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Finding geodesics on graphs using reinforcement learning

Daniel Kious* Cécile Mailler*[†] Bruno Schapira[‡]

October 9, 2020

Abstract

It is well-known in biology that ants are able to find shortest paths between their nest and the food by successive random explorations, without any mean of communication other than the pheromones they leave behind them. This striking phenomenon has been observed experimentally and modelled by different mean-field reinforcement-learning models in the biology literature.

In this paper, we introduce the first probabilistic reinforcement-learning model for this phenomenon. In this model, the ants explore a finite graph in which two nodes are distinguished as the nest and the source of food. The ants perform successive random walks on this graph, starting from the nest and stopped when first reaching the food, and the transition probabilities of each random walk depend on the realizations of all previous walks through some dynamic weighting of the graph. We discuss different variants of this model based on different reinforcement rules and show that slight changes in this reinforcement rule can lead to drastically different outcomes.

We prove that, in two variants of this model and when the underlying graph is, respectively, any series-parallel graph and a 5-edge non-series-parallel *losange* graph, the ants indeed *eventually find the shortest path(s)* between their nest and the food. Both proofs rely on the electrical network method for random walks on weighted graphs and on Rubin’s embedding in continuous time. The proof in the series-parallel cases uses the recursive nature of this family of graphs, while the proof in the seemingly-simpler losange case turns out to be quite intricate: it relies on a fine analysis of some stochastic approximation, and on various couplings with standard and generalised Pólya urns.

1 Introduction and main results

1.1 Context and motivation

In this paper, we introduce and analyse two variants of a stochastic, unsupervised, reinforcement-learning algorithm, which, given as an input a graph in which two nodes are marked, gives as output the shortest path(s) between the two marked nodes. This algorithm is inspired by mean-field models introduced in the biology literature as models for the behavior of foraging ants (see, e.g. [DS04, MJT⁺13]): it has been widely empirically observed (see, e.g., [GADP89, MJT⁺13] for experiments) that a colony of ants is able to find shortest paths between their nest and the food. Unsupervised reinforcement learning is widely proposed as a model for this phenomenon in the biology literature. Our contribution is to introduce a new probabilistic reinforcement-learning model for this phenomenon and prove that, in this model, the ants indeed find the shortest path between their nest and the food.

We consider a sequence of random walkers on a finite graph $\mathcal{G} = (V, \mathcal{E})$ with two distinguished nodes N and F (for “nest” and “food” when the walkers are interpreted as ants). At the beginning of time, all edges of \mathcal{G} are given weight 1. The idea is that the walkers explore the graph from N to F one after

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each other, and the weights of the edges are updated after each walker reaches F . More precisely, for all $n \geq 1$, the n -th walker starts a random walk from N and walks randomly on the graph until it reaches F . At every step, the walker chooses one of the neighboring edges with probability proportional to their weights and crosses the chosen edge to the next vertex. Once the n -th walker has reached F , we update the weights of the edges by adding 1 to a subset of the trace of this walker. In this paper, we look at two possible rules for the choice of this subset of edges to reinforce:

- In the *loop-erased* version of the model, we reinforce the loop-erased time-reversed trace of walker n . This corresponds to how a hiker without a map would go back from F to N by walking backwards on their own trace, but avoiding unnecessary loops: when facing a choice between several edges they crossed on their way to F , they choose the edge that they crossed the earliest on their way forward.
- In the *geodesic* version of the model, we reinforce the shortest path from N to F inside the trace of the walker (i.e. we only look at the subgraph of all edges that were crossed by this specific walker). The case when there are several shortest paths presents some subtleties, on which we will come back when we will define more formally the model in Subsection 1.2 and when discussing our main results (see Subsection 1.3).

We call this stochastic process the loop-erased or geodesic ant process.

The interpretation of the model in terms of ants is as follows: (1) the ants only lay pheromones behind them on their way back from the food to the nest, (2) each ant goes back to the nest either following the loop-erasure of their forward trajectory reversed in time (for the loop-erased ant process), or following the shortest path in the subgraph that they have explored on the way forward (for the geodesic ant process), and (3) each ant can sense from the amount of pheromones how many of its predecessors have crossed an edge on their way back to the nest, and crosses each neighboring edge with probability proportional to this number. We conjecture that, following this simple unsupervised reinforcement-learning algorithm, the colony of ants *eventually finds the shortest path(s) between the nest and the food*, more precisely, asymptotically when time goes to infinity, a proportion 1 of all ants go from the nest to the food following a geodesic.

The difficulty of our analysis comes from different factors: (i) This is a linear reinforcement model: indeed, each ant chooses the next edge to cross with probability proportional to the number of previous ants that laid pheromones on it on their way back to the nest. Interestingly, the assumption that ants react linearly to pheromones is supported in the biology literature (see, e.g. [PGG⁺12, VPFV13]). In fact, one can easily find counter-examples that show that the same algorithm with super- or sub-linear reinforcement would not find the shortest path (see Subsection 1.3). (ii) The algorithm is a sequence of interacting reinforced random walks, and the reinforcement of the n -th random walk depends from the realisations of all previous ones.

Our main contribution is to prove that, as conjectured, the ants indeed find the shortest path if we assume that the underlying graph is either a series-parallel graph (as in [HJ04]) whose “source” is the nest and whose “sink” is the source of food, or the 5-edge losange graph of Figure 4. Surprisingly, the proof for the 5-edge losange graph is more intricate than the proof for the whole class of series-parallel graphs; we therefore expect that finding a proof that would hold for any underlying graph is a very challenging and interesting problem. Both our proofs rely heavily on the electric network method for random walks on graphs (see, e.g., [LP05] for an introduction to this method), and Rubin’s embedding in continuous time (first introduced in [Dav90]). The proof for series-parallel graphs also uses the inductive nature of this family of graphs; the proof for the losange graph relies on the fine analysis of different stochastic approximations (see, e.g., [Duf97, Pem07]). Interestingly, we show that the losange case can be seen as an intricate coupling between two types of Pólya urns (see, e.g., [Pem07] for a survey); a fact that is reminiscent of the proof of Pemantle and Volkov [PV99] of the localisation on five sites with positive probability of the vertex-reinforced random walk (see also [Tar11, Tar04]).

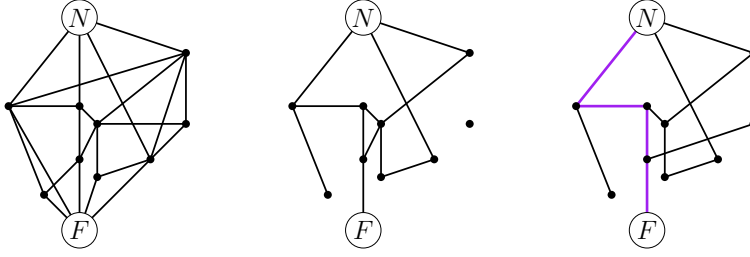


Figure 1: First visual aid for the definition of the uniform-geodesic ant process. On the left is pictured a graph \mathcal{G} . In the middle is a possible realization of a graph $\mathcal{G}^{(n)}$, the trace of the n -th random walk, and on the right is $\gamma^{(n)}$ the unique geodesic from N to F in $\mathcal{G}^{(n)}$.

84 1.2 Mathematical description of the model and statement of the main results

85 Let $\mathcal{G} = (V, E)$ be a finite graph with vertex set V and edge set E . Let N (the nest) and F (the food)
 86 be two distinct vertices in V . In this paper we consider two versions of the same model, which differ by
 87 their reinforcement rules.

88 We define the sequence $(\mathbf{W}(n) = (W_e(n) : e \in E))_{n \geq 0}$ recursively as follows: $W_e(0) = 1$ for all $e \in E$,
 89 and, for all $n \geq 1$:

- 90 • We sample a random walk $X^{(n)} = (X_i^{(n)})_{i \geq 0}$ on \mathcal{G} that starts at N , is killed when first reaching F ,
 91 and whose transition probabilities are: for all $i \geq 1$, for all $u, v \in V$,

$$92 \quad \mathbb{P}(X_i^{(n)} = v \mid X_{i-1}^{(n)} = u, \mathbf{W}(n-1)) = \frac{W_{\{u,v\}}(n-1) \mathbf{1}_{u \sim v}}{\sum_{w \sim u} W_{\{u,w\}}(n-1)},$$

93 where $\{u, v\}$ is the (unoriented) edge between u and v , and $u \sim v$ if and only if the edge $\{u, v\}$ is
 94 in E .

- 95 • Let $\mathcal{G}^{(n)}$ be the trace of $X^{(n)}$, that is the subgraph of \mathcal{G} obtained when removing from \mathcal{G} all edges
 96 that the random walk $X^{(n)}$ did not cross, and choose a path of edges γ_n as follows:

- 97 – For the loop-erased ant process, we imagine that the walker goes back from F to N by following
 98 its trajectory $X^{(n)}$ backwards and avoiding loops as follows: when the walker is at a vertex that
 99 was visited several times on the way forward, possibly coming from different edges at different
 100 times, it chooses to cross the edge that was crossed the earliest on the way forward. We define
 101 $\gamma^{(n)}$ as the set of edges crossed by the walker on its way back to the nest.

102 **Remark.** Note that this construction selects a self-avoiding path between F and N , which is
 103 in fact the loop-erased version of the backward trajectory. Indeed, if we assume that $X^{(n)} =$
 104 $(X_0^{(n)} = N, X_1^{(n)}, \dots, X_{K_n}^{(n)} = F)$, for some $K_n \geq 1$, and define the time-reversed trajectory $\overline{X}^{(n)} =$
 105 $(X_{K_n-i}^{(n)}, 0 \leq i \leq K_n)$, then, by definition, we have that $\gamma_i^{(n)} = \overline{X}_{j_i}^{(n)}$ for $0 \leq i \leq k_n$ for some
 106 $1 \leq k_n \leq K_n$, where $j_0 = 0$ and $\gamma_{k_n}^{(n)} = F$, for $0 \leq i \leq k_n - 1$, $j_{i+1} = \max\{j+1 : \overline{X}_j^{(n)} = \overline{X}_{j_i}^{(n)}\}$. This
 107 corresponds to the loop-erasure of $\overline{X}^{(n)}$, as defined in [LL10].

- 108 – In the uniform-geodesic version of the model, we define $\gamma^{(n)}$ as the shortest path from N to F
 109 in $\mathcal{G}^{(n)}$; if there are several shortest path, we choose one of them uniformly at random.

- 110 • For all $e \in E$, set $W_e(n+1) = W_e(n) + \mathbf{1}_{e \in \gamma_n}$.

111 We conjecture that

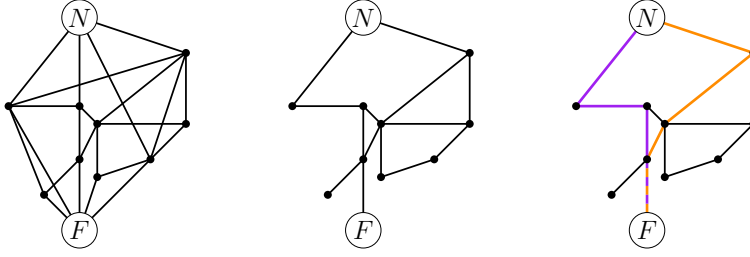


Figure 2: Second visual aid for the definition of the uniform-geodesic ant process. On the left is pictured a graph \mathcal{G} . In the middle is a possible realization of a graph $\mathcal{G}^{(n)}$, the trace of the n -th random walk. On the right, in orange and purple, are the two geodesics from N to F in $\mathcal{G}^{(n)}$, and thus the two possible choices for $\gamma^{(n)}$.

112 **Conjecture 1.1.** *Let $\mathcal{G} = (V, E)$ be any finite graph in which two distinct nodes have been marked as N*
 113 *and F . Almost surely when $n \rightarrow +\infty$, for all $e \in E$,*

114
$$\frac{W_e(n)}{n} \rightarrow \chi_e,$$

115 *where $(\chi_e)_{e \in E}$ is a random vector such that:*

116 (1) *For the loop-erased ant process, $\chi_e \neq 0$ almost surely **if and only if** the edge e belongs to at least*
 117 *one of the geodesics from N to F .*

118 (2) *For the uniform-geodesic ant process, $\chi_e \neq 0$ almost surely **only if** the edge e belongs to at least one*
 119 *of the geodesics from N to F .*

120 *Thus, if there is a unique geodesic γ from N to F in \mathcal{G} , then almost surely $\chi_e = \mathbf{1}_{e \in \gamma}$, for all $e \in E$, in*
 121 *the two versions of the model.*

122 This indeed means that the ants *eventually find the shortest paths* between their nest and the source
 123 of food, because it implies that the probability that the n -th ant goes from the nest to the food through
 124 a geodesic path converges to 1 when $n \rightarrow +\infty$.

125 The difference between (1) and (2) is that, in the uniform-geodesic ant process, edges that belong to a
 126 geodesic may have limiting normalised weight χ_e that equal zero with positive probability: *the ants find*
 127 *at least one of the geodesics, but maybe not all of them.* In Proposition 1.5, we provide an example of a
 128 series-parallel graph where $\chi_e = 0$ with positive probability for some edge e on a geodesic path.

129 Our first main contribution is to prove that this conjecture is true for all series-parallel graphs for
 130 the loop-erased ant process. As their name suggests, series-parallel graphs are classical in electricity;
 131 in probability theory, they are the object of a famous and still-open conjecture of Hambly and
 132 Jordan [HJ04]. They have two distinguished nodes called the “source” and the “sink”, which we can
 133 naturally see as the the nest N and the source of food F in our context.

134 **Definition 1.2** (See Figure 3). We define series-parallel (SP) graphs recursively as follows: a series-parallel
 135 graph is

- 136 • either the single-edge graph (graph made of two vertices joined by one edge) with one node marked
 137 as the source and the other as the sink,
- 138 • or two series-parallel graphs in series (i.e. we merge the sink of the first graph and the source of the
 139 second),
- 140 • or two series-parallel graphs in parallel (i.e. we merge the two sources and the two sinks).

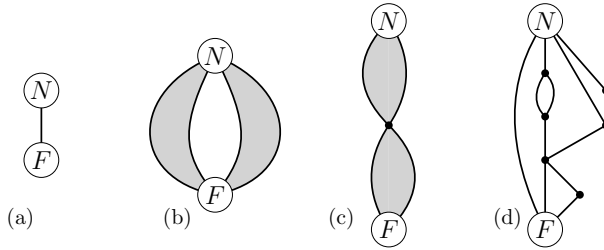


Figure 3: The definition of series-parallel graphs: a SP graph is either (a) the base case, or (b) two SP graphs in parallel, or (c) two SP graphs in series. (d) is an example.

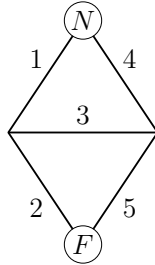


Figure 4: The losange graph

141 **Theorem 1.3.** For any SP graph whose source and sink are respectively marked as N and F , and for the
 142 loop-erased ant process, Conjecture 1.1 is true, i.e. almost surely when $n \rightarrow +\infty$, for all $e \in E$,

143
$$\frac{W_e(n)}{n} \rightarrow \chi_e,$$

144 where $(\chi_e)_{e \in E}$ is a random vector, such that $\chi_e \neq 0$ almost surely if and only if the edge e belongs to at
 145 least one of the geodesics from N to F .

146 Interestingly, the analysis of the loop-erased ant process outside the family of series-parallel graphs
 147 turns out to be very challenging. To illustrate this, we consider one of the simplest non-series-parallel
 148 graph one could think of, which is the 5-edge losange of Figure 4, which we call “the losange graph”: even
 149 on this simple graph, we are not able to prove convergence of the loop-erased ant process. However, we
 150 are able to prove convergence of the uniform-geodesic ant process, which turns out to be simpler in this
 151 setting (see the remark before Lemma 3.6).

152 We number the edges of the losange graph from 1 to 5 as in Figure 4. Our second main result is the
 153 following.

154 **Theorem 1.4.** For all $1 \leq i \leq 5$ and $n \geq 0$, we denote by $W_i(n)$ ($\forall 1 \leq i \leq 5$) the weight of edge
 155 number i after the n -th walker has reached the food in the uniform-geodesic ant process on the losange
 156 graph. (Recall that $W_i(0) = 1$, by definition.) Almost surely as $n \rightarrow +\infty$,

157
$$\frac{W_i(n)}{n} \rightarrow \chi_i, \quad \text{for all } 1 \leq i \leq 5,$$

158 where $(\chi_i)_{1 \leq i \leq 5}$ is a random vector such that almost surely $\chi_1 = \chi_2 = 1 - \chi_4 = 1 - \chi_5 \in (0, 1)$ and $\chi_3 = 0$.

159 1.3 Discussion

160 **Discussion on the loop-erased vs. uniform-geodesic reinforcement rules:** While we believe the
 161 result on the losange graph is also true for the loop-erased ant process, we think the proof would be more
 162 involved than with the uniform-geodesic ant process.

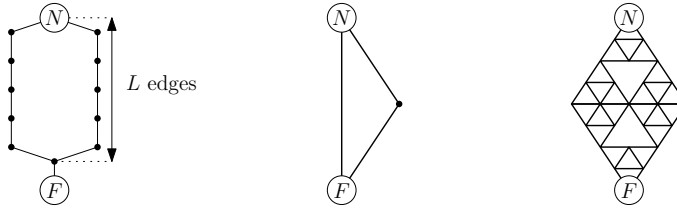


Figure 5: Graphs used in the discussions of Subsection 1.3.

163 The first of three steps in the proof in the uniform-geodesic case is to show that the normalised weight
 164 of the middle edge (edge number 3) converges to zero, and then use this convergence to zero to prove that
 165 the speed of convergence to zero is polynomial. Although proving convergence of the normalised weight
 166 of edge 3 would be similar (and in fact almost identical) in the loop-erased case, proving that the speed of
 167 convergence is polynomial is, we believe, much harder, and could in fact be wrong. Intuitively, it should
 168 not be surprising that the weight of edge 3 could be bigger in the loop-erased than in the uniform-geodesic
 169 version of the model: this comes from the fact that reinforcing the edge 3 is more likely at every step
 170 in the loop-erased version of the model. Since the proof in the uniform-geodesic case is already quite
 171 involved, we leave the case of the loop-erased ant process on the losange open.

172 Conversely, the analysis of the uniform-geodesic ant process (and all its variants - see discussion
 173 below) on series-parallel graphs seems to be a challenging problem, which we also leave for further work.
 174 In summary, it seems that neither of the two versions of the process is easier to analyse than the other in
 175 general, but that this depends on the underlying (family of) graph(s).

176 **Discussion on the (uniform-)geodesic version of the model:** First note that on the losange
 177 graph, the trace of a walker can contain at most one geodesic, and thus the rule of choosing the subset of
 178 edges to reinforce uniformly among all geodesics in the trace is irrelevant in this case. In fact, we believe
 179 that the way we choose which shortest path to reinforce when there are several in the trace can have a
 180 significant impact on the behaviour of the system.

181 Indeed, we first observe that, in the uniform-geodesic version of the model, there could exist an edge
 182 that belongs to a geodesic between N and F whose normalised weight converges to zero:

183 **Proposition 1.5.** *If \mathcal{G} is the graph on the left-hand side of Figure 5, then the uniform-geodesic version*
 184 *of the model satisfies: there exists $e \in E$ such that e lies on a geodesic between N and F (in fact, all edges*
 185 *lie on such a geodesic in this graph) and, for all L large enough (see Figure 5 for the definition of L),*

$$186 \quad \mathbb{P}(W_e(n)/n \rightarrow 0) > 0.$$

187 This proposition also holds (with an almost identical proof) when the choice of the geodesic is not
 188 uniform as long as any geodesic within the trace is chosen with a probability bounded away from 0.

189 Another rule for the choice of $\gamma^{(n)}$ when there are several shortest paths in $\mathcal{G}^{(n)}$ is the following:
 190 Consider $\mathcal{G}_0^{(n)}$ the subgraph of $\mathcal{G}^{(n)}$ obtained by removing all the edges and vertices that do not belong to
 191 any of the shortest paths from N to F in $\mathcal{G}^{(n)}$. As in the loop-erased version of the model, imagine that
 192 the walker walks back from F to N , by only crossing edges from $\mathcal{G}_0^{(n)}$, and, when faced with a choice,
 193 choosing the edge it crossed the earliest on the way forward. Define $\gamma^{(n)}$ as the set of edges crossed by the
 194 walker on its way back to the nest. We believe that the same conjecture as for the loop-erased version of
 195 the model should be true for this version of geodesic ant process.

196 **Other possible reinforcement rules:** An alternative reinforcement rule could be to reinforce all
 197 edges that the n -th walker crossed, i.e. all edges in $\mathcal{G}^{(n)}$, instead of only reinforce the edges of $\gamma^{(n)}$.
 198 Intuitively, this would mean that ants lay pheromones on their way to the food instead of laying them on
 199 their way back to the nest. A mean-field version of this alternative model is also considered in the biology

200 literature (see, e.g., [MJT⁺13]). Preliminary work on this alternative reinforcement rule suggests that it
 201 could lead to surprisingly different results and that the ants may not always find the shortest path, we
 202 leave this for further work.

203 In this alternative reinforcement rule where ants lay pheromones on their way to the food, one could
 204 consider that ants cannot sense from the pheromones laid on an edge how many different ants have crossed
 205 this edge, but rather how many times this edge has been crossed by an ant. This would mean that if
 206 the n -th ant crossed an edge k times the weight of this edge is increased by k when updating the weights
 207 after the n -th ant has reached the food. Finally, one could wonder how the results are impacted if the ants
 208 are sensitive to their own pheromones, i.e. if the weights are updated during the random walks after every
 209 steps of the ants, and not after each ant reaches the food. Each ant would then perform a (self-)reinforced
 210 random walk that starts on an already-weighted graph. We believe that these variants could lead to
 211 different asymptotic behaviours and raise various interesting mathematical challenges.

212 **Discussion on linear vs. sub- or super-linear reinforcement:** As mentioned in the introduction,
 213 Conjecture 1.1 would no longer be true if we considered super- or sub-linear reinforcement instead of linear
 214 reinforcement. Indeed, consider the graph in the middle of Figure 5, and imagine that all the ants perform
 215 weighted random walks on the graph \mathcal{G} , but according to the weights $W_e(n)^\alpha$ ($\forall e \in E$), for some $\alpha > 0$.
 216 One can check that if $\alpha > 1$ (i.e. in the super-linear case), then, almost surely, the subset of edges from E
 217 such that $\liminf_n W_e(n)/n \neq 0$ is either $\{N, F\}$ or $E \setminus \{\{N, F\}\}$, each with positive probability. Also, if
 218 $\alpha < 1$ (i.e. in the sub-linear case), the subset of all edges from E such that $\liminf_n W_e(n)/n \neq 0$ is almost
 219 surely equal to E itself.

220 **Discussion on the underlying graph:** Theorems 1.3 and 1.4 confirm Conjecture 1.1 in the cases
 221 when \mathcal{G} is a series-parallel graph or when \mathcal{G} is the losange graph, which is the simplest non-series parallel
 222 graph. In the proof for series-parallel graph the iterative nature of this family of graph allows us to
 223 reason by induction. An iterative family of graphs that builds on the losange example is the “double
 224 Sierpiński gasket” graph, which consists of two Sierpiński gaskets of the same fractal depth whose bases
 225 have been merged (see the right-hand side of Figure 5 where a double Sierpiński gasket graph of depth 3
 226 is represented). Interestingly, a version of this graph has been considered in the biology literature under
 227 the name “tower of Hanoi” (see [MJT⁺13, RSB11]).

228 **Other models of path and network formation by reinforcement:** Our model can be seen as a
 229 reinforcement path formation model. The idea is that we start from a weighted graph \mathcal{G} where all edges
 230 have the same weight 1, and we look at the graph $\mathcal{G}^{(\infty)}$ of all edges whose normalised weight does not
 231 tend to zero when time goes to infinity. In the language of Conjecture 1.1, $\mathcal{G}^\infty = (V, E^\infty)$ where $e \in E^\infty$ if
 232 and only if $e \in E$ and $\chi_e > 0$. The fact that $\mathcal{G}^\infty \neq \mathcal{G}$ means that some path or some network has been
 233 selected by the dynamics: in our case, we conjecture (and prove for series parallel graphs or the losange
 234 graph) that the dynamics selects the shortest paths between the nest and the food.

235 Other related models of path formation by reinforcement exist in the literature: for example, Le
 236 Goff and Raimond [LGR18] look at a model of non-backtracking vertex-reinforced random walk, also
 237 inspired from ant behaviour. They show that, in this model, with positive probability, the ant eventually
 238 walks along a cycle of finite edges. This model is very different from ours: the reinforcement is super-
 239 linear instead of linear, there is one ant as opposed to several ants walking successively in the graph, the
 240 underlying graph is infinite (although locally finite), and there is no nest or food and thus no geodesics
 241 involved.

242 Another related model of network formation is the WARM model of van der Hofstad, Holmes, Kuznetsov
 243 and Ruszel [VDHKK⁺16], where, at every time step, an edge is chosen at random and its weight increased
 244 by one (see also [HK17]). The choice of the edge to reinforce at each step is done according to a two-step
 245 procedure that involves super-linear reinforcement. Van der Hofstad et al. prove that the limiting graph
 246 (i.e. the graph consisting of all edges whose normalised weight does not go to zero) is a linearly-stable
 247 equilibrium with positive probability. They conjecture that, if the reinforcement is strong enough, all

248 linearly-stable configuration is a union of trees of diameter at most 3. They prove this conjecture in the
 249 simple case of a triangle graph, i.e. the complete graph on three vertices.

250 A model of network formation with linear reinforcement is the “signaling game” of [HST11, KT16],
 251 where at every time step, “Nature” decides which pairs of neighbours are allowed to communicate for this
 252 round, and each vertex chooses a neighbour with probability proportional to the number of times they
 253 have communicated in the past, and they communicate if they both choose each other and if Nature
 254 allows it. In [KT16], the authors show that the limiting graph (consisting of edges between two vertices
 255 that communicate asymptotically a positive proportion of rounds) is star-shaped with positive probability.

256 **Plan of the paper:** Section 2 contains the proof of Theorem 1.3 (i.e. the series-parallel case), and
 257 Section 3 the proof of Theorem 1.4 (i.e. the losange case). These two sections can be read independently.
 258 Finally we prove Proposition 1.5 in Section 4.

259 2 The loop-erased ant process on series-parallel graphs

260 In this section, we only consider the loop-erased ant process. We define the size of a graph as its number
 261 of edges. For a series-parallel graph G , we define its height, which we denote by $h_{\min}(G)$, as the length
 262 of a shortest path from the source to the sink.

263 2.1 Preliminary lemmas

264 We start with two simple observations. The first one is a direct consequence of the definition of series-
 265 parallel graphs:

266 **Lemma 2.1.** *Let G be a nonempty series-parallel graph. Then, either G is reduced to a single edge (it
 267 has size one), or one can find two non-empty series-parallel subgraphs G_1 and G_2 , such that G is obtained
 268 by merging G_1 and G_2 , either in series or in parallel.*

269 The second observation is the following:

270 **Lemma 2.2.** *Let $\varphi : (0, +\infty)^2 \rightarrow (0, \infty)$ be the function defined by*

$$271 \quad \varphi(x, y) = \frac{1}{\frac{1}{x} + \frac{1}{y}} \quad \text{for all } (x, y) \in (0, +\infty)^2.$$

272 *Then, for all $(x, y), (x', y') \in (0, +\infty)^2$, one has*

273 **(a)** $\varphi(x + x', y + y') \geq \varphi(x, y) + \varphi(x', y')$, and

274 **(b)** $\varphi(x + 1, y + 1) \leq \varphi(x, y) + 1$.

275 *Proof.* Since

$$276 \quad \varphi\left(\frac{x + x'}{2}, \frac{y + y'}{2}\right) = \frac{1}{2} \cdot \varphi(x + x', y + y'),$$

277 proving that φ is concave is enough to prove **(a)**. A simple calculation shows that

$$278 \quad \frac{\partial^2 \varphi}{\partial x^2}(x, y) = \frac{-2y^2}{(x + y)^3}, \quad \frac{\partial^2 \varphi}{\partial y^2}(x, y) = \frac{-2x^2}{(x + y)^3}, \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial x \partial y}(x, y) = \frac{2xy}{(x + y)^3}.$$

279 This implies that the Hessian of φ is everywhere non-positive, and thus that φ is concave as claimed,
 280 which concludes the proof of **(a)**.

281 To prove **(b)**, fix $x > 0$ and set $H(y) := 1 + \varphi(x, y) - \varphi(x + 1, y + 1)$ for all $y > 0$. Using the definition
 282 of φ , we can calculate

$$283 \quad H'(y) = \frac{1}{\left(\frac{y}{x} + 1\right)^2} - \frac{1}{\left(\frac{y+1}{x+1} + 1\right)^2},$$

284 implying that the function H is increasing on $[0, x)$ and decreasing on $(x, +\infty)$. Since $H(0) > 0$, and
 285 $\lim_{y \rightarrow \infty} H(y) = 0$, it follows that $H(y) > 0$ for all positive y , which concludes the proof of **(b)**. \square

286 **Remark.** Item (a) has another simple proof in terms of conductances. Indeed one could note that by
 287 Rayleigh’s monotonicity’s principle, putting first (x, x') in parallel and (y, y') in parallel, and then putting
 288 the two of them in series has a better conductance than first putting (x, y) in series and (x', y') in series,
 289 and then putting them in parallel (to go to the first one, we need to add an edge with infinite conductance).

290 2.2 Our main result in terms of effective conductances

291 The main idea to prove Theorem 1.3 is to reason in terms of the “effective conductance” of the graph.
 292 We interpret the weight of an edge as its “conductance” and let $\mathcal{C}_G(n)$ be the effective conductance (from
 293 the source to the sink) after the n -th walk has reached the sink, and simply write \mathcal{C}_G for the initial
 294 effective conductance. In order to compute the effective conductance of a series parallel graph, one can
 295 use Lemma 2.1 and the two following rules:

- 296 • If G is composed of two graphs G_1 and G_2 merged in parallel, then $\mathcal{C}_G = \mathcal{C}_{G_1} + \mathcal{C}_{G_2}$.
- 297 • If G is composed of two graphs G_1 and G_2 merged in series, then $\mathcal{C}_G = \varphi(\mathcal{C}_{G_1}, \mathcal{C}_{G_2})$.

298 Our main result in terms of effective conductances reads as follows.

299 **Theorem 2.3.** *If G is a series-parallel graph, and $\mathcal{C}_G(n)$ is its conductance after the n -th walker has*
 300 *reached the sink, then, almost surely when $n \rightarrow +\infty$,*

$$301 \frac{\mathcal{C}_G(n)}{n} \rightarrow \frac{1}{h_{\min}(G)},$$

302 where $h_{\min}(G)$ is the graph distance between the source and the sink in G .

303 2.3 Deterministic bounds for the effective conductance of a series-parallel graph after 304 n walks

305 The first step towards proving Theorems 1.3 and 2.3 is the following (deterministic) lemma.

306 **Lemma 2.4. (a)** *Let G be a series-parallel graph with weighted edges and let \mathcal{C}_G be its effective con-*
 307 *ductance from the source to the sink. Consider a self-avoiding path from the source to the sink of length*
 308 *L , and denote by \mathcal{C}'_G the effective conductance of G after the weights of all edges on this path have been*
 309 *increased by one. Then,*

$$310 \frac{1}{L} \leq \mathcal{C}'_G - \mathcal{C}_G \leq 1.$$

311 **(b)** *Let G be a series-parallel graph and consider the loop-erased ant process on G . There exists a constant*
 312 *$C > 0$ depending only on G , such that, almost surely,*

$$313 \mathcal{C}_G(n) \leq \frac{n + C}{h_{\min}(G)}, \quad \text{for all } n \geq 0.$$

314 *Proof.* We first prove **(a)** by induction on the size of the graph. If G has size one, then the result is
 315 immediate since $\mathcal{C}'_G = \mathcal{C}_G + 1$. Now assume that the result holds for all series-parallel graphs with size at
 316 most N (for some integer $N \geq 1$) and consider a graph G of size $N + 1$. By Lemma 2.1 we know that G
 317 is the merging of two non-empty subgraphs G_1 and G_2 , either in parallel or in series. Note that G_1 and
 318 G_2 both have size at most N and thus that the induction hypothesis applies to them.

319 If G_1 and G_2 are in parallel, then $\mathcal{C}_G = \mathcal{C}_{G_1} + \mathcal{C}_{G_2}$. Now since the chosen path is self-avoiding, it either
 320 lies entirely in G_1 or in G_2 . Assume for instance that it lies in G_1 : using the induction hypothesis, we
 321 get that $1 \geq \mathcal{C}'_{G_1} - \mathcal{C}_{G_1} \geq 1/L$, which concludes the proof since $\mathcal{C}'_G = \mathcal{C}'_{G_1} + \mathcal{C}_{G_2}$.

322 If G_1 and G_2 are in series, then first observe that one can write $L = L_1 + L_2$, with L_i the length of
 323 the restriction of the path to G_i , for $i = 1, 2$. Then,

$$324 \quad \mathcal{C}'_G = \frac{1}{\frac{1}{\mathcal{C}'_{G_1}} + \frac{1}{\mathcal{C}'_{G_2}}} \geq \frac{1}{\frac{1}{\mathcal{C}_{G_1 + L_1}} + \frac{1}{\mathcal{C}_{G_2 + L_2}}} \geq \frac{1}{\frac{1}{\mathcal{C}_{G_1}} + \frac{1}{\mathcal{C}_{G_2}}} + \frac{1}{L_1 + L_2} = \mathcal{C}_G + \frac{1}{L},$$

325 using the induction hypothesis for the first inequality and Lemma 2.2(a) for the second one. This
 326 concludes the proof of the lower bound of (a). The proof of the upper bound is entirely similar, using
 327 this time Lemma 2.2(b) instead of Lemma 2.2(a).

328 Let us now prove (b) by induction on the size of the graph again. If G has only one edge (which
 329 connects the source and the sink), then $\mathcal{C}_G(n) = 1 + n$, which proves the result in this case. Assume
 330 by induction that the upper bound holds for all graphs with at most N edges, and assume that G has
 331 $N + 1$ edges. By Lemma 2.1, G consists of two nonempty graphs G_1 and G_2 which are merged either in
 332 parallel or in series, and such that both G_1 and G_2 have at most N edges. By hypothesis, there exist two
 333 constants C_1 and C_2 , such that for all $n \geq 0$,

$$334 \quad \mathcal{C}_{G_1}(n) \leq \frac{n + C_1}{h_{\min}(G_1)}, \quad \text{and} \quad \mathcal{C}_{G_2}(n) \leq \frac{n + C_2}{h_{\min}(G_2)}. \quad (1)$$

335 If G_1 and G_2 are in parallel, then $\mathcal{C}_G(n) = \mathcal{C}_{G_1}(n_1) + \mathcal{C}_{G_2}(n - n_1)$, for some (random) integer $0 \leq n_1 \leq n$,
 336 and the result follows immediately from (1), with the constant $C := C_1 + C_2$, using that $h_{\min}(G) =$
 337 $\min(h_{\min}(G_1), h_{\min}(G_2))$. If G_1 and G_2 are in series, then noting that $h_{\min}(G) = h_{\min}(G_1) + h_{\min}(G_2)$,
 338 we get

$$339 \quad \mathcal{C}_G(n) = \frac{1}{\frac{1}{\mathcal{C}_{G_1}(n)} + \frac{1}{\mathcal{C}_{G_2}(n)}} \stackrel{(1)}{\leq} \frac{1}{\frac{h_{\min}(G_1)}{n + C_1} + \frac{h_{\min}(G_2)}{n + C_2}} \leq \frac{n + \max(C_1, C_2)}{h_{\min}(G_1) + h_{\min}(G_2)} = \frac{n + \max(C_1, C_2)}{h_{\min}(G)}.$$

340 This proves the induction step when G_1 and G_2 are in series, and concludes the proof of the lemma. \square

341 A consequence of this lemma is that one has the deterministic bounds

$$342 \quad \frac{n}{h_{\max}(G)} \leq \mathcal{C}_G(n) - \mathcal{C}_G(0) \leq \frac{n + C}{h_{\min}(G)}, \quad \text{for all } n \geq 0, \quad (2)$$

343 for some constant $C > 0$ and where $h_{\max}(G)$ is the length of the longest self-avoiding path from the source
 344 to the sink of G . In particular, almost surely $\mathcal{C}_G(n) \rightarrow \infty$, as $n \rightarrow \infty$. Note also that if the ants were
 345 always choosing the shortest path, then we would have

$$346 \quad \frac{n}{h_{\min}(G)} \leq \mathcal{C}_G(n) \leq \frac{n + C}{h_{\min}(G)},$$

347 for some constant $C > 0$, for all $n \geq 0$. While the ants usually do not make this optimal choice, we will
 348 see that almost surely the asymptotic behavior of the effective conductance of the graph is still of this
 349 order (with a weaker control on the error term for the lower bound).

350 2.4 Bounds for a generalised version of the model

351 In the following, for any series-parallel graph G , any (series-parallel) subgraph $H \subseteq G$, and any $n \geq 0$,
 352 we let $\mathbf{W}_H^G(n)$ denote the set of weights on the edges of G after the n -th time a path in H has been
 353 reinforced. We also simply write $\mathbf{W}_G(n)$, when $H = G$.

354 In order to implement an induction argument, we need to consider a generalisation of the loop-erased
 355 ant process. The reason for this is that we want the law of the process to be stable under restriction to a
 356 subgraph. Unfortunately, the loop-erased ant process does not fulfil this: for instance if G is the merging

357 of two subgraphs G_1 and G_2 in parallel, then when reinforcing a path in G_1 , an ant on G tends to visit
 358 the source less often than an ant restricted to G_1 . We now explain how we go around this problem.

359 In the original model on a graph G , when the n -th ant starts its random walk from N , it comes back
 360 to N a random geometric number of times, say B_n , and then goes from N to F without returning to N .
 361 We say that the n -th ant did B_n *unsuccessful excursions* in G (i.e. going from N to N without hitting
 362 F), and one *successful excursion* (i.e. going from N to F without returning to N).

363 In the original model, for all $n \geq 1$, B_n is measurable with respect to $\mathcal{F}_{n-1}(G) := \sigma(\mathbf{W}_G(0), \dots, \mathbf{W}_G(n-1))$.
 364 In the generalised model, we allow B_n and its law to be different and to depend on a larger sigma-
 365 field. More precisely, given $\mathcal{F}_{n-1}(G)$ and given some additional integer-valued random variable B_n , we
 366 condition the n -th ant on performing B_n unsuccessful excursions before hitting F , and then reinforce a
 367 path in its range according to the same rule as for the loop-erased ant process, i.e. we increase by one the
 368 weights of the edges along the loop-erasure of the backwards trajectory of the n -th ant. The only case of
 369 interest is when B_n is measurable with respect to some sigma-field of the type $\sigma(\mathbf{W}_{G'}^G(0), \dots, \mathbf{W}_{G'}^G(n-1))$,
 370 where G' is some series-parallel graph containing G ; however the proofs of the next results work in full
 371 generality, without assuming anything on the random variables B_n .

372 For a series-parallel graph G , we still let $\mathcal{C}_G(n)$ denote the effective conductance of graph G after n
 373 walkers have performed their walks and updated the weights in the generalised version of the loop-erased
 374 ant process described above. We set

$$375 \quad \alpha(G) := \frac{h_{\min}(G)}{h_{\min}(G) + 1}. \quad (3)$$

376 The following proposition, together with Lemma 2.4, implies Theorem 2.3:

377 **Proposition 2.5.** *Consider a generalised version of the loop-erased ant process on a series-parallel*
 378 *graph G , and let $\alpha = \alpha(G)$. There exists a real random variable K_G , such that almost surely K_G is*
 379 *finite, and for all $n \geq 1$,*

$$380 \quad (i) \quad \mathcal{C}_G(n) \geq \frac{n - K_G \cdot n^\alpha}{h_{\min}(G)};$$

381 *(ii) after n steps, the conditional probability that the $(n + 1)$ -th walk reinforces a geodesic path is larger*
 382 *than $1 - K_G \cdot n^{\alpha-1}$.*

383 *Proof.* We reason by induction on the size of G : if G has size 1, then $\mathcal{C}_G(n) = n + 1$ almost surely, implying
 384 that the result holds. Let us now assume that the result holds for all series-parallel graphs of size at most
 385 N , and consider a graph G of size $N + 1$. By Lemma 2.1 we know that G is the merging of two nonempty
 386 subgraphs G_1 and G_2 , either in parallel or in series. We denote N_1 and N_2 the sources of G_1 and G_2 , F_1
 387 and F_2 their sinks.

388 **Case 1: G_1 and G_2 are in series.** Assume without loss of generality that G_1 is on the top of G_2
 389 (meaning that the sink F_1 of G_1 coincides with the source N_2 of G_2). First note that each ant performing
 390 its walk in G will reinforce one path in G_1 and one path in G_2 . Moreover, by definition of the loop-erasure
 391 process, the path that is reinforced in G_1 is entirely determined by the trajectory of the walk up to its first
 392 hitting time of $F_1 = N_2$, while the path that is reinforced in G_2 is entirely determined by the trajectory
 393 of the ants after this hitting time of N_2 . As a consequence, conditionally on the number of times the
 394 walker returns to N before first hitting N_2 , the laws of the two paths that are reinforced in G_1 and G_2
 395 are independent.

396 Furthermore, for each n , the number of unsuccessful excursions in G_1 (resp. G_2) that are made by
 397 the n -th walk before first hitting N_2 (resp. after first hitting N_2) is a measurable function of $\mathbf{W}_G(n - 1)$
 398 and the number B_n of unsuccessful excursions that are prescribed in G . Therefore, the restrictions
 399 of the process to G_1 and G_2 are generalised versions of the loop-erased ant process, as defined before
 400 Proposition 2.5. Therefore, we can use the induction hypothesis for G_1 and G_2 : there exist two random

401 variables $K_1, K_2 \in (0, \infty)$, such that with $\alpha_1 = \alpha(G_1)$, and $\alpha_2 = \alpha(G_2)$,

$$402 \quad \mathcal{C}_{G_1}(n) \geq \frac{n - K_1 n^{\alpha_1}}{h_{\min}(G_1)} \quad \text{and} \quad \mathcal{C}_{G_2}(n) \geq \frac{n - K_2 n^{\alpha_2}}{h_{\min}(G_2)}.$$

403 If we denote by $\beta = \max(\alpha_1, \alpha_2)$, and by $K = \max(K_1, K_2)$, then

$$404 \quad \mathcal{C}_G(n) = \frac{1}{\frac{1}{\mathcal{C}_{G_1}(n)} + \frac{1}{\mathcal{C}_{G_2}(n)}} \geq \frac{1}{\frac{h_{\min}(G_1)}{n - K_1 n^{\alpha_1}} + \frac{h_{\min}(G_2)}{n - K_2 n^{\alpha_2}}} \\ 405 \quad \geq \frac{n - K n^\beta}{h_{\min}(G_1) + h_{\min}(G_2)} = \frac{n - K n^\beta}{h_{\min}(G)},$$

407 since $h_{\min}(G) = h_{\min}(G_1) + h_{\min}(G_2)$; which concludes the induction argument for Part (i) because, by
408 definition, $\beta \leq \alpha(G)$.

409 For Part (ii) we just observe that, by the induction hypothesis and a union bound, the conditional
410 probability that the n -th walker does not reinforce a geodesic path is smaller than $K_1 n^{\alpha_1 - 1} + K_2 n^{\alpha_2 - 1} \leq$
411 $K n^{\beta - 1}$, which concludes the induction argument in the case when G_1 and G_2 are merged in series.

412 **Case 2: G_1 and G_2 are in parallel.** We start again by showing that the restrictions of the process
413 on G_1 and G_2 are generalised versions of the loop-erased model as defined before Proposition 2.5. For all
414 integers n , we denote by $N_i(n)$ the number of times a path in G_i have been reinforced after n ants have
415 performed their walks in G : one has

$$416 \quad \mathcal{C}_G(n) = \mathcal{C}_{G_1}(N_1(n)) + \mathcal{C}_{G_2}(N_2(n)). \quad (4)$$

417 We also let $(\tau_k^{(i)})_{k \geq 1}$ be the random times when the process N_i increases by one, i.e. the times when an
418 ant reinforces a path in G_i . For all $n \geq 1, k \geq 0, i \in \{1, 2\}$, given $\tau_{k-1}^{(i)}$, the time to wait until another ant
419 reinforces a path in G_i (i.e. $\tau_k^{(i)} - \tau_{k-1}^{(i)}$) and the number $B_k^{(i)}$ of unsuccessful excursions made by this ant
420 (the $\tau_k^{(i)}$ -th ant) in G_i are both measurable functions of $\mathbf{W}(\tau_{k-1}^{(i)})$ and of the total number of unsuccessful
421 excursions performed in G by all ants between times $\tau_{k-1}^{(i)} + 1$ and $\tau_k^{(i)}$. Moreover, by definition, given this
422 information, the reinforced path in G_i is chosen by performing $B_k^{(i)}$ independent unsuccessful excursions,
423 plus one additional independent successful excursion, and using the loop-erasure rule. Thus we can use
424 the induction hypothesis for G_1 and G_2 .

425 In the following, we use the fact that, at any time n , the $(n + 1)$ -th walker performs its successful
426 excursion in G_i with probability $\mathcal{C}_{G_i}(n)/(\mathcal{C}_{G_1}(n) + \mathcal{C}_{G_2}(n))$, for $i = 1, 2$. Indeed, this follows from the fact
427 the law of the successful excursion of each ant walking on G is by definition independent of the number of
428 unsuccessful excursions performed by this ant and of their trajectories. Moreover, for the simple random
429 walk in G (that is if we were considering the original model), the probability to reinforce a path in G_i is
430 given by the ratio of the effective conductances, and this happens if and only if the successful excursion
431 belongs to G_i .

432 **Case 2.1:** We first assume that $h_{\min}(G_1) = h_{\min}(G_2)$. Using the induction hypothesis, there exist
433 two random variables $K_1, K_2 \in (0, \infty)$ such that, almost surely,

$$434 \quad \mathcal{C}_G(n) \geq \frac{N_1(n) - K_1 N_1(n)^\alpha}{h_{\min}(G_1)} + \frac{N_2(n) - K_2 N_2(n)^\alpha}{h_{\min}(G_2)} \\ 435 \quad \geq \frac{N_1(n) + N_2(n)}{h_{\min}(G)} - \frac{K(N_1(n)^\alpha + N_2(n)^\alpha)}{h_{\min}(G)},$$

437 with $\alpha = \alpha(G)$ (see Equation (3) for the definition of $\alpha(G)$) and $K = K_1 + K_2$. This concludes the
438 induction argument for Part (i), since by concavity of the map $x \mapsto x^\alpha$, we have

$$439 \quad N_1(n)^\alpha + N_2(n)^\alpha \leq 2^{1-\alpha} n^\alpha. \quad (5)$$

Concerning Part (ii), note that using the induction hypothesis, if the $(n+1)$ -th walker makes its successful excursion in G_1 , then the probability that its range contains a geodesic path of G_1 is larger than $1 - K_1 N_1(n)^{\alpha-1}$, and similarly for G_2 . Considering the complement and using a union bound, we deduce that the probability that the $(n+1)$ -th walker reinforces a geodesic path of G is at least

$$1 - K_1 N_1(n)^{\alpha-1} - K_2 N_2(n)^{\alpha-1} \geq 1 - Kn^{\alpha-1},$$

which concludes the proof of the induction argument in the case when $h_{\min}(G_1) = h_{\min}(G_2)$.

Case 2.2: We now assume that $h_{\min}(G_1) \neq h_{\min}(G_2)$, and without loss of generality $h_{\min}(G_1) < h_{\min}(G_2)$, which implies $\alpha(G) = \alpha(G_1)$ (see Equation (3) for the definition of $\alpha(G)$). Using the induction hypothesis, we have that there exists a random variable $K_1 \in (0, \infty)$, such that

$$\mathcal{C}_{G_1}(N_1(n)) \geq \frac{N_1(n)}{h_{\min}(G_1)} \cdot (1 - K_1 N_1(n)^{\alpha-1}). \quad (6)$$

For small values of $N_1(n)$, this lower bound can be negative (recall that, by definition, $\alpha = \alpha(G) < 1$; see Equation (3)); a better lower bound for small values of $N_1(n)$ is given by

$$\mathcal{C}_{G_1}(N_1(n)) \geq \mathcal{C}_{G_1}(0). \quad (7)$$

By Lemma 2.4(b), there exists a constant $C_2 > 0$ (only depending on G_2), such that

$$\mathcal{C}_{G_2}(n - N_1(n)) \leq \frac{n - N_1(n) + C_2}{h_{\min}(G_2)} \leq \frac{n - N_1(n) + C_2}{h_{\min}(G_1) + 1}, \quad (8)$$

because, by assumption, $h_{\min}(G_2) \geq h_{\min}(G_1) + 1$. For all $b > 0$, we define the function φ_b such that, for all $i \geq 0$,

$$\varphi_b(i) := \max\left(\mathcal{C}_{G_1}(0), \frac{i - bi^\alpha}{h_{\min}(G_1)}\right). \quad (9)$$

We also define the function ψ such that, for all $i \geq 0$,

$$\psi(i) := \frac{\alpha(i + C_2)}{h_{\min}(G_1)}. \quad (10)$$

By Equations (6), (7), and (8) we get that the probability p_n that the $(n+1)$ -th reinforces a geodesic path of G_1 , conditionally on $\mathbf{W}_G(n)$, satisfies

$$p_n = \frac{\mathcal{C}_{G_1}(N_1(n))}{\mathcal{C}_{G_1}(N_1(n)) + \mathcal{C}_{G_2}(n - N_1(n))} \geq \frac{\varphi_{K_1}(N_1(n))}{\varphi_{K_1}(N_1(n)) + \psi(n - N_1(n))}, \quad (11)$$

for all $n \geq 0$. We now prove that, almost surely, there exists a finite random variable $K > 0$, such that

$$N_1(n) \geq n - Kn^\alpha, \quad \text{for all } n \geq 1. \quad (12)$$

This is enough to conclude the proofs of the induction step for both Parts (i) and (ii). Indeed, on the one hand, we get that, for all $n \geq 1$,

$$\mathcal{C}_G(n) \geq \mathcal{C}_{G_1}(N_1(n)) \geq \frac{N_1(n)(1 - K_1 N_1(n)^{\alpha-1})}{h_{\min}(G_1)} \geq \frac{n - (K + K_1)n^\alpha}{h_{\min}(G)},$$

which concludes the proof of the induction step of Part (i). And, on the other hand, using (2), we get that the probability of not reinforcing a geodesic path in G is smaller than

$$K_1 N_1(n)^{\alpha-1} + (1 - p_n) \leq (K_1 + h_{\max}(G)K)n^{\alpha-1},$$

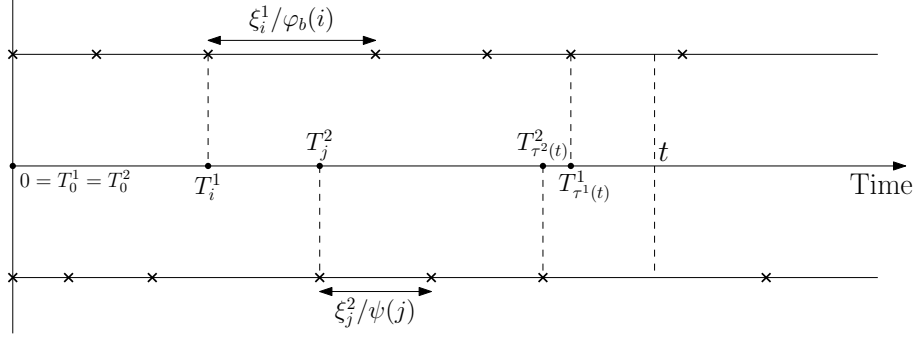


Figure 6: Rubin's construction for the proof of Proposition 2.5 (Case 2.2 in the proof). On the top line, the intervals between crosses are the $\xi_i^1/\varphi_b(i)$ and, similarly, on the bottom line, the intervals between crosses are the $\xi_j^2/\psi(j)$. On the middle line, we show how the T_i^1 's and T_i^2 's are defined as the partial sums of these interval lengths and how τ_t^1 and τ_t^2 are defined for a given time $t > 0$.

472 which concludes the proof of the induction step of Part (ii). Therefore, to conclude the proof, it only
 473 remains to prove Equation (12).

474 If K_1 was a fixed constant, the conclusion would come by simply analysing the generalised urn process
 475 associated to φ_{K_1} and ψ . But here, K_1 is a random variable that depends on the whole history of the
 476 process. To go around this issue, we are going to define a family of generalised Pólya urns, and couple all
 477 of them with the process $(N_1(n))_{n \geq 0}$, in such a way that almost surely $(N_1(n))_{n \geq 0}$ will dominate at least
 478 one of those urns. To be more precise, for all $b > 0$, we define the Markov process $(R_n^b)_{n \geq 0}$, by $R_0^b = 0$,
 479 and for all $n \geq 0$,

$$480 \quad q_n^b := \mathbb{P}(R_{n+1}^b = R_n^b + 1 \mid R_n^b) = 1 - \mathbb{P}(R_{n+1}^b = R_n^b \mid R_n^b) = \frac{\varphi_b(R_n^b)}{\varphi_b(R_n^b) + \psi(n - R_n^b)}, \quad (13)$$

481 where φ_b and ψ are defined in Equations (9) and (10) respectively. We now fix some $b > 0$ and show that
 482 there exists an almost surely finite random variable C_b , such that

$$483 \quad n - R_n^b \leq C_b n^\alpha, \quad \text{for all } n \geq 0. \quad (14)$$

484 To prove Equation (14), it is convenient to use Rubin's algorithm, which was introduced in Davis's
 485 paper on reinforced random walks [Dav90]. Consider $\{\xi_i^1\}_{i \geq 0}$ and $\{\xi_i^2\}_{i \geq 0}$ two independent sequences of
 486 independent mean-one exponential random variables, and define, for all $n \geq 1$,

$$487 \quad T_n^1 := \sum_{k=0}^{n-1} \frac{\xi_k^1}{\varphi_b(k)}, \quad \text{and} \quad T_n^2 := \sum_{k=0}^{n-1} \frac{\xi_k^2}{\psi(k)}.$$

488 Set also $T_0^1 = T_0^2 = 0$ and, for all $t > 0$ (see Figure 6),

$$489 \quad \tau^1(t) := \sup\{n \geq 0 : T_n^1 \leq t\}, \quad \text{and} \quad \tau^2(t) := \sup\{n \geq 0 : T_n^2 \leq t\}.$$

490 It follows from standard properties of independent exponential random variables that, for any $t > 0$,
 491 conditionally on the fact that $\tau^1(t) = n_1$, and $\tau^2(t) = n_2$, the probability $q_{n_1+n_2}^b$ that $R_{n_1+n_2}^b$ increases
 492 by one at the next step is also equal to the probability of $T_{n_1+1}^1$ being smaller than $T_{n_2+1}^2$.

493 As a consequence if we let $t_n = \inf\{t \geq 0 : \tau^1(t) + \tau^2(t) \geq n\}$, then the process $(\tau^1(t_n))_{n \geq 0}$ has the
 494 same law as $(R_n^b)_{n \geq 0}$. Note that, since they are bounded in L^2 , the series

$$495 \quad \sum_{k=0}^{\infty} \frac{\xi_k^1 - 1}{\varphi_b(k)} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\xi_k^2 - 1}{\psi(k)}$$

496 converge almost surely. In particular

$$497 \quad T_n^1 = \sum_{k=0}^{n-1} \frac{1}{\varphi_b(k)} + \mathcal{O}(1) = \log n + \mathcal{O}(1), \quad \text{and} \quad T_n^2 = \sum_{k=0}^{n-1} \frac{1}{\psi(k)} + \mathcal{O}(1) = \frac{1}{\alpha} \log n + \mathcal{O}(1), \quad (15)$$

498 where the $\mathcal{O}(1)$ are almost surely bounded. Moreover, by definition, one has

$$499 \quad \sup_{n \geq 0} |T_{\tau^1(t_n)}^1 - T_{\tau^2(t_n)}^2| \leq \sup_{n \geq 0} \max \left(\frac{\xi_n^1}{\varphi_b(n)}, \frac{\xi_n^2}{\psi(n)} \right),$$

500 from which it follows that

$$501 \quad T_{\tau^1(t_n)}^1 = T_{\tau^2(t_n)}^2 + \mathcal{O}(1),$$

502 where $\mathcal{O}(1)$ stands for an almost surely finite random variable. Together with (15), this entails $\tau^2(t_n) \leq$
 503 $C_b n^\alpha$, for all $n \geq 0$, and some almost surely finite random variable C_b , which concludes the proof of
 504 Equation (14).

505 To conclude the proof of Equation (12), we only need to couple the family of processes $(R_n^b)_{n \geq 0}$, $b > 0$,
 506 with $(N_1(n))_{n \geq 0}$ so that, almost surely, there exists $K > 0$ such that $N_1(n) \geq R_n^K$ and $p_n \geq q_n^K$, for
 507 all $n \geq 0$. To do this coupling, we use a sequence $(U_n)_{n \geq 1}$ of i.i.d. uniform random variables on $[0, 1]$,
 508 independent of everything else. We start the processes so that $N_1(0) = 0$ and $R_0^b = 0$ for all $b > 0$. Then,
 509 at each time step $n \geq 0$, set $N_1(n+1) = N_1(n) + 1$ if and only if $p_n \geq U_{n+1}$ and, similarly for all $b > 0$,
 510 $R_{n+1}^b = R_n^b + 1$ if and only if $q_n^b \geq U_{n+1}$. By induction on n , we can prove that, in this coupling, for all
 511 $b \geq K_1$, for all $n \geq 1$, $N_1(n) \geq R_n^b$. Indeed, first note that, by Equation (11), for all $b \geq K_1$, $N_1(n) \geq R_n^b$
 512 implies $p_n \geq q_n^b$. Moreover, if $N_1(n) \geq R_n^b$ and $p_n \geq q_n^b$, then $N_1(n+1) \geq R_{n+1}^b$, which concludes the
 513 proof by induction: we get that, for all $b \geq K_1$, $N_1(n) \geq R_n^b \geq n - C_b n^\alpha$. This concludes the proof of
 514 Equation (12), thus the proof of the induction step in Case 2.2, and thus the proof of Proposition 2.5
 515 altogether. \square

516 2.5 Proof of Theorem 1.3

517 Since the original model is a particular case of the generalised model of Section 2.4, it is enough to prove
 518 that Theorem 1.3 holds in the generalised model.

519 By Proposition 2.5, for any edge e that is not contained in a geodesic path, one has $W_e(n)/n \rightarrow 0$,
 520 when $n \rightarrow +\infty$. Thus it only remains to show that, for every edge e that lies on a geodesic path, $W_e(n)/n$
 521 converges to some random variable χ_e , which is almost surely non-zero.

522 The proof is done by induction on the size of G . If G has size one, the result is straightforward.
 523 We now assume that the result holds for all series-parallel graphs of size at most N , and consider a
 524 series-parallel graph G of size $N+1$. Once again, by Lemma 2.1 we know that G is the merging of two
 525 non-empty subgraphs G_1 and G_2 . If G_1 and G_2 are in series, then the result for G follows immediately
 526 from the induction hypothesis.

527 Let us now assume that G_1 and G_2 are merged in parallel. If $h_{\min}(G_1) \neq h_{\min}(G_2)$, and for instance
 528 if $h_{\min}(G_1) < h_{\min}(G_2)$, then the proof in the previous subsection shows that a fraction $1 - o(1)$ of the
 529 ants chooses a path in G_1 , and then the result follows from the induction hypothesis.

530 If $h_{\min}(G_1) = h_{\min}(G_2)$, we first show that $\liminf N_i(n)/n > 0$, almost surely for all $i \in \{1, 2\}$. To do
 531 this, we use again Rubin's construction; the argument is very similar to the one given in Case 2.2 of the
 532 proof of Proposition 2.5. We only briefly indicate how to adapt the proof to show that $\liminf N_i(n)/n > 0$
 533 in the present case. We aim at coupling the process $(N_1(n))_{n \geq 0}$ with a family of processes $(R_n^b)_{n \geq 0}$, $b > 0$.
 534 We define φ_b as in Equation (9) and set $\psi(i) = (i + C_2)/h_{\min}(G_1)$ for all integers i (compare with
 535 Equation (10)). We then define R_n^b as in Equation (13). One can show that, on the one hand, for any
 536 $b > 0$, there exists a random variable $c_b > 0$, such that almost surely for all $n \geq 1$, $R_n^b \geq c_b n$. And, on
 537 the other hand, there exists a random $b > 0$ such that $N_1(n) \geq R_n^b$ for all $n \geq 0$ almost surely. Hence,

538 we deduce that almost surely $\liminf N_1(n)/n > 0$, as claimed. In other words, almost surely, a positive
 539 fraction of the ants chooses a path in G_1 , and by symmetry the same holds for G_2 .

540 We now show that $N_1(n)/n$ converges almost surely when n tends to infinity. To do this, we show
 541 that $X(n) := N_1(n)/n$ is a stochastic approximation. Indeed, we have, for all $n \geq 1$,

$$542 \quad X(n+1) = X(n) + \frac{\Delta M_n + h_n}{n+1},$$

543 where $\Delta M_n = N_1(n+1) - N_1(n) - p_n$, with p_n as defined in (11), and $h_n := p_n - X(n)$. Iterating the
 544 above equation, we get that, for all $n \geq 1$,

$$545 \quad X(n+1) = X(1) + \sum_{k=1}^n \frac{\Delta M_k + h_k}{k}.$$

546 Note that, by definition, the martingale increment ΔM_k is bounded by 1 in absolute value, and thus the
 547 martingale $\sum_{k=1}^n \Delta M_k/k$, is bounded in L^2 , and hence almost surely convergent. Using the definition of p_n
 548 (see Equation (11)), together with Lemma 2.4(b), Proposition 2.5, and the fact that $\liminf N_i(n)/n > 0$,
 549 for $i = 1, 2$, one can show that, almost surely when n tends to infinity, $h_n = \mathcal{O}(n^{\alpha(G_1)-1})$, where we
 550 recall that, by definition (see Equation (3)), $\alpha(G_1) < 1$. This implies that the sum $\sum_{k=1}^n h_k/k$ is almost
 551 surely convergent, and thus that $X(n)$ converges almost surely, as claimed. Together with the induction
 552 hypothesis applied to G_1 and G_2 , this allows us to conclude the induction step for that last case (G_1 and
 553 G_2 merged in parallel and $h_{\min}(G_1) = h_{\min}(G_2)$).

554 Altogether, this concludes the proof of Theorem 1.3.

555 3 The geodesic ant process on the losange graph

556 We prove here Theorem 1.4 concerning the losange graph; in this section, we thus only consider the
 557 (uniform-)geodesic version of the model (as discussed in Section 1.3, the rule about how to choose the
 558 geodesic to reinforce when there are several in the trace of the walker is irrelevant here since the trace
 559 of a walker can only contain one geodesic). The proof relies primarily on the fact that the sequence of
 560 weights is the solution of a certain stochastic recursion formula, which we state in Lemma 3.1 below.

561 Recall Figure 4 of the losange graph, and define for $n \geq 0$,

$$562 \quad \mathbf{W}(n) := (W_1(n), W_2(n), W_3(n), W_4(n), W_5(n)), \quad \text{and} \quad \hat{\mathbf{W}}(n) = \frac{\mathbf{W}(n)}{n+2}, \quad (16)$$

563 where $W_i(n)$ denotes the weight of edge i after n walkers (or ants) have reached the food. Then for
 564 $w = (w_1, \dots, w_5) \in [0, 1]^5$, denote by $p_{12}(w)$ the probability that a walker reinforces edges 1 and 2, when
 565 the weights of the five edges of the losange graph are respectively w_1, \dots, w_5 . Define similarly $p_{135}(w)$,
 566 $p_{234}(w)$, and $p_{45}(w)$, and set

$$567 \quad F(w) := p_{12}(w)(1, 1, 0, 0, 0) + p_{135}(w)(1, 0, 1, 0, 1) + p_{45}(w)(0, 0, 0, 1, 1) + p_{234}(w)(0, 1, 1, 1, 0) - w. \quad (17)$$

568 Lemma 3.1 expresses the fact that the whole study of the process $(\mathbf{W}(n))_{n \geq 0}$ takes place in the subset of
 569 $[0, 1]^5$, defined as

$$570 \quad \mathcal{E} := \left\{ (w_1, w_2, w_3, w_4, w_5) \in [0, 1]^5 : \begin{array}{ll} w_1 + w_4 = 1, & \text{and } w_2 + w_5 = 1 \\ |w_1 - w_2| \leq w_3 & \text{and } |w_5 - w_4| \leq w_3 \\ w_1 + w_2 \geq w_3 & \text{and } w_4 + w_5 \geq w_3 \end{array} \right\}. \quad (18)$$

571 Let us briefly explain the restrictions above. Note that each walk can only reinforce one of the following
 572 sets of edges:

- 573 (i) edge 1 and edge 2;
574 (ii) edge 4 and edge 5;
575 (iii) edge 1, edge 3 and edge 5;
576 (iv) edge 4, edge 3 and edge 2.

577 One can see above that, at each round, precisely one of edge 1 or edge 4 is reinforced and precisely one
578 of edge 2 or edge 5 is reinforced. Hence, we have that $w_1 + w_4 = w_2 + w_5 = 1$. Next, the only cases
579 where edge 1 is reinforced but not edge 2, or edge 2 is reinforced but not edge 1 are in scenarios (iii) and
580 (iv), in which cases edge 3 is reinforced. Therefore, $|w_1 - w_2| \leq w_3$, and by symmetry $|w_4 - w_5| \leq w_3$.
581 Finally, again using (iii) and (iv), every time edge 3 is reinforced edge 1 or edge 2 is reinforced. Therefore
582 $w_1 + w_2 \geq w_3$ and by symmetry $w_4 + w_5 \geq w_3$.

583 Using further the definition of the ant process, we obtain Lemma 3.1 below. We use now the shorthand
584 notation \mathbb{E}_n to denote the conditional expectation with respect to the sigma-field \mathcal{F}_n (where $(\mathcal{F}_n)_{n \geq 0}$ is
585 the natural filtration of the process).

586 **Lemma 3.1.** *For all $n \geq 0$, $\hat{\mathbf{W}}(n) \in \mathcal{E}$. Furthermore,*

$$587 \quad \hat{\mathbf{W}}(n+1) = \hat{\mathbf{W}}(n) + \frac{1}{n+3} (F(\hat{\mathbf{W}}(n)) + \Delta \mathbf{M}(n+1)), \quad (19)$$

588 where $\Delta \mathbf{M}(n+1) = Y(n+1) - \mathbb{E}_n[Y(n+1)]$, and $Y(n+1) := \mathbf{W}(n+1) - \mathbf{W}(n)$.

589 As mentioned in the introduction, this losange case can be seen as an intricate coupling between a
590 biased urn (the ants that reinforce edge 3 versus all others, i.e. $W_3(n)$ vs. $n - W_3(n)$) and a standard Pólya
591 urn (the ants that reinforce edges 1 and 2 vs. the ants that reinforce edges 4 and 5). In Subsection 3.1
592 we treat the first urn by proving that $W_3(n)/n$ converges to 0 almost surely, at a polynomial speed. The
593 ‘‘Pólya’’ part is treated in two additional steps: In Subsection 3.2 we show that $\hat{\mathbf{W}}(n)$ converges almost
594 surely to some limit in $[0, 1]^5$, and, in Subsection 3.3, we prove that the limit is non-degenerate, in the
595 sense that it does not charge the extremal points $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$. In terms of the ants, this
596 means that the ants *find both geodesics* and not just one of them. Interestingly, ruling out these extremal
597 cases is the most delicate part of the proof.

598 3.1 On the convergence of $W_3(n)/n$ to 0

599 In this section, we prove here the following result.

600 **Proposition 3.2.** *Almost surely, as $n \rightarrow +\infty$, one has $W_3(n)/n \rightarrow 0$. More precisely, there exists*
601 *$\alpha \in (0, 1)$, such that almost surely,*

$$602 \quad \lim_{n \rightarrow \infty} \frac{W_3(n)}{n^\alpha} = 0.$$

603 The first idea of the proof is to compare $W_3(n)$ with the number of red balls in a two-colour Friedman-
604 like urn defined as follows:

605 **Lemma 3.3.** *We define a Markov process $(R_n)_{n \geq 0}$ as follows: first $R_0 = 1$, and for all $n \geq 0$, we set*
606 *$R_{n+1} = R_n + A_{n+1}$, where*

$$607 \quad \mathbb{P}(A_{n+1} = 1 \mid R_n) = 1 - \mathbb{P}(A_{n+1} = 0 \mid R_n) := \frac{R_n}{n+2} \cdot \frac{\left(\frac{R_n}{n+2}\right)^2 + \frac{1}{2}}{\frac{R_n}{n+2} + \frac{1}{2}}.$$

608 *Then almost surely when $n \rightarrow +\infty$, we have $R_n/n \rightarrow 0$.*

609 *Proof.* Let us define $Z_n := R_n/(n+2)$, for all $n \geq 0$. We use stochastic approximation: by definition, we
 610 have that, for all $n \geq 0$,

$$611 \quad Z_{n+1} = \frac{R_{n+1}}{n+3} = \frac{R_n + A_{n+1}}{n+3} = \frac{R_n}{n+2} \cdot \frac{n+2}{n+3} + \frac{A_{n+1}}{n+3} = Z_n + \frac{1}{n+3}(A_{n+1} - Z_n).$$

612 For $n \geq 0$, set $\Delta M_{n+1} = A_{n+1} - \mathbb{E}[A_{n+1} | R_n]$. By definition of the model, we have

$$613 \quad \mathbb{E}[A_{n+1} | R_n] = Z_n \cdot \frac{Z_n^2 + \frac{1}{2}}{Z_n + \frac{1}{2}},$$

614 implying that

$$615 \quad Z_{n+1} = Z_n + \frac{1}{n+3}(G(Z_n) + \Delta M_{n+1}),$$

616 where, for all $x \in [0, 1]$,

$$617 \quad G(x) = x \cdot \frac{x^2 + \frac{1}{2}}{x + \frac{1}{2}} - x.$$

618 Note that $G(x) \leq 0$ for all $x \in [0, 1]$. Thus $(Z_n)_{n \geq 0}$ is a non-negative supermartingale, and converges
 619 almost surely. Moreover, by definition $|\Delta M_n| \leq 1$, for all $n \geq 0$, and thus the martingale

$$620 \quad \widetilde{M}_n := \sum_{i=1}^{n-1} \frac{\Delta M_{i+1}}{i+3},$$

621 converges almost surely, since it is bounded in L^2 . It follows that the series $\sum G(Z_n)/n$ also converges
 622 almost surely, which implies that the limit of $(Z_n)_{n \geq 0}$ is necessarily a zero of G , that is either 0 or 1. To
 623 see that $Z_n \rightarrow 0$ almost surely, we couple $(Z_n)_{n \geq 0}$ with a Pólya urn: this coupling is based on the fact
 624 that, by definition and because $\frac{x^2+1}{x+1} \leq 1$ for all $x \in [0, 1]$, we have

$$625 \quad \mathbb{P}(A_{n+1} = 1 | Z_n) \leq Z_n.$$

626 Thus if we define a process $(U_n)_{n \geq 0}$ such that $U_0 = Z_0$ and, for all $n \geq 0$,

$$627 \quad \mathbb{P}(U_{n+1} = U_n + 1 | U_n) = 1 - \mathbb{P}(U_{n+1} = U_n | U_n) = U_n,$$

628 then $(U_n)_{n \geq 0}$ and $(Z_n)_{n \geq 0}$ can be coupled in a way that $Z_n \leq U_n$ almost surely for all $n \geq 0$. It is known
 629 that $U_n \rightarrow U$ almost surely when $n \rightarrow +\infty$, where U is uniform on $[0, 1]$. Thus Z_n cannot converge to 1
 630 and thus converges to 0 almost surely when $n \rightarrow +\infty$. \square

631 The next step to prove Proposition 3.2 is to compute the probability that a walker reinforces the
 632 middle edge 3. Recall the definition (18) of the set \mathcal{E} .

633 **Lemma 3.4.** *One has for all $w \in \mathcal{E}$,*

$$634 \quad p_{135}(w) = \frac{w_1 w_3 w_5}{(w_2 + w_3 + w_1 w_4)(w_4 + w_5) + w_2 w_3 + w_1 w_3 w_4}.$$

635 *Proof.* We call “left” vertex the vertex linked to edges 1, 2 and 3, and “right” vertex the vertex between
 636 edges 3, 4 and 5. To reinforce edges 1, 3 and 5, a walker has to

- 637 (i) go through edge 1 in its first step,
- 638 (ii) then, from the left vertex, reach the right vertex before going through edge 2,
- 639 (iii) finally, from the right vertex, reach the food before going through edge 2 or 4.

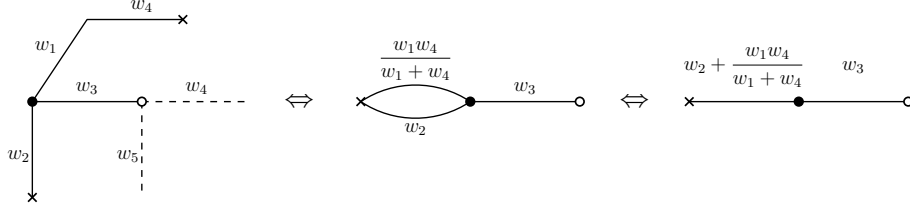


Figure 7: Calculation of the probability of (ii) , the event that a random walker starting at the black dot reaches the white dot before reaching the crosses. The dashed edges in the left-hand side picture have no effect on the calculation and can be removed. In terms of effective conductances between the black dot and the crosses and the black dot and white dot, these three graphs are equivalent.

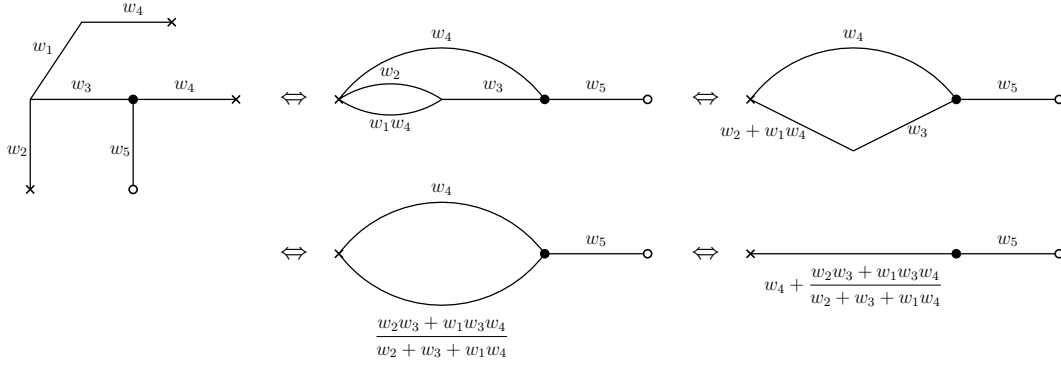


Figure 8: Calculation of the probability of (iii) in the case $w_1 + w_4 = 1$, the event that a random walker starting at the black dot reaches the white dot before reaching the crosses. In terms of effective conductances between the black dot and the crosses and the black dot and white dot, these five graphs are equivalent.

640 Let us denote by p_i , p_{ii} and p_{iii} the respective probabilities of these three events; we thus have $p_{135}(w) =$
 641 $p_i p_{ii} p_{iii}$. First note that

$$642 \quad p_i = \frac{w_1}{w_1 + w_4} = w_1,$$

643 using for the last equality that $w \in \mathcal{E}$. To calculate p_{ii} and p_{iii} , we use effective conductances. One can
 644 check that p_{ii} is the probability that a random walker starting from the black dot in the left-hand side
 645 of Figure 7 reached the white dot before reaching one of the crosses. In Figure 7, we use the parallel
 646 and series formulas for effective conductances to simplify the left-hand side graph into the equivalent (in
 647 terms of effective conductances) right-hand side graph. In the right-hand side graph, it is easy to see that
 648 the probability to reach the white dot before the cross starting from the black dot is

$$649 \quad p_{ii} = \frac{w_3}{w_2 + w_3 + \frac{w_1 w_4}{w_1 + w_4}} = \frac{w_3}{w_2 + w_3 + w_1 w_4},$$

650 using again that $w \in \mathcal{E}$ for the last equality. Similarly, one can check that p_{iii} is the probability that a
 651 walker starting from the black dot in the left-hand side of Figure 8 reaches the white dot before reaching
 652 one of the crosses. Using the calculation of effective conductances done in Figure 8, we eventually get
 653 that

$$654 \quad p_{iii} = \frac{w_5}{w_4 + w_5 + \frac{w_2 w_3 (w_1 + w_4) + w_1 w_3 w_4}{(w_2 + w_3)(w_1 + w_4) + w_1 w_4}} = \frac{w_5}{w_4 + w_5 + \frac{w_2 w_3 + w_1 w_3 w_4}{w_2 + w_3 + w_1 w_4}},$$

655 which concludes the proof, since, for all $w \in \mathcal{E}$, $w_1 + w_4 = 1$. \square

656 We deduce the following result, proving the first part of Proposition 3.2.

657 **Lemma 3.5.** *One has for all $w \in \mathcal{E}$,*

$$658 \quad p_{135}(w) + p_{234}(w) \leq w_3 \cdot \frac{w_3^2 + \frac{1}{2}}{w_3 + \frac{1}{2}},$$

659 *and as a consequence almost surely,*

$$660 \quad \lim_{n \rightarrow \infty} \frac{W_3(n)}{n} = 0.$$

661 *Proof.* The idea is the following: we run the ants walk from time 0, and simultaneously, we consider an
 662 urn that contains black and red balls. We call this urn the “ants urn”. At time zero, we put one black
 663 ball and one red ball in the urn, and everytime an ant reaches the food in the ants walk process, we add
 664 a ball into the urn: this ball is red if edge number 3 has been reinforced by this ant, black otherwise. The
 665 first part of the lemma will show that this urn can be coupled with a Friedman-like urn of Lemma 3.3 so
 666 that there are always more red balls in the Friedman-like urn.

667 By Lemmas 3.1 and 3.4 we have for all $w \in \mathcal{E}$,

$$668 \quad p_{135}(w) = \frac{w_1 w_3 w_5}{(w_2 + w_3 + w_1 w_4)(w_4 + w_5) + w_2 w_3 + w_1 w_3 w_4} \leq \frac{w_1 w_3 w_5}{(w_3 + w_2)(w_4 + w_5) + w_3 w_2}. \quad (20)$$

669 Using that $w_3 + w_2 \geq w_1$, and $w_2 + w_5 = 1$, we deduce

$$670 \quad p_{135}(w) \leq \frac{w_1 w_3 w_5}{w_3 + w_1 w_4 + w_2 w_5}. \quad (21)$$

671 By symmetry, we have that

$$672 \quad p_{234}(w) \leq \frac{w_2 w_3 w_4}{w_3 + w_1 w_4 + w_2 w_5},$$

673 and thus, the probability that the n -th walker reinforces edge 3 is at most

$$674 \quad p_{135}(w) + p_{234}(w) \leq w_3 \cdot \frac{w_1 w_5 + w_2 w_4}{w_3 + w_1 w_4 + w_2 w_5}.$$

675 Finally, we note that

$$676 \quad w_1 w_5 + w_2 w_4 = w_1 w_4 + w_2 w_5 + (w_1 - w_2)(w_5 - w_4) \leq w_1 w_4 + w_2 w_5 + w_3^2,$$

677 which entails

$$678 \quad p_{135}(w) + p_{234}(w) \leq w_3 \cdot \frac{w_1 w_4 + w_2 w_5 + w_3^2}{w_1 w_4 + w_2 w_5 + w_3} \leq w_3 \left(1 - \frac{w_3(1 - w_3)}{w_1 w_4 + w_2 w_5 + w_3} \right).$$

679 Recalling next that, for all $x \in [0, 1]$, $x(1 - x) \leq 1/4$ and that $w_1 + w_4 = w_2 + w_5 = 1$, we have that
 680 $w_1 w_4 + w_2 w_5 \leq 1/2$, which implies

$$681 \quad p_{135}(w) + p_{234}(w) \leq w_3 \left(1 - \frac{w_3(1 - w_3)}{w_3 + \frac{1}{2}} \right) = w_3 \cdot \frac{w_3^2 + \frac{1}{2}}{w_3 + \frac{1}{2}},$$

682 proving the first part of the lemma. Applying this with $w = \hat{\mathbf{W}}(n)$, we thus have proved that, at every
 683 time step n , the probability to add a red ball in the ants-urn is at most the probability to add a red ball
 684 in the Friedman-like urn of Lemma 3.3. Therefore, the number of red balls in the ants urn (i.e. $W_3(n)$)
 685 is at most R_n at time n (for all $n \geq 0$), where R_n is the quantity defined in Lemma 3.3. Thus the result
 686 follows from Lemma 3.3. \square

687 **Remark.** It is interesting to note that, in the loop-erased ant process, one has

$$688 \quad p_{135} = \frac{w_1 w_3 w_5}{w_3 + w_1 w_4 + w_2 w_5},$$

689 to compare with Equation (21). This means that Lemma 3.5 holds in this case too. However, to show
 690 almost-sure convergence of $\hat{\mathbf{W}}(n)$, we need to know that the convergence of $\hat{W}_3(n)$ to zero has polynomial
 691 speed. This is done in the following lemma, whose proof relies on a better bound, using the equality in
 692 Equation (20). Therefore, the fact that this better bound does not hold in the loop-erased case is the
 693 reason why we believe that the proof of Conjecture 1.1 in that case is more intricate.

694 We now aim at bootstrapping the previous result to get a polynomial speed of convergence. For this
 695 we will need the following fact.

696 **Lemma 3.6.** *For any $\rho \in (0, 1/6)$, there exists $\varepsilon > 0$ such that for any $w \in \mathcal{E}$ satisfying $w_3 \leq \varepsilon$,*

$$697 \quad p_{135}(w) + p_{234}(w) \leq (1 - \rho)w_3.$$

698 *Proof.* By Lemma 3.4, for any $w \in \mathcal{E}$,

$$699 \quad p_{135}(w) = \frac{w_1 w_3 w_5}{w_3(1 + w_4 + w_1 w_4) + w_2 w_4 + w_2 w_5 + w_1 w_4^2 + w_1 w_4 w_5}. \quad (22)$$

700 Assume that $w_3 < 1/4$. Let us first prove a lower bound on the denominator of (22). This denominator
 701 is at least equal to $w_3(1 + w_4) + w_2 w_4 + w_2 w_5$, and we would like to prove that

$$702 \quad w_3(1 + w_4) + w_2 w_4 + w_2 w_5 \geq -w_3^2 + 2w_1 w_5. \quad (23)$$

703 Indeed, first using the fact that, for all $w \in \mathcal{E}$, $w_4 \geq w_5 - w_3$, we get

$$704 \quad w_3(1 + w_4) + w_2 w_4 + w_2 w_5 \geq w_3(1 + w_4) + w_2(w_5 - w_3) + w_2 w_5 \geq w_3(1 + w_4 - w_2) + 2w_2 w_5.$$

705 Now, using the facts that, for all $w \in \mathcal{E}$, $w_2 \geq w_1 - w_3$, $w_4 - w_5 \geq -w_3$, and $1 - w_2 = w_5$, we get that

$$706 \quad w_3(1 + w_4) + w_2 w_4 + w_2 w_5 \geq w_3(w_4 + w_5) + 2(w_1 - w_3)w_5 \geq w_3(w_4 - w_5) + 2w_1 w_5 \geq -w_3^2 + 2w_1 w_5,$$

707 which concludes the proof of (23).

708 Next we distinguish two cases: either $w_2 \geq w_1$ or $w_2 < w_1$.

709 • We first treat the case when $w_1 \geq w_2$ and, as a consequence, $w_5 \geq w_4$. Plugging Equation (23) into
 710 Equation (22), we thus get

$$711 \quad p_{135}(w) \leq \frac{w_1 w_3 w_5}{2w_1 w_5 - w_3^2}.$$

712 Since $w \in \mathcal{E}$, we have $w_1 + w_2 \geq w_3$, which, since $w_1 \geq w_2$ implies $w_1 \geq w_3/2$. Similarly, the facts that
 713 $w_4 + w_5 \geq w_3$ and $w_5 \geq w_4$ imply that $w_5 \geq w_3/2$. Moreover, since $w \in \mathcal{E}$, we have $w_1 + w_4 = 1$, and
 714 thus either $w_1 \geq 1/2$ or $w_4 \geq 1/2$. If $w_1 \geq 1/2$ then we conclude that $w_1 w_5 \geq w_5/2 \geq w_3/4$. If $w_4 \geq 1/2$,
 715 then $w_5 \geq w_4 \geq 1/2$, and we also get $w_1 w_5 \geq w_3/4$ in this case. Therefore, in both cases ($w_1 \geq 1/2$ and
 716 $w_4 \geq 1/2$), using the fact that $\frac{1}{1-x} \leq 1 + 2x$ for all $0 \leq x \leq 1/2$, we get

$$717 \quad p_{135}(w) \leq \frac{w_3}{2(1 - 2w_3)} \leq \frac{w_3}{2} + 2w_3^2,$$

718 as long as $w_3 < 1/4$.

719 • We now treat the case when $w_2 \geq w_1$, which implies $w_4 \geq w_5$. In that case, it is straightforward to
 720 see that the denominator in (22) is at least $w_2(w_4 + w_5) \geq 2w_1 w_5$, which implies $p_{135}(w) \leq w_3/2$.

721 By the two cases above, we have thus proved that, for all $w \in \mathcal{E}$ such that $w_3 \leq 1/4$,

$$722 \quad p_{135}(w) \leq \frac{w_3}{2} + 2w_3^2.$$

723 Note that by symmetry the same inequality holds for $p_{234}(w)$, i.e. for all $w \in \mathcal{E}$,

$$724 \quad \max(p_{135}(w), p_{234}(w)) \leq \frac{w_3}{2} + 2w_3^2, \quad (24)$$

725 but this is not yet enough to conclude the proof: we need to get a better upper bound by taking into
726 account the terms in the denominator of Equation (22) that we previously neglected.

727 To do that, we again distinguish two cases: first assume that $w_4 \geq 1/2$. In this case, using the fact
728 that for all $w \in \mathcal{E}$, $w_4 \geq w_5 - w_3$, we get

$$729 \quad w_1 w_4^2 \geq w_1 w_4 w_5 - w_1 w_3 w_4 \geq \frac{1}{2} w_1 w_5 - w_1 w_3 w_4,$$

730 and since we assume that $w_4 \geq 1/2$, we also get

$$731 \quad w_1 w_4 w_5 \geq \frac{1}{2} w_1 w_5.$$

732 Now if in addition $w_5 \geq w_4$, one has $w_1 \geq w_2$ and thus $w_1 \geq w_3/2$, as well as $w_5 - w_4 \leq w_3 \leq 4w_1 w_5$.
733 This, together with the last two displays and (22) implies

$$734 \quad p_{135}(w) \leq \frac{w_1 w_3 w_5}{3w_1 w_5 + w_3(w_4 - w_5)} \leq \frac{w_3}{3} \cdot \frac{1}{1 - \frac{4w_3}{3}} \leq \frac{w_3}{3} \left(1 + \frac{8w_3}{3}\right),$$

735 as long as $w_3 \leq 3/8$. We thus get that, for all $w \in \mathcal{E}$ such that $w_3 \leq 3/8$, and $w_4 \geq 1/2$,

$$736 \quad p_{135}(w) \leq \frac{w_3}{3} + w_3^2.$$

737 We now need to treat the case when $w_4 \leq 1/2$. In that case, $w_1 \geq 1/2$ and we get, by symmetry,

$$738 \quad p_{234}(w) \leq \frac{w_3}{3} + w_3^2.$$

739 In both cases ($w_4 \geq 1/2$ and $w_1 \geq 1/2$), using Equation (24), we get

$$740 \quad p_{135}(w) + p_{234}(w) \leq \left(\frac{1}{2} + \frac{1}{3}\right)w_3 + 3w_3^2 = \frac{5w_3}{6} + 3w_3^2,$$

741 as long as $w_3 \leq 1/4$, and the lemma follows. □

742 **Lemma 3.7.** *Almost surely, for any $\alpha > 5/6$,*

$$743 \quad \lim_{n \rightarrow \infty} \frac{W_3(n)}{n^\alpha} = 0.$$

744 *Proof.* Fix $\alpha > 5/6$, and set $Z_n := n^{-\alpha} \cdot W_3(n)$. Using Equation (19), we get, for all $n \geq 1$,

$$745 \quad Z_{n+1} = Z_n \cdot \left(1 - \frac{1}{n+1}\right)^\alpha + \frac{W_3(n+1) - W_3(n)}{(n+1)^\alpha} = Z_n + \frac{r_n}{n+1} + \frac{\Delta M_3(n+1)}{(n+1)^\alpha},$$

746 with $r_n = (n+1)^{1-\alpha} \mathbb{E}_n Y_3(n+1) - \alpha Z_n + \mathcal{O}(Z_n/n)$, almost surely when $n \rightarrow +\infty$. Recall that $Y(n+1) =$
747 $\mathbf{W}(n+1) - \mathbf{W}(n)$, and thus

$$748 \quad \mathbb{E}_n Y_3(n+1) = p_{135}(\hat{\mathbf{W}}(n)) + p_{234}(\hat{\mathbf{W}}(n)) \leq \alpha \hat{W}_3(n),$$

749 almost surely for all n large enough, by Lemmas 3.5 and 3.6. Therefore, almost surely for n large enough,
750 $r_n \leq -\delta Z_n$, for some constant $\delta > 0$. As a consequence, almost surely there exists $m \geq 1$, such that for
751 all $n > m$,

$$752 \quad Z_n \leq \gamma_{m,n} \cdot Z_m + \sum_{i=m+1}^n \frac{\gamma_{i,n} \cdot \Delta M_3(i)}{i^\alpha},$$

753 where $\gamma_{i,n} := \prod_{j=i+1}^n (1 - \frac{\delta}{j})$, for all $i \leq n$ (with the convention that $\gamma_{n,n} = 1$). Recall that by definition
754 $|\Delta M_3(i)| \leq 1$, almost surely for all $i \geq 1$. Thus, by Doob's L^2 -inequality, one has as $m \rightarrow +\infty$,

$$755 \quad \mathbb{P} \left(\sup_{n \geq m} \left| \sum_{i=m+1}^n \frac{\gamma_{i,n} \cdot \Delta M_3(i)}{i^\alpha} \right| \geq \frac{1}{m^{2\alpha-1-\frac{3}{5}}} \right) = \mathcal{O} \left(\frac{1}{m^{6/5}} \right).$$

756 By Borel-Cantelli, we deduce that almost surely, one has for all m large enough,

$$757 \quad \sup_{n \geq m} \left| \sum_{i=m+1}^n \frac{\gamma_{i,n} \cdot \Delta M_3(i)}{i^\alpha} \right| \leq \frac{1}{m^{2\alpha-1-\frac{3}{5}}}.$$

758 The lemma follows, since $2\alpha - 1 - \frac{3}{5} > 0$, and for any fixed $m \geq 1$, $\gamma_{m,n} \rightarrow 0$, as $n \rightarrow \infty$. □

759 3.2 Convergence of $\hat{W}(n)$

760 Our next goal is to prove the following proposition.

761 **Proposition 3.8.** *Almost surely, there exists some (random) real $\chi \in [0, 1]$, such that as $n \rightarrow \infty$,*

$$762 \quad \frac{W_i(n)}{n} \rightarrow \chi, \quad \forall i = 1, 2, \quad \text{and} \quad \frac{W_i(n)}{n} \rightarrow 1 - \chi, \quad \forall i = 4, 5.$$

763 We start with a computation giving the probability to reinforce edge 2, which is similar to Lemma 3.4.

764 **Lemma 3.9.** *One has for all $w \in \mathcal{E}$,*

$$765 \quad p_{12}(w) + p_{234}(w) = \frac{w_2 w_3 + w_1 w_2 w_5 + w_1 w_2 w_4}{w_3 + w_2 w_5 + w_1 w_4}.$$

767 *Proof.* Note that $p_{12}(w) + p_{234}(w)$ is equal to the probability that the last step before reaching the vertex
768 F is through edge 2. Let us compute this probability by decomposing with respect to the first step, which
769 is either through edge 1 (jumping on the left vertex), or through edge 4 (jumping on the right vertex),
770 hence we will write

$$771 \quad p_{12}(w) + p_{234}(w) = p^\ell(w) + p^r(w). \quad (25)$$

772 For $w \in \mathcal{E}$, the probability to jump on the left vertex is w_1 , and once on the left vertex, we need
773 to compute the probability to cross edge 2 before crossing edge 5, which is easily done through graph
774 transformations similar to those done in the proof of Lemma 3.4; see Figure 9. One obtains:

$$775 \quad p^\ell(w) = w_1 \times \frac{w_2(w_3 + w_5 + w_1 w_4)}{w_2(w_3 + w_5 + w_1 w_4) + w_5(w_3 + w_1 w_4)} \quad (26)$$

$$776 \quad = w_1 \times \frac{w_2 w_3 + w_2 w_5 + w_1 w_2 w_4}{w_3 + w_2 w_5 + w_1 w_4}, \quad (27)$$

778 where we used that $w_2 + w_5 = 1$.

779 Now, using symmetry, one has

$$780 \quad p^r(w) = w_4 \times \left(1 - \frac{w_5 w_3 + w_2 w_5 + w_1 w_5 w_4}{w_3 + w_2 w_5 + w_1 w_4} \right) \quad (28)$$

$$781 \quad = w_4 \times \frac{w_2 w_3 + w_1 w_2 w_4}{w_3 + w_2 w_5 + w_1 w_4}. \quad (29)$$

783 One can now easily conclude using (25), by adding up (27) with (29) and using that $w_1 + w_4 = 1$. □

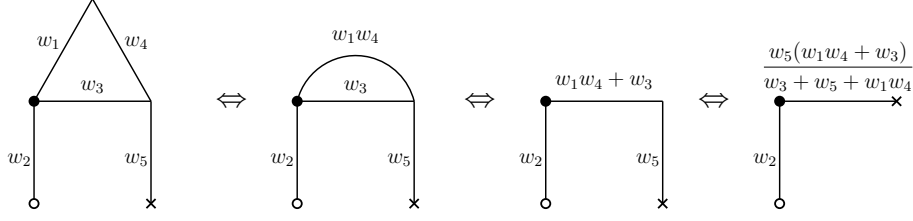


Figure 9: Calculation of $p^\ell(w)/w_1$ (i.e. the probability to reach the circled vertex before the crossed vertex starting from the black vertex) for $w \in \mathcal{E}$ (in particular, we use $w_1 + w_4 = 1$).

784 We next deduce the following bound on $F_2(w)$ (the second coordinate of the function $F(w)$ from (17)).

785 **Lemma 3.10.** *For any $w \in \mathcal{E}$, we have*

786
$$|F_2(w)| \leq \frac{w_3}{2}.$$

787 *Proof.* By Lemma 3.9, for any $w \in \mathcal{E}$,

788
$$F_2(w) = p_{12}(w) + p_{234}(w) - w_2 = \frac{(w_1 - w_2)w_2w_5}{w_3 + w_2w_5 + w_1w_4}. \quad (30)$$

789 Note now that since $w_1 - w_2 = w_5 - w_4$, either $w_1 \geq w_2$, or $w_4 \geq w_5$. In the first case, using also that
790 $w_4 \geq w_5 - w_3$, we deduce $w_3 + w_1w_4 \geq w_2w_5$. By symmetry, the same holds when $w_4 \geq w_5$. We thus get

791
$$|F_2(w)| \leq \frac{|w_1 - w_2|}{2} \leq \frac{w_3}{2}, \quad \text{for all } w \in \mathcal{E},$$

792 where we have used $|w_1 - w_2| \leq w_3$ in the second inequality. □

793 *Proof of Proposition 3.8.* Iterating Equation (19), we get that, for all $n \geq 0$

794
$$\hat{\mathbf{W}}(n) = \hat{\mathbf{W}}(0) + \sum_{i=0}^{n-1} \frac{1}{i+3} (F(\hat{\mathbf{W}}(i)) + \Delta \mathbf{M}(i+1)). \quad (31)$$

795 where we recall that $\Delta \mathbf{M}(n+1) := Y(n+1) - \mathbb{E}_n Y(n+1)$ with $Y(n+1) := \mathbf{W}(n+1) - \mathbf{W}(n)$, and
796 where F is defined in Equation (17). By definition of the model, $\|Y(n+1)\|_1 \leq 3$ almost surely, and thus
797 $\|\Delta \mathbf{M}(i+1)\|_1 \leq 3$ almost surely, which implies that the martingale

798
$$\hat{\mathbf{M}}(n) := \sum_{i=0}^{n-1} \frac{\Delta \mathbf{M}(i+1)}{i+3}$$

799 is bounded in L^2 and thus converges almost surely when $n \rightarrow +\infty$. By Lemma 3.1, $\hat{\mathbf{W}}(n) \in \mathcal{E}$, for all
800 $n \geq 0$. Thus Lemma 3.10 gives $|F_2(\hat{\mathbf{W}}(n))| \leq \hat{W}_3(n)/2$, for all $n \geq 0$, which implies using Lemma 3.7
801 that

802
$$\hat{W}_2(n) = \hat{W}_2(0) + \sum_{i=0}^{n-1} \frac{F_2(\hat{\mathbf{W}}(i))}{i+3} + \sum_{i=0}^{n-1} \frac{\Delta M_2(i+1)}{i+3},$$

803 converges almost surely when $n \rightarrow +\infty$. The proposition follows, since by Lemma 3.5, one has $\hat{W}_1(n) -$
804 $\hat{W}_2(n) \rightarrow 0$, and by Lemma 3.1, one has $\hat{W}_4(n) = 1 - \hat{W}_1(n)$, and $\hat{W}_5(n) = 1 - \hat{W}_2(n)$, for all $n \geq 0$. □

805 **3.3 On the absence of convergence to 0 or 1**

806 The last step of the proof is to exclude the convergence toward an extremal point, that is we prove the
807 following proposition.

808 **Proposition 3.11.** *Almost surely,*

809
$$\lim_{n \rightarrow \infty} \frac{W_1(n)}{n} \notin \{0, 1\}.$$

810 Note that by symmetry it suffices to exclude the possibility of converging to 1. We prove this by
811 contradiction, and start with the following fact.

812 **Lemma 3.12.** *For all $\alpha \in (0, 1)$, on the event where*

813
$$\lim_{n \rightarrow \infty} \frac{W_1(n)}{n} = 1, \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{W_3(n)}{n^\alpha} = 0,$$

814 *both hold, we have almost surely for any $\beta > \alpha$,*

815
$$\lim_{n \rightarrow +\infty} \frac{W_5(n)}{n^\beta} = 0.$$

816 *Proof.* Fix $\alpha \in (0, 1)$ and assume that both $W_1(n)/n \rightarrow 1$ and $W_3(n)/n^\alpha \rightarrow 0$ when $n \rightarrow +\infty$. Assume by
817 contradiction that there exists $\beta > \alpha$, such that $\limsup_{n \rightarrow +\infty} W_5(n)/n^\beta > 0$. Without loss of generality
818 one can even assume that $\limsup_{n \rightarrow +\infty} W_5(n)/n^\beta > 1$, by taking a smaller β if necessary. In other words,
819 letting

820
$$E := \left\{ \lim_{n \rightarrow \infty} \frac{W_3(n)}{n^\alpha} = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{W_1(n)}{n} = 1 \right\}, \quad \text{and} \quad E' := E \cap \left\{ \limsup_{n \rightarrow +\infty} \frac{W_5(n)}{n^\beta} > 1 \right\},$$

821 our aim is to show that $\mathbb{P}(E') = 0$.

822 For $m \geq 1$ integer, define

823
$$E_m := \{W_3(n) \leq n^\alpha, \text{ and } W_2(n) \geq 3(n+2)/4 \text{ for all } n \geq m\}.$$

824 By definition, and using that $W_2(m) \geq W_1(m) - W_3(m)$, for all $m \geq 0$, one has that $E \subset \cup_m E_m$, and
825 therefore

826
$$\lim_{m \rightarrow \infty} \mathbb{P}(E \cap E_m^c) = 0.$$

827 Thus it amounts to show that

828
$$\lim_{m \rightarrow \infty} \mathbb{P}(E_m \cap E') = 0.$$

829 Note now that by conditioning with respect to the first time $n \geq m$ when $W_5(n) \geq n^\beta$, it suffices in fact
830 to show that almost surely

831
$$\lim_{m \rightarrow \infty} \mathbb{P}(E_m \cap E \mid \mathcal{F}_m) \cdot \mathbf{1}\{W_5(m) \geq m^\beta\} = 0, \tag{32}$$

832 where $\mathcal{F}_m = \sigma(\mathbf{W}(0), \dots, \mathbf{W}(m))$. Thus the rest of the proof consists in proving (32). The idea is to show
833 that for any integer $m \geq 1$, on the event that $\{W_5(m) \geq m^\beta\}$, the process $(W_2(n))_{n \geq m}$ can be coupled
834 with another process $(R_n)_{n \geq m}$, in a way that outside an event with vanishing probability as $m \rightarrow \infty$, one
835 has $W_2(n) \leq R_n$ for all $n \geq m$, and $\limsup_{n \rightarrow \infty} R_n/n < 1$, from which (32) follows.

836 We proceed with the details now. Fix $\gamma \in (0, 1)$, such that $1 + \alpha < \beta + \gamma$. Let $m \geq 1$ be given, and
837 conditionally on \mathcal{F}_m , we define the process $(R_n)_{n \geq m}$ as follows: $R_m = W_2(m)$, and for all $n \geq m$,

838
$$q_n := \mathbb{P}(R_{n+1} = R_n + 1 \mid \mathcal{G}_n) = 1 - \mathbb{P}(R_{n+1} = R_n \mid \mathcal{G}_n) = \frac{R_n + R_n^\gamma}{n + R_n^\gamma}, \tag{33}$$

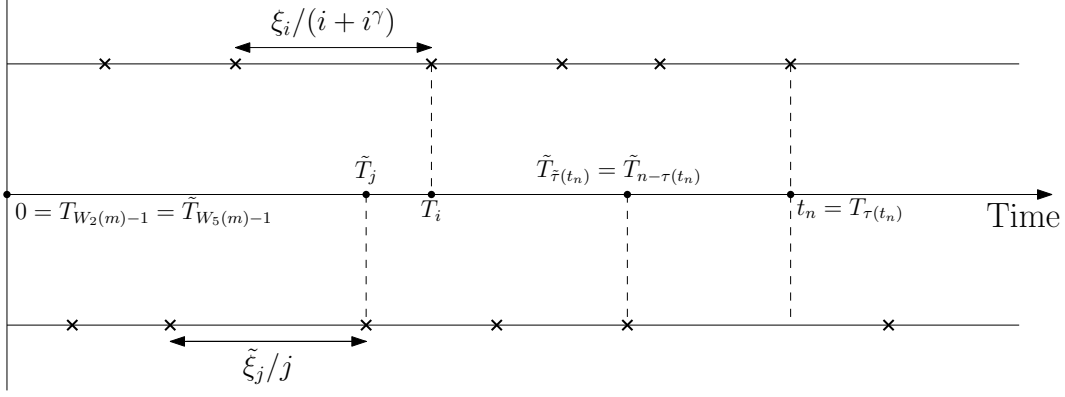


Figure 10: Rubin's construction for the proof of Lemma 3.12.

839 where $\mathcal{G}_n = \mathcal{F}_m \vee \sigma(R_m, \dots, R_n)$.

840 • First, we prove that, for all $m \geq 1$, if we set

$$841 \quad \mathcal{A}_m := \left\{ \inf_{n \geq m} \frac{R_n}{n} > \frac{3}{5} \right\} \cap \left\{ \inf_{n \geq m} \frac{n - R_n}{n} > \frac{3}{5m^{1-\beta}} \right\}, \quad (34)$$

842 then almost surely on the event $\{W_5(m) \geq m^\beta\} \cap \{W_2(m) \geq 3(m+2)/4\}$, one has

$$843 \quad \mathbb{P}(\mathcal{A}_m^c \mid \mathcal{F}_m) = \mathcal{O}(m^{-\delta}), \quad (35)$$

844 where the implicit constant in the \mathcal{O} is deterministic, and $\delta = \delta(\beta, \gamma)$ is some positive constant depending
 845 only on β and γ . To do this, we use again Rubin's construction; see Figure 10: Let $(\xi_i)_{i \geq 1}$ and $(\tilde{\xi}_i)_{i \geq 1}$
 846 be two independent sequences of independent exponential random variables with mean 1 (also independent
 847 of the process $(W_2(n))_{n \geq 1}$). For all $m \geq 0$ and $i \geq 0$, set

$$848 \quad T_i := \sum_{j=W_2(m)}^i \frac{\xi_j}{j + j^\gamma}, \quad \text{and} \quad \tilde{T}_i := \sum_{j=W_5(m)}^i \frac{\tilde{\xi}_j}{j},$$

849 with the convention that $T_i = 0$ for $i < W_2(m)$, and $\tilde{T}_i = 0$, for $i < W_5(m)$. For all $t \geq 0$, set

$$850 \quad \tau(t) := \sup\{i \geq 0 : T_i \leq t\}, \quad \text{and} \quad \tilde{\tau}(t) := \sup\{i \geq 0 : \tilde{T}_i \leq t\},$$

851 and for all $n \geq m$,

$$852 \quad t_n := \inf\{t \geq m : \tau(t) + \tilde{\tau}(t) \geq n\}.$$

853 Standard properties of independent exponential random variables imply that $(\tau(t_n))_{n \geq m}$ and $(R_n)_{n \geq m}$
 854 have the same law. Note that for all $m \geq 1$, and $i \geq W_2(m)$,

$$855 \quad T_i = M_i + \log\left(\frac{i}{W_2(m)}\right) + \mathcal{O}\left(\frac{1}{W_2(m)^{1-\gamma}}\right), \quad \text{with} \quad M_i := \sum_{j=W_2(m)}^i \frac{\xi_j - 1}{j + j^\gamma}, \quad (36)$$

856 when $m \rightarrow +\infty$, and for all $i \geq W_5(m)$,

$$857 \quad \tilde{T}_i = \tilde{M}_i + \log\left(\frac{i}{W_5(m)}\right) + \mathcal{O}\left(\frac{1}{W_5(m)}\right), \quad \text{with} \quad \tilde{M}_i := \sum_{j=W_5(m)}^i \frac{\tilde{\xi}_j - 1}{j}, \quad (37)$$

858 when $m \rightarrow +\infty$. By Doob's L^2 -maximal inequality, we get that, almost surely

$$859 \quad \mathbb{P}\left(\sup_{i \geq W_2(m)} |M_i| > \frac{1}{W_2(m)^{1/4}} \mid W_2(m)\right) \leq 4W_2(m)^{1/2} \sum_{i \geq W_2(m)} \frac{1}{i^2} = \mathcal{O}\left(\frac{1}{W_2(m)^{1/2}}\right), \quad (38)$$

860 when $m \rightarrow +\infty$, and similarly,

$$861 \quad \mathbb{P}\left(\sup_{i \geq W_5(m)} |\tilde{M}_i| > \frac{1}{W_5(m)^{1/4}} \mid W_5(m)\right) = \mathcal{O}\left(\frac{1}{W_5(m)^{1/2}}\right), \quad (39)$$

862 when $m \rightarrow +\infty$. Moreover, by definition

$$863 \quad T_{\tau(t_n)} = \tilde{T}_{n-\tau(t_n)} + \mathcal{O}(\Gamma_m + \tilde{\Gamma}_m), \quad (40)$$

864 with

$$865 \quad \Gamma_m = \sup_{j \geq W_2(m)} \xi_j/j, \quad \text{and} \quad \tilde{\Gamma}_m = \sup_{j \geq W_5(m)} \tilde{\xi}_j/j.$$

866 Note that, for all m large enough,

$$867 \quad \mathbb{P}\left(\Gamma_m > \frac{1}{W_2(m)^{1/2}} \mid W_2(m)\right) \leq \exp\left(-\frac{\sqrt{W_2(m)}}{2}\right) \quad \text{and} \quad \mathbb{P}\left(\tilde{\Gamma}_m > \frac{1}{W_5(m)^{1/2}} \mid W_5(m)\right) \leq \exp\left(-\frac{\sqrt{W_5(m)}}{2}\right). \quad (41)$$

868 Taking the exponential in Equation (40) gives

$$869 \quad \frac{\tau(t_n)}{n - \tau(t_n)} \cdot \frac{W_5(m)}{W_2(m)} = \exp(\varepsilon_n), \quad (42)$$

870 where, by Equations (36), (37), (38), (39) and (41), there exists $\delta = \delta(\beta, \gamma) > 0$ such that almost surely
871 on the event $\{W_5(m) \geq m^\beta\} \cap \{W_2(m) \geq 3(m+2)/4\}$,

$$872 \quad \mathbb{P}(\sup_{n \geq m} |\varepsilon_n| > m^{-\delta} \mid \mathcal{F}_m) = \mathcal{O}(m^{-\delta}).$$

873 Since, by Lemma 3.1, $W_5(m) = m + 2 - W_2(m)$, we get that on the event $\{W_2(m) \geq 3(m+2)/4\}$, one
874 has $W_5(m)/W_2(m) \leq 1/3$, and thus, by Equation (42),

$$875 \quad \tau(t_n) \geq (n - \tau(t_n))3e^{\varepsilon_n} \quad \implies \quad \tau(t_n) \geq \frac{3e^{\varepsilon_n}n}{1 + 3e^{\varepsilon_n}} = \frac{3n(1 - \mathcal{O}(m^{-\delta}))}{4}, \quad (43)$$

876 where the last equality holds on an event of probability at least $1 - \mathcal{O}(m^{-\delta})$ when $m \rightarrow +\infty$. Similarly,
877 on the event $\{W_5(m) \geq m^\beta\}$, we have $W_5(m)/W_2(m) \geq m^{\beta-1}$, and thus, by Equation (42),

$$878 \quad (n - \tau(t_n))m^{1-\beta}e^{\varepsilon_n} \geq \tau(t_n) \geq \frac{3n(1 - \mathcal{O}(m^{-\delta}))}{4},$$

879 where the last inequality comes from Equation (43). This implies

$$880 \quad n - \tau(t_n) \geq \frac{3n}{4m^{1-\beta}}(1 - \mathcal{O}(m^{-\delta})),$$

881 when $m \rightarrow +\infty$, on an event of probability at least $1 - \mathcal{O}(m^{-\delta})$. Since $(R_n)_{n \geq m}$ and $(\tau(t_n))_{n \geq m}$ have the
882 same law by construction, this concludes the proof of (35).

883 • To conclude we just need to show that there exists a coupling of $(W_2(n))_{n \geq m}$ and $(R_n)_{n \geq m}$, such
884 that almost surely on the event $\mathcal{A}_m \cap E_m$ (see Equation (34) for the definition of \mathcal{A}_m), one has $W_2(n) \leq R_n$

885 for all $n \geq m$, at least for m large enough. Indeed, this would prove that on $\mathcal{A}_m \cap E_m$, the sequence
 886 $(W_1(n)/n)_{n \geq m}$ cannot converge to 1, or otherwise stated that for all m large enough, almost surely,

$$887 \quad \mathbb{P}(\mathcal{A}_m \cap E_m \cap E \mid \mathcal{F}_m) = 0.$$

888 Together with (35), this would conclude the proof of (32). So let us prove the existence of the desired
 889 coupling now.

890 Recall that, by Lemma 3.10, for all $n \geq m$, on the event $\{W_3(n) \leq n^\alpha\}$,

$$891 \quad p_n := \mathbb{P}(W_2(n+1) = W_2(n) + 1 \mid \mathcal{F}_n) \leq \frac{W_2(n) + n^\alpha}{n+2}.$$

892 To show that our coupling exists, it is enough to prove that, for all $n \geq m$, if $W_2(n) \leq R_n$, then $p_n \leq q_n$,
 893 where q_n is defined in Equation (33). Indeed, if $W_2(n) \leq R_n$ and $p_n \leq q_n$, then there exists a one-step
 894 coupling such that $W_2(n+1) \leq R_{n+1}$, and we can proceed by induction. Note that $q_n \geq p_n$ is implied by

$$895 \quad (n+2)(R_n + R_n^\gamma) \geq (n + R_n^\gamma)(W_2(n) + n^\alpha).$$

896 Developing and using the induction hypothesis (i.e. $W_2(n) \leq R_n$), it suffices to show that

$$897 \quad R_n^\gamma(n - R_n - n^\alpha) \geq n^{1+\alpha},$$

898 which is indeed true on \mathcal{A}_m , since on this event

$$899 \quad R_n^\gamma(n - R_n - n^\alpha) \geq \frac{3}{5}n^\gamma \left(\frac{3}{5}n^\beta - n^\alpha \right),$$

900 which is well larger than $n^{1+\alpha}$, for all n large enough, since by hypothesis $\gamma \in (0, 1)$ and $\gamma + \beta > 1 + \alpha$.
 901 This concludes the proof of (32), and of the lemma. \square

902 **Lemma 3.13.** *Let $\alpha \in (0, 1)$ be given. On the event*

$$903 \quad \mathcal{A}(\alpha) = \{W_i(n) = \mathcal{O}(n^\alpha), \text{ for } i = 3, 5\}$$

904 *one has almost surely for any $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$,*

$$905 \quad W_4(n) + W_5(n) \leq W_3(n) + \mathcal{O}(n^\beta).$$

906 For the proof of this lemma we need some intermediate results. The first one gives a lower bound on
 907 $F_4 + F_5 - F_3$.

908 **Lemma 3.14.** *For all $w \in \mathcal{E}$, such that $w_5 \leq 1/2$, we have*

$$909 \quad F_4(w) + F_5(w) - F_3(w) \geq -8(w_3^2 + w_5^2).$$

910 *Proof.* Note that

$$911 \quad F_4(w) + F_5(w) - F_3(w) = 2F_5(w) - 2p_{135}(w) + w_5 - w_4 + w_3.$$

912 Recall that by Equation (30) (using the symmetry of the model), we have

$$913 \quad F_5(w) = \frac{(w_4 - w_5)w_2w_5}{w_3 + w_1w_4 + w_2w_5}.$$

914 Also recall that (see Equation (22)),

$$915 \quad p_{135}(w) = \frac{w_1w_3w_5}{w_3(1 + w_4 + w_1w_4) + w_2w_4 + w_2w_5 + w_1w_4^2 + w_1w_4w_5}$$

$$916 \quad \leq \frac{w_1w_3w_5}{w_3 + (w_3 + w_2)w_4 + w_2w_5} \leq \frac{w_1w_3w_5}{w_3 + w_1w_4 + w_2w_5},$$

917

918 where we have used in the last inequality that $w_1 \leq w_2 + w_3$ for all $w \in \mathcal{E}$. Using again that $w_1 \leq w_2 + w_3$
919 for all $w \in \mathcal{E}$, and the fact that $w_3 + w_1w_4 + w_2w_5 \geq w_5(1 - w_5) \geq w_5/2$, for all $w \in \mathcal{E}$ such that $w_5 \leq 1/2$,
920 we get that

$$921 \quad p_{135}(w) \leq \frac{w_2w_3w_5}{w_3 + w_1w_4 + w_2w_5} + 2w_3^2.$$

922 Therefore,

$$923 \quad F_5(w) - p_{135}(w) \geq -\frac{w_5w_2(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} - 2w_3^2,$$

924 and thus

$$925 \quad \begin{aligned} F_4(w) + F_5(w) - F_3(w) &\geq \frac{(w_3 + w_1w_4 - w_2w_5)(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} - 4w_3^2 \\ 926 \quad &\geq \frac{(w_3 + w_4 - w_5)(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} + \frac{(w_5(1 - w_2) - w_4(1 - w_1))(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} - 4w_3^2, \end{aligned} \quad (44)$$

928 Recall that, for all $w \in \mathcal{E}$, $w_1 + w_4 = w_2 + w_5 = 1$, and thus $w_5(1 - w_2) - w_4(1 - w_1) = w_5^2 - w_4^2 \geq -w_4^2$.
929 As a consequence, for all $w_5 \leq 1/2$,

$$930 \quad \frac{(w_5(1 - w_2) - w_4(1 - w_1))(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} \geq -\frac{w_4^2(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} \geq -\frac{w_4^2(w_3 + w_5)}{w_3 + w_1w_4 + w_5 - w_5^2} \geq -2w_4^2,$$

931 where we used that, as $w_5 \leq 1/2$, $w_3 + w_1w_4 + w_5 - w_5^2 \geq w_3 + w_5(1 - w_5) \geq (w_3 + w_5)/2$. Then from
932 Equation (44), we get

$$933 \quad \begin{aligned} F_4(w) + F_5(w) - F_3(w) &\geq \frac{(w_3 + w_4 - w_5)(w_3 - w_4 + w_5)}{w_3 + w_1w_4 + w_2w_5} - 4w_3^2 - 2w_4^2 \\ 934 \quad &\geq \frac{w_3^2 - (w_4 - w_5)^2}{w_3 + w_1w_4 + w_2w_5} - 4w_3^2 - 2w_4^2 \geq -4w_3^2 - 2(w_3 + w_5)^2, \end{aligned}$$

936 using that $|w_4 - w_5| \leq w_3$, for all $w \in \mathcal{E}$. This concludes the proof because $(w_3 + w_5)^2 \leq 2w_3^2 + 2w_5^2$. \square

937 The second result we shall need is the following general fact, which will be used at several places
938 during the rest of the proof. For a process $(M_n)_{n \geq 0}$, we write $\Delta M_n := M_{n+1} - M_n$, for all $n \geq 0$.

939 **Lemma 3.15.** *Let $a, b, c \in (0, 1)$, be such that $b < a$ and $1 < 2a + c$.*

940 **(i)** *Let $(A_n)_{n \geq 1}$ be a sequence of real random variables. On the event $\{A_n = \mathcal{O}(n^{b-1})\}$, we have almost
941 surely as $m \rightarrow +\infty$,*

$$942 \quad \sup_{n \geq m} \sum_{i=m}^n \frac{A_i}{(i+3)^a} = \mathcal{O}(m^{b-a}).$$

943 **(ii)** *Let $(M_n)_{n \leq 0}$ be a real martingale such that $|\Delta M_n| \leq 1$, almost surely for all $n \geq 0$. On the event
944 $\{\mathbb{E}_n[(\Delta M_n)^2] = \mathcal{O}(n^{-c})\}$, we have almost surely when $m \rightarrow +\infty$,*

$$945 \quad \sup_{n \geq m} \left| \sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a} \right| = \mathcal{O}(m^{\kappa-a}),$$

946 for all $\kappa \in (\frac{1-c}{2}, a)$.

947 *Proof.* **(i)** is straightforward. For **(ii)**, we fix $\frac{1-c}{2} < \kappa < a$, and then $\varepsilon > 0$, such that $\frac{1-c+\varepsilon}{2} < \kappa$, and
948 $\hat{\kappa} := \kappa + \varepsilon/2 < a$. For $m \leq n$, define the event

$$949 \quad \mathcal{A}_{m,n} := \{\mathbb{E}_i[(\Delta M_i)^2] \leq i^{-c+\varepsilon} \quad \forall m \leq i \leq n\}.$$

950 We have, for all $n \geq m \geq 1$,

$$951 \quad \mathbb{P}\left(\sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a} \geq m^{\hat{\kappa}-a}, \mathcal{A}_{m,n}\right) \leq \exp(-m^{\varepsilon/2}) \mathbb{E}\left[\prod_{i=m}^n \exp\left(m^{a-\kappa} \frac{\Delta M_i}{(i+3)^a}\right) \mathbf{1}_{\mathcal{A}_{m,n}}\right].$$

953 Using the bound $|\Delta M_n| \leq 1$, and a Taylor expansion, we get for all $n \geq m$, on the event $\mathcal{A}_{m,n}$,

$$954 \quad \mathbb{E}_n\left[\exp\left(m^{a-\kappa} \frac{\Delta M_n}{(n+3)^a}\right)\right] = 1 + \mathcal{O}\left(m^{2a-2\kappa} \frac{\mathbb{E}_n[(\Delta M_n)^2]}{(n+3)^{2a}}\right) \leq 1 + \mathcal{O}(n^{-2\kappa-c+\varepsilon}),$$

955 where the constant in the \mathcal{O} -term is deterministic. By induction, and since $2\kappa + c - \varepsilon > 1$, we get that
956 for all m large enough,

$$957 \quad \mathbb{E}\left[\prod_{i=m}^n \exp\left(m^{a-\kappa} \frac{\Delta M_i}{(i+3)^a}\right) \mathbf{1}_{\mathcal{A}_{m,n}}\right] \leq 2,$$

958 and thus for all $1 \leq m \leq n$, with m large enough,

$$959 \quad \mathbb{P}\left(\sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a} \geq m^{\hat{\kappa}-a}, \mathcal{A}_{m,n}\right) \leq 2 \exp(-m^{\varepsilon/2}).$$

960 By symmetry and a union bound, we deduce that for all m large enough,

$$961 \quad \mathbb{P}\left(\sup_{m \leq n \leq 2m} \left|\sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a}\right| \geq m^{\hat{\kappa}-a}, \mathcal{A}_{m,2m}\right) \leq 4m \exp(-m^{\varepsilon/2}).$$

962 Next, another union bound gives, for all m large enough,

$$963 \quad \mathbb{P}\left(\sup_{n \geq m} \left|\sum_{i=m}^n \frac{\Delta M_i}{(i+3)^a}\right| \geq R \cdot m^{\hat{\kappa}-a}, \mathcal{A}_{m,\infty}\right) \leq \exp\left(-\frac{1}{2} \cdot m^{\varepsilon/2}\right),$$

964 with $R := \sum_{i \geq 0} 2^{(\hat{\kappa}-a)i}$, which is finite since $\hat{\kappa} < a$. Then the result follows from Borel-Cantelli's lemma,
965 since on the event $\{\mathbb{E}_n[(\Delta M_n)^2] = \mathcal{O}(n^{-c})\}$, almost surely $\mathcal{A}_{m,\infty}$ holds for all m large enough. \square

966 We now prove Lemma 3.13.

967 *Proof of Lemma 3.13.* Consider the process $U(n) = W_5(n) + W_4(n) - W_3(n)$, and set $\hat{U}(n) := \frac{U(n)}{n+2}$. One
968 has for any integers $m < n$,

$$969 \quad \hat{U}(n) = \hat{U}(m) + \sum_{i=m}^{n-1} \frac{G(\hat{\mathbf{W}}(i))}{i+3} + \sum_{i=m}^{n-1} \frac{\Delta\Phi(i)}{i+3}, \quad (45)$$

970 where $G(w) = F_5(w) + F_4(w) - F_3(w)$, and $\Delta\Phi(i) = Y(i+1) - \mathbb{E}_i Y(i+1)$, with $Y(i+1) = U(i+1) - U(i)$
971 for all $i \geq 0$.

972 Note first that, $|Y(n+1)| \leq 2$ almost surely for all $n \geq 0$, by definition of the model, and thus also
973 $|\Delta\Phi(n)| \leq 4$. Note furthermore, that on $\mathcal{A}(\alpha)$, one has $W_4(n) = \mathcal{O}(n^\alpha)$, since $W_4(n) \leq W_5(n) + W_3(n)$
974 (recall Lemma 3.1), and thus $|\hat{U}(n)| = \mathcal{O}(n^{\alpha-1})$. Moreover, using Lemmas 3.5 and 3.10 (and the fact that
975 if at some time n , W_4 increases by one unit, then either W_3 or W_5 also), we deduce that on $\mathcal{A}(\alpha)$,

$$976 \quad \mathbb{E}_n[|\Delta\Phi(n)|^2] \leq \mathbb{E}_n[Y(n+1)^2] \leq 4 \cdot \mathbb{P}_n(Y(n+1) \neq 0) = \mathcal{O}(\hat{W}_5(n) + \hat{W}_3(n)) = \mathcal{O}(n^{\alpha-1}).$$

977 On the other hand, by Lemma 3.14, on $\mathcal{A}(\alpha)$, we have almost surely $G(\hat{\mathbf{W}}(i)) \geq -\mathcal{O}(i^{2\alpha-2})$. Thus
978 Lemma 3.15 (applied with $a = 1$, $b = 2\alpha - 1$, and $c = 1 - \alpha$) and Equation (45) (with n taken large
979 enough) give $\hat{U}(m) \leq \mathcal{O}(m^{\beta-1})$, for any $\beta > \max(2\alpha - 1, \alpha/2)$, which proves the desired result. \square

980 We deduce the following fact (recall the definition of the events $\mathcal{A}(\alpha)$ from Lemma 3.13).

981 **Lemma 3.16.** *Let $\alpha \in (0, 1)$ be given. One has almost surely,*

$$982 \quad \mathcal{A}(\alpha) \subseteq \mathcal{A}(\beta),$$

983 *for any $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$.*

984 For the proof of this result, we will need some intermediate result.

985 **Lemma 3.17.** **(a)** *For all $c \in (1/2, 3/4)$, there exist positive constants ε and C , such that for all $r \in [0, 1)$,*
 986 *and all $w \in \mathcal{E}$, with $w_3 \leq \varepsilon$ and $w_4 + w_5 \leq w_3 + r$, one has*

$$987 \quad F_5(w) - F_4(w) \geq c(w_4 - w_5) - Cr.$$

988 **(b)** *There exist positive constants ε and C , such that for any $w \in \mathcal{E}$, with $w_3 \leq \varepsilon$, $w_4 + w_5 \leq w_3 + r$,*
 989 *and $w_4 \leq w_5$, one has*

$$990 \quad \frac{9}{2}F_4(w) - F_3(w) \geq -Cr.$$

991 **(c)** *Let $\rho \in (0, 1/4)$ be given. There exist positive constants ε and C , such that for any $r \in [0, 1)$, and*
 992 *any $w \in \mathcal{E}$, with $w_3 \leq \varepsilon$, $w_4 + w_5 \leq w_3 + r$, and $w_4 \leq \rho w_3 + r$, one has*

$$993 \quad F_4(w) \geq -Cr.$$

994 *Proof.* Let us start with Part **(a)**. Note that, if, under the assumption of the lemma, we have $w_4 + w_5 - w_3 \geq$
 995 $2w_3$, then $3w_3 \leq w_4 + w_5 \leq w_3 + r$, which implies $w_3 \leq r/2$, and thus $w_4 + w_5 \leq 3r/2$. In particular, we
 996 have that $w_3, w_4, w_5 \in [0, 2r]$. Recall that, $F_5(w) - F_4(w) = p_{135}(w) - p_{234}(w) + w_4 - w_5 \geq -p_{234}(w) - w_5$,
 997 and, by Equation (22) (using the symmetry of the model), we have that, for all $w \in \mathcal{E}$, $p_{234}(w) \leq w_4$.
 998 Thus $F_5(w) - F_4(w) \geq -w_4 - w_5 \geq -4r$, which, using that $w_4 \leq 2r$, concludes the proof of **(a)** in the
 999 case when $w_4 + w_5 - w_3 \geq 2w_3$. We now assume that $w_4 + w_5 - w_3 < 2w_3$. This implies

$$1000 \quad \frac{1}{2w_3} \geq \frac{1}{w_3 + w_4 + w_5} \geq \frac{1}{2w_3(1 + \frac{w_4 + w_5 - w_3}{2w_3})} \geq \frac{1}{2w_3} - \frac{r}{4w_3^2}, \quad (46)$$

1001 using that $w_4 + w_5 \geq w_3$, for all $w \in \mathcal{E}$, for the first inequality. Using again (22), we get that, when
 1002 $w_3, w_4, w_5 \rightarrow 0$, with $(w_4 + w_5)/3 \leq w_3 \leq w_4 + w_5$,

$$\begin{aligned} 1003 \quad F_5(w) - F_4(w) &= p_{135}(w) - p_{234}(w) + w_4 - w_5 \\ 1004 \quad &= w_4 - w_5 + \frac{w_3 w_5}{w_3 + w_4 + w_5} (1 - o(1)) - \frac{w_3 w_4}{2(w_3 + w_4 + w_5)} (1 + o(1)) \\ 1005 \quad &\geq w_4 - w_5 + \frac{w_5}{2} (1 - o(1)) - \frac{w_4}{4} (1 + o(1)) - \frac{r w_5 (1 + o(1))}{4w_3} \\ 1006 \quad &\geq \frac{3w_4}{4} (1 - o(1)) - \frac{w_5}{2} (1 + o(1)) - \frac{3r(1 + o(1))}{4}, \end{aligned}$$

1008 because $w_4 + w_5 - w_3 < 2w_3$ implies $w_5 \leq 3w_3$. This concludes the proof of **(a)**.

1009 We prove now Part **(b)**. First note that if $w_4 + w_5 - w_3 \geq 2w_3$, then we have as in Part **(a)** that
 1010 $w_3, w_4, w_5 \in [0, 2r]$, and since $F_3(w) \leq 0$ by Lemma 3.5, we deduce that $9/2 \cdot F_4(w) - F_3(w) \geq -9w_4/2 \geq$
 1011 $-9r$, proving the result. So we may assume now that $w_4 + w_5 - w_3 < 2w_3$. In this case

$$1012 \quad \frac{9}{2}F_4(w) - F_3(w) = \frac{9}{2}(F_5(w) + w_5 - w_4 + p_{234}(w) - p_{135}(w)) - F_3(w).$$

1013 Using Equation (30), we have, when $w_3, w_4, w_5 \rightarrow 0$,

$$1014 \quad F_5(w) + w_5 - w_4 = \frac{(w_5 - w_4)(w_3 + w_1w_4)}{w_3 + w_1w_4 + w_2w_5} = \frac{(w_5 - w_4)(w_3 + w_4)(1 + o(1))}{w_3 + w_4 + w_5}.$$

1015 Using Equation (46), and the fact that $w_4 \leq w_5$, we get

$$1016 \quad F_5(w) + w_5 - w_4 \geq \frac{(w_5 - w_4)(1 + o(1))}{2} \left(1 - \frac{r}{2w_3}\right) \geq \frac{(w_5 - w_4)(1 + o(1))}{2} - \frac{r}{4}(1 + o(1)),$$

1017 using that $w_5 - w_4 \leq w_3$. In the proof of **(a)**, we have shown that

$$1018 \quad p_{234} - p_{135} = \frac{w_3w_4(1 + o(1))}{2(w_3 + w_4 + w_5)} - \frac{w_3w_5(1 + o(1))}{w_3 + w_4 + w_5} \geq \frac{(w_4 - 2w_5)(1 + o(1))}{4}.$$

1019 Using in addition that by assumption $w_3 \geq w_4 + w_5 - r$, we get

$$1020 \quad F_3(w) = p_{135} + p_{234} - w_3 = \frac{w_3w_5(1 + o(1))}{w_3 + w_4 + w_5} + \frac{w_3w_4(1 + o(1))}{2(w_3 + w_4 + w_5)} - w_3 \quad (47)$$

$$1021 \quad \leq -\frac{w_5(1 + o(1))}{2} - \frac{3w_4(1 + o(1))}{4} + r.$$

1023 In total, we thus get

$$1024 \quad \frac{9}{2}F_4(w) - F_3(w) \geq -\frac{9w_4(1 + o(1))}{8} - \frac{9r(1 + o(1))}{8} + \left(\frac{w_5}{2} + \frac{3w_4}{4} - r\right)(1 + o(1))$$

$$1025 \quad \geq -\frac{3w_4}{8}(1 + o(1)) + \frac{w_5}{2}(1 - o(1)) - \frac{17r(1 + o(1))}{8},$$

$$1026 \quad \geq \frac{w_5(1 + o(1))}{8} - \frac{17r(1 + o(1))}{8},$$

1028 because $w_4 \leq w_5$, which concludes the proof of **(b)**.

1029 Finally we prove **(c)**. Assuming again that $w_4 + w_5 - w_3 < 2w_3$ (as otherwise we conclude as in Part
1030 **(b)**), we get when $w_3 \rightarrow 0$ (and as consequence $w_4, w_5 \rightarrow 0$ also),

$$1031 \quad F_4(w) = F_5(w) + p_{234}(w) - p_{135}(w) + w_5 - w_4$$

$$1032 \quad = \frac{(w_5 - w_4)(w_3 + w_1w_4)}{w_3 + w_1w_4 + w_2w_5} + p_{234}(w) - p_{135}(w)$$

$$1033 \quad \geq \frac{(w_5 - w_4)(w_3 + w_1w_4)}{w_3 + w_1w_4 + w_2w_5} + \frac{w_3w_4(1 - o(1))}{2(w_3 + w_4 + w_5)} - \frac{w_3w_5}{w_3 + w_1w_4 + w_2w_5}$$

$$1034 \quad \geq -w_4(1 + o(1)) \cdot \frac{w_3 + w_1w_4 - w_1w_5}{w_3 + w_4 + w_5} + \frac{w_3w_4(1 - o(1))}{2(w_3 + w_4 + w_5)}$$

$$1035 \quad = \frac{w_4 \left[w_5(1 - o(1)) - \left(\frac{w_3}{2} + w_4\right)(1 + o(1)) \right]}{w_3 + w_4 + w_5}.$$

1037 Using the fact that, for all $w \in \mathcal{E}$, $w_5 \geq w_3 - w_4$, and the fact that, by assumption, $w_4 \leq \rho w_3 + r$, we get

$$1038 \quad w_5 - \frac{w_3}{2} - w_4 \geq \frac{w_3}{2} - 2w_4 \geq \frac{w_3}{2} - 2\rho w_3 - 2r = \frac{w_3}{2}(1 - 4\rho) - 2r,$$

1039 which implies

$$1040 \quad F_4(w) \geq \frac{w_4w_3(1 - 4\rho)}{2(w_3 + w_4 + w_5)}(1 - o(1)) - 2r(1 + o(1)) \geq -2r(1 + o(1)),$$

1041 since $\rho < 1/4$ by assumption. □

1042 We are now ready to prove Lemma 3.16.

1043 *Proof of Lemma 3.16.* We fix $\alpha \in (0, 1)$. Recall that

$$1044 \quad \mathcal{A}(\alpha) = \{W_3(n) = \mathcal{O}(n^\alpha) \text{ and } W_5(n) = \mathcal{O}(n^\alpha)\}.$$

1045 The proof is divided in three steps.

1046 First step. Fix $c \in (3/4, 1)$. Consider the process $U(n) = \frac{W_5(n) - W_4(n)}{(n+2)^{1-c}}$. By Equation (19), we have for
1047 $n \geq 1$,

$$1048 \quad U(n+1) = U(n) + \frac{r_n}{(n+3)^{1-c}} + \frac{\Delta\Psi(n)}{(n+3)^{1-c}}, \quad (48)$$

1049 where (when $n \rightarrow +\infty$ in the second equality)

$$1050 \quad r_n = F_5(\hat{\mathbf{W}}(n)) + \hat{W}_5(n) - F_4(\hat{\mathbf{W}}(n)) - \hat{W}_4(n) + ((n+2)^{1-c} - (n+3)^{1-c})U(n) \\ 1051 \quad = F_5(\hat{\mathbf{W}}(n)) - F_4(\hat{\mathbf{W}}(n)) + \hat{W}_5(n) - \hat{W}_4(n) + (c-1+o(1))n^{-c}U(n),$$

1053 and $\Delta\Psi(n) := \Delta M_5(n+1) - \Delta M_4(n+1)$. By Lemma 3.13, for all $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$, almost surely on
1054 $\mathcal{A}(\alpha)$, we have $\hat{W}_4(n) + \hat{W}_5(n) \leq \hat{W}_3(n) + \mathcal{O}(n^{\beta-1})$ when $n \rightarrow +\infty$. Using Lemma 3.17(a), this implies
1055 that, almost surely on $\mathcal{A}(\alpha)$, for all n large enough,

$$1056 \quad F_5(\hat{\mathbf{W}}(n)) - F_4(\hat{\mathbf{W}}(n)) \geq (c-1/4)(\hat{W}_4(n) - \hat{W}_5(n)) - \mathcal{O}(n^{\beta-1}),$$

1057 and thus (using the fact that $U(n) = (\hat{W}_5(n) - \hat{W}_4(n))(n+2)^c$)

$$1058 \quad r_n \geq (1-c+1/4)(\hat{W}_5(n) - \hat{W}_4(n)) - \mathcal{O}(n^{\beta-1}) + (c-1+o(1))n^{-c}U(n) = \left(1/4+o(1)\right)n^{-c}U(n) - \mathcal{O}(n^{\beta-1}), \quad (49)$$

1059 for all $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$. Note that $\Delta U(n) \neq 0$ implies $W_3(n+1) - W_3(n) = 1$, and thus by Lemma 3.5
1060 on $\mathcal{A}(\alpha)$, one has

$$1061 \quad \mathbb{E}_n[|\Delta\Psi(n)|^2] \leq \mathbb{P}(\Delta U(n) \neq 0) = \mathcal{O}(n^{\alpha-1}).$$

1062 Using next Lemma 3.15, we deduce that for any $\beta > \alpha/2$,

$$1063 \quad \sup_{n \geq m} \left| \sum_{i=m}^n \frac{\Delta\Psi(i)}{(i+3)^{1-c}} \right| = \mathcal{O}(m^{-1+c+\beta}). \quad (50)$$

1064 By Equation (48), we have, for all $n > m$,

$$1065 \quad U(n) = U(m) + \sum_{i=m}^{n-1} \frac{r_i}{(i+3)^{1-c}} + \sum_{i=m}^{n-1} \frac{\Delta\Psi(i)}{(i+3)^{1-c}}. \quad (51)$$

1066 Now fix $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$. Observe that if for some $\varepsilon > 0$, $\limsup_{m \rightarrow +\infty} (W_5(m) - W_4(m))/m^{\beta+\varepsilon} > 0$,
1067 then Equations (49), (50), (51) and Lemma 3.15 imply, by induction, that $U(n) \geq 0$, for all n large enough.
1068 Thus on $\mathcal{A}(\alpha)$, there are only two possibilities: either $W_5(n) \leq W_4(n) + \mathcal{O}(n^\beta)$, for all $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$,
1069 or $W_5(n) \geq W_4(n)$ for all large enough n .

1070 Second step. Consider first the case when $W_5(n) \leq W_4(n) + \mathcal{O}(n^\beta)$, for any $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$. Note
1071 that, for all $w \in \mathcal{E}$ and asymptotically when $w_3, w_4, w_5 \rightarrow 0$, we have, using Equation (30),

$$1072 \quad F_5(w) = \frac{(w_4 - w_5)w_2w_5}{w_3 + w_1w_4 + w_2w_5} = \frac{(w_4 - w_5)w_5}{w_3 + w_4 + w_5}(1 + o(1)). \quad (52)$$

1073 By Lemma 3.13, on $\mathcal{A}(\alpha)$, we have $W_3(n) \leq W_4(n) + W_5(n) \leq W_3(n) + \mathcal{O}(n^\beta)$. Since we also assume, in
 1074 this second step, that $W_5(n) \leq W_4(n) + \mathcal{O}(n^\beta)$, this implies $2W_5(n) \leq W_3(n) + \mathcal{O}(n^\beta)$. Therefore, using
 1075 that $W_4(n) - W_5(n) \geq -W_3(n) \wedge \mathcal{O}(n^\beta)$ we get $F_5(\hat{\mathbf{W}}(n)) \geq -\mathcal{O}(n^{\beta-1})$. Next, let us prove that for any
 1076 $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$, $W_5(n) = \mathcal{O}(n^\beta)$. Using (19), we have, for $n \geq m$,

$$1077 \quad \hat{W}_5(n) = \hat{W}_5(m) + \sum_{i=m}^{n-1} \frac{F_5(\hat{\mathbf{W}}(i))}{i+3} + \sum_{i=m}^{n-1} \frac{\Delta M_5(i+1)}{i+3},$$

1078 where by Lemma 3.15 the two sums are greater than $-\mathcal{O}(m^{\beta-1})$. On $\mathcal{A}(\alpha)$, if $\limsup_m \hat{W}_5(m)/m^{\beta-1} = \infty$,
 1079 then the equation above would contradict that $\hat{W}_5(n)$ goes to zero, when $n \rightarrow \infty$. Thus $W_5(n) = \mathcal{O}(n^\beta)$,
 1080 as claimed.

1081 Now note that, for all $w \in \mathcal{E}$, asymptotically when $w_3, w_4, w_5 \rightarrow 0$,

$$1082 \quad F_3(w) = p_{135}(w) + p_{234}(w) - w_3 = \frac{w_3(w_5 + \frac{w_4}{2})}{w_3 + w_4 + w_5}(1 + o(1)) - w_3 \leq \frac{2w_5 + w_4}{4}(1 + o(1)) - w_3,$$

1083 where we have used Equation (47) and the fact that $w_4 + w_5 \geq w_3$ for all $w \in \mathcal{E}$. Since, for all $w \in \mathcal{E}$,
 1084 $w_4 \leq w_3 + w_5$, we get

$$1085 \quad F_3(w) \leq \frac{3(w_5 - w_3)}{4}(1 + o(1)).$$

1086 Applying this to $w = \hat{\mathbf{W}}(n)$ (which belongs to \mathcal{E} by Lemma 3.1), we get that on $\mathcal{A}(\alpha)$, almost surely
 1087 when $n \rightarrow +\infty$,

$$1088 \quad F_3(\hat{\mathbf{W}}(n)) \leq \mathcal{O}(n^{\beta-1}).$$

1089 Now if $\liminf W_3(n)/n^\beta \leq 1$, for some $\beta > \max(2\alpha - 1, \alpha/2)$, then Lemma 3.15 gives $W_3(n) = \mathcal{O}(n^\beta)$
 1090 following an argument very similar to the one above for $\hat{\mathbf{W}}_5(n)$. On the other hand, if $\liminf W_3(n)/n^\beta =$
 1091 $+\infty$, for any $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$, then, as $W_4(n) \geq W_3(n) - W_5(n) \geq W_3(n) - \mathcal{O}(n^\beta)$, we have $W_4(n) -$
 1092 $W_5(n) \geq 0$, for all n large enough, which implies $F_5(\hat{\mathbf{W}}(n)) \geq 0$. From (19), this means that for m
 1093 large enough, the process $(W_5(n))_{n \geq m}$ stochastically dominates a Pólya urn process $(R_n)_{n \geq m}$ defined by
 1094 $\mathbb{P}(R_{n+1} = R_n + 1 \mid R_n) = 1 - P(R_{n+1} = R_n \mid R_n) = \frac{R_n}{n+2}$, which is well known to grow almost surely
 1095 linearly in n (this can be seen using Rubin's construction as in the proof of Lemma 3.12). Thus $W_5(n)$
 1096 would also grow linearly in n , and we would get a contradiction. Therefore, necessarily $W_3(n) = \mathcal{O}(n^\beta)$,
 1097 as wanted.

1098 Third step. Consider next the case when $W_5(n) \geq W_4(n)$, for all n large enough. Define $V(n) =$
 1099 $\frac{9}{2}W_4(n) - W_3(n)$, and $\hat{V}(n) = \frac{V(n)}{n+2}$. One has for any $n \geq 1$,

$$1100 \quad \hat{V}(n+1) = \hat{V}(n) + \frac{H(\hat{\mathbf{W}}(n))}{n+3} + \frac{\Delta\Theta(n)}{n+3},$$

1101 with again $\Delta\Theta(n)$ the increment of some martingale, and $H(w) = \frac{9}{2}F_4(w) - F_3(w)$. Using Lemmas 3.17(b)
 1102 and 3.15 (with arguments similar to those in the second step), we deduce that $V(n) \leq \mathcal{O}(n^\beta)$, for any
 1103 $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$. We note finally that by Lemma 3.17(c) this entails $F_4(\hat{\mathbf{W}}(n)) \geq -\mathcal{O}(n^{\beta-1})$, and thus
 1104 by another application of Lemma 3.15, we conclude that $W_4(n) = \mathcal{O}(n^\beta)$, for any $\beta > \max(2\alpha - 1, \frac{\alpha}{2})$.
 1105 Then we can use the same argument as in step 2: we first observe that this entails

$$1106 \quad F_3(\hat{\mathbf{W}}(n)) \leq \frac{\hat{W}_5(n)}{2}(1 - o(1)) - \hat{W}_3(n) + \mathcal{O}(n^{\beta-1}) \leq -\frac{\hat{W}_3(n)}{4} + \mathcal{O}(n^{\beta-1}) \leq \mathcal{O}(n^{\beta-1}).$$

1107 Therefore, if $\liminf \frac{W_3(n)}{n^\beta} = \infty$, then $W_3(n) \sim W_5(n)$, and $W_4(n) = o(W_3(n))$. Thus by Lemma 3.17(c)
 1108 again (applied with $r = 0$), we get $F_4(\hat{\mathbf{W}}(n)) \geq 0$, for all n large enough, which leads to a contradiction
 1109 as in step 2. We conclude that $W_3(n) = \mathcal{O}(n^\beta)$, as wanted. This concludes the proof of the lemma. \square

1110 An immediate corollary of the results obtained so far is the following fact.

1111 **Corollary 3.18.** *On the event when $W_1(n)/n \rightarrow 1$, one has almost surely for any $\varepsilon > 0$,*

$$1112 \quad W_3(n) = \mathcal{O}(n^\varepsilon).$$

1113 *Proof.* It suffices to combine Lemmas 3.7 and 3.12 with Lemma 3.16, which we can iterate as much as
 1114 needed. Indeed, the map $\varphi : \alpha \mapsto \max(2\alpha - 1, \frac{\alpha}{2})$ is decreasing, with 0 as unique fixed point in $[0, 1)$,
 1115 which implies that any sequence defined by $\alpha_{n+1} = \varphi(\alpha_n)$, with $\alpha_0 < 1$, converges to 0. Lemmas 3.7
 1116 and 3.12 give the existence of $\alpha_0 < 1$ such that, on the event $W_1(n)/n \rightarrow 1$, $\mathcal{A}(\alpha_0)$ has probability 1.
 1117 Lemma 3.16 then implies that for all $n \geq 0$, on the event $W_1(n)/n \rightarrow 1$, $\mathcal{A}(\alpha_n)$ has probability 1. We
 1118 then choose n large enough so that $\alpha_n < \varepsilon$. \square

1119 The final step is the following result, which together with Corollary 3.18 brings a contradiction, if
 1120 $W_1(n)/n \rightarrow 1$, and therefore concludes the proof of Proposition 3.11.

1121 **Lemma 3.19.** *On the event when $W_1(n)/n \rightarrow 1$, one has almost surely for any $c \in (0, 1/5)$,*

$$1122 \quad \lim_{n \rightarrow \infty} \frac{W_3(n)}{n^c} = +\infty.$$

1123 *Proof.* Recall that when w_3, w_4 and w_5 go to 0, one has for $w \in \mathcal{E}$,

$$1124 \quad p_{135}(w) + p_{234}(w) = \frac{w_3(w_5 + \frac{w_4}{2})}{w_3 + w_4 + w_5} (1 + o(1)).$$

1125 Using now that $w_4 + w_5 \geq w_3$, we get that

$$1126 \quad p_{135}(w) + p_{234}(w) \geq \frac{w_3(1 + o(1))}{4} = \frac{w_3/4(1 + o(1))}{1 - w_3 + w_3/4}.$$

1127 Thus there exists $\varepsilon > 0$, such that for any $w \in \mathcal{E}$, with $w_3, w_4, w_5 \leq \varepsilon$,

$$1128 \quad p_{135}(w) + p_{234}(w) \geq \frac{w_3/5}{1 - w_3 + w_3/5}.$$

1129 By Proposition 3.2 and Lemma 3.12, we deduce that almost surely on the event when $W_1(n)/n \rightarrow 1$,
 1130 there exists a random integer n_0 such that, for all $n \geq n_0$,

$$1131 \quad p_{135}(\hat{\mathbf{W}}(n)) + p_{234}(\hat{\mathbf{W}}(n)) \geq \frac{\hat{W}_3(n)/5}{1 - \hat{W}_3(n) + \hat{W}_3(n)/5} = \frac{W_3(n)/5}{n + 2 - W_3(n) + W_3(n)/5}.$$

1132 Therefore, after some (random) time n_0 , the process $(W_3(n))_{n \geq n_0}$ stochastically dominates an urn process
 1133 $(U(n))_{n \geq n_0}$, defined by $U(n_0) = 1$ and, for all $n \geq n_0$,

$$1134 \quad \mathbb{P}\left(U(n+1) = U(n) + 1 \mid U(n)\right) = 1 - \mathbb{P}\left(U(n+1) = U(n) \mid U(n)\right) = \frac{U(n)/5}{n + n_0 + 2 - U(n) + U(n)/5}.$$

1135 For any fixed n_0 , the urn process $(U(n))_{n \geq n_0}$ is studied for instance in Janson [Jan06] (see in particular
 1136 Theorem 1.4 and Remark 1.12 there), which provides a precise asymptotic behavior of $n^{-1/5}U(n)$: it
 1137 converges in law towards some non-degenerate random positive variable. But here one can simply rely
 1138 again on Rubin's construction, which covers our needs. It shows that for any fixed n_0 , almost surely there
 1139 exists a constant $c > 0$, such that $U(n) \geq cn^{1/5}$, for any $n \geq n_0$, and the lemma follows. \square

1140 The proofs of Proposition 3.11 and Theorem 1.4 are now complete.

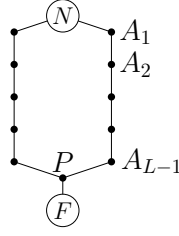


Figure 11: The graph of Proposition 1.5 and the notations used in Section 4.

1141 4 Proof of Proposition 1.5

1142 We fix an integer L and look at the uniform-geodesic version of the model on the graph on the left-hand
 1143 side of Figure 5. Note that each ant reinforces either the L edges on the left (and the edge linked to F)
 1144 or the L edges on the right (and the edge linked to F). Thus, the L edges on the left have all the same
 1145 weight at all times, and similarly for the the L edges on the right. For all integers n , we set $N_1(n)$ to
 1146 be the weight of the L left-edges at time n , and by $N_2(n)$ the weights of the right-edges at time n . By
 1147 definition, we have that $N_1(n) + N_2(n) = n + 2$, which is also the weight of the edge linked to F .

1148 To prove Proposition 1.5, we apply a result of [HLS80] (see [Pem07, Theorem 2.8]):

1149 **Theorem 4.1.** *Let $(Z_n)_{n \geq 0}$ be a sequence of random variables taking values in $[0, 1]$ satisfying, for all*
 1150 *$n \geq 0$,*

$$1151 \quad Z_{n+1} = Z_n + \frac{1}{n} (F(Z_n) + \Delta M_{n+1}), \quad (53)$$

1152 *where $F : [0, 1] \rightarrow [0, 1]$ and ΔM_{n+1} is a martingale increment. If there exists $\varepsilon > 0$ such that $F < 0$ on*
 1153 *$[0, \varepsilon]$, then $\mathbb{P}(Z_n \rightarrow 0) > 0$.*

1154 First note that, if $Z_n = N_1(n)/(n + 2)$ for all $n \geq 0$, then Z_n satisfies Equation (53) with $F(x) =$
 1155 $p(x) - x$, where $p(x)$ is the probability that an ant reinforces the left-hand side geodesic after performing
 1156 a random walk on \mathcal{G} with weights x on all edges on the left, $1 - x$ on all edges on the right, and 1 on
 1157 the edge linked to F . We let $\mathcal{G}(x)$ denote the graph \mathcal{G} equipped with these weights, P denote the unique
 1158 vertex neighbouring F , and for all $k \in \{0, \dots, L - 1\}$, A_k denote the vertex at distance k of N on the
 1159 right-hand-side geodesic (with $A_0 = N$). See Figure 11 where the notations are illustrated.

1160 We now calculate $p(x)$ when $x \rightarrow 0$ to show that $p(x) < x$ in a neighbourhood of zero; this implies
 1161 that $F < 0$ in a neighbourhood of zero and thus that Theorem 4.1 applies. Asymptotically when $x \rightarrow 0$,

$$1162 \quad p(x) = \sum_{k=0}^{L-1} \left(p_k^{(1)}(x) + \frac{1}{2} p_k^{(2)}(x) \right) + \frac{1}{2} p^{(3)}(x) + \frac{1}{2} p^{(4)}(x) + \mathcal{O}(x^2), \quad (54)$$

1163 where, for all $k \in \{0, \dots, L - 1\}$,

- 1164 • $p_k^{(1)}(x)$ the probability that a walker on the weighted graph $\mathcal{G}(x)$ goes from N to A_k using only
 1165 edges on the right-hand-side geodesic, then goes from A_k to N without reaching A_{k+1} , then goes
 1166 from N to P without reaching A_{k+1} , and, finally, goes from P to F without using the left-hand-side
 1167 geodesic or reaching A_k ;
- 1168 • $p_k^{(2)}(x)$ the probability that a walker on the weighted graph $\mathcal{G}(x)$ goes from N to A_k using only edges
 1169 on the right-hand-side geodesic, then goes from A_k to P without reaching A_{k+1} (thus using edges
 1170 on the left-hand-side geodesic), and, finally, goes from P to A_k using only edges on the right-hand
 1171 side geodesic;

- 1172 • $p^{(3)}(x)$ the probability that a walker on the weighted graph $\mathcal{G}(x)$ first goes from N to P only using
 1173 edges on the right-hand-side geodesic, then goes from P to N using edges on the left-hand-side
 1174 geodesic, and before entering the left-hand geodesic from N ;
- 1175 • $p^{(4)}(x)$ the probability that a walker on the weighted graph $\mathcal{G}(x)$ first goes from N to P only using
 1176 edges on the right-hand-side geodesic, then goes back from P to N only using edges on the right-
 1177 hand-side geodesic, and, finally, goes from N to P using edges on the left-hand-side geodesic and
 1178 before entering the left-hand-side geodesic from P or hitting F .

1179 The $\mathcal{O}(x^2)$ -term in Equation (54) stands for all trajectories of the walker that leave N or P at least twice
 1180 towards the left. We have, if $k \in \{1, \dots, L-1\}$,

$$1181 \quad p_k^{(1)}(x) = \frac{\frac{1-x}{k}}{\frac{1-x}{k} + x} \cdot \frac{\frac{1-x}{k}}{\frac{1-x}{k} + (1-x)} \cdot \frac{\frac{x}{L}}{\frac{x}{L} + \frac{(1-x)}{k+1}} \cdot \frac{1}{1+x + \frac{1-x}{L-k}} = \frac{x}{L} \cdot \frac{L-k}{L-k+1} + \mathcal{O}(x^2).$$

1182 We also have

$$1183 \quad p_0^{(1)}(x) = \frac{\frac{x}{L}}{\frac{x}{L} + (1-x)} \cdot \frac{1}{1+x + \frac{1-x}{L}} = \frac{x}{L} \cdot \frac{L}{L+1} + \mathcal{O}(x^2).$$

1184 Using the fact that $\sum_{i=1}^n 1/i = \log n + \mathcal{O}(1)$ when $n \rightarrow +\infty$, we get

$$1185 \quad \sum_{k=0}^{L-1} p_k^{(1)}(x) = \frac{x}{L} \sum_{k=0}^{L-1} \left(1 - \frac{1}{L-k+1}\right) + \mathcal{O}(x^2) = x \left(1 - \frac{\log L}{L} + \mathcal{O}_{L \rightarrow +\infty}(1)\right) + \mathcal{O}(x^2), \quad (55)$$

1186 where the $\mathcal{O}_{L \rightarrow +\infty}(1)$ -term does not depend on x and corresponds to the $L \rightarrow +\infty$ limit, while the
 1187 $\mathcal{O}(x^2)$ -term depends on L and refers to the $x \rightarrow 0$ limit. Similarly, for all $k \in \{1, \dots, L-1\}$, we have

$$1188 \quad p_k^{(2)}(x) = \frac{\frac{1-x}{k}}{\frac{1-x}{k} + x} \cdot \frac{\frac{1-x}{k}}{\frac{1-x}{k} + 1-x} \cdot \frac{\frac{x}{L}}{\frac{x}{L} + \frac{1-x}{k+1}} \cdot \frac{\frac{1-x}{L-k}}{\frac{1-x}{L-k} + 1+x} = \frac{x}{L} \cdot \frac{1}{L-k+1},$$

1189 and

$$1190 \quad p_0^{(2)}(x) = \frac{\frac{x}{L}}{\frac{x}{L} + (1-x)} \cdot \frac{\frac{1-x}{L}}{\frac{1-x}{L} + 1+x} = \frac{x}{L} \cdot \frac{1}{L+1} + \mathcal{O}(x^2).$$

1191 Using again the asymptotic behaviour of the harmonic sum, we get

$$1192 \quad \frac{1}{2} \sum_{k=0}^{L-1} p_k^{(1)}(x) = \frac{x}{L} \sum_{k=0}^{L-1} \frac{1}{L-k+1} + \mathcal{O}(x^2) = x \left(\frac{\log L}{2L} (1 + \mathcal{O}_{L \rightarrow +\infty}(1)) \right) + \mathcal{O}(x^2). \quad (56)$$

1193 We also have, when $x \rightarrow 0$,

$$1194 \quad p^{(3)}(x) = \frac{\frac{1-x}{L}}{\frac{1-x}{L} + x} \cdot \frac{\frac{x}{L}}{\frac{x}{L} + 1 + \frac{x \frac{1-x}{L}}{x + \frac{1-x}{L}}} = \frac{x}{L} + \mathcal{O}(x^2),$$

1195 and

$$1196 \quad p^{(4)}(x) = \frac{\frac{1-x}{L}}{\frac{1-x}{L} + x} \cdot \frac{\frac{1-x}{L}}{\frac{1-x}{L} + 1+x} \cdot \frac{\frac{x}{L}}{\frac{x}{L} + \frac{(1+x)\frac{1-x}{L}}{1+x + \frac{1-x}{L}}} = \frac{x}{L} + \mathcal{O}(x^2).$$

1197 Using these last equations together with (55) and (56) into Equation (54), we get that, in total,

$$1198 \quad p(x) = x \left(1 - \frac{\log L}{2L} (1 + \mathcal{O}_{L \rightarrow +\infty}(1))\right) + \mathcal{O}(x^2).$$

1199 Therefore,

$$1200 \quad F(x) = p(x) - x = -x \left(\frac{\log L}{2L} (1 + \mathcal{O}_{L \rightarrow +\infty}(1)) \right) + \mathcal{O}(x^2),$$

1201 implying that for all L large enough, F is indeed negative in a right-neighbourhood of 0. Hence, Theorem
 1202 4.1 applies and we conclude that $\mathbb{P}(Z_n \rightarrow 0) > 0$, which concludes the proof of Proposition 1.5.

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