On spectral gaps of growth-fragmentation semigroups with mass loss or death

Mustapha Mokhtar-Kharroubi

On spectral gaps of growth-fragmentation semigroups with mass loss or death

Mustapha Mokhtar-Kharroubi
Laboratoire de Mathématiques, CNRS-UMR 6623
Université de Bourgogne Franche-Comté
16 Route de Gray, 25030 Besançon, France.
E-mail: mmokhtar@univ-fcomte.fr

Abstract

We give a general theory on well-posedness and time asymptotics for growth fragmentation equations. These linear kinetic (integro-differential) equations arise in the modeling of various physical or biological phenomena involving concentration of agregates which experience both growth and fragmentation. We prove first generation of C_0-semigroups $(V(t))_{t\geq 0}$ governing them for unbounded total fragmentation rate and fragmentation kernel $b(.,.)$ such that $\int_0^y xb(x,y)dx = y-\eta(y)y$ ($0 \leq \eta(y) \leq 1$ expresses the mass loss) and continuous growth rate $r(.)$ such that $\int_0^\infty \frac{1}{r(\tau)}d\tau = +\infty$. This is done in three natural functional spaces $L^1(\mathbb{R}_+, \mu(dx))$ ($\mu(dx) = dx,\ xdx$ or $(1+x)\,dx$) which correspond respectively to finite number of agregates, finite mass or finite mass and number of agregates. The mass loss or death assumptions are needed only in the vicinity of points where the total fragmentation rate gets infinite. The analysis relies on unbounded perturbation theory peculiar to positive semigroups in L^1 spaces. Secondly, we show when the resolvent of the generator is compact and the semigroup has a spectral gap, i.e. $r_{ess}(V(t)) < r_\sigma(V(t))$ (r_{ess} is the essential spectral radius), and an asynchronous exponential growth. The analysis relies on weak compactness tools and Frobenius theory of positive operators. A systematic functional analytic construction is provided.

Contents

1 Introduction

This paper provides a general theory on well-posedness (in the sense of C_0-semigroups) and time asymptotics of growth-fragmentation equations

$$\begin{aligned} &\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left(r(x)u(x,t)\right) + (a(x)+d(x))u(x,t) \\ = &\int_x^{+\infty} a(y)b(x,y)u(y,t)dy, \quad u(x,0) = u_0(x), \quad x,t>0 \qquad (1) \end{aligned}$$

with measurable nonnegative fragmentation kernel $b(.,.)$ and positive growth rate $r(.)$ satisfying the general structural assumptions

$$\int_0^y xb(x,y)dx = y\left(1-\eta(y)\right), \quad 0 \leq \eta(y) \leq 1 \; (y \geq 0) \tag{2}$$

$$b(x,y) > 0 \text{ if } 0 \leq x < y \tag{3}$$

$$r(.) \in C\left(0,+\infty\right), \; \int_0^\infty \frac{1}{r(\tau)} d\tau = +\infty; \tag{4}$$

while the total fragmentation rate $a(.)$ and the death (or degradation) rate $d(.)$ are measurable nonnegative and

$$\beta := a + d \in L^1_{loc}(0,+\infty). \tag{5}$$

Among the physical examples of growth rates we can find in the literature, note for instance the typical ones

$$r(x) = 1 \;\text{ or } r(x) = x, \quad (x > 0)\,. \tag{6}$$

The kinetic equations (1) arise in the modeling of various physical or biological phenomena involving concentration of agregates which experience both growth and fragmentation. Typical biological examples are provided by phytoplankton dynamics [1][2] or by prions dynamics [15]; we refer to [16] and references therein for a lot of contexts where these equations arise; see also the monographs [20][5][34] for more information. The unknown $u(x,t)$ represents the concentration at time t of "agregates" with mass $x > 0$ while $b(x,y)$ $(x < y)$ describes the distribution of mass x agregates, called daughter agregates, spawned by the fragmentation of a mass y agregates. The local mass conservation in the fragmentation process corresponds to

$$\frac{1}{y}\int_0^y xb(x,y)dx = 1$$

i.e. to $\eta(.) = 0$. In this case, we say that the kernel $b(.,.)$ is *conservative*. Most of the literature is concerned with conservative fragmentation kernels. On the other hand

$$\eta(.) \neq 0 \tag{7}$$

amounts to saying that a *mass loss* takes place in the fragmentation process, i.e.

$$\frac{1}{y}\int_0^y xb(x,y)dx \leq 1$$

where

$$\eta(y) = 1 - \frac{1}{y}\int_0^y xb(x,y)dx \tag{8}$$

quantifies this mass loss ([3] Chapter 9).

Notice that these fragmentation kernels contain for example homogeneous kernels

$$b(x,y) = \frac{1}{y}h(\frac{x}{y}) \text{ with } \int_0^1 zh(z)dz \leq 1 \tag{9}$$

(for some $h \in L^1_+((0,1);xdx)$) since

$$\int_0^y xb(x,y)dx = \int_0^y \frac{x}{y}h(\frac{x}{y})dx = y\int_0^1 zh(z)dz = y\,(1-\eta)$$

where

$$\eta = 1 - \int_0^1 zh(z)dz.$$

More generally, for any conservative fragmentation kernel $\widehat{b}(x,y)$ and

$$0 \leq \zeta(x,y) \leq 1, \tag{10}$$

the kernel

$$b(x,y) := \zeta(x,y)\widehat{b}(x,y) \tag{11}$$

satisfies (2) with

$$\eta(y) = \frac{1}{y}\int_0^y x\,(1-\zeta(x,y))\,\widehat{b}(x,y)dx. \tag{12}$$

Conversely, any fragmentation kernel $b(.,.)$ satisfying (2) is of the form (11) with

$$\zeta(x,y) = \frac{1}{y}\int_0^y xb(x,y)dx \text{ and } \widehat{b}(x,y) = \frac{b(x,y)}{\zeta(x,y)}.$$

We have thus a full description of the fragmentation kernels considered in this paper by means of conservative kernels.

We point out that for a given concentration $u(.,.)$,

$$\int_0^{+\infty} u(x,t)xdx \quad \text{and} \quad \int_0^{+\infty} u(x,t)dx$$

are respectively the total mass and the total number of agregates at time $t \geq 0$. Thus, three natural functional spaces are of particular interest: the "finite mass" space

$$X_1 := L^1(\mathbb{R}_+;\ xdx) \text{ with norm } \|\|_{X_1},$$

the "finite agregates number" space

$$X_0 := L^1(\mathbb{R}_+;\ dx) \text{ with norm } \|\|_{X_0}$$

and the "finite mass and agregates number" space

$$X_{0,1} := L^1(\mathbb{R}_+;\ (1+x)\,dx) \text{ with norm } \|\|_{X_{0,1}}.$$

Because of their physical relevance and for the sake of completeness, we consider the growth-fragmentation equations in *each* of the functional spaces above and for the *different types* of divergence (4) (see (21)(22) below). The technical details (and assumptions) may vary a bit from one context to another but overall the general mathematical arguments are similar.

We recall that semigroup generation for growth (i.e. transport) equations

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left(r(x)u(x,t)\right) + \beta(x)u(x,t) = 0 \tag{13}$$

is known in various functional settings and under various assumptions on the growth rate r (see e.g. [5][7][8] for a resolvent approach via Hille-Yosida theory). The first object of this paper is well-posedness of (1) in the sense of C_0-semigroups. To this end, we give first a direct and systematic construction of explicit growth semigroups $(U(t))_{t\geq 0}$ governing (13) with continuous growth rates $r(.)$ in the functional spaces above under "optimal" (i.e. sufficient and "necessary") assumptions. We consider then the fragmentation operator

$$B : \varphi \in D(T) \to \int_x^{+\infty} a(y)b(x,y)\varphi(y)dy \tag{14}$$

(T is the generator of $(U(t))_{t\geq 0}$) as a perturbation and show a generation of a C_0-semigroup $(V(t))_{t\geq 0}$ by $T+B$ with domain

$$D(T+B) = D(T) \tag{15}$$

under suitable assumptions depending on the functional space we consider. To this end, we use a perturbation theorem *peculiar* to positive C_0-semigroups in L^1-spaces by W. Desch [13]:

Theorem 1 *([13]; see also [36] or [23] Chapter 8) Let $(U(t))_{t\geq 0}$ be a positive C_0-semigroup with generator T on some $L^1(\mu)$-space and let*

$$B : \varphi \in D(T) \subset L^1(\mu) \to L^1(\mu)$$

be continuous on $D(T)$ (endowed with the graph norm) and positive (i.e. $B : D(T) \cap L^1_+(\mu) \to L^1_+(\mu)$). Then

$$T + B : D(T) \subset L^1(\mu) \to L^1(\mu)$$

generates a positive C_0-semigroup on $L^1(\mu)$ if and only if

$$\lim_{\lambda \to +\infty} r_\sigma \left(B (\lambda - T)^{-1} \right) < 1.$$

Based on a systematic use of weak compactness arguments, the second object of this paper is to analyze, in each of the above functional spaces and for different types of divergence (4) (see (21)(22) below), the existence of a *spectral gap*

$$r_{ess}(V(t)) < r_\sigma(V(t)) \tag{16}$$

(r_{ess} and r_σ refer respectively to the essential spectral radius and the spectral radius) or equivalently

$$\omega_{ess}(V) < \omega(V)$$

where $\omega_{ess}(V)$ and $\omega(V)$ denote respectively the essential type and the type of $(V(t))_{t\geq 0}$, (see e.g. [23] Chapter 2). This property is related to *stability of essential type* under suitable perturbations, (see below). Note that the expression "spectral gap" is widely used in the mathematical literature but has not always a univocal meaning; we use it here in the sense above. The combination of (16) with some irreducibility condition implies the so-called asynchronous exponential growth of $(V(t))_{t\geq 0}$

$$\left\| e^{-\lambda t} V(t) - P \right\| = O(e^{-\varepsilon t}) \tag{17}$$

(for some $\varepsilon > 0$) where P is a one-dimensional spectral projection relative to the leading isolated algebraically simple dominant eigenvalue λ (Malthus parameter) of the generator, (see e.g. [37]). More precisely

$$P\varphi = \left(\int_0^{+\infty} \varphi(x) u_*(x) \mu(dx) \right) u$$

where $\mu(dx) = dx,\ xdx$ or $(1+x)\,dx$ (depending on the choice of $X_0,\ X_1$ or $X_{0,1}$), u is the nonnegative eigenfunction associated to the leading isolated eigenvalue λ of the generator and u_* is the dual nonnegative eigenfunction (associated to the leading isolated eigenvalue λ) with a normalization

$$\int_0^{+\infty} u(x) u_*(x) \mu(dx) = 1.$$

We point out that the existence of such Perron eigenelements is a consequence of the spectral gap (16).

The existence of Perron eigenvectors, *regardless* of the occurence of a spectral gap, and their asymptotic stability in weighted L^1 spaces (the weight being the dual eigenvector) rely on different tools and have been the subject of rich works in the last decade. Without pretense to completeness, we refer e.g. to [18][15][21][16][22][7][8][9][10] where some results combine relative entropy techniques; see also [11] for a probabilistic approach. We refer to the introductions of [22][16][8][10] for a comprehensive review of the existing tools and results. In particular, we point out that asymptotic stability need not be uniform with respect to initial data and, at least in suitable weighted spaces, we cannot expect the existence of a spectral gap for *bounded* total fragmentation rates $a(.)$ even if Perron eigenvectors can exist [7]. (We do not comment here on the case of bounded state spaces which goes back to the pioneer paper [14]; see [4] and references therein for more recent works in this direction.)

Our paper is rather in the same spirit as [8]. The latter deals with asynchronous exponential growth (17) under the divergence (22) below in *higher* moment spaces

$$L^1(\mathbb{R}_+;\ (1+x)^{\alpha}\,dx)\ \ (\alpha > \alpha_*) \tag{18}$$

for a suitable threshold $\alpha_* \geq 1$ (see also [22][10]). We deal here with the asynchronous exponential growth (17) in the natural spaces X_0, $X_1, X_{0,1}$ under the general divergence (4) but at the expense of the additional assumption

$$\eta(.) \neq 0 \text{ or } d(.) \neq 0 \tag{19}$$

(*mass loss* or *death*) which does not occur in the literature on the subject.

Assumption (19) opens new mathematical perspectives and allows a systematic functional analytic construction which is the object of this paper. This general theory is based on few structural assumptions only. Besides the main results on spectral gaps, many preliminary results of independent interest are also given and the role of unboundedness of total fragmentation rates $a(.)$ *is fully highlighted.*

Our construction, inspired by recent contributions to other structured models [29][30][31], relies on three key mathematical ingredients:

(i) The weak compactness tools, for absorption semigroups in L^1 spaces, introduced in [28].

(ii) The convex (weak) compactness property of the strong operator topology in Banach spaces [35] (see also [24] for an elementary proof in $L^1(\nu)$ spaces).

(iii) Strict comparison of spectral radii in Frobenius theory [19].

Among linear kinetic equations, growth-fragmentation equations present a very particular trait: the state-variable is *one-dimensional*. This gives them a *local* regularizing effect that does not exist in usual kinetic theory, e.g. in neutron transport, where the transport part has no local regularizing effect and the perturbation (the collision operator) is non-local with respect to another (velocity) v-variable; this second state variable has a regularizing (local compactness) effect with respect to space x-variable and induces the stability of the essential type [25][26]. On the other hand, for growth-fragmentation equations, the compactness results (i.e. the key point behind the spectral gap property) are consequences of the local regularizing effect we alluded to and of the confining role of singular absorptions [28], hence the key role, in our construction, of *unboundedness* of total fragmentation rates $a(.)$. We point out that (19) is needed *only* in the vicinity of points where $a(.)$ gets infinite. Finally, we note that for bounded total fragmentation rates, no spectral gap can exist in the weighted spaces

$$L^1(\mathbb{R}_+;\ (1+x)^\alpha\, dx) \ \ (\alpha < 1), \tag{20}$$

see [7]; we conjecture that we cannot expect spectral gaps in X_1, X_0 or $X_{0,1}$ if the total fragmentation rate is bounded.

Our paper is organized as follows:

We provide first an explicit construction of growth C_0-semigroups governing (13) by the method of characteristics. Two different growth C_0-semigroups occur according as

$$\int_0^1 \frac{1}{r(\tau)} d\tau = +\infty, \ \int_1^\infty \frac{1}{r(\tau)} d\tau = +\infty \tag{21}$$

or

$$\int_0^1 \frac{1}{r(\tau)} d\tau < +\infty, \ \int_1^\infty \frac{1}{r(\tau)} d\tau = +\infty \tag{22}$$

to cover e.g. the examples (6). (For the sake of simplicity, we ignore the case $\int_0^1 \frac{1}{r(\tau)} d\tau = +\infty,\ \int_1^\infty \frac{1}{r(\tau)} d\tau < +\infty$.) Note that (22) is complemented by a boundary condition, see (31) below. Our main results in the spaces X_1 and $X_{0,1}$ under Assumption (21) are the following:

A transport C_0-semigroup $(U(t))_{t\geq 0}$ governing (13) exists in X_1 (resp. in $X_{0,1}$) and is given by

$$U(t)f = e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)} dp} f(X(y,t)) \frac{\partial X(y,t)}{\partial y}$$

($X(y,t)$ is defined by $\int_{X(y,t)}^{y}\frac{1}{r(\tau)}d\tau=t$) provided that

$$\alpha:=\sup_{z>0}\frac{r(z)}{z}<+\infty \ \ (\text{resp. } \sup_{z>1}\frac{r(z)}{z}<+\infty); \tag{23}$$

(see Proposition 11 and Proposition 32). In addition, the assumptions (23) are "necessary" to a generation theory, (see Proposition 6 and Remark 26). Note that under (21), the generation theory in $X_{0,1}$ needs *no* condition on the growth rate at the origin. Note also that the C_0-semigroup $(U_0(t))_{t\geq 0}$ corresponding to $\beta=0$ is not contractive, (see Remarks 5 and 27).

We also "compute" the spectral bound

$$s(T):=\sup\{\operatorname{Re}\nu;\ \nu\in\sigma(T)\}$$

of its generator T; (see Proposition 13 and Proposition 34). We recall that $s(T)$ *coincides* with the type of $(U(t))_{t\geq 0}$; (this is a general property of positive semigroups in Lebesgue spaces, see e.g. [38]). The resolvent of T is given by

$$\left((\lambda-T)^{-1}f\right)(y)=\frac{1}{r(y)}\int_0^y e^{-\int_x^y\frac{\lambda+\beta(\tau)}{r(\tau)}d\tau}f(x)dx\ (\operatorname{Re}\lambda>s(T))$$

in both spaces X_1 and $X_{0,1}$; (see Proposition 12 and Proposition 33). Note the domination

$$U(t)\leq U_0(t)\ (t\geq 0)\text{ and }(\lambda-T)^{-1}\leq(\lambda-T_0)^{-1}\ \ (\lambda>s(T_0))$$

where T_0 is the generator of $(U_0(t))_{t\geq 0}$. We show the *pointwise* a priori estimate in X_1

$$\left|(\lambda-T_0)^{-1}f\right|(y)\leq\frac{1}{yr(y)}\|f\|_{X_1}\ \ (f\in X_1)\ \ (\lambda>\alpha);$$

(see Lemma 10).

In $X_{0,1}$, if we replace the natural condition $\sup_{z>1}\frac{r(z)}{z}<+\infty$ by the stronger one

$$C:=\sup_{z>0}\frac{r(z)}{1+z}<+\infty, \tag{24}$$

we show the *pointwise* a priori estimate in $X_{0,1}$

$$\left|(\lambda-T_0)^{-1}f\right|(y)\leq\frac{1}{(1+y)\,r(y)}\|f\|_{X_{0,1}}\ \ \ (f\in X_{0,1})\ \ (\lambda>C),$$

(see Lemma 30). Note that by domination, the pointwise estimates above are inherited by $(\lambda - T)^{-1}$.

We show that T has a *smoothing* effect in X_1 for $\lambda > \alpha$

$$\int_0^{+\infty} \left|(\lambda - T)^{-1} f\right|(y)\beta(y)ydy \leq \int_0^{+\infty} |f(y)|\, ydy \quad (f \in X_1),$$

(see Lemma 14).

In $X_{0,1}$, if we replace the natural condition $\sup_{z>1} \frac{r(z)}{z} < +\infty$ by the stronger one (24), we show the smoothing effect for $\lambda > C$

$$\int_0^{+\infty} \left|(\lambda - T)^{-1} f\right|(y)\beta(y)(1+y)dy \leq \int_0^{+\infty} |f(y)|\,(1+y)dy \quad (f \in X_{0,1}),$$

(see Lemma 35). The above estimates, combined to the general theory [28] on compactness properties in L^1 spaces induced by the *confining effect of singular absorptions,* show that if the sublevel sets of β

$$\Omega_c = \{x > 0; \beta(x) < c\} \quad (c > 0)$$

are "thin near zero and near infinity relatively to r" in the sense

$$\int_0^{+\infty} \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \quad (c > 0) \tag{25}$$

where 1_{Ω_c} is the indicator function of Ω_c (note that $\frac{1}{r(z)} \notin L^1(0, +\infty)$) then T is resolvent compact in both spaces X_1 and $X_{0,1}$, i.e. $(\lambda - T)^{-1}$ is compact in X_1 and $X_{0,1}$; (see Theorem 20 and Theorem 42). This occurs for instance if

$$\lim_{y \to 0_+} \beta(y) = +\infty, \quad \lim_{y \to +\infty} \beta(y) = +\infty.$$

Note that if Ω_c has finite Lebesgue measure then $\int_1^{+\infty} \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty$ provided that $\frac{1}{r(\tau)} \in L^p(1, +\infty)$ for some $p > 1$.

One shows that the fragmentation operator B given par (14) is T-bounded in X_1 and

$$\lim_{\lambda \to +\infty} \left\| B(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_1)} \leq \lim \sup_{a(y) \to +\infty} \frac{1 - \eta(y)}{1 + \frac{d(y)}{a(y)}};$$

in particular, by W. Desch's perturbation theorem (Theorem 1),

$$T + B : D(T) \subset X_1 \to X_1$$

generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ on X_1 provided that

$$\gamma := \lim \sup_{a(y)\to+\infty} \frac{1-\eta(y)}{1+\frac{d(y)}{a(y)}} < 1; \tag{26}$$

(see Theorem 16). Note that (26) is satisfied e.g. if $\lim\inf_{a(y)\to+\infty} \frac{d(y)}{a(y)} > 0$ or if

$$\lim \inf_{a(y)\to+\infty} \eta(y) > 0. \tag{27}$$

This explains why we need *mass loss* or *death* assumptions and why these are needed *only* in the vicinity of points where $a(.)$ gets infinite. We note that for homogeneous kernels (9), the condition (27) amounts to

$$\int_0^1 zh(z)dz < 1.$$

More generally, in the case (11), the condition (27) holds if

$$\lim \sup_{a(y)\to+\infty} \zeta(x,y) < 1,$$

(see Remark 18).

Under (26),

$$T + B : D(T) \subset X_1 \to X_1$$

is resolvent compact provided that $T : D(T) \to X_1$ is; (see Corollary 21). We build a C_0-semigroup $\left(\widehat{V}(t)\right)_{t\geq 0}$ on X_1 such that

$$U(t) \leq \widehat{V}(t) \leq V(t) \quad (t \geq 0).$$

By using the convex (weak) compactness property of the strong operator topology [35][24] (see below), we show that $\left(\widehat{V}(t)\right)_{t\geq 0}$ and $(V(t))_{t\geq 0}$ have the same essential type

$$\omega_{ess}(\widehat{V}) = \omega_{ess}(V).$$

The resolvent compactness of their generators and the *strict* comparison results of spectral radii of positive compact operators in domination contexts [19] imply the strict comparison of the types

$$\omega(\widehat{V}) < \omega(V)$$

and consequently $(V(t))_{t\geq 0}$ has a spectral gap (16) and exhibits the asynchronous exponential growth (17) in X_1 provided that the support of $a(.)$ is not bounded; (see Theorem 23).

The analysis in $X_{0,1}$ is similar but needs a different assumption. Indeed, one shows that the fragmentation operator (14) is T-bounded in $X_{0,1}$ and

$$\lim_{\lambda\to+\infty}\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_1)}\leq \lim\sup_{a(y)\to+\infty}\frac{[(y-\eta(y)y)+n(y)]}{(1+y)\left(1+\frac{d(y)}{a(y)}\right)}$$

provided that

$$n(y):=\int_0^y b(x,y)dx,$$

(the expected number of daughter agregates spawned by a mother agregates of mass y) is such that

$$\sup_{y>0}\frac{n(y)}{1+y}<+\infty.$$

By appealing again to W. Desch's perturbation theorem,

$$T+B:D(T)\subset X_{0,1}\to X_{0,1}$$

generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ on $X_{0,1}$ provided that

$$\lim\sup_{a(y)\to+\infty}\frac{[(y-\eta(y)y)+n(y)]}{(1+y)\left(1+\frac{d(y)}{a(y)}\right)}<1; \tag{28}$$

(see Theorem 38). As previously, mass loss or death are needed only in the vicinity of points where $a(.)$ gets infinite. Note that

$$\frac{[(y-\eta(y)y)+n(y)]}{(1+y)\left(1+\frac{d(y)}{a(y)}\right)}=\frac{\left[y+\frac{n(y)}{(1-\eta(y))}\right]}{(y+1)}\frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}}$$

so (28) occurs provided that

$$\lim\sup_{a(y)\to+\infty}\frac{\left[y+\frac{n(y)}{(1-\eta(y))}\right]}{(y+1)}<\gamma^{-1}.$$

In particular, if $\gamma<1$ (i.e. under the generation criterion in X_1) and if $a(.)$ is unbounded at zero and at infinity only, then (28) (i.e. the generation criterion in $X_{0,1}$) occurs provided that

$$\max\left\{\lim\sup_{y\to 0}\frac{n(y)}{(1-\eta(y))},\ 1+\lim\sup_{y\to+\infty}\frac{n(y)}{y\,(1-\eta(y))}\right\}<\gamma^{-1},$$

(see Corollary 39). This is the case e.g. if $\eta(.) = 0$ and

$$\max\left\{\limsup_{y\to 0} n(y),\ 1 + \lim\sup_{y\to+\infty} \frac{n(y)}{y}\right\} < 1 + \lim\inf_{a(y)\to+\infty} \frac{d(y)}{a(y)}$$

or if $d(.) = 0$ and

$$\begin{aligned} &\max\left\{\limsup_{y\to 0} \frac{n(y)}{(1-\eta(y))},\ 1 + \lim\sup_{y\to+\infty} \frac{n(y)}{y\,(1-\eta(y))}\right\} \\ < \quad &\left(1 - \lim\inf_{a(y)\to+\infty} \eta(y)\right)^{-1}. \end{aligned} \tag{29}$$

We note that for homogeneous kernels (9),

$$n(y) = \int_0^y \frac{1}{y} h(\frac{x}{y}) dx = \int_0^1 h(z) dz$$

and the condition (29) amounts to

$$\int_0^1 h(z) dz < 1.$$

More generally, in the case (11), the condition (29) holds if $\zeta_\infty^+ < 1$ and

$$\limsup_{y\to 0} \widehat{n}(y) < \frac{\zeta_\infty^-}{\left(\zeta_\infty^+\right)^2} \text{ and } \lim\sup_{y\to+\infty} \frac{\widehat{n}(y)}{y} < \frac{\left(1-\zeta_\infty^+\right)\zeta_\infty^-}{\left(\zeta_\infty^+\right)^2}$$

where $\widehat{n}(y) = \int_0^y \widehat{b}(x,y)dx$ and

$$\zeta_\infty^- = \lim\inf_{a(y)\to+\infty} \zeta(x,v) \text{ and } \zeta_\infty^+ = \lim\sup_{a(y)\to+\infty} \zeta(x,v), \tag{30}$$

(see Proposition 41).

Under Assumption (28),

$$T + B : D(T) \subset X_{0,1} \to X_{0,1} \text{ is resolvent compact}$$

provided that $T : D(T) \to X_{0,1}$ is; (see Corollary 43). As previously, we deduce that $(V(t))_{t\geq 0}$ has a spectral gap (16) and exhibits the asynchronous exponential growth (17) in $X_{0,1}$ provided that the support of $a(.)$ is not bounded; (see Theorem 45). We conjecture that in $X_{0,1}$, the results hold under the natural assumption $\sup_{z>1} \frac{r(z)}{z} < +\infty$ instead of (24), (see Remark 46).

Let us describe now very briefly the situation under Assumption (22); (see Section 3 for the different statements). First of all, we cannot expect a generation theory in X_1 under (22), (see Remark 51). A growth C_0-semigroup $(U(t))_{t\geq 0}$ governing (13) with boundary condition

$$\lim_{x\to 0} r(x)u(x,t) = 0 \tag{31}$$

exists in the space $X_{0,1}$ and is given by

$$U(t)f = \chi_{\left\{\int_0^y \frac{1}{r(\tau)}d\tau > t\right\}} e^{-\int_{X(y,t)}^y \frac{\beta(p)}{r(p)}dp} f(X(y,t))\frac{\partial X(y,t)}{\partial y} \tag{32}$$

($X(y,t)$ is defined by $\int_{X(y,t)}^y \frac{1}{r(\tau)}d\tau = t$ for $\int_0^y \frac{1}{r(\tau)}d\tau > t$) provided that (24) is satisfied. This sufficient condition for a generation theory in $X_{0,1}$ under (22) is "partly necessary", (see Remark 50). The mathematical analysis is the same as in the previous case (21) in $X_{0,1}$. The only different result is that the resolvent compactness of T holds once the sublevel sets of β

$$\Omega_c = \{x > 0; \beta(x) < c\} \quad (c > 0)$$

are "thin near *infinity* relatively to r" in the sense

$$\int_1^{+\infty} \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \ \ (c > 0),$$

e.g. if $\lim_{y\to+\infty}\beta(y) = +\infty$; (no condition at $y = 0$ is needed). In particular, $(V(t))_{t\geq 0}$ has a spectral gap and exhibits the asynchronous exponential growth (17) in $X_{0,1}$ provided that the support of $a(.)$ is not bounded; (see Theorem 64).

Section 4 is devoted to the "finite agregates number" space

$$X_0 = L^1(\mathbb{R}_+;\ dx)$$

under Assumption (22); (a similar construction could also be done under Assumption (21)). For simplicity, we restrict ourselves to

$$d(.) = 0.$$

A growth C_0-semigroup $(U(t))_{t\geq 0}$ governing (13) with boundary condition (31) exists in the space X_0 and is given by (32) but without any further condition on the growth rate $r(.)$. As in X_1 or $X_{0,1}$, its generator T satisfies a smoothing effect and the pointwise estimate and its resolvent is compact if the sublevel sets of $a(.)$ are "thin near *infinity* relatively to r", e.g. if

$$\lim_{y\to+\infty} a(y) = +\infty;$$

(see Theorem 67). We show also that the fragmentation operator (14) is T-bounded in X_0 and

$$\lim_{\lambda\to+\infty} \left\| B(\lambda-T)^{-1} \right\|_{\mathcal{L}(X_0)} \leq \lim \sup_{a(y)\to+\infty} n(y)$$

provided that

$$n(.) := \int_0^y b(x,.)dx \in L^\infty(0,+\infty).$$

In particular, by W. Desch's perturbation theorem again,

$$T + B : D(T) \subset X_0 \to X_0$$

generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ on X_0 provided that

$$\lim \sup_{a(y)\to+\infty} n(y) < 1; \tag{33}$$

(see Theorem 68). Note that (33) *cannot* hold for conservative fragmentation kernels, hence the necessity of the mass loss condition. In particular, for homogeneous kernels (9), it amounts to $\int_0^1 h(z)dz < 1$. More generally, in the case (11), the condition (33) is satisfied if

$$\lim \sup_{a(y)\to+\infty} \widehat{n}(y) < \frac{1}{\zeta_\infty^+},$$

(see Remark 70).

We show that $(V(t))_{t\geq 0}$ has a spectral gap and exhibits the asynchronous exponential growth (17) in X_0 provided that the support of $a(.)$ is not bounded; (see Theorem 71). If $a(.)$ is unbounded at zero or at infinity only, then (33) expresses a smallness condition on $n(.)$ at zero or at infinity.

As far as we know, our results are new and appear here for the first time. The role of mass loss or death assumptions appears in the spaces $X = X_1, X_0$ or $X_{0,1}$ at two key places : In the proof that

$$T + B : D(T) \to X \tag{34}$$

is a generator via W. Desch's perturbation theorem *and* (consequently) in the fact that the resolvent compactness of T implies the resolvent compactness of $T + B$. A priori, we can overcome the first point. Indeed, if we consider for instance the space X_1, by adapting honesty theory (see e.g. [27]), without mass loss or death assumptions (i.e. $\eta(.) = d(.) = 0$), (34) need *not* be a generator but there exists a unique *extension*

$$T_B \supset T + B$$

of (34) which generates a positive C_0-semigroup $(V(t))_{t\geq 0}$ in X_1. Unfortunately, even in the honest case (i.e. $T_B = \overline{T+B}$), if $T+B$ is not closed, a priori we cannot infer that T_B is resolvent compact when T is. This is the main obstruction to build a general theory of asynchronous exponential growth in X_1 without mass loss or death conditions. The same observation can also be made for the other spaces.

In [8], under Assumption (22), where no mass loss or death condition is assumed and where $T_B = \overline{T+B}$ (and a priori $T_B \neq T + B$), the asynchronous exponential growth is *not* obtained in the natural space $X_{0,1}$ but in higher moment spaces (18) ([8] Theorem 1.2). We note that a general construction similar to the present one holds without mass loss or death assumptions provided the growth-fragmentation equations are considered in higher moment spaces

$$L^1(\mathbb{R}_+;\ (1+x)^{\alpha}\, dx),\ L^1(\mathbb{R}_+;\ x^{\alpha} dx) \quad (\alpha > \alpha_*)$$

for a suitable threshold $\alpha_* \geq 1$ depending on the functional space [32]; (this is due to the fact that W. Desch's perturbation theorem can apply in higher moment spaces without resorting to mass loss or death assumptions [6]). This strongly suggests the conjecture that we cannot expect the asynchronous exponential growth (17) in X_1, X_0 or $X_{0,1}$ without mass loss or death assumptions.

2 The first construction

We deal first with the case

$$\int_0^1 \frac{1}{r(\tau)} d\tau = +\infty \text{ and } \int_1^{\infty} \frac{1}{r(\tau)} d\tau = +\infty \tag{35}$$

and start with:

Proposition 2 *Let (35) be satisfied. The partial differential equation*

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x}\left[r(x)u(x,t)\right] = 0,\ (x, t > 0)$$

with initial condition $u(x,0) = f(x)$ has a unique solution given by

$$u(y,t) = \frac{r(X(y,t))f(X(y,t))}{r(y)}$$

where $X(y,t) > 0$ is defined by

$$\int_{X(y,t)}^{y} \frac{1}{r(\tau)} d\tau = t \quad (t > 0). \tag{36}$$

Proof. We solve

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} [r(x)u(x,t)] = 0$$

with intial data $u(x,0) = f(x)$ by the method of characteristics. Making the change

$$r(x)u(x,t) = \varphi(x,t)$$

this amounts to solving

$$\frac{1}{r(x)} \frac{\partial}{\partial t} \varphi(x,t) + \frac{\partial}{\partial x} [\varphi(x,t)] = 0; \quad \varphi(x,0) = r(x)f(x).$$

We introduce the characteristic equations

$$\frac{dt}{ds} = \frac{1}{r(x(s))}, \ \frac{dx}{ds} = 1$$

with "initial" conditions

$$x(0) = x, \ t(0) = 0 \quad (x > 0)$$

i.e. $x(s) = s + x$ and

$$t(s) = \int_0^s \frac{1}{r(\tau + x)} d\tau = \int_x^{s+x} \frac{1}{r(\tau)} d\tau \ (s \geqslant 0).$$

Thus

$$[0, +\infty) \ni s \to r(s+x)u(s+x, \int_x^{s+x} \frac{1}{r(\tau)} d\tau) \text{ is constant}$$

and then

$$r(s+x)u(s+x, \int_x^{s+x} \frac{1}{r(\tau)} d\tau) = r(x)u(x,0) = r(x)f(x) \ \ \forall s \geqslant 0.$$

For $t>0$ and $y>0$ given, we set

$$\int_x^{s+x} \frac{1}{r(\tau)} d\tau = t,\ s+x=y$$

i.e. $\int_x^y \frac{1}{r(\tau)} d\tau = t$. Since $x \in [0,y] \to \int_x^y \frac{1}{r(\tau)} d\tau$ is (a continuous function) strictly decreasing from $+\infty$ to 0, then we denote by $X(y,t)$ the *unique* $x \in (0,y)$ such that $\int_x^y \frac{1}{r(\tau)} d\tau = t$. Thus $r(y)u(y,t) = r(X(y,t))f(X(y,t))$ which ends the proof. ■

2.1 Theory in the space X_1

Within assumption (35), we first develop a general theory on well-posedness and spectral analysis in the "finite mass" space

$$X_1 := L^1(\mathbb{R}_+;\ xdx).$$

2.1.1 An unperturbed semigroup $(U_0(t))_{t\geqslant 0}$

We put now the solution given in Proposition 2 in the functional space X_1. Let

$$\left(U_0(t)f\right)(y) := \frac{r(X(y,t))f(X(y,t))}{r(y)}$$

where $X(y,t)$ $(t>0)$ is defined by (36).

Theorem 3 *Let (35) be satisfied. Then*

$$\left(U_0(t)f\right)(y) = f(X(y,t))\frac{\partial X(y,t)}{\partial y}$$

and $(U_0(t))_{t\geqslant 0}$ *is a positive* C_0*-semigroup on the space* X_1 *if and only if* $\sup_{x>0} \frac{y(x,t)}{x} < +\infty \ \ \forall t \geqslant 0$ *and*

$$[0,+\infty) \ni t \to \sup_{x>0} \frac{y(x,t)}{x} \quad \textit{is locally bounded} \tag{37}$$

where $y(x,t)$ *is defined by*

$$\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t \quad (t>0). \tag{38}$$

If

$$\alpha := \sup_{z>0} \frac{r(z)}{z} < +\infty \tag{39}$$

then (37) is satisfied; more precisely $\frac{y(x,t)}{x} \leq e^{\alpha t}$.

Proof. Let us check that $U_0(t)$ is a bounded operator on X_1. Note that (36) shows that (for $t>0$ fixed) $X(y,t)$ is strictly increasing in y and tends to 0 as $y \to 0$ (because $X(y,t) \leq y$). Note that

$$(0,+\infty) \ni y \to X(y,t) \in (0,+\infty)$$

is continuous. Since (for $t>0$ fixed)

$$U(y,z) := \int_z^y \frac{1}{r(\tau)} d\tau - t$$

is of class C^1 in (y,z) with

$$\frac{\partial U(y,z)}{\partial z} = -\frac{1}{r(z)} \neq 0$$

then the implicit function theorem shows that $X(y,t)$ is a C^1 function in $y \in (0,+\infty)$ so that differentiating (36) in $y \in (0,+\infty)$

$$\frac{1}{r(y)} - \frac{1}{r(X(y,t))} \frac{\partial X(y,t)}{\partial y} = 0$$

i.e.

$$\frac{\partial X(y,t)}{\partial y} = \frac{r(X(y,t))}{r(y)}$$

and

$$(U_0(t)f)(y) = f(X(y,t)) \frac{\partial X(y,t)}{\partial y}; \quad y \in (0,+\infty).$$

Thus

$$\|u(.,t)\|_{X_1} = \int_0^{+\infty} |u(y,t)|\, y dy = \int_0^{+\infty} |f(X(y,t))| \frac{\partial X(y,t)}{\partial y} y dy.$$

Note that $\int_{X(y,t)}^y \frac{1}{r(\tau)} d\tau = t$ shows that $\lim_{y \to +\infty} X(y,t) = +\infty$. The change of variable $x = X(y,t)$ gives

$$\|u(.,t)\|_{X_1} = \int_0^{+\infty} |f(x)|\, y(x,t) dx$$

where $y(x,t)$ is the unique $y > x$ such that $x = X(y,t)$, see (38). Thus

$$\|u(.,t)\|_{X_1} = \int_0^{+\infty} \frac{y(x,t)}{x} |f(x)|\, x dx$$

and $f(.) \to u(.,t)$ defines a bounded linear operator on X_1 if and only if $\sup_{x>0} \frac{y(x,t)}{x} < +\infty$. Hence

$$U_0(t) : X_1 \ni f \to \frac{r(X(y,t))f(X(y,t))}{r(y)} \in X_1$$

is bounded with

$$\|U_0(t)\|_{\mathcal{L}(X_1)} = \sup_{x>0} \frac{y(x,t)}{x}$$

and then $[0,+\infty) \ni t \to U_0(t) \in \mathcal{L}(X_1)$ is locally bounded if and only if

$$[0,+\infty) \ni t \to \sup_{x>0} \frac{y(x,t)}{x}$$

is. It follows (see e.g. [12]) that $(U_0(t))_{t\geqslant 0}$ is exponentially bounded. In this case, to show that $(U_0(t))_{t\geqslant 0}$ is strongly continuous on X_1 it suffices to check that

$$U_0(t)f \to f \text{ in } L^1(\mathbb{R}_+;\ xdx) \text{ as } t \to 0$$

on a *dense* subspace of $L^1(\mathbb{R}_+;\ xdx)$, e.g. for f continuous with compact support in $(0,+\infty)$. Note that (36) shows that $X(y,t) \to y$ as $t \to 0$ uniformly on compact sets of $(0,+\infty)$. Let the support of f be included in a set $\left[c, c^{-1}\right]$ with $0 < c < 1$. Since

$$X(\frac{c}{2},t) \to \frac{c}{2} \text{ and } X(2c^{-1},t) \to 2c^{-1} \text{ as } t \to 0$$

there exists $t_c > 0$ such that

$$X(\frac{c}{2},t) < c, \quad X(2c^{-1},t) > c^{-1}$$

once $t < t_c$ and consequently, for $t < t_c$,

$$X(y,t) < c\ \forall y \leq \frac{c}{2},\ X(y,t) > c^{-1}\ \forall y > 2c^{-1}$$

because $y \to X(y,t)$ is strictly increasing. Hence

$$U_0(t)f = 0 \text{ on } (0,\frac{c}{2}) \cup (2c^{-1},+\infty), \quad t < t_c.$$

Since $X(y,t) \to y$ as $t \to 0$ uniformly on $\left[\frac{c}{2}, 2c^{-1}\right]$, $r(X(y,t))f(X(y,t))$ is uniformly bounded in $y \in \left[\frac{c}{2}, 2c^{-1}\right]$ as $t \to 0$ and $\frac{1}{r}$ is integrable on $\left[\frac{c}{2}, 2c^{-1}\right]$ so

$$\frac{r(X(y,t))f(X(y,t))}{r(y)} \to f \quad \text{in } L^1(\left[\frac{c}{2}, 2c^{-1}\right],\ dx)$$

as $t \to 0$ by the dominated convergence theorem. In particular $U_0(t)f \to f$ in $L^1(\mathbb{R}_+;\ xdx)$ as $t \to 0$.

It follows from (39) that

$$r(z) \leq \alpha z \quad \forall z > 0. \tag{40}$$

We differentiate (38) in t to obtain

$$\frac{\partial y(x,t)}{\partial t} = r(y(x,t)) \quad \forall t > 0 \tag{41}$$

so

$$y(x,t) = x + \int_0^t r(y(x,s))ds \leq x + \int_0^t \alpha y(x,s)ds.$$

Gronwall's lemma gives $y(x,t) \leq xe^{\alpha t}$ so $\|u(.,t)\|_{X_1} \leq e^{\alpha t} \int_0^{+\infty} |f(x)|\, xdx$ and finally $\|U_0(t)\|_{\mathcal{L}(X_1)} \leq e^{\alpha t}$. ■

Summarising:

Corollary 4 *Let (35)(39) be satisfied. Let $X(y,t)$ be defined by (36). Then*

$$\left(U_0(t)f\right)(y) := \frac{r(X(y,t))f(X(y,t))}{r(y)} = f(X(y,t))\frac{\partial X(y,t)}{\partial y}$$

defines a C_0-semigroup $(U_0(t))_{t\geqslant 0}$ on X_1 such that $\|U_0(t)\|_{\mathcal{L}(X_1)} \leq e^{\alpha t}$ where $\alpha = \sup_{z>0} \frac{r(z)}{z}$.

Remark 5 *Note that the fact that $x < y(x,t)$ (for $t > 0$) implies that $\sup_x \left(x^{-1}y(x,t)\right) > 1$ and then $\|U_0(t)\|_{\mathcal{L}(X_1)} > 1$ for $t > 0$, i.e. $(U_0(t))_{t\geqslant 0}$ is not contractive.*

We strongly suspect that the sufficient condition (39) for (37) is actually *necessary*. Indeed, we have:

Proposition 6 *Let (35) be satisfied. If*

$$\lim_{z\to 0} \frac{r(z)}{z} = +\infty \ \text{or} \lim_{z\to +\infty} \frac{r(z)}{z} = +\infty$$

then $\sup_{x>0} \frac{y(x,t)}{x} = +\infty$. where $y(x,t)$ is defined by (38). In particular, the generation theory in X_1 fails.

Proof. We have $\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t$ so the change of variable $\frac{\tau}{x} = s$ gives

$$\int_1^{\frac{y(x,t)}{x}} \frac{xs}{r(xs)} \frac{1}{s} ds = t \quad (t > 0).$$

Arguing by contradiction, suppose that $C := \sup_{x>0} \frac{y(x,t)}{x} < +\infty$. Then

$$0 < t \leq \int_1^C \frac{xs}{r(xs)} \frac{1}{s} ds \quad (x > 0). \tag{42}$$

If $\lim_{z \to +\infty} \frac{r(z)}{z} = +\infty$ then $\lim_{z \to +\infty} \frac{z}{r(z)} = 0$ and consequently, for any sequence $(x_n)_n$ such that $x_n \to +\infty$ there exists a positive constant $\widehat{C}$ such that

$$\frac{x_n s}{r(x_n s)} \frac{1}{s} \leq \frac{\widehat{C}}{s} \quad (s \in (1, C))$$

for n large enough. Since $\frac{x_n s}{r(x_n s)} \frac{1}{s} \to 0 \quad (n \to \infty) \ (s \in (1, C))$ then the dominated convergence theorem implies

$$\int_1^C \frac{xs}{r(xs)} \frac{1}{s} ds \to 0$$

which contradicts (42). We argue similarly if $\lim_{z \to 0} \frac{r(z)}{z} = +\infty$. ■

2.1.2 On the generator of $(U_0(t))_{t \geqslant 0}$

We identify now the resolvent of the generator.

Proposition 7 *Let (35)(39) be satisfied. Let T_0 be the generator of $(U_0(t))_{t \geqslant 0}$ in X_1. Then*

$$\left((\lambda - T_0)^{-1} f\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda}{r(s)} ds} f(x) dx, \quad \operatorname{Re} \lambda > s(T_0)$$

where $s(T_0)$ is the spectral bound of T_0

Proof. We recall that the spectral bound of T_0 is nothing but the type of $(U_0(t))_{t \geqslant 0}$, see e.g. [33]. Note first that

$$\left((\lambda - T_0)^{-1} f\right)(y) = \int_0^{+\infty} e^{-\lambda t} \frac{r(X(y,t)) f(X(y,t))}{r(y)} dt \quad (\operatorname{Re} \lambda > s(T_0)).$$

Note that (36) shows that $t \in (0, +\infty) \to x := X(y,t)$ is strictly decreasing from y to 0. Differentiating (36) in t we get

$$-\frac{1}{r(X(y,t))}\frac{\partial X(y,t)}{\partial t} = 1$$

so the change of variable $x = X(y,t)$ gives

$$dx = -r(X(y,t))dt$$

and

$$\int_0^{+\infty} e^{-\lambda t}\frac{r(X(y,t))f(X(y,t))}{r(y)}dt = \frac{1}{r(y)}\int_0^y e^{-\lambda X_{-1}(y,x)}f(x)dx$$

where $X_{-1}(y,x)$ is the inverse of $t \to x = X(y,t)$. Observe that this inverse is nothing but

$$x \to t = -\int_y^x \frac{1}{r(\tau)}d\tau$$

so

$$\left((\lambda - T_0)^{-1}f\right)(y) = \frac{1}{r(y)}\int_0^y e^{-\lambda\int_x^y \frac{1}{r(\tau)}d\tau}f(x)dx$$

and this ends the proof. ■

Remark 8 *It is possible to characterize* T_0

$$D(T_0) = \left\{f \in X_1;\ \frac{\partial\,(rf)}{\partial y} \in X_1\right\},\quad T_0 = -\frac{\partial\,(rf)}{\partial y}$$

where $\frac{\partial(rf)}{\partial y}$ *is the derivative (in the sense of distributions on* $(0,+\infty)$*) of the locally integrable function* rf *on* $(0,+\infty)$*; see [5].*

We characterize the spectral bound of T_0.

Proposition 9 *We assume that (35)(39) are satisfied. The spectral bound of* T_0 *(or equivalently the type of* $(U_0(t))_{t\geqslant 0}$*) is given by*

$$s(T_0) = \lim_{t\to+\infty}\frac{1}{t}\ln\left[\sup_x\left(x^{-1}y(x,t)\right)\right] = \inf_{t>0}\frac{1}{t}\ln\left[\sup_x\left(x^{-1}y(x,t)\right)\right] \geqslant 0$$

where $y(x,t)$ *is defined by (38).*

Proof. In the proof of Theorem 3, $\|U_0(t)\|_{\mathcal{L}(X_1)} = \sup_{x>0} \frac{y(x,t)}{x}$ and the type of $(U_0(t))_{t\geqslant 0}$ is given by $\lim_{t\to+\infty} \frac{1}{t} \ln \|U_0(t)\|_{\mathcal{L}(X_1)}$. ■

We give now a key *pointwise* estimate in X_1.

Lemma 10 *Let (35)(39) be satisfied. Let $\lambda > \alpha$. Then*

$$\left|(\lambda - T_0)^{-1} f\right|(y) \leq \frac{1}{yr(y)} \|f\|_{X_1} \quad (f \in X_1).$$

Proof. Since $r(x) \leq \alpha x$ then

$$\begin{aligned}
\left|(\lambda - T_0)^{-1} f\right| &\leq \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{\alpha\tau} d\tau} |f(x)|\, dx \\
&= \frac{1}{r(y)} \int_0^y \frac{1}{x} e^{-\lambda \int_x^y \frac{1}{\alpha\tau} d\tau} |f(x)|\, x dx.
\end{aligned}$$

On the other hand, if $\lambda > \alpha$ then $\frac{\lambda}{\alpha} - 1 > 0$ and

$$\begin{aligned}
\frac{1}{x} e^{-\frac{\lambda}{\alpha} \int_x^y \frac{1}{\tau} d\tau} &= \frac{1}{x} e^{-\frac{\lambda}{\alpha} \ln(\frac{y}{x})} = \frac{1}{x} e^{\ln(\frac{x}{y})^{\frac{\lambda}{\alpha}}} = \frac{1}{x} \frac{x^{\frac{\lambda}{\alpha}}}{y^{\frac{\lambda}{\alpha}}} \\
&= \frac{x^{\frac{\lambda}{\alpha}-1}}{y^{\frac{\lambda}{\alpha}}} \leq \frac{y^{\frac{\lambda}{\alpha}-1}}{y^{\frac{\lambda}{\alpha}}} = \frac{1}{y} \quad (\forall x \leq y)
\end{aligned}$$

so that

$$\left|(\lambda - T_0)^{-1} f\right| \leq \frac{1}{yr(y)} \int_0^y |f(x)|\, x dx \leq \frac{1}{yr(y)} \|f\|_{X_1}.$$

This ends the proof. ■

2.1.3 A first perturbed semigroup $(U(t))_{t\geqslant 0}$

Arguing as previously, we solve

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} \left[r(x) u(x,t)\right] + \beta(x) u(x,t) = 0, \; u(x,0) = f(x)$$

by the method of characteristics and get

$$u(y,t) = \frac{e^{-\int_{X(y,t)}^y \frac{\beta(p)}{r(p)} dp} r(X(y,t)) f(X(y,t))}{r(y)}$$

where $X(y,t)$ is defined by (36).

Proposition 11 *Let (35)(39) be satisfied. Let $X(y,t)$ be defined by (36). Then*

$$\begin{aligned} U(t)f \quad &: \quad = e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)} dp} \frac{r(X(y,t))f(X(y,t))}{r(y)} \\ &= \quad e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)} dp} f(X(y,t)) \frac{\partial X(y,t)}{\partial y} = e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)} dp} U_0(t)f \end{aligned}$$

defines a positive C_0-semigroup $(U(t))_{t\geqslant 0}$ on X_1.

Proposition 12 *Let (35)(39) be satisfied. Let T be the generator of $(U(t))_{t\geqslant 0}$. Then*

$$\left((\lambda - T)^{-1} f\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda+\beta(\tau)}{r(\tau)} d\tau} f(x)dx.$$

for $\operatorname{Re}\lambda > s(T)$, where $s(T)$ is the spectral bound of T.

Proof. We note that for $\operatorname{Re}\lambda > s(T)$

$$\left((\lambda - T)^{-1} f\right)(y) = \int_0^{+\infty} e^{-\lambda t} e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)} dp} \frac{r(X(y,t))f(X(y,t))}{r(y)} dt$$

where $X(y,t)$ is defined by (36). Arguing as in the proof of Proposition 7, the change of variable $x = X(y,t)$ gives

$$\left((\lambda - T)^{-1} f\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} f(x)dx = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda+\beta(\tau)}{r(\tau)} d\tau} f(x)dx$$

and ends the proof. ■

We study the spectral bound of T.

Proposition 13 *Let (35)(39) be satisfied. The spectral bound of T is given by*

$$s(T) = \lim_{t\to\infty} \sup_{x>0} \left[\left(-t^{-1} \int_x^{y(x,t)} \frac{\beta(p)}{r(p)} dp \right) + t^{-1} \left(\ln \frac{y(x,t)}{x} \right) \right].$$

In particular

$$s(T) \leq -\lim_{t\to\infty} \inf_{x>0} t^{-1} \int_x^{y(x,t)} \frac{\beta(p)}{r(p)} dp + s(T_0).$$

If $\widehat{\alpha} := \inf \frac{p\beta(p)}{r(p)} > 0$ (e.g. if $\inf \beta > 0$) then

$$s(T) \leq (1 - \widehat{\alpha})\, s(T_0).$$

Proof. We have

$$\|U(t)f\|_{X_1} = \int_0^{+\infty} e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)}dp} f(X(y,t)) \frac{\partial X(y,t)}{\partial y} y dy.$$

The change of variable $x = X(y,t)$ (i.e. $\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t$) gives

$$\|U(t)f\|_{X_1} = \int_0^{+\infty} e^{-\int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp} f(x) y(x,t) dx = \int_0^{+\infty} e^{-\int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp} \frac{y(x,t)}{x} f(x) x dx$$

and

$$\|U(t)\|_{\mathcal{L}(X_1)} = \sup_{x>0} \left(e^{-\int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp} \frac{y(x,t)}{x} \right).$$

Hence

$$\begin{aligned}
\ln\left(\|U(t)\|_{\mathcal{L}(X_1)}\right) &= \ln\left(\sup_{x>0}\left(e^{-\int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp} \frac{y(x,t)}{x}\right)\right) = \sup_{x>0} \ln\left(\left(e^{-\int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp} \frac{y(x,t)}{x}\right)\right) \\
&= \sup_{x>0}\left[\left(-\int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp\right) + \left(\ln \frac{y(x,t)}{x}\right)\right] \\
&\leq \sup_{x>0}\left(-\int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp\right) + \sup_{x>0}\left(\ln \frac{y(x,t)}{x}\right) \\
&= -\inf_{x>0} \int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp + \sup_{x>0}\left(\ln \frac{y(x,t)}{x}\right).
\end{aligned}$$

This ends the first claim.

Note that if $\inf \beta > 0$ then $\widehat{\alpha} := \inf \frac{p\beta(p)}{r(p)} > 0$ since $\sup \frac{r(p)}{p} < +\infty$. Thus

$$\int_x^{y(x,t)} \frac{\beta(p)}{r(p)}dp \geq \widehat{\alpha} \int_x^{y(x,t)} \frac{1}{p}dp = \widehat{\alpha} \ln \frac{y(x,t)}{x}$$

and

$$\begin{aligned}
\ln\left(\|U(t)\|_{\mathcal{L}(X_1)}\right) &\leq \sup_{x>0}\left(-\widehat{\alpha} \ln \frac{y(x,t)}{x} + \ln \frac{y(x,t)}{x}\right) \\
&= (1-\widehat{\alpha}) \sup_{x>0}\left(\ln \frac{y(x,t)}{x}\right) \\
&= (1-\widehat{\alpha}) \ln\left(\sup_{x>0} \frac{y(x,t)}{x}\right) = (1-\widehat{\alpha}) \ln\left(\|U_0(t)\|_{\mathcal{L}(X_1)}\right)
\end{aligned}$$

implies

$$s(T) = \lim_{t\to+\infty} t^{-1} \ln\left(\|U(t)\|_{\mathcal{L}(X_1)}\right) \leq (1-\widehat{\alpha}) \lim_{t\to+\infty} t^{-1} \ln\left(\|U_0(t)\|_{\mathcal{L}(X_1)}\right)$$

and ends the proof. ■

2.1.4 A smoothing effect of the perturbed resolvent

We give now another key estimate in X_1.

Lemma 14 *Let (35)(39) be satisfied. Let* $\lambda > \alpha$. *Then*

$$\int_0^{+\infty} \left| \left((\lambda - T)^{-1} f \right)(y) \right| \beta(y) y dy \leq \int_0^{+\infty} \left| (f(y) \right| y dy, \ \left(f \in X_1 \right).$$

Proof. We note first that

$$e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} \leq e^{-\frac{\lambda}{\alpha} \int_x^y \frac{1}{\tau} d\tau} = e^{-\frac{\lambda}{\alpha} \ln(\frac{y}{x})} = (\frac{x}{y})^{\frac{\lambda}{\alpha}}$$

so

$$\begin{aligned}
& \int_0^{+\infty} \left| \left((\lambda - T)^{-1} f \right)(y) \right| \beta(y) y dy \\
\leq & \int_0^{+\infty} \frac{\beta(y) y}{r(y)} \left(\int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \left| f(x) \right| dx \right) dy \\
= & \int_0^{+\infty} \left(\frac{1}{x} \int_x^{+\infty} (\frac{x}{y})^{\frac{\lambda}{\alpha}} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \frac{\beta(y) y}{r(y)} dy \right) \left| f(x) \right| x dx \\
= & \int_0^{+\infty} \left(\int_x^{+\infty} (\frac{x}{y})^{\frac{\lambda}{\alpha} - 1} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \frac{\beta(y)}{r(y)} dy \right) \left| f(x) \right| x dx \\
\leq & \int_0^{+\infty} \left(\int_x^{+\infty} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \frac{\beta(y)}{r(y)} dy \right) \left| f(x) \right| x dx.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_x^{+\infty} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \frac{\beta(y)}{r(y)} dy &= -\int_x^{+\infty} \frac{d}{dy} \left(e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \right) dy \\
&= -\left[e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \right]_{y=x}^{y=+\infty} \leq 1
\end{aligned}$$

whence

$$\int_0^{+\infty} \left| \left((\lambda - T)^{-1} f \right)(y) \right| \beta(y) y dy \leq \int_0^{+\infty} \left| f(x) \right| x dx$$

and we are done. ■

Remark 15 *One can deduce from Lemma 14 that*

$$D(T) = \left\{ f \in D(T_0);\ \beta f \in X_1 \right\}, \quad T f = T_0 f - \beta f.$$

2.1.5 On the full perturbed semigroup $(V(t))_{t\geqslant 0}$

We give now a second perturbation theorem in X_1.

Theorem 16 *Let (35)(39) be satisfied. Then the fragmentation operator (14) is T-bounded in X_1 and*

$$\lim_{\lambda\to+\infty}\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_1)}\leq \lim\sup_{a(y)\to+\infty}\frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}}.$$

In particular,

$$T+B:D(T)\subset X_1\to X_1$$

generates a positive semigroup $(V(t))_{t\geqslant 0}$ in X_1 if

$$\lim\sup_{a(y)\to+\infty}\frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}}<1. \tag{43}$$

Proof. We observe that

$$\begin{aligned}
\|B\varphi\|_{X_1} &\leq \int_0^{+\infty}\left(\int_x^{+\infty}a(y)b(x,y)\left|\varphi(y)\right|dy\right)xdx\\
&= \int_0^{+\infty}a(y)\left(\int_0^{y}xb(x,y)dx\right)\left|\varphi(y)\right|dy\\
&= \int_0^{+\infty}a(y)\left(y-\eta(y)y\right)\left|\varphi(y)\right|dy\\
&= \int_0^{+\infty}a(y)\left(1-\eta(y)\right)\left|\varphi(y)\right|ydy
\end{aligned}$$

so

$$\begin{aligned}
&\left\|B(\lambda-T)^{-1}f\right\|_{X_1}\\
=&\int_0^{+\infty}a(y)\left(1-\eta(y)\right)\left((\lambda-T)^{-1}\left|f\right|\right)ydy\\
=&\int_{\{a\leq c\}}a(y)\left(1-\eta(y)\right)\left((\lambda-T)^{-1}\left|f\right|\right)ydy\\
&+\int_{\{a>c\}}a(y)\left(1-\eta(y)\right)\left((\lambda-T)^{-1}\left|f\right|\right)ydy.
\end{aligned}$$

We have

$$\int_{\{a\leq c\}}a(y)\left(1-\eta(y)\right)\left((\lambda-T)^{-1}\left|f\right|\right)ydy\leq c\left\|(\lambda-T)^{-1}f\right\|_{X_1}$$

and

$$
\begin{aligned}
& \int_{\{a>c\}} a(y)\,(1-\eta(y))\left((\lambda-T)^{-1}|f|\right) ydy \\
= & \int_{\{a>c\}} \frac{a(y)\,(1-\eta(y))}{a(y)+d(y)}\,(a(y)+d(y))\left((\lambda-T)^{-1}|f|\right) ydy \\
\leq & \sup_{\{a>c\}} \frac{a(y)\,(1-\eta(y))}{a(y)+d(y)} \int_0^{+\infty} (a(y)+d(y))\left((\lambda-T)^{-1}|f|\right) ydy \\
= & \sup_{\{a>c\}} \frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}} \int_0^{+\infty} (a(y)+d(y))\left((\lambda-T)^{-1}|f|\right) ydy.
\end{aligned}
$$

On the other hand, according to Lemma 14, for $\lambda>\alpha$

$$
\int_0^{+\infty} \left|\left((\lambda-T)^{-1}f\right)(x)\right| (a(x)+d(x))\,xdx \leq \int_0^{+\infty} |f(x)|\,xdx
$$

whence

$$
\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_1)} \leq c\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_1)} + \sup_{\{a>c\}} \frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}}
$$

and

$$
\lim_{\lambda\to+\infty} \left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_1)} \leq \sup_{\{a>c\}} \frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}} \quad (\forall c>0).
$$

Fnally

$$
\lim_{\lambda\to+\infty} \left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_1)} \leq \lim \sup_{a(y)\to+\infty} \frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}}
$$

and this ends the proof by invoking W. Desch's theorem (i.e. Theorem 1) since T generates a positive semigroup $(U(t))_{t\geqslant 0}$. ■

Remark 17 *The well posedness via W. Desch's theorem depends on the existence of an amount of mass loss or death in the system. For instance (43) is satisfied if*

$$
d=0 \text{ and } \lim \inf_{a(y)\to+\infty} \eta(y)>0 \text{ or } \eta=0 \text{ and } \lim \inf_{a(y)\to+\infty} \frac{d(y)}{a(y)}>0.
$$

Remark 18 *For homogeneous kernels (9),* $\lim\inf_{a(y)\to+\infty}\eta(y)>0$ *amounts to* $\int_0^1 zh(z)dz<1$*. More generally, in the case (11),*

$$
\lim \inf_{a(y)\to+\infty} \eta(y) \geq 1-\zeta_\infty^+
$$

where

$$\zeta_\infty^+ = \lim \sup_{a(y)\to+\infty} \zeta(x,y) := \lim_{m\to+\infty} \sup_{\{(x,v);\ a(y)\geq m\}} \zeta(x,y)$$

and $\liminf_{a(y)\to+\infty} \eta(y) > 0$ *holds if* $\zeta_\infty^+ < 1$.

2.1.6 Compactness results in X_1

Let

$$\Omega_c = \{x > 0; \beta(x) < c\} \quad (c > 0)$$

be the sublevel sets of β.

Definition 19 *If* $\int_0^1 \frac{1}{r(\tau)} d\tau = +\infty$ *we say that the sublevel sets of* β *are thin near zero relatively to* r *(thin near zero for short) if*

$$\int_0^1 \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \ \ (c > 0)$$

where 1_{Ω_c} *is the indicator function of* Ω_c. *In particular, if* $\lim_{y\to 0} \beta(y) = +\infty$ *then the sublevel sets of* β *are automatically thin near zero.*

Similarly, if $\int_1^\infty \frac{1}{r(\tau)} d\tau = +\infty$ *we say that the sublevel sets of* β *are thin near infinity relatively to* r *(thin near infinity for short) if*

$$\int_1^\infty \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \ \ (c > 0).$$

In particular, if $\lim_{y\to+\infty} \beta(y) = +\infty$ *then the sublevel sets of* β *are automatically thin near infinity.*

The confining role of *singular* absorption potentials in compactness properties of (perturbed) positive contraction semigroups in abstract L^1 spaces has been systematically analyzed in [28]. Note that growth-semigroups are not contractive and we provide here a direct analysis adapted to them.

Theorem 20 *Let (35)(39) be satisfied. If*

$$\text{the sublevel sets of } \beta \text{ are thin near zero and infinity} \tag{44}$$

then T *is resolvent compact on* X_1.

Proof. Let $\lambda > \alpha$ and f in the unit ball of X_1, i.e.

$$\int_0^{+\infty} |f(x)|\, x dx \leq 1.$$

According to Lemma 14

$$\int_0^{+\infty} \left| \left((\lambda - T)^{-1} f \right)(x) \right| \beta(x) x dx \leq 1.$$

Let $c > 0$ and $\varepsilon > 0$ be arbitrary. We have

$$\begin{aligned} 1 &\geq \int_0^{\varepsilon} \left| \left((\lambda - T)^{-1} f \right)(x) \right| \beta(x) x dx = \int_0^{\varepsilon} 1_{\{\beta < c\}} \left| \left((\lambda - T)^{-1} f \right)(x) \right| \beta(x) x dx \\ &\quad + \int_0^{\varepsilon} 1_{\{\beta \geq c\}} \left| \left((\lambda - T)^{-1} f \right)(x) \right| \beta(x) x dx \end{aligned}$$

so

$$\sup_{\|f\|_{X_1} \leq 1} \int_0^{\varepsilon} 1_{\{\beta \geq c\}} \left| \left((\lambda - T)^{-1} f \right)(x) \right| x dx \leq \frac{1}{c}.$$

On the other hand, according to Lemma 10,

$$\left| \left((\lambda - T)^{-1} f \right)(x) \right| \leq \frac{1}{x r(x)} \quad (x > 0)$$

uniformly in $\|f\|_{X_1} \leq 1$ so

$$\int_0^{\varepsilon} 1_{\{\beta < c\}} \left| \left((\lambda - T)^{-1} f \right)(x) \right| \beta(x) x dx \leq c \int_0^{\varepsilon} 1_{\{\beta < c\}} \frac{1}{x r(x)} x dx = c \int_0^{\varepsilon} 1_{\{\beta < c\}} \frac{1}{r(x)} dx$$

and

$$\sup_{\|f\|_{X_1} \leq 1} \int_0^{\varepsilon} \left| \left((\lambda - T)^{-1} f \right)(x) \right| x dx \leq \frac{1}{c} + c \int_0^{\varepsilon} 1_{\{\beta < c\}} \frac{1}{r(x)} dx$$

can be made arbitrarily small by choosing first c large enough and then ε small enough.

Similarly,

$$\begin{aligned} 1 &\geq \int_{\varepsilon^{-1}}^{+\infty} \left| \left((\lambda - T)^{-1} f \right)(x) \right| \beta(x) x dx = \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta < c\}} \left| \left((\lambda - T)^{-1} f \right)(x) \right| \beta(x) x dx \\ &\quad + \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta \geq c\}} \left| \left((\lambda - T)^{-1} f \right)(x) \right| \beta(x) x dx \end{aligned}$$

so

$$\sup_{\|f\|_{X_1}\leq 1}\int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta\geq c\}}\left|\left((\lambda-T)^{-1}f\right)(x)\right| xdx \leq \frac{1}{c}.$$

We have also

$$\int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta<c\}}\left|\left((\lambda-T)^{-1}f\right)(x)\right| \beta(x)xdx \leq c\int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta<c\}}\frac{1}{xr(x)}xdx = c\int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta<c\}}\frac{1}{r(x)}dx$$

and

$$\sup_{\|f\|_{X_1}\leq 1}\int_0^{\varepsilon}\left|\left((\lambda-T)^{-1}f\right)(x)\right| xdx \leq \frac{1}{c}+c\int_0^{\varepsilon} 1_{\{\beta<c\}}\frac{1}{r(x)}dx$$

can be made arbitrarily small by choosing first c large enough and then ε small enough. Finally, the uniform estimate

$$\left|\left((\lambda-T)^{-1}f\right)(x)\right| \leq \frac{1}{xr(x)} \quad (x>0) \quad (\|f\|_{X_1}\leq 1)$$

gives a uniform domination by $\frac{1_{(\varepsilon,\varepsilon^{-1})}}{xr(x)} \in X_1$

$$1_{(\varepsilon,\varepsilon^{-1})}\left|\left((\lambda-T)^{-1}f\right)(x)\right| \leq \frac{1_{(\varepsilon,\varepsilon^{-1})}}{xr(x)} \qquad (\|f\|_{X_1}\leq 1)$$

so

$$\left\{1_{(\varepsilon,\varepsilon^{-1})}\left|\left((\lambda-T)^{-1}f\right)\right|;\ \|f\|_{X_1}\leq 1\right\} \text{ is relatively weakly compact.}$$

Finally, $\left\{(\lambda-T)^{-1}f;\ \|f\|_{X_1}\leq 1\right\}$ is as close to a relatively weakly compact set as we want and consequently is weakly compact. Hence $(\lambda-T)^{-1}$ is weakly compact operator and consequently (see [28] Lemma 14) $(\lambda-T)^{-1}$ is compact. ■

We can state:

Corollary 21 *Let (35)(39)(44)(43) be satisfied. Then*

$$T+B : D(T)\subset X_1 \to X_1$$

is resolvent compact in X_1.

Proof. Theorem 16 implies $\sum_{j=0}^{+\infty}(B(\lambda-T)^{-1})^j \in \mathcal{L}(X_1)$ and

$$(\lambda-T-B)^{-1} = (\lambda-T)^{-1}\sum_{j=0}^{+\infty}(B(\lambda-T)^{-1})^j$$

so Theorem 20 ends the proof. ■

2.1.7 Spectral gap of the full semigroup $(V(t))_{t\geqslant 0}$ in X_1

We start with:

Lemma 22 *Let (35)(39) be satisfied. We assume that the support of $a(.)$ is not bounded. Then $(\lambda - T - B)^{-1}$ is positivity improving, i.e.*

$$(\lambda - T - B)^{-1} f > 0 \text{ a.e.}$$

for any nontrivial nonnegative $f \in X_1$, or equivalently $(V(t))_{t\geqslant 0}$ is irreducible in X_1.

Proof. Note that

$$\begin{aligned}(\lambda - T - B)^{-1} f &= (\lambda - T)^{-1} \sum_{n=0}^{+\infty} \left[B(\lambda - T)^{-1}\right]^n f \\ &\geqslant (\lambda - T)^{-1} \sum_{n=1}^{+\infty} \left[B(\lambda - T)^{-1}\right]^n f\end{aligned}$$

and

$$\left((\lambda - T)^{-1}\varphi\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \varphi(x) dx.$$

Note also that

$$\begin{aligned}&B(\lambda - T)^{-1} f \\ = &\int_0^{+\infty} 1_{\{x<y\}} a(y) b(x,y) \left[\frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} f(x) dx\right] dy \\ = &\int_0^{+\infty} \left[\int_x^{+\infty} \frac{1}{r(y)} 1_{\{x<y\}} a(y) b(x,y) e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} dy\right] f(x) dx \\ = &\int_0^{+\infty} \left[\int_x^{+\infty} \frac{1}{r(y)} a(y) b(x,y) e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} dy\right] f(x) dx\end{aligned}$$

and

$$\int_x^{+\infty} \frac{1}{r(y)} a(y) b(x,y) e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} dy > 0 \ \forall x > 0$$

so that $B(\lambda - T)^{-1} f > 0$ a.e. for any nontrivial nonnegative f. It follows that

$$\sum_{n=1}^{+\infty} \left[B(\lambda - T)^{-1}\right]^n f \ > 0 \text{ a.e.}$$

and consequently

$$(\lambda - T)^{-1} \sum_{n=1}^{+\infty} \left[B(\lambda - T)^{-1}\right]^n f > 0 \text{ a.e.}$$

for any nontrivial nonnegative f. This ends the proof. ■

We are now ready to show the main result of Subsection 2.1.

Theorem 23 *We assume that (35)(39)(43)(44) are satisfied and that the support of* $a(.)$ *is not bounded. Then* $(V(t))_{t \geqslant 0}$ *has a spectral gap in* X_1*, i.e.*

$$r_{ess}(V(t)) < r_\sigma(V(t)),$$

and satisfies the asynchronous exponential growth.

Proof. Let

$$k(x,y) := 1_{\{x<y\}} a(y) b(x,y)$$

be the kernel of B. Let

$$\overline{k}(x,y) := k(x,y) \wedge 1$$

and

$$\overline{k}_c(x,y) := \overline{k}(x,y) p(x) p(y)$$

where $p \in C(0,+\infty)$ has a compact support in $(0,+\infty)$ and $0 \leq p(x) \leq 1$. Note that $k(x,y) \geqslant \overline{k}_c(x,y)$ and

$$\begin{aligned} k(x,y) &= \big(k(x,y) - \overline{k}_c(x,y)\big) + \overline{k}_c(x,y) \\ &= \widehat{k}(x,y) + \overline{k}_c(x,y) \end{aligned}$$

where $\widehat{k}(x,y) := k(x,y) - \overline{k}_c(x,y)$. Let $\overline{B}$ be the integral operator with kernel $\overline{k}_c(x,y)$ and $\widehat{B}$ be the integral operator with kernel $\widehat{k}(x,y)$. Since $\widehat{k}(x,y) \leq k(x,y)$ then

$$\left\| \widehat{B}(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_1)} \leq \left\| B(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_1)} < 1$$

for λ large enough so $T + \widehat{B} : D(T) \to X_1$ generates a positive semigroup $(\widehat{V}(t))_{t \geqslant 0}$. Note that $(V(t))_{t \geqslant 0}$ is generated by

$$\left(T + \widehat{B}\right) + \overline{B}$$

where $\overline{B}$ is a *bounded* operator on X_1. Actually the kernel of $\overline{B}$ is compactly supported in $(0,+\infty)\times(0,+\infty)$ and bounded and consequently $\overline{B}$ is a *weakly compact* operator on X_1. On the other hand

$$V(t) = \widehat{V}(t) + \int_0^t \widehat{V}(t-s)\overline{B}\widehat{V}(s)ds$$

and, by the convex (weak) compactness property of the strong operator topology (see [35] or [24]), the *strong* integral (*not* a Bochner integral)

$$\int_0^t \widehat{V}(t-s)\overline{B}\widehat{V}(s)ds$$

defined just *strongly*, i.e. by

$$X_1 \ni \varphi \to \int_0^t \widehat{V}(t-s)\overline{B}\widehat{V}(s)\varphi ds \in X_1,$$

is a weakly compact operator. It follows that $\widehat{V}(t)$ and $V(t)$ have the same essential spectrum [17] and therefore

$$r_{ess}(\widehat{V}(t)) = r_{ess}(V(t)) \ (t > 0) \tag{45}$$

or, equivalently, the identity of their essential types

$$\omega_{ess}(\widehat{V}) = \omega_{ess}(V).$$

On the other hand $\widehat{V}(t) \leq V(t)$

$$(\lambda - T - \widehat{B})^{-1} \leq (\lambda - T - B)^{-1}$$

and

$$(\lambda - T - \widehat{B})^{-1} \neq (\lambda - T - B)^{-1}$$

because $\overline{B} \neq 0$. Since, by Lemma 22, $(\lambda - T - B)^{-1}$ is positivity improving (and thus irreducible) and compact (by Corollary 21) then

$$r_\sigma\left[(\lambda - T - \widehat{B})^{-1}\right] < r_\sigma\left[(\lambda - T - B)^{-1}\right]$$

see [19]. Moreover

$$r_\sigma\left[(\lambda - T - \widehat{B})^{-1}\right] = \frac{1}{\lambda - s(T+\widehat{B})} \text{ and } r_\sigma\left[(\lambda - T - B)^{-1}\right] = \frac{1}{\lambda - s(T+B)}$$

(see e.g. [33]) whence $s(T+\widehat{B}) < s(T+B)$. This implies in particular that

$$s(T+B) > -\infty.$$

Note that the type of a positive semigroup on L^1 coincides with the spectral bound of its generator so that

$$r_\sigma(\widehat{V}(t)) = e^{s(T+\widehat{B})t} < e^{s(T+B)t} = r_\sigma(V(t)).$$

Since $r_{ess}(\widehat{V}(t)) \leq r_\sigma(\widehat{V}(t))$ then (45) gives $r_{ess}(V(t)) < r_\sigma(V(t))$ i.e. $(V(t))_{t\geqslant 0}$ has a spectral gap. Finally, the asynchronous exponential growth follows from the irreducibility of $(V(t))_{t\geqslant 0}$. ■

Remark 24 *Note that if the sublevel sets of $a(.)$ are thin at infinity then the support of $a(.)$ is not bounded.*

2.2 Theory in the space $X_{0,1}$

Within assumption (35), we develop now a general theory on well-posedness and spectral analysis in the "finite mass and number of agregates" space

$$X_{0,1} := L^1\left(\mathbb{R}_+,\ (1+x)dx\right).$$

2.2.1 First generation result in $X_{0,1}$

We put now the solution given in Proposition 2 in the functional space $X_{0,1}$.

Theorem 25 *Let (35) be satisfied and $X(y,t)$ be given by (36). Then*

$$\left(U_0(t)f\right)(y) := \frac{r(X(y,t))f(X(y,t))}{r(y)} = f(X(y,t))\frac{\partial X(y,t)}{\partial y}$$

defines a stongly contiuous semigroup $(U_0(t))_{t\geqslant 0}$ on $X_{0,1}$ if and only if

$$\sup_{x>0}\frac{1+y(x,t)}{1+x} < +\infty \quad \forall t \geqslant 0 \tag{46}$$

and

$$[0,+\infty) \ni t \to \sup_{x>0}\frac{1+y(x,t)}{1+x} \quad \textit{is locally bounded} \tag{47}$$

where $y(x,0)=x$ and $y(x,t)$ is defined for $t>0$ by (38). If

$$\alpha_1 := \sup_{z>1}\frac{r(z)}{z} < +\infty \tag{48}$$

then (47) is satisfied.

Proof. We have

$$\|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} |u(y,t)|\,(1+y)\,dy = \int_0^{+\infty} |f(X(y,t))|\,\frac{\partial X(y,t)}{\partial y}\,(1+y)\,dy.$$

By the change of variable $x = X(y,t)$, we have

$$\begin{aligned}\|U_0(t)f\|_{X_{0,1}} &= \int_0^{+\infty} |f(x)|\,(1+y(x,t))\,dx \\ &= \int_0^{+\infty} \frac{1+y(x,t)}{1+x}\,|f(x)|\,(1+x)dx\end{aligned}$$

so $U_0(t)$ is a bounded operator on $X_{0,1}$ if and only if (46) holds; in this case,

$$\|U_0(t)\|_{\mathcal{L}(X_{0,1})} = \sup_{x>0} \frac{1+y(x,t)}{1+x}. \tag{49}$$

This shows the first claim. Note that under (35), $\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t$ implies that $\lim_{x\to 0} y(x,t) = 0$ *uniformly* in t bounded so

$$\lim_{x\to 0} \frac{1+y(x,t)}{1+x} = 1 \quad (t \geqslant 0)$$

uniformly in t bounded. In particular (46) holds if and only if

$$\sup_{x>1} \frac{1+y(x,t)}{1+x} < +\infty \quad (t \geqslant 0)$$

and

$$[0,+\infty) \ni t \to \sup_{x>1} \frac{1+y(x,t)}{1+x} \quad \text{is locally bounded}$$

or equivalently if

$$\sup_{x>1} \frac{y(x,t)}{x} < +\infty \quad (t \geqslant 0)$$

and

$$[0,+\infty) \ni t \to \sup_{x>1} \frac{y(x,t)}{x} \quad \text{is locally bounded.}$$

It follows from (41) that

$$y(x,t) = x + \int_0^t r(y(x,s))ds.$$

Since $y(x,t) > x$ then (48) gives

$$r(y(x,s)) \leq \alpha_1 y(x,s) \quad (x \geq 1)$$

so

$$y(x,t) \ \leq x + \int_0^t \alpha_1 y(x,s) ds, \quad (x \geq 1)$$

and Gronwall's lemma gives $\sup_{x>1} \frac{y(x,t)}{x} \leq e^{\alpha_1 t}$. The strong continuity at the origin can be dealt with as in the space X_1. ■

Remark 26 *By arguing as in Proposition 6 one can check that if*

$$\lim_{z \to +\infty} \frac{r(z)}{z} = +\infty$$

then $\sup_{x>1} \frac{y(x,t)}{x} = +\infty$ *and consequently the generation theory in* $X_{0,1}$ *fails.*

Remark 27 *We observe that in contrast to the* X_1*-generation theory, we need no assumption on the growth rate function at the origin. The fact that* $y(x,t) > x$ *and (49) show that* $(U_0(t))_{t \geqslant 0}$ *is not contractive in* $X_{0,1}$*.*

2.2.2 On the generator of $(U_0(t))_{t \geqslant 0}$ in $X_{0,1}$

As in X_1, the resolvent of the generator T_0 in $X_{0,1}$ is characterized by:

Proposition 28 *Let (35)(48) be satisfied. Let* T_0 *be the generator of* $(U_0(t))_{t \geqslant 0}$ *in* $X_{0,1}$*. Then*

$$\left((\lambda - T_0)^{-1} f\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda}{r(s)} ds} f(x) dx, \quad \operatorname{Re} \lambda > s(T_0)$$

where $s(T_0)$ *is the spectral bound of* T_0*. Moreover,*

$$D(T_0) = \left\{ f \in X_{0,1};\ \frac{\partial (rf)}{\partial y} \in X_{0,1} \right\}, \quad T_0 = -\frac{\partial (rf)}{\partial y}$$

where $\frac{\partial (rf)}{\partial y}$ *is the derivative (in the sense of distributions on* $(0,+\infty)$*) of the locally integrable function* rf *on* $(0,+\infty)$*.*

By using (49) we get:

Proposition 29 *Let (48) be satisfied. The spectral bound of* T_0 *(or equivalently the type of* $(U_0(t))_{t\geqslant 0}$*) in* $X_{0,1}$ *is given by*

$$\widehat{s}(T_0) = \lim_{t\to+\infty} \frac{1}{t} \ln\left[\sup_{x>0} \frac{1+y(x,t)}{1+x}\right] = \inf_{t>0} \frac{1}{t} \ln\left[\sup_{x>0} \frac{1+y(x,t)}{1+x}\right] \geqslant 0$$

where $y(x,t)$ *is defined by (38).*

We can recover the previous pointwise estimate but under an assumption stronger than (48).

Lemma 30 *Let (35) be satisfied. If* $C := \sup_{z>0} \frac{r(z)}{1+z} < +\infty$ *then*

$$\left|(\lambda - T_0)^{-1} f\right|(y) \leq \frac{1}{(1+y)\,r(y)} \left\|f\right\|_{X_{0,1}}, \quad (f \in X_{0,1}), \quad (\lambda > C).$$

Proof. Since $r(z) \leq C(z+1)$ $(\forall z > 0)$ then

$$\frac{1}{r(z)} \geq \frac{C^{-1}}{z+1}$$

and

$$e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} \leq e^{-\frac{\lambda}{C} \int_x^y \frac{1}{\tau+1} d\tau} = e^{-\frac{\lambda}{C} \ln(\frac{y+1}{x+1})} = (\frac{x+1}{y+1})^{\frac{\lambda}{C}}. \tag{50}$$

It follows that

$$\begin{aligned}
\left|(\lambda - T_0)^{-1} f(y)\right| &\leq \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} \left|f(x)\right| dx \\
&\leq \frac{1}{r(y)} \int_0^y (\frac{x+1}{y+1})^{\frac{\lambda}{C}} \left|f(x)\right| dx \\
&= \frac{1}{(1+y)\,r(y)} \int_0^y (\frac{x+1}{y+1})^{\frac{\lambda}{C}-1} \left|f(x)\right| (1+x)\, dx \\
&\leq \frac{1}{(1+y)\,r(y)} \int_0^{+\infty} \left|f(x)\right| (1+x)\, dx
\end{aligned}$$

because $\frac{x+1}{y+1} \leq 1$ and $\frac{\lambda}{C} - 1 > 0$. This ends the proof. ■

Remark 31 *(Open question) We suspect that a similar statement should hold under the general assumption (48).*

2.2.3 A first perturbed semigroup in $X_{0,1}$

As previously in X_1 we have:

Proposition 32 *Let (35)(48) be satisfied. Let $X(y,t)$ be defined by (36). Then*

$$\begin{aligned} U(t)f \quad &: = e^{-\int_{X(y,t)}^{y}\frac{\beta(p)}{r(p)}dp}\frac{r(X(y,t))f(X(y,t))}{r(y)} \\ &= e^{-\int_{X(y,t)}^{y}\frac{\beta(p)}{r(p)}dp}f(X(y,t))\frac{\partial X(y,t)}{\partial y} = e^{-\int_{X(y,t)}^{y}\frac{\beta(p)}{r(p)}dp}U_0(t)f \end{aligned}$$

defines a positive C_0-semigroup $(U(t))_{t\geqslant 0}$ on $X_{0,1}$.

and

Proposition 33 *Let (35)(48) be satisfied. Let T be the generator of $(U(t))_{t\geqslant 0}$ in $X_{0,1}$. Then*

$$\begin{aligned} \left((\lambda - T)^{-1}f\right)(y) &= \frac{1}{r(y)}\int_0^y e^{-\lambda\int_x^y \frac{1}{r(\tau)}d\tau}e^{-\int_x^y \frac{\beta(\tau)}{r(\tau)}d\tau}f(x)dx \\ &= \frac{1}{r(y)}\int_0^y e^{-\int_x^y \frac{\lambda+\beta(\tau)}{r(\tau)}d\tau}f(x)dx. \end{aligned}$$

for $\operatorname{Re}\lambda > s(T)$, where $s(T)$ is the spectral bound of T.

Arguing as in Proposition 13 we get:

Proposition 34 *Let (35)(48) be satisfied. The spectral bound of T in $X_{0,1}$ is given by*

$$\widehat{s}(T) = \lim_{t\to\infty}\sup_{x>0}\left[\left(-t^{-1}\int_x^{y(x,t)}\frac{\beta(p)}{r(p)}dp\right) + t^{-1}\left(\ln\frac{1+y(x,t)}{1+x}\right)\right].$$

In particular

$$\widehat{s}(T) \leq -\lim_{t\to\infty}\inf_{x>0} t^{-1}\int_x^{y(x,t)}\frac{\beta(p)}{r(p)}dp + \widehat{s}(T_0).$$

If $\widehat{\alpha} := \inf\frac{(1+p)\beta(p)}{r(p)} > 0$ (e.g. if $\inf\beta > 0$) then

$$\widehat{s}(T) \leq (1-\widehat{\alpha})\,\widehat{s}(T_0).$$

2.2.4 A smoothing effect of the perturbed resolvent in $X_{0,1}$

As in X_1, we show now a smoothing effect in $X_{0,1}$ but we have to replace the natural assumption (48) by a stronger one.

Lemma 35 *Let (35) be satisfied. If* $C := \sup_{z>0} \frac{r(z)}{1+z} < +\infty$ *then, for* $\lambda > C$,

$$\int_0^{+\infty} \left| \left((\lambda - T)^{-1} f \right)(y) \right| \beta(y) (1+y)\, dy \leq \int_0^{+\infty} |(f(y)| (1+y)\, dy,\ (f \in X_{0,1}).$$

Proof. By using (50) we have for $\lambda > C$

$$\begin{aligned}
& \int_0^{+\infty} \left| \left((\lambda - T)^{-1} f \right)(y) \right| \beta(y) (1+y)\, dy \\
\leq\ & \int_0^{+\infty} \frac{\beta(y)(1+y)}{r(y)} \left(\int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} |f(x)|\, dx \right) dy \\
=\ & \int_0^{+\infty} \left(\int_x^{+\infty} \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \frac{\beta(y)(1+y)}{r(y)} dy \right) |f(x)|\, dx \\
=\ & \int_0^{+\infty} \left(\int_x^{+\infty} \left(\frac{x+1}{y+1}\right)^{\frac{\lambda}{C}-1} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \frac{\beta(y)}{r(y)} dy \right) |f(x)| (1+x)\, dx \\
\leq\ & \int_0^{+\infty} \left(\int_x^{+\infty} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \frac{\beta(y)}{r(y)} dy \right) |f(x)| (1+x)\, dx
\end{aligned}$$

where we have used in the last step that $\frac{x}{y} \leq 1$ and $\frac{\lambda}{C} - 1 > 0$. We already know that

$$\int_x^{+\infty} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \frac{\beta(y)}{r(y)} dy = -\int_x^{+\infty} \frac{d}{dy} \left(e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \right) dy = -\left[e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \right]_{y=x}^{y=+\infty} \leq 1$$

whence

$$\int_0^{+\infty} \left| \left((\lambda - T)^{-1} f \right)(y) \right| \beta(y) (1+y)\, dy \leq \int_0^{+\infty} |f(x)| (1+x)\, dx$$

and we are done. ■

Remark 36 *(Open question) We suspect that a similar smoothing effect should hold under (48).*

Remark 37 *One can deduce from Lemma 35 that*

$$D(T) = \{ f \in D(T_0);\ \beta f \in X_{0,1} \},\quad Tf = T_0 f - \beta f.$$

2.2.5 The full perturbed semigroup in $X_{0,1}$

We give now a second perturbation theorem in $X_{0,1}$.

Theorem 38 *Let (35)(48) be satisfied. We assume that*

$$n(y) := \int_0^y b(x,y)dx$$

is such that $\widehat{C} := \sup_{y>0} \frac{n(y)}{1+y} < +\infty$. *Then the fragmentation operator (14) is* T*-bounded in* $X_{0,1}$ *and*

$$\lim_{\lambda\to+\infty} \left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,1})} \leq \lim \sup_{a(y)\to+\infty} \frac{[(y-\eta(y)y)+n(y)]}{(1+y)\left(1+\frac{d(y)}{a(y)}\right)}.$$

In particular, if

$$\lim \sup_{a(y)\to+\infty} \frac{[(y-\eta(y)y)+n(y)]}{(1+y)\left(1+\frac{d(y)}{a(y)}\right)} < 1 \tag{51}$$

then

$$T+B : D(T) \subset X_{0,1} \to X_{0,1}$$

generates a positive semigroup $(V(t))_{t\geqslant 0}$ *in* $X_{0,1}$.

Proof. We have

$$\begin{aligned}
\|B\varphi\|_{X_{0,1}} &\leq \int_0^{+\infty}\left(\int_x^{+\infty} a(y)b(x,y)\left|\varphi(y)\right|dy\right)(1+x)\,dx \\
&= \int_0^{+\infty} a(y)\left(\int_0^y (1+x)\,b(x,y)dx\right)\left|\varphi(y)\right|dy \\
&= \int_0^{+\infty} a(y)\left[(y-\eta(y)y)+n(y)\right]\left|\varphi(y)\right|dy \\
&= \int_0^{+\infty} \frac{a(y)\left[(y-\eta(y)y)+n(y)\right]}{1+y}\left|\varphi(y)\right|(1+y)\,dy
\end{aligned}$$

so

$$\begin{aligned}
&\left\|B(\lambda-T)^{-1}f\right\|_{X_{0,1}} \\
= &\int_{\{a\leq c\}} \frac{a(y)\left[(y-\eta(y)y)+n(y)\right]}{1+y}\left|\left((\lambda-T)^{-1}f\right)\right|(1+y)\,dy \\
&+\int_{\{a>c\}} \frac{a(y)\left[(y-\eta(y)y)+n(y)\right]}{1+y}\left|\left((\lambda-T)^{-1}f\right)\right|(1+y)\,dy.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\{a\le c\}} \frac{a(y)\left[(y-\eta(y)y)+n(y)\right]}{1+y}\left|\left((\lambda-T)^{-1}f\right)\right|(1+y)\,dy \\
\le\ & c\left(1+\widehat{C}\right)\left\|(\lambda-T)^{-1}f\right\|_{X_{0,1}}
\end{aligned}$$

while

$$\begin{aligned}
&\int_{\{a>c\}} \frac{a(y)\left[(y-\eta(y)y)+n(y)\right]}{1+y}\left|\left((\lambda-T)^{-1}f\right)\right|(1+y)\,dy \\
=\ & \int_{\{a>c\}} \frac{a(y)\left[(y-\eta(y)y)+n(y)\right]}{(1+y)\left(a(y)+d(y)\right)}\left(a(y)+d(y)\right)\left((\lambda-T)^{-1}|f|\right)(1+y)\,dy \\
\le\ & \sup_{\{a>c\}} \frac{a(y)\left[(y-\eta(y)y)+n(y)\right]}{(1+y)\left(a(y)+d(y)\right)}\int_0^{+\infty}\left(a(y)+d(y)\right)\left((\lambda-T)^{-1}|f|\right)(1+y)\,dy \\
\le\ & \sup_{\{a>c\}} \frac{a(y)\left[(y-\eta(y)y)+n(y)\right]}{(1+y)\left(a(y)+d(y)\right)}\int_0^{+\infty}|f(y)|\,(1+y)\,dy
\end{aligned}$$

(Lemma 35 is used in the last step) so

$$\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,1})}\le c\left(1+\widehat{C}\right)\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,1})}+\sup_{\{a>c\}}\frac{\left[(y-\eta(y)y)+n(y)\right]}{(1+y)\left(1+\frac{d(y)}{a(y)}\right)}$$

for arbitrary $c>0$ and consequently

$$\lim_{\lambda\to+\infty}\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_{0,1})}\le\lim\sup_{a(y)\to+\infty}\frac{\left[(y-\eta(y)y)+n(y)\right]}{(1+y)\left(1+\frac{d(y)}{a(y)}\right)}$$

which ends the proof by invoking W. Desch's theorem (i.e. Theorem 1). ■

Let us check Assumption (51).

Corollary 39 *We assume that $a(.)\in L^{\infty}_{loc}(0,+\infty)$ and*

$$\gamma:=\lim\sup_{a(y)\to+\infty}\frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}}<1.$$

Then Assumption (51) is satisfied in the following cases:

(i) $a(.)$ is unbounded at zero and at infinity and

$$\max\left\{\lim\sup_{y\to 0}\frac{n(y)}{(1-\eta(y))},\ 1+\lim\sup_{y\to+\infty}\frac{n(y)}{y\,(1-\eta(y))}\right\}<\gamma^{-1}. \tag{52}$$

(ii) $a(.)$ is unbounded at zero only and

$$\limsup_{y\to 0} \frac{n(y)}{(1-\eta(y))} < \gamma^{-1}.$$

(iii) $a(.)$ is unbounded at infinity only and

$$1 + \limsup_{y\to +\infty} \frac{n(y)}{y\,(1-\eta(y))} < \gamma^{-1}.$$

Proof. Note that

$$\frac{[(y-\eta(y)y)+n(y)]}{(1+y)\left(1+\frac{d(y)}{a(y)}\right)} = \frac{y+\frac{n(y)}{(1-\eta(y))}}{(y+1)} \frac{(1-\eta(y))}{1+\frac{d(y)}{a(y)}}$$

so that (51) is satisfied if

$$\lim \sup_{a(y)\to +\infty} \frac{y+\frac{n(y)}{(1-\eta(y))}}{(y+1)} < \gamma^{-1}.$$

This ends the proof since

$$\limsup_{y\to 0} \frac{y+\frac{n(y)}{(1-\eta(y))}}{(y+1)} = \limsup_{y\to 0} \frac{n(y)}{(1-\eta(y))}$$

and

$$\lim \sup_{y\to +\infty} \frac{y+\frac{n(y)}{(1-\eta(y))}}{(y+1)} = 1 + \lim \sup_{y\to +\infty} \frac{n(y)}{y\,(1-\eta(y))}.$$

■

Remark 40 *As noted in the Introduction, for homogeneous fragmentation kernels (9), the above conditions are satisfied if $\int_0^1 h(z)dz < 1$.*

Let us give more general examples.

Proposition 41 *Let the fragmentation kernel be given by (11). Let $a(.)$ be unbounded at zero and at infinity only. Then (52) holds if $\zeta_\infty^+ < 1$ and*

$$\limsup_{y\to 0} \widehat{n}(y) < \frac{\zeta_\infty^-}{\left(\zeta_\infty^+\right)^2} \text{ and } \lim \sup_{y\to +\infty} \frac{\widehat{n}(y)}{y} < \frac{\left(1-\zeta_\infty^+\right)\zeta_\infty^-}{\left(\zeta_\infty^+\right)^2}.$$

where $\widehat{n}(y) = \int_0^y \widehat{b}(x,y)dx$ and ζ_∞^-, ζ_∞^+ are given by (30).

Proof. Note that

$$1-\eta(y)=\frac{1}{y}\int_0^y x\zeta(x,y)\widehat{b}(x,y)dx$$

so

$$1-\lim\inf_{a(y)\to+\infty}\eta(y)\leq\lim\sup_{a(y)\to+\infty}\frac{1}{y}\int_0^y x\zeta(x,y)\widehat{b}(x,y)dx\leq\zeta_\infty^+$$

and

$$1-\lim\sup_{a(y)\to+\infty}\eta(y)\geq\lim\inf_{a(y)\to+\infty}\frac{1}{y}\int_0^y x\zeta(x,y)\widehat{b}(x,y)dx\geq\zeta_\infty^-.$$

Since

$$\left(1-\lim\inf_{a(y)\to+\infty}\eta(y)\right)^{-1}\geq\left(\zeta_\infty^+\right)^{-1}$$

and

$$\begin{aligned}&\max\left\{\limsup_{y\to0}\frac{n(y)}{(1-\eta(y))},\ 1+\lim\sup_{y\to+\infty}\frac{n(y)}{y\,(1-\eta(y))}\right\}\\ \leq\ &\max\left\{\limsup_{y\to0}\frac{\zeta_\infty^+\widehat{n}(y)}{\zeta_\infty^-},\ 1+\lim\sup_{y\to+\infty}\frac{\zeta_\infty^+\widehat{n}(y)}{y\zeta_\infty^-}\right\}\end{aligned}$$

then (52) holds if

$$\max\left\{\frac{\zeta_\infty^+}{\zeta_\infty^-}\limsup_{y\to0}\widehat{n}(y),\ 1+\frac{\zeta_\infty^+}{\zeta_\infty^-}\lim\sup_{y\to+\infty}\frac{\widehat{n}(y)}{y}\right\}<\frac{1}{\zeta_\infty^+}$$

which is equivalent to $\zeta_\infty^+<1$ and

$$\limsup_{y\to0}\widehat{n}(y)<\frac{\zeta_\infty^-}{\left(\zeta_\infty^+\right)^2}\text{ and }\lim\sup_{y\to+\infty}\frac{\widehat{n}(y)}{y}<\frac{\left(1-\zeta_\infty^+\right)\zeta_\infty^-}{\left(\zeta_\infty^+\right)^2}.$$

■

2.2.6 Compactness results in $X_{0,1}$

By replacing the natural assumption (48) by a stronger one, we can show:

Theorem 42 *Let (35)(44) be satisfied. If* $C:=\sup_{z>0}\frac{r(z)}{1+z}<+\infty$ *then* T *is resolvent compact on* $X_{0,1}$.

Proof. Let $\lambda > C$ and f be in the unit ball of $X_{0,1}$, i.e.

$$\int_0^{+\infty} |f(x)|\,(1+x)\,dx \leq 1.$$

According to Lemma 35

$$\int_0^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(x)\right| \beta(x)\,(1+x)\,dx \leq 1.$$

Let $c > 0$ and $\varepsilon > 0$ be arbitrary. We have

$$\begin{aligned} 1 \quad \geq \quad & \int_{\varepsilon^{-1}}^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(x)\right| \beta(x)\,(1+x)\,dx = \\ & \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta<c\}} \left|\left((\lambda - T)^{-1} f\right)(x)\right| \beta(x)\,(1+x)\,dx \\ & + \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta\geq c\}} \left|\left((\lambda - T)^{-1} f\right)(x)\right| \beta(x)\,(1+x)\,dx \end{aligned}$$

so

$$\sup_{\|f\|_{X_{0,1}}\leq 1} \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta\geq c\}} \left|\left((\lambda - T)^{-1} f\right)(x)\right| (1+x)\,dx \leq \frac{1}{c}.$$

On the other hand, according to Lemma 30,

$$\left|(\lambda - T)^{-1} f\right| \leq \left|(\lambda - T_0)^{-1} f\right| \leq \frac{1}{(1+x)\,r(x)}\,\|f\|_{X_{0,1}}$$

so

$$\int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta<c\}} \left|\left((\lambda - T)^{-1} f\right)(x)\right| \beta(x)\,(1+x)\,dx \leq c \int_{\varepsilon^{-1}}^{+\infty} \frac{1_{\{\beta<c\}}}{r(x)}dx$$

and then

$$\sup_{\|f\|_{X_{0,1}}\leq 1} \int_{\varepsilon^{-1}}^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(x)\right| (1+x)\,dx \leq \frac{1}{c} + c \int_{\varepsilon^{-1}}^{+\infty} \frac{1_{\{\beta<c\}}}{r(x)}dx$$

can be made arbitrarily small by choosing first c large enough and then ε small enough.

Similarly, we have

$$\begin{aligned} 1 \quad \geq \quad & \int_0^{\varepsilon} \left|\left((\lambda - T)^{-1} f\right)(x)\right| \beta(x)\,(1+x)\,dx \\ = \quad & \int_0^{\varepsilon} 1_{\{\beta<c\}} \left|\left((\lambda - T)^{-1} f\right)(x)\right| \beta(x)\,(1+x)\,dx \\ & + \int_0^{\varepsilon} 1_{\{\beta\geq c\}} \left|\left((\lambda - T)^{-1} f\right)(x)\right| \beta(x)\,(1+x)\,dx \end{aligned}$$

so

$$\sup_{\|f\|_{X_{0,1}}\leq 1}\int_0^{\varepsilon} 1_{\{\beta\geq c\}}\left|\left((\lambda-T)^{-1}f\right)(x)\right|(1+x)\,dx\leq \frac{1}{c}.$$

As previously

$$\int_0^{\varepsilon} 1_{\{\beta<c\}}\left|\left((\lambda-T)^{-1}f\right)(x)\right|\beta(x)\,(1+x)\,dx\leq c\int_0^{\varepsilon}\frac{1_{\{\beta<c\}}}{r(x)}dx$$

so

$$\sup_{\|f\|_{X_{0,1}}\leq 1}\int_0^{\varepsilon}\left|\left((\lambda-T)^{-1}f\right)(x)\right|(1+x)\,dx\leq \frac{1}{c}+c\int_0^{\varepsilon}\frac{1_{\{\beta<c\}}}{r(x)}dx$$

can be made arbitrarily small by choosing first c large enough and then ε small enough.

On $\left(\varepsilon,\varepsilon^{-1}\right)$ we have the uniform domination

$$\left|(\lambda-T)^{-1}f\right|\leq\left|(\lambda-T_0)^{-1}f\right|\leq\frac{1_{(\varepsilon,\varepsilon^{-1})}(x)}{(1+x)\,r(x)}\in X_{0,1}\quad (\|f\|_{X_{0,1}}\leq 1)$$

so the restriction of the set $\left\{\left|(\lambda-T)^{-1}f\right|,\ \|f\|_{X_{0,1}}\leq 1\right\}$ to the set $\left(\varepsilon,\varepsilon^{-1}\right)$ is is relatively weakly compact on $X_{0,1}$. Finally

$$\left\{(\lambda-T)^{-1}f;\ \|f\|_{X_{0,1}}\leq 1\right\}$$

is as close to a relatively weakly compact set as we want and consequently is weakly compact. This shows that $(\lambda-T)^{-1}$ is weakly compact operator and consequently (see [28] Lemma 14) $(\lambda-T)^{-1}$ is compact. ■

As in Corollary 21 we have:

Corollary 43 *Let (35)(51)(44) be satisfied. If* $C:=\sup_{z>0}\frac{r(z)}{1+z}<+\infty$ *then*

$$T+B:D(T)\subset X_{0,1}\to X_{0,1}$$

is resolvent compact.

2.2.7 Spectral gap of the full semigroup $(V(t))_{t\geqslant 0}$ in $X_{0,1}$

The same arguments as in the proof of Lemma 22 give:

Lemma 44 *Let (35)(48) be satisfied. We assume that the support of* $a(.)$ *is not bounded. Then* $(\lambda - T - B)^{-1}$ *is positivity improving, i.e.*

$$(\lambda - T - B)^{-1} f > 0 \ a.e.$$

for any nontrivial nonnegative $f \in X_{0,1}$*, or equivalently* $(V(t))_{t\geqslant 0}$ *is irreducible in* $X_{0,1}$*.*

Arguing as in the proof of Theorem 23 we get the main result of Subsection 2.2.

Theorem 45 *Let (35)(51)(44) be satisfied. If* $\sup_{z>0} \frac{r(z)}{1+z} < +\infty$ *and the support of* $a(.)$ *is not bounded then* $(V(t))_{t\geqslant 0}$ *has a spectral gap in* $X_{0,1}$*, i.e.*

$$r_{ess}(V(t)) < r_\sigma(V(t)),$$

and satisfies the asynchronous exponential growth.

Remark 46 *(Open questions) Following Remarks 31 and 36, we suspect that the different statements of this subsection 2.2, should hold under Assumption (48) instead of* $\sup_{z>0} \frac{r(z)}{1+z} < +\infty$.

3 The second construction

We consider now the case

$$\int_0^1 \frac{1}{r(\tau)} d\tau < +\infty \text{ and } \int_1^\infty \frac{1}{r(\tau)} d\tau = +\infty. \tag{53}$$

It turns out that we cannot expect a generation theory in the space $X_1 = L^1(\mathbb{R}_+,\ xdx)$, see Remark 51 below. So we restrict ourselves to the "finite mass and number of agregates" space

$$X_{0,1} = L^1(\mathbb{R}_+,\ (1+x)dx).$$

We start with:

Proposition 47 *Let (53) be satisfied. Then the partial differential equation*

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x}[r(x)u(x,t)] = 0,\ (x, t > 0)$$

with initial condition $u(x,0) = f(x)$ *and boundary condition*

$$\lim_{y\to 0} r(y)u(y,t) = 0 \quad (t > 0)$$

has a unique solution given by

$$u(y,t) = \begin{cases} \frac{r(X(y,t))f(X(y,t))}{r(y)} \quad if \quad \int_0^y \frac{1}{r(\tau)} d\tau > t \\ = 0 \;\; if \quad \int_0^y \frac{1}{r(\tau)} d\tau < t \end{cases}$$

where $X(y,t)$ $(t > 0)$ is defined by

$$\int_{X(y,t)}^{y} \frac{1}{r(\tau)} d\tau = t, \quad \left(\int_0^y \frac{1}{r(\tau)} d\tau > t \right).$$

Proof. We solve

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} \left[r(x) u(x,t) \right] = 0$$

with intial data

$$u(x,0) = f(x)$$

and boundary condition

$$\lim_{x \to 0} r(x) u(x,t) = 0 \quad (t > 0)$$

by the method of characteristics. This amounts to solving

$$\frac{1}{r(x)} \frac{\partial}{\partial t} \varphi(x,t) + \frac{\partial}{\partial x} \left[\varphi(x,t) \right] = 0; \quad \varphi(x,0) = r(x) f(x).$$

We introduce the characteristic equations

$$\frac{dt}{ds} = \frac{1}{r(x(s))}, \; \frac{dx}{ds} = 1$$

with "initial" conditions

$$x(0) = x, \; t(0) = 0 \quad (x > 0)$$

i.e. $x(s) = s + x$ and

$$t(s) = \int_0^s \frac{1}{r(\tau + x)} d\tau = \int_x^{s+x} \frac{1}{r(\tau)} d\tau \; (s \geqslant 0).$$

Thus

$$[0, +\infty) \ni s \to r(s+x) u(s+x, \int_x^{s+x} \frac{1}{r(\tau)} d\tau) \text{ is constant}$$

and then

$$r(s+x)u(s+x, \int_x^{s+x} \frac{1}{r(\tau)} d\tau) = r(x)u(x,0) = r(x)f(x) \quad \forall s \geqslant 0.$$

For $t > 0$ and $y > 0$ given, we set

$$\int_x^{s+x} \frac{1}{r(\tau)} d\tau = t, \ s + x = y$$

i.e. $\int_x^y \frac{1}{r(\tau)} d\tau = t$. Since $\int_0^y \frac{1}{r(\tau)} d\tau < +\infty$ let $y_0(t) > 0$ be defined by

$$\int_0^{y_0(t)} \frac{1}{r(\tau)} d\tau = t. \tag{54}$$

Hence there exists a unique $X(y,t) < y$ such that

$$\int_{X(y,t)}^y \frac{1}{r(\tau)} d\tau = t, \ (y > y_0(t)). \tag{55}$$

We denote by $X(y,.)$ the *continuous* function which gives $x \in (0,y)$ from t (given $y > 0$). Thus, for $y > y_0(t)$

$$r(y)u(y,t) = r(X(y,t))f(X(y,t))$$

i.e.

$$u(y,t) = \frac{r(X(y,t))f(X(y,t))}{r(y)} \quad (y > y_0(t)).$$

On the other hand, for $y < y_0(t)$

$$\int_0^y \frac{1}{r(\tau)} d\tau < t.$$

We introduce the characteristic equations

$$\frac{dt}{ds} = \frac{1}{r(x(s))}, \ \frac{dx}{ds} = 1$$

with "initial" conditions

$$x(0) = 0, \ t(0) = \bar{t} > 0$$

i.e. $x(s) = s$ and

$$t(s) = \bar{t} \ + \int_0^s \frac{1}{r(\tau)} d\tau.$$

Note the constancy of

$$r(s)u(s,\bar{t}\ +\int_0^s \frac{1}{r(\tau)}d\tau)\quad (s \geqslant 0)$$

amounts to

$$r(y)u(y,\bar{t}\ +\int_0^y \frac{1}{r(\tau)}d\tau)\ (y>0) \text{ is constant}$$

i.e. (formally)

$$r(y)u(y,\bar{t}\ +\int_0^y \frac{1}{r(\tau)}d\tau) = r(0)u(0,\bar{t}\ ) = 0$$

Thus

$$r(y)u(y,\bar{t}\ +\int_0^y \frac{1}{r(\tau)}d\tau) = 0 \quad \forall y>0,\ \bar{t}\ >0$$

Thus for any $t>0$ and $y>0$ such that

$$\int_0^y \frac{1}{r(\tau)}d\tau < t$$

we can *choose* $\bar{t}\ >0$ such that

$$\bar{t}\ +\int_0^y \frac{1}{r(\tau)}d\tau = t$$

namely

$$\bar{t}\ = t-\int_0^y \frac{1}{r(\tau)}d\tau.$$

Finally, $u(y,t)=0$ if $\int_0^y \frac{1}{r(\tau)}d\tau < t$. ■

3.1 Theory in the space $X_{0,1}$

As previously, we develop a general theory on well-posedness and spectral analysis.

3.1.1 The first generation result

Theorem 48 *Let (53) be satisfied. Let $X(y,t)$ be defined by (55). Then*

$$U_0(t)f := \chi_{\left\{\int_0^y \frac{1}{r(\tau)}d\tau > t\right\}} \frac{r(X(y,t))f(X(y,t))}{r(y)}$$

defines a positive C_0-semigroup on $X_{0,1}$ if and only if

$$[0,+\infty) \ni t \to \sup_{x>0} \frac{1+y(x,t)}{1+x} \tag{56}$$

is locally bounded where $y(x,t)$ is defined by (59). This occurs if

$$r(z) \leq C(z+1) \quad (\forall z > 0); \tag{57}$$

in this case

$$\frac{1+y(x,t)}{x+1} \leq e^{Ct} \quad (x > 0).$$

Proof. Let us check that $U_0(t)$ is a bounded operator on $X_{0,1}$. Note that for $\int_0^y \frac{1}{r(\tau)} d\tau > t$ (i.e. if $y > y_0(t)$) we have

$$\int_{X(y,t)}^{y} \frac{1}{r(\tau)} d\tau = t \tag{58}$$

which shows that (for $t > 0$ *fixed)* $X(y,t)$ is strictly increasing in y and tends to 0 as $y \to y_0(t)$. Note that

$$(y_0(t), +\infty) \ni y \to X(y,t) \in (0,+\infty)$$

is continuous. By arguing as previously we show that

$$\frac{1}{r(y)} = \frac{1}{r(X(y,t))} \frac{\partial X(y,t)}{\partial y}$$

and

$$(U_0(t)f)(y) = f(X(y,t)) \frac{\partial X(y,t)}{\partial y}; \quad y \in (y_0(t), +\infty).$$

Thus

$$\|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} |(U_0(t)f)(y)| (1+y) dy = \int_{y_0(t)}^{+\infty} |f(X(y,t))| \frac{\partial X(y,t)}{\partial y} (1+y) dy$$

and the change of variable $x = X(y,t)$ gives

$$\|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} |f(x)| (1 + y(x,t)) dx$$

where $y(x,t)$ is the unique $y > x$ such that $x = X(y,t)$ i.e.

$$\int_x^{y(x,t)} \frac{1}{r(\tau)} d\tau = t. \tag{59}$$

Since

$$\|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} \frac{1+y(x,t)}{1+x}\,|f(x)|\,(1+x)\,dx$$

then $U_0(t)$ is a bounded linear operator in $X_{0,1}$ if and only if

$$\sup_{x>0} \frac{1+y(x,t)}{1+x} < +\infty.$$

In such a case

$$\|U_0(t)\|_{\mathcal{L}(X_{0,1})} = \sup_{x>0} \frac{1+y(x,t)}{1+x}$$

and

$$[0,+\infty) \ni t \to U_0(t) \in \mathcal{L}(X_{0,1})$$

is locally bounded if and only if

$$[0,+\infty) \ni t \to \sup_{x>0} \frac{1+y(x,t)}{1+x}$$

is. It follows (see e.g. [12]) that $(U_0(t))_{t\geqslant 0}$ is exponentially bounded. As previously, to show that $(U_0(t))_{t\geqslant 0}$ is strongly continuous on $X_{0,1}$ it suffices to check that

$$U_0(t)f \to f \text{ in } L^1(\mathbb{R}_+;\ (1+x)\,dx) \text{ as } t \to 0$$

on a *dense* subspace of $L^1(\mathbb{R}_+;\ (1+x)\,dx)$, e.g. for f continuous with compact support in $(0,+\infty)$. Note that for *any* compact set $\left[c,c^{-1}\right]$

$$\int_0^y \frac{1}{r(\tau)} d\tau > t$$

for t small enough uniformly in $y \in \left[c,c^{-1}\right]$ so

$$\left(U_0(t)f\right)(y) = f(X(y,t))\frac{\partial X(y,t)}{\partial y} \quad \forall y \in \left[c,c^{-1}\right]$$

for t small enough. In particular

$$\int_{X(y,t)}^y \frac{1}{r(\tau)} d\tau = t \quad \forall y \in \left[c,c^{-1}\right]$$

and

$$U_0(t)f = \frac{r(X(y,t))f(X(y,t))}{r(y)} \quad \forall y \in \left[c,c^{-1}\right].$$

for t small enough. We note that $X(y,t) \to y$ as $t \to 0$ for any $y > 0$ and uniformly in $y \in \left[\frac{c}{2}, 2c^{-1}\right]$. Hence

$$U_0(t)f = \frac{r(X(y,t))f(X(y,t))}{r(y)} \to f(y) \quad (t \to 0)$$

and, by the dominated convergence theorem, $U_0(t)f \to f$ in $L^1(\mathbb{R}_+;\ (1+x)\,dx)$ as $t \to 0$. It follows from (59) that

$$\frac{\partial y(x,t)}{\partial t} = r(y(x,t)) \quad \forall t > 0.$$

so, using (57),

$$y(x,t) = x + \int_0^t r(y(x,s))ds \leq x + \int_0^t C\,(y(x,s)+1)\,ds$$

and

$$y(x,t) + 1 \leq x + 1 + \int_0^t C\,(y(x,s)+1)\,ds.$$

Gronwall's lemma gives $y(x,t)+1 \leq (x+1)\,e^{Ct}$. Finally $\frac{1+y(x,t)}{x+1} \leq e^{Ct}$ $(x > 0)$ and $\|U_0(t)\|_{\mathcal{L}(X_{0,1})} \leq e^{Ct}$. ∎

Remark 49 *The proof above shows that $(U_0(t))_{t\geqslant 0}$ is not contractive in $X_{0,1}$.*

Remark 50 *As in Remark 26 on checks that if $\lim_{z\to+\infty} \frac{r(z)}{z} = +\infty$ then the generation theory fails. Hence the sufficient condition (57) is partly necessary. The necessity of boundedness of the growth rate at the origin is unclear.*

Remark 51 *Note that $\|U_0(t)f\|_{X_1} = \int_0^{+\infty} \frac{y(x,t)}{x}\,|f(x)|\,xdx$ so that the boundedness of $U_0(t)$ on X_1 amounts to*

$$\sup_{x>0} \frac{y(x,t)}{x} < +\infty. \tag{60}$$

But (54) and (59) imply that $\lim_{x\to 0} y(x,t) = y_0(t) > 0$ $(t > 0)$ so that (60) is violated and we cannot expect a generation theory in X_1 under assumption (53).

3.1.2 On the generator of $(U_0(t))_{t\geqslant 0}$

We identify now the resolvent of the generator.

Proposition 52 *Let (53)(57) be satisfied. Let T_0 be the generator of $(U_0(t))_{t\geqslant 0}$. Then*

$$\left((\lambda - T_0)^{-1} f\right)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda}{r(s)} ds} f(x) dx, \quad \operatorname{Re}\lambda > s(T_0)$$

where $s(T_0)$ is the spectral bound of T_0

Proof. We know that

$$U_0(t)f = \chi_{\left\{\int_0^y \frac{1}{r(\tau)} d\tau > t\right\}} \frac{r(X(y,t)) f(X(y,t))}{r(y)}$$

so

$$\begin{aligned} \left((\lambda - T_0)^{-1} f\right)(y) &= \int_0^{+\infty} e^{-\lambda t} \chi_{\left\{\int_0^y \frac{1}{r(\tau)} d\tau < t\right\}} \frac{r(X(y,t)) f(X(y,t))}{r(y)} dt \\ &= \int_0^{\int_0^y \frac{1}{r(\tau)} d\tau} e^{-\lambda t} \frac{r(X(y,t)) f(X(y,t))}{r(y)} dt \end{aligned}$$

Note that for any fixed $y > 0$ and $t < \int_0^y \frac{1}{r(\tau)} d\tau$ we have $\int_{X(y,t)}^y \frac{1}{r(\tau)} d\tau = t$ and

$$-\frac{1}{r(X(y,t))} \frac{\partial X(y,t)}{\partial t} = 1.$$

One sees that

$$t \in (0, \int_0^y \frac{1}{r(\tau)} d\tau) \to x := X(y,t)$$

is strictly decreasing from y to 0 so the change of variable $t \to x = X(y,t)$ gives

$$\int_0^{\int_0^y \frac{1}{r(\tau)} d\tau} e^{-\lambda t} \frac{r(X(y,t)) f(X(y,t))}{r(y)} dt = \int_0^y e^{-\lambda X_{-1}(y,x)} \frac{f(x)}{r(y)} dx$$

where $X_{-1}(y,x)$ is the inverse of $t \to x = X(y,t)$. This inverse is nothing but $x \to t = \int_x^y \frac{1}{r(\tau)} d\tau$ so

$$\left((\lambda - T_0)^{-1} f\right)(y)) = \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} f(x) dx$$

and this ends the proof. ■

Remark 53 *Note that $X_{0,1} \subset L^1(\mathbb{R}_+;\ dx)$. We can check that*

$$D(T_0) = \left\{ f \in X_{0,1};\ \frac{\partial\,(rf)}{\partial y} \in X_{0,1},\ \lim_{y\to 0} r(y)f(y) = 0 \right\},\quad T_0 = -\frac{\partial\,(rf)}{\partial y}$$

where $\frac{\partial(rf)}{\partial y}$ is the derivative (in the sens of distributions on $(0,+\infty)$) of the function $rf \in L^1(\mathbb{R}_+;\ dx)$. Note that $rf \in W^{1,1}(\mathbb{R}_+)$ so that $\lim_{y\to 0} r(y)f(y)$ exists.

The same proof as in Lemma 30 gives a *pointwise* estimate in $X_{0,1}$.

Lemma 54 *Let (53)(57) be satisfied. Let $\lambda > C$. Then*

$$\left|(\lambda - T_0)^{-1} f\right|(y) \le \frac{1}{(1+y)\,r(y)} \|f\|_{X_{0,1}} \ \ (f \in X_{0,1}).$$

3.1.3 The first perturbed semigroup

We build now a second explicit C_0-semigroup by the method of characteristics. We solve

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left[r(x)u(x,t)\right] + \beta(x)u(x,t) = 0$$

with boundary condition $\lim_{x\to 0} r(x)u(x,t) = 0$ and intial data $u(x,0) = f(x)$. By arguing as in subsection 2.2 we show that the solution is given by

$$U(t)f = \chi_{\left\{\int_0^y \frac{1}{r(\tau)}d\tau > t\right\}} e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)}dp} \frac{r(X(y,t))f(X(y,t))}{r(y)} = e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)}dp} U_0(t)f$$

($X(y,t)$ is given by (36)) and defines a C_0-semigroup $(U(t))_{t\ge 0}$ on $X_{0,1}$ while the resolvent of its generator is given by:

Proposition 55 *Let (53)(57) be satisfied. The resolvent of the generator T of $(U(t))_{t\ge 0}$ in $X_{0,1}$ is given by*

$$(\lambda - T)^{-1} f = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda+\beta(p)}{r(\tau)}d\tau} f(x)dx.$$

Remark 56 *We can "compute" the spectral bound of T_0 and T in $X_{0,1}$ as in Proposition 29 and Proposition 34.*

The same proof as in Lemma 35 gives a smoothing effect in $X_{0,1}$.

Lemma 57 *Let (53)(57) be satisfied. Let* $\lambda > C$. *Then*

$$\int_0^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(y)\right| \beta(y)\,(1+y)\,dy \leq \int_0^{+\infty} |(f(y)|\,(1+y)\,dy,\ (f \in X_{0,1})\,.$$

Remark 58 *One can deduce from Lemma 57 that*

$$D(T) = \{f \in D(T_0);\ \beta f \in X_{0,1}\}\,,\quad Tf =\ T_0 f - \beta f.$$

3.1.4 The second perturbed semigroup

The proof of the following theorem relying on W. Desch's perturbation theorem is identical to that of Theorem 38.

Theorem 59 *Let (53)(57) be satisfied. We assume that*

$$n(y) := \int_0^y b(x,y)dx$$

is such that $\sup_{y>0} \frac{n(y)}{1+y} < +\infty$. *Then the fragmentation operator (14) is* T*-bounded in* $X_{0,1}$ *and*

$$\lim_{\lambda \to +\infty} \left\| B(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_{0,1})} \leq \lim \sup_{a(y) \to +\infty} \frac{[(y - \eta(y)y) + n(y)]}{(1+y)\left(1 + \frac{d(y)}{a(y)}\right)}.$$

In particular, if

$$\lim \sup_{a(y) \to +\infty} \frac{[(y - \eta(y)y) + n(y)]}{(1+y)\left(1 + \frac{d(y)}{a(y)}\right)} < 1. \tag{61}$$

then

$$T + B : D(T) \subset X_{0,1} \to X_{0,1}$$

generates a positive semigroup $(V(t))_{t \geqslant 0}$ *in* $X_{0,1}$.

Remark 60 *We can state results similar to those given in Proposition 41.*

3.1.5 Compactness results in $X_{0,1}$

We are ready to show:

Theorem 61 *Let (53)(57) be satisfied. Let the sublevel sets of β be thin at infinity in the sense that*

$$\int_1^{+\infty} \frac{1_{\{\beta<c\}}}{r(x)} dx < +\infty \quad (c>0) \tag{62}$$

(e.g. let $\lim_{x\to+\infty}\beta(x)=+\infty$). Then T is resolvent compact on $X_{0,1}$.

Proof. Let $\lambda > C$ and f be in the unit ball of $X_{0,1}$, i.e.

$$\int_0^{+\infty} |f(x)|\,(1+x)\,dx \leq 1.$$

According to Lemma 57

$$\int_0^{+\infty} \left|\left((\lambda-T)^{-1}f\right)(x)\right| \beta(x)\,(1+x)\,dx \leq 1.$$

Let $c>0$ and $\varepsilon>0$ be arbitrary. We have

$$\begin{aligned} 1 \quad &\geq \quad \int_{\varepsilon^{-1}}^{+\infty} \left|\left((\lambda-T)^{-1}f\right)(x)\right| \beta(x)\,(1+x)\,dx \\ &= \quad \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta<c\}} \left|\left((\lambda-T)^{-1}f\right)(x)\right| \beta(x)\,(1+x)\,dx \\ &\quad + \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta\geq c\}} \left|\left((\lambda-T)^{-1}f\right)(x)\right| \beta(x)\,(1+x)\,dx \end{aligned}$$

so

$$\sup_{\|f\|_{X_{0,1}}\leq 1} \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta\geq c\}} \left|\left((\lambda-T)^{-1}f\right)(x)\right| (1+x)\,dx \leq \frac{1}{c}.$$

On the other hand, according to Lemma 54,

$$\left|(\lambda-T)^{-1}f\right| \leq \left|(\lambda-T_0)^{-1}f\right| \leq \frac{1}{(1+x)\,r(x)}\,\|f\|_{X_{0,1}}$$

so

$$\int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta<c\}} \left|\left((\lambda-T)^{-1}f\right)(x)\right| \beta(x)\,(1+x)\,dx \leq c\int_{\varepsilon^{-1}}^{+\infty} \frac{1_{\{\beta<c\}}}{r(x)} dx$$

and then

$$\sup_{\|f\|_E\leq 1} \int_{\varepsilon^{-1}}^{+\infty} \left|\left((\lambda-T)^{-1}f\right)(x)\right| (1+x)\,dx \leq \frac{1}{c} + c\int_{\varepsilon^{-1}}^{+\infty} \frac{1_{\{\beta<c\}}}{r(x)} dx$$

can be made arbitrarily small by choosing first c large enough and then ε small enough.

On the other hand on $\left(0, \varepsilon^{-1}\right)$ we have the uniform domination

$$\left|(\lambda - T)^{-1} f\right| \leq \frac{1_{(0,\varepsilon^{-1})}(x)}{(1+x)\, r(x)} \in X_{0,1} \quad (\|f\|_{X_{0,1}} \leq 1)$$

because

$$\int_0^{\varepsilon^{-1}} \frac{1}{r(x)} dx < +\infty.$$

Finally $\left\{(\lambda - T)^{-1} f;\ \|f\|_{X_{0,1}} \leq 1\right\}$ is as close to a relatively weakly compact set as we want and consequently is weakly compact so $(\lambda - T)^{-1}$ is weakly compact operator and consequently (see [28] Lemma 14) $(\lambda - T)^{-1}$ is compact. ■

As in Corollary 43, we have:

Corollary 62 *Let (53)(57)(61)(62) be satisfied. Then $T+B : D(T) \to X_{0,1}$ is resolvent compact.*

3.1.6 Spectral gap of the full semigroup $(V(t))_{t\geqslant 0}$ in $X_{0,1}$

The same arguments as in Lemma 22 give:

Lemma 63 *Let (53)(57)(61) be satisfied. We assume that the support of $a(.)$ is not bounded. Then $(\lambda - T - B)^{-1}$ is positivity improving, i.e.*

$$(\lambda - T - B)^{-1} f > 0 \ a.e.$$

for any nontrivial nonnegative $f \in X_{0,1}$, or equivalently $(V(t))_{t\geqslant 0}$ is irreducible in $X_{0,1}$.

The same arguments as in Theorem 23 give the main result of Section 3.

Theorem 64 *Let (53)(57)(61)(62) be satisfied. We assume that the support of $a(.)$ is not bounded. Then $(V(t))_{t\geqslant 0}$ has a spectral gap in $X_{0,1}$, i.e.*

$$r_{ess}(V(t)) < r_\sigma(V(t)),$$

and satisfies the asynchronous exponential growth.

4 Theory in the space X_0

This last section is devoted to growth-fragmentation equations in the "finite agregates number" space

$$X_0 := L^1(\mathbb{R}_+, dx)$$

under (53). For simplicity, we restrict ourselves to the case

$$d(.) = 0.$$

By resuming the proof of Theorem 48 (the sublinearity condition (57) is *no longer* necessary here) one sees that

$$(U_0(t)f)(y) = \begin{cases} f(X(y,t))\frac{\partial X(y,t)}{\partial y}; \quad y \in (y_0(t), +\infty) \\ 0 \quad (y < y_0(t)) \end{cases}$$

so

$$\|U_0(t)f\|_{X_0} = \int_0^{+\infty} |(U_0(t)f)(y)|\, dy = \int_{y_0(t)}^{+\infty} |f(X(y,t))| \frac{\partial X(y,t)}{\partial y} dy$$

and the change of variable $x = X(y,t)$ gives

$$\|U_0(t)f\|_{X_0} = \int_0^{+\infty} |f(x)|\, dx = \|f\|_{X_0}.$$

Hence we have:

Theorem 65 *Let (53) be satisfied. Let $X(y,t)$ be defined by (55). Then*

$$U_0(t)f := \chi_{\left\{\int_0^y \frac{1}{r(\tau)} d\tau > t\right\}} \frac{r(X(y,t))f(X(y,t))}{r(y)}$$

defines a stochastic C_0-semigroup $(U_0(t))_{t\geqslant 0}$ on X_0.

As previously, we show that

$$U(t)f = \chi_{\left\{\int_0^y \frac{1}{r(\tau)} d\tau > t\right\}} e^{-\int_{X(y,t)}^y \frac{a(p)}{r(p)} dp} \frac{r(X(y,t))f(X(y,t))}{r(y)} = e^{-\int_{X(y,t)}^y \frac{a(p)}{r(p)} dp} U_0(t)f$$

($X(y,t)$ is given by (55)) defines a contraction C_0-semigroup $(U(t))_{t\geq 0}$ on X_0 and the resolvent of its generator T is given by

$$(\lambda - T)^{-1} f = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda + a(p)}{r(\tau)} d\tau} f(x) dx. \tag{63}$$

We have a smoothing effect in X_0.

Lemma 66 *Let (53) be satisfied. Then*

$$\int_0^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(y)\right| a(y) dy \leq \int_0^{+\infty} |f(x)|\, dx \quad (f \in X_0),\ (\lambda > 0).$$

Proof. One sees that for $\lambda > 0$

$$\begin{aligned}
& \int_0^{+\infty} \left|\left((\lambda - T)^{-1} f\right)(y)\right| a(y) dy \\
\leq & \int_0^{+\infty} \frac{a(y)}{r(y)} \left(\int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{a(p)}{r(p)} dp} |f(x)|\, dx \right) dy \\
= & \int_0^{+\infty} \left(\int_x^{+\infty} \frac{a(y)}{r(y)} e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy \right) |f(x)|\, dx \\
\leq & \int_0^{+\infty} \left(\int_x^{+\infty} \frac{a(y)}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy \right) |f(x)|\, dx \leq \int_0^{+\infty} |f(x)|\, dx
\end{aligned}$$

since

$$\int_x^{+\infty} \frac{a(y)}{r(y)} e^{-\int_x^y \frac{a(p)}{r(p)} dp} dy = -\int_x^{+\infty} \frac{d}{dy} \left(e^{-\int_x^y \frac{a(p)}{r(p)} dp} \right) dy = -\left[e^{-\int_x^y \frac{a(p)}{r(p)} dp} \right]_{y=x}^{y=+\infty} \leq 1.$$

■

A trivial consequence of (63) is the pointwise estimate

$$\left|(\lambda - T)^{-1} f\right|(y) \leq \frac{1}{r(y)} \int_0^{+\infty} |f(x)|\, dx \quad (f \in X_0),\ (\lambda > 0). \tag{64}$$

By combining the above smoothing effect and (64) and arguing as in the proof of Theorem 61, we get.

Theorem 67 *Let (53) be satisfied. Let the sublevel sets of a be thin at infinity in the sense that*

$$\int_1^{+\infty} \frac{1_{\{a<c\}}}{r(x)} dx < +\infty \quad (c > 0) \tag{65}$$

(e.g. let $\lim_{x \to +\infty} a(x) = +\infty$). Then T is resolvent compact on X_0.

We give now the full generation result.

Theorem 68 *Let (53) be satisfied and let*

$$n(.) := \int_0^y b(x, .) dx \in L^\infty(0, +\infty). \tag{66}$$

Then the fragmentation operator (14) is T-bounded in X_0 and

$$\lim_{\lambda\to+\infty}\left\|B(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_0)}\le\lim\sup_{a(y)\to+\infty}n(y).$$

In particular, if

$$\lim\sup_{a(y)\to+\infty}n(y)<1 \tag{67}$$

then

$$T+B:D(T)\subset X_0\to X_0$$

generates a positive semigroup $(V(t))_{t\geqslant 0}$ in X_0.

Proof. We note that

$$\begin{aligned}\|B\varphi\|_{X_0} &\le \int_0^{+\infty}\left(\int_x^{+\infty}a(y)b(x,y)\,|\varphi(y)|\,dy\right)dx\\ &= \int_0^{+\infty}a(y)\left(\int_0^y b(x,y)dx\right)|\varphi(y)|\,dy\\ &= \int_0^{+\infty}a(y)n(y)\,|\varphi(y)|\,dy\end{aligned}$$

and

$$\begin{aligned}&\left\|B(\lambda-T)^{-1}f\right\|_{X_0}=\int_0^{+\infty}a(y)n(y)\left|\left((\lambda-T)^{-1}f\right)\right|(y)dy\\ =\;&\int_{\{a\le c\}}a(y)n(y)\left|\left((\lambda-T)^{-1}f\right)\right|(y)dy\\ &+\int_{\{a>c\}}a(y)n(y)\left|\left((\lambda-T)^{-1}f\right)\right|(y)dy.\end{aligned}$$

Since

$$\int_{\{a\le c\}}a(y)n(y)\left|\left((\lambda-T)^{-1}f\right)\right|(y)dy\le c\,\|n\|_{L^\infty}\left\|(\lambda-T)^{-1}\right\|_{\mathcal{L}(X_0)}\|f\|_{X_0}$$

and

$$\begin{aligned}\int_{\{a>c\}}a(y)n(y)\left|\left((\lambda-T)^{-1}f\right)\right|(y)dy &\le \sup_{\{a>c\}}n(y)\int_0^{+\infty}a(y)\left|\left((\lambda-T)^{-1}f\right)\right|(y)dy\\ &\le \sup_{\{a>c\}}n(y)\int_0^{+\infty}|f(x)|\,dx\end{aligned}$$

(the smoothing effect is used in the last step), then

$$\left\| B(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_0)} \leq c \left\| n \right\|_{L^\infty} \left\| (\lambda - T)^{-1} \right\|_{\mathcal{L}(X_0)} + \sup_{\{a>c\}} n(y) \quad (c > 0)$$

and

$$\lim_{\lambda \to +\infty} \left\| B(\lambda - T)^{-1} \right\|_{\mathcal{L}(X_0)} \leq \sup_{\{a>c\}} n(y) \quad (\forall c > 0).$$

Finally, W. Desch's theorem ends the proof. ■

Remark 69 *Note that $n(y) \geq 1$ for conservative fragmentation kernels. This shows the key role of the mass loss assumption.*

Remark 70 *For homogeneous kernels (9) with mass loss, (67) amounts to $\int_0^1 h(z)dz < 1$. More generally, for a fragmentation kernel given by (11) the condition (67) holds if $\zeta_\infty^+ \limsup_{a(y) \to +\infty} \widehat{n}(y) < 1$ where $\widehat{n}(y) = \int_0^y \widehat{b}(x, y)dx$.*

As previously, $T + B$ is resolvent compact and, arguing as in the proof of Theorem 23, we obtain the main result of Section 4.

Theorem 71 *Let (53)(65)(66)(67) be satisfied. We assume that the support of $a(.)$ is not bounded. Then $(V(t))_{t \geqslant 0}$ has a spectral gap in X_0, i.e.*

$$r_{ess}(V(t)) < r_\sigma(V(t)),$$

and satisfies the asynchronous exponential growth in X_0.

Remark 72 *We could build a similar theory in X_0 under (35). We leave the details to the interested reader.*

References

[1] O. Arino and R. Rudnicki. Stability of phytoplankton dynamics. *C. R. Biologies,* 327 (2004) 961–969.

[2] J. Banasiak. On conservativity and shattering for an equation of phytoplankton dynamics. C. R. Biologies 327 (2004) 1025–1036.

[3] J. Banasiak and L. Arlotti. *Perturbations of positive semigroups with applications*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2006.

[4] J. Banasiak, K. Pichór, R. Rudnicki, Asynchronous exponential growth of a general structured population model. *Acta. Appl. Math*, **119** (2012) 149–166.

[5] J. Banasiak, W. Lamb and Ph. Laurençot. *Analytic Methods for Coagulation-Fragmentation Models*, Vol I, 2019, CRC Press.

[6] J. Banasiak and W. Lamb. Growth-fragmentation-coagulation equations with unbounded coagulation kernels. To appear in *Philosophical Transactions of the Royal Society A*, (2020).

[7] E. Bernard and P. Gabriel. Asymptotic behavior of the growth-fragmentation equation with bounded fragmentation rate. *J. Funct. Anal*, **272** (2017) 3455–3485.

[8] E. Bernard and P. Gabriel. Asynchronous exponential growth of the growth-fragmentation equation with unbounded fragmentation rate. *J. Evol. Equ,* **20** (2020), 375–401.

[9] M. J. Cáceres, J. A. Cañizo and S. Mischler. Rate of convergence to an asymptotic profile for the self-similar fragmentation and growth-fragmentation equations. *J. Math. Pures Appl.*, **96**(4) (2011) 334 – 362.

[10] J. A. Cañizo, P. Gabriel and H. Yoldasz. Spectral gap for the growth-fragmentation equation via Harris's Theorem. *arXiv:2004.08343v1* [math.AP], (2020).

[11] J. Bertoin and A. R. Watson. A probabilistic approach to spectral analysis of growth-fragmentation equations. *J. Funct. Anal*, **274** (2018) 2163–2204.

[12] B. Davies. *One-parameter Semigroups.* Academic Press, 1980.

[13] W. Desch. Perturbations of positive semigroups in AL-spaces; unpublished manuscript, 1988.

[14] O. Diekmann, H.J.A.M. Heijmans and H.R. Thieme. On the stability of the cell size distribution. *J. Math. Biol,* **19**(2) (1984) 227–248.

[15] H. Engler, J. Prüss and G. F. Webb. Analysis of a model for the dynamics of prions. II. *J. Math. Anal. Appl*, **324** (1) (2006) 98–117.

[16] P. Gabriel and M. Doumic Jauffret. Eigenelements of a general agregation-fragmentation model. *Math Models and Methods in Appl Sci*, **20**(5) (2010), 757-783.

[17] T. Kato. Perturbation theory for nullity, deficiency and other quantities of linear operators. *J. Anal. Math*, **6** (1958) 261–322.

[18] Ph. Laurençot and B. Perthame. Exponential decay for the growth-fragmentation/ Cell-Division equation. *Commun. Math. Sci,* **7**(2) (2009) 503–510.

[19] I. Marek. Frobenius theory of positive operators: Comparison theorems and applications. *SIAM Journal on Appl Math*, **19** (1970), 607-628.

[20] J.A.J. Metz and O. Diekmann (eds.). *The Dynamics of Physiologically Structured Populations.* Springer Lecture Notes in Biomathematics, vol. 68. Springer, New York, 1986.

[21] P. Michel, S. Mischler and B. Perthame. General relative entropy inequality: an illustration on growth models. *J. Math. Pures Appl,* **84**(9) (2005) 1235–1260.

[22] S. Mischler and J. Scher. Spectral analysis of semigroups and growth-fragmentation equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **33**(3) (2016), 849 – 898.

[23] M. Mokhtar-Kharroubi. *Mathematical Topics in Neutron Transport Theory. New Aspects*, Series on Adv in Math for Appl Sci, 46, World Scientific, 1997.

[24] M. Mokhtar-Kharroubi. On the convex compactness property for the strong operator topology and related topics. *Math. Methods Appl. Sci.*, **27**(6) (2004), 687–701.

[25] M. Mokhtar-Kharroubi. Optimal spectral theory of the linear Boltzmann equation. *J. Funct. Anal*, **226** (2005) 21-47.

[26] M. Mokhtar-Kharroubi. On L^1 spectral theory of neutron transport. *Differ. Integral Equ*, **11** (2005) 1221-1242.

[27] M. Mokhtar-Kharroubi and J. Voigt. On honesty of perturbed substochastic C_0-semigroups in L^1-spaces. *J. Operator Theory*, **64**(1) (2010) 131-147.

[28] M. Mokhtar-Kharroubi. Compactness properties of perturbed substochastic C_0-semigroups on $L^1(\mu)$ with applications to discreteness and spectral gaps. *Mém. Soc. Math. Fr*, 2016, no. 148.

[29] M. Mokhtar-Kharroubi and Q. Richard. Time asymptotics of structured populations with diffusion and dynamic boundary conditions. *Discrete and Cont Dyn Syst, Series B*, **23**(10) (2018), 4087-4116.

[30] M. Mokhtar-Kharroubi and Q. Richard. Spectral theory and time asymptotics of size-structured two-phase population models. *Discrete and Cont Dyn Syst, Series B*, **25**(8) (2020), 2969-3004.

[31] M. Mokhtar-Kharroubi. Spectra of structured diffusive population equations with generalized Wentzell-Robin boundary conditions and related topics. *Discrete and Cont Dyn Syst, Series S*, (2020), doi:10.3934/dcdss.2020244.

[32] M. Mokhtar-Kharroubi and J. Banasiak. On spectral gaps of growth-fragmentation semigroups in higher moment spaces. Work in preparation.

[33] R. Nagel (Ed). *One-Parameter Semigroups of Positive Operators*, vol. 1184, Springer-Verlag Berlin, 1986.

[34] B. Perthame. *Transport Equations in Biology*. Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007.

[35] G. Schluchtermann. On weakly compact operators. *Math. Ann*, **292** (1992), 263-266.

[36] J. Voigt. On resolvent positive operators and positive C_0-semigroups on AL-spaces. *Semigroup Forum*, **38** (1989), 263-266.

[37] G. F. Webb. An operator-theoretic formulation of asynchronous exponential growth. *Transactions of the Amer Math Soc*, **303** (1987), 751-763.

[38] L. Weis. A short proof for the stability theorem for positive semigroups on $L^p(\mu)$. *Proc. Amer. Math. Soc*, **126** (1998) 3253-3256.