version of the loop-erased  
 374 ant process described above. We set

$$375 \quad \alpha(G) := \frac{h_{\min}(G)}{h_{\min}(G) + 1}. \quad (3)$$

376 The following proposition, together with Lemma 2.4, implies Theorem 2.3:

377 **Proposition 2.5.** *Consider a generalised version of the loop-erased ant process on a series-parallel*  
 378 *graph  $G$ , and let  $\alpha = \alpha(G)$ . There exists a real random variable  $K_G$ , such that almost surely  $K_G$  is*  
 379 *finite, and for all  $n \geq 1$ ,*

$$380 \quad (i) \quad \mathcal{C}_G(n) \geq \frac{n - K_G \cdot n^\alpha}{h_{\min}(G)};$$

381 *(ii) after  $n$  steps, the conditional probability that the  $(n + 1)$ -th walk reinforces a geodesic path is larger*  
 382 *than  $1 - K_G \cdot n^{\alpha-1}$ .*

383 *Proof.* We reason by induction on the size of  $G$ : if  $G$  has size 1, then  $\mathcal{C}_G(n) = n + 1$  almost surely, implying  
 384 that the result holds. Let us now assume that the result holds for all series-parallel graphs of size at most  
 385  $N$ , and consider a graph  $G$  of size  $N + 1$ . By Lemma 2.1 we know that  $G$  is the merging of two nonempty  
 386 subgraphs  $G_1$  and  $G_2$ , either in parallel or in series. We denote  $N_1$  and  $N_2$  the sources of  $G_1$  and  $G_2$ ,  $F_1$   
 387 and  $F_2$  their sinks.

388 **Case 1:  $G_1$  and  $G_2$  are in series.** Assume without loss of generality that  $G_1$  is on the top of  $G_2$   
 389 (meaning that the sink  $F_1$  of  $G_1$  coincides with the source  $N_2$  of  $G_2$ ). First note that each ant performing  
 390 its walk in  $G$  will reinforce one path in  $G_1$  and one path in  $G_2$ . Moreover, by definition of the loop-erasure  
 391 process, the path that is reinforced in  $G_1$  is entirely determined by the trajectory of the walk up to its first  
 392 hitting time of  $F_1 = N_2$ , while the path that is reinforced in  $G_2$  is entirely determined by the trajectory  
 393 of the ants after this hitting time of  $N_2$ . As a consequence, conditionally on the number of times the  
 394 walker returns to  $N$  before first hitting  $N_2$ , the laws of the two paths that are reinforced in  $G_1$  and  $G_2$   
 395 are independent.

396 Furthermore, for each  $n$ , the number of unsuccessful excursions in  $G_1$  (resp.  $G_2$ ) that are made by  
 397 the  $n$ -th walk before first hitting  $N_2$  (resp. after first hitting  $N_2$ ) is a measurable function of  $\mathbf{W}_G(n - 1)$   
 398 and the number  $B_n$  of unsuccessful excursions that are prescribed in  $G$ . Therefore, the restrictions  
 399 of the process to  $G_1$  and  $G_2$  are generalised versions of the loop-erased ant process, as defined before  
 400 Proposition 2.5. Therefore, we can use the induction hypothesis for  $G_1$  and  $G_2$ : there exist two random

401 variables  $K_1, K_2 \in (0, \infty)$ , such that with  $\alpha_1 = \alpha(G_1)$ , and  $\alpha_2 = \alpha(G_2)$ ,

$$402 \quad \mathcal{C}_{G_1}(n) \geq \frac{n - K_1 n^{\alpha_1}}{h_{\min}(G_1)} \quad \text{and} \quad \mathcal{C}_{G_2}(n) \geq \frac{n - K_2 n^{\alpha_2}}{h_{\min}(G_2)}.$$

403 If we denote by  $\beta = \max(\alpha_1, \alpha_2)$ , and by  $K = \max(K_1, K_2)$ , then

$$404 \quad \mathcal{C}_G(n) = \frac{1}{\frac{1}{\mathcal{C}_{G_1}(n)} + \frac{1}{\mathcal{C}_{G_2}(n)}} \geq \frac{1}{\frac{h_{\min}(G_1)}{n - K_1 n^{\alpha_1}} + \frac{h_{\min}(G_2)}{n - K_2 n^{\alpha_2}}} \\ 405 \quad \geq \frac{n - K n^\beta}{h_{\min}(G_1) + h_{\min}(G_2)} = \frac{n - K n^\beta}{h_{\min}(G)},$$

407 since  $h_{\min}(G) = h_{\min}(G_1) + h_{\min}(G_2)$ ; which concludes the induction argument for Part (i) because, by  
408 definition,  $\beta \leq \alpha(G)$ .

409 For Part (ii) we just observe that, by the induction hypothesis and a union bound, the conditional  
410 probability that the  $n$ -th walker does not reinforce a geodesic path is smaller than  $K_1 n^{\alpha_1 - 1} + K_2 n^{\alpha_2 - 1} \leq$   
411  $K n^{\beta - 1}$ , which concludes the induction argument in the case when  $G_1$  and  $G_2$  are merged in series.

412 **Case 2:  $G_1$  and  $G_2$  are in parallel.** We start again by showing that the restrictions of the process  
413 on  $G_1$  and  $G_2$  are generalised versions of the loop-erased model as defined before Proposition 2.5. For all  
414 integers  $n$ , we denote by  $N_i(n)$  the number of times a path in  $G_i$  have been reinforced after  $n$  ants have  
415 performed their walks in  $G$ : one has

$$416 \quad \mathcal{C}_G(n) = \mathcal{C}_{G_1}(N_1(n)) + \mathcal{C}_{G_2}(N_2(n)). \quad (4)$$

417 We also let  $(\tau_k^{(i)})_{k \geq 1}$  be the random times when the process  $N_i$  increases by one, i.e. the times when an  
418 ant reinforces a path in  $G_i$ . For all  $n \geq 1, k \geq 0, i \in \{1, 2\}$ , given  $\tau_{k-1}^{(i)}$ , the time to wait until another ant  
419 reinforces a path in  $G_i$  (i.e.  $\tau_k^{(i)} - \tau_{k-1}^{(i)}$ ) and the number  $B_k^{(i)}$  of unsuccessful excursions made by this ant  
420 (the  $\tau_k^{(i)}$ -th ant) in  $G_i$  are both measurable functions of  $\mathbf{W}(\tau_{k-1}^{(i)})$  and of the total number of unsuccessful  
421 excursions performed in  $G$  by all ants between times  $\tau_{k-1}^{(i)} + 1$  and  $\tau_k^{(i)}$ . Moreover, by definition, given this  
422 information, the reinforced path in  $G_i$  is chosen by performing  $B_k^{(i)}$  independent unsuccessful excursions,  
423 plus one additional independent successful excursion, and using the loop-erasure rule. Thus we can use  
424 the induction hypothesis for  $G_1$  and  $G_2$ .

425 In the following, we use the fact that, at any time  $n$ , the  $(n + 1)$ -th walker performs its successful  
426 excursion in  $G_i$  with probability  $\mathcal{C}_{G_i}(n)/(\mathcal{C}_{G_1}(n) + \mathcal{C}_{G_2}(n))$ , for  $i = 1, 2$ . Indeed, this follows from the fact  
427 the law of the successful excursion of each ant walking on  $G$  is by definition independent of the number of  
428 unsuccessful excursions performed by this ant and of their trajectories. Moreover, for the simple random  
429 walk in  $G$  (that is if we were considering the original model), the probability to reinforce a path in  $G_i$  is  
430 given by the ratio of the effective conductances, and this happens if and only if the successful excursion  
431 belongs to  $G_i$ .

432 **Case 2.1:** We first assume that  $h_{\min}(G_1) = h_{\min}(G_2)$ . Using the induction hypothesis, there exist  
433 two random variables  $K_1, K_2 \in (0, \infty)$  such that, almost surely,

$$434 \quad \mathcal{C}_G(n) \geq \frac{N_1(n) - K_1 N_1(n)^\alpha}{h_{\min}(G_1)} + \frac{N_2(n) - K_2 N_2(n)^\alpha}{h_{\min}(G_2)} \\ 435 \quad \geq \frac{N_1(n) + N_2(n)}{h_{\min}(G)} - \frac{K(N_1(n)^\alpha + N_2(n)^\alpha)}{h_{\min}(G)},$$

437 with  $\alpha = \alpha(G)$  (see Equation (3) for the definition of  $\alpha(G)$ ) and  $K = K_1 + K_2$ . This concludes the  
438 induction argument for Part (i), since by concavity of the map  $x \mapsto x^\alpha$ , we have

$$439 \quad N_1(n)^\alpha + N_2(n)^\alpha \leq 2^{1-\alpha} n^\alpha. \quad (5)$$

Concerning Part (ii), note that using the induction hypothesis, if the  $(n+1)$ -th walker makes its successful excursion in  $G_1$ , then the probability that its range contains a geodesic path of  $G_1$  is larger than  $1 - K_1 N_1(n)^{\alpha-1}$ , and similarly for  $G_2$ . Considering the complement and using a union bound, we deduce that the probability that the  $(n+1)$ -th walker reinforces a geodesic path of  $G$  is at least

$$1 - K_1 N_1(n)^{\alpha-1} - K_2 N_2(n)^{\alpha-1} \geq 1 - Kn^{\alpha-1},$$

which concludes the proof of the induction argument in the case when  $h_{\min}(G_1) = h_{\min}(G_2)$ .

**Case 2.2:** We now assume that  $h_{\min}(G_1) \neq h_{\min}(G_2)$ , and without loss of generality  $h_{\min}(G_1) < h_{\min}(G_2)$ , which implies  $\alpha(G) = \alpha(G_1)$  (see Equation (3) for the definition of  $\alpha(G)$ ). Using the induction hypothesis, we have that there exists a random variable  $K_1 \in (0, \infty)$ , such that

$$\mathcal{C}_{G_1}(N_1(n)) \geq \frac{N_1(n)}{h_{\min}(G_1)} \cdot (1 - K_1 N_1(n)^{\alpha-1}). \quad (6)$$

For small values of  $N_1(n)$ , this lower bound can be negative (recall that, by definition,  $\alpha = \alpha(G) < 1$ ; see Equation (3)); a better lower bound for small values of  $N_1(n)$  is given by

$$\mathcal{C}_{G_1}(N_1(n)) \geq \mathcal{C}_{G_1}(0). \quad (7)$$

By Lemma 2.4(b), there exists a constant  $C_2 > 0$  (only depending on  $G_2$ ), such that

$$\mathcal{C}_{G_2}(n - N_1(n)) \leq \frac{n - N_1(n) + C_2}{h_{\min}(G_2)} \leq \frac{n - N_1(n) + C_2}{h_{\min}(G_1) + 1}, \quad (8)$$

because, by assumption,  $h_{\min}(G_2) \geq h_{\min}(G_1) + 1$ . For all  $b > 0$ , we define the function  $\varphi_b$  such that, for all  $i \geq 0$ ,

$$\varphi_b(i) := \max\left(\mathcal{C}_{G_1}(0), \frac{i - bi^\alpha}{h_{\min}(G_1)}\right). \quad (9)$$

We also define the function  $\psi$  such that, for all  $i \geq 0$ ,

$$\psi(i) := \frac{\alpha(i + C_2)}{h_{\min}(G_1)}. \quad (10)$$

By Equations (6), (7), and (8) we get that the probability  $p_n$  that the  $(n+1)$ -th reinforces a geodesic path of  $G_1$ , conditionally on  $\mathbf{W}_G(n)$ , satisfies

$$p_n = \frac{\mathcal{C}_{G_1}(N_1(n))}{\mathcal{C}_{G_1}(N_1(n)) + \mathcal{C}_{G_2}(n - N_1(n))} \geq \frac{\varphi_{K_1}(N_1(n))}{\varphi_{K_1}(N_1(n)) + \psi(n - N_1(n))}, \quad (11)$$

for all  $n \geq 0$ . We now prove that, almost surely, there exists a finite random variable  $K > 0$ , such that

$$N_1(n) \geq n - Kn^\alpha, \quad \text{for all } n \geq 1. \quad (12)$$

This is enough to conclude the proofs of the induction step for both Parts (i) and (ii). Indeed, on the one hand, we get that, for all  $n \geq 1$ ,

$$\mathcal{C}_G(n) \geq \mathcal{C}_{G_1}(N_1(n)) \geq \frac{N_1(n)(1 - K_1 N_1(n)^{\alpha-1})}{h_{\min}(G_1)} \geq \frac{n - (K + K_1)n^\alpha}{h_{\min}(G)},$$

which concludes the proof of the induction step of Part (i). And, on the other hand, using (2), we get that the probability of not reinforcing a geodesic path in  $G$  is smaller than

$$K_1 N_1(n)^{\alpha-1} + (1 - p_n) \leq (K_1 + h_{\max}(G)K)n^{\alpha-1},$$

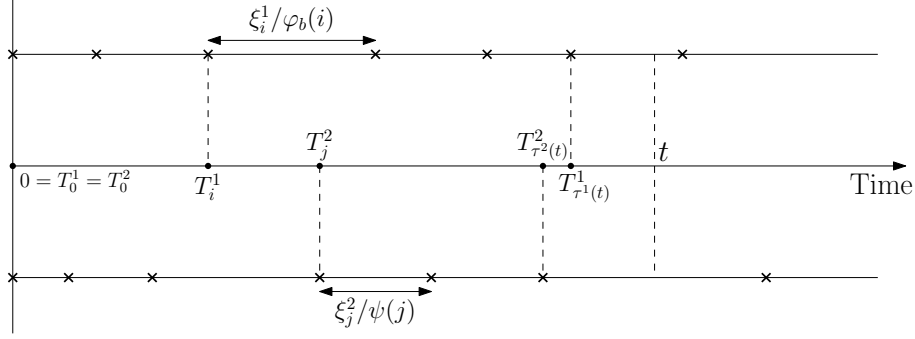


Figure 6: Rubin's construction for the proof of Proposition 2.5 (Case 2.2 in the proof). On the top line, the intervals between crosses are the  $\xi_i^1/\varphi_b(i)$  and, similarly, on the bottom line, the intervals between crosses are the  $\xi_i^2/\psi(i)$ . On the middle line, we show how the  $T_i^1$ 's and  $T_i^2$ 's are defined as the partial sums of these interval lengths and how  $\tau_t^1$  and  $\tau_t^2$  are defined for a given time  $t > 0$ .

472 which concludes the proof of the induction step of Part (ii). Therefore, to conclude the proof, it only  
 473 remains to prove Equation (12).

474 If  $K_1$  was a fixed constant, the conclusion would come by simply analysing the generalised urn process  
 475 associated to  $\varphi_{K_1}$  and  $\psi$ . But here,  $K_1$  is a random variable that depends on the whole history of the  
 476 process. To go around this issue, we are going to define a family of generalised Pólya urns, and couple all  
 477 of them with the process  $(N_1(n))_{n \geq 0}$ , in such a way that almost surely  $(N_1(n))_{n \geq 0}$  will dominate at least  
 478 one of those urns. To be more precise, for all  $b > 0$ , we define the Markov process  $(R_n^b)_{n \geq 0}$ , by  $R_0^b = 0$ ,  
 479 and for all  $n \geq 0$ ,

$$480 \quad q_n^b := \mathbb{P}(R_{n+1}^b = R_n^b + 1 \mid R_n^b) = 1 - \mathbb{P}(R_{n+1}^b = R_n^b \mid R_n^b) = \frac{\varphi_b(R_n^b)}{\varphi_b(R_n^b) + \psi(n - R_n^b)}, \quad (13)$$

481 where  $\varphi_b$  and  $\psi$  are defined in Equations (9) and (10) respectively. We now fix some  $b > 0$  and show that  
 482 there exists an almost surely finite random variable  $C_b$ , such that

$$483 \quad n - R_n^b \leq C_b n^\alpha, \quad \text{for all } n \geq 0. \quad (14)$$

484 To prove Equation (14), it is convenient to use Rubin's algorithm, which was introduced in Davis's  
 485 paper on reinforced random walks [Dav90]. Consider  $\{\xi_i^1\}_{i \geq 0}$  and  $\{\xi_i^2\}_{i \geq 0}$  two independent sequences of  
 486 independent mean-one exponential random variables, and define, for all  $n \geq 1$ ,

$$487 \quad T_n^1 := \sum_{k=0}^{n-1} \frac{\xi_k^1}{\varphi_b(k)}, \quad \text{and} \quad T_n^2 := \sum_{k=0}^{n-1} \frac{\xi_k^2}{\psi(k)}.$$

488 Set also  $T_0^1 = T_0^2 = 0$  and, for all  $t > 0$  (see Figure 6),

$$489 \quad \tau^1(t) := \sup\{n \geq 0 : T_n^1 \leq t\}, \quad \text{and} \quad \tau^2(t) := \sup\{n \geq 0 : T_n^2 \leq t\}.$$

490 It follows from standard properties of independent exponential random variables that, for any  $t > 0$ ,  
 491 conditionally on the fact that  $\tau^1(t) = n_1$ , and  $\tau^2(t) = n_2$ , the probability  $q_{n_1+n_2}^b$  that  $R_{n_1+n_2}^b$  increases  
 492 by one at the next step is also equal to the probability of  $T_{n_1+1}^1$  being smaller than  $T_{n_2+1}^2$ .

493 As a consequence if we let  $t_n = \inf\{t \geq 0 : \tau^1(t) + \tau^2(t) \geq n\}$ , then the process  $(\tau^1(t_n))_{n \geq 0}$  has the  
 494 same law as  $(R_n^b)_{n \geq 0}$ . Note that, since they are bounded in  $L^2$ , the series

$$495 \quad \sum_{k=0}^{\infty} \frac{\xi_k^1 - 1}{\varphi_b(k)} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\xi_k^2 - 1}{\psi(k)}$$

496 converge almost surely. In particular

$$497 \quad T_n^1 = \sum_{k=0}^{n-1} \frac{1}{\varphi_b(k)} + \mathcal{O}(1) = \log n + \mathcal{O}(1), \quad \text{and} \quad T_n^2 = \sum_{k=0}^{n-1} \frac{1}{\psi(k)} + \mathcal{O}(1) = \frac{1}{\alpha} \log n + \mathcal{O}(1), \quad (15)$$

498 where the  $\mathcal{O}(1)$  are almost surely bounded. Moreover, by definition, one has

$$499 \quad \sup_{n \geq 0} |T_{\tau^1(t_n)}^1 - T_{\tau^2(t_n)}^2| \leq \sup_{n \geq 0} \max \left( \frac{\xi_n^1}{\varphi_b(n)}, \frac{\xi_n^2}{\psi(n)} \right),$$

500 from which it follows that

$$501 \quad T_{\tau^1(t_n)}^1 = T_{\tau^2(t_n)}^2 + \mathcal{O}(1),$$

502 where  $\mathcal{O}(1)$  stands for an almost surely finite random variable. Together with (15), this entails  $\tau^2(t_n) \leq$   
 503  $C_b n^\alpha$ , for all  $n \geq 0$ , and some almost surely finite random variable  $C_b$ , which concludes the proof of  
 504 Equation (14).

505 To conclude the proof of Equation (12), we only need to couple the family of processes  $(R_n^b)_{n \geq 0}$ ,  $b > 0$ ,  
 506 with  $(N_1(n))_{n \geq 0}$  so that, almost surely, there exists  $K > 0$  such that  $N_1(n) \geq R_n^K$  and  $p_n \geq q_n^K$ , for  
 507 all  $n \geq 0$ . To do this coupling, we use a sequence  $(U_n)_{n \geq 1}$  of i.i.d. uniform random variables on  $[0, 1]$ ,  
 508 independent of everything else. We start the processes so that  $N_1(0) = 0$  and  $R_0^b = 0$  for all  $b > 0$ . Then,  
 509 at each time step  $n \geq 0$ , set  $N_1(n+1) = N_1(n) + 1$  if and only if  $p_n \geq U_{n+1}$  and, similarly for all  $b > 0$ ,  
 510  $R_{n+1}^b = R_n^b + 1$  if and only if  $q_n^b \geq U_{n+1}$ . By induction on  $n$ , we can prove that, in this coupling, for all  
 511  $b \geq K_1$ , for all  $n \geq 1$ ,  $N_1(n) \geq R_n^b$ . Indeed, first note that, by Equation (11), for all  $b \geq K_1$ ,  $N_1(n) \geq R_n^b$   
 512 implies  $p_n \geq q_n^b$ . Moreover, if  $N_1(n) \geq R_n^b$  and  $p_n \geq q_n^b$ , then  $N_1(n+1) \geq R_{n+1}^b$ , which concludes the  
 513 proof by induction: we get that, for all  $b \geq K_1$ ,  $N_1(n) \geq R_n^b \geq n - C_b n^\alpha$ . This concludes the proof of  
 514 Equation (12), thus the proof of the induction step in Case 2.2, and thus the proof of Proposition 2.5  
 515 altogether.  $\square$

## 516 2.5 Proof of Theorem 1.3

517 Since the original model is a particular case of the generalised model of Section 2.4, it is enough to prove  
 518 that Theorem 1.3 holds in the generalised model.

519 By Proposition 2.5, for any edge  $e$  that is not contained in a geodesic path, one has  $W_e(n)/n \rightarrow 0$ ,  
 520 when  $n \rightarrow +\infty$ . Thus it only remains to show that, for every edge  $e$  that lies on a geodesic path,  $W_e(n)/n$   
 521 converges to some random variable  $\chi_e$ , which is almost surely non-zero.

522 The proof is done by induction on the size of  $G$ . If  $G$  has size one, the result is straightforward.  
 523 We now assume that the result holds for all series-parallel graphs of size at most  $N$ , and consider a  
 524 series-parallel graph  $G$  of size  $N+1$ . Once again, by Lemma 2.1 we know that  $G$  is the merging of two  
 525 non-empty subgraphs  $G_1$  and  $G_2$ . If  $G_1$  and  $G_2$  are in series, then the result for  $G$  follows immediately  
 526 from the induction hypothesis.

527 Let us now assume that  $G_1$  and  $G_2$  are merged in parallel. If  $h_{\min}(G_1) \neq h_{\min}(G_2)$ , and for instance  
 528 if  $h_{\min}(G_1) < h_{\min}(G_2)$ , then the proof in the previous subsection shows that a fraction  $1 - o(1)$  of the  
 529 ants chooses a path in  $G_1$ , and then the result follows from the induction hypothesis.

530 If  $h_{\min}(G_1) = h_{\min}(G_2)$ , we first show that  $\liminf N_i(n)/n > 0$ , almost surely for all  $i \in \{1, 2\}$ . To do  
 531 this, we use again Rubin's construction; the argument is very similar to the one given in Case 2.2 of the  
 532 proof of Proposition 2.5. We only briefly indicate how to adapt the proof to show that  $\liminf N_i(n)/n > 0$   
 533 in the present case. We aim at coupling the process  $(N_1(n))_{n \geq 0}$  with a family of processes  $(R_n^b)_{n \geq 0}$ ,  $b > 0$ .  
 534 We define  $\varphi_b$  as in Equation (9) and set  $\psi(i) = (i + C_2)/h_{\min}(G_1)$  for all integers  $i$  (compare with  
 535 Equation (10)). We then define  $R_n^b$  as in Equation (13). One can show that, on the one hand, for any  
 536  $b > 0$ , there exists a random variable  $c_b > 0$ , such that almost surely for all  $n \geq 1$ ,  $R_n^b \geq c_b n$ . And, on  
 537 the other hand, there exists a random  $b > 0$  such that  $N_1(n) \geq R_n^b$  for all  $n \geq 0$  almost surely. Hence,



538 we deduce that almost surely  $\liminf N_1(n)/n > 0$ , as claimed. In other words, almost surely, a positive  
 539 fraction of the ants chooses a path in  $G_1$ , and by symmetry the same holds for  $G_2$ .

540 We now show that  $N_1(n)/n$  converges almost surely when  $n$  tends to infinity. To do this, we show  
 541 that  $X(n) := N_1(n)/n$  is a stochastic approximation. Indeed, we have, for all  $n \geq 1$ ,

$$542 \quad X(n+1) = X(n) + \frac{\Delta M_n + h_n}{n+1},$$

543 where  $\Delta M_n = N_1(n+1) - N_1(n) - p_n$ , with  $p_n$  as defined in (11), and  $h_n := p_n - X(n)$ . Iterating the  
 544 above equation, we get that, for all  $n \geq 1$ ,

$$545 \quad X(n+1) = X(1) + \sum_{k=1}^n \frac{\Delta M_k + h_k}{k}.$$

546 Note that, by definition, the martingale increment  $\Delta M_k$  is bounded by 1 in absolute value, and thus the  
 547 martingale  $\sum_{k=1}^n \Delta M_k/k$ , is bounded in  $L^2$ , and hence almost surely convergent. Using the definition of  $p_n$   
 548 (see Equation (11)), together with Lemma 2.4(b), Proposition 2.5, and the fact that  $\liminf N_i(n)/n > 0$ ,  
 549 for  $i = 1, 2$ , one can show that, almost surely when  $n$  tends to infinity,  $h_n = \mathcal{O}(n^{\alpha(G_1)-1})$ , where we  
 550 recall that, by definition (see Equation (3)),  $\alpha(G_1) < 1$ . This implies that the sum  $\sum_{k=1}^n h_k/k$  is almost  
 551 surely convergent, and thus that  $X(n)$  converges almost surely, as claimed. Together with the induction  
 552 hypothesis applied to  $G_1$  and  $G_2$ , this allows us to conclude the induction step for that last case ( $G_1$  and  
 553  $G_2$  merged in parallel and  $h_{\min}(G_1) = h_{\min}(G_2)$ ).

554 Altogether, this concludes the proof of Theorem 1.3.

### 555 3 The geodesic ant process on the losange graph

556 We prove here Theorem 1.4 concerning the losange graph; in this section, we thus only consider the  
 557 (uniform-)geodesic version of the model (as discussed in Section 1.3, the rule about how to choose the  
 558 geodesic to reinforce when there are several in the trace of the walker is irrelevant here since the trace  
 559 of a walker can only contain one geodesic). The proof relies primarily on the fact that the sequence of  
 560 weights is the solution of a certain stochastic recursion formula, which we state in Lemma 3.1 below.

561 Recall Figure 4 of the losange graph, and define for  $n \geq 0$ ,

$$562 \quad \mathbf{W}(n) := (W_1(n), W_2(n), W_3(n), W_4(n), W_5(n)), \quad \text{and} \quad \hat{\mathbf{W}}(n) = \frac{\mathbf{W}(n)}{n+2}, \quad (16)$$

563 where  $W_i(n)$  denotes the weight of edge  $i$  after  $n$  walkers (or ants) have reached the food. Then for  
 564  $w = (w_1, \dots, w_5) \in [0, 1]^5$ , denote by  $p_{12}(w)$  the probability that a walker reinforces edges 1 and 2, when  
 565 the weights of the five edges of the losange graph are respectively  $w_1, \dots, w_5$ . Define similarly  $p_{135}(w)$ ,  
 566  $p_{234}(w)$ , and  $p_{45}(w)$ , and set

$$567 \quad F(w) := p_{12}(w)(1, 1, 0, 0, 0) + p_{135}(w)(1, 0, 1, 0, 1) + p_{45}(w)(0, 0, 0, 1, 1) + p_{234}(w)(0, 1, 1, 1, 0) - w. \quad (17)$$

568 Lemma 3.1 expresses the fact that the whole study of the process  $(\mathbf{W}(n))_{n \geq 0}$  takes place in the subset of  
 569  $[0, 1]^5$ , defined as

$$570 \quad \mathcal{E} := \left\{ (w_1, w_2, w_3, w_4, w_5) \in [0, 1]^5 : \begin{array}{ll} w_1 + w_4 = 1, & \text{and} \quad w_2 + w_5 = 1 \\ |w_1 - w_2| \leq w_3 & \text{and} \quad |w_5 - w_4| \leq w_3 \\ w_1 + w_2 \geq w_3 & \text{and} \quad w_4 + w_5 \geq w_3 \end{array} \right\}. \quad (18)$$

571 Let us briefly explain the restrictions above. Note that each walk can only reinforce one of the following  
 572 sets of edges:

- 573 (i) edge 1 and edge 2;  
574 (ii) edge 4 and edge 5;  
575 (iii) edge 1, edge 3 and edge 5;  
576 (iv) edge 4, edge 3 and edge 2.

577 One can see above that, at each round, precisely one of edge 1 or edge 4 is reinforced and precisely one  
578 of edge 2 or edge 5 is reinforced. Hence, we have that  $w_1 + w_4 = w_2 + w_5 = 1$ . Next, the only cases  
579 where edge 1 is reinforced but not edge 2, or edge 2 is reinforced but not edge 1 are in scenarios (iii) and  
580 (iv), in which cases edge 3 is reinforced. Therefore,  $|w_1 - w_2| \leq w_3$ , and by symmetry  $|w_4 - w_5| \leq w_3$ .  
581 Finally, again using (iii) and (iv), every time edge 3 is reinforced edge 1 or edge 2 is reinforced. Therefore  
582  $w_1 + w_2 \geq w_3$  and by symmetry  $w_4 + w_5 \geq w_3$ .

583 Using further the definition of the ant process, we obtain Lemma 3.1 below. We use now the shorthand  
584 notation  $\mathbb{E}_n$  to denote the conditional expectation with respect to the sigma-field  $\mathcal{F}_n$  (where  $(\mathcal{F}_n)_{n \geq 0}$  is  
585 the natural filtration of the process).

586 **Lemma 3.1.** *For all  $n \geq 0$ ,  $\hat{\mathbf{W}}(n) \in \mathcal{E}$ . Furthermore,*

$$587 \quad \hat{\mathbf{W}}(n+1) = \hat{\mathbf{W}}(n) + \frac{1}{n+3} (F(\hat{\mathbf{W}}(n)) + \Delta \mathbf{M}(n+1)), \quad (19)$$

588 where  $\Delta \mathbf{M}(n+1) = Y(n+1) - \mathbb{E}_n[Y(n+1)]$ , and  $Y(n+1) := \mathbf{W}(n+1) - \mathbf{W}(n)$ .

589 As mentioned in the introduction, this losange case can be seen as an intricate coupling between a  
590 biased urn (the ants that reinforce edge 3 versus all others, i.e.  $W_3(n)$  vs.  $n - W_3(n)$ ) and a standard Pólya  
591 urn (the ants that reinforce edges 1 and 2 vs. the ants that reinforce edges 4 and 5). In Subsection 3.1  
592 we treat the first urn by proving that  $W_3(n)/n$  converges to 0 almost surely, at a polynomial speed. The  
593 ‘‘Pólya’’ part is treated in two additional steps: In Subsection 3.2 we show that  $\hat{\mathbf{W}}(n)$  converges almost  
594 surely to some limit in  $[0, 1]^5$ , and, in Subsection 3.3, we prove that the limit is non-degenerate, in the  
595 sense that it does not charge the extremal points  $(1, 1, 0, 0, 0)$  and  $(0, 0, 0, 1, 1)$ . In terms of the ants, this  
596 means that the ants *find both geodesics* and not just one of them. Interestingly, ruling out these extremal  
597 cases is the most delicate part of the proof.

### 598 3.1 On the convergence of $W_3(n)/n$ to 0

599 In this section, we prove here the following result.

600 **Proposition 3.2.** *Almost surely, as  $n \rightarrow +\infty$ , one has  $W_3(n)/n \rightarrow 0$ . More precisely, there exists*  
601  *$\alpha \in (0, 1)$ , such that almost surely,*

$$602 \quad \lim_{n \rightarrow \infty} \frac{W_3(n)}{n^\alpha} = 0.$$

603 The first idea of the proof is to compare  $W_3(n)$  with the number of red balls in a two-colour Friedman-  
604 like urn defined as follows:

605 **Lemma 3.3.** *We define a Markov process  $(R_n)_{n \geq 0}$  as follows: first  $R_0 = 1$ , and for all  $n \geq 0$ , we set*  
606  *$R_{n+1} = R_n + A_{n+1}$ , where*

$$607 \quad \mathbb{P}(A_{n+1} = 1 \mid R_n) = 1 - \mathbb{P}(A_{n+1} = 0 \mid R_n) := \frac{R_n}{n+2} \cdot \frac{\left(\frac{R_n}{n+2}\right)^2 + \frac{1}{2}}{\frac{R_n}{n+2} + \frac{1}{2}}.$$

608 *Then almost surely when  $n \rightarrow +\infty$ , we have  $R_n/n \rightarrow 0$ .*

609 *Proof.* Let us define  $Z_n := R_n/(n+2)$ , for all  $n \geq 0$ . We use stochastic approximation: by definition, we  
 610 have that, for all  $n \geq 0$ ,

$$611 \quad Z_{n+1} = \frac{R_{n+1}}{n+3} = \frac{R_n + A_{n+1}}{n+3} = \frac{R_n}{n+2} \cdot \frac{n+2}{n+3} + \frac{A_{n+1}}{n+3} = Z_n + \frac{1}{n+3}(A_{n+1} - Z_n).$$

612 For  $n \geq 0$ , set  $\Delta M_{n+1} = A_{n+1} - \mathbb{E}[A_{n+1} | R_n]$ . By definition of the model, we have

$$613 \quad \mathbb{E}[A_{n+1} | R_n] = Z_n \cdot \frac{Z_n^2 + \frac{1}{2}}{Z_n + \frac{1}{2}},$$

614 implying that

$$615 \quad Z_{n+1} = Z_n + \frac{1}{n+3}(G(Z_n) + \Delta M_{n+1}),$$

616 where, for all  $x \in [0, 1]$ ,

$$617 \quad G(x) = x \cdot \frac{x^2 + \frac{1}{2}}{x + \frac{1}{2}} - x.$$

618 Note that  $G(x) \leq 0$  for all  $x \in [0, 1]$ . Thus  $(Z_n)_{n \geq 0}$  is a non-negative supermartingale, and converges  
 619 almost surely. Moreover, by definition  $|\Delta M_n| \leq 1$ , for all  $n \geq 0$ , and thus the martingale

$$620 \quad \widetilde{M}_n := \sum_{i=1}^{n-1} \frac{\Delta M_{i+1}}{i+3},$$

621 converges almost surely, since it is bounded in  $L^2$ . It follows that the series  $\sum G(Z_n)/n$  also converges  
 622 almost surely, which implies that the limit of  $(Z_n)_{n \geq 0}$  is necessarily a zero of  $G$ , that is either 0 or 1. To  
 623 see that  $Z_n \rightarrow 0$  almost surely, we couple  $(Z_n)_{n \geq 0}$  with a Pólya urn: this coupling is based on the fact  
 624 that, by definition and because  $\frac{x^2+1}{x+1} \leq 1$  for all  $x \in [0, 1]$ , we have

$$625 \quad \mathbb{P}(A_{n+1} = 1 | Z_n) \leq Z_n.$$

626 Thus if we define a process  $(U_n)_{n \geq 0}$  such that  $U_0 = Z_0$  and, for all  $n \geq 0$ ,

$$627 \quad \mathbb{P}(U_{n+1} = U_n + 1 | U_n) = 1 - \mathbb{P}(U_{n+1} = U_n | U_n) = U_n,$$

628 then  $(U_n)_{n \geq 0}$  and  $(Z_n)_{n \geq 0}$  can be coupled in a way that  $Z_n \leq U_n$  almost surely for all  $n \geq 0$ . It is known  
 629 that  $U_n \rightarrow U$  almost surely when  $n \rightarrow +\infty$ , where  $U$  is uniform on  $[0, 1]$ . Thus  $Z_n$  cannot converge to 1  
 630 and thus converges to 0 almost surely when  $n \rightarrow +\infty$ .  $\square$

631 The next step to prove Proposition 3.2 is to compute the probability that a walker reinforces the  
 632 middle edge 3. Recall the definition (18) of the set  $\mathcal{E}$ .

633 **Lemma 3.4.** *One has for all  $w \in \mathcal{E}$ ,*

$$634 \quad p_{135}(w) = \frac{w_1 w_3 w_5}{(w_2 + w_3 + w_1 w_4)(w_4 + w_5) + w_2 w_3 + w_1 w_3 w_4}.$$

635 *Proof.* We call “left” vertex the vertex linked to edges 1, 2 and 3, and “right” vertex the vertex between  
 636 edges 3, 4 and 5. To reinforce edges 1, 3 and 5, a walker has to

- 637 (i) go through edge 1 in its first step,
- 638 (ii) then, from the left vertex, reach the right vertex before going through edge 2,
- 639 (iii) finally, from the right vertex, reach the food before going through edge 2 or 4.

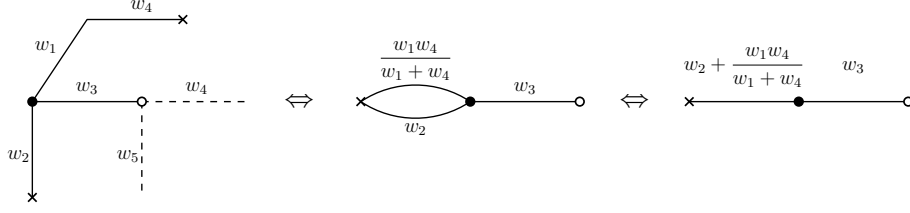


Figure 7: Calculation of the probability of  $(ii)$ , the event that a random walker starting at the black dot reaches the white dot before reaching the crosses. The dashed edges in the left-hand side picture have no effect on the calculation and can be removed. In terms of effective conductances between the black dot and the crosses and the black dot and white dot, these three graphs are equivalent.

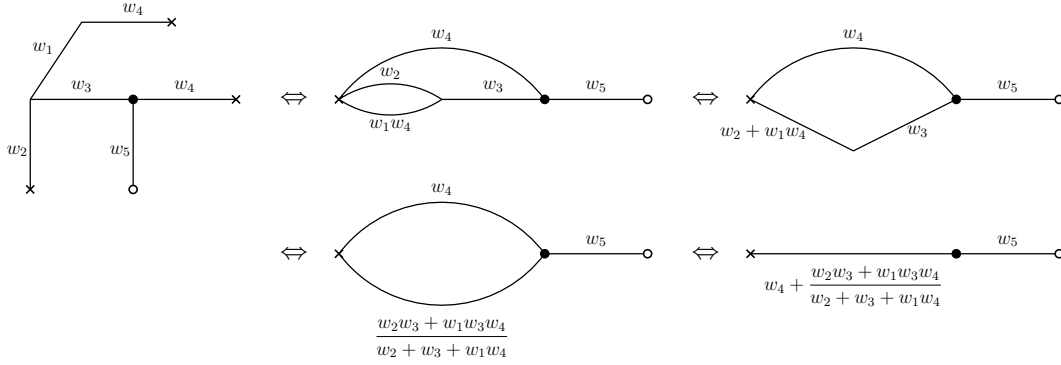


Figure 8: Calculation of the probability of  $(iii)$  in the case  $w_1 + w_4 = 1$ , the event that a random walker starting at the black dot reaches the white dot before reaching the crosses. In terms of effective conductances between the black dot and the crosses and the black dot and white dot, these five graphs are equivalent.

640 Let us denote by  $p_i$ ,  $p_{ii}$  and  $p_{iii}$  the respective probabilities of these three events; we thus have  $p_{135}(w) =$   
 641  $p_i p_{ii} p_{iii}$ . First note that

$$642 \quad p_i = \frac{w_1}{w_1 + w_4} = w_1,$$

643 using for the last equality that  $w \in \mathcal{E}$ . To calculate  $p_{ii}$  and  $p_{iii}$ , we use effective conductances. One can  
 644 check that  $p_{ii}$  is the probability that a random walker starting from the black dot in the left-hand side  
 645 of Figure 7 reached the white dot before reaching one of the crosses. In Figure 7, we use the parallel  
 646 and series formulas for effective conductances to simplify the left-hand side graph into the equivalent (in  
 647 terms of effective conductances) right-hand side graph. In the right-hand side graph, it is easy to see that  
 648 the probability to reach the white dot before the cross starting from the black dot is

$$649 \quad p_{ii} = \frac{w_3}{w_2 + w_3 + \frac{w_1 w_4}{w_1 + w_4}} = \frac{w_3}{w_2 + w_3 + w_1 w_4},$$

650 using again that  $w \in \mathcal{E}$  for the last equality. Similarly, one can check that  $p_{iii}$  is the probability that a  
 651 walker starting from the black dot in the left-hand side of Figure 8 reaches the white dot before reaching  
 652 one of the crosses. Using the calculation of effective conductances done in Figure 8, we eventually get  
 653 that

$$654 \quad p_{iii} = \frac{w_5}{w_4 + w_5 + \frac{w_2 w_3 (w_1 + w_4) + w_1 w_3 w_4}{(w_2 + w_3)(w_1 + w_4) + w_1 w_4}} = \frac{w_5}{w_4 + w_5 + \frac{w_2 w_3 + w_1 w_3 w_4}{w_2 + w_3 + w_1 w_4}},$$

655 which concludes the proof, since, for all  $w \in \mathcal{E}$ ,  $w_1 + w_4 = 1$ . □

656 We deduce the following result, proving the first part of Proposition 3.2.

657 **Lemma 3.5.** *One has for all  $w \in \mathcal{E}$ ,*

$$658 \quad p_{135}(w) + p_{234}(w) \leq w_3 \cdot \frac{w_3^2 + \frac{1}{2}}{w_3 + \frac{1}{2}},$$

659 *and as a consequence almost surely,*

$$660 \quad \lim_{n \rightarrow \infty} \frac{W_3(n)}{n} = 0.$$

661 *Proof.* The idea is the following: we run the ants walk from time 0, and simultaneously, we consider an  
 662 urn that contains black and red balls. We call this urn the “ants urn”. At time zero, we put one black  
 663 ball and one red ball in the urn, and everytime an ant reaches the food in the ants walk process, we add  
 664 a ball into the urn: this ball is red if edge number 3 has been reinforced by this ant, black otherwise. The  
 665 first part of the lemma will show that this urn can be coupled with a Friedman-like urn of Lemma 3.3 so  
 666 that there are always more red balls in the Friedman-like urn.

667 By Lemmas 3.1 and 3.4 we have for all  $w \in \mathcal{E}$ ,

$$668 \quad p_{135}(w) = \frac{w_1 w_3 w_5}{(w_2 + w_3 + w_1 w_4)(w_4 + w_5) + w_2 w_3 + w_1 w_3 w_4} \leq \frac{w_1 w_3 w_5}{(w_3 + w_2)(w_4 + w_5) + w_3 w_2}. \quad (20)$$

669 Using that  $w_3 + w_2 \geq w_1$ , and  $w_2 + w_5 = 1$ , we deduce

$$670 \quad p_{135}(w) \leq \frac{w_1 w_3 w_5}{w_3 + w_1 w_4 + w_2 w_5}. \quad (21)$$

671 By symmetry, we have that

$$672 \quad p_{234}(w) \leq \frac{w_2 w_3 w_4}{w_3 + w_1 w_4 + w_2 w_5},$$

673 and thus, the probability that the  $n$ -th walker reinforces edge 3 is at most

$$674 \quad p_{135}(w) + p_{234}(w) \leq w_3 \cdot \frac{w_1 w_5 + w_2 w_4}{w_3 + w_1 w_4 + w_2 w_5}.$$

675 Finally, we note that

$$676 \quad w_1 w_5 + w_2 w_4 = w_1 w_4 + w_2 w_5 + (w_1 - w_2)(w_5 - w_4) \leq w_1 w_4 + w_2 w_5 + w_3^2,$$

677 which entails

$$678 \quad p_{135}(w) + p_{234}(w) \leq w_3 \cdot \frac{w_1 w_4 + w_2 w_5 + w_3^2}{w_1 w_4 + w_2 w_5 + w_3} \leq w_3 \left( 1 - \frac{w_3(1 - w_3)}{w_1 w_4 + w_2 w_5 + w_3} \right).$$

679 Recalling next that, for all  $x \in [0, 1]$ ,  $x(1 - x) \leq 1/4$  and that  $w_1 + w_4 = w_2 + w_5 = 1$ , we have that  
 680  $w_1 w_4 + w_2 w_5 \leq 1/2$ , which implies

$$681 \quad p_{135}(w) + p_{234}(w) \leq w_3 \left( 1 - \frac{w_3(1 - w_3)}{w_3 + \frac{1}{2}} \right) = w_3 \cdot \frac{w_3^2 + \frac{1}{2}}{w_3 + \frac{1}{2}},$$

682 proving the first part of the lemma. Applying this with  $w = \hat{\mathbf{W}}(n)$ , we thus have proved that, at every  
 683 time step  $n$ , the probability to add a red ball in the ants-urn is at most the probability to add a red ball  
 684 in the Friedman-like urn of Lemma 3.3. Therefore, the number of red balls in the ants urn (i.e.  $W_3(n)$ )  
 685 is at most  $R_n$  at time  $n$  (for all  $n \geq 0$ ), where  $R_n$  is the quantity defined in Lemma 3.3. Thus the result  
 686 follows from Lemma 3.3.  $\square$

687 **Remark.** It is interesting to note that, in the loop-erased ant process, one has

$$688 \quad p_{135} = \frac{w_1 w_3 w_5}{w_3 + w_1 w_4 + w_2 w_5},$$

689 to compare with Equation (21). This means that Lemma 3.5 holds in this case too. However, to show  
 690 almost-sure convergence of  $\hat{\mathbf{W}}(n)$ , we need to know that the convergence of  $\hat{W}_3(n)$  to zero has polynomial  
 691 speed. This is done in the following lemma, whose proof relies on a better bound, using the equality in  
 692 Equation (20). Therefore, the fact that this better bound does not hold in the loop-erased case is the  
 693 reason why we believe that the proof of Conjecture 1.1 in that case is more intricate.

694 We now aim at bootstrapping the previous result to get a polynomial speed of convergence. For this  
 695 we will need the following fact.

696 **Lemma 3.6.** *For any  $\rho \in (0, 1/6)$ , there exists  $\varepsilon > 0$  such that for any  $w \in \mathcal{E}$  satisfying  $w_3 \leq \varepsilon$ ,*

$$697 \quad p_{135}(w) + p_{234}(w) \leq (1 - \rho)w_3.$$

698 *Proof.* By Lemma 3.4, for any  $w \in \mathcal{E}$ ,

$$699 \quad p_{135}(w) = \frac{w_1 w_3 w_5}{w_3(1 + w_4 + w_1 w_4) + w_2 w_4 + w_2 w_5 + w_1 w_4^2 + w_1 w_4 w_5}. \quad (22)$$

700 Assume that  $w_3 < 1/4$ . Let us first prove a lower bound on the denominator of (22). This denominator  
 701 is at least equal to  $w_3(1 + w_4) + w_2 w_4 + w_2 w_5$ , and we would like to prove that

$$702 \quad w_3(1 + w_4) + w_2 w_4 + w_2 w_5 \geq -w_3^2 + 2w_1 w_5. \quad (23)$$

703 Indeed, first using the fact that, for all  $w \in \mathcal{E}$ ,  $w_4 \geq w_5 - w_3$ , we get

$$704 \quad w_3(1 + w_4) + w_2 w_4 + w_2 w_5 \geq w_3(1 + w_4) + w_2(w_5 - w_3) + w_2 w_5 \geq w_3(1 + w_4 - w_2) + 2w_2 w_5.$$

705 Now, using the facts that, for all  $w \in \mathcal{E}$ ,  $w_2 \geq w_1 - w_3$ ,  $w_4 - w_5 \geq -w_3$ , and  $1 - w_2 = w_5$ , we get that

$$706 \quad w_3(1 + w_4) + w_2 w_4 + w_2 w_5 \geq w_3(w_4 + w_5) + 2(w_1 - w_3)w_5 \geq w_3(w_4 - w_5) + 2w_1 w_5 \geq -w_3^2 + 2w_1 w_5,$$

707 which concludes the proof of (23).

708 Next we distinguish two cases: either  $w_2 \geq w_1$  or  $w_2 < w_1$ .

709 • We first treat the case when  $w_1 \geq w_2$  and, as a consequence,  $w_5 \geq w_4$ . Plugging Equation (23) into  
 710 Equation (22), we thus get

$$711 \quad p_{135}(w) \leq \frac{w_1 w_3 w_5}{2w_1 w_5 - w_3^2}.$$

712 Since  $w \in \mathcal{E}$ , we have  $w_1 + w_2 \geq w_3$ , which, since  $w_1 \geq w_2$  implies  $w_1 \geq w_3/2$ . Similarly, the facts that  
 713  $w_4 + w_5 \geq w_3$  and  $w_5 \geq w_4$  imply that  $w_5 \geq w_3/2$ . Moreover, since  $w \in \mathcal{E}$ , we have  $w_1 + w_4 = 1$ , and  
 714 thus either  $w_1 \geq 1/2$  or  $w_4 \geq 1/2$ . If  $w_1 \geq 1/2$  then we conclude that  $w_1 w_5 \geq w_5/2 \geq w_3/4$ . If  $w_4 \geq 1/2$ ,  
 715 then  $w_5 \geq w_4 \geq 1/2$ , and we also get  $w_1 w_5 \geq w_3/4$  in this case. Therefore, in both cases ( $w_1 \geq 1/2$  and  
 716  $w_4 \geq 1/2$ ), using the fact that  $\frac{1}{1-x} \leq 1 + 2x$  for all  $0 \leq x \leq 1/2$ , we get

$$717 \quad p_{135}(w) \leq \frac{w_3}{2(1 - 2w_3)} \leq \frac{w_3}{2} + 2w_3^2,$$

718 as long as  $w_3 < 1/4$ .

719 • We now treat the case when  $w_2 \geq w_1$ , which implies  $w_4 \geq w_5$ . In that case, it is straightforward to  
 720 see that the denominator in (22) is at least  $w_2(w_4 + w_5) \geq 2w_1 w_5$ , which implies  $p_{135}(w) \leq w_3/2$ .

721 By the two cases above, we have thus proved that, for all  $w \in \mathcal{E}$  such that  $w_3 \leq 1/4$ ,

$$722 \quad p_{135}(w) \leq \frac{w_3}{2} + 2w_3^2.$$

723 Note that by symmetry the same inequality holds for  $p_{234}(w)$ , i.e. for all  $w \in \mathcal{E}$ ,

$$724 \quad \max(p_{135}(w), p_{234}(w)) \leq \frac{w_3}{2} + 2w_3^2, \quad (24)$$

725 but this is not yet enough to conclude the proof: we need to get a better upper bound by taking into  
726 account the terms in the denominator of Equation (22) that we previously neglected.

727 To do that, we again distinguish two cases: first assume that  $w_4 \geq 1/2$ . In this case, using the fact  
728 that for all  $w \in \mathcal{E}$ ,  $w_4 \geq w_5 - w_3$ , we get

$$729 \quad w_1 w_4^2 \geq w_1 w_4 w_5 - w_1 w_3 w_4 \geq \frac{1}{2} w_1 w_5 - w_1 w_3 w_4,$$

730 and since we assume that  $w_4 \geq 1/2$ , we also get

$$731 \quad w_1 w_4 w_5 \geq \frac{1}{2} w_1 w_5.$$

732 Now if in addition  $w_5 \geq w_4$ , one has  $w_1 \geq w_2$  and thus  $w_1 \geq w_3/2$ , as well as  $w_5 - w_4 \leq w_3 \leq 4w_1 w_5$ .  
733 This, together with the last two displays and (22) implies

$$734 \quad p_{135}(w) \leq \frac{w_1 w_3 w_5}{3w_1 w_5 + w_3(w_4 - w_5)} \leq \frac{w_3}{3} \cdot \frac{1}{1 - \frac{4w_3}{3}} \leq \frac{w_3}{3} \left(1 + \frac{8w_3}{3}\right),$$

735 as long as  $w_3 \leq 3/8$ . We thus get that, for all  $w \in \mathcal{E}$  such that  $w_3 \leq 3/8$ , and  $w_4 \geq 1/2$ ,

$$736 \quad p_{135}(w) \leq \frac{w_3}{3} + w_3^2.$$

737 We now need to treat the case when  $w_4 \leq 1/2$ . In that case,  $w_1 \geq 1/2$  and we get, by symmetry,

$$738 \quad p_{234}(w) \leq \frac{w_3}{3} + w_3^2.$$

739 In both cases ( $w_4 \geq 1/2$  and  $w_1 \geq 1/2$ ), using Equation (24), we get

$$740 \quad p_{135}(w) + p_{234}(w) \leq \left(\frac{1}{2} + \frac{1}{3}\right)w_3 + 3w_3^2 = \frac{5w_3}{6} + 3w_3^2,$$

741 as long as  $w_3 \leq 1/6$ , and the lemma follows. □

742 **Lemma 3.7.** *Almost surely, for any  $\alpha > 5/6$ ,*

$$743 \quad \lim_{n \rightarrow \infty} \frac{W_3(n)}{n^\alpha} = 0.$$

744 *Proof.* Fix  $\alpha > 5/6$ , and set  $Z_n := n^{-\alpha} \cdot W_3(n)$ . Using Equation (19), we get, for all  $n \geq 1$ ,

$$745 \quad Z_{n+1} = Z_n \cdot \left(1 - \frac{1}{n+1}\right)^\alpha + \frac{W_3(n+1) - W_3(n)}{(n+1)^\alpha} = Z_n + \frac{r_n}{n+1} + \frac{\Delta M_3(n+1)}{(n+1)^\alpha},$$

746 with  $r_n = (n+1)^{1-\alpha} \mathbb{E}_n Y_3(n+1) - \alpha Z_n + \mathcal{O}(Z_n/n)$ , almost surely when  $n \rightarrow +\infty$ . Recall that  $Y(n+1) =$   
747  $\mathbf{W}(n+1) - \mathbf{W}(n)$ , and thus

$$748 \quad \mathbb{E}_n Y_3(n+1) = p_{135}(\hat{\mathbf{W}}(n)) + p_{234}(\hat{\mathbf{W}}(n)) \leq \alpha \hat{W}_3(n),$$

749 almost surely for all  $n$  large enough, by Lemmas 3.5 and 3.6. Therefore, almost surely for  $n$  large enough,  
 750  $r_n \leq -\delta Z_n$ , for some constant  $\delta > 0$ . As a consequence, almost surely there exists  $m \geq 1$ , such that for  
 751 all  $n > m$ ,

$$752 \quad Z_n \leq \gamma_{m,n} \cdot Z_m + \sum_{i=m+1}^n \frac{\gamma_{i,n} \cdot \Delta M_3(i)}{i^\alpha},$$

753 where  $\gamma_{i,n} := \prod_{j=i+1}^n (1 - \frac{\delta}{j})$ , for all  $i \leq n$  (with the convention that  $\gamma_{n,n} = 1$ ). Recall that by definition  
 754  $|\Delta M_3(i)| \leq 1$ , almost surely for all  $i \geq 1$ . Thus, by Doob's  $L^2$ -inequality, one has as  $m \rightarrow +\infty$ ,

$$755 \quad \mathbb{P} \left( \sup_{n \geq m} \left| \sum_{i=m+1}^n \frac{\gamma_{i,n} \cdot \Delta M_3(i)}{i^\alpha} \right| \geq \frac{1}{m^{2\alpha-1-\frac{3}{5}}} \right) = \mathcal{O} \left( \frac{1}{m^{6/5}} \right).$$

756 By Borel-Cantelli, we deduce that almost surely, one has for all  $m$  large enough,

$$757 \quad \sup_{n \geq m} \left| \sum_{i=m+1}^n \frac{\gamma_{i,n} \cdot \Delta M_3(i)}{i^\alpha} \right| \leq \frac{1}{m^{2\alpha-1-\frac{3}{5}}}.$$

758 The lemma follows, since  $2\alpha - 1 - \frac{3}{5} > 0$ , and for any fixed  $m \geq 1$ ,  $\gamma_{m,n} \rightarrow 0$ , as  $n \rightarrow \infty$ . □

### 759 3.2 Convergence of $\hat{W}(n)$

760 Our next goal is to prove the following proposition.

761 **Proposition 3.8.** *Almost surely, there exists some (random) real  $\chi \in [0, 1]$ , such that as  $n \rightarrow \infty$ ,*

$$762 \quad \frac{W_i(n)}{n} \rightarrow \chi, \quad \forall i = 1, 2, \quad \text{and} \quad \frac{W_i(n)}{n} \rightarrow 1 - \chi, \quad \forall i = 4, 5.$$

763 We start with a computation giving the probability to reinforce edge 2, which is similar to Lemma 3.4.

764 **Lemma 3.9.** *One has for all  $w \in \mathcal{E}$ ,*

$$765 \quad p_{12}(w) + p_{234}(w) = \frac{w_2 w_3 + w_1 w_2 w_5 + w_1 w_2 w_4}{w_3 + w_2 w_5 + w_1 w_4}.$$

767 *Proof.* Note that  $p_{12}(w) + p_{234}(w)$  is equal to the probability that the last step before reaching the vertex  
 768  $F$  is through edge 2. Let us compute this probability by decomposing with respect to the first step, which  
 769 is either through edge 1 (jumping on the left vertex), or through edge 4 (jumping on the right vertex),  
 770 hence we will write

$$771 \quad p_{12}(w) + p_{234}(w) = p^\ell(w) + p^r(w). \quad (25)$$

772 For  $w \in \mathcal{E}$ , the probability to jump on the left vertex is  $w_1$ , and once on the left vertex, we need  
 773 to compute the probability to cross edge 2 before crossing edge 5, which is easily done through graph  
 774 transformations similar to those done in the proof of Lemma 3.4; see Figure 9. One obtains:

$$775 \quad p^\ell(w) = w_1 \times \frac{w_2(w_3 + w_5 + w_1 w_4)}{w_2(w_3 + w_5 + w_1 w_4) + w_5(w_3 + w_1 w_4)} \quad (26)$$

$$776 \quad = w_1 \times \frac{w_2 w_3 + w_2 w_5 + w_1 w_2 w_4}{w_3 + w_2 w_5 + w_1 w_4}, \quad (27)$$

778 where we used that  $w_2 + w_5 = 1$ .

779 Now, using symmetry, one has

$$780 \quad p^r(w) = w_4 \times \left( 1 - \frac{w_5 w_3 + w_2 w_5 + w_1 w_5 w_4}{w_3 + w_2 w_5 + w_1 w_4} \right) \quad (28)$$

$$781 \quad = w_4 \times \frac{w_2 w_3 + w_1 w_2 w_4}{w_3 + w_2 w_5 + w_1 w_4}. \quad (29)$$

783 One can now easily conclude using (25), by adding up (27) with (29) and using that  $w_1 + w_4 = 1$ . □



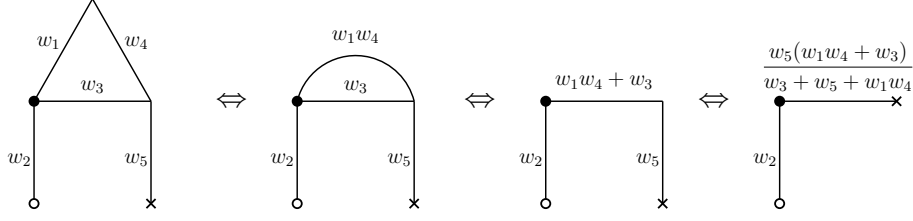


Figure 9: Calculation of  $p^\ell(w)/w_1$  (i.e. the probability to reach the circled vertex before the crossed vertex starting from the black vertex) for  $w \in \mathcal{E}$  (in particular, we use  $w_1 + w_4 = 1$ ).

784 We next deduce the following bound on  $F_2(w)$  (the second coordinate of the function  $F(w)$  from (17)).

785 **Lemma 3.10.** *For any  $w \in \mathcal{E}$ , we have*

786 
$$|F_2(w)| \leq \frac{w_3}{2}.$$

787 *Proof.* By Lemma 3.9, for any  $w \in \mathcal{E}$ ,

788 
$$F_2(w) = p_{12}(w) + p_{234}(w) - w_2 = \frac{(w_1 - w_2)w_2w_5}{w_3 + w_2w_5 + w_1w_4}. \quad (30)$$

789 Note now that since  $w_1 - w_2 = w_5 - w_4$ , either  $w_1 \geq w_2$ , or  $w_4 \geq w_5$ . In the first case, using also that  
790  $w_4 \geq w_5 - w_3$ , we deduce  $w_3 + w_1w_4 \geq w_2w_5$ . By symmetry, the same holds when  $w_4 \geq w_5$ . We thus get

791 
$$|F_2(w)| \leq \frac{|w_1 - w_2|}{2} \leq \frac{w_3}{2}, \quad \text{for all } w \in \mathcal{E},$$

792 where we have used  $|w_1 - w_2| \leq w_3$  in the second inequality. □

793 *Proof of Proposition 3.8.* Iterating Equation (19), we get that, for all  $n \geq 0$

794 
$$\hat{\mathbf{W}}(n) = \hat{\mathbf{W}}(0) + \sum_{i=0}^{n-1} \frac{1}{i+3} (F(\hat{\mathbf{W}}(i)) + \Delta \mathbf{M}(i+1)). \quad (31)$$

795 where we recall that  $\Delta \mathbf{M}(n+1) := Y(n+1) - \mathbb{E}_n Y(n+1)$  with  $Y(n+1) := \mathbf{W}(n+1) - \mathbf{W}(n)$ , and  
796 where  $F$  is defined in Equation (17). By definition of the model,  $\|Y(n+1)\|_1 \leq 3$  almost surely, and thus  
797  $\|\Delta \mathbf{M}(i+1)\|_1 \leq 3$  almost surely, which implies that the martingale

798 
$$\hat{\mathbf{M}}(n) := \sum_{i=0}^{n-1} \frac{\Delta \mathbf{M}(i+1)}{i+3}$$

799 is bounded in  $L^2$  and thus converges almost surely when  $n \rightarrow +\infty$ . By Lemma 3.1,  $\hat{\mathbf{W}}(n) \in \mathcal{E}$ , for all  
800  $n \geq 0$ . Thus Lemma 3.10 gives  $|F_2(\hat{\mathbf{W}}(n))| \leq \hat{W}_3(n)/2$ , for all  $n \geq 0$ , which implies using Lemma 3.7  
801 that

802 
$$\hat{W}_2(n) = \hat{W}_2(0) + \sum_{i=0}^{n-1} \frac{F_2(\hat{\mathbf{W}}(i))}{i+3} + \sum_{i=0}^{n-1} \frac{\Delta M_2(i+1)}{i+3},$$

803 converges almost surely when  $n \rightarrow +\infty$ . The proposition follows, since by Lemma 3.5, one has  $\hat{W}_1(n) -$   
804  $\hat{W}_2(n) \rightarrow 0$ , and by Lemma 3.1, one has  $\hat{W}_4(n) = 1 - \hat{W}_1(n)$ , and  $\hat{W}_5(n) = 1 - \hat{W}_2(n)$ , for all  $n \geq 0$ . □

805 **3.3 On the absence of convergence to 0 or 1**

806 The last step of the proof is to exclude the convergence toward an extremal point, that is we prove the  
807 following proposition.

808 **Proposition 3.11.** *Almost surely,*

809 
$$\lim_{n \rightarrow \infty} \frac{W_1(n)}{n} \notin \{0, 1\}.$$

810 Note that by symmetry it suffices to exclude the possibility of converging to 1. We prove this by  
811 contradiction, and start with the following fact.

812 **Lemma 3.12.** *For all  $\alpha \in (0, 1)$ , on the event where*

813 
$$\lim_{n \rightarrow \infty} \frac{W_1(n)}{n} = 1, \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{W_3(n)}{n^\alpha} = 0,$$

814 *both hold, we have almost surely for any  $\beta > \alpha$ ,*

815 
$$\lim_{n \rightarrow +\infty} \frac{W_5(n)}{n^\beta} = 0.$$

816 *Proof.* Fix  $\alpha \in (0, 1)$  and assume that both  $W_1(n)/n \rightarrow 1$  and  $W_3(n)/n^\alpha \rightarrow 0$  when  $n \rightarrow +\infty$ . Assume by  
817 contradiction that there exists  $\beta > \alpha$ , such that  $\limsup_{n \rightarrow +\infty} W_5(n)/n^\beta > 0$ . Without loss of generality  
818 one can even assume that  $\limsup_{n \rightarrow +\infty} W_5(n)/n^\beta > 1$ , by taking a smaller  $\beta$  if necessary. In other words,  
819 letting

820 
$$E := \left\{ \lim_{n \rightarrow \infty} \frac{W_3(n)}{n^\alpha} = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{W_1(n)}{n} = 1 \right\}, \quad \text{and} \quad E' := E \cap \left\{ \limsup_{n \rightarrow +\infty} \frac{W_5(n)}{n^\beta} > 1 \right\},$$

821 our aim is to show that  $\mathbb{P}(E') = 0$ .

822 For  $m \geq 1$  integer, define

823 
$$E_m := \{W_3(n) \leq n^\alpha, \text{ and } W_2(n) \geq 3(n+2)/4 \text{ for all } n \geq m\}.$$

824 By definition, and using that  $W_2(m) \geq W_1(m) - W_3(m)$ , for all  $m \geq 0$ , one has that  $E \subset \cup_m E_m$ , and  
825 therefore

826 
$$\lim_{m \rightarrow \infty} \mathbb{P}(E \cap E_m^c) = 0.$$

827 Thus it amounts to show that

828 
$$\lim_{m \rightarrow \infty} \mathbb{P}(E_m \cap E') = 0.$$

829 Note now that by conditioning with respect to the first time  $n \geq m$  when  $W_5(n) \geq n^\beta$ , it suffices in fact  
830 to show that almost surely

831 
$$\lim_{m \rightarrow \infty} \mathbb{P}(E_m \cap E \mid \mathcal{F}_m) \cdot \mathbf{1}\{W_5(m) \geq m^\beta\} = 0, \tag{32}$$

832 where  $\mathcal{F}_m = \sigma(\mathbf{W}(0), \dots, \mathbf{W}(m))$ . Thus the rest of the proof consists in proving (32). The idea is to show  
833 that for any integer  $m \geq 1$ , on the event that  $\{W_5(m) \geq m^\beta\}$ , the process  $(W_2(n))_{n \geq m}$  can be coupled  
834 with another process  $(R_n)_{n \geq m}$ , in a way that outside an event with vanishing probability as  $m \rightarrow \infty$ , one  
835 has  $W_2(n) \leq R_n$  for all  $n \geq m$ , and  $\limsup_{n \rightarrow \infty} R_n/n < 1$ , from which (32) follows.

836 We proceed with the details now. Fix  $\gamma \in (0, 1)$ , such that  $1 + \alpha < \beta + \gamma$ . Let  $m \geq 1$  be given, and  
837 conditionally on  $\mathcal{F}_m$ , we define the process  $(R_n)_{n \geq m}$  as follows:  $R_m = W_2(m)$ , and for all  $n \geq m$ ,

838 
$$q_n := \mathbb{P}(R_{n+1} = R_n + 1 \mid \mathcal{G}_n) = 1 - \mathbb{P}(R_{n+1} = R_n \mid \mathcal{G}_n) = \frac{R_n + R_n^\gamma}{n + R_n^\gamma}, \tag{33}$$

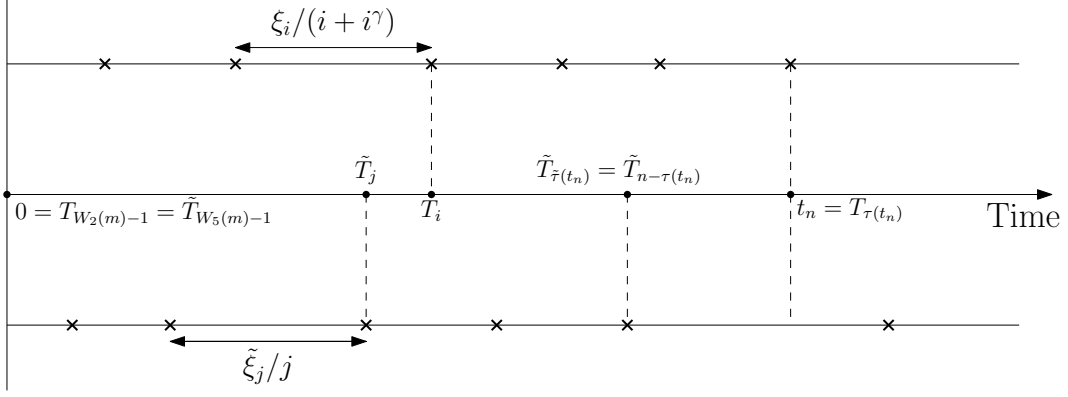


Figure 10: Rubin's construction for the proof of Lemma 3.12.

839 where  $\mathcal{G}_n = \mathcal{F}_m \vee \sigma(R_m, \dots, R_n)$ .

840 • First, we prove that, for all  $m \geq 1$ , if we set

$$841 \quad \mathcal{A}_m := \left\{ \inf_{n \geq m} \frac{R_n}{n} > \frac{3}{5} \right\} \cap \left\{ \inf_{n \geq m} \frac{n - R_n}{n} > \frac{3}{5m^{1-\beta}} \right\}, \quad (34)$$

842 then almost surely on the event  $\{W_5(m) \geq m^\beta\} \cap \{W_2(m) \geq 3(m+2)/4\}$ , one has

$$843 \quad \mathbb{P}(\mathcal{A}_m^c \mid \mathcal{F}_m) = \mathcal{O}(m^{-\delta}), \quad (35)$$

844 where the implicit constant in the  $\mathcal{O}$  is deterministic, and  $\delta = \delta(\beta, \gamma)$  is some positive constant depending  
 845 only on  $\beta$  and  $\gamma$ . To do this, we use again Rubin's construction; see Figure 10: Let  $(\xi_i)_{i \geq 1}$  and  $(\tilde{\xi}_i)_{i \geq 1}$  be  
 846 two independent sequences of independent exponential random variables with mean 1 (also independent  
 847 of the process  $(W_2(n))_{n \geq 1}$ ). For all  $m \geq 0$  and  $i \geq 0$ , set

$$848 \quad T_i := \sum_{j=W_2(m)}^i \frac{\xi_j}{j + j^\gamma}, \quad \text{and} \quad \tilde{T}_i := \sum_{j=W_5(m)}^i \frac{\tilde{\xi}_j}{j},$$

849 with the convention that  $T_i = 0$  for  $i < W_2(m)$ , and  $\tilde{T}_i = 0$ , for  $i < W_5(m)$ . For all  $t \geq 0$ , set

$$850 \quad \tau(t) := \sup\{i \geq 0 : T_i \leq t\}, \quad \text{and} \quad \tilde{\tau}(t) := \sup\{i \geq 0 : \tilde{T}_i \leq t\},$$

851 and for all  $n \geq m$ ,

$$852 \quad t_n := \inf\{t \geq m : \tau(t) + \tilde{\tau}(t) \geq n\}.$$

853 Standard properties of independent exponential random variables imply that  $(\tau(t_n))_{n \geq m}$  and  $(R_n)_{n \geq m}$   
 854 have the same law. Note that for all  $m \geq 1$ , and  $i \geq W_2(m)$ ,

$$855 \quad T_i = M_i + \log\left(\frac{i}{W_2(m)}\right) + \mathcal{O}\left(\frac{1}{W_2(m)^{1-\gamma}}\right), \quad \text{with} \quad M_i := \sum_{j=W_2(m)}^i \frac{\xi_j - 1}{j + j^\gamma}, \quad (36)$$

856 when  $m \rightarrow +\infty$ , and for all  $i \geq W_5(m)$ ,

$$857 \quad \tilde{T}_i = \tilde{M}_i + \log\left(\frac{i}{W_5(m)}\right) + \mathcal{O}\left(\frac{1}{W_5(m)}\right), \quad \text{with} \quad \tilde{M}_i := \sum_{j=W_5(m)}^i \frac{\tilde{\xi}_j - 1}{j}, \quad (37)$$

858 when  $m \rightarrow +\infty$ . By Doob's  $L^2$ -maximal inequality, we get that, almost surely

$$859 \quad \mathbb{P}\left(\sup_{i \geq W_2(m)} |M_i| > \frac{1}{W_2(m)^{1/4}} \mid W_2(m)\right) \leq 4W_2(m)^{1/2} \sum_{i \geq W_2(m)} \frac{1}{i^2} = \mathcal{O}\left(\frac{1}{W_2(m)^{1/2}}\right), \quad (38)$$

860 when  $m \rightarrow +\infty$ , and similarly,

$$861 \quad \mathbb{P}\left(\sup_{i \geq W_5(m)} |\tilde{M}_i| > \frac{1}{W_5(m)^{1/4}} \mid W_5(m)\right) = \mathcal{O}\left(\frac{1}{W_5(m)^{1/2}}\right), \quad (39)$$

862 when  $m \rightarrow +\infty$ . Moreover, by definition

$$863 \quad T_{\tau(t_n)} = \tilde{T}_{n-\tau(t_n)} + \mathcal{O}(\Gamma_m + \tilde{\Gamma}_m), \quad (40)$$

864 with

$$865 \quad \Gamma_m = \sup_{j \geq W_2(m)} \xi_j/j, \quad \text{and} \quad \tilde{\Gamma}_m = \sup_{j \geq W_5(m)} \tilde{\xi}_j/j.$$

866 Note that, for all  $m$  large enough,

$$867 \quad \mathbb{P}\left(\Gamma_m > \frac{1}{W_2(m)^{1/2}} \mid W_2(m)\right) \leq \exp\left(-\frac{\sqrt{W_2(m)}}{2}\right) \quad \text{and} \quad \mathbb{P}\left(\tilde{\Gamma}_m > \frac{1}{W_5(m)^{1/2}} \mid W_5(m)\right) \leq \exp\left(-\frac{\sqrt{W_5(m)}}{2}\right). \quad (41)$$

868 Taking the exponential in Equation (40) gives

$$869 \quad \frac{\tau(t_n)}{n - \tau(t_n)} \cdot \frac{W_5(m)}{W_2(m)} = \exp(\varepsilon_n), \quad (42)$$

870 where, by Equations (36), (37), (38), (39) and (41), there exists  $\delta = \delta(\beta, \gamma) > 0$  such that almost surely  
871 on the event  $\{W_5(m) \geq m^\beta\} \cap \{W_2(m) \geq 3(m+2)/4\}$ ,

$$872 \quad \mathbb{P}(\sup_{n \geq m} |\varepsilon_n| > m^{-\delta} \mid \mathcal{F}_m) = \mathcal{O}(m^{-\delta}).$$

873 Since, by Lemma 3.1,  $W_5(m) = m + 2 - W_2(m)$ , we get that on the event  $\{W_2(m) \geq 3(m+2)/4\}$ , one  
874 has  $W_5(m)/W_2(m) \leq 1/3$ , and thus, by Equation (42),

$$875 \quad \tau(t_n) \geq (n - \tau(t_n))3e^{\varepsilon_n} \quad \implies \quad \tau(t_n) \geq \frac{3e^{\varepsilon_n}n}{1 + 3e^{\varepsilon_n}} = \frac{3n(1 - \mathcal{O}(m^{-\delta}))}{4}, \quad (43)$$

876 where the last equality holds on an event of probability at least  $1 - \mathcal{O}(m^{-\delta})$  when  $m \rightarrow +\infty$ . Similarly,  
877 on the event  $\{W_5(m) \geq m^\beta\}$ , we have  $W_5(m)/W_2(m) \geq m^{\beta-1}$ , and thus, by Equation (42),

$$878 \quad (n - \tau(t_n))m^{1-\beta}e^{\varepsilon_n} \geq \tau(t_n) \geq \frac{3n(1 - \mathcal{O}(m^{-\delta}))}{4},$$

879 where the last inequality comes from Equation (43). This implies

$$880 \quad n - \tau(t_n) \geq \frac{3n}{4m^{1-\beta}}(1 - \mathcal{O}(m^{-\delta})),$$

881 when  $m \rightarrow +\infty$ , on an event of probability at least  $1 - \mathcal{O}(m^{-\delta})$ . Since  $(R_n)_{n \geq m}$  and  $(\tau(t_n))_{n \geq m}$  have the  
882 same law by construction, this concludes the proof of (35).

883 • To conclude we just need to show that there exists a coupling of  $(W_2(n))_{n \geq m}$  and  $(R_n)_{n \geq m}$ , such  
884 that almost surely on the event  $\mathcal{A}_m \cap E_m$  (see Equation (34) for the definition of  $\mathcal{A}_m$ ), one has  $W_2(n) \leq R_n$

885 for all  $n \geq m$ , at least for  $m$  large enough. Indeed, this would prove that on  $\mathcal{A}_m \cap E_m$ , the sequence  
 886  $(W_1(n)/n)_{n \geq m}$  cannot converge to 1, or otherwise stated that for all  $m$  large enough, almost surely,

$$887 \quad \mathbb{P}(\mathcal{A}_m \cap E_m \cap E \mid \mathcal{F}_m) = 0.$$

888 Together with (35), this would conclude the proof of (32). So let us prove the existence of the desired  
 889 coupling now.

890 Recall that, by Lemma 3.10, for all  $n \geq m$ , on the event  $\{W_3(n) \leq n^\alpha\}$ ,

$$891 \quad p_n := \mathbb{P}(W_2(n+1) = W_2(n) + 1 \mid \mathcal{F}_n) \leq \frac{W_2(n) + n^\alpha}{n+2}.$$

892 To show that our coupling exists, it is enough to prove that, for all  $n \geq m$ , if  $W_2(n) \leq R_n$ , then  $p_n \leq q_n$ ,  
 893 where  $q_n$  is defined in Equation (33). Indeed, if  $W_2(n) \leq R_n$  and  $p_n \leq q_n$ , then there exists a one-step  
 894 coupling such that  $W_2(n+1) \leq R_{n+1}$ , and we can proceed by induction. Note that  $q_n \geq p_n$  is implied by

$$895 \quad (n+2)(R_n + R_n^\gamma) \geq (n + R_n^\gamma)(W_2(n) + n^\alpha).$$

896 Developing and using the induction hypothesis (i.e.  $W_2(n) \leq R_n$ ), it suffices to show that

$$897 \quad R_n^\gamma(n - R_n - n^\alpha) \geq n^{1+\alpha},$$

898 which is indeed true on  $\mathcal{A}_m$ , since on this event

$$899 \quad R_n^\gamma(n - R_n - n^\alpha) \geq \frac{3}{5}n^\gamma \left( \frac{3}{5}n^\beta - n^\alpha \right),$$

900 which is well larger than  $n^{1+\alpha}$ , for all  $n$  large enough, since by hypothesis  $\gamma \in (0, 1)$  and  $\gamma + \beta > 1 + \alpha$ .  
 901 This concludes the proof of (32), and of the lemma.  $\square$

902 **Lemma 3.13.** *Let  $\alpha \in (0, 1)$  be given. On the event*

$$903 \quad \mathcal{A}(\alpha) = \{W_i(n) = \mathcal{O}(n^\alpha), \text{ for } i = 3, 5\}$$

904 *one has almost surely for any  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ ,*

$$905 \quad W_4(n) + W_5(n) \leq W_3(n) + \mathcal{O}(n^\beta).$$

906 For the proof of this lemma we need some intermediate results. The first one gives a lower bound on  
 907  $F_4 + F_5 - F_3$ .

908 **Lemma 3.14.** *For all  $w \in \mathcal{E}$ , such that  $w_5 \leq 1/2$ , we have*

$$909 \quad F_4(w) + F_5(w) - F_3(w) \geq -8(w_3^2 + w_5^2).$$

910 *Proof.* Note that

$$911 \quad F_4(w) + F_5(w) - F_3(w) = 2F_5(w) - 2p_{135}(w) + w_5 - w_4 + w_3.$$

912 Recall that by Equation (30) (using the symmetry of the model), we have

$$913 \quad F_5(w) = \frac{(w_4 - w_5)w_2w_5}{w_3 + w_1w_4 + w_2w_5}.$$

914 Also recall that (see Equation (22)),

$$915 \quad p_{135}(w) = \frac{w_1w_3w_5}{w_3(1 + w_4 + w_1w_4) + w_2w_4 + w_2w_5 + w_1w_4^2 + w_1w_4w_5}$$

$$916 \quad \leq \frac{w_1w_3w_5}{w_3 + (w_3 + w_2)w_4 + w_2w_5} \leq \frac{w_1w_3w_5}{w_3 + w_1w_4 + w_2w_5},$$

$$917$$

918 where we have used in the last inequality that  $w_1 \leq w_2 + w_3$  for all  $w \in \mathcal{E}$ . Using again that  $w_1 \leq w_2 + w_3$   
919 for all  $w \in \mathcal{E}$ , and the fact that  $w_3 + w_1w_4 + w_2w_5 \geq w_5(1 - w_5) \geq w_5/2$ , for all  $w \in \mathcal{E}$  such that  $w_5 \leq 1/2$ ,  
920 we get that

$$921 \quad p_{135}(w) \leq \frac{w_2w_3w_5}{w_3 + w_1w_4 + w_2w_5} + 2w_3^2.$$

922 Therefore,

$$923 \quad F_5(w) - p_{135}(w) \geq -\frac{w_5w_2(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} - 2w_3^2,$$

924 and thus

$$925 \quad \begin{aligned} F_4(w) + F_5(w) - F_3(w) &\geq \frac{(w_3 + w_1w_4 - w_2w_5)(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} - 4w_3^2 \\ 926 \quad &\geq \frac{(w_3 + w_4 - w_5)(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} + \frac{(w_5(1 - w_2) - w_4(1 - w_1))(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} - 4w_3^2, \end{aligned} \quad (44)$$

928 Recall that, for all  $w \in \mathcal{E}$ ,  $w_1 + w_4 = w_2 + w_5 = 1$ , and thus  $w_5(1 - w_2) - w_4(1 - w_1) = w_5^2 - w_4^2 \geq -w_4^2$ .  
929 As a consequence, for all  $w_5 \leq 1/2$ ,

$$930 \quad \frac{(w_5(1 - w_2) - w_4(1 - w_1))(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} \geq -\frac{w_4^2(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} \geq -\frac{w_4^2(w_3 + w_5)}{w_3 + w_1w_4 + w_5 - w_5^2} \geq -2w_4^2,$$

931 where we used that, as  $w_5 \leq 1/2$ ,  $w_3 + w_1w_4 + w_5 - w_5^2 \geq w_3 + w_5(1 - w_5) \geq (w_3 + w_5)/2$ . Then from  
932 Equation (44), we get

$$933 \quad \begin{aligned} F_4(w) + F_5(w) - F_3(w) &\geq \frac{(w_3 + w_4 - w_5)(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} - 4w_3^2 - 2w_4^2 \\ 934 \quad &\geq \frac{w_3^2 - (w_4 - w_5)^2}{w_3 + w_1w_4 + w_2w_5} - 4w_3^2 - 2w_4^2 \geq -4w_3^2 - 2(w_3 + w_5)^2, \end{aligned}$$

936 using that  $|w_4 - w_5| \leq w_3$ , for all  $w \in \mathcal{E}$ . This concludes the proof because  $(w_3 + w_5)^2 \leq 2w_3^2 + 2w_5^2$ .  $\square$

937 The second result we shall need is the following general fact, which will be used at several places  
938 during the rest of the proof. For a process  $(M_n)_{n \geq 0}$ , we write  $\Delta M_n := M_{n+1} - M_n$ , for all  $n \geq 0$ .

939 **Lemma 3.15.** *Let  $a, b, c \in (0, 1)$ , be such that  $b < a$  and  $1 < 2a + c$ .*

940 **(i)** *Let  $(A_n)_{n \geq 1}$  be a sequence of real random variables. On the event  $\{A_n = \mathcal{O}(n^{b-1})\}$ , we have almost  
941 surely as  $m \rightarrow +\infty$ ,*

$$942 \quad \sup_{n \geq m} \sum_{i=m}^n \frac{A_i}{(i+3)^a} = \mathcal{O}(m^{b-a}).$$

943 **(ii)** *Let  $(M_n)_{n \leq 0}$  be a real martingale such that  $|\Delta M_n| \leq 1$ , almost surely for all  $n \geq 0$ . On the event  
944  $\{\mathbb{E}_n[(\Delta M_n)^2] = \mathcal{O}(n^{-c})\}$ , we have almost surely when  $m \rightarrow +\infty$ ,*

$$945 \quad \sup_{n \geq m} \left| \sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a} \right| = \mathcal{O}(m^{\kappa-a}),$$

946 for all  $\kappa \in (\frac{1-c}{2}, a)$ .

947 *Proof.* **(i)** is straightforward. For **(ii)**, we fix  $\frac{1-c}{2} < \kappa < a$ , and then  $\varepsilon > 0$ , such that  $\frac{1-c+\varepsilon}{2} < \kappa$ , and  
948  $\hat{\kappa} := \kappa + \varepsilon/2 < a$ . For  $m \leq n$ , define the event

$$949 \quad \mathcal{A}_{m,n} := \{\mathbb{E}_i[(\Delta M_i)^2] \leq i^{-c+\varepsilon} \quad \forall m \leq i \leq n\}.$$

950 We have, for all  $n \geq m \geq 1$ ,

$$951 \quad \mathbb{P}\left(\sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a} \geq m^{\hat{\kappa}-a}, \mathcal{A}_{m,n}\right) \leq \exp(-m^{\varepsilon/2}) \mathbb{E}\left[\prod_{i=m}^n \exp\left(m^{a-\kappa} \frac{\Delta M_i}{(i+3)^a}\right) \mathbf{1}_{\mathcal{A}_{m,n}}\right].$$

953 Using the bound  $|\Delta M_n| \leq 1$ , and a Taylor expansion, we get for all  $n \geq m$ , on the event  $\mathcal{A}_{m,n}$ ,

$$954 \quad \mathbb{E}_n\left[\exp\left(m^{a-\kappa} \frac{\Delta M_n}{(n+3)^a}\right)\right] = 1 + \mathcal{O}\left(m^{2a-2\kappa} \frac{\mathbb{E}_n[(\Delta M_n)^2]}{(n+3)^{2a}}\right) \leq 1 + \mathcal{O}(n^{-2\kappa-c+\varepsilon}),$$

955 where the constant in the  $\mathcal{O}$ -term is deterministic. By induction, and since  $2\kappa + c - \varepsilon > 1$ , we get that  
956 for all  $m$  large enough,

$$957 \quad \mathbb{E}\left[\prod_{i=m}^n \exp\left(m^{a-\kappa} \frac{\Delta M_i}{(i+3)^a}\right) \mathbf{1}_{\mathcal{A}_{m,n}}\right] \leq 2,$$

958 and thus for all  $1 \leq m \leq n$ , with  $m$  large enough,

$$959 \quad \mathbb{P}\left(\sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a} \geq m^{\hat{\kappa}-a}, \mathcal{A}_{m,n}\right) \leq 2 \exp(-m^{\varepsilon/2}).$$

960 By symmetry and a union bound, we deduce that for all  $m$  large enough,

$$961 \quad \mathbb{P}\left(\sup_{m \leq n \leq 2m} \left|\sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a}\right| \geq m^{\hat{\kappa}-a}, \mathcal{A}_{m,2m}\right) \leq 4m \exp(-m^{\varepsilon/2}).$$

962 Next, another union bound gives, for all  $m$  large enough,

$$963 \quad \mathbb{P}\left(\sup_{n \geq m} \left|\sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a}\right| \geq R \cdot m^{\hat{\kappa}-a}, \mathcal{A}_{m,\infty}\right) \leq \exp\left(-\frac{1}{2} \cdot m^{\varepsilon/2}\right),$$

964 with  $R := \sum_{i \geq 0} 2^{(\hat{\kappa}-a)i}$ , which is finite since  $\hat{\kappa} < a$ . Then the result follows from Borel-Cantelli's lemma,  
965 since on the event  $\{\mathbb{E}_n[(\Delta M_n)^2] = \mathcal{O}(n^{-c})\}$ , almost surely  $\mathcal{A}_{m,\infty}$  holds for all  $m$  large enough.  $\square$

966 We now prove Lemma 3.13.

967 *Proof of Lemma 3.13.* Consider the process  $U(n) = W_5(n) + W_4(n) - W_3(n)$ , and set  $\hat{U}(n) := \frac{U(n)}{n+2}$ . One  
968 has for any integers  $m < n$ ,

$$969 \quad \hat{U}(n) = \hat{U}(m) + \sum_{i=m}^{n-1} \frac{G(\hat{\mathbf{W}}(i))}{i+3} + \sum_{i=m}^{n-1} \frac{\Delta\Phi(i)}{i+3}, \quad (45)$$

970 where  $G(w) = F_5(w) + F_4(w) - F_3(w)$ , and  $\Delta\Phi(i) = Y(i+1) - \mathbb{E}_i Y(i+1)$ , with  $Y(i+1) = U(i+1) - U(i)$   
971 for all  $i \geq 0$ .

972 Note first that,  $|Y(n+1)| \leq 2$  almost surely for all  $n \geq 0$ , by definition of the model, and thus also  
973  $|\Delta\Phi(n)| \leq 4$ . Note furthermore, that on  $\mathcal{A}(\alpha)$ , one has  $W_4(n) = \mathcal{O}(n^\alpha)$ , since  $W_4(n) \leq W_5(n) + W_3(n)$   
974 (recall Lemma 3.1), and thus  $|\hat{U}(n)| = \mathcal{O}(n^{\alpha-1})$ . Moreover, using Lemmas 3.5 and 3.10 (and the fact that  
975 if at some time  $n$ ,  $W_4$  increases by one unit, then either  $W_3$  or  $W_5$  also), we deduce that on  $\mathcal{A}(\alpha)$ ,

$$976 \quad \mathbb{E}_n[|\Delta\Phi(n)|^2] \leq \mathbb{E}_n[Y(n+1)^2] \leq 4 \cdot \mathbb{P}_n(Y(n+1) \neq 0) = \mathcal{O}(\hat{W}_5(n) + \hat{W}_3(n)) = \mathcal{O}(n^{\alpha-1}).$$

977 On the other hand, by Lemma 3.14, on  $\mathcal{A}(\alpha)$ , we have almost surely  $G(\hat{\mathbf{W}}(i)) \geq -\mathcal{O}(i^{2\alpha-2})$ . Thus  
978 Lemma 3.15 (applied with  $a = 1$ ,  $b = 2\alpha - 1$ , and  $c = 1 - \alpha$ ) and Equation (45) (with  $n$  taken large  
979 enough) give  $\hat{U}(m) \leq \mathcal{O}(m^{\beta-1})$ , for any  $\beta > \max(2\alpha - 1, \alpha/2)$ , which proves the desired result.  $\square$

980 We deduce the following fact (recall the definition of the events  $\mathcal{A}(\alpha)$  from Lemma 3.13).

981 **Lemma 3.16.** *Let  $\alpha \in (0, 1)$  be given. One has almost surely,*

$$982 \quad \mathcal{A}(\alpha) \subseteq \mathcal{A}(\beta),$$

983 *for any  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ .*

984 For the proof of this result, we will need some intermediate result.

985 **Lemma 3.17.** **(a)** *For all  $c \in (1/2, 3/4)$ , there exist positive constants  $\varepsilon$  and  $C$ , such that for all  $r \in [0, 1)$ ,*  
 986 *and all  $w \in \mathcal{E}$ , with  $w_3 \leq \varepsilon$  and  $w_4 + w_5 \leq w_3 + r$ , one has*

$$987 \quad F_5(w) - F_4(w) \geq c(w_4 - w_5) - Cr.$$

988 **(b)** *There exist positive constants  $\varepsilon$  and  $C$ , such that for any  $w \in \mathcal{E}$ , with  $w_3 \leq \varepsilon$ ,  $w_4 + w_5 \leq w_3 + r$ ,*  
 989 *and  $w_4 \leq w_5$ , one has*

$$990 \quad \frac{9}{2}F_4(w) - F_3(w) \geq -Cr.$$

991 **(c)** *Let  $\rho \in (0, 1/4)$  be given. There exist positive constants  $\varepsilon$  and  $C$ , such that for any  $r \in [0, 1)$ , and*  
 992 *any  $w \in \mathcal{E}$ , with  $w_3 \leq \varepsilon$ ,  $w_4 + w_5 \leq w_3 + r$ , and  $w_4 \leq \rho w_3 + r$ , one has*

$$993 \quad F_4(w) \geq -Cr.$$

994 *Proof.* Let us start with Part **(a)**. Note that, if, under the assumption of the lemma, we have  $w_4 + w_5 - w_3 \geq$   
 995  $2w_3$ , then  $3w_3 \leq w_4 + w_5 \leq w_3 + r$ , which implies  $w_3 \leq r/2$ , and thus  $w_4 + w_5 \leq 3r/2$ . In particular, we  
 996 have that  $w_3, w_4, w_5 \in [0, 2r)$ . Recall that,  $F_5(w) - F_4(w) = p_{135}(w) - p_{234}(w) + w_4 - w_5 \geq -p_{234}(w) - w_5$ ,  
 997 and, by Equation (22) (using the symmetry of the model), we have that, for all  $w \in \mathcal{E}$ ,  $p_{234}(w) \leq w_4$ .  
 998 Thus  $F_5(w) - F_4(w) \geq -w_4 - w_5 \geq -4r$ , which, using that  $w_4 \leq 2r$ , concludes the proof of **(a)** in the  
 999 case when  $w_4 + w_5 - w_3 \geq 2w_3$ . We now assume that  $w_4 + w_5 - w_3 < 2w_3$ . This implies

$$1000 \quad \frac{1}{2w_3} \geq \frac{1}{w_3 + w_4 + w_5} \geq \frac{1}{2w_3(1 + \frac{w_4 + w_5 - w_3}{2w_3})} \geq \frac{1}{2w_3} - \frac{r}{4w_3^2}, \quad (46)$$

1001 using that  $w_4 + w_5 \geq w_3$ , for all  $w \in \mathcal{E}$ , for the first inequality. Using again (22), we get that, when  
 1002  $w_3, w_4, w_5 \rightarrow 0$ , with  $(w_4 + w_5)/3 \leq w_3 \leq w_4 + w_5$ ,

$$\begin{aligned} 1003 \quad F_5(w) - F_4(w) &= p_{135}(w) - p_{234}(w) + w_4 - w_5 \\ 1004 \quad &= w_4 - w_5 + \frac{w_3 w_5}{w_3 + w_4 + w_5} (1 - o(1)) - \frac{w_3 w_4}{2(w_3 + w_4 + w_5)} (1 + o(1)) \\ 1005 \quad &\geq w_4 - w_5 + \frac{w_5}{2} (1 - o(1)) - \frac{w_4}{4} (1 + o(1)) - \frac{r w_5 (1 + o(1))}{4w_3} \\ 1006 \quad &\geq \frac{3w_4}{4} (1 - o(1)) - \frac{w_5}{2} (1 + o(1)) - \frac{3r(1 + o(1))}{4}, \end{aligned}$$

1008 because  $w_4 + w_5 - w_3 < 2w_3$  implies  $w_5 \leq 3w_3$ . This concludes the proof of **(a)**.

1009 We prove now Part **(b)**. First note that if  $w_4 + w_5 - w_3 \geq 2w_3$ , then we have as in Part **(a)** that  
 1010  $w_3, w_4, w_5 \in [0, 2r]$ , and since  $F_3(w) \leq 0$  by Lemma 3.5, we deduce that  $9/2 \cdot F_4(w) - F_3(w) \geq -9w_4/2 \geq$   
 1011  $-9r$ , proving the result. So we may assume now that  $w_4 + w_5 - w_3 < 2w_3$ . In this case

$$1012 \quad \frac{9}{2}F_4(w) - F_3(w) = \frac{9}{2}(F_5(w) + w_5 - w_4 + p_{234}(w) - p_{135}(w)) - F_3(w).$$



1013 Using Equation (30), we have, when  $w_3, w_4, w_5 \rightarrow 0$ ,

$$1014 \quad F_5(w) + w_5 - w_4 = \frac{(w_5 - w_4)(w_3 + w_1w_4)}{w_3 + w_1w_4 + w_2w_5} = \frac{(w_5 - w_4)(w_3 + w_4)(1 + o(1))}{w_3 + w_4 + w_5}.$$

1015 Using Equation (46), and the fact that  $w_4 \leq w_5$ , we get

$$1016 \quad F_5(w) + w_5 - w_4 \geq \frac{(w_5 - w_4)(1 + o(1))}{2} \left(1 - \frac{r}{2w_3}\right) \geq \frac{(w_5 - w_4)(1 + o(1))}{2} - \frac{r}{4}(1 + o(1)),$$

1017 using that  $w_5 - w_4 \leq w_3$ . In the proof of **(a)**, we have shown that

$$1018 \quad p_{234} - p_{135} = \frac{w_3w_4(1 + o(1))}{2(w_3 + w_4 + w_5)} - \frac{w_3w_5(1 + o(1))}{w_3 + w_4 + w_5} \geq \frac{(w_4 - 2w_5)(1 + o(1))}{4}.$$

1019 Using in addition that by assumption  $w_3 \geq w_4 + w_5 - r$ , we get

$$1020 \quad F_3(w) = p_{135} + p_{234} - w_3 = \frac{w_3w_5(1 + o(1))}{w_3 + w_4 + w_5} + \frac{w_3w_4(1 + o(1))}{2(w_3 + w_4 + w_5)} - w_3 \quad (47)$$

$$1021 \quad \leq -\frac{w_5(1 + o(1))}{2} - \frac{3w_4(1 + o(1))}{4} + r.$$

1022 In total, we thus get

$$1023 \quad \frac{9}{2}F_4(w) - F_3(w) \geq -\frac{9w_4(1 + o(1))}{8} - \frac{9r(1 + o(1))}{8} + \left(\frac{w_5}{2} + \frac{3w_4}{4} - r\right)(1 + o(1))$$

$$1024 \quad \geq -\frac{3w_4}{8}(1 + o(1)) + \frac{w_5}{2}(1 - o(1)) - \frac{17r(1 + o(1))}{8},$$

$$1025 \quad \geq \frac{w_5(1 + o(1))}{8} - \frac{17r(1 + o(1))}{8},$$

1026 because  $w_4 \leq w_5$ , which concludes the proof of **(b)**.

1027 Finally we prove **(c)**. Assuming again that  $w_4 + w_5 - w_3 < 2w_3$  (as otherwise we conclude as in Part **(b)**), we get when  $w_3 \rightarrow 0$  (and as consequence  $w_4, w_5 \rightarrow 0$  also),

$$1028 \quad F_4(w) = F_5(w) + p_{234}(w) - p_{135}(w) + w_5 - w_4$$

$$1029 \quad = \frac{(w_5 - w_4)(w_3 + w_1w_4)}{w_3 + w_1w_4 + w_2w_5} + p_{234}(w) - p_{135}(w)$$

$$1030 \quad \geq \frac{(w_5 - w_4)(w_3 + w_1w_4)}{w_3 + w_1w_4 + w_2w_5} + \frac{w_3w_4(1 - o(1))}{2(w_3 + w_4 + w_5)} - \frac{w_3w_5}{w_3 + w_1w_4 + w_2w_5}$$

$$1031 \quad \geq -w_4(1 + o(1)) \cdot \frac{w_3 + w_1w_4 - w_1w_5}{w_3 + w_4 + w_5} + \frac{w_3w_4(1 - o(1))}{2(w_3 + w_4 + w_5)}$$

$$1032 \quad = \frac{w_4 \left[ w_5(1 - o(1)) - \left(\frac{w_3}{2} + w_4\right)(1 + o(1)) \right]}{w_3 + w_4 + w_5}.$$

1033 Using the fact that, for all  $w \in \mathcal{E}$ ,  $w_5 \geq w_3 - w_4$ , and the fact that, by assumption,  $w_4 \leq \rho w_3 + r$ , we get

$$1034 \quad w_5 - \frac{w_3}{2} - w_4 \geq \frac{w_3}{2} - 2w_4 \geq \frac{w_3}{2} - 2\rho w_3 - 2r = \frac{w_3}{2}(1 - 4\rho) - 2r,$$

1035 which implies

$$1036 \quad F_4(w) \geq \frac{w_4w_3(1 - 4\rho)}{2(w_3 + w_4 + w_5)}(1 - o(1)) - 2r(1 + o(1)) \geq -2r(1 + o(1)),$$

1037 since  $\rho < 1/4$  by assumption. □

1042 We are now ready to prove Lemma 3.16.

1043 *Proof of Lemma 3.16.* We fix  $\alpha \in (0, 1)$ . Recall that

$$1044 \quad \mathcal{A}(\alpha) = \{W_3(n) = \mathcal{O}(n^\alpha) \text{ and } W_5(n) = \mathcal{O}(n^\alpha)\}.$$

1045 The proof is divided in three steps.

1046 First step. Fix  $c \in (3/4, 1)$ . Consider the process  $U(n) = \frac{W_5(n) - W_4(n)}{(n+2)^{1-c}}$ . By Equation (19), we have for  
1047  $n \geq 1$ ,

$$1048 \quad U(n+1) = U(n) + \frac{r_n}{(n+3)^{1-c}} + \frac{\Delta\Psi(n)}{(n+3)^{1-c}}, \quad (48)$$

1049 where (when  $n \rightarrow +\infty$  in the second equality)

$$1050 \quad r_n = F_5(\hat{\mathbf{W}}(n)) + \hat{W}_5(n) - F_4(\hat{\mathbf{W}}(n)) - \hat{W}_4(n) + ((n+2)^{1-c} - (n+3)^{1-c})U(n) \\ 1051 \quad = F_5(\hat{\mathbf{W}}(n)) - F_4(\hat{\mathbf{W}}(n)) + \hat{W}_5(n) - \hat{W}_4(n) + (c-1+o(1))n^{-c}U(n),$$

1053 and  $\Delta\Psi(n) := \Delta M_5(n+1) - \Delta M_4(n+1)$ . By Lemma 3.13, for all  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ , almost surely on  
1054  $\mathcal{A}(\alpha)$ , we have  $\hat{W}_4(n) + \hat{W}_5(n) \leq \hat{W}_3(n) + \mathcal{O}(n^{\beta-1})$  when  $n \rightarrow +\infty$ . Using Lemma 3.17(a), this implies  
1055 that, almost surely on  $\mathcal{A}(\alpha)$ , for all  $n$  large enough,

$$1056 \quad F_5(\hat{\mathbf{W}}(n)) - F_4(\hat{\mathbf{W}}(n)) \geq (c-1/4)(\hat{W}_4(n) - \hat{W}_5(n)) - \mathcal{O}(n^{\beta-1}),$$

1057 and thus (using the fact that  $U(n) = (\hat{W}_5(n) - \hat{W}_4(n))(n+2)^c$ )

$$1058 \quad r_n \geq (1-c+1/4)(\hat{W}_5(n) - \hat{W}_4(n)) - \mathcal{O}(n^{\beta-1}) + (c-1+o(1))n^{-c}U(n) = \left(1/4+o(1)\right)n^{-c}U(n) - \mathcal{O}(n^{\beta-1}), \quad (49)$$

1059 for all  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ . Note that  $\Delta U(n) \neq 0$  implies  $W_3(n+1) - W_3(n) = 1$ , and thus by Lemma 3.5  
1060 on  $\mathcal{A}(\alpha)$ , one has

$$1061 \quad \mathbb{E}_n[|\Delta\Psi(n)|^2] \leq \mathbb{P}(\Delta U(n) \neq 0) = \mathcal{O}(n^{\alpha-1}).$$

1062 Using next Lemma 3.15, we deduce that for any  $\beta > \alpha/2$ ,

$$1063 \quad \sup_{n \geq m} \left| \sum_{i=m}^n \frac{\Delta\Psi(i)}{(i+3)^{1-c}} \right| = \mathcal{O}(m^{-1+c+\beta}). \quad (50)$$

1064 By Equation (48), we have, for all  $n > m$ ,

$$1065 \quad U(n) = U(m) + \sum_{i=m}^{n-1} \frac{r_i}{(i+3)^{1-c}} + \sum_{i=m}^{n-1} \frac{\Delta\Psi(i)}{(i+3)^{1-c}}. \quad (51)$$

1066 Now fix  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ . Observe that if for some  $\varepsilon > 0$ ,  $\limsup_{m \rightarrow +\infty} (W_5(m) - W_4(m))/m^{\beta+\varepsilon} > 0$ ,  
1067 then Equations (49), (50), (51) and Lemma 3.15 imply, by induction, that  $U(n) \geq 0$ , for all  $n$  large enough.  
1068 Thus on  $\mathcal{A}(\alpha)$ , there are only two possibilities: either  $W_5(n) \leq W_4(n) + \mathcal{O}(n^\beta)$ , for all  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ ,  
1069 or  $W_5(n) \geq W_4(n)$  for all large enough  $n$ .

1070 Second step. Consider first the case when  $W_5(n) \leq W_4(n) + \mathcal{O}(n^\beta)$ , for any  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ . Note  
1071 that, for all  $w \in \mathcal{E}$  and asymptotically when  $w_3, w_4, w_5 \rightarrow 0$ , we have, using Equation (30),

$$1072 \quad F_5(w) = \frac{(w_4 - w_5)w_2w_5}{w_3 + w_1w_4 + w_2w_5} = \frac{(w_4 - w_5)w_5}{w_3 + w_4 + w_5}(1 + o(1)). \quad (52)$$

1073 By Lemma 3.13, on  $\mathcal{A}(\alpha)$ , we have  $W_3(n) \leq W_4(n) + W_5(n) \leq W_3(n) + \mathcal{O}(n^\beta)$ . Since we also assume, in  
 1074 this second step, that  $W_5(n) \leq W_4(n) + \mathcal{O}(n^\beta)$ , this implies  $2W_5(n) \leq W_3(n) + \mathcal{O}(n^\beta)$ . Therefore, using  
 1075 that  $W_4(n) - W_5(n) \geq -W_3(n) \wedge \mathcal{O}(n^\beta)$  we get  $F_5(\hat{\mathbf{W}}(n)) \geq -\mathcal{O}(n^{\beta-1})$ . Next, let us prove that for any  
 1076  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ ,  $W_5(n) = \mathcal{O}(n^\beta)$ . Using (19), we have, for  $n \geq m$ ,

$$1077 \quad \hat{W}_5(n) = \hat{W}_5(m) + \sum_{i=m}^{n-1} \frac{F_5(\hat{\mathbf{W}}(i))}{i+3} + \sum_{i=m}^{n-1} \frac{\Delta M_5(i+1)}{i+3},$$

1078 where by Lemma 3.15 the two sums are greater than  $-\mathcal{O}(m^{\beta-1})$ . On  $\mathcal{A}(\alpha)$ , if  $\limsup_m \hat{W}_5(m)/m^{\beta-1} = \infty$ ,  
 1079 then the equation above would contradict that  $\hat{W}_5(n)$  goes to zero, when  $n \rightarrow \infty$ . Thus  $W_5(n) = \mathcal{O}(n^\beta)$ ,  
 1080 as claimed.

1081 Now note that, for all  $w \in \mathcal{E}$ , asymptotically when  $w_3, w_4, w_5 \rightarrow 0$ ,

$$1082 \quad F_3(w) = p_{135}(w) + p_{234}(w) - w_3 = \frac{w_3(w_5 + \frac{w_4}{2})}{w_3 + w_4 + w_5}(1 + o(1)) - w_3 \leq \frac{2w_5 + w_4}{4}(1 + o(1)) - w_3,$$

1083 where we have used Equation (47) and the fact that  $w_4 + w_5 \geq w_3$  for all  $w \in \mathcal{E}$ . Since, for all  $w \in \mathcal{E}$ ,  
 1084  $w_4 \leq w_3 + w_5$ , we get

$$1085 \quad F_3(w) \leq \frac{3(w_5 - w_3)}{4}(1 + o(1)).$$

1086 Applying this to  $w = \hat{\mathbf{W}}(n)$  (which belongs to  $\mathcal{E}$  by Lemma 3.1), we get that on  $\mathcal{A}(\alpha)$ , almost surely  
 1087 when  $n \rightarrow +\infty$ ,

$$1088 \quad F_3(\hat{\mathbf{W}}(n)) \leq \mathcal{O}(n^{\beta-1}).$$

1089 Now if  $\liminf W_3(n)/n^\beta \leq 1$ , for some  $\beta > \max(2\alpha - 1, \alpha/2)$ , then Lemma 3.15 gives  $W_3(n) = \mathcal{O}(n^\beta)$   
 1090 following an argument very similar to the one above for  $\hat{\mathbf{W}}_5(n)$ . On the other hand, if  $\liminf W_3(n)/n^\beta =$   
 1091  $+\infty$ , for any  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ , then, as  $W_4(n) \geq W_3(n) - W_5(n) \geq W_3(n) - \mathcal{O}(n^\beta)$ , we have  $W_4(n) -$   
 1092  $W_5(n) \geq 0$ , for all  $n$  large enough, which implies  $F_5(\hat{\mathbf{W}}(n)) \geq 0$ . From (19), this means that for  $m$   
 1093 large enough, the process  $(W_5(n))_{n \geq m}$  stochastically dominates a Pólya urn process  $(R_n)_{n \geq m}$  defined by  
 1094  $\mathbb{P}(R_{n+1} = R_n + 1 \mid R_n) = 1 - P(R_{n+1} = R_n \mid R_n) = \frac{R_n}{n+2}$ , which is well known to grow almost surely  
 1095 linearly in  $n$  (this can be seen using Rubin's construction as in the proof of Lemma 3.12). Thus  $W_5(n)$   
 1096 would also grow linearly in  $n$ , and we would get a contradiction. Therefore, necessarily  $W_3(n) = \mathcal{O}(n^\beta)$ ,  
 1097 as wanted.

1098 Third step. Consider next the case when  $W_5(n) \geq W_4(n)$ , for all  $n$  large enough. Define  $V(n) =$   
 1099  $\frac{9}{2}W_4(n) - W_3(n)$ , and  $\hat{V}(n) = \frac{V(n)}{n+2}$ . One has for any  $n \geq 1$ ,

$$1100 \quad \hat{V}(n+1) = \hat{V}(n) + \frac{H(\hat{\mathbf{W}}(n))}{n+3} + \frac{\Delta\Theta(n)}{n+3},$$

1101 with again  $\Delta\Theta(n)$  the increment of some martingale, and  $H(w) = \frac{9}{2}F_4(w) - F_3(w)$ . Using Lemmas 3.17(b)  
 1102 and 3.15 (with arguments similar to those in the second step), we deduce that  $V(n) \leq \mathcal{O}(n^\beta)$ , for any  
 1103  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ . We note finally that by Lemma 3.17(c) this entails  $F_4(\hat{\mathbf{W}}(n)) \geq -\mathcal{O}(n^{\beta-1})$ , and thus  
 1104 by another application of Lemma 3.15, we conclude that  $W_4(n) = \mathcal{O}(n^\beta)$ , for any  $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$ .  
 1105 Then we can use the same argument as in step 2: we first observe that this entails

$$1106 \quad F_3(\hat{\mathbf{W}}(n)) \leq \frac{\hat{W}_5(n)}{2}(1 - o(1)) - \hat{W}_3(n) + \mathcal{O}(n^{\beta-1}) \leq -\frac{\hat{W}_3(n)}{4} + \mathcal{O}(n^{\beta-1}) \leq \mathcal{O}(n^{\beta-1}).$$

1107 Therefore, if  $\liminf \frac{W_3(n)}{n^\beta} = \infty$ , then  $W_3(n) \sim W_5(n)$ , and  $W_4(n) = o(W_3(n))$ . Thus by Lemma 3.17(c)  
 1108 again (applied with  $r = 0$ ), we get  $F_4(\hat{\mathbf{W}}(n)) \geq 0$ , for all  $n$  large enough, which leads to a contradiction  
 1109 as in step 2. We conclude that  $W_3(n) = \mathcal{O}(n^\beta)$ , as wanted. This concludes the proof of the lemma.  $\square$

1110 An immediate corollary of the results obtained so far is the following fact.

1111 **Corollary 3.18.** *On the event when  $W_1(n)/n \rightarrow 1$ , one has almost surely for any  $\varepsilon > 0$ ,*

$$1112 \quad W_3(n) = \mathcal{O}(n^\varepsilon).$$

1113 *Proof.* It suffices to combine Lemmas 3.7 and 3.12 with Lemma 3.16, which we can iterate as much as  
 1114 needed. Indeed, the map  $\varphi : \alpha \mapsto \max(2\alpha - 1, \frac{\alpha}{2})$  is decreasing, with 0 as unique fixed point in  $[0, 1)$ ,  
 1115 which implies that any sequence defined by  $\alpha_{n+1} = \varphi(\alpha_n)$ , with  $\alpha_0 < 1$ , converges to 0. Lemmas 3.7  
 1116 and 3.12 give the existence of  $\alpha_0 < 1$  such that, on the event  $W_1(n)/n \rightarrow 1$ ,  $\mathcal{A}(\alpha_0)$  has probability 1.  
 1117 Lemma 3.16 then implies that for all  $n \geq 0$ , on the event  $W_1(n)/n \rightarrow 1$ ,  $\mathcal{A}(\alpha_n)$  has probability 1. We  
 1118 then choose  $n$  large enough so that  $\alpha_n < \varepsilon$ .  $\square$

1119 The final step is the following result, which together with Corollary 3.18 brings a contradiction, if  
 1120  $W_1(n)/n \rightarrow 1$ , and therefore concludes the proof of Proposition 3.11.

1121 **Lemma 3.19.** *On the event when  $W_1(n)/n \rightarrow 1$ , one has almost surely for any  $c \in (0, 1/5)$ ,*

$$1122 \quad \lim_{n \rightarrow \infty} \frac{W_3(n)}{n^c} = +\infty.$$

1123 *Proof.* Recall that when  $w_3, w_4$  and  $w_5$  go to 0, one has for  $w \in \mathcal{E}$ ,

$$1124 \quad p_{135}(w) + p_{234}(w) = \frac{w_3(w_5 + \frac{w_4}{2})}{w_3 + w_4 + w_5} (1 + o(1)).$$

1125 Using now that  $w_4 + w_5 \geq w_3$ , we get that

$$1126 \quad p_{135}(w) + p_{234}(w) \geq \frac{w_3(1 + o(1))}{4} = \frac{w_3/4(1 + o(1))}{1 - w_3 + w_3/4}.$$

1127 Thus there exists  $\varepsilon > 0$ , such that for any  $w \in \mathcal{E}$ , with  $w_3, w_4, w_5 \leq \varepsilon$ ,

$$1128 \quad p_{135}(w) + p_{234}(w) \geq \frac{w_3/5}{1 - w_3 + w_3/5}.$$

1129 By Proposition 3.2 and Lemma 3.12, we deduce that almost surely on the event when  $W_1(n)/n \rightarrow 1$ ,  
 1130 there exists a random integer  $n_0$  such that, for all  $n \geq n_0$ ,

$$1131 \quad p_{135}(\hat{\mathbf{W}}(n)) + p_{234}(\hat{\mathbf{W}}(n)) \geq \frac{\hat{W}_3(n)/5}{1 - \hat{W}_3(n) + \hat{W}_3(n)/5} = \frac{W_3(n)/5}{n + 2 - W_3(n) + W_3(n)/5}.$$

1132 Therefore, after some (random) time  $n_0$ , the process  $(W_3(n))_{n \geq n_0}$  stochastically dominates an urn process  
 1133  $(U(n))_{n \geq n_0}$ , defined by  $U(n_0) = 1$  and, for all  $n \geq n_0$ ,

$$1134 \quad \mathbb{P}(U(n+1) = U(n) + 1 \mid U(n)) = 1 - \mathbb{P}(U(n+1) = U(n) \mid U(n)) = \frac{U(n)/5}{n + n_0 + 2 - U(n) + U(n)/5}.$$

1135 For any fixed  $n_0$ , the urn process  $(U(n))_{n \geq n_0}$  is studied for instance in Janson [Jan06] (see in particular  
 1136 Theorem 1.4 and Remark 1.12 there), which provides a precise asymptotic behavior of  $n^{-1/5}U(n)$ : it  
 1137 converges in law towards some non-degenerate random positive variable. But here one can simply rely  
 1138 again on Rubin's construction, which covers our needs. It shows that for any fixed  $n_0$ , almost surely there  
 1139 exists a constant  $c > 0$ , such that  $U(n) \geq cn^{1/5}$ , for any  $n \geq n_0$ , and the lemma follows.  $\square$

1140 The proofs of Proposition 3.11 and Theorem 1.4 are now complete.

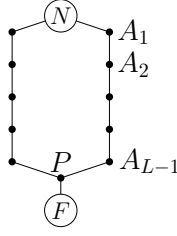


Figure 11: The graph of Proposition 1.5 and the notations used in Section 4.

## 1141 4 Proof of Proposition 1.5

1142 We fix an integer  $L$  and look at the uniform-geodesic version of the model on the graph on the left-hand  
 1143 side of Figure 5. Note that each ant reinforces either the  $L$  edges on the left (and the edge linked to  $F$ )  
 1144 or the  $L$  edges on the right (and the edge linked to  $F$ ). Thus, the  $L$  edges on the left have all the same  
 1145 weight at all times, and similarly for the the  $L$  edges on the right. For all integers  $n$ , we set  $N_1(n)$  to  
 1146 be the weight of the  $L$  left-edges at time  $n$ , and by  $N_2(n)$  the weights of the right-edges at time  $n$ . By  
 1147 definition, we have that  $N_1(n) + N_2(n) = n + 2$ , which is also the weight of the edge linked to  $F$ .

1148 To prove Proposition 1.5, we apply a result of [HLS80] (see [Pem07, Theorem 2.8]):

1149 **Theorem 4.1.** *Let  $(Z_n)_{n \geq 0}$  be a sequence of random variables taking values in  $[0, 1]$  satisfying, for all*  
 1150  *$n \geq 0$ ,*

$$1151 \quad Z_{n+1} = Z_n + \frac{1}{n} (F(Z_n) + \Delta M_{n+1}), \quad (53)$$

1152 *where  $F : [0, 1] \rightarrow [0, 1]$  and  $\Delta M_{n+1}$  is a martingale increment. If there exists  $\varepsilon > 0$  such that  $F < 0$  on*  
 1153  *$[0, \varepsilon]$ , then  $\mathbb{P}(Z_n \rightarrow 0) > 0$ .*

1154 First note that, if  $Z_n = N_1(n)/(n + 2)$  for all  $n \geq 0$ , then  $Z_n$  satisfies Equation (53) with  $F(x) =$   
 1155  $p(x) - x$ , where  $p(x)$  is the probability that an ant reinforces the left-hand side geodesic after performing  
 1156 a random walk on  $\mathcal{G}$  with weights  $x$  on all edges on the left,  $1 - x$  on all edges on the right, and 1 on  
 1157 the edge linked to  $F$ . We let  $\mathcal{G}(x)$  denote the graph  $\mathcal{G}$  equipped with these weights,  $P$  denote the unique  
 1158 vertex neighbouring  $F$ , and for all  $k \in \{0, \dots, L - 1\}$ ,  $A_k$  denote the vertex at distance  $k$  of  $N$  on the  
 1159 right-hand-side geodesic (with  $A_0 = N$ ). See Figure 11 where the notations are illustrated.

1160 We now calculate  $p(x)$  when  $x \rightarrow 0$  to show that  $p(x) < x$  in a neighbourhood of zero; this implies  
 1161 that  $F < 0$  in a neighbourhood of zero and thus that Theorem 4.1 applies. Asymptotically when  $x \rightarrow 0$ ,

$$1162 \quad p(x) = \sum_{k=0}^{L-1} \left( p_k^{(1)}(x) + \frac{1}{2} p_k^{(2)}(x) \right) + \frac{1}{2} p^{(3)}(x) + \frac{1}{2} p^{(4)}(x) + \mathcal{O}(x^2), \quad (54)$$

1163 where, for all  $k \in \{0, \dots, L - 1\}$ ,

- 1164 •  $p_k^{(1)}(x)$  the probability that a walker on the weighted graph  $\mathcal{G}(x)$  goes from  $N$  to  $A_k$  using only  
 1165 edges on the right-hand-side geodesic, then goes from  $A_k$  to  $N$  without reaching  $A_{k+1}$ , then goes  
 1166 from  $N$  to  $P$  without reaching  $A_{k+1}$ , and, finally, goes from  $P$  to  $F$  without using the left-hand-side  
 1167 geodesic or reaching  $A_k$ ;
- 1168 •  $p_k^{(2)}(x)$  the probability that a walker on the weighted graph  $\mathcal{G}(x)$  goes from  $N$  to  $A_k$  using only edges  
 1169 on the right-hand-side geodesic, then goes from  $A_k$  to  $P$  without reaching  $A_{k+1}$  (thus using edges  
 1170 on the left-hand-side geodesic), and, finally, goes from  $P$  to  $A_k$  using only edges on the right-hand  
 1171 side geodesic;

- 1172 •  $p^{(3)}(x)$  the probability that a walker on the weighted graph  $\mathcal{G}(x)$  first goes from  $N$  to  $P$  only using  
1173 edges on the right-hand-side geodesic, then goes from  $P$  to  $N$  using edges on the left-hand-side  
1174 geodesic, and before entering the left-hand geodesic from  $N$ ;
- 1175 •  $p^{(4)}(x)$  the probability that a walker on the weighted graph  $\mathcal{G}(x)$  first goes from  $N$  to  $P$  only using  
1176 edges on the right-hand-side geodesic, then goes back from  $P$  to  $N$  only using edges on the right-  
1177 hand-side geodesic, and, finally, goes from  $N$  to  $P$  using edges on the left-hand-side geodesic and  
1178 before entering the left-hand-side geodesic from  $P$  or hitting  $F$ .

1179 The  $\mathcal{O}(x^2)$ -term in Equation (54) stands for all trajectories of the walker that leave  $N$  or  $P$  at least twice  
1180 towards the left. We have, if  $k \in \{1, \dots, L-1\}$ ,

$$1181 \quad p_k^{(1)}(x) = \frac{\frac{1-x}{k}}{\frac{1-x}{k} + x} \cdot \frac{\frac{1-x}{k}}{\frac{1-x}{k} + (1-x)} \cdot \frac{\frac{x}{L}}{\frac{x}{L} + \frac{(1-x)}{k+1}} \cdot \frac{1}{1+x + \frac{1-x}{L-k}} = \frac{x}{L} \cdot \frac{L-k}{L-k+1} + \mathcal{O}(x^2).$$

1182 We also have

$$1183 \quad p_0^{(1)}(x) = \frac{\frac{x}{L}}{\frac{x}{L} + (1-x)} \cdot \frac{1}{1+x + \frac{1-x}{L}} = \frac{x}{L} \cdot \frac{L}{L+1} + \mathcal{O}(x^2).$$

1184 Using the fact that  $\sum_{i=1}^n 1/i = \log n + \mathcal{O}(1)$  when  $n \rightarrow +\infty$ , we get

$$1185 \quad \sum_{k=0}^{L-1} p_k^{(1)}(x) = \frac{x}{L} \sum_{k=0}^{L-1} \left(1 - \frac{1}{L-k+1}\right) + \mathcal{O}(x^2) = x \left(1 - \frac{\log L}{L} + \mathcal{O}_{L \rightarrow +\infty}(1)\right) + \mathcal{O}(x^2), \quad (55)$$

1186 where the  $\mathcal{O}_{L \rightarrow +\infty}(1)$ -term does not depend on  $x$  and corresponds to the  $L \rightarrow +\infty$  limit, while the  
1187  $\mathcal{O}(x^2)$ -term depends on  $L$  and refers to the  $x \rightarrow 0$  limit. Similarly, for all  $k \in \{1, \dots, L-1\}$ , we have

$$1188 \quad p_k^{(2)}(x) = \frac{\frac{1-x}{k}}{\frac{1-x}{k} + x} \cdot \frac{\frac{1-x}{k}}{\frac{1-x}{k} + 1-x} \cdot \frac{\frac{x}{L}}{\frac{x}{L} + \frac{1-x}{k+1}} \cdot \frac{\frac{1-x}{L-k}}{\frac{1-x}{L-k} + 1+x} = \frac{x}{L} \cdot \frac{1}{L-k+1},$$

1189 and

$$1190 \quad p_0^{(2)}(x) = \frac{\frac{x}{L}}{\frac{x}{L} + (1-x)} \cdot \frac{\frac{1-x}{L}}{\frac{1-x}{L} + 1+x} = \frac{x}{L} \cdot \frac{1}{L+1} + \mathcal{O}(x^2).$$

1191 Using again the asymptotic behaviour of the harmonic sum, we get

$$1192 \quad \frac{1}{2} \sum_{k=0}^{L-1} p_k^{(1)}(x) = \frac{x}{L} \sum_{k=0}^{L-1} \frac{1}{L-k+1} + \mathcal{O}(x^2) = x \left( \frac{\log L}{2L} (1 + \mathcal{O}_{L \rightarrow +\infty}(1)) \right) + \mathcal{O}(x^2). \quad (56)$$

1193 We also have, when  $x \rightarrow 0$ ,

$$1194 \quad p^{(3)}(x) = \frac{\frac{1-x}{L}}{\frac{1-x}{L} + x} \cdot \frac{\frac{x}{L}}{\frac{x}{L} + 1 + \frac{x \frac{1-x}{L}}{x + \frac{1-x}{L}}} = \frac{x}{L} + \mathcal{O}(x^2),$$

1195 and

$$1196 \quad p^{(4)}(x) = \frac{\frac{1-x}{L}}{\frac{1-x}{L} + x} \cdot \frac{\frac{1-x}{L}}{\frac{1-x}{L} + 1+x} \cdot \frac{\frac{x}{L}}{\frac{x}{L} + \frac{(1+x)\frac{1-x}{L}}{1+x + \frac{1-x}{L}}} = \frac{x}{L} + \mathcal{O}(x^2).$$

1197 Using these last equations together with (55) and (56) into Equation (54), we get that, in total,

$$1198 \quad p(x) = x \left(1 - \frac{\log L}{2L} (1 + \mathcal{O}_{L \rightarrow +\infty}(1))\right) + \mathcal{O}(x^2).$$

1199 Therefore,

$$1200 \quad F(x) = p(x) - x = -x \left( \frac{\log L}{2L} (1 + \mathcal{O}_{L \rightarrow +\infty}(1)) \right) + \mathcal{O}(x^2),$$

1201 implying that for all  $L$  large enough,  $F$  is indeed negative in a right-neighbourhood of 0. Hence, Theorem  
1202 4.1 applies and we conclude that  $\mathbb{P}(Z_n \rightarrow 0) > 0$ , which concludes the proof of Proposition 1.5.

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