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Thermal boundaries in kinetic and hydrodynamic limits

Tomasz Komorowski and Stefano Olla

Abstract We investigate how a thermal boundary, modelled by a Langevin dynamics, affect the macroscopic evolution of the energy at different space-time scales.

1 Introduction

Chains of an-harmonic oscillators are commonly used models in non-equilibrium statistical mechanics, in particular to study macroscopic energy transport. To treat mathematically non-linear dynamics is a very hard task, even for a small non linear perturbation of the harmonic chain, see [15]. In the purely harmonic chain the energy transport is ballistic, see [14]. Numerical evidence, see e.g. [13], shows that non-linear perturbations can cause the transport in a one-dimensional system to become diffusive, in case of optical chains and superdiffusive for acoustic chains. Replacing the non-linearity by a stochastic exchange of momenta between neighboring particles makes the problem mathematically treatable (see the review [1] and the references therein). This stochastic exchange can be modelled in various ways: e.g. for each pair of the nearest neighbor particles the exchange of their momenta can occur independently at an exponential rate (which models their elastic collision). Otherwise, for each triple of consecutive particles, exchange of momenta can be performed in a continuous, diffusive fashion, so that its energy and momentum are preserved. The energy transport proven for such stochastic dynamics is qualitatively similar to the one expected in the case of the non-linear deterministic dynamics. In particular, for a one-dimensional acoustic chain it could be proved that the macroscopic thermal

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energy density evolves according to a fractional heat equation corresponding to the fractional laplacian $(-\Delta)^{3/4}$, see [5].

In the recent years we have been interested in the macroscopic effects of a heat bath in contact with the chain at a point.

In Section 3 we consider first a purely harmonic chain in contact with a stochastic Langevin thermostat at temperature T , see (6) and (5). In the absence of a thermostat, its dynamics is completely integrable and the energy of each frequency (the Fourier mode k) is conserved. Rescaling space-time by the same parameter, the energy of mode k , localized by Wigner distribution $W(t, y, k)$, evolves according to a linear transport equation

$$\partial_t W + \bar{\omega}'(k) \partial_y W = 0,$$

with velocity $\bar{\omega}'(k) = \omega'(k)/2\pi$, where $\omega(k)$ is the dispersion relation of the harmonic chain. We can interpret $W(t, y, k)$ as the energy density of *phonons* of mode k at time t in the position y . The presence of a Langevin thermostat results in the emergence of a boundary (interface) condition at $y = 0$ ([7]):

$$W(t, 0^+, k) = p_-(k)W(t, 0^+, -k) + p_+(k)W(t, 0^-, k) + g(k)T, \quad \text{for } k > 0 \quad (1)$$

$$W(t, 0^-, k) = p_-(k)W(t, 0^-, -k) + p_+(k)W(t, 0^+, k) + g(k)T, \quad \text{for } k < 0.$$

The coefficients appearing in the boundary condition correspond to probabilities of the phonon transmission $p_+(k)$, reflection $p_-(k)$ and absorption $g(k)$. These parameters are non-negative and satisfy $p_+(k) + p_-(k) + g(k) = 1$. In addition, $T g(k)$ is the intensity of the phonon creations. The transmission, reflection and absorption parameters depend in a quite complicated way on the dispersion relation $\omega(\cdot)$ and the strength of the thermostat $\gamma > 0$ (cf. (39), (42), (50)). Some of their properties and an explicit calculation for the nearest-neighbor interactions are presented in Appendix A (see Section 7, the results contained there are original and are not part of [7]).

It is somewhat surprising that an incident *phonon* of mode k after scattering, if not absorbed, can produce only an identical transmitted phonon, or a reflected phonon of mode $-k$ (at least for a unimodal dispersion relation). This stands in contrast with what takes place at the microscopic scale. Then, an incident wave of frequency k scatters and produces waves of all possible frequencies. In the macroscopic limit, all frequencies produced by the scattering on the thermostat, except those corresponding to $\pm k$, are damped by oscillations.

In [9] we have considered the same problem after adding the bulk noise that conserves energy and momentum, see Section 4. The noise is properly rescaled in such a way that finite total amount of momentum is exchanged locally in the macroscopic unit time (in analogy to a kinetic limit). The effect of the bulk noise is to add a macroscopic scattering term to the transport equation :

$$\partial_t W + \bar{\omega}'(k) \partial_y W = \gamma_0 \int_{\mathbb{T}} R(k, k') (W(k') - W(k)) dk', \quad (2)$$

i.e. a phonon of mode k changes to the one of mode k' , with intensity $\gamma_0 R(k, k')$, given by (58). The case when no the heat bath is present, has been studied in [3]. In [9] we prove that the heat bath adds to (2) the same boundary condition (1), see section 4.

Since in (2) the energies of the different modes are mixed up by the bulk scattering, we can further rescale space-time in this equation in order to obtain an autonomous equation for the evolution of the total energy. Without the thermal bath this case has been studied in [4] and the following results have been obtained:

- For *optical chains* the velocity of the phonon behaves like $\bar{\omega}'(k) \sim k$ for small k , while for the total scattering rate $R(k) = \int R(k, k') dk' \sim k^2$. This means that phonons of low frequency rarely scatter, but move very slowly so they have the time to diffuse under an appropriate (diffusive) scaling. The phonons corresponding to other modes also behave diffusively at the respective scales. Consequently, in the optical chain, under diffusive space-time scaling, all modes homogenize equally, contributing to the macroscopic evolution of the energy $e(t, y)$, i.e. $W(t/\delta^2, y/\delta, k) \xrightarrow{\delta \rightarrow 0} e(t, y)$, that follows the linear heat equation $(\partial_t - D\partial_y^2)e(t, y) = 0$ with an explicitly given $D > 0$, see (86).
- In *acoustic chains*, the bulk scattering rate is the same, but $\bar{\omega}'(k) \sim O(1)$ for small k . Consequently low frequency phonons scatter rarely but they still move with velocities of order 1, and the resulting macroscopic limit is superdiffusive. In particular, in this case the low frequency modes are responsible for the macroscopic transport of the energy. The respective superdiffusive space-time scaling limit $W(t/\delta^{3/2}, y/\delta, k) \xrightarrow{\delta \rightarrow 0} e(t, y)$, described by the solution of a fractional heat equation $(\partial_t - \hat{c}|\partial_y^2|^{3/4})e(t, y) = 0$, with $\hat{c} > 0$ given by (94).

The thermal bath adds a boundary condition at $y = 0$ to the diffusive, or superdiffusive equations described above. More precisely the situation is as follows.

- In the optical chain we obtain a Dirichlet boundary condition $e(t, 0) = T$ for the respective heat equation (see (85)): phonons trajectories behave like Brownian motions, and since they can cross the boundary infinitely many times, they are absorbed almost surely. In effect, there is no energy *transfer* through the boundary at the macroscopic scale. This is proven in [2] using analytic techniques, see Section 5.1.
- In the acoustic chain, the long wave phonons, are responsible for the macroscopic energy transport. Their trajectories behave in the limit like superdiffusive symmetric, $3/2$ -stable Levy processes: they can jump over the boundary on the macroscopic scale and there is a positive probability of survival, i.e. of energy macroscopic transmission across the thermal boundary. Since the absorption probability $g(k)$ remains strictly positive as $k \rightarrow 0$ (as we prove here in appendix A, at least for the nearest neighbor acoustic chain, see (119)), the thermal boundary affects the transport. The macroscopic energy evolution is given by a fractional heat equation with boundary defined by (93) and (87). This is proven in [8] using probabilistic techniques, see Section 5.2.

In Section 2 we review the results of [7] for the harmonic chain with the thermostat attached at a point. Since the calculations for the transmission and reflection

scattering of [7] are quite complex, we present their somewhat simplified version (conveying nevertheless their gist) in Appendix B (Section 8). We hope that the outline would help the reader to understand how the macroscopic scattering emerges.

In Section 3 we review the results of [9] in the presence of the conservative bulk noise. Section 4 contains the review of the diffusive and superdiffusive limits proven in [2] and [8]. In Section 5 we mention some open problems, in particular the question of the direct hydrodynamic limit, without passing through the kinetic limit, in the spirit of [5].

Appendix A (Section 7) contains some original results that are not present in the discussed articles, concerning the properties of the scattering coefficients and their behaviour for $k \rightarrow 0$.

2 Notation

Given $a > 0$ by \mathbb{T}_a we denote the torus of size $a > 0$, i.e. the interval $[-a/2, a/2]$ with identified endpoints. When $a = 1$ we shall write $\mathbb{T} := \mathbb{T}_1$ for the unit torus. Let $\mathbb{T}_\pm := [k \in \mathbb{T} : 1/2 > \pm k > 0]$. We also let $\mathbb{R}_+ := (0, +\infty)$, $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$ and $\mathbb{T}_* := \mathbb{T} \setminus \{0\}$.

By $\ell^p(\mathbb{Z})$, $L^p(\mathbb{T})$, where $p \geq 1$, we denote the spaces of all complex valued sequences $(f_x)_{x \in \mathbb{Z}}$ and functions $f : \mathbb{T} \rightarrow \mathbb{C}$ that are summable with p -th power, respectively. The Fourier transform of $(f_x)_{x \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ and the inverse Fourier transform of $\hat{f} \in L^2(\mathbb{T})$ are given by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}} f_x \exp\{-2\pi i x k\}, \quad f_x = \int_{\mathbb{T}} \hat{f}(k) \exp\{2\pi i x k\} dk, \quad x \in \mathbb{Z}, \quad k \in \mathbb{T}. \quad (3)$$

We use the notation

$$(f \star g)_y = \sum_{y' \in \mathbb{Z}} f_{y-y'} g_{y'}$$

for the convolution of two sequences $(f_x)_{x \in \mathbb{Z}}, (g_x)_{x \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ that belong to appropriate spaces $\ell^p(\mathbb{Z})$. In most cases we shall assume that one of the sequences rapidly decays, while the other belongs to $\ell^2(\mathbb{Z})$.

For a function $G : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{C}$ that is either L^1 , or L^2 -summable, we denote by $\hat{G} : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{C}$ its Fourier transform, in the first variable, defined as

$$\hat{G}(\eta, k) := \int_{\mathbb{R}} e^{-2\pi i \eta x} G(x, k) dx, \quad (\eta, k) \in \mathbb{R} \times \mathbb{T}. \quad (4)$$

Denote by $C_0(\mathbb{R} \times \mathbb{T})$ the class of functions G that are continuous and satisfy $\lim_{|y| \rightarrow +\infty} \sup_{k \in \mathbb{T}} |G(y, k)| = 0$.

3 Harmonic chain in contact with a Langevin thermostat

We consider the evolution of an infinite particle system governed by the Hamiltonian

$$\mathcal{H}(\mathfrak{p}, \mathfrak{q}) := \frac{1}{2} \sum_{y \in \mathbb{Z}} p_y^2 + \frac{1}{2} \sum_{y, y' \in \mathbb{Z}} \alpha_{y-y'} q_y q_{y'}. \quad (5)$$

Here, the particle label is $y \in \mathbb{Z}$, (q_y, p_y) is the position and momentum of the y 's particle, respectively, and $(\mathfrak{q}, \mathfrak{p}) = \{(q_y, p_y), y \in \mathbb{Z}\}$ denotes the entire configuration. The coupling coefficients α_y are assumed to have exponential decay and chosen positive definite such that the energy is positive. We couple the particle with label 0 to a Langevin thermostat at temperature T . Then the evolution equation then writes as the stochastic differential equations:

$$\begin{aligned} \dot{q}_y(t) &= p_y(t), \\ dp_y(t) &= -(\alpha \star \mathfrak{q}(t))_y dt + (-\gamma p_0(t) dt + \sqrt{2\gamma T} dw(t)) \delta_{0,y}, \quad y \in \mathbb{Z}. \end{aligned} \quad (6)$$

Here, $\{w(t), t \geq 0\}$ is a standard Wiener process, while $\gamma > 0$ is a coupling parameter with the thermostat.

Assumptions on the dispersion relation and its basic properties

We assume (cf [3]) that the coupling constants $(\alpha_x)_{x \in \mathbb{Z}}$ satisfy the following:

- a1) they are real valued and there exists $C > 0$ such that $|\alpha_x| \leq C e^{-|x|/C}$ for all $x \in \mathbb{Z}$,
- a2) $\hat{\alpha}(k) = \sum_{x \in \mathbb{Z}} \alpha_x e^{-2\pi i k x}$ is also real valued and $\hat{\alpha}(k) > 0$ for $k \neq 0$ and in case $\hat{\alpha}(0) = 0$ we have $\hat{\alpha}''(0) > 0$.

The above conditions imply that both functions $x \mapsto \alpha_x$ and $k \mapsto \hat{\alpha}(k)$ are even. In addition, $\hat{\alpha} \in C^\infty(\mathbb{T})$ and in case $\hat{\alpha}(0) = 0$ we have $\hat{\alpha}(k) = k^2 \phi(k^2)$ for some strictly positive $\phi \in C^\infty(\mathbb{T})$. The dispersion relation $\omega : \mathbb{T} \rightarrow \bar{\mathbb{R}}_+$, given by

$$\omega(k) := \sqrt{\hat{\alpha}(k)}, \quad k \in \mathbb{T}. \quad (7)$$

is obviously also even. Throughout the paper it is assumed to be unimodal, i.e. increasing on $\bar{\mathbb{T}}_+$ and then, in consequence, decreasing on $\bar{\mathbb{T}}_-$. Its unique minimum and maximum are attained at $k = 0$, $k = 1/2$, respectively. They are denoted by $\omega_{\min} \geq 0$ and ω_{\max} , correspondingly. Denote the two branches of its inverse by $\omega_\pm : [\omega_{\min}, \omega_{\max}] \rightarrow \bar{\mathbb{T}}_\pm$.

In order to avoid technical problems with the definition of the dynamics, we assume that the initial conditions are random but with finite energy: $\mathcal{H}(\mathfrak{p}, \mathfrak{q}) < \infty$. This property will be conserved in time. For such configurations we can define the complex wave function

$$\psi_y(t) := (\tilde{\omega} \star q(t))_y + ip_y(t) \quad (8)$$

where $(\tilde{\omega}_y)_{y \in \mathbb{Z}}$ is the inverse Fourier transform of the dispersion relation. We have $\mathcal{H}(p(t), q(t)) = \sum_y |\psi_y(t)|^2$.

The Fourier transform of the wave function is given by

$$\hat{\psi}(t, k) := \omega(k)\hat{q}(t, k) + i\hat{p}(t, k), \quad k \in \mathbb{T}, \quad (9)$$

so that

$$\hat{p}(t, k) = \frac{1}{2i}[\hat{\psi}(t, k) - \hat{\psi}^*(t, -k)], \quad p_0(t) = \int_{\mathbb{T}} \text{Im} \hat{\psi}(t, k) dk.$$

Using (6), it is easy to verify that the wave function evolves according to

$$d\hat{\psi}(t, k) = (-i\omega(k)\hat{\psi}(t, k) - i\gamma p_0(t))dt + i\sqrt{2\gamma T}dw(t). \quad (10)$$

Introducing a (small) parameter $\varepsilon \in (0, 1)$, we wish to study the behaviour of the distribution of the energy at a large space-time scale, i.e. for the wave function $\psi_{[x/\varepsilon]}(t/\varepsilon)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, when $\varepsilon \rightarrow 0$. In this scaling limit we would like to maintain each particle contribution to the total energy to be of order $O(1)$, on the average, and therefore keep the total energy of the chain to be order ε^{-1} . For this reason, we choose random initial data that is distributed by probability measures μ_ε defined on the phase space (p, q) , in such a way that

$$\sup_{\varepsilon \in (0, 1)} \varepsilon \langle \mathcal{H}(p, q) \rangle_{\mu_\varepsilon} = \sup_{\varepsilon \in (0, 1)} \sum_{y \in \mathbb{Z}} \varepsilon \langle |\psi_y|^2 \rangle_{\mu_\varepsilon} = \sup_{\varepsilon \in (0, 1)} \varepsilon \langle \|\hat{\psi}\|_{L^2(\mathbb{T})}^2 \rangle_{\mu_\varepsilon} < \infty. \quad (11)$$

The symbol $\langle \cdot \rangle_{\mu_\varepsilon}$ denotes, as usual, the average with respect to measure μ_ε . To simplify our calculations we will also assume that

$$\langle \hat{\psi}(k)\hat{\psi}(\ell) \rangle_{\mu_\varepsilon} = 0, \quad k, \ell \in \mathbb{T}. \quad (12)$$

This condition is easily satisfied by local Gibbs measures like

$$\prod_{y \in \mathbb{Z}} \frac{e^{-\beta_y^\varepsilon |\psi_y|^2/2}}{Z_{\beta_y^\varepsilon}} d\psi_y \quad (13)$$

for a proper choice of temperature profiles $(\beta_y^\varepsilon)^{-1} > 0$, decaying fast enough to 0, as $|y| \rightarrow +\infty$. Here $Z_{\beta_y^\varepsilon}$ is the normalizing constant.

Wigner distributions

Wigner distributions provide an effective tool to localize in space energy per frequency, separating microscopic from macroscopic scale. The (averaged) Wigner distribution (or Wigner transform) is defined by its action on a test function $G \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ as

$$\langle G, W^{(\varepsilon)}(t) \rangle := \frac{\varepsilon}{2} \sum_{y, y' \in \mathbb{Z}} \int_{\mathbb{T}} e^{2\pi i k(y' - y)} \mathbb{E}_{\varepsilon} \left[\psi_y \left(\frac{t}{\varepsilon} \right) (\psi_{y'})^* \left(\frac{t}{\varepsilon} \right) \right] G^* \left(\varepsilon \frac{y + y'}{2}, k \right) dk. \quad (14)$$

The Fourier transform of the Wigner distribution, or the Fourier-Wigner function is defined as

$$\widehat{W}_{\varepsilon}(t, \eta, k) := \frac{\varepsilon}{2} \mathbb{E}_{\varepsilon} \left[\widehat{\psi}^* \left(\frac{t}{\varepsilon}, k - \frac{\varepsilon \eta}{2} \right) \widehat{\psi} \left(\frac{t}{\varepsilon}, k + \frac{\varepsilon \eta}{2} \right) \right], \quad (t, \eta, k) \in [0, \infty) \times \mathbb{T}_{2/\varepsilon} \times \mathbb{T}, \quad (15)$$

so that

$$\langle G, W^{(\varepsilon)}(t) \rangle = \int_{\mathbb{T} \times \mathbb{R}} \widehat{W}_{\varepsilon}(t, \eta, k) \widehat{G}^*(\eta, k) d\eta dk, \quad G \in \mathcal{S}(\mathbb{R} \times \mathbb{T}). \quad (16)$$

Taking $G(x, k) := G(x)$ in (14) we obtain

$$\langle G, W^{(\varepsilon)}(t) \rangle = \frac{\varepsilon}{2} \sum_{y \in \mathbb{Z}} \mathbb{E}_{\varepsilon} \left[\left| \psi_y \left(\frac{t}{\varepsilon} \right) \right|^2 \right] G(\varepsilon y). \quad (17)$$

In what follows we assume that the initial data, after averaging, leads to a sufficiently fast decaying (in η) Fourier-Wigner function. More precisely, we suppose that there exist $C, \kappa > 0$ such that

$$|\widehat{W}_{\varepsilon}(0, \eta, k)| \leq \frac{C}{(1 + \eta^2)^{3/2 + \kappa}}, \quad (\eta, k) \in \mathbb{T}_{2/\varepsilon} \times \mathbb{T}, \quad \varepsilon \in (0, 1). \quad (18)$$

In addition, we assume that there exists a distribution $W_0 \in \mathcal{S}'(\mathbb{R} \times \mathbb{T})$ such that for any $G \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$

$$\lim_{\varepsilon \rightarrow 0^+} \langle G, W^{(\varepsilon)}(0) \rangle = \langle G, W_0 \rangle. \quad (19)$$

Note that, thanks to (18), distribution W_0 is in fact a function that belongs to $C_0(\mathbb{R} \times \mathbb{T}) \cap L^2(\mathbb{R} \times \mathbb{T})$.

3.1 The thermostat free case: $\gamma = 0$

If the thermostat is not present ($\gamma = 0$), the equation of motion (10) can be explicitly solved and the solution is $\widehat{\psi}(t, k) = \widehat{\psi}(k) e^{-i\omega(k)t}$. Defining

$$\delta_{\varepsilon} \omega(k, \eta) := \frac{1}{\varepsilon} \left[\omega \left(k + \frac{\varepsilon \eta}{2} \right) - \omega \left(k - \frac{\varepsilon \eta}{2} \right) \right], \quad (20)$$

we can compute explicitly the Wigner transform:

$$\widehat{W}_{\varepsilon}(t, \eta, k) = e^{-i\delta_{\varepsilon} \omega(k, \eta)t/\varepsilon} \widehat{W}_{\varepsilon}(0, \eta, k) \xrightarrow{\varepsilon \rightarrow 0} e^{-i\omega'(k)\eta t} \widehat{W}_0(\eta, k), \quad (21)$$

assuming the corresponding convergence at initial time, see (19). The inverse Fourier transform gives

$$W(t, y, k) = W_0(y - \bar{\omega}'(k)t, k), \quad (22)$$

where $\bar{\omega}'(k) := \omega'(k)/(2\pi)$, i.e. it solves the simple linear transport equation

$$\partial_t W(t, y, k) + \bar{\omega}'(k) \partial_y W(t, y, k) = 0, \quad W(0, y, k) = W_0(y, k). \quad (23)$$

We can view this equation as the evolution of the density in independent particles (phonons), labelled by the frequency mode $k \in \mathbb{T}$, and moving with velocity $\bar{\omega}'(k)$.

3.2 The evolution with the Langevin thermostat: $\gamma > 0$

We use the mild formulation of (10):

$$\hat{\psi}(t, k) = e^{-i\omega(k)t} \hat{\psi}(0, k) - i\gamma \int_0^t e^{-i\omega(k)(t-s)} \mathfrak{p}_0(s) ds + i\sqrt{2\gamma T} \int_0^t e^{-i\omega(k)(t-s)} dw(s). \quad (24)$$

Integrating both sides in the k -variable and taking the imaginary part in both sides, we obtain a closed equation for $\mathfrak{p}_0(t)$:

$$\mathfrak{p}_0(t) = \mathfrak{p}_0^0(t) - \gamma \int_0^t J(t-s) \mathfrak{p}_0(s) ds + \sqrt{2\gamma T} \int_0^t J(t-s) dw(s), \quad (25)$$

where

$$J(t) = \int_{\mathbb{T}} \cos(\omega(k)t) dk, \quad (26)$$

and

$$\mathfrak{p}_0^0(t) = \int_{\mathbb{T}} \text{Im} \left(\hat{\psi}(0, k) e^{-i\omega(k)t} \right) dk, \quad (27)$$

is the momentum at $y = 0$ for the free evolution with $\gamma = 0$ (without the thermostat).

Taking the Laplace transform

$$\tilde{\mathfrak{p}}_0(\lambda) = \int_0^{+\infty} e^{-\lambda t} \mathfrak{p}_0(t) dt, \quad \text{Re } \lambda > 0,$$

in (25) we obtain

$$\tilde{\mathfrak{p}}_0(\lambda) = \tilde{g}(\lambda) \tilde{\mathfrak{p}}_0^0(\lambda) + \sqrt{2\gamma T} \tilde{g}(\lambda) \tilde{J}(\lambda) \tilde{w}(\lambda). \quad (28)$$

Here, $\tilde{g}(\lambda)$ is given by

$$\tilde{g}(\lambda) := (1 + \gamma \tilde{J}(\lambda))^{-1}. \quad (29)$$

and

$$\tilde{J}(\lambda) := \int_0^\infty e^{-\lambda t} J(t) dt = \int_{\mathbb{T}} \frac{\lambda}{\lambda^2 + \omega^2(k)} dk, \quad \text{Re } \lambda > 0. \quad (30)$$

We will show below that $\tilde{g}(\lambda)$ is the Laplace transform of a signed locally finite measure $g(d\tau)$. Then, the term $(\lambda + i\omega(k))^{-1} \tilde{g}(\lambda) \tilde{p}_0^0(\lambda)$, that appears in (33), is the Laplace transform of the convolution

$$\int_0^t \phi(t-s, k) \tilde{p}_0^0(s) ds, \quad (31)$$

where

$$\phi(t, k) = \int_0^t e^{-i\omega(k)(t-\tau)} g(d\tau). \quad (32)$$

Next, taking the Laplace transform of both sides of (24) and using (28), we arrive at an explicit formula for the Fourier-Laplace transform of $\psi_y(t)$:

$$\begin{aligned} \tilde{\psi}(\lambda, k) &= \frac{\hat{\psi}(0, k) - i\gamma \tilde{p}_0(\lambda) + i\sqrt{2\gamma T} \tilde{w}(\lambda)}{\lambda + i\omega(k)} \\ &= \frac{\hat{\psi}(0, k) - i\gamma \tilde{g}(\lambda) (\tilde{p}_0^0(\lambda) + \sqrt{2\gamma T} \tilde{J}(\lambda) \tilde{w}(\lambda)) + i\sqrt{2\gamma T} \tilde{w}(\lambda)}{\lambda + i\omega(k)} \\ &= \frac{\hat{\psi}(0, k) - i\gamma \tilde{g}(\lambda) \tilde{p}_0^0(\lambda) + i\tilde{g}(\lambda) \sqrt{2\gamma T} \tilde{w}(\lambda)}{\lambda + i\omega(k)}. \end{aligned} \quad (33)$$

The Laplace inversion of (33) yields an explicit expression for $\hat{\psi}(t, k)$:

$$\begin{aligned} \hat{\psi}(t, k) &= e^{-i\omega(k)t} \hat{\psi}(0, k) - i\gamma \int_0^t \phi(t-s, k) \tilde{p}_0^0(s) ds \\ &\quad + i\sqrt{2\gamma T} \int_0^t \phi(t-s, k) dw(s). \end{aligned} \quad (34)$$

3.3 Phonon creation by the heat bath

Since the contribution to the energy given by the thermal term and the initial energy are completely separate, we can assume first that $\widehat{W}_0 = 0$. In this case $\hat{\psi}(0, k) = 0$ and (34) reduces to a stochastic convolution:

$$\hat{\psi}(t, k) = i\sqrt{2\gamma T} \int_0^t \phi(t-s, k) dw(s). \quad (35)$$

To shorten the notation, denote

$$\tilde{\phi}(t, k) = \int_0^t e^{i\omega(k)\tau} g(d\tau) = e^{i\omega(k)t} \phi(t, k),$$

We can compute directly the Fourier-Wigner function

$$\widehat{W}_\varepsilon(t, \eta, k) = \gamma T \int_0^t e^{-i\delta_\varepsilon \omega(k, \eta)s} \tilde{\phi}\left(s/\varepsilon, k + \frac{\varepsilon\eta}{2}\right) \tilde{\phi}^*\left(s/\varepsilon, k - \frac{\varepsilon\eta}{2}\right) ds.$$

Taking its Laplace transform we obtain

$$\begin{aligned} \widehat{w}_\varepsilon(\lambda, \eta, k) &= \gamma T \int_0^\infty dt e^{-\lambda t} \int_0^t ds e^{-i\delta_\varepsilon \omega(k, \eta)s} \tilde{\phi}\left(\varepsilon^{-1}s, k + \frac{\varepsilon\eta}{2}\right) \tilde{\phi}^*\left(\varepsilon^{-1}s, k - \frac{\varepsilon\eta}{2}\right) \\ &= \frac{\gamma T}{\lambda} \int_0^\infty ds e^{-(\lambda + i\delta_\varepsilon \omega(k, \eta))s} \tilde{\phi}\left(\varepsilon^{-1}s, k + \frac{\varepsilon\eta}{2}\right) \tilde{\phi}^*\left(\varepsilon^{-1}s, k - \frac{\varepsilon\eta}{2}\right). \end{aligned} \quad (36)$$

Using the inverse Laplace formula for the product of functions we obtain, for any $c > 0$,

$$\begin{aligned} \widehat{w}_\varepsilon(\lambda, \eta, k) &= \frac{\gamma T}{\lambda} \frac{1}{2\pi i} \lim_{\ell \rightarrow \infty} \int_{c-i\ell}^{c+i\ell} \{\sigma(\lambda + i\delta_\varepsilon \omega(k, \eta) - \sigma)\}^{-1} \\ &\quad \times \tilde{g}\left(\varepsilon\sigma - i\omega(k + \frac{\varepsilon\eta}{2})\right) \tilde{g}^*\left(\varepsilon(\lambda + i\delta_\varepsilon \omega(k, \eta) - \sigma) - i\omega(k - \frac{\varepsilon\eta}{2})\right) d\sigma. \end{aligned} \quad (37)$$

Since \tilde{g} is bounded and $\operatorname{Re}\lambda > 0$, we can take the limit as $\varepsilon \rightarrow 0$, obtaining

$$\widehat{w}(\lambda, \eta, k) = \frac{\gamma T |\nu(k)|^2}{\lambda(\lambda + i\omega'(k)\eta)}, \quad (38)$$

where

$$\nu(k) := \lim_{\varepsilon \rightarrow 0} \tilde{g}(\varepsilon - i\omega(k)). \quad (39)$$

The limit in (39) is well defined everywhere, see Section 7 for details.

The inverse Laplace transform of (38) gives

$$\widehat{W}(t, \eta, k) = \frac{1 - e^{-i\omega'(k)\eta t}}{i\omega'(k)\eta} \gamma T |\nu(k)|^2. \quad (40)$$

Performing the inverse Fourier transform, according to (30), we obtain

$$W(t, y, k) = T \mathfrak{g}(k) 1_{[[0, \bar{\omega}'(k)t]]}(y) \quad (41)$$

where $\bar{\omega}(k) = \omega(k)/2\pi$,

$$\mathfrak{g}(k) := \frac{\gamma |\nu(k)|^2}{|\bar{\omega}'(k)|} \quad (42)$$

and

$$[[0, a]] := \begin{cases} [0, a], & \text{if } a > 0 \\ [a, 0], & \text{if } a < 0. \end{cases}$$

We can interpret (41) as the energy density of k -phonons that are created at the interface $y = 0$ by the heat bath with intensity $T \mathfrak{g}(k)$ and then move with velocity $\bar{\omega}'(k)$. The Wigner function $W(t, y, k)$ can be viewed as a formal solution of

$$\partial_t W(t, y, k) + \bar{\omega}'(k) \partial_y W(t, y, k) = |\bar{\omega}'(k)| T \mathfrak{g}(k) \delta(y), \quad W(0, y, k) = 0. \quad (43)$$

3.4 Phonon scattering and absorption by the heat bath

The scattering of incoming waves can be studied at temperature $T = 0$, by looking at the deterministic equation

$$\hat{\psi}(t, k) = e^{-i\omega(k)t} \hat{\psi}(0, k) - i\gamma \int_0^t \phi(t-s, k) \mathfrak{p}_0^0(s) ds, \quad (44)$$

Proceeding along the lines of the calculation of the previous section we obtain

$$\widehat{W}_\varepsilon(t, \eta, k) = \widehat{W}_\varepsilon^0(t, \eta, k) + \widehat{W}_\varepsilon^1(t, \eta, k) + \widehat{W}_\varepsilon^2(t, \eta, k), \quad (45)$$

where $\widehat{W}_\varepsilon(t, \eta, k)$ is given by (15),

$$\begin{aligned} \widehat{W}_\varepsilon^0(t, \eta, k) &:= e^{-i\delta_\varepsilon \omega(k, \eta)t/\varepsilon} \widehat{W}_\varepsilon(0, \eta, k), \\ \widehat{W}_\varepsilon^1(t, \eta, k) &:= -i \frac{\varepsilon \gamma}{2} \int_0^{t/\varepsilon} \left\{ \mathbb{E}_\varepsilon \left[\hat{\psi} \left(0, k - \frac{\varepsilon \eta}{2} \right)^* \mathfrak{p}_0^0(s) \right] e^{i\omega(k - \frac{\varepsilon \eta}{2}) \frac{t}{\varepsilon}} \phi \left(\frac{t}{\varepsilon} - s, k + \frac{\varepsilon \eta}{2} \right) \right. \\ &\quad \left. - \mathbb{E}_\varepsilon \left[\hat{\psi} \left(0, k + \frac{\varepsilon \eta}{2} \right) \mathfrak{p}_0^0(s) \right] e^{-i\omega(k + \frac{\varepsilon \eta}{2}) \frac{t}{\varepsilon}} \phi \left(\frac{t}{\varepsilon} - s, k - \frac{\varepsilon \eta}{2} \right)^* \right\} ds, \\ \widehat{W}_\varepsilon^2(t, \eta, k) &:= \frac{\varepsilon \gamma^2}{2} \int_0^{t/\varepsilon} ds_1 \int_0^{t/\varepsilon} ds_2 \mathbb{E}_\varepsilon \left[\mathfrak{p}_0^0(s_1) \mathfrak{p}_0^0(s_2) \right] \\ &\quad \times \phi \left(\frac{t}{\varepsilon} - s_1, k - \frac{\varepsilon \eta}{2} \right)^* \phi \left(\frac{t}{\varepsilon} - s_2, k - \frac{\varepsilon \eta}{2} \right). \end{aligned} \quad (46)$$

The limit behavior of $\widehat{W}_\varepsilon^0(t, \eta, k)$ is already described by (21). The calculations for the other two terms $\widehat{W}_\varepsilon^1$ and $\widehat{W}_\varepsilon^2$ are more involved. Their outline is presented in Section 8 below. We have

$$\lim_{\varepsilon \rightarrow 0^+} \widehat{W}_\varepsilon^1(t, \eta, k) = -\gamma \operatorname{Re} \nu(k) e^{-i\omega'(k)t} \int_{\mathbb{R}} \frac{1 - e^{-i\omega'(k)(\eta' - \eta)t}}{i\omega'(k)(\eta' - \eta)} \widehat{W}(0, \eta', k) d\eta' \quad (47)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \widehat{W}_\varepsilon^2(t, \eta, k) = \frac{|\nu(k)|^2}{4|\bar{\omega}'(k)|} \sum_{\iota = \pm} \int_{\mathbb{R}} \frac{d\eta}{\lambda + i\omega'(k)\eta} \int_{\mathbb{R}} \frac{\widehat{W}(0, \eta', \iota k) d\eta'}{\lambda + i\omega'(\iota k)\eta'}. \quad (48)$$

Putting together the limits of the solutions for $T \geq 0$ and $\widehat{W}_0 = 0$, given by (41), and the solution of the deterministic equation for $T = 0$ and a non-vanishing \widehat{W}_0 obtained by taking the sum of the limits of $\widehat{W}_\varepsilon^j(t, \eta, k)$, $j = 0, 1, 2$ we conclude the formula for the limit as $ps \rightarrow 0$, of Wigner function $\widehat{W}_\varepsilon(t, \eta, k)$, see (15), for $\widehat{\phi}(t, k)$, given by (34), equals:

$$\begin{aligned} W(t, y, k) &= 1_{[[0, \bar{\omega}'(k)t]^c]}(y)W(0, y - \bar{\omega}'(k)t, k) \\ &+ p_+(k)1_{[[0, \bar{\omega}'(k)t]]}(y)W(0, y - \bar{\omega}'(k)t, k) \\ &+ p_-(k)1_{[[0, \bar{\omega}'(k)t]]}(y)W(0, -y + \bar{\omega}'(k)t, -k) + T \mathfrak{g}(k)1_{[[0, \bar{\omega}'(k)t]]}(y), \end{aligned} \quad (49)$$

where the coefficient $\mathfrak{g}(k)$ is given by (42) and

$$p_+(k) := \left| 1 - \frac{\gamma v(k)}{2|\bar{\omega}'(k)|} \right|^2, \quad p_-(k) := \left(\frac{\gamma |v(k)|}{2|\bar{\omega}'(k)|} \right)^2. \quad (50)$$

By a direct inspection we can verify that $W(t, y, k)$, given by (49), solves the transport equation

$$\partial_t W(t, y, k) + \bar{\omega}'(k) \partial_y W(t, y, k) = 0, \quad y \neq 0, \quad (51)$$

with the transmission/reflection and phonon creation boundary condition at $y = 0$:

$$W(t, 0^+, k) = p_-(k)W(t, 0^+, -k) + p_+(k)W(t, 0^-, k) + \mathfrak{g}(k)T, \quad \text{for } k \in \mathbb{T}_+ \quad (52)$$

$$W(t, 0^-, k) = p_-(k)W(t, 0^-, -k) + p_+(k)W(t, 0^+, k) + \mathfrak{g}(k)T, \quad \text{for } k \in \mathbb{T}_-.$$

In Section 7 below we show that

$$p_+(k) + p_-(k) + \mathfrak{g}(k) = 1. \quad (53)$$

Coefficients $p_+(k)$ and $p_-(k)$ can be interpreted therefore as the probabilities of phonon transmission and reflection, respectively. Since $p_+(k) + p_-(k) = 1 - \mathfrak{g}(k)$, the coefficient $\mathfrak{g}(k)$ is the phonon absorption probability at the interface.

4 Harmonic chain with bulk conservative noise in contact with Langevin thermostat

4.1 The model and the statement of the result

In [9] we consider a stochastically perturbed chain of harmonic oscillators thermostatted at a fixed temperature $T \geq 0$ at $x = 0$. Its dynamics is described by the system of Itô stochastic differential equations

$$\begin{aligned}
dq_x(t) &= p_x(t)dt, \quad x \in \mathbb{Z}, \\
dp_x(t) &= \left[-(\alpha \star q(t))_x - \frac{\varepsilon\gamma_0}{2}(\theta \star p(t))_x \right] dt \\
&+ \sqrt{\varepsilon\gamma_0} \sum_{k=-1,0,1} (Y_{x+k} p_x(t)) dw_{x+k}(t) + \left(-\gamma p_0(t)dt + \sqrt{2\gamma T} dw(t) \right) \delta_{0,x}.
\end{aligned} \tag{54}$$

Here the coupling constants $(\alpha_x)_{x \in \mathbb{Z}}$ are as in (5),

$$Y_x := (p_x - p_{x+1})\partial_{p_{x-1}} + (p_{x+1} - p_{x-1})\partial_{p_x} + (p_{x-1} - p_x)\partial_{p_{x+1}} \tag{55}$$

and $(w_x(t))_{t \geq 0}$, $x \in \mathbb{Z}$ with $(w(t))_{t \geq 0}$, are i.i.d. one dimensional independent Brownian motions. In addition,

$$\theta_x = \Delta\theta_x^{(0)} := \theta_{x+1}^{(0)} + \theta_{x-1}^{(0)} - 2\theta_x^{(0)}$$

with

$$\theta_x^{(0)} = \begin{cases} -4, & x = 0 \\ -1, & x = \pm 1 \\ 0, & \text{if otherwise.} \end{cases}$$

Parameters $\varepsilon\gamma_0 > 0$, γ describe the strength of the inter-particle and thermostat noises, respectively. In what follows we shall assume that $\varepsilon > 0$ is small, that corresponds to the low density hypothesis that results in atoms suffering finitely many "collisions" in a macroscopic unit of time (the Boltzmann-Grad limit). Although the noise considered here is continuous we believe that the results extend to other type of conservative noises, such as e.g. Poisson exchanges of velocities between nearest neighbor particles.

Since the vector field Y_x is orthogonal both to a sphere $p_{x-1}^2 + p_x^2 + p_{x+1}^2 \equiv \text{const}$ and plane $p_{x-1} + p_x + p_{x+1} \equiv \text{const}$, the inter-particle noise conserves locally the kinetic energy and momentum. Because these conservation laws are common also for chaotic hamiltonian system, this model has been used to understand energy transport in presence of momentum conservation, see [1] and references there.

The case without the Langevin thermostat, i.e. with $\gamma = 0$, was studied in [3], where it is proved that

$$W_\varepsilon(t, y, k) \xrightarrow{\varepsilon \rightarrow 0} W(t, y, k) \tag{56}$$

where $W(t, y, k)$ is the solution of the transport equation

$$\partial_t W(t, y, k) + \bar{\omega}'(k)\partial_y W(t, y, k) = \gamma_0 \int_{\mathbb{T}} R(k, k') (W(t, y, k') - W(t, y, k)) dk', \tag{57}$$

where

$$R(k, k') = 32 \sin^2(\pi k) \sin^2(\pi k') \{ \sin^2(\pi k) \cos^2(\pi k') + \sin^2(\pi k') \cos^2(\pi k) \}. \tag{58}$$

We have therefore

$$R(k) = \int_{\mathbb{T}} R(k, k') dk' = 4 \sin^2(\pi k)(1 + 3 \cos^2(\pi k)). \quad (59)$$

In [9] we have proved the following result.

Theorem 1 *Suppose that $\gamma_0, \gamma > 0$ and the initial data satisfies (18), Then,*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_0^{+\infty} dt \iint_{\mathbb{T}_{2/\varepsilon} \times \mathbb{T}} W_\varepsilon(t, y, k) G(t, y, k) dy dk \\ = \int_0^{+\infty} dt \iint_{\mathbb{R} \times \mathbb{T}} W(t, y, k) G(t, y, k) dy dk \end{aligned} \quad (60)$$

for any $G \in C_0^\infty([0, +\infty) \times \mathbb{R} \times \mathbb{T})$, where the limiting Wigner $W(t, y, k)$ function satisfies (57) for $(t, y, k) \in \mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{T}_*$, with the boundary conditions (52), at the interface $y = 0$.

4.2 A sketch of the proof of Theorem 1

Consider the wave function $\psi(t)$ that corresponds to the dynamics (54) via (8) and $\hat{\psi}(t)$ its Fourier transform. In contrast with the situation described in Section 3.2 (the case $\gamma_0 = 0$) we no longer have an explicit expression for the solution of the equation for $\hat{\psi}(t)$, see (24), so we cannot proceed by a direct calculation of the Wigner distributions as in Sections 3 and 8.

In order to close the dynamics of the Fourier-Wigner function, we shall need all the components of the full covariance tensor of the Fourier transform of the wave field. Define therefore the Wigner distribution tensor $\mathbf{W}_\varepsilon(t)$, as a 2×2 -matrix tensor, whose entries are distributions, given by their respective Fourier transforms

$$\widehat{\mathbf{W}}_\varepsilon(t, \eta, k) := \begin{bmatrix} \widehat{W}_{\varepsilon,+}(t, \eta, k) & \widehat{Y}_{\varepsilon,+}(t, \eta, k) \\ \widehat{Y}_{\varepsilon,-}(t, \eta, k) & \widehat{W}_{\varepsilon,-}(t, \eta, k) \end{bmatrix}, \quad (\eta, k) \in \mathbb{T}_{2/\varepsilon} \times \mathbb{T}, \quad (61)$$

with

$$\begin{aligned} \widehat{W}_{\varepsilon,+}(t, \eta, k) &:= \widehat{W}_\varepsilon(t, \eta, k) = \frac{\varepsilon}{2} \mathbb{E}_\varepsilon \left[\hat{\psi} \left(t/\varepsilon, k + \frac{\varepsilon\eta}{2} \right) \hat{\psi}^* \left(t/\varepsilon, k - \frac{\varepsilon\eta}{2} \right) \right], \\ \widehat{Y}_{\varepsilon,+}(t, \eta, k) &:= \frac{\varepsilon}{2} \mathbb{E}_\varepsilon \left[\hat{\psi} \left(t/\varepsilon, k + \frac{\varepsilon\eta}{2} \right) \hat{\psi} \left(t/\varepsilon, -k + \frac{\varepsilon\eta}{2} \right) \right], \\ \widehat{Y}_{\varepsilon,-}(t, \eta, k) &:= \widehat{Y}_{\varepsilon,+}^*(t, -\eta, k), \quad \widehat{W}_{\varepsilon,-}(t, \eta, k) := \widehat{W}_{\varepsilon,+}(t, \eta, -k). \end{aligned}$$

By a direct calculation we show that the following energy bound is satisfied, see Proposition 2.1 of [9]

$$\sup_{\varepsilon \in (0,1]} \frac{\varepsilon}{2} \mathbb{E}_\varepsilon \|\hat{\psi}(t/\varepsilon)\|_{L^2(\mathbb{T})}^2 \leq \sup_{\varepsilon \in (0,1]} \frac{\varepsilon}{2} \mathbb{E}_\varepsilon \|\hat{\psi}(0)\|_{L^2(\mathbb{T})}^2 + \gamma T t, \quad t \geq 0. \quad (62)$$

The above estimate implies in particular that

$$\sup_{\varepsilon \in (0,1]} \|W_{\varepsilon,+}\|_{L^\infty([0,\tau];\mathcal{A}')} < +\infty, \quad \text{for any } \tau > 0, \quad (63)$$

where \mathcal{A}' is the dual to \mathcal{A} - the Banach space obtained by the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ in the norm

$$\|G\|_{\mathcal{A}} := \int_{\mathbb{R}} \sup_{k \in \mathbb{T}} |\widehat{G}(\eta, k)| d\eta, \quad G \in \mathcal{S}(\mathbb{R} \times \mathbb{T}). \quad (64)$$

Similar estimates hold also for the remaining entries of $\mathbf{W}_\varepsilon(t)$.

In consequence $(\mathbf{W}_\varepsilon(\cdot))$ is sequentially \star -weakly compact in $L_{\text{loc}}^\infty([0, +\infty), \mathcal{A}')$ and the problem of proving its \star -weak convergence reduces to the limit identification.

4.2.1 The case of zero temperature at the thermostat

Suppose first that $T = 0$. We can treat then the microscopic dynamics (54) as a small (stochastic) perturbation of the purely deterministic dynamics when all the terms containing the noises $(w_x(t))$ and $(w(t))$ are omitted. Denote by $\hat{\phi}^{(\varepsilon)}(t, k)$ the Fourier transform of the wave function corresponding to the latter dynamics, cf (9). We consider then the respective Wigner distribution tensor $\mathbf{W}_\varepsilon^{\text{un}}(t, y, k)$ whose Fourier transform is given by an analogue of (61), where the wave function of the "true" (perturbed) dynamics $\hat{\psi}(t, k)$ is replaced by $\hat{\phi}(t, k)$, which corresponds to the deterministic dynamics:

$$\begin{aligned} dq_x(t) &= p_x(t)dt, & x \in \mathbb{Z}, \\ dp_x(t) &= \left[-(\alpha \star q(t))_x - \frac{\varepsilon\gamma_0}{2}(\theta \star p(t))_x \right] dt - \gamma p_0(t)\delta_{0,x}dt. \end{aligned} \quad (65)$$

Denote by $\mathcal{L}_{2,\varepsilon}$ the Hilbert space made of the 2×2 matrix valued distributions on $\mathbb{R} \times \mathbb{T}$, such that the Fourier transforms of their entries belong to $L^2(\mathbb{T}_{2/\varepsilon} \times \mathbb{T})$. The Hilbert norm on $\mathcal{L}_{2,\varepsilon}$ is defined in an obvious way using the L^2 norms of the Fourier transforms.

Using the equations for the microscopic dynamics of $\hat{\phi}(t, k)$ we conclude that the tensor $\mathbf{W}_\varepsilon^{\text{un}}(t)$ can be described by an $\mathcal{L}_{2,\varepsilon}$ strongly continuous semigroup $(\mathfrak{B}_\varepsilon^{\text{un}}(t))$, i.e.

$$\mathbf{W}_\varepsilon^{\text{un}}(t) = \mathfrak{B}_\varepsilon^{\text{un}}(t) \left(\mathbf{W}_\varepsilon^{\text{un}}(0) \right), \quad t \geq 0. \quad (66)$$

Using a very similar argument to that used in the case of $\gamma_0 = 0$ (remember no noise is present in the unperturbed dynamics) we can prove, see Theorem 5.7 of [9], that

Theorem 2 *Under the assumptions on the initial data made in(18) and (19), we have*

$$\lim_{\varepsilon \rightarrow 0^+} \langle G, \mathbf{W}_\varepsilon^{\text{un}}(t) \rangle = \langle G, \mathbf{W}^{\text{un}}(t) \rangle, \quad t \geq 0, G \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$$

where

$$\mathbf{W}^{\text{un}}(t, y, k) = \begin{bmatrix} W_+^{\text{un}}(t, y, k) & 0 \\ 0 & W_+^{\text{un}}(t, y, -k) \end{bmatrix}, \quad (y, k) \in \mathbb{R} \times \mathbb{T},$$

and

$$\partial_t W^{\text{un}}(t, y, k) + \bar{\omega}'(k) \partial_y W^{\text{un}}(t, y, k) = -\gamma_0 R(k) W^{\text{un}}(t, y, k), \quad (t, y, k) \in \mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{T}_*, \quad (67)$$

with the interface conditions (52) for $T = 0$.

Similarly to (51) equation (67) can be solved explicitly and we obtain $W^{\text{un}}(t) = \mathfrak{W}_t^{\text{un}}(W^{\text{un}}(0))$, where $W^{\text{un}}(t, y, k) = e^{-\gamma_0 R(k)t} \tilde{W}^{\text{un}}(t, y, k)$ and $\tilde{W}^{\text{un}}(t, y, k)$ is given by (49). Consider a semigroup defined by

$$\mathfrak{W}_t^{\text{un}}(W^{\text{un}}(0))(y, k) := W^{\text{un}}(t, y, k). \quad (68)$$

One can show that $(\mathfrak{W}_t^{\text{un}})_{t \geq 0}$ forms a strongly continuous semigroup of contractions on any $L^p(\mathbb{R} \times \mathbb{T})$, $1 \leq p < +\infty$.

We can use the semigroup $\mathfrak{W}_\varepsilon^{\text{un}}(t)$ to write a Duhamel type equation for

$$\mathbf{w}_\varepsilon(\lambda) = \begin{bmatrix} \widehat{w}_{\varepsilon,+}(\lambda, \eta, k) & \widehat{y}_{\varepsilon,+}(\lambda, \eta, k) \\ \widehat{y}_{\varepsilon,-}(\lambda, \eta, k) & \widehat{w}_{\varepsilon,-}(\lambda, \eta, k) \end{bmatrix} = \int_0^{+\infty} e^{-\lambda t} \mathbf{W}_\varepsilon(t) dt$$

- the Laplace transform of the Wigner tensor of (61) defined for $\text{Re } \lambda > \lambda_0$ and some sufficiently large $\lambda_0 > 0$. It reads

$$\mathbf{w}_\varepsilon(\lambda) = \mathfrak{W}_\varepsilon^{\text{un}}(\lambda) \mathbf{W}_\varepsilon(0) + \frac{\gamma_0}{2} \mathfrak{W}_\varepsilon^{\text{un}}(\lambda) \mathbf{v}_\varepsilon(\lambda), \quad \text{Re } \lambda > \lambda_0. \quad (69)$$

Here

$$\mathfrak{W}_\varepsilon^{\text{un}}(\lambda) := \int_0^{+\infty} e^{-\lambda t} \mathfrak{W}_\varepsilon^{\text{un}}(t) dt, \quad (70)$$

and $\mathbf{v}_\varepsilon(\lambda) := \mathfrak{R}_\varepsilon \mathbf{w}_\varepsilon(\lambda)$. The operator \mathfrak{R}_ε acts on 2×2 matrix valued \mathbf{w} whose entries belong to $\mathcal{L}_{2,\varepsilon}$ and whose Fourier transform (in y) is given by

$$\widehat{\mathbf{w}}(\eta, k) := \begin{bmatrix} \widehat{w}_+(\eta, k) & \widehat{y}_+(\eta, k) \\ \widehat{y}_-(\eta, k) & \widehat{w}_-(\eta, k) \end{bmatrix}, \quad (\eta, k) \in \mathbb{T}_{2/\varepsilon} \times \mathbb{T}$$

as follows. The Fourier transform $\widehat{\mathfrak{R}_\varepsilon \mathbf{w}}$ have entries of the form

$$\pm \int_{\mathbb{T}} r_0(k, k', \varepsilon \eta) [\widehat{w}_{\varepsilon,+}(\eta, k') + \widehat{w}_{\varepsilon,-}(\eta, k') - \widehat{y}_{\varepsilon,+}(\eta, k') - \widehat{y}_{\varepsilon,-}(\eta, k')] dk'.$$

Here $r_0(k, k', \varepsilon \eta)$ is a scattering kernel that satisfies $R(k, k') = r_0(k, k', 0) + r_0(k, -k', 0)$, with $R(k, k')$ given by (58). Suppose now that we test both sides of (69) against a 2×2 -matrix valued smooth function \mathbf{G} whose entries have compactly supported Fourier transforms in the y variable, say in the interval $[-K, K]$ for some $K > 0$. Denote by $(\mathfrak{W}_\varepsilon^{\text{un}}(\lambda))^*$ and $\mathcal{R}_\varepsilon^*$ the adjoints of the respective operators in $\mathcal{L}_{2,\varepsilon}$.

We already know that from any sequence $(\mathbf{w}_{\varepsilon_n}(\lambda))$, where $\varepsilon_n \rightarrow 0+$ we can choose a subsequence, that will be denoted by the same symbol, converging \star -weakly in \mathcal{A}' to some $\mathbf{w}(\lambda)$. Using Theorem 2 and the strong convergence of the sequence

$1_{[-K,K]}(\eta)\mathcal{R}_{\varepsilon_n}^*(\mathfrak{W}_{\varepsilon_n}^{\text{un}}(\lambda))^*\mathbf{G}$, $n \rightarrow +\infty$ in $L^2(\mathbb{R} \times \mathbb{T})$ we can prove the following, see Theorem 5.7 of [9].

Theorem 3 *Suppose that \mathbf{W} is the \star -weak limit of $(\mathbf{W}_{\varepsilon_n})$ in $(L^1([0, +\infty); \mathcal{A}))'$ for some sequence $\varepsilon_n \rightarrow 0+$. Then, it has to be of the form*

$$\mathbf{W}(t, y, k) = \begin{bmatrix} W(t, y, k) & 0 \\ 0 & W(t, y, -k) \end{bmatrix}, \quad (y, k) \in L^2(\mathbb{R} \times \mathbb{T}), \quad (71)$$

where $W(t, y, k)$ satisfies the equation

$$W(t) = \mathfrak{W}_t^{\text{un}}(W(0)) + \gamma_0 \int_0^t \mathfrak{W}_{t-s}^{\text{un}}(\mathcal{R}W_s) ds, \quad (72)$$

and $\mathcal{R} : L^2(\mathbb{R} \times \mathbb{T}) \rightarrow L^2(\mathbb{R} \times \mathbb{T})$ is given by

$$\mathcal{R}F(y, k) := \int_{\mathbb{T}} R(k, k') F(y, k') dk', \quad (y, k) \in \mathbb{R} \times \mathbb{T}, \quad F \in L^2(\mathbb{R} \times \mathbb{T}). \quad (73)$$

The convergence claimed in Theorem 1 is then a direct consequence of Theorems 2 and 3.

It turns out that the microscopic evolution of the Wigner transform given by (61) allows us to define a strongly continuous semigroup on $L^2(\mathbb{T}_{2/\varepsilon} \times \mathbb{T})$ by letting $\mathfrak{W}_\varepsilon(t)(\mathbf{W}_\varepsilon(0)) := \mathbf{W}_\varepsilon(t)$. The norms of the semigroups $L^2(\mathbb{T}_{2/\varepsilon} \times \mathbb{T})$ stay bounded with $\varepsilon \in (0, 1]$, see Corollary 4.2 of [9]. Using Theorems 2 and 3 we can show therefore that for any $\mathbf{G} \in L^1([0, +\infty), \mathcal{A})$ we have

$$\lim_{\varepsilon \rightarrow 0+} \int_0^{+\infty} \langle \mathfrak{W}_\varepsilon(t) \mathbf{W}_\varepsilon(0), \mathbf{G}(t) \rangle dt = \int_0^{+\infty} \langle \mathfrak{W}_t \mathbf{W}(0), \mathbf{G}(t) \rangle dt, \quad (74)$$

where $\mathfrak{W}_t \mathbf{W}(0) := \mathbf{W}(t)$ is given by (71).

4.2.2 The case of positive temperature at the thermostat

Finally we consider the case $T > 0$. Suppose that $\chi \in C_c^\infty(\mathbb{R})$ is an arbitrary real valued, even function satisfying

$$\chi(y) = \begin{cases} 1, & \text{for } |y| \leq 1/2, \\ 0, & \text{for } |y| \geq 1, \\ \text{belongs to } [0, 1], & \text{if otherwise.} \end{cases} \quad (75)$$

Then its Fourier transform $\widehat{\chi} \in \mathcal{S}(\mathbb{R})$. Let $\widehat{\chi}_\varepsilon \in C^\infty(\mathbb{T}_{2/\varepsilon})$ be given by

$$\widehat{\chi}_\varepsilon(\eta) := \sum_{n \in \mathbb{Z}} \widehat{\chi} \left(\eta + \frac{2n}{\varepsilon} \right), \quad \eta \in \mathbb{T}_{2/\varepsilon}.$$

and

$$\widehat{\mathbf{V}}_\varepsilon(t, \eta, k) := \widehat{\mathbf{W}}_\varepsilon(t, \eta, k) - T\widehat{\chi}_\varepsilon(\eta)\mathbf{I}_2,$$

where \mathbf{I}_2 is the 2×2 identity matrix. In fact, $\mathbf{V}_\varepsilon(t)$ is a solution of the equation

$$\mathbf{V}_\varepsilon(t) = \mathfrak{B}_\varepsilon(t)\mathbf{V}_\varepsilon(0) + \int_0^t \mathfrak{B}_\varepsilon(s)\left(\mathbf{F}_\varepsilon\right)ds, \quad (76)$$

where

$$\widehat{\mathbf{F}}_\varepsilon(\eta, k) := -\frac{iT\widehat{\chi}_\varepsilon(\eta)}{\varepsilon} \left[\omega\left(k + \frac{\varepsilon\eta}{2}\right) - \omega\left(k - \frac{\varepsilon\eta}{2}\right) \right] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Using the convergence of (74) we conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^{+\infty} \langle \mathbf{V}_\varepsilon(t), \mathbf{G}(t) \rangle dt = \int_0^{+\infty} \langle \mathbf{V}(t), \mathbf{G}(t) \rangle dt, \quad (77)$$

where

$$\mathbf{V}(t, y, k) = \begin{bmatrix} V(t, y, k) & 0 \\ 0 & V(t, y, -k) \end{bmatrix}$$

and

$$V(t, y, k) := W(t, y, k) + T \int_0^t \mathfrak{B}_s(\bar{\omega}'(k)\chi'(y))ds. \quad (78)$$

We can identify therefore $W(t, y, k)$ with the solution of the equation

$$W(t, y, k) := \mathfrak{B}_t(\widetilde{W}_0)(y, k) + \int_0^t \mathfrak{B}_s(F)(y, k)ds + T\chi(y), \quad (t, y, k) \in \bar{\mathbb{R}}_+ \times \mathbb{R} \times \mathbb{T}. \quad (79)$$

Here

$$F(y, k) := -T\bar{\omega}'(k)\chi'(y), \quad \widetilde{W}_0(y, k) := W_0(y, k) - T\chi(y). \quad (80)$$

This ends the proof of Theorem 1 for an arbitrary $T \geq 0$.

5 Diffusive and Superdiffusive limit from the kinetic equation with boundary thermostat

We are now interested in the space-time rescaling of the solution of the equation (57) with the boundary condition (52). We should distinguish here two cases:

- the *optical chain*, when $\omega(k) \sim k^2$ for small k ,
- the *acoustic chain*, when $\omega(k) \sim |k|$ for small k .

In the optical chain, the long-wave phonons (corresponding to small k) have a small velocity, consequently even if the bulk scattering rate is small ($R(k) \sim k^2$), they still have time to diffuse. In fact, all other phonons (i.e. those corresponding to other k) have their non-trivial contribution to the diffusive limit.

In the acoustic chain the long-wave phonons move with the speed that is bounded away from 0 and rarely scatter. Therefore, they are responsible for a superdiffusion of the Levy type arising in the macroscopic limit. In the superdiffusive time-scale all other phonons (corresponding to non-vanishing k) do not yet move, their contribution to the asymptotic limit is therefore negligible.

5.1 The optical chain: diffusive behavior.

This case was studied in [2]. The diffusive rescaling of the solution of (57) is defined by

$$W^\delta(t, y, k) = W(t/\delta^2, y/\delta, k), \quad (81)$$

with an initial condition that varying in the *macroscopic* space scale

$$W^\delta(0, y, k) = W_0(y, k). \quad (82)$$

We assume here that $W_0(y, k) = T + \tilde{W}_0(y, k)$, with $\tilde{W}_0 \in L^2(\mathbb{R} \times \mathbb{T})$. This rescaled solution solves

$$\partial_t W^\delta(t, y, k) + \frac{1}{\delta} \bar{\omega}'(k) \partial_y W^\delta(t, y, k) = \frac{\gamma_0}{\delta^2} \int_{\mathbb{T}} R(k, k') (W^\delta(t, y, k') - W^\delta(t, y, k)) dk', \quad (83)$$

with the boundary condition (52) in $y = 0$.

In [2] it is proven that, for any test function $\varphi(t, y, k) \in C_0^\infty([0, +\infty) \times \mathbb{R} \times \mathbb{T})$,

$$\lim_{\delta \rightarrow 0} \int_0^{+\infty} dt \iint_{\mathbb{R} \times \mathbb{T}} W^\delta(t, y, k) \varphi(t, y, k) dy dk = \int_0^{+\infty} dt \iint_{\mathbb{R} \times \mathbb{T}} \rho(t, y) \varphi(t, y, k) dy dk, \quad (84)$$

where $\rho(t, y)$ is the solution of the heat equation

$$\begin{aligned} \partial_t \rho(t, y) &= D \partial_y^2 \rho(t, y), & y \neq 0, \\ \rho(t, 0) &= T, & \forall t > 0, \\ \rho(0, y) &= \rho_0(y) := \int_{\mathbb{T}} W_0(y, k) dk. \end{aligned} \quad (85)$$

The diffusion coefficient is given by

$$D := \frac{1}{\gamma_0} \int_{\mathbb{T}} \frac{\bar{\omega}'(k)^2}{R(k)} dk. \quad (86)$$

Notice that, under the condition of the optical dispersion relation, $D < +\infty$. The proof in [2] follows a classical Hilbert expansion method, with a modification needed to account for the boundary condition.

Intuitively, the result can be explained in the following way: phonons of all frequencies behave diffusively, under the scaling they converge to Brownian motions

with diffusion D , that has continuous path. As they get close to the thermostat boundary, they cross it many times till they get absorbed with probability 1 in the macroscopic time scale. Consequently there is no (macroscopic) transmission of energy from one side to the other. Phonons are created with intensity T , and this explains the value at the boundary $y = 0$.

5.2 The acoustic chain: superdiffusive behavior

This limit was studied in [8], while the case without thermostat had been previously considered in [4]. In a one dimensional acoustic chain, long wave phonons (small k) move with finite velocities but still scatter very rarely. Consequently these long wave phonons on the microscopic scale move ballistically with some rare scattering of their velocities. Under the superdiffusive rescaling $\delta^{-3/2}t, \delta^{-1}y$ they converge to corresponding Levy processes, generated by the fractional laplacian $-|\Delta|^{3/4}$. The effect of the thermal boundary is more complex than in the diffusive case, as now the phonons have a positive probability to cross the boundary without absorption and jump at a macroscopic distance on the other side. This causes a particular boundary condition for the fractional laplacian at the interface $y = 0$, that we explain below. Let us define the fractional laplacian $-|\Delta|^{3/4}$, admitting an interface value T , with absorption g_0 , transmission p_+ and reflection p_- , as the L^2 closure of the singular integral operator

$$\begin{aligned} \Lambda_{3/4}F(y) = & \text{p.v.} \int_{yy'>0} q(y-y')[F(y')-F(y)]dy' \\ & + g_0[T-F(y)] \int_{yy'<0} q(y-y')dy' \\ & + p_- \int_{yy'<0} q(y-y')[F(-y')-F(y)]dy' \\ & + p_+ \int_{yy'<0} q(y-y')[F(y')-F(y)]dy', \quad y \neq 0, F \in C_0^\infty(\mathbb{R}), \end{aligned} \quad (87)$$

where, cf [12, Theorem 1.1 e)],

$$q(y) = \frac{c_{3/4}}{|y|^{5/2}}, \quad c_{3/4} = \frac{2^{3/2}\Gamma(5/4)}{\sqrt{\pi}|\Gamma(-3/4)|} = \frac{3}{2^{5/2}\sqrt{\pi}}. \quad (88)$$

The first integral appearing in the right hand side of (87) is understood in the principal value (p.v.) sense. The choice of constant $c_{3/4}$ is made in such a way that the "free" fractional laplacian, defined by the kernel $q(\cdot)$, coincides with the definition using the "usual" Fourier symbol, see [12, Theorem 1.1 a)]. To define $\Lambda_{3/4}F(0)$, note that, due to the fact that $g_0 > 0$, the finiteness of the second integral forces the condition $F(0) = T$ on any function belonging to the domain of the generator. We can define

$\Lambda_{3/4}F(0)$ using (87) for any continuous function that satisfies $F(0) = T$, for which the integrals appearing in the right hand side (without the principal value) converge.

Notice that in the case without thermal interface, $g_0 = 0, p_- = 0, p_+ = 1$, and we recover the usual "free" fractional laplacian on the real line. The absorption, transmission and reflection coefficients that arise here are given by

$$g_0 = \lim_{k \rightarrow 0} g(k), \quad p_{\pm} = \lim_{k \rightarrow 0} p_{\pm}(k). \quad (89)$$

For the nearest neighbor acoustic chain, with the dispersion relation $\omega(k) := \omega_a |\sin(\pi k)|$ (cf (111)) it turns out that, see (119),

$$p_+ = \left(\frac{\omega_a}{\omega_a + \gamma} \right)^2, \quad p_-(k) := \left(\frac{\gamma}{\omega_a + \gamma} \right)^2, \quad g_0 = \frac{2\gamma\omega_a}{(\omega_a + \gamma)^2}. \quad (90)$$

The rescaled solution of the kinetic equation, see (57), is defined now by

$$W^\delta(t, y, k) = W(t/\delta^{3/2}, y/\delta, k) \quad (91)$$

In [8] it is proven that for any $t > 0$ and $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{T})$

$$\lim_{\delta \rightarrow 0} \iint_{\mathbb{R} \times \mathbb{T}} W^\delta(t, y, k) \varphi(y, k) dy dk = \iint_{\mathbb{R} \times \mathbb{T}} \rho(t, y) \varphi(y, k) dy dk, \quad (92)$$

where $\rho(t, y)$ is the solution of

$$\partial_t \rho(t, y) = \hat{c} \Lambda_{3/4} \rho(t, y), \quad (93)$$

where

$$\hat{c} := \frac{\pi^2 \omega_a^{3/2}}{(2^5 \gamma_0)^{1/2}} \int_0^{+\infty} \frac{(1 - \cos \lambda) d\lambda}{\lambda^{5/2}} = \left(\frac{\pi^5 \omega_a^3}{6\gamma_0} \right)^{1/2}, \quad (94)$$

cf [6, formula 3.762, 1, p. 437]

The proof of (92), presented in [8], is based on the probabilistic representation of the phonon trajectory process associated with the kinetic equation (57). It is shown that superdiffusively scaled trajectories of the process converge in law to those of a Levy process, with corresponding probabilities to be absorbed, transmitted or reflected when crossing $y = 0$, with a creation in the same point (its generator is given by (87)).

Remark

Notice that the convergence in (92) holds for every time $t > 0$, while in the diffusive case it is only weakly in time (cf (84)). The explanation comes from different methods adopted in the respective proofs. The proof of (92) is of probabilistic nature, and uses the fact that the corresponding limiting transmitted/reflected/absorbed process jumps over the thermostat interface only finitely many times before being absorbed. On the other hand, the proof of (84) is analytic, and it would be difficult to establish, by a probabilistic method, a result for every time, since the corresponding Brownian motion crosses the thermostat infinitely many times before being absorbed by it.

6 Perspectives and open problems

6.1 Direct hydrodynamic limit

The results presented in the previous sections are obtained in the typical two-step procedure: we first take a kinetic limit (rarefied collisions) and obtain a kinetic equation with a boundary condition for the thermostat, next we rescale (diffusively or superdiffusively) this equation getting a diffusive or superdiffusive equation with an appropriate boundary condition.

It would be interesting to obtain a direct hydrodynamic limit, rescaling diffusively or superdiffusively the microscopic dynamics, without rarefaction of the random collision in the bulk. This means considering the evolution equations (54) with $\varepsilon = 1$, then setting a scale parameter δ (that does not appear in the evolution equations) and define the Wigner distribution by

$$\langle G, W^{(\delta)}(t) \rangle := \frac{\delta}{2} \sum_{y, y' \in \mathbb{Z}} \int_{\mathbb{T}} e^{2\pi i k(y' - y)} \mathbb{E} \left[\psi_y \left(\frac{t}{\delta^\alpha} \right) (\psi_{y'})^* \left(\frac{t}{\delta^\alpha} \right) \right] G^* \left(\delta \frac{y + y'}{2}, k \right) dk. \quad (95)$$

with $\alpha = 2$, or $\alpha = 3/2$ in the diffusive, or superdiffusive case, respectively. Then one would like to show that, in some sense,

$$W^{(\delta)}(t, y, k) \xrightarrow{\delta \rightarrow 0} \rho(t, y), \quad (96)$$

where $\rho(t, y)$ is solution of (85) or (93), depending on the scaling. In absence of a thermostat, this has been proved in [5].

6.2 More thermostats

In non-equilibrium statistical mechanics it is always interesting to put the system in contact with a number of heat baths at various temperatures. If, in the case of dynamics defined by (6) or by (54), we add another Langevin thermostat at the site $[\varepsilon^{-1}y_0]$ with $y_0 \neq 0$, at a temperature T_1 , we expect to obtain the same kinetic equations with added boundary conditions at the point y_0 analogous to (52) but of course the phonon production rate $g(k)T_1$. The difficulty in constructing the proof, lies in the fact that we no longer have an explicit formula for a solution in the case the inter-particle scattering is absent, that has been quite essential in our argument.

6.3 Poisson thermostat

A different model for a heat bath at temperature T is given by a renewal of the velocity $p_0(t)$ at random times given by a Poisson process of intensity γ : each time the Poisson clock rings, the velocity is renewed with value chosen with a Gaussian distribution of variance T , independently of anything else. This mechanism represents the interaction with an infinitely extended reservoir of independent particles in equilibrium at temperature T and uniform density.

From a preliminary calculation (cf [10]) it seems that the scattering rates in the high frequency-kinetic limit are different, implying their dependence on the microscopic model of the thermostat. Obviously, in the hydrodynamic limit, diffusive or superdiffusive, we expect that these boundary conditions will not depend anymore on the microscopic model of the thermostat.

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7 Appendix A: Properties of the scattering coefficients

7.1 Some properties of the scattering coefficient for a unimodal dispersion relation

Recall that $v(k)$ is defined by (39). From (30), we have

$$\lim_{\varepsilon \rightarrow 0} \tilde{J}(\varepsilon - i\omega(k)) = iG(\omega(k)) + iH(\omega(k)), \quad \text{for } \omega(k) \neq 0$$

where

$$G(u) := \int_{\mathbb{T}_+} \frac{d\ell}{u + \omega(\ell)}, \quad H(u) := \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}} \frac{d\ell}{i\varepsilon + u - \omega(\ell)}. \quad (97)$$

If $\omega(k) = 0$, then $k = 0$ and, according to (30),

$$\lim_{\varepsilon \rightarrow 0} \tilde{J}(\varepsilon) = \frac{\pi}{|\omega'(0+)|}. \quad (98)$$

If the dispersion relation $\omega(k)$ is unimodal and $\omega_{\min} := \omega(0)$, $\omega_{\max} := \omega(1/2)$, then we can write $H(\omega(k)) = H^r(\omega(k)) + iH^i(\omega(k))$, with $H^r(u)$, $H^i(u)$ real valued functions equal

$$H^r(u) := \lim_{\varepsilon \rightarrow 0} \int_{\omega_{\min}}^{\omega_{\max}} \frac{(u - v)dv}{|\omega'(\omega_+^{-1}(v))|[\varepsilon^2 + (u - v)^2]} \quad (99)$$

and

$$H^i(u) := - \lim_{\varepsilon \rightarrow 0} \int_{\omega_{\min}}^{\omega_{\max}} \frac{\varepsilon dv}{|\omega'(\omega_+^{-1}(v))|[\varepsilon^2 + (u-v)^2]} = - \frac{\pi}{|\omega'(\omega_+^{-1}(u))|}. \quad (100)$$

Here $\omega_+^{-1} : [\omega_{\min}, \omega_{\max}] \rightarrow [0, 1/2]$ is the inverse of the increasing branch of $\omega(\cdot)$. For $u \in (\omega_{\min}, \omega_{\max})$ we can write

$$H^r(u) = \frac{1}{\omega'(\omega_+^{-1}(u))} \log \frac{\omega_{\max} - u}{u - \omega_{\min}} + \int_{\omega_{\min}}^{\omega_{\max}} \frac{[(\omega_+^{-1})'(v) - (\omega_+^{-1})'(u)] dv}{u - v}. \quad (101)$$

According to (29) and (39)

$$v(k) = \begin{cases} \{1 - \gamma H^i(\omega(k)) + i\gamma[G(\omega(k)) + H^r(\omega(k))]\}^{-1}, & \text{if } \omega(k) \neq 0, \\ \frac{2|\bar{\omega}'(0+)|}{2|\bar{\omega}'(0+)| + \gamma}, & \text{if } \omega(k) = 0. \end{cases} \quad (102)$$

Summarizing, from the above argument we conclude the following.

Theorem 4 For a unimodal dispersion relation $\omega(\cdot)$ the following are true:

i) we have

$$|v(k)| \leq \frac{2|\bar{\omega}'(k)|}{\gamma + 2|\bar{\omega}'(k)|}, \quad k \in \mathbb{T}, \quad (103)$$

ii) if k_* is such that $\omega'(k_*) = 0$, then

$$\lim_{k \rightarrow k_*} v(k) = 0 \quad \text{and} \quad \lim_{k \rightarrow k_*} g(k) = 0, \quad (104)$$

iii)

$$\operatorname{Re} v(k) > 0, \quad \text{for all } k \in \mathbb{T} \setminus \{0, 1/2\}, \quad (105)$$

iv)

$$p_+(k) > 0 \quad \text{and} \quad p_-(k) < 1 \quad \text{for all } k \text{ such that } \omega'(k) \neq 0 \quad (106)$$

and

$$p_-(k) > 0 \quad \text{for all } k \in \mathbb{T} \setminus \{0, 1/2\}, \quad (107)$$

v) we have the formula

$$\operatorname{Re} v(k) = \left(1 + \frac{\gamma}{2|\bar{\omega}'(k)|}\right) |v(k)|^2, \quad k \in \mathbb{T}. \quad (108)$$

Proof Substituting into (102) from (99) and (100) immediately yields (108). Estimate (105) follows directly from (102), formulas (97), (100) and (101).

Statement (109) is a consequence of (100) and (102). Part ii) follows from part i), cf (42). Estimates (106) follow directly from (109), while (107) is a straightforward consequence of part iii), cf (50). \square

From part v) of Theorem 4 we immediately conclude the following.

Corollary 1 Suppose that $v(k) \neq 0$ is real valued. Then,

$$v(k) = \frac{|\omega'(k)|}{|\omega'(k)| + \gamma\pi}. \quad (109)$$

7.2 Proof of (53)

To conclude (53) we invoke (50). Then, thanks to (108), we can write

$$p_+(k) + p_-(k) + g(k) = 1 + \frac{\gamma}{|\bar{\omega}'(k)|} \left[|v(k)|^2 \left(1 + \frac{\gamma}{2|\bar{\omega}'(k)|} \right) - \operatorname{Re} v(k) \right] = 1$$

and (53) follows.

7.3 An example: scattering coefficient $v(k)$ for a nearest neighbor interaction harmonic chain - computation of $\tilde{J}(\lambda)$ using contour integration

Assume that $\omega(k)$ is the dispersion relation of a nearest neighbor interaction harmonic chain. We let $\alpha_0 := (\omega_0^2 + \omega_a^2)/2$ and $\alpha_{\pm 1} := -\omega_a^2/4$, and $\omega_0 \geq 0$, $\omega_a > 0$. Then, see Section 3,

$$\hat{\alpha}(k) = \frac{\omega_0^2 + \omega_a^2}{2} - \frac{\omega_a^2}{4} \left(e^{2\pi i k} + e^{-2\pi i k} \right) = \frac{\omega_0^2}{2} + \omega_a^2 \sin^2(\pi k) \quad (110)$$

and, according to (7),

$$\omega(k) := \sqrt{\frac{\omega_0^2}{2} + \omega_a^2 \sin^2(\pi k)}. \quad (111)$$

Using the definition of $\tilde{J}(\lambda)$, see (30), and (110) for any $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > 0$ we can write

$$\tilde{J}(\lambda) = \int_{\mathbb{T}} \frac{\lambda d\ell}{\lambda^2 + \hat{\alpha}(\ell)} = -\frac{4\lambda}{\omega_a^2} \int_{-1/2}^{1/2} \frac{e^{2\pi i \ell} d\ell}{e^{4\pi i \ell} - 2W(\lambda)e^{2\pi i \ell} + 1}, \quad (112)$$

where

$$W(\lambda) = 1 + \left(\frac{\omega_0}{\omega_a} \right)^2 + 2 \left(\frac{\lambda}{\omega_a} \right)^2. \quad (113)$$

Note that $W(\lambda) \in \mathbb{C} \setminus [-1, 1]$, if $\operatorname{Re} \lambda > 0$.

The expression for $\tilde{J}(\lambda)$ can be rewritten using the contour integral over the unit circle $C(1)$ on the complex plane oriented counterclockwise and

$$\tilde{J}(\lambda) = -\frac{2\lambda}{i\pi\omega_a^2} \int_{C(1)} \frac{d\zeta}{\zeta^2 - 2W(\lambda)\zeta + 1}. \quad (114)$$

When $w \in \mathbb{C} \setminus [-1, 1]$ the equation

$$z^2 - 2wz + 1 = 0$$

has two roots. They are given by Φ_+, Φ_- , holomorphic functions on $\mathbb{C} \setminus [-1, 1]$, that are the inverse branches of the Joukowski function $\mathfrak{J}(z) = 1/2(z + z^{-1})$, $z \in \mathbb{C}$ taking values in $\bar{\mathbb{D}}^c$ and \mathbb{D} , respectively. Here $\mathbb{D} := [z \in \mathbb{C} : |z| < 1]$ is the unit disc. We have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{2} (\Phi_+(a - \varepsilon i) - \Phi_-(a - \varepsilon i)) = -i\sqrt{1 - a^2}, \quad \text{for } a \in [-1, 1]. \quad (115)$$

Using the Cauchy formula for contour integrals, from (114) we obtain

$$\tilde{J}(\lambda) = \frac{2\lambda}{\omega_a^2 (\Phi_+(W(\lambda)) - \Phi_-(W(\lambda)))}. \quad (116)$$

For the dispersion relation $\omega(k)$ given by (111) and $\varepsilon > 0$ we have, cf (113),

$$W(\varepsilon - i\omega(k)) = \cos(2\pi k) + 2 \left(\frac{\varepsilon}{\omega_a} \right)^2 - 4i \frac{\omega(k)\varepsilon}{\omega_a^2}.$$

As a result we get, cf (115) and (116),

$$\lim_{\varepsilon \rightarrow 0^+} \tilde{J}(\varepsilon - i\omega(k)) = \frac{2}{\omega_a^2 \sin(2\pi|k|)} \sqrt{\frac{\omega_0^2}{2} + \omega_a^2 \sin^2(\pi k)}$$

and the following result holds.

Theorem 5 *For the dispersion relation given by (111) we have*

$$v(k) = \omega_a^2 \sin(2\pi|k|) \left\{ \omega_a^2 \sin(2\pi|k|) + 2\gamma \sqrt{\frac{\omega_0^2}{2} + \omega_a^2 \sin^2(\pi k)} \right\}^{-1}, \quad k \in \mathbb{T}. \quad (117)$$

In particular, if $\omega_0 = 0$ (the acoustic case) we have, cf (42) and (50),

$$v(k) = \frac{\omega_a \cos(\pi k)}{\omega_a \cos(\pi k) + \gamma}, \quad k \in \mathbb{T} \quad (118)$$

and

$$\begin{aligned}
p_+(k) &:= \left(\frac{\omega_a \cos(\pi k)}{\omega_a \cos(\pi k) + \gamma} \right)^2, & p_-(k) &:= \left(\frac{\gamma}{\omega_a \cos(\pi k) + \gamma} \right)^2 \\
g(k) &= \frac{2\gamma\omega_a \cos(\pi k)}{(\omega_a \cos(\pi k) + \gamma)^2}, & k &\in \mathbb{T}.
\end{aligned} \tag{119}$$

8 Appendix B: proofs of (47) and (48)

8.1 Proof of (47)

Using (27) and (12) we can write

$$\widehat{W}_\varepsilon^1(t, \eta, k) = -\frac{\gamma}{2} \left\{ \mathcal{I} \left(\frac{t}{\varepsilon}, \eta, k \right) + \mathcal{I}^* \left(\frac{t}{\varepsilon}, -\eta, k \right) \right\}, \tag{120}$$

where

$$\mathcal{I}(t, \eta, k) := \frac{\varepsilon}{2} \int_{\mathbb{T}} dk' \mathbb{E}_\varepsilon \left[\widehat{\psi} \left(0, k - \frac{\varepsilon\eta}{2} \right)^* \widehat{\psi} (0, k') \right] \int_0^t e^{i[\omega(k - \frac{\varepsilon\eta}{2})t - \omega(k')s]} \phi \left(t - s, k + \frac{\varepsilon\eta}{2} \right) ds. \tag{121}$$

The Laplace transform of $\mathcal{I} \left(\frac{t}{\varepsilon}, \eta, k \right)$ equals

$$\begin{aligned}
\tilde{\mathcal{I}}_\varepsilon(\lambda, \eta, k) &:= \frac{\varepsilon^2}{2} \int_0^{+\infty} e^{-\varepsilon\lambda t} \mathcal{I}(t, \eta, k) dt \\
&= \frac{\varepsilon^2}{2} \int_0^{+\infty} e^{i\omega(k + \frac{\varepsilon\eta}{2})\tau} g(d\tau) \int_\tau^{+\infty} e^{i[\omega(k') - \omega(k + \frac{\varepsilon\eta}{2})]s} ds \int_s^{+\infty} e^{-\{\varepsilon\lambda + i[\omega(k') - \omega(k - \frac{\varepsilon\eta}{2})]\}t} dt \\
&\times \int_{\mathbb{T}} dk' \mathbb{E}_\varepsilon \left[\widehat{\psi} \left(0, k - \frac{\varepsilon\eta}{2} \right)^* \widehat{\psi} (0, k') \right].
\end{aligned} \tag{122}$$

Performing the integration over the temporal variables we conclude that

$$\begin{aligned}
\tilde{\mathcal{I}}_\varepsilon(\lambda, \eta, k) &= \int_{\mathbb{T}} \mathbb{E}_\varepsilon \left[\widehat{\psi} \left(0, k - \frac{\varepsilon\eta}{2} \right)^* \widehat{\psi} (0, k') \right] \\
&\times \frac{\tilde{g}(\lambda\varepsilon - i\omega(k - \frac{\varepsilon\eta}{2})) dk'}{2\{\lambda + i\varepsilon^{-1}[\omega(k - \frac{\varepsilon\eta}{2}) - \omega(k')]\}\{\lambda + i\varepsilon^{-1}[\omega(k + \frac{\varepsilon\eta}{2}) - \omega(k - \frac{\varepsilon\eta}{2})]\}}.
\end{aligned} \tag{123}$$

Using (39) we conclude that for any test function $G \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$

$$\begin{aligned}
& \int_{\mathbb{R} \times \mathbb{T}} \tilde{I}_\varepsilon(\lambda, \eta, k) \hat{G}^*(\eta, k) d\eta dk \\
& \approx \int_{\mathbb{R} \times \mathbb{T}^2} \frac{\mathbb{E}_\varepsilon \left[\hat{\psi}(0, k)^* \hat{\psi}(0, k') \right] \hat{G}^*(\eta, k) v(k) d\eta dk dk'}{2\{\lambda + i\varepsilon^{-1}[\omega(k') - \omega(k)]\} \{\lambda + i\varepsilon^{-1}[\omega(k + \varepsilon\eta) - \omega(k)]\}},
\end{aligned} \tag{124}$$

as $\varepsilon \ll 1$. Changing variables $k := \ell - \varepsilon\eta'/2$ and $k' := \ell + \varepsilon\eta'/2$ we obtain that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R} \times \mathbb{T}} \tilde{I}_\varepsilon(\lambda, \eta, k) \hat{G}^*(\eta, k) d\eta dk \\
& = \int_{\mathbb{R}^2 \times \mathbb{T}} \frac{\widehat{W}(0, \eta', \ell) \hat{G}^*(\eta, \ell) v(\ell) d\eta d\eta' d\ell}{(\lambda + i\omega'(\ell)\eta')(\lambda + i\omega'(\ell)\eta)}.
\end{aligned} \tag{125}$$

The limit of $\widehat{w}_\varepsilon^1(\lambda, \eta, k)$ - the Laplace transform of $\widehat{W}_\varepsilon^1(t, \eta, k)$ - is therefore given by

$$\widehat{w}^1(\lambda, \eta, k) = -\frac{\gamma \operatorname{Re} v(k)}{\lambda + i\omega'(k)\eta} \int_{\mathbb{R}} \frac{\widehat{W}(0, \eta', k) d\eta'}{\lambda + i\omega'(k)\eta'} \tag{126}$$

therefore

$$\lim_{\varepsilon \rightarrow 0^+} \widehat{W}_\varepsilon^1(t, \eta, k) = -\gamma \operatorname{Re} v(k) e^{-i\omega'(k)t} \int_{\mathbb{R}} \frac{1 - e^{-i\omega'(k)(\eta' - \eta)t}}{i\omega'(k)(\eta' - \eta)} \widehat{W}(0, \eta', k) d\eta' \tag{127}$$

and, performing the inverse Fourier transform, (47) follows.

8.2 Proof of (48)

Concerning the term $\widehat{W}_\varepsilon^2(t, \eta, k)$, from the third formula of (46), (12) and (27) we obtain

$$\widehat{W}_\varepsilon^2(t, \eta, k) = \frac{\gamma^2}{2} \left\{ \mathcal{J} \left(\frac{t}{\varepsilon}, \eta, k \right) + \mathcal{R} \left(\frac{t}{\varepsilon}, \eta, k \right) \right\}, \tag{128}$$

where

$$\begin{aligned}
\mathcal{J}(t, \eta, k) & := \frac{\varepsilon}{4} \int_{[0, t]^2} ds ds' \int_{\mathbb{T}^2} d\ell d\ell' \phi \left(t - s, k - \frac{\varepsilon\eta}{2} \right)^* \phi \left(t - s', k + \frac{\varepsilon\eta}{2} \right) \\
& \times e^{i[\omega(\ell)s - \omega(\ell')s']} \mathbb{E}_\varepsilon \left[\hat{\psi}(0, \ell)^* \hat{\psi}(0, \ell') \right], \\
\mathcal{R}(t, \eta, k) & := \frac{\varepsilon}{4} \int_{[0, t]^2} ds ds' \int_{\mathbb{T}^2} d\ell d\ell' \phi \left(t - s, k - \frac{\varepsilon\eta}{2} \right)^* \phi \left(t - s', k + \frac{\varepsilon\eta}{2} \right) \\
& \times e^{-i[\omega(\ell)s - \omega(\ell')s']} \mathbb{E}_\varepsilon \left[\hat{\psi}(0, \ell) \hat{\psi}(0, \ell')^* \right].
\end{aligned} \tag{129}$$

A simple computation shows that

$$\begin{aligned} \mathcal{J}(t, \eta, k) &= \frac{\varepsilon}{4} \int_{[0,t]^2} ds ds' \int_0^s g(d\tau)g(d\tau') \int_{\mathbb{T}^2} d\ell d\ell' \mathbb{E}_\varepsilon [\hat{\psi}(0, \ell)^* \hat{\psi}(0, \ell')] \\ &\times e^{i[\omega(k - \varepsilon\eta/2)(s - \tau) - \omega(k + \varepsilon\eta/2)(s' - \tau')]} e^{-i[\omega(\ell')(t - s') - \omega(\ell)(t - s)]}. \end{aligned} \quad (130)$$

The respective Laplace transform equals

$$\begin{aligned} \tilde{\mathcal{J}}_\varepsilon(\lambda, \eta, k) &:= \varepsilon \int_0^{+\infty} e^{-\varepsilon\lambda\tau_0} \mathcal{J}(\tau_0, \eta, k) d\tau_0 \\ &= \varepsilon \int_0^{+\infty} \int_0^{+\infty} \delta(\tau_0 - \tau'_0) e^{-\varepsilon\lambda\tau_0/2} e^{-\varepsilon\lambda\tau'_0/2} \mathcal{J}(\tau_0, \eta, k) \mathcal{J}(\tau'_0, \eta, k) d\tau_0 d\tau'_0 \\ &= \frac{\varepsilon^2}{4} \int_{\bar{\mathbb{R}}_+^4 \times \bar{\mathbb{R}}_+^4} d\tau_{0,2} g(d\tau_3) d\tau'_{0,2} g(d\tau'_3) \int_{\mathbb{T}^2} d\ell d\ell' \\ &\times \delta(\tau_0 - \tau'_0) \delta\left(\tau_0 - \sum_{j=1}^3 \tau_j\right) \delta\left(\tau'_0 - \sum_{j=1}^3 \tau'_j\right) e^{-\varepsilon\lambda \sum_{j=0}^3 \tau_j/4} e^{-\varepsilon\lambda \sum_{j=0}^3 \tau'_j/4} \\ &\times e^{i[\omega(k - \varepsilon\eta/2)\tau_2 + \omega(\ell)\tau_1]} e^{-i[\omega(k + \varepsilon\eta/2)\tau'_2 + \omega(\ell')\tau'_1]} \mathbb{E}_\varepsilon [\hat{\psi}(0, \ell)^* \hat{\psi}(0, \ell')]. \end{aligned} \quad (131)$$

Here, for abbreviation sake we write $d\tau_{0,3} = d\tau_0 d\tau_1 d\tau_2$ and likewise for the prime variables. Using the identity $\delta(t) = (2\pi)^{-1} \int_{\mathbb{R}} e^{i\beta t} d\beta$ and integrating out the τ and τ' variables we obtain

$$\begin{aligned} \tilde{\mathcal{J}}_\varepsilon(\lambda, \eta, k) &= \frac{\varepsilon^2}{2^5 \pi^3} \int_{\mathbb{R}^3} d\beta_0 d\beta_1 d\beta'_1 \int_{\mathbb{T}^2} d\ell d\ell' \mathbb{E}_\varepsilon [\hat{\psi}(0, \ell)^* \hat{\psi}(0, \ell')] \\ &\times \frac{\tilde{g}(\varepsilon\lambda/4 + i\beta_1)}{[\varepsilon\lambda/4 - i(\beta_0 + \beta_1)][\varepsilon\lambda/4 + i(\beta_1 - \omega(\ell))][\varepsilon\lambda/4 + i(\beta_1 - \omega(k - \varepsilon\eta/2))]} \\ &\times \frac{\tilde{g}(\varepsilon\lambda/4 + i\beta'_1)}{[\varepsilon\lambda/4 + i(\beta_0 - \beta'_1)][\varepsilon\lambda/4 + i(\beta'_1 + \omega(\ell'))][\varepsilon\lambda/4 + i(\beta'_1 + \omega(k + \varepsilon\eta/2))]} \end{aligned} \quad (132)$$

We integrate β_1 and β'_1 variables using the Cauchy integral formula

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(i\beta) d\beta}{z - i\beta} = f(z), \quad z \in \mathbb{H}, \quad (133)$$

valid for any holomorphic function f on the right half-plane $\mathbb{H} := [z \in \mathbb{C} : \text{Re } z > 0]$ that belongs to the Hardy class $H^p(\mathbb{H})$ for some $p \geq 1$, see e.g. [11, p. 113]. Performing the above integration and, subsequently, changing variables $\varepsilon\beta'_0 := \beta_0 + \omega(k - \varepsilon\eta/2)$ we get

$$\begin{aligned}
\tilde{\mathcal{J}}_\varepsilon(\lambda, \eta, k) &= \frac{1}{2^3 \pi \varepsilon} \int_{\mathbb{R}} \frac{d\beta_0}{\lambda/2 - i\beta_0} \int_{\mathbb{T}^2} d\ell d\ell' \mathbb{E}_\varepsilon [\hat{\psi}(0, \ell)^* \hat{\psi}(0, \ell')] \\
&\times \frac{|\tilde{g}(\varepsilon\lambda/2 - i\varepsilon\beta_0 + i\omega(k - \varepsilon\eta/2))|^2}{\lambda/2 - i\varepsilon^{-1}(\omega(\ell) - \omega(k - \varepsilon\eta/2)) - i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(\ell') - \omega(k - \varepsilon\eta/2)) + i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(k + \varepsilon\eta/2) - \omega(k - \varepsilon\eta/2)) + i\beta_0}.
\end{aligned} \tag{134}$$

Change variables ℓ, ℓ' according to the formulas $\ell := k' - \varepsilon\eta'/2$ and $\ell' := k' + \varepsilon\eta'/2$ and use (cf (39))

$$|\tilde{g}(\varepsilon\lambda/2 - i\varepsilon\beta_0 + i\omega(k - \varepsilon\eta/2))|^2 \approx |v(k)|^2, \quad \text{as } \varepsilon \ll 1.$$

We obtain then

$$\begin{aligned}
\tilde{\mathcal{J}}_\varepsilon(\lambda, \eta, k) &\approx \frac{|v(k)|^2}{2^3 \pi \varepsilon} \int_{\mathbb{R}} \frac{d\beta_0}{\lambda/2 - i\beta_0} \int_{\mathbb{R} \times \mathbb{T}} \widehat{W}(0, \eta', k') d\eta' dk' \\
&\times \frac{1}{\lambda/2 - i\varepsilon^{-1}(\omega(k' - \varepsilon\eta'/2) - \omega(k - \varepsilon\eta/2)) - i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(k' + \varepsilon\eta'/2) - \omega(k - \varepsilon\eta/2)) + i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(k + \varepsilon\eta/2) - \omega(k - \varepsilon\eta/2)) + i\beta_0}.
\end{aligned} \tag{135}$$

Since $\omega(k)$ is unimodal we can write

$$\begin{aligned}
\tilde{\mathcal{J}}_\varepsilon(\lambda, \eta, k) &\approx \frac{|v(k)|^2}{2^3 \pi \varepsilon} \sum_{i=\pm} \int_{\mathbb{R}} \frac{d\beta_0}{\lambda/2 - i\beta_0} \int_{\mathbb{R} \times [ik - \delta, ik + \delta]} \widehat{W}(0, \eta', k') d\eta' dk' \\
&\times \frac{1}{\lambda/2 - i\varepsilon^{-1}(\omega(k' - \varepsilon\eta'/2) - \omega(k - \varepsilon\eta/2)) - i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(k' + \varepsilon\eta'/2) - \omega(k - \varepsilon\eta/2)) + i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(k + \varepsilon\eta/2) - \omega(k - \varepsilon\eta/2)) + i\beta_0}.
\end{aligned} \tag{136}$$

for a (small) fixed $\delta > 0$. Changing variables $k' = k + \varepsilon\eta''$ and using the approximations $\varepsilon^{-1}[\omega(k + \varepsilon\xi) - \omega(k)] \approx \omega'(k)\xi$ and $\widehat{W}(0, \eta', ik + \varepsilon\eta'') \approx \widehat{W}(0, \eta', ik)$ we conclude that

$$\begin{aligned}
\tilde{\mathcal{J}}_\varepsilon(\lambda, \eta, k) &\approx \frac{|v(k)|^2}{2^3 \pi} \sum_{\iota=\pm} \int_{\mathbb{R}} \frac{d\beta_0}{\lambda/2 - i\beta_0} \int_{\mathbb{R}^2} \widehat{W}(0, \eta', \iota k + \varepsilon \eta'') d\eta' d\eta'' \\
&\times \frac{1}{\lambda/2 - i\varepsilon^{-1}(\omega(\iota(k + \varepsilon \eta'') - \varepsilon \eta'/2) - \omega(k - \varepsilon \eta/2)) - i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(\iota(k + \varepsilon \eta'') + \varepsilon \eta'/2) - \omega(k - \varepsilon \eta/2)) + i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(k + \varepsilon \eta/2) - \omega(k - \varepsilon \eta/2)) + i\beta_0} \\
&\approx \frac{|v(k)|^2}{2^3 \pi} \sum_{\iota=\pm} \int_{\mathbb{R}} \frac{d\beta_0}{(\lambda/2 - i\beta_0)(\lambda/2 + i\omega'(k)\eta + i\beta_0)} \int_{\mathbb{R}} \widehat{W}(0, \eta', \iota k) d\eta' \\
&\times \int_{\mathbb{R}} \frac{1}{\lambda/2 - i\omega'(k)(\eta'' + \eta/2 - \iota \eta'/2) - i\beta_0} \\
&\times \frac{d\eta''}{\lambda/2 + i\omega'(k)(\eta'' + \eta/2 + \iota \eta'/2) + i\beta_0}.
\end{aligned} \tag{137}$$

Integrating, first with respect to η'' and then β_0 variables, using e.g. (133), we get

$$\lim_{\varepsilon \rightarrow 0^+} \tilde{\mathcal{J}}_\varepsilon(\lambda, \eta, k) = \frac{|v(k)|^2}{4|\bar{\omega}'(k)|} \sum_{\iota=\pm} \int_{\mathbb{R}} \frac{d\eta}{\lambda + i\omega'(k)\eta} \int_{\mathbb{R}} \frac{\widehat{W}(0, \eta', \iota k) d\eta'}{\lambda + i\omega'(k)\eta'} \tag{138}$$

From the second equality of (129) we can see that formula for $\tilde{\mathcal{R}}_\varepsilon(\lambda, \eta, k)$ can be obtained from (135) by changing $\omega(\ell)$ and $\omega(\ell')$ to $-\omega(\ell)$ and $-\omega(\ell')$ respectively and altering the complex conjugation by the wave functions. It yields

$$\begin{aligned}
\tilde{\mathcal{R}}_\varepsilon(\lambda, \eta, k) &\approx \frac{|v(k)|^2}{2^3 \pi \varepsilon} \sum_{\iota=\pm} \int_{\mathbb{R}} \frac{d\beta_0}{\lambda/2 - i\beta_0} \int_{\mathbb{R} \times [\iota k - \delta, \iota k + \delta]} \widehat{W}(0, \eta', k') d\eta' dk' \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(k' - \varepsilon \eta'/2) + \omega(k - \varepsilon \eta/2)) - i\beta_0} \\
&\times \frac{1}{\lambda/2 - i\varepsilon^{-1}(\omega(k' + \varepsilon \eta'/2) + \omega(k - \varepsilon \eta/2)) + i\beta_0} \\
&\times \frac{1}{\lambda/2 + i\varepsilon^{-1}(\omega(k + \varepsilon \eta/2) - \omega(k - \varepsilon \eta/2)) + i\beta_0} \approx 0,
\end{aligned} \tag{139}$$

as both the second and third lines are of order ε , while the fourth one is of order 1. Summarizing, we have shown that (see [7] for a rigorous derivation)

$$\begin{aligned}
&\frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} e^{i\eta y} \widehat{W}_\varepsilon^2(t, \eta, k) d\eta \\
&= \frac{\gamma^2 |v(k)|^2}{4|\bar{\omega}'(k)|^2} \mathbf{1}_{[[0, \bar{\omega}'(k)t]]}(y) (W(0, y - \bar{\omega}'(k)t, k) + W(0, -y + \bar{\omega}'(k)t, -k))
\end{aligned} \tag{140}$$

and (48) follows.

References

1. G. Basile, C. Bernardin, M. Jara, T. Komorowski, S. Olla, *Thermal conductivity in harmonic lattices with random collisions*, in “Thermal transport in low dimensions: from statistical physics to nanoscale heat transfer”, S. Lepri ed., Lecture Notes in Physics 921, chapter 5, Springer 2016. DOI 10.1007/978-3-319-29261-8-5.
2. G. Basile, T. Komorowski, S. Olla, *Diffusive limits for a kinetic equation with a thermostatted interface*, Kinetic and Related Models, AIMS, (2019), 12(5): 1185-1196, doi: 10.3934/krm.2019045.
3. G. Basile, S. Olla, H. Spohn, *Energy transport in stochastically perturbed lattice dynamics*, Arch. Rat. Mech., Vol. 195, no. 1, 171-203, 2009. DOI 10.1007/s00205-008-0205-6.
4. M. Jara, T. Komorowski, S. Olla, *A limit theorem for an additive functionals of Markov chains*, Annals of Applied Probability, Vol. 19, No. 6, 2270-2300, 2009. DOI: 10.1214/09-AAP610
5. M. Jara, T. Komorowski, S. Olla, *Superdiffusion of Energy in a system of harmonic oscillators with noise*, Commun. Math. Phys., 339, 407-453 (2015), DOI 10.1007/s00220-015-2417-6
6. Gradshteyn, I. S.; Ryzhik, I. M. *Table of integrals, series, and products*. Transl. from the Russian. Translation edited and with a preface by A. Jeffrey and D. Zwillinger. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007.
7. T. Komorowski, S. Olla, L. Ryzhik, H Spohn *High frequency limit for a chain of harmonic oscillators with a point Langevin thermostat.*, Arch. Rational Mech. An., 237, 497-543 (2020). <https://doi.org/10.1007/s00205-020-01513-7>.
8. T. Komorowski, S. Olla, L. Ryzhik, *Fractional Diffusion limit for a kinetic equation with an interface*, Annals of Probability, Vol. 48, No. 5, 2290-2322, (2020), <https://doi.org/10.1214/20-AOP1423>
9. T. Komorowski, S. Olla, *Kinetic limit for a chain of harmonic oscillators with a point Langevin thermostat*, J.Funct.Anal., 279, n.12, 108764, (2020), <https://doi.org/10.1016/j.jfa.2020.108764>
10. T. Komorowski, S. Olla, *In preparation*.
11. Koosis, P., *Introduction to H^p spaces*. Cambridge Univ. Press, 1980.
12. Kwaśnicki, M. *Ten equivalent definitions of the fractional Laplace operator*. Fract. Calc. Appl. Anal. 20 (2017), no. 1, 7-51.
13. Lepri, S., Livi, R., Politi, A.: *Thermal conduction in classical low-dimensional lattices*. Phys. Rep. 377, 1 – 80 (2003)
14. Rieder, Z., Lebowitz, J.L., Lieb, E.: Properties of harmonic crystal in a stationary non- equilibrium state. J. Math. Phys. 8, 1073-1078 (1967)
15. Spohn, H., *The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics*, J. Stat. Phys. 124, no.2-4, 1041-1104 (2006).